School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 1

PARTIAL DIFFERENTIATION

Milan Pahor

MATH1231 CALCULUS
PARTIAL DIFFERENTIATION

Suppose $z = f(x,y)$. Define

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \longrightarrow 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \lim_{\Delta y \longrightarrow 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Notation

$$\frac{\partial f}{\partial x} = f_x = z_x, \qquad \frac{\partial f}{\partial y} = f_y = z_y$$

In your previous studies the focus was on functions of a single variable $y = f(x)$ and their rates of change $\frac{dy}{dx}$. It is however quite rare for a quantity of interest to depend on only one variable and in complicated physical systems it may be the case that the variable you are concerned with may depend upon dozens of other variables. Partial differentiation is the extension of our usual calculus to functions of several variables.
Given a function of two variables $z = f(x,y)$ we denote the rates of change in the x and y directions as $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ (note the curly d's) or sometimes simply as z_x and z_y. The formal definitions of these derivatives are presented above however in reality we only need to remember a few things to differentiate partially:

The old specific rules of differentiation

y	y'
x^n	nx^{n-1}
e^x	e^x
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\ln(x)$	$\frac{1}{x}$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$

The old general rules of differentiation

$$(uv)' = u'v + v'u \qquad \text{Product Rule}$$

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2} \qquad \text{Quotient Rule}$$

The only extra issue that needs to be kept in mind is that when you are differentiating in a particular direction you treat all other variables *exactly* as if they were constant.

Example 1 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = x^2 + y^5 + 7$.

$$\frac{\partial z}{\partial x} = 2x + 0 + 0 = 2x$$

$$\frac{\partial z}{\partial y} = 0 + 5y^4 + 0 = \underline{5y^4}$$

★ $\frac{\partial z}{\partial x} = 2x$, $\frac{\partial z}{\partial y} = 5y^4$ ★

Example 2 Suppose that $z = f(x, y) = x^3y^5 + 3x - 8y + 2$. Find the function value and the rate of change of f in the x direction at the point $(1, 2)$.

$$f(1,2) = 1^3 2^5 + 3(1) - 8(2) + 2$$
$$= 32 + 3 - 16 + 2 = 21$$

$$\frac{\partial f}{\partial x} = 3x^2y^5 + 3 - 0 + 0$$
$$= 3(1)^2 2^5 + 3 = 3 \times 32 + 3$$
$$= \underline{99}$$

★ $f(1, 2) = 21$, $\frac{\partial z}{\partial x}(1, 2) = 99$ ★

Example 3 Find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ if $w = u^3v^4 + \sinh(v^9)$.

$$\frac{\partial w}{\partial u} = 3u^2v^4 + 0 = 3u^2v^4$$

$$\frac{\partial w}{\partial v} = u^3 4v^3 + \cosh(v^9)(9v^8)$$

$$= 4u^3v^3 + 9v^8\cosh(v^9)$$

★ $\frac{\partial w}{\partial u} = 3u^2v^4$, $\frac{\partial w}{\partial v} = 4u^3v^3 + 9v^8\cosh(v^9)$ ★

Example 4 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = \frac{e^{7y}}{x^3+1}$.

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

$$\frac{\partial z}{\partial y} = \frac{7e^{7y}}{x^3+1}$$

$$\frac{\partial z}{\partial x} = \frac{(x^3+1)(0) - e^{7y}(3x^2)}{(x^3+1)^2}$$

$$= \frac{-3x^2e^{7y}}{(x^3+1)^2}$$

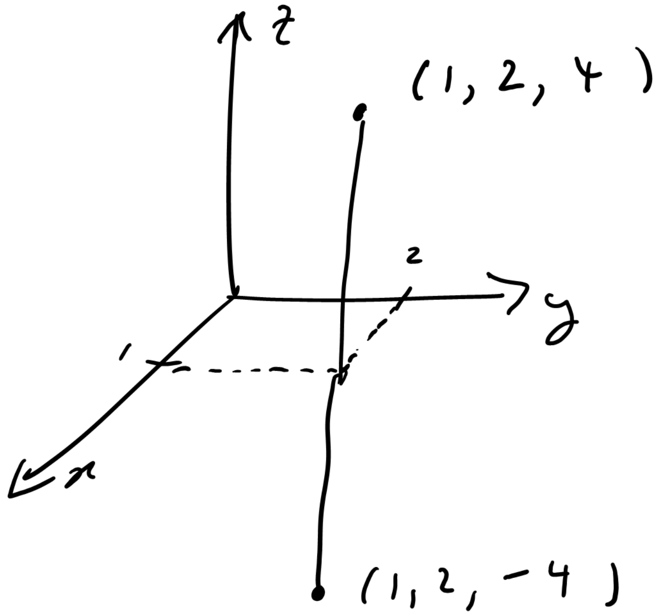
★ $\frac{\partial z}{\partial x} = \frac{-3e^{7y}x^2}{(x^3+1)^2}$, $\frac{\partial z}{\partial y} = \frac{7e^{7y}}{x^3+1}$ ★

A

Plotting in Space

Before examining partial derivatives from a geometrical point of view let us consider the issue of sketching in higher dimensions.

