LINEARIZED SPLIT BREGMAN

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1. PROXIMITY OPERATOR

Firstly, Consider the following problem

$$prox_x = \arg\min_z \|x-z\|_2^2 + \lambda\|z\|_1 \tag{1}$$

since all the functions in the problem are convex, the optimality condition is

$$0 \in \nabla(\|x-z\|_2^2) + \lambda\partial\|z\|_1$$

Notice that the above equation is just summation of each component, it is equivalent to considering an arbitrary i-th component. if $\|z_i\| \neq 0$, the subgradient of $|z_i|$ is just the gradient, which is $sgn(z_i)$, where z_i is one component of z, thus the above relation becomes the equality

$$0 = 2(z_i - x_i) + \lambda sgn(z_i) \tag{2}$$

$$z_i = x_i - \frac{1}{2}\lambda sgn(z_i) \tag{3}$$

from above equation it is clear that if $z_i > 0$, x_i must be greater then 0, and if $z_i < 0$, $x_i < 0$ also holds. Hence $sgn(z_i) = sgn(x_i)$, in turn the following is true

$$z_i = x_i - \frac{1}{2}\lambda sgn(x_i) \tag{4}$$

For z_i equals 0 the subgraidient of $|z_i|$is the interval $[-1, 1]$,

$$0 \in 2(z_i - x_i) + \lambda[-1, 1] => x_i \in [-\frac{\lambda}{2}, \frac{\lambda}{2}]$$

Combine two cases, for each z_i the minimizer satisfy the following relation

$$z_i = \begin{cases} 0 & |x_i| \leq \frac{\lambda}{2} \\ x_i - \frac{1}{2}\lambda sgn(x_i) & |x_i| > \frac{\lambda}{2} \end{cases} \tag{5}$$

2. General linear L_1 minimization

In many applications, the problem can be formulized as

$$\arg\min_z \|Az - x\|_2^2 + \lambda\|z\|_1 \tag{6}$$

where A is a linear operator. Approximate the above equation by Taylor expansion at z^k, the linearized verion of the problme becomes an iterative shceme

$$z^{k+1} = \arg\min_z \lambda\|z\|_1 + \|Az^k - x\|_2^2 + (A^tAz^k - A^tx, z - z^k) + \frac{1}{2\delta}\|z - z^k\|^2 \tag{7}$$

The last penalty term is to make sure z_{k+1} and z_k not too far from each other so that Taylor expansion is still valid. Throw away some constants, the scheme can be simplified as

$$z^{k+1} = \arg\min_z \lambda\|z\|_1 + \frac{1}{2\delta}\|z - (z^k - \delta(A^tAz^k - A^tx))\|^2 \tag{8}$$

Note that the right hand side is of the form derived in Section 1, hence z^{k+1} can be solved as

$$z_i = \begin{cases} 0 & |y_i| \leq \lambda\delta \\ y_i - \frac{1}{2}\lambda sgn(y_i) & |y_i| > \lambda\delta \end{cases} \tag{9}$$

where $y_i = z_i^k - \delta(A^tAz_i^k - A^tx_i)$

3. Linearized split Bregman and Augmented Lagrange method

4. Appendix

subderivative