QCQI Chapter 2

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Exercise 1

$$(1,-1)+(1,2)-(2,1)=(0,0)$$

Hence the three vectors are linearly dependent.

Exercise 2

A matrix representation of A is:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where we take $\{|0\rangle, |1\rangle\}$ as the input and output bases.

If we change the output bases to $\{|+\rangle, |-\rangle\}$ (input bases remains), note that $A|0\rangle = |1\rangle = \frac{|+\rangle-|-\rangle}{\sqrt{2}}$ and $A|1\rangle = |0\rangle = \frac{|+\rangle+|-\rangle}{\sqrt{2}}$, there is a different matrix representation:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Exercise 3

For simplicity, take $\{|v_i\rangle\}$, $\{|w_j\rangle\}$ and $\{|x_k\rangle\}$ as the corresponding input and output bases.

Given that

$$\forall i.\ A|v_i\rangle = \sum_j A_{ji}|w_j\rangle$$

and

$$\forall j.\ B|w_j\rangle = \sum_k B_{kj}|x_k\rangle,$$

we have

$$
\begin{aligned}
\forall i.\ BA\,|v_i\rangle &= B\sum_j A_{ji}\,|w_j\rangle \\
&= \sum_j A_{ji}B\,|w_j\rangle \\
&= \sum_j \sum_k A_{ji}B_{kj}\,|x_k\rangle \\
&= \sum_k \sum_j B_{kj}A_{ji}\,|x_k\rangle
\end{aligned}
$$

Hence for the new operator BA, there is a matrix representation denoted as $(BA)_{\mathrm{rank}(X)\times\mathrm{rank}(V)}$, which satisfies:

$$
\forall k, i.\ (BA)_{k,i} = \sum_k \sum_j B_{kj}A_{ji}
$$

i.e., $(BA)_{\mathrm{rank}(X)\times\mathrm{rank}(V)}$ is the matrix product of $B_{\mathrm{rank}(X)\times\mathrm{rank}(W)}$ and $A_{\mathrm{rank}(W)\times\mathrm{rank}(V)}$.

Exercise 4

Let n denotes the rank of V and $\{|v\rangle_i\}$ as a set of bases of V.

Have the identity operator I written in a matrix form as follows:

$$
\forall j = 0, 1, 2, \cdots, n.\ I\,|v_j\rangle = |v_j\rangle = \sum_{i=0,1,\cdots,n} k_{ij}\,|v_i\rangle
$$

Assume that

$$
\exists m \neq n.\ k_{mn} \neq 0,
$$

There are only two possible cases: Either $\{|v_i\rangle\}$ are linearly dependent, or $|v_m\rangle = \vec{0}$,

which leads to contradiction with the fact that $\{|v_i\rangle\}$ forms a set of bases of V.

Therefore we have

$$
\forall i \neq j.\ k_{ij} = 0
$$

and hence

$$
\forall i.k_{ii} = 1
$$

So the one and only matrix representation of I is $\mathrm{diag}(\underbrace{1, 1, 1, \cdots, 1}_{\text{n}})$.

Exercise 5

Let $\vec{x} = (x_1, \cdots, x_n) \in \mathbb{C}^n$, $\vec{y} = (y_1, \cdots, y_n) \in \mathbb{C}^n$ and $\vec{z} = (z_1, \cdots, z_n) \in \mathbb{C}^n$.

Check the requirements of a inner product:

- linear in the second argument:

$$\begin{aligned}
(\vec{x}, \lambda_y \vec{y} + \lambda_z \vec{z}) &= (\vec{x}, (\lambda_y y_1 + \lambda_z z_1, \cdots, \lambda_y y_n + \lambda_z z_n)) \\
&= \sum_i x_i^\star (\lambda_y y_i + \lambda_z z_i) \\
&= \lambda_y \sum_i x_i^\star y_i + \lambda_z \sum_i x_i^\star z_i \\
&= \lambda_y (\vec{x}, \vec{y}) + \lambda_z (\vec{x}, \vec{z})
\end{aligned}$$

- Note that $\forall a, b \in \mathbb{C}.\ (ab)^\star = a^\star b^\star = b^\star a^\star$ and $a^\star + b^\star = (a+b)^\star$,

  then we have:

$$(\vec{x}, \vec{y}) = \sum_i x_i^\star y_i = \sum_i (y_i^\star x_i)^\star = (\sum_i y_i^\star x_i)^\star = (\vec{y}, \vec{x})^\star$$

- Note that $\forall a \in \mathbb{C}.\ a^\star a \geq 0$ with equality if and only if $a = 0$,

  then we have

$$(\vec{x}, \vec{x}) = \sum_i x_i^\star x_i \geq 0,$$

  with equality if and only if all $x_i = 0$, i.e., $\vec{x} = \vec{0}$.

So $(\cdot, \cdot)$ is an inner product on $\mathbb{C}^n$.

Exercise 6

Let $(\cdot, \cdot)$ be inner product from $V \times V$ to $\mathbb{C}$, and $|x\rangle, |y\rangle, |z\rangle \in V$.