Example 5 Plot the points $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ in $\mathbb{R}^3$.



You will observe that plotting in $\mathbb{R}^3$ is somewhat problematic as you are trying to squeeze three dimensions onto a two dimensional page. It gets worse!

Example 6 Plot the point $\begin{pmatrix} 1 \\ 2 \\ 4 \\ 7 \end{pmatrix}$ in $\mathbb{R}^4$.

?

Surfaces in Space

You will recall that the graph of $y = f(x)$ is generally a curve in $\mathbb{R}^2$......lines, parabolas, hyperbolas etc. The graph of $z = f(x, y)$ is always a *surface* in $\mathbb{R}^3$.

When attempting to sketch a complicated surface in $\mathbb{R}^3$ we:

- Determine how the surface intersects the $x - y$ plane (**by setting** $z = 0$).
- Find the shape of some curves of intersection (level curves) (**by setting** $z = 1, 2....$**).**
- If necessary repeat the above cycling through the x, y and z variables.

Example 7 Sketch each of the following surfaces in space:

a) $3x + 4y + 6z = 12$

b) $x^2 + y^2 + z^2 = 25$

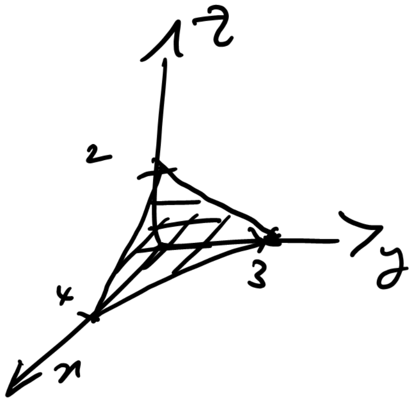
c) $z = x^2 + y^2$

d) $x = y^2 + z^2$

e) $z = 3\sqrt{x^2 + y^2}$

f) $x^2 + y^2 = 9$

Linear in all variables ∴ plane!



a) $3x + 4y + 6z = 12$

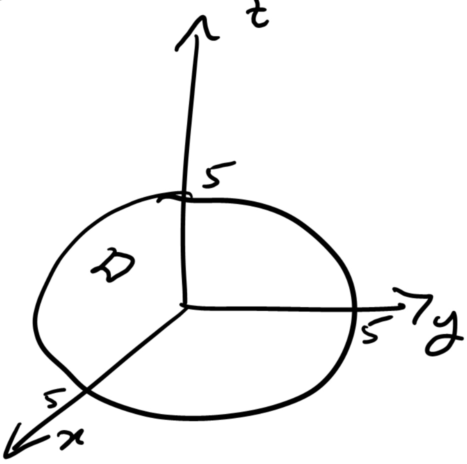
$y = z = 0 \Rightarrow 3x = 12 \Rightarrow x = 4$

$x = z = 0 \Rightarrow 4y = 12 \Rightarrow y = 3$

$x = y = 0 \Rightarrow 6z = 12 \Rightarrow z = 2$

b) $x^2 + y^2 + z^2 = 25$

r^2



c) $z = x^2 + y^2$

$z = 1 \Rightarrow x^2 + y^2 = 1 \rightarrow$ circle

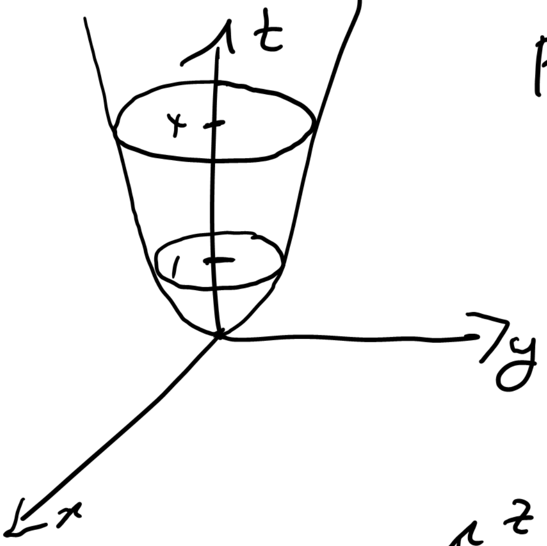
$z = 4 \Rightarrow x^2 + y^2 = 4 \rightarrow$ circle.

