QCQI Chapter 2

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Exercise 1

The exercises are automatically numbered, starting from one. Packages such as `amsmath` and `hyperref` are included by default.

Paragraphs are not indented, but are instead separated by some vertical space.

As an example: the *standard inner product* on $\mathbb{R}^n$ is defined as

$$\vec{a} \cdot \vec{b} := x_1 y_1 + \cdots + x_n y_n \qquad \text{for } \vec{a}, \vec{b} \in \mathbb{R}^n.$$

Note that `*` can be used instead of `\cdot`, and `\R` instead of `\mathbb{R}`. (For a normal asterisk, use `\ast`.) Of course, there are macros for the natural numbers etc. too. Commands such as `\abs{}` and `\Set{}` can be used to easily create (scaled) delimiters. For example,

$$\left|\frac{1}{1-\lambda h}\right| \leq 1 \qquad \text{and} \qquad \left\{ x \in \mathbb{R} \;\middle|\; 1 < \sqrt{x^3+2} < \frac{3}{2} \right\}.$$

The starred version of these commands disables the auto-scaling.

Exercise 2

A matrix representation of A is:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where we take $\{|0\rangle, |1\rangle\}$ as the input and output bases.

If we change the output bases to $\{|+\rangle, |-\rangle\}$ (input bases remains), note that $A|0\rangle = |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$ and $A|1\rangle = |0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$, there is a different matrix representation:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Exercise 3

For simplicity, take $\{|v_i\rangle\}$, $\{|w_j\rangle\}$ and $\{|x_k\rangle\}$ as the corresponding input and output bases.

Given that

$$\forall i.\ A\,|v_i\rangle = \sum_j A_{ji}\,|w_j\rangle$$

and

$$\forall j.\ B\,|w_j\rangle = \sum_k B_{kj}\,|x_k\rangle\,,$$

we have

$$\begin{aligned}\forall i.\ BA\,|v_i\rangle &= B\sum_j A_{ji}\,|w_j\rangle \\ &= \sum_j A_{ji}B\,|w_j\rangle \\ &= \sum_j\sum_k A_{ji}B_{kj}\,|x_k\rangle \\ &= \sum_k\sum_j B_{kj}A_{ji}\,|x_k\rangle\end{aligned}$$

Hence for the new operator BA, there is a matrix representation denoted as $(BA)_{\mathrm{rank}(X)\times\mathrm{rank}(V)}$, which satisfies:

$$\forall k, i.\ (BA)_{k,i} = \sum_k\sum_j B_{kj}A_{ji}$$

i.e., $(BA)_{\mathrm{rank}(X)\times\mathrm{rank}(V)}$ is the matrix product of $B_{\mathrm{rank}(X)\times\mathrm{rank}(W)}$ and $A_{\mathrm{rank}(W)\times\mathrm{rank}(V)}$.

Exercise 4

Let n denotes the rank of V and $\{|v\rangle_i\}$ as a set of bases of V.

Have the identity operator I written in a matrix form as follows:

$$\forall j = 0, 1, 2, \cdots, n.\ I\,|v_j\rangle = |v_j\rangle = \sum_{i=0,1,\cdots,n} k_{ij}\,|v_i\rangle$$

Assume that

$$\exists m \neq n.\ k_{mn} \neq 0,$$

There are only two possible cases: Either $\{|v_i\rangle\}$ are linearly dependent, or $|v_m\rangle = \vec{0}$, which leads to contradiction with the fact that $\{|v_i\rangle\}$ forms a set of bases of V.

Therefore we have

$$\forall i \neq j.\ k_{ij} = 0$$

and hence

$$\forall i. k_{ii} = 1$$

So the one and only matrix representation of I is $\mathrm{diag}(\underbrace{1, 1, 1, \cdots, 1}_{\mathrm{n}})$.

Exercise 5

Let $\vec{x} = (x_1, \cdots, x_n) \in \mathbb{C}^n$, $\vec{y} = (y_1, \cdots, y_n) \in \mathbb{C}^n$ and $\vec{z} = (z_1, \cdots, z_n) \in \mathbb{C}^n$.