Based on the second property (conjugate-symmetry) and apply linearity in the second argument, we have

$$\begin{aligned}
(\lambda_x |x\rangle + \lambda_y |y\rangle, |z\rangle) &= (|z\rangle, \lambda_x |x\rangle + \lambda_y |y\rangle)^\star \\
&= (\lambda_x (|z\rangle, |x\rangle) + \lambda_y (|z\rangle, |y\rangle))^\star \\
&= \lambda_x^\star (|z\rangle, |x\rangle)^\star + \lambda_y^\star (|z\rangle, |y\rangle)^\star \\
&= \lambda_x^\star (|x\rangle, |z\rangle) + \lambda_y^\star (|y\rangle, |z\rangle)
\end{aligned}$$

i.e., an inner product is conjugate-linear in the first argument.

Exercise 7

Let $\{|0\rangle, |1\rangle\}$ denotes the orthonormal bases of the vector representation.

We have

$$\begin{aligned}\langle w|v\rangle &= (\langle 0| + \langle 1|)(|0\rangle - |1\rangle) \\ &= \langle 0|0\rangle - \langle 1|1\rangle + \langle 1|0\rangle - \langle 0|1\rangle \\ &= 1 - 1 + 0 - 0 \\ &= 0\end{aligned}$$

So $|w\rangle$ and $|v\rangle$ are orthogonal.

Their normalized forms are $|w'\rangle = \frac{|w\rangle}{\||w\rangle\|} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $|v'\rangle = \frac{|v\rangle}{\||v\rangle\|} = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.

Exercise 8

Let $\{|w_i\rangle\}$ denotes the original basis and $\{|v_i\rangle\}$ as the basis given by the Gram-Schmidt procedure.

According to definition, it is clear that they are normalized, i.e.,

$$\forall i.\ \langle v_i|v_i\rangle = \|v_i\| = 1$$

Note that the inner product is conjugate-symmetry, we just need to check whether for all $1 \le k \le d - 1$ and $1 \le l \le k$, $|v_{k+1}\rangle$ and $|v_l\rangle$ are orthogonal. And for simplicity we check numerators of the Gram-Schmidt form.

- $|v_2\rangle$ and $|v_1\rangle$ are orthogonal because

$$(\langle w_2| - \langle v_1|w_2\rangle\langle v_1|)\,|v_1\rangle = \langle w_2|v_1\rangle - \langle v_1|w_2\rangle\langle v_1|v_1\rangle = 0$$

- Assume that for all $1 \le i \le k$ $(2 \le k \le d - 1)$, $\{|v_i\rangle\}$ are orthonormal, then we have

$$\begin{aligned}\forall 1 \le i \le k.\ (\langle w_{k+1}| - \sum_{j=1}^{k}\langle v_j|w_{k+1}\rangle\langle v_j|)\,|v_i\rangle &= \langle w_{k+1}|v_i\rangle - \sum_{j=1}^{k}\langle v_j|w_{k+1}\rangle\,\delta_{ij} \\ &= \langle w_{k+1}|v_i\rangle - \langle v_i|w_{k+1}\rangle \\ &= 0\end{aligned}$$

  i.e., $\{|v_i\rangle\} \cup \{|v_{k+1}\rangle\}$ are orthonormal.

By induction we prove that $\{|v_i\rangle\}$ is an an orthonormal basis.

Exercise 9

- $X = |0\rangle\langle 1| + |1\rangle\langle 0|$
- $Y = -i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|$
- $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

Exercise 10

Let A be the matrix representation for the operator $|v_j\rangle\langle v_k|$, we have

$$\forall m \neq j \text{ or } n \neq k.\ \langle v_m| A |v_n\rangle = 0$$

Note that $A|v_n\rangle$ is a linear combination of $\{|v_i\rangle\}$, hence $\langle v_m| A |v_n\rangle = 0$ if and only if $A_{mn} = 0$.

Furthermore,

$$\langle v_j| A |v_k\rangle = A_{jk} \langle v_j|v_j\rangle = A_{jk} = 1$$

So A is a rank$(V) \times$ rank(V) matrix, with $A_{jk} = 1$ and other entries set to 0.

Exercise 11

- $\det|X - \lambda I| = \lambda^2 - 1 = 0 \Longrightarrow \lambda_{1,2} = \pm 1$

  $\lambda_1 = 1$: $(X - I)|x\rangle = 0 \Longrightarrow$ a normalized eigenvector $|x_1\rangle = \frac{\sqrt{2}}{2}(1, 1)$

  $\lambda_2 = -1$: $(X + I)|x\rangle = 0 \Longrightarrow$ a normalized eigenvector $|x_2\rangle = \frac{\sqrt{2}}{2}(1, -1)$

  Hence a diagonal representation is $X = |x_1\rangle\langle x_1| - |x_2\rangle\langle x_2|$
- $\det|Y - \lambda I| = \lambda^2 - 1 = 0 \Longrightarrow \lambda_{1,2} = \pm 1$

  $\lambda_1 = 1$: $(Y - I)|y\rangle = 0 \Longrightarrow$ a normalized eigenvector $|y_1\rangle = \frac{\sqrt{2}}{2}(1, i)$

  $\lambda_2 = -1$: $(Y + I)|y\rangle = 0 \Longrightarrow$ a normalized eigenvector $|y_2\rangle = \frac{\sqrt{2}}{2}(1, -i)$

  Hence a diagonal representation is $Y = |y_1\rangle\langle y_1| - |y_2\rangle\langle y_2|$
- $\det|Z - \lambda I| = \lambda^2 - 1 = 0 \Longrightarrow \lambda_{1,2} = \pm 1$

  $\lambda_1 = 1$: $(Z - I)|z\rangle = 0 \Longrightarrow$ a normalized eigenvector $|z_1\rangle = (1, 0)$

  $\lambda_2 = -1$: $(Z + I)|z\rangle = 0 \Longrightarrow$ a normalized eigenvector $|z_2\rangle = (0, 1)$

  Hence a diagonal representation is $Y = |z_1\rangle\langle z_1| - |z_2\rangle\langle z_2|$

Exercise 12

$$\det(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \lambda I) = (1 - \lambda)^2 = 0$$

We get the only eigenvalue $\lambda = 1$.