$x = 0 \rightarrow z = y^2$

$y = 0 \rightarrow z = x^2$

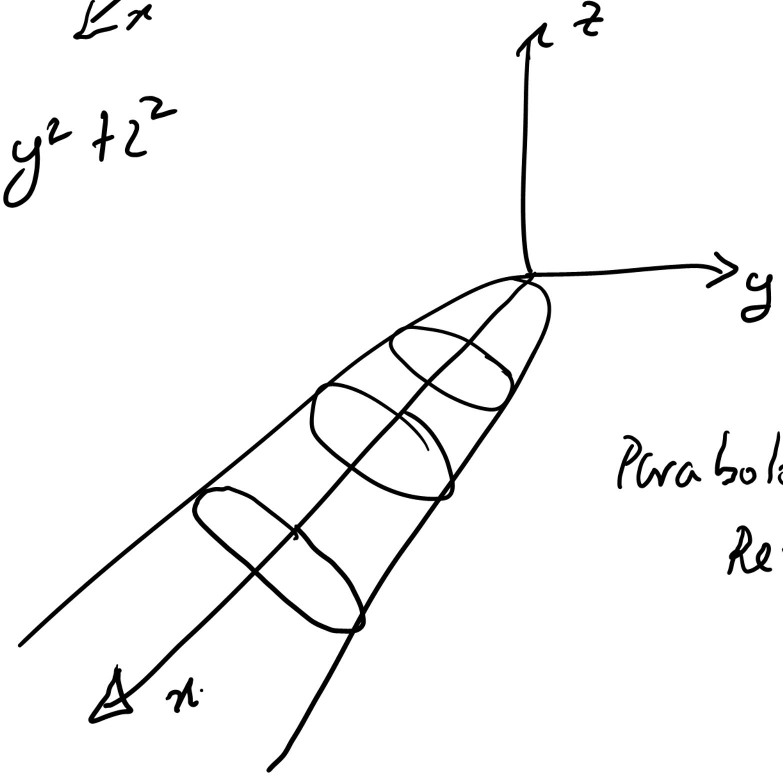
$z=-1 \Rightarrow x^2+y^2=-1$

X X



Paraboloid of Revolution

d) $x = y^2 + z^2$



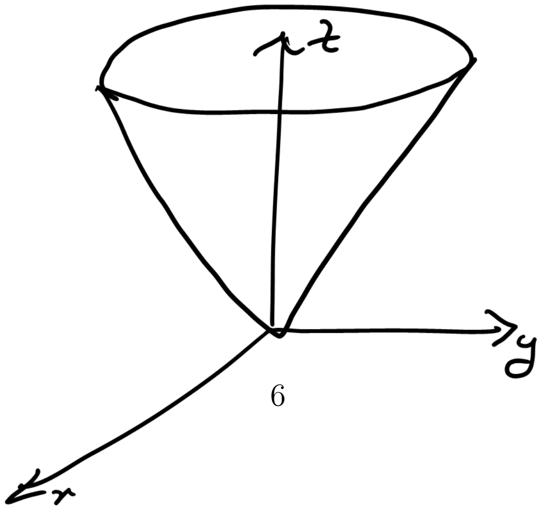
Paraboloid of Rev^{n}

e) $z = 3\sqrt{x^2+y^2}$

$y=0$: $z = 3\sqrt{x^2} = 3|x|$ (x-z plane)

$x=0$: $z = 3\sqrt{y^2} = 3|y|$ (y-z plane)

$z=1$: $1 = 3\sqrt{x^2+y^2} \Rightarrow \sqrt{x^2+y^2} = \frac{1}{3} \Rightarrow x^2+y^2 = \frac{1}{9} \Rightarrow$ circle



OR

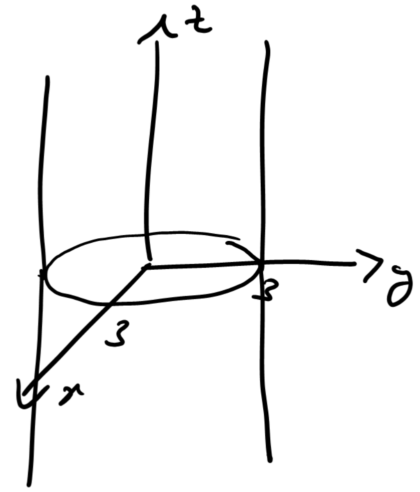
$z = 3\sqrt{x^2+y^2}$

$z = 3r$

(f) $x^2 + y^2 = 9$ in $\mathbb{R}^3$

z variable absent

$\therefore z$ can be anything



$\therefore$ cylinder

SUMMARY

a) $ax + by + cz = d$ is a plane.