Check the requirements of a inner product:

- linear in the second argument:

$$\begin{aligned}
(\vec{x}, \lambda_y \vec{y} + \lambda_z \vec{z}) &= (\vec{x}, (\lambda_y y_1 + \lambda_z z_1, \cdots, \lambda_y y_n + \lambda_z z_n)) \\
&= \sum_i x_i^\star (\lambda_y y_i + \lambda_z z_i) \\
&= \lambda_y \sum_i x_i^\star y_i + \lambda_z \sum_i x_i^\star z_i \\
&= \lambda_y (\vec{x}, \vec{y}) + \lambda_z (\vec{x}, \vec{z})
\end{aligned}$$

- Note that $\forall a, b \in \mathbb{C}.\ (ab)^\star = a^\star b^\star = b^\star a^\star$ and $a^\star + b^\star = (a + b)^\star$,

  then we have:

$$(\vec{x}, \vec{y}) = \sum_i x_i^\star y_i = \sum_i (y_i^\star x_i)^\star = (\sum_i y_i^\star x_i)^\star = (\vec{y}, \vec{x})^\star$$

- Note that $\forall a \in \mathbb{C}.\ a^\star a \geq 0$ with equality if and only if $a = 0$,

  then we have

$$(\vec{x}, \vec{x}) = \sum_i x_i^\star x_i \geq 0,$$

  with equality if and only if all $x_i = 0$, i.e., $\vec{x} = \vec{0}$.

So $(\cdot, \cdot)$ is an inner product on $\mathbb{C}^n$.

Exercise 6

Let $(\cdot, \cdot)$ be inner product from $V \times V$ to $\mathbb{C}$, and $|x\rangle, |y\rangle, |z\rangle \in V$.

Based on the second property (conjugate-symmetry) and apply linearity in the second argument, we have

$$\begin{aligned}(\lambda_x |x\rangle + \lambda_y |y\rangle, |z\rangle) &= (|z\rangle, \lambda_x |x\rangle + \lambda_y |y\rangle)^\star \\ &= (\lambda_x(|z\rangle, |x\rangle) + \lambda_y(|z\rangle, |y\rangle))^\star \\ &= \lambda_x^\star(|z\rangle, |x\rangle)^\star + \lambda_y^\star(|z\rangle, |y\rangle)^\star \\ &= \lambda_x^\star(|x\rangle, |z\rangle) + \lambda_y^\star(|y\rangle, |z\rangle)\end{aligned}$$

i.e., an inner product is conjugate-linear in the first argument.

Exercise 7

Let $\{|0\rangle, |1\rangle\}$ denotes the orthonormal bases of the vector representation.

We have

$$\begin{aligned}\langle w|v\rangle &= (\langle 0| + \langle 1|)(|0\rangle - |1\rangle) \\ &= \langle 0|0\rangle - \langle 1|1\rangle + \langle 1|0\rangle - \langle 0|1\rangle \\ &= 1 - 1 + 0 - 0 \\ &= 0\end{aligned}$$

So $|w\rangle$ and $|v\rangle$ are orthogonal.

Their normalized forms are $|w'\rangle = \frac{|w\rangle}{\||w\rangle\|} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $|v'\rangle = \frac{|v\rangle}{\||v\rangle\|} = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.

Exercise 8

Let $\{|w_i\rangle\}$ denotes the original basis and $\{|v_i\rangle\}$ as the basis given by the Gram-Schmidt procedure.

According to definition, it is clear that they are normalized, i.e.,

$$\forall i.\ \langle v_i|v_i\rangle = \|v_i\| = 1$$

Note that the inner product is conjugate-symmetry, we just need to check whether for all $1 \le k \le d-1$ and $1 \le l \le k$, $|v_{k+1}\rangle$ and $|v_l\rangle$ are orthogonal. And for simplicity we check numerators of the Gram-Schmidt form.

- $|v_2\rangle$ and $|v_1\rangle$ are orthogonal because

$$(\langle w_2| - \langle v_1|w_2\rangle\langle v_1|)\,|v_1\rangle = \langle w_2|v_1\rangle - \langle v_1|w_2\rangle\langle v_1|v_1\rangle = 0$$

- Assume that for all $1 \le i \le k$ ($2 \le k \le d-1$), $\{|v_i\rangle\}$ are orthonormal, then we have

$$\begin{aligned}\forall 1 \le i \le k.\ (\langle w_{k+1}| - \sum_{j=1}^{k}\langle v_j|w_{k+1}\rangle\langle v_j|)\,|v_i\rangle &= \langle w_{k+1}|v_i\rangle - \sum_{j=1}^{k}\langle v_j|w_{k+1}\rangle\,\delta_{ij} \\ &= \langle w_{k+1}|v_i\rangle - \langle v_i|w_{k+1}\rangle \\ &= 0\end{aligned}$$

i.e., $\{|v_i\rangle\} \cup \{|v_{k+1}\rangle\}$ are orthonormal.