Let $(A - I)|x\rangle = 0$, then get an eigenvector $|x\rangle = (0, 1)$ and the rank of the eigenspace w.r.t. $\lambda = 1$ is 1.

Note that

$$k|x\rangle\langle x| = \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Hence the matrix is not diagonalizable.

Exercise 13

Suppose $|w\rangle \in W$ and $|v\rangle \in V$.

Let operator $A = |w\rangle\langle v|$ and $B = |v\rangle\langle w|$. It is not difficult to see that A maps from V to W and B maps from W to V.

$$\forall |i\rangle \in V.\ (A|i\rangle)^\dagger = \langle v|i\rangle^\star \langle w| = \langle i|v\rangle\langle w| = \langle i| A^\dagger = \langle i| B$$

Hence $B = A^\dagger$, i.e., $(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$.

Exercise 14

Suppose A_i maps from V to W.

For all $|v\rangle \in V, |w\rangle \in W$,

$$\begin{aligned}((\sum_i a_i A_i)^\dagger |v\rangle, |w\rangle) &= (|v\rangle, \sum_i a_i A_i |w\rangle) \\ &= \sum_i a_i (|v\rangle, A_i |w\rangle) \\ &= \sum_i a_i (A_i^\dagger |v\rangle, |w\rangle) \\ &= ((\sum_i a_i^\star A_i^\dagger) |v\rangle, |w\rangle)\end{aligned}$$

Hence $(\sum_i a_i A_i)^\dagger = \sum_i a_i^\star A_i^\dagger$.

Exercise 15

Suppose A maps from V to W.

For all $|v\rangle \in V, |w\rangle \in W$,

$$((A^\dagger)^\dagger |v\rangle, |w\rangle) = (|v\rangle, A^\dagger |w\rangle) = (A^\dagger |w\rangle, |v\rangle)^\star = (|w\rangle, A|v\rangle)^\star = (A|v\rangle, |w\rangle)$$

Hence $(A^\dagger)^\dagger = A$.

Exercise 16

Suppose P maps from V to its subspace W:

$$P = \sum_{i=1}^{k} |i\rangle\langle i|,$$

where $\{|i\rangle\}$ is an orthonormal basis for W.

$$\begin{aligned}
\forall |v\rangle \in V, P^2 &= \sum_i |i\rangle\langle i| \sum_j |j\rangle\langle j| \\
&= \sum_{i,j} |i\rangle\langle i|j\rangle\langle j| \\
&= \sum_{i,j} \delta_{ij} |i\rangle\langle j| \\
&= \sum_i |i\rangle\langle i| \\
&= P
\end{aligned}$$

Exercise 17

- $\Longrightarrow$:

  For any Hermitian matrix A, suppose it has an eigenvalue λ and a corresponding eigenvector $|x\rangle$.

  We have

  $$\langle x| A |x\rangle = \langle x| \lambda |x\rangle) = \lambda \langle x|x\rangle$$

  On the other side,

  $$\langle x| A |x\rangle = \langle x| A^\dagger |x\rangle) = \langle x| A |x\rangle^\star = \lambda^\star \langle x|x\rangle$$

  Note that $\langle x|x\rangle > 0$, so $\lambda = \lambda^\star$.i.e., λ is real.

- $\Longleftarrow$:

  Let A be any normal matrix with real eigenvalues. As it is normal, it can be diagonalize as follows:

  $$A = \sum_i \lambda_i |i\rangle\langle i|$$

  where $\{|i\rangle\}$ are eigenvectors corresponding to eigenvalues $\{\lambda_i\}$.

  (Notice that the rank of an eigenspace may be greater than 1, but it does not matter.)

  Given that λ_i is real,

  $$A^\dagger = (A^\star)^T = \sum_i \lambda_i^\star |i\rangle\langle i| = \sum_i \lambda_i |i\rangle\langle i| = A$$

  i.e., A is Hermitian.

Combining the two parts above, a normal matrix is Hermitian if and only if it has real eigenvalues.

Exercise 18

Let U be a unitary matrix, and λ as any of its eigenvalues. Then we have

$$(U|v\rangle, U|w\rangle) = (\lambda|v\rangle, \lambda|w\rangle) = \lambda\lambda^\star(|v\rangle, |w\rangle)$$

Also by definition

$$(U|v\rangle, U|w\rangle) = (|v\rangle, |w\rangle)$$

Hence

$$\lambda\lambda^\star = 1$$

Note that $\|\lambda\|\,\|\lambda^\star\| = 1$ and $\|\lambda\| = \|\lambda^\star\| \geq 0$, so $\|\lambda\| = 1$.

In conclusion, all eigenvalues of U have modulus 1.