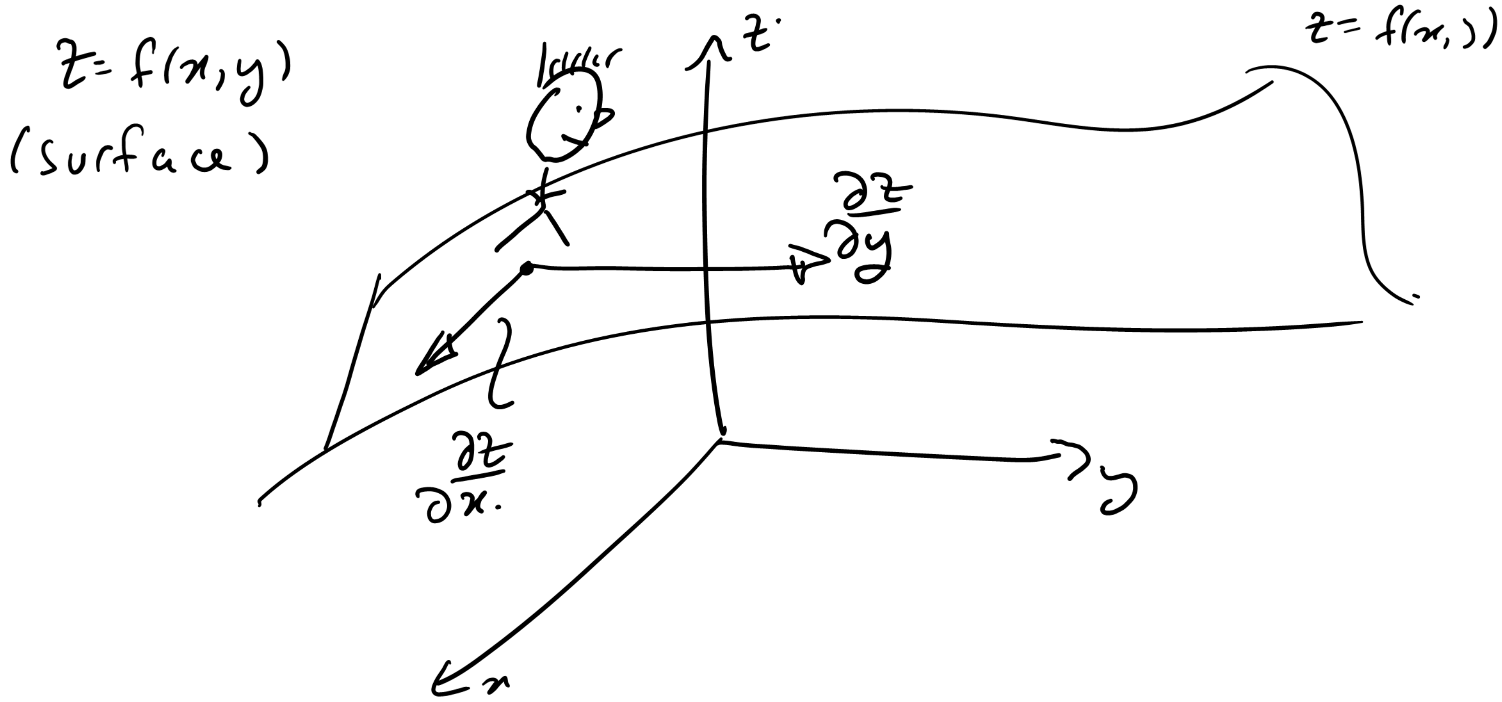
b) $x^2 + y^2 + z^2 = r^2$ is a sphere centre the origin radius r.

c) $z = \alpha(x^2 + y^2)$ is a paraboloid of revolution.