By induction we prove that $\{|v_i\rangle\}$ is an an orthonormal basis.

Exercise 9

- $X = |0\rangle\langle 1| + |1\rangle\langle 0|$
- $Y = -i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|$
- $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

Exercise 10

Let A be the matrix representation for the operator $|v_j\rangle\langle v_k|$, we have

$$\forall m \neq j \text{ or } n \neq k.\ \langle v_m| A |v_n\rangle = 0$$

Note that $A|v_n\rangle$ is a linear combination of $\{|v_i\rangle\}$, hence $\langle v_m| A |v_n\rangle = 0$ if and only if $A_{mn} = 0$.

Furthermore,

$$\langle v_j| A |v_k\rangle = A_{jk} \langle v_j|v_j\rangle = A_{jk} = 1$$

So A is a $\text{rank}(V) \times \text{rank}(V)$ matrix, with $A_{jk} = 1$ and other entries set to 0.

Exercise 11

- $\det|X - \lambda I| = \lambda^2 - 1 = 0 \Longrightarrow \lambda_{1,2} = \pm 1$

  $\lambda_1 = 1$: $(X - I)|x\rangle = 0 \Longrightarrow$ a normalized eigenvector $|x_1\rangle = \frac{\sqrt{2}}{2}(1, 1)$

  $\lambda_2 = -1$: $(X + I)|x\rangle = 0 \Longrightarrow$ a normalized eigenvector $|x_2\rangle = \frac{\sqrt{2}}{2}(1, -1)$

  Hence a diagonal representation is $X = |x_1\rangle\langle x_1| - |x_2\rangle\langle x_2|$
- $\det|Y - \lambda I| = \lambda^2 - 1 = 0 \Longrightarrow \lambda_{1,2} = \pm 1$

  $\lambda_1 = 1$: $(Y - I)|y\rangle = 0 \Longrightarrow$ a normalized eigenvector $|y_1\rangle = \frac{\sqrt{2}}{2}(1, i)$

  $\lambda_2 = -1$: $(Y + I)|y\rangle = 0 \Longrightarrow$ a normalized eigenvector $|y_2\rangle = \frac{\sqrt{2}}{2}(1, -i)$

  Hence a diagonal representation is $Y = |y_1\rangle\langle y_1| - |y_2\rangle\langle y_2|$
- $\det|Z - \lambda I| = \lambda^2 - 1 = 0 \Longrightarrow \lambda_{1,2} = \pm 1$

  $\lambda_1 = 1$: $(Z - I)|z\rangle = 0 \Longrightarrow$ a normalized eigenvector $|z_1\rangle = (1, 0)$

  $\lambda_2 = -1$: $(Z + I)|z\rangle = 0 \Longrightarrow$ a normalized eigenvector $|z_2\rangle = (0, 1)$

  Hence a diagonal representation is $Y = |z_1\rangle\langle z_1| - |z_2\rangle\langle z_2|$

Exercise 12

$$\det(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \lambda I) = (1 - \lambda)^2 = 0$$

We get the only eigenvalue $\lambda = 1$.

Let $(A - I)|x\rangle = 0$, then get an eigenvector $|x\rangle = (0, 1)$ and the rank of the eigenspace w.r.t. $\lambda = 1$ is 1.

Note that

$$k|x\rangle\langle x| = \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Hence the matrix is not diagonalizable.

Exercise 13

Suppose $|w\rangle \in W$ and $|v\rangle \in V$.

Let operator $A = |w\rangle\langle v|$ and $B = |v\rangle\langle w|$. It is not difficult to see that A maps from V to W and B maps from W to V.

$$\forall |i\rangle \in V.\ (A|i\rangle)^\dagger = \langle v|i\rangle^\star \langle w| = \langle i|v\rangle\langle w| = \langle i| A^\dagger = \langle i| B$$

Hence $B = A^\dagger$, i.e., $(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$.