Exercise 19

- I: It is clear that I is Hermitian and unitary.
- X:

$$X^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X$$

$$X^\dagger X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

- Y:

$$Y^\dagger = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y$$

$$Y^\dagger Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

- Z:

$$Z^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z$$

$$Z^\dagger Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Exercise 20

Let $U = \sum_i |w_i\rangle\langle v_i|$.

Note that $|v_i\rangle$ and $|w_i\rangle$ are both orthonormal bases, hence the operator U is unitary.

$$\begin{aligned}
A'_{ij} = \langle v_i| A |v_j\rangle = \langle v_i| UU^\dagger AUU^\dagger |v_j\rangle \\
&= \sum_k \langle v_i|w_k\rangle \langle v_k| U^\dagger AU \sum_l \langle w_l|v_j\rangle |v_l\rangle \\
&= \sum_k \langle v_i|w_k\rangle \langle v_k| \sum_m |v_m\rangle \langle w_m| A \sum_n |w_n\rangle \langle v_n| \sum_l \langle w_l|v_j\rangle |v_l\rangle \\
&= \sum_{k,l} \langle v_i|w_k\rangle \langle w_l|v_j\rangle \sum_{m,n} \delta_{km} A''_{mn} \delta_{nl} \\
&= \sum_{k,l} \langle v_i|w_k\rangle \langle w_l|v_j\rangle A''_{kl}
\end{aligned}$$

which characterizes the relationship between A' and A''.

Exercise 21

- $\Rightarrow$: We also prove by induction on the dimension d of V. The case $d = 1$ is also trivial.

  Let λ be an eigenvalue of M, P the projector onto the λ eigenspace, and Q the projector onto the orthogonal complement. Then

  $$M = (P + Q)M(P + Q) = PMP + QMP + PMQ + QMQ$$

  Note that $QMP = 0$ as M takes the subspace P into itself. Also we have

  $$PMQ = (QM^\dagger P)^\star = (QMP)^\star = 0$$

  Next we prove that QMQ is Hermitian:

  $$(QMQ)^\dagger = QM^\dagger Q = QMQ$$

  By induction, QMQ is diagonal w.r.t. some orthonormal basis for the subspace Q, and PMP is already diagonal w.r.t. some orthonormal basis for the λ eigenspace.

  It follows that $M = PMP + QMQ$ is also diagonal, w.r.t. some orthonormal basis for V.

- $\Leftarrow$: It turns out that the statement does not hold conversely, in the Hermitian case.

Exercise 22

Let A by any Hermitian operator with eigenvalues λ_1 and λ_2, and corresponding eigenvectors $|\lambda_1\rangle$ and $|\lambda_2\rangle$. (λ_1 and λ_2 are real as A is Hermitian.)

Note that $A|\lambda_2\rangle = \lambda_2 |\lambda_2\rangle$ and $\langle\lambda_1| A^\dagger = \lambda_1^\star \langle\lambda_1| = \lambda_1 \langle\lambda_1|$,

$$\langle\lambda_1| A |\lambda_2\rangle = \lambda_2 \langle\lambda_1|\lambda_2\rangle$$

Also we have

$$\langle\lambda_1| A |\lambda_2\rangle = \langle\lambda_1| A^\dagger |\lambda_2\rangle = \lambda_1 \langle\lambda_1|\lambda_2\rangle$$

Combine the two equations above we get

$$(\lambda_1 - \lambda_2) \langle\lambda_1|\lambda_2| = 0$$

As $\lambda_1 \neq \lambda_2$, $|\lambda_1\rangle$ and $|\lambda_2\rangle$ are necessarily orthogonal.

Exercise 23

For simplicity we use P to denote both the projector and the associated subspace.

Let V be the total vector space and Q the orthogonal complement of P.

For any non-zero vector $|v\rangle \in V$, it could be written in the following form:

$$|v\rangle = |v_P\rangle + |v_Q\rangle$$

where $|v_P\rangle \in P$ and $|v_Q\rangle \in Q$.

If $P|v\rangle = \lambda|v\rangle$ holds for some λ, we have

$$\begin{aligned} P|v\rangle &= P|v_P\rangle + P|v_Q\rangle \\ &= |v_P\rangle \\ &= \lambda|v_P\rangle + \lambda|v_Q\rangle \end{aligned}$$

Hence

$$(\lambda - 1)|v_P\rangle + \lambda|v_Q\rangle = \vec{0}$$

From exercise 2.22 we know that $|v_P\rangle$ and $|v_Q\rangle$ are orthonormal and they cannot be zero vectors at the same time, so there are only two possible cases:

- $\lambda = 0$: In this case $|v\rangle \in Q$
- $\lambda = 1$: In this case $|v\rangle \in P$

In conclusion, the eigenvalues of P are all either 0 or 1.

Exercise 24

For any positive operator A, we have

$$A = \frac{A + A^\dagger}{2} + i\frac{A - A^\dagger}{2i}$$

Let $B = \frac{A+A^\dagger}{2}$ and $C = \frac{A-A^\dagger}{2i}$, and it is clear that they are both Hermitian. Hence we have

$$\forall |v\rangle.\ \langle v|A|v\rangle = \langle v|B + iC|v\rangle = \langle v|B|v\rangle + i\langle v|C|v\rangle \geq 0$$

Note that B and C are Hermitian, $\langle v|B|v\rangle$ and $\langle v|C|v\rangle$ should both be real. Since the equation above holds for all $|v\rangle$, C must be a zero operator. Therefore

$$A = A^\dagger$$

i.e., A is Hermitian.