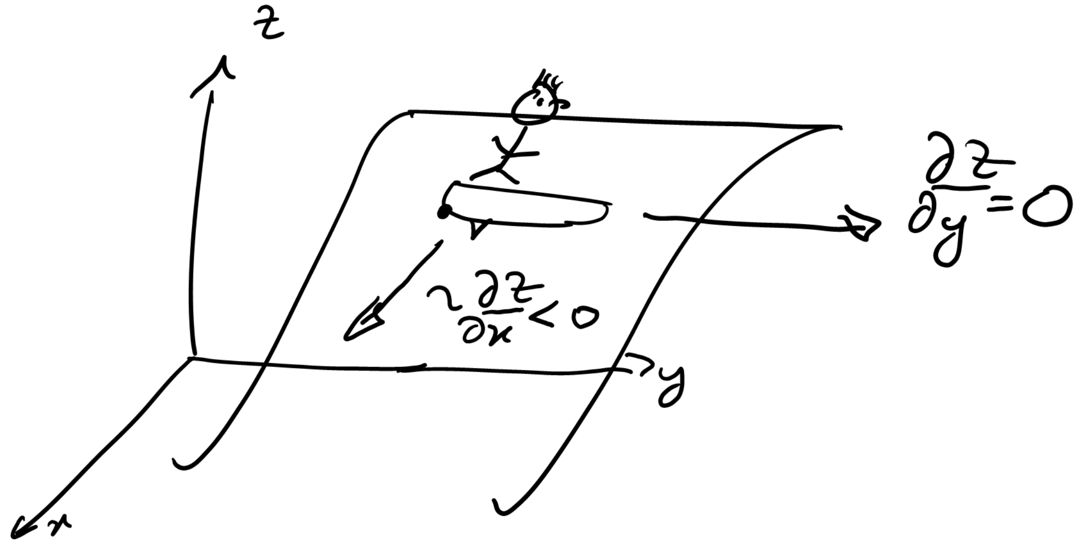
d) $x = \alpha(y^2 + z^2)$ is a sideways paraboloid of revolution.

e) $z = \alpha\sqrt{x^2 + y^2}$ is a cone with semi-vertical angle $\tan^{-1}(\frac{1}{\alpha})$.

f) If a variable is absent it is **!unrestricted!** Extrude the two dimensional curve into the missing direction.



$z = f(x,y)$
(surface)
z
$z = f(x,y)$
$\frac{\partial z}{\partial y}$
$\frac{\partial z}{\partial x}$
y
x



z
$\frac{\partial z}{\partial y} = 0$
$\frac{\partial z}{\partial x} < 0$
y
x

There is of course no reason why we must restrict ourselves to two independent variables!!

Example 8 If $f(x_1, x_2, x_3, x_4, x_5, x_6) = x_1^3 x_3^4 + \dfrac{x_5}{\sin(x_6)} + \ln(x_4) - \dfrac{\sinh(x_2)}{e^{x_1}}$ find $\dfrac{\partial f}{\partial x_4}$.

$$\frac{\partial f}{\partial x_4} = \frac{1}{x_4}$$

$\bigstar \quad \dfrac{1}{x_4} \quad \bigstar$

As in single variable calculus we make extensive use of second and higher order derivatives. However with partial differentiation we have many more options!

Example 9 If $z = f(x, y) = x^2 \sin(y) + x^3 y$ then:

$$\frac{\partial z}{\partial x} = 2x\sin y + 3x^2 y$$

$$\frac{\partial z}{\partial y} = x^2\cos(y) + x^3$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^2\cos(y) + x^3\right) = 2x\cos(y) + 3x^2$$

equal ||

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(2x\sin(y) + 3x^2 y\right) = 2x\cos(y) + 3x^2$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x\sin y + 3x^2 y\right)$$
$$= 2\sin(y) + 6xy$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^2\cos(y) + x^3\right)$$
$$= -x^2\sin(y) + 0$$

You will observe in the above example that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \quad \checkmark$$

This is always true for most reasonably well behaved functions.

Note however that in general $\dfrac{\partial^2 z}{\partial x^2} \neq \dfrac{\partial^2 z}{\partial y^2}$. ✗

Note also that $\dfrac{\partial^2 z}{\partial x \partial y}$ is most definitely not equal to $\left(\dfrac{\partial z}{\partial x}\right) \times \left(\dfrac{\partial z}{\partial y}\right)$.

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 2

TANGENT PLANES AND ERROR ESTIMATES

Milan Pahor

MATH1231 CALCULUS
TANGENT PLANES AND ERROR ESTIMATES

The Cartesian equation of a plane in $\mathbb{R}^3$ takes the form

$$ax + by + cz = d \quad \text{where} \quad a, b, c, d \in \mathbb{R}.$$

The plane $ax + by + cz = d$ has normal vector $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

At any point P on the surface $z = f(x, y)$ a normal vector to the tangent plane (and hence a normal vector to the surface) is given by

$$\mathbf{n} = \begin{pmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ -1 \end{pmatrix}.$$

For error estimates use

$$|\Delta f| \le |\tfrac{\partial f}{\partial x}||\Delta x| + |\tfrac{\partial f}{\partial y}||\Delta y|.$$

This lecture will be devoted to two major applications of partial differentiation:

(I) The calculation of tangent planes to the surface $z = f(x, y)$, and

(II) Error estimates for $z = f(x, y)$.

