Barycentric Subdivision

This note parallels material on pages 119–123 of Hatcher.

Linear simplices Let Y be a convex subspace of $\mathbb{R}^q$. Given points v_0, v_1, …, v_n in Y (not necessarily independent), recall the *linear n-simplex*

$$\lambda = [v_0, v_1, \dots, v_n]\colon \Delta^n \longrightarrow Y, \tag{1}$$

which is the restriction to Δ^n of the linear map $\mathbb{R}^{n+1} \to \mathbb{R}^q$ that sends the standard basis vector e_i to v_i for each i. The *barycenter* or *centroid* of this linear simplex is the point $b(\lambda) = \sum_i v_i/(n+1)$. The linear simplices generate the group $LC_n(Y)$ of *linear chains* on Y.

Cone operator Given a point $y \in Y$, we define the *cone operator* homomorphism $C_y\colon LC_n(Y) \to LC_{n+1}(Y)$ by

$$C_y\lambda = C_y[v_0, v_1, \dots, v_n] = [y, v_0, v_1, \dots, v_n]. \tag{2}$$

Geometrically, we join everything to the point y.

We compute

$$\partial C_y\lambda = [v_0, v_1, \dots, v_n] - \sum_{i=0}^{n}[y, v_0, \dots, \widehat{v_i}, \dots, v_n] = \lambda - C_y\partial\lambda,$$

assuming that $n > 0$. If $n = 0$, this calculation is *not valid* (unless one introduces the empty (-1)-simplex [ ] as in Hatcher); instead we find $\partial C_y[v_0] = \partial[y, v_0] = [v_0] - [y]$. We combine these as

$$\partial C_y\lambda + C_y\partial\lambda = \lambda - r_y\lambda \qquad \text{in } LC_n(Y) \tag{3}$$

for all n, where the chain map $r_y\colon LC(Y) \to LC(Y)$ is given by $r_y\lambda = 0$ for $n > 0$ and $r_y[v_0] = [y]$. Thus C_y is a contracting chain homotopy for $LC(Y)$, which expresses algebraically the contraction of the convex subspace Y to the point y.

Barycentric subdivision We define the barycentric subdivision first on linear simplices, to produce a chain map $S_n\colon LC_n(Y) \to LC_n(Y)$. Geometrically, we proceed by induction; once the faces of λ have been subdivided, we join everything to the barycenter $b(\lambda)$ of λ.

Algebraically, we begin the induction with $S_0 = \mathbf{1}$, and continue with

$$S_n\lambda = C_{b(\lambda)}S_{n-1}\partial\lambda \qquad \text{for } n > 0. \tag{4}$$

(Of course, $S_n = 0$ for $n < 0$.) The form of this definition, with equation (3), implies that S is a chain map. For $n \geq 2$ we compute

$$\partial S_n\lambda = \partial C_{b(\lambda)}S_{n-1}\partial\lambda = S_{n-1}\partial\lambda - C_{b(\lambda)}\partial S_{n-1}\partial\lambda = S_{n-1}\partial\lambda,$$

since by induction $\partial S_{n-1}\partial\lambda = S_{n-2}\partial\partial\lambda = 0$. For $n = 1$, there is an extra term $r_yS_0\partial\lambda = r_y\partial\lambda$, which vanishes.

Chain homotopy We need a chain homotopy T between S and the identity chain map $\mathbf{1}$, i.e. $T_n\colon LC_n(Y) \to LC_{n+1}(Y)$ that satisfies

$$\partial \circ T_n + T_{n-1} \circ \partial = \mathbf{1} - S_n \qquad \text{for all } n. \tag{5}$$

Geometrically (see the picture on page 122), we subdivide the face $\Delta^n \times 0$ of the prism $\Delta^n \times I$, leave the face $\Delta^n \times 1$ alone, and join the barycenter $(b(\lambda), 0)$ to the already subdivided faces $\partial\Delta^n \times I$ of the prism.

Algebraically, we begin the induction with $T_0[v_0] = 0$. (Of course, $T_n = 0$ for $n < 0$. Hatcher uses in effect $T_0[v_0] = [v_0, v_0]$, but both choices satisfy equation (5) for $n = 0$.) We continue with

$$T_n\lambda = C_{b(\lambda)}(\lambda - T_{n-1}\partial\lambda) \qquad \text{for } n > 0. \tag{6}$$

We verify equation (5) for $n > 0$ by using equation (3),

$$\partial T_n\lambda = \partial C_{b(\lambda)}(\lambda - T_{n-1}\lambda) = \lambda - T_{n-1}\partial\lambda - C_{b(\lambda)}\partial\lambda + C_{b(\lambda)}\partial T_{n-1}\partial\lambda.$$

By induction, we have, from equation (5) for $n-1$,

$$C_{b(\lambda)}\partial T_{n-1}\partial\lambda + C_{b(\lambda)}T_{n-2}\partial\partial\lambda = C_{b(\lambda)}\partial\lambda - C_{b(\lambda)}S_{n-1}\partial\lambda.$$

The second term on the left vanishes, and the second term on the right is $S_n\lambda$, by definition.

The following property of S_n and T_n is immediate.

LEMMA 7 *Let $A\colon \mathbb{R}^q \to \mathbb{R}^r$ be a linear map, and Y' a convex subspace of $\mathbb{R}^r$ such that $A(Y) \subset Y'$. Then the chain map $A_\#\colon LC_n(Y) \to LC_n(Y')$ commutes with S_n and T_n, $S_n \circ A_\# = A_\# \circ S_n$ and $T_n \circ A_\# = A_\# \circ T_n$.* □

General singular simplices We extend the definition of S_n and T_n to a singular n-simplex $\sigma\colon \Delta^n \to X$ of any space X by using the chain map $\sigma_\#\colon LC_n(\Delta^n) \subset C_n(\Delta^n) \to C_n(X)$ and noting that $\sigma = \sigma_\#[e_0, e_1, \ldots, e_n]$. For all n, we define

$$S_n\sigma = \sigma_\# S_n[e_0, e_1, \ldots, e_n] \tag{8}$$

and

$$T_n\sigma = \sigma_\# T_n[e_0, e_1, \ldots, e_n]. \tag{9}$$

These are consistent with previous definitions by Lemma 7 if X happens to be a real vector space and σ is a linear map.

To verify that S remains a chain map, we compute

$$\partial S_n\sigma = \partial\sigma_\# S_n[e_0, e_1, \ldots, e_n] = \sigma_\# S_{n-1}\partial[e_0, e_1, \ldots, e_n],$$

since $\sigma_\#$ and the linear version of S are chain maps. On the other side, equation (8) for the face $d_i\sigma = \sigma \circ \eta_i$ gives

$$S_{n-1}d_i\sigma = \sigma_\#\eta_{i\#}S_{n-1}[e_0, e_1, \ldots, e_{n-1}].$$

By Lemma 7, we may rewrite this as

$$\sigma_\# S_{n-1}\eta_{i\#}[e_0, e_1, \ldots, e_{n-1}] = \sigma_\# S_{n-1}[e_0, \ldots, \widehat{e_i}, \ldots, e_n].$$

Now we take alternating sums over i.

A similar proof shows that T continues to satisfy equation (5).