Exercise 25

For any $|v\rangle$, suppose A maps from $|v\rangle$ to $|w\rangle$.

We have

$$(|v\rangle, A^\dagger A|v\rangle) = \langle v|A^\dagger A|v\rangle = \langle w|w\rangle \geq 0$$

By definition $A^\dagger A$ is positive.

Exercise 26

- $$|\psi\rangle^{\otimes 2} = (\frac{|0\rangle + |1\rangle}{\sqrt{2}}) \otimes (\frac{|0\rangle + |1\rangle}{\sqrt{2}}) = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$
  $$|\psi\rangle^{\otimes 2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
- $$\begin{aligned} |\psi\rangle^{\otimes 3} =& (\frac{|0\rangle + |1\rangle}{\sqrt{2}}) \otimes (\frac{|0\rangle + |1\rangle}{\sqrt{2}}) \otimes (\frac{|0\rangle + |1\rangle}{\sqrt{2}}) \\ =& \frac{\sqrt{2}}{4}|000\rangle + \frac{\sqrt{2}}{4}|001\rangle + \frac{\sqrt{2}}{4}|010\rangle + \frac{\sqrt{2}}{4}|011\rangle + \frac{\sqrt{2}}{4}|100\rangle + \frac{\sqrt{2}}{4}|101\rangle \\ &+ \frac{\sqrt{2}}{4}|110\rangle + \frac{\sqrt{2}}{4}|111\rangle \end{aligned}$$
  $$|\psi\rangle^{\otimes 3} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{\sqrt{2}}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Exercise 27

- $$X \otimes Z = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$
- $$I \otimes X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
- $$X \otimes I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Note that $I \otimes X \neq X \otimes I$, hence the tensor product is not commutative, in general.

Exercise 28

- Distribution of transposition:

$$(A \otimes B)^{\mathrm{T}} = \begin{bmatrix} A_{11}B^{\mathrm{T}} & A_{21}B^{\mathrm{T}} & \cdots & A_{m1}B^{\mathrm{T}} \\ A_{12}B^{\mathrm{T}} & A_{22}B^{\mathrm{T}} & \cdots & A_{m2}B^{\mathrm{T}} \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n}B^{\mathrm{T}} & A_{2n}B^{\mathrm{T}} & \cdots & A_{mn}B^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{m1} \\ A_{12} & A_{22} & \cdots & A_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{mn} \end{bmatrix} \otimes B^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}$$

- Distribution of complex conjugation:

$$(A \otimes B)^{\star} = \begin{bmatrix} A_{11}^{\star}B^{\star} & A_{12}^{\star}B^{\star} & \cdots & A_{1n}^{\star}B^{\star} \\ A_{21}^{\star}B^{\star} & A_{22}^{\star}B^{\star} & \cdots & A_{2n}^{\star}B^{\star} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}^{\star}B^{\star} & A_{m2}B^{\star} & \cdots & A_{mn}^{\star}B^{\star} \end{bmatrix} = \begin{bmatrix} A_{11}^{\star} & A_{12}^{\star} & \cdots & A_{1n}^{\star} \\ A_{21}^{\star} & A_{22}^{\star} & \cdots & A_{2n}^{\star} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}^{\star} & A_{2n}^{\star} & \cdots & A_{mn}^{\star} \end{bmatrix} \otimes B^{\star} = A^{\star} \otimes B^{\star}$$

- Distribution of adjoint operation:

  Based on the two distribution laws above, we have

$$(A \otimes B)^{\dagger} = ((A \otimes B)^{\star})^{\mathrm{T}} = ((A^{\star} \otimes B^{\star})^{\mathrm{T}} = A^{\dagger} \otimes B^{\dagger}$$

Exercise 29

Let $U_1 : V \longrightarrow V'$ be a unitary operator and $U_2 : W \longrightarrow W'$ as another unitary operator.

Based on exercise 2.28 we know that $(U_1 \otimes U_2)^{\dagger} = U_1^{\dagger} \otimes U_2^{\dagger}$. So

$$(U_1 \otimes U_2)^{\dagger}(U_1 \otimes U_2) = U_1^{\dagger}U_1 \otimes U_2^{\dagger}U_2 = I_1 \otimes I_2 = I_{V \otimes W}$$

Similarly we have $(U_1 \otimes U_2)(U_1 \otimes U_2)^{\dagger} = I_{V' \otimes W'}$, i.e., $U_1 \otimes U_2$ is also unitary.

Exercise 30

Let A be a Hermitian operator on V and B as another Hermitian operator on W.

Based on exercise 2.28 we know that

$$(A \otimes B)^{\dagger} = A^{\dagger} \otimes B^{\dagger} = A \otimes B$$

i.e., $A \otimes B$ is also Hermitian.

Exercise 31

Let A be a positive operator on V and B as another positive operator on W.