Exercise 14

Suppose A_i maps from V to W.

For all $|v\rangle \in V, |w\rangle \in W$,

$$\begin{aligned} ((\sum_i a_i A_i)^\dagger |v\rangle, |w\rangle) &= (|v\rangle, \sum_i a_i A_i |w\rangle) \\ &= \sum_i a_i (|v\rangle, A_i |w\rangle) \\ &= \sum_i a_i (A_i^\dagger |v\rangle, |w\rangle) \\ &= ((\sum_i a_i^\star A_i^\dagger) |v\rangle, |w\rangle) \end{aligned}$$

Hence $(\sum_i a_i A_i)^\dagger = \sum_i a_i^\star A_i^\dagger$.

Exercise 15

Suppose A maps from V to W.

For all $|v\rangle \in V, |w\rangle \in W$,

$$((A^\dagger)^\dagger |v\rangle, |w\rangle) = (|v\rangle, A^\dagger |w\rangle) = (A^\dagger |w\rangle, |v\rangle)^\star = (|w\rangle, A|v\rangle)^\star = (A|v\rangle, |w\rangle)$$

Hence $(A^\dagger)^\dagger = A$.

Exercise 16

Suppose P maps from V to its subspace W:

$$P = \sum_{i=1}^{k} |i\rangle\langle i|,$$

where $\{|i\rangle\}$ is an orthonormal basis for W.

$$\begin{aligned}\forall |v\rangle \in V, P^2 &= \sum_i |i\rangle\langle i| \sum_j |j\rangle\langle j| \\ &= \sum_{i,j} |i\rangle\langle i|j\rangle\langle j| \\ &= \sum_{i,j} \delta_{ij} |i\rangle\langle j| \\ &= \sum_i |i\rangle\langle i| \\ &= P\end{aligned}$$

Exercise 17

- $\Longrightarrow$:

  For any Hermitian matrix A, suppose it has an eigenvalue λ and a corresponding eigenvector $|x\rangle$.

  We have

  $$\langle x| A |x\rangle = \langle x| \lambda |x\rangle) = \lambda \langle x|x\rangle$$

  On the other side,

  $$\langle x| A |x\rangle = \langle x| A^\dagger |x\rangle) = \langle x| A |x\rangle^\star = \lambda^\star \langle x|x\rangle$$

  Note that $\langle x|x\rangle > 0$, so $\lambda = \lambda^\star$.i.e., λ is real.

- $\Longleftarrow$:

  Let A be any normal matrix with real eigenvalues. As it is normal, it can be diagonalize as follows:

  $$A = \sum_i \lambda_i |i\rangle\langle i|$$

  where $\{|i\rangle\}$ are eigenvectors corresponding to eigenvalues $\{\lambda_i\}$.

  (Notice that the rank of an eigenspace may be greater than 1, but it does not matter.)

  Given that λ_i is real,

  $$A^\dagger = (A^\star)^T = \sum_i \lambda_i^\star |i\rangle\langle i| = \sum_i \lambda_i |i\rangle\langle i| = A$$

  i.e., A is Hermitian.

Combining the two parts above, a normal matrix is Hermitian if and only if it has real eigenvalues.

Exercise 18

Let U be a unitary matrix, and λ as any of its eigenvalues. Then we have

$$(U|v\rangle, U|w\rangle) = (\lambda|v\rangle, \lambda|w\rangle) = \lambda\lambda^\star(|v\rangle, |w\rangle)$$

Also by definition

$$(U|v\rangle, U|w\rangle) = (|v\rangle, |w\rangle)$$

Hence

$$\lambda\lambda^\star = 1$$

Note that $\|\lambda\| \|\lambda^\star\| = 1$ and $\|\lambda\| = \|\lambda^\star\| \geq 0$, so $\|\lambda\| = 1$.

In conclusion, all eigenvalues of U have modulus 1.

Exercise 19

- I: It is clear that I is Hermitian and unitary.
- X:

$$X^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X$$

$$X^\dagger X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

- Y:

$$Y^\dagger = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y$$

$$Y^\dagger Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

- Z:

$$Z^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z$$

$$Z^\dagger Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Exercise 20