TANGENT PLANES

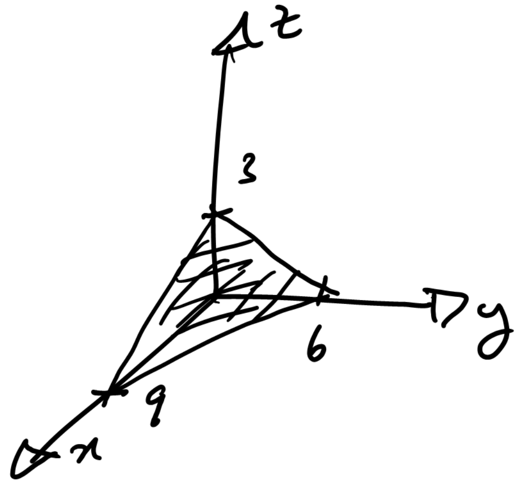
On of the main applications of calculus in two dimensions is the calculation of the tangent line to a curve $y = f(x)$. In 3-D we have surfaces rather than curves and tangent **planes** rather than tangent lines. In this lecture we will investigate how to find the Cartesian equation of a tangent plane to any surface at any point on the surface. But first some revision on the theory of planes from 1131 algebra.

Revision: The Cartesian Equation of a plane

The Cartesian equation of a plane in $\mathbb{R}^3$ is linear in all variables and takes the form

$$ax + by + cz = d \quad \text{where} \quad a, b, c, d \in \mathbb{R}.$$

Example 1 Sketch the plane $2x + 3y + 6z = 18$ in $\mathbb{R}^3$. Find two points on the plane.



$y = z = 0$: $2x = 18 \Rightarrow x = 9$

$x = z = 0$: $3y = 18 \Rightarrow y = 6$

$x = y = 0$: $6z = 18 \Rightarrow z = 3$

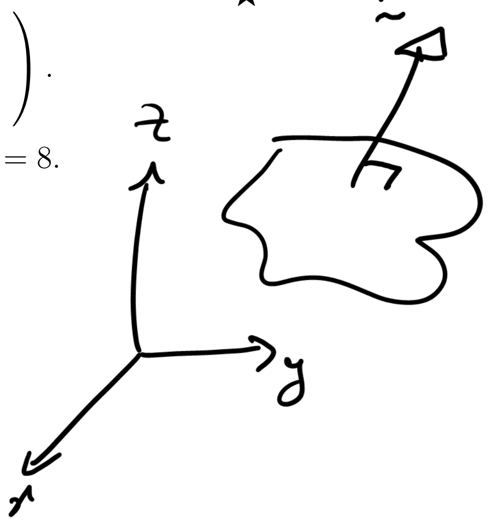
Points <u>on</u> plane

$$\begin{pmatrix} 9 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{9}{2} \\ 1 \\ 1 \end{pmatrix}$$

OR

$(9, 0, 0)$, $(9/2, 1, 1)$

★



Fact: The plane $ax + by + cz = d$ has normal vector $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Example 2 Find a unit vector normal to the plane $6x + 2y + 3z = 8$.

Normal vector: $\underset{\sim}{\hat{n}} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$

$|\underset{\sim}{n}| = \sqrt{36 + 4 + 9}$

$= 7$

$\underset{\sim}{\hat{n}} = \frac{1}{7}\begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$

★ $\begin{pmatrix} \frac{6}{7} \\ \frac{2}{7} \\ \frac{3}{7} \end{pmatrix}$ ★

To find the Cartesian equation of a plane all we need is a perpendicular vector to the plane and a point on the plane. Perpendicular vectors are also called normal vectors.

Example 3 Find a Cartesian equation of the plane passing through the point $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ and perpendicular to the vector $\begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$.

Observe how easy this is!