Note that $|v_i\rangle \otimes |w_j\rangle$ forms an orthonormal basis for $V \otimes W$, hence for any $|u\rangle$ in $V \otimes W$, it is a linear combination of $\{|v_i\rangle \otimes |w_j\rangle\}$:

$$|u\rangle = \sum_{k,l} a_{kl} |v_k\rangle \otimes |w_l\rangle$$

Consider the natural inner product on $V \otimes W$ and apply linearity:

$$\begin{aligned}(|u\rangle, (A \otimes B)|u\rangle) &= (\sum_{k,l} a_{kl}|v_k\rangle \otimes |w_l\rangle, \sum_{k,l} a_{kl}A|v_k\rangle \otimes B|w_l\rangle) \\ &= \sum_{k,l} \|a_{kl}\|^2 \langle v_k| A |v_k\rangle \langle w_l| B |w_l\rangle \\ &\geq 0\end{aligned}$$

Hence $A \otimes B$ is also positive.

Exercise 32

Let $P_V = \sum_i |v_i\rangle\langle v_i|$ be a projector on V and $P_W = \sum_j |w_j\rangle\langle w_j|$ as another projector on W. For simplicity we have P_V and P_W also denote the corresponding subspace.

It is clear that

$$P_V \otimes P_W = \sum_{i,j} |v_i\rangle \otimes |w_j\rangle\langle v_i| \otimes \langle w_j|$$

Note that $|v_i\rangle \otimes |w_j\rangle$ forms a orthonormal basis for $V \otimes W$, therefore $P_V \otimes P_W$ is also a projector on $V \otimes W$, w.r.t. the subspace $P_V \otimes P_W$.

Exercise 33

We prove the statement inductively.

- When $n = 1$, it holds that

$$H^{\otimes 1} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) = \frac{1}{\sqrt{2}} \sum_{x_1, y_1 = 0,1} (-1)^{x \cdot y} |x_1\rangle\langle y_1|)$$

- Assume that for any $n \geq 1$,

$$H^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x_n, y_n = 0,1,\ldots,2^n - 1} (-1)^{x_n \cdot y_n} |x_n\rangle\langle y_n|$$

  Note that every time we tensor with H, the basis changes from $|x_n\rangle$ to $|x_{n+1}\rangle$, with $|i\rangle \longleftarrow |i\rangle \otimes |0\rangle$ and $|i + 2^n\rangle \longleftarrow |i\rangle \otimes |1\rangle$ for all $0 \leq i < 2^n$.

  Hence we have

$$H^{\otimes n+1} = H^{\otimes n} \otimes H = \frac{1}{\sqrt{2^{n+1}}} \sum_{x_{n+1}, y_{n+1} = 0,1,\ldots,2^{n+1}-1} (-1)^{x_{n+1} \cdot y_{n+1}} |x_{n+1}\rangle\langle y_{n+1}|$$

By induction we can prove that the equation of $H^{\otimes n}$ always holds.

The matrix representation for $H^{\otimes 2}$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Exercise 34

Let $A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$.

$$\det(A - \lambda I) = 0 \implies \lambda_1 = 1, \lambda_2 = 7$$

The corresponding normalized eigenvectors are

$$|\lambda_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$|\lambda_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then

$$A = |\lambda_1\rangle\langle\lambda_1| + 7\,|\lambda_2\rangle\langle\lambda_2|$$

The square root of A is

$$\sqrt{A} = |\lambda_1\rangle\langle\lambda_1| + \sqrt{7}\,|\lambda_2\rangle\langle\lambda_2| = \frac{1}{2} \begin{bmatrix} 1 + \sqrt{7} & -1 + \sqrt{7} \\ -1 + \sqrt{7} & 1 + \sqrt{7} \end{bmatrix}$$

The logarithm of A is

$$\log(A) = \log(0)\,|\lambda_1\rangle\langle\lambda_1| + \log(7)\,|\lambda_2\rangle\langle\lambda_2| = \frac{\log(7)}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Exercise 35

By definition we get

$$\vec{v} \cdot \vec{\sigma} = \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix}$$

Note that $\vec{v}$ is a unit vector,

$$\begin{aligned} \det(\vec{v} \cdot \vec{\sigma} - \lambda I) = 0 &\implies \lambda^2 = v_1^2 + v_2^2 + v_3^2 = 1 \\ &\implies \lambda_{1,2} = \pm 1 \end{aligned}$$

Note that $\vec{v} \cdot \vec{\sigma}$ is Hermitian, hence it could be diagonalized as follows:

$$\vec{v} \cdot \vec{\sigma} = |\lambda_1\rangle\langle\lambda_1| - |\lambda_2\rangle\langle\lambda_2|$$

where $|\lambda_{1,2}\rangle$ are the corresponding normalized eigenvectors. (Do not need to explicitly solve them since we already know that $\vec{v} \cdot \vec{\sigma}$ is diagonalizable.)

Furthermore,

$$|\lambda_1\rangle\langle\lambda_1| - |\lambda_2\rangle\langle\lambda_2| = I$$

Based on the definition of operator functions and applying Euler's equation, we have

$$\begin{aligned} \exp(i\theta\vec{v} \cdot \vec{\sigma}) &= e^{i\theta}\,|\lambda_1\rangle\langle\lambda_1| + e^{-i\theta}\,|\lambda_2\rangle\langle\lambda_2| \\ &= (\cos\theta + i\sin\theta)\,|\lambda_1\rangle\langle\lambda_1| + (\cos\theta - i\sin\theta)\,|\lambda_2\rangle\langle\lambda_2| \\ &= \cos\theta(|\lambda_1\rangle\langle\lambda_1| + |\lambda_2\rangle\langle\lambda_2|) + i\sin\theta(|\lambda_1\rangle\langle\lambda_1| - |\lambda_2\rangle\langle\lambda_2|) \\ &= \cos\theta I + i\sin\theta\vec{v} \cdot \vec{\sigma} \end{aligned}$$

