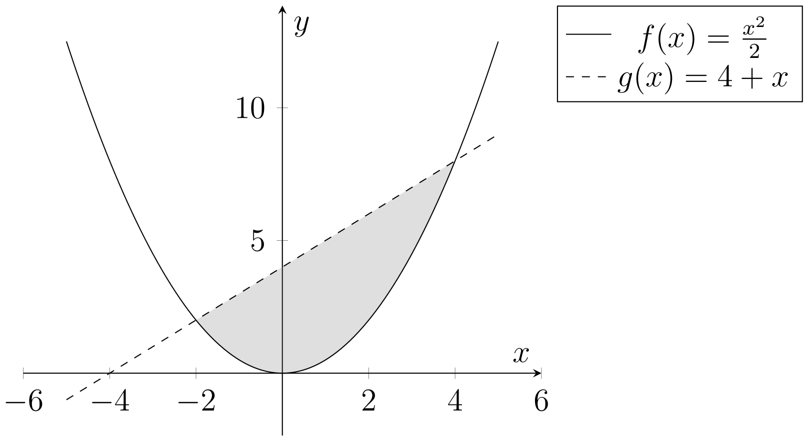
1. (k points) How big is the parabolic segment between the parabola $f(x) = \frac{x^2}{2}$ and the line $g(x) = 4 + x$?

   Sketch a graph to visualize the desired area.

Solution: The functions intersect at $P_1(-2, 2)^T$ and at $P_2(4, 8)^T$. Thus, the area is

$$A = \int_{-2}^{4} g(x) - f(x)\,\mathrm{d}x = \int_{-2}^{4} 4 + x + \left(\frac{-1}{2}\right) x^2\,\mathrm{d}x = \left[\frac{1}{6}x\big(24 + 3x - x^2\big)\right]_{-2}^{4} = -16+3x^2 - 0.5x^3$$



2. (k points) Given the function

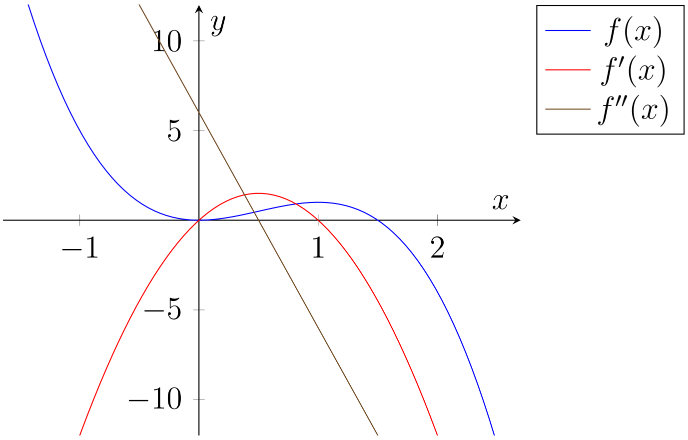
$$f(x) = 3x^2 - 2x^3$$

   (a) Sketch f, f' and f'' in one coordinate system.
   (b) Identify all of the minimum and maximum points and find its inflection points.

Solution:

(a) First, calculate the derivatives

$$\begin{aligned} f(x) &= 3x^2 - 2x^3 \\ f'(x) &= 6x - 6x^2 \\ f''(x) &= 6 - 12x \\ f'''(x) &= -12 \end{aligned}$$



(b) The function f has zeros at $x_1 = \frac{3}{2}$ and at $x_2 = 0$. The function f' has zeros at $x_3 = 1$ and at $x_4 = 0$. The function f has a maximum at $(1, 1)$ because $f''(x_3) < 0$ and a minimum at $(0, 0)$ because $f''(x_4) > 0$.

3. Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -10 & 5 \\ 60 & -15 \end{bmatrix}.$$

Solution:

Calculate $A - \lambda I_2$:

$$A - \lambda I_2 = \begin{bmatrix} -10 & 5 \\ 60 & -15 \end{bmatrix} - \lambda \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} = \begin{bmatrix} -10 - 1.0\lambda & 5.0 \\ 60.0 & -15 - 1.0\lambda \end{bmatrix}.$$

Then, calculate $\det(A - \lambda I_2)$.

$$\det(A - \lambda I_2) = -300.0 + (-10 - 1.0\lambda)(-15 - 1.0\lambda)$$

Now, we solve $\det(A - \lambda I_2) = 0$.

The matrix A has the eigenvalues $\lambda_1 = -30$ and $\lambda_2 = 5$.