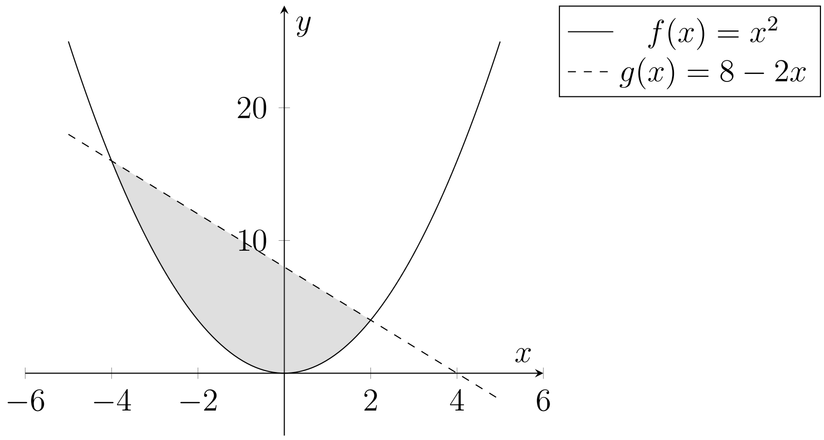
1. (k points) How big is the parabolic segment between the parabola $f(x) = x^2$ and the line $g(x) = 8 - 2x$?

   Sketch a graph to visualize the desired area.

Solution: The functions intersect at $P_1(-4, 16)^T$ and at $P_2(2, 4)^T$. Thus, the area is

$$A = \int_{-4}^{2} g(x) - f(x)\,\mathrm{d}x = \int_{-4}^{2} 8 - 2x - x^2\,\mathrm{d}x = \left[\frac{1}{3}x\big(24 - 3x - x^2\big)\right]_{-4}^{2} = -16+12x - x^3$$



2. (k points) Given the function

$$f(x) = x^2 + 3x^3$$

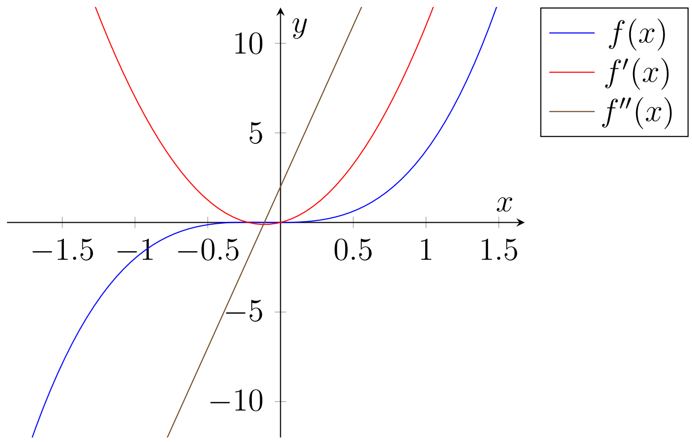
   (a) Sketch f, f' and f'' in one coordinate system.

   (b) Identify all of the minimum and maximum points and find its inflection points.

Solution:

(a) First, calculate the derivatives

$$\begin{aligned} f(x) &= x^2 + 3x^3 \\ f'(x) &= 2x + 9x^2 \\ f''(x) &= 2 + 18x \\ f'''(x) &= 18 \end{aligned}$$



(b) The function f has zeros at $x_1 = \frac{-1}{3}$ and at $x_2 = 0$. The function f' has zeros at $x_3 = \frac{-2}{9}$ and at $x_4 = 0$. The function f has a maximum at $(\frac{-2}{9}, 0.0164609053497942)$ because $f''(x_3) < 0$ and a minimum at $(0, 0)$ because $f''(x_4) > 0$.

3. Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -12 & 15 \\ 15 & -12 \end{bmatrix}.$$

Solution:

Calculate $A - \lambda I_2$:

$$A - \lambda I_2 = \begin{bmatrix} -12 & 15 \\ 15 & -12 \end{bmatrix} - \lambda \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} = \begin{bmatrix} -12 - 1.0\lambda & 15.0 \\ 15.0 & -12 - 1.0\lambda \end{bmatrix}.$$

Then, calculate $\det(A - \lambda I_2)$.

$$\det(A - \lambda I_2) = -225.0 + (-12 - 1.0\lambda)^2$$

Now, we solve $\det(A - \lambda I_2) = 0$.

The matrix A has the eigenvalues $\lambda_1 = -27$ and $\lambda_2 = 3$.