Exercise 36

- $\mathrm{tr}(X) = 0$
- $\mathrm{tr}(Y) = 0$
- $\mathrm{tr}(Z) = 1 - 1 = 0$

Exercise 37

Suppose both A and B act on V which has a orthonormal basis $|v_i\rangle$.

$$\begin{aligned}
\mathrm{tr}(AB) &= \sum_i \langle v_i| AB |v_i\rangle \\
&= \sum_i \langle v_i| AIB |v_i\rangle \\
&= \sum_{i,j} \langle v_i| A |v_j\rangle \langle v_j| B |v_i\rangle \\
&= \sum_{i,j} \langle v_j| B |v_i\rangle \langle v_i| A |v_j\rangle \\
&= \sum_j \langle v_j| BA |v_j\rangle \\
&= \mathrm{tr}(BA)
\end{aligned}$$

Exercise 38

Suppose both A and B act on V which has an orthonormal basis $|v_i\rangle$.

$$\begin{aligned}
\mathrm{tr}(A + B) &= \sum_i \langle v_i| A + B |v_i\rangle \\
&= \sum_i \langle v_i| A |v_i\rangle + \sum_i \langle v_i| B |v_i\rangle \\
&= \mathrm{tr}(A) + \mathrm{tr}(B)
\end{aligned}$$

Note that an inner product is linear in its second argument,

$$\begin{aligned}
\mathrm{tr}(zA) &= \sum_i \langle v_i| zA |v_i\rangle \\
&= z \sum_i \langle v_i| A |v_i\rangle \\
&= z\mathrm{tr}(A)
\end{aligned}$$

Exercise 39

39.1

Let A, B, C be operators in L_V.

- Based on exercise 2.38 we have

$$\begin{aligned}(A, z_b B + z_c C) &= \mathrm{tr}(A^\dagger(z_b B + z_c C)) \\ &= \mathrm{tr}(z_b A^\dagger B) + \mathrm{tr}(z_c A^\dagger C) \\ &= z_b \mathrm{tr}(A^\dagger B) + z_c \mathrm{tr}(A^\dagger C) \\ &= z_b(A, B) + z_c(A, C)\end{aligned}$$

  where z_b and z_c are arbitary complex numbers.

  i.e., $(\cdot, \cdot)$ is linear in its second argument.

- Conjugate-symmetry:

$$(A, B) = \mathrm{tr}(A^\dagger B) = \sum_i \langle i| A^\dagger B |i\rangle = (\sum_i \langle i| B^\dagger A |i\rangle)^\star = (B, A)^\star$$

-

$$\begin{aligned}(A, A) = \mathrm{tr}(A^\dagger A) &= \sum_i \langle i| A^\dagger A |i\rangle \\ &= \sum_i \langle i| A^\dagger I A |i\rangle \\ &= \sum_i \langle i| A^\dagger (\sum_j |j\rangle \langle j|) A |i\rangle \\ &= \sum_{i,j} \langle i| A^\dagger |j\rangle \langle j| A |i\rangle \\ &= \sum_{i,j} \langle j| A |i\rangle^\star \langle j| A |i\rangle \\ &= \sum_i \|\langle i| A |i\rangle\|^2 \\ &\geq 0\end{aligned}$$

  with equality if and only if A is a zero operator.

39.2

Let $|i\rangle$ be an orthonormal basis for V.

It is clear that $|i\rangle \langle j|$ are linearly independent. And for any operator in L_V, it could be written in the outer product form, i.e., linear combination of $|i\rangle \langle j|$.

Hence $|i\rangle \langle j|$ forms a basis for L_V. So

$$\mathrm{rank}(L_V) = \|\{|i\rangle \langle j|\}\| = d^2$$

39.3

Note that the basis given above is countable, we get a set of basis namely $|l_k\rangle$, where

$$|l_k\rangle \equiv |i\rangle \langle j| \quad (i = \lfloor k/d \rfloor - 1,\ j = k \bmod n)$$

Apply the Gram–Schmidt procedure:

- Define $|l'_1\rangle = \frac{|l_1\rangle}{\||l_1\rangle\|}$
- For $1 \le k \le d^2 - 1$ define $|l'_{k+1}\rangle$ inductively by

$$|l'_{k+1}\rangle = \frac{|l_{k+1}\rangle - \sum_{i=1}^{k} \langle l'_i|l_{k+1}\rangle |l'_i\rangle}{\| |l_{k+1}\rangle - \sum_{i=1}^{k} \langle l'_i|l_{k+1}\rangle |l'_i\rangle \|}$$

The vectors $|l'_1\rangle, |l'_2\rangle, \dots, |l'_{d^2}\rangle$ form an orthonormal basis for L_V.