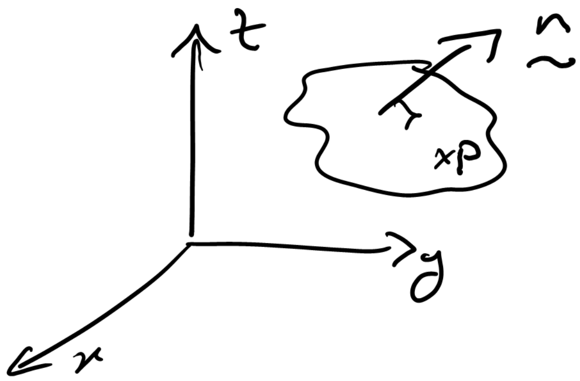
$$4x - 2y + 6z = \#$$

Sub in $(1, 2, 5)$

$$\therefore \; 4(1) - 2(2) + 6(5) = \#$$

$$\therefore \; 30 = \#$$

$$\therefore \; \underline{4x - 2y + 6z = 30}$$



$$\bigstar \quad 2x - y + 3z = 15 \quad \bigstar$$

$\bigstar$ End of revision $\bigstar$

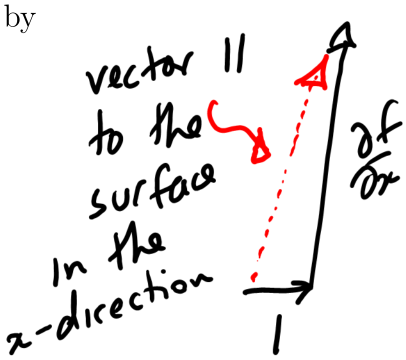
Tangent Planes

In the previous lecture we looked at various surfaces $z = f(x, y)$ including spheres, paraboloids and cones. At any point P on a surface there exists a unique plane which sits tangentially to the surface at the point. This plane is called the tangent plane and it is the three dimensional analog to the tangent line to a curve. A simple little fact will open the door to the calculation of tangent planes.

perpendicular

Fact: At any point P on the surface $z = f(x, y)$ a normal vector to the tangent plane (and hence a perpendicular vector to the surface) is given by

$$\mathbf{n} = \begin{pmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ -1 \end{pmatrix}.$$



Proof: Note that our approach here is very different from that used in the printed algebra notes.

We saw in the previous lecture that $\dfrac{\partial f}{\partial x}$ measures the gradient to the surface $z = f(x, y)$ in the x direction. Thus at any point P on the surface the vector $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ \dfrac{\partial f}{\partial x} \end{pmatrix}$ is parallel to the surface in the x direction since this vector rises by $\dfrac{\partial f}{\partial x}$ units in the vertical z direction for every unit travelled in the x direction. Similarly $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ \dfrac{\partial f}{\partial y} \end{pmatrix}$ is parallel to the surface in the y direction since this vector rises by $\dfrac{\partial f}{\partial y}$ units in the vertical z direction for every unit travelled in the y direction. Thus $\mathbf{u}$ and $\mathbf{v}$ are both parallel to the tangent plane at P. Using the cross product we can now find a normal vector $\mathbf{n} = \mathbf{v} \times \mathbf{u}$ to the plane at P.

$$\mathbf{n} = \mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & \frac{\partial f}{\partial y} \\ 1 & 0 & \frac{\partial f}{\partial x} \end{vmatrix} = \underset{\sim}{i} \begin{vmatrix} 1 & \frac{\partial f}{\partial y} \\ 0 & \frac{\partial f}{\partial x} \end{vmatrix} - \underset{\sim}{j} \begin{vmatrix} 0 & \frac{\partial f}{\partial y} \\ 1 & \frac{\partial f}{\partial x} \end{vmatrix} + \underset{\sim}{k} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= i\left(\frac{\partial f}{\partial x} - 0\right) - j\left(0 - \frac{\partial f}{\partial y}\right) + k(0 - 1)$$

Note that $\mathbf{u} \times \mathbf{v}$ would also work yielding an equivalent negative normal vector, pointing in the opposite direction.

$$\underset{\sim}{n} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ -1 \end{pmatrix} \quad \text{as required}$$

Example 4 Consider the paraboloid

$$z = f(x, y) = 1 + x^2 + y^2.$$

a) Sketch the paraboloid in $\mathbb{R}^3$.

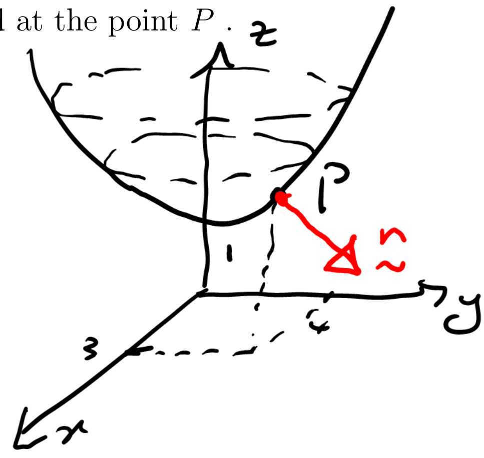
b) Prove that $P(3, 4, 26)$ is a point on the paraboloid and add P to your sketch.

b) Calculate both $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at the point P.

d) Find a normal vector $\mathbf{n}$ to the paraboloid at P, and add $\mathbf{n}$ to your sketch.

e) Find the equation of the tangent plane to the paraboloid at the point P.