Exercise 40

- $$[X, Y] = XY - YX = 2\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = 2iZ$$
- $$[Y, Z] = YZ - ZY = 2\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = 2iX$$
- $$[Z, X] = ZX - XZ = 2\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = 2iY$$

Exercise 41

Exercise 42

$$\frac{[A, B] + \{A, B\}}{2} = \frac{AB - BA + AB + BA}{2} = AB$$

Exercise 43

Based on exercise 2.41, we have

$$\{\sigma_j, \sigma_k\} = 2\delta_{jk} I$$

Hence

$$\begin{aligned} \sigma_j \sigma_k &= \frac{\{\sigma_j, \sigma_k\} + [\sigma_j, \sigma_k]}{2} \\ &= \frac{2\delta_{jk} I + 2i \sum_{l=1}^{3} \varepsilon_{jkl} \sigma_l}{2} \\ &= \delta_{jk} I + i \sum_{l=1}^{3} \varepsilon_{jkl} \sigma_l \end{aligned}$$

Exercise 44

Given that $[A, B] = 0$ and $\{A, B\} = 0$, we have

$$AB = BA$$

$$AB = -BA$$

Multiply by A^{-1}, we get

$$ABA^{-1} = BAA^{-1} = B$$

$$ABA^{-1} = -BAA^{-1} = -B$$

Note that $B = -B$, so B must be 0.

Exercise 45

$$[A, B]^\dagger = (AB - BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = [B^\dagger, A^\dagger]$$

Exercise 46

$$[A, B] = AB - BA = -(BA - AB) = -[B, A]$$

Exercise 47

$$i[A, B] = i(AB - BA)$$

Given that $A = A^\dagger$ and $B = B^\dagger$ as they are Hermitian,

$$(i[A, B])^\dagger = -i[A, B]^\dagger = -i(B^\dagger A^\dagger - A^\dagger B^\dagger) = i(BA - AB) = i[A, B]$$

Hence $i[A, B]$ is also Hermitian.

Exercise 48

Exercise 49

Exercise 50

Exercise 51

$$H^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H$$

$$H^\dagger H = HH^\dagger = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I$$

Hence H is unitary.

Exercise 52

Note that $H = H^\dagger$, so

$$H^2 = HH^\dagger = I$$

Exercise 53

$$\det(H - \lambda I) = 0 \Longrightarrow \lambda_{1,2} = \pm\frac{\sqrt{5}}{2}$$

Exercise 54

Note that A and B commute, so they could be diagonalized w.r.t. an orthonormal basis:

$$A = \sum_i a_i |i\rangle\langle i|$$

$$B = \sum_i b_i |i\rangle\langle i|$$

By definition of operator functions, we have

$$\begin{aligned}
\exp(A)\exp(B) &= \sum_i \exp(a_i)|i\rangle\langle i| \sum_j \exp(b_j)|j\rangle\langle j| \\
&= \sum_{i,j} \exp(a_i + b_j)|i\rangle \delta_{ij} \langle j| \\
&= \sum_i \exp(a_i + b_i)|i\rangle\langle i| \\
&= \exp(A + B)
\end{aligned}$$

Exercise 55

Note that the Hamiltonian is Hermitian and hence diagonalizable,

$$H = \sum_E E|E\rangle\langle E|$$

where E are real eigenvalues.

We get

$$U(t_1, t_2) = \exp[\frac{-iH(t_2 - t_1)}{\hbar}] = \sum_E \exp[\frac{-iE(t_2 - t_1)}{\hbar}]|E\rangle\langle E|$$

and

$$U^\dagger(t_1, t_2) = \sum_E \exp[\frac{iE(t_2 - t_1)}{\hbar}]|E\rangle\langle E|$$

Therefore

$$U(t_1, t_2)U^\dagger(t_1, t_2) = \sum_{E_1, E_2} \exp[\frac{iE_2(t_2 - t_1) - iE_1(t_2 - t_1)}{\hbar}]|E_1\rangle \delta_{E_1,E_2} \langle E_2| = \sum_E |E\rangle\langle E| = I$$

and similarly $U^\dagger(t_1, t_2)U(t_1, t_2) = I$. i.e., $U(t_1, t_2)$ is unitary.

Exercise 56

- $\Longrightarrow$:

  Note that U is normal ($UU^\dagger = U^\dagger U = I$) and hence diagonalizable,

  $$U = \sum_E E \left|E\right\rangle \left\langle E\right|$$

  where all eigenvalues E (not necessarily real) have modulus 1 and can be written in the following form:

  $$E = \exp(i\theta_E)$$

  for some real θ_E.

  We have

  $$K = -i\log(U) = \sum_E -i\log(E) \left|E\right\rangle \left\langle E\right| = \sum_E \theta_E \left|E\right\rangle \left\langle E\right|$$

  $$K^\dagger = \sum_E \theta_E \left|E\right\rangle \left\langle E\right| = K$$

  Therefore $K \equiv -i\log(U)$ is Hermitian for any unitary U.

- $\Longleftarrow$:

  Note that K is Hermitian and hence diagonalizable,

  $$K = \sum_E E \left|E\right\rangle \left\langle E\right|$$

  where all eigenvalues E are real numbers and eigenvectors $\left|E\right\rangle$ are normalized.

  We have

  $$U = \exp(iK) = \sum_E \exp(iE) \left|E\right\rangle \left\langle E\right|$$

  $$U^\dagger = \sum_E \exp(-iE) \left|E\right\rangle \left\langle E\right|$$

  $$U^\dagger U = UU^\dagger = \sum_E \exp(iE - iE) \left|E\right\rangle \left\langle E\right| = \sum_E \left|E\right\rangle \left\langle E\right| = I$$

  Therefore $U \equiv \exp(iK)$ is unitary for any Hermitian K.