a)



b) $z = 1 + x^2 + y^2$

$x = 3, y = 4 \Rightarrow z = 1 + 9 + 16 = 26$ ✓

$\therefore \begin{pmatrix} 3 \\ 4 \\ 26 \end{pmatrix}$ is <u>on</u> the surface

c) $\dfrac{\partial f}{\partial x} = 2x = 2(3) = 6$ @P

$\dfrac{\partial f}{\partial y} = 2y = 2(4) = 8$ @P

d) $\underset{\sim}{n} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ -1 \end{pmatrix}$

e) Tangent Plane : $6x + 8y - z = \#$

sub in $(3, 4, 26)$: $18 + 32 - 26 = \#$

$\# = 24$

$\therefore 6x + 8y - z = 24$

★ $\mathbf{n} = \begin{pmatrix} 6 \\ 8 \\ -1 \end{pmatrix}$, $\quad 6x + 8y - z = 24$ ★

B

ERROR ESTIMATES

It is often the case that the independent variables x and y for $z = f(x, y)$ have a degree of uncertainty in their values. That is we need to cope with small errors Δx and Δy in x and y respectively. You will recall that in two dimensions we have

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} \implies \Delta y \approx \frac{dy}{dx}\Delta x$$

Jumping up a dimension to $z = f(x, y)$ we need to account for error in both the x and y directions yielding the lovely little formula

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

This little equation gives you the error in f in terms of the errors in x and y.

It is rare however that we know the exact value and the exact direction of an error!

Usually the error could be $\pm$ up to some maximum. Via the triangle inequality we then use

$$|\Delta f| \leq \left|\frac{\partial f}{\partial x}\right| |\Delta x| + \left|\frac{\partial f}{\partial y}\right| |\Delta y|.$$

to generate the maximum possible error in f under the given conditions.

Example 5 Suppose that a quantity T in an laboratory experiment depends upon two measurements x and y via the formula

$$T = x^2 + y^3.$$

During the experiment x and y are measured to be $x = 1$ and $y = 2$, with maximum possible errors of $|\Delta x| \leq 0.1$ and $|\Delta y| \leq 0.3$. Find an approximation for the:

a) the maximum possible absolute error in T.

b) the maximum relative absolute error in T.

c) the maximum percentage absolute error in T.

a) $$\Delta T = \underline{\frac{\partial T}{\partial x}\Delta x} + \underline{\frac{\partial T}{\partial y}\Delta y}$$

$$|\Delta T| \leq \left|\frac{\partial T}{\partial x}\right||\Delta x| + \left|\frac{\partial T}{\partial y}\right||\Delta y|$$
$$= |2x||\Delta x| + |3y^2||\Delta y|$$
$$= (2\times 1)(.01) + (3\times 4)(0.3)$$
$$= 3.8$$

b) $T(1,2) = 1^2 + 2^3 = 9$

Relative max. abs. error $\frac{3.8}{9} = 0.42$

c) $0.42 \times 100 = 42\%$

★ *a)* 3.8 *b)* $\frac{3.8}{9} \approx 0.42$ *c)* 42% ★

Example 6 In an Ohm's law experiment the resistance R is given by

$$R = \frac{V}{I} = VI^{-1}$$

where $V = 5 \pm \frac{1}{10}$ volts and $I = 2 \pm \frac{1}{100}$ amps.

Use the error estimate formula to calculate the maximum possible percentage error in the calculated value of the resistance.

$$\Delta R = \frac{\partial R}{\partial V}\Delta V + \frac{\partial R}{\partial I}\Delta I$$

$$= \frac{1}{I}\Delta V + (-VI^{-2})\Delta I$$

$$= \frac{\Delta V}{I} - \frac{V\Delta I}{I^2}$$

$V=5$, $|\Delta V| \le \frac{1}{10}$, $I=2$, $|\Delta I| \le \frac{1}{100}$

$$|\Delta R| \le \left|\frac{\Delta V}{I}\right| + \left|\frac{V\Delta I}{I^2}\right|$$

$$= \left|\frac{\frac{1}{10}}{2}\right| + \left|\frac{5(\frac{1}{100})}{4}\right|$$

$$= \frac{1}{16}$$

When $I=2$, $V=5$ we have

$$R = \frac{V}{I} = \frac{5}{2} = 2.5.$$

$$\text{max \% error} = \frac{\Delta R}{R}\times 100 = \left(\frac{\frac{1}{16}}{2.5}\right)\times 100$$

$$= 2.5\%$$

★ 2.5% ★

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 3

THE CHAIN RULE

Milan Pahor

MATH1231 CALCULUS
CHAIN RULE

If $z = f(x, y)$ and $x = x(t)$ and $y = y(t)$ then

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

If $z = f(x, y)$ and $x = x(u, v)$ and $y = y(u, v)$ then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

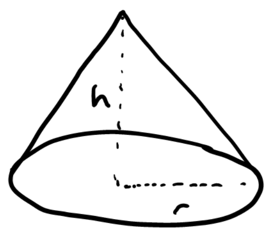
Error Estimates Continued

Example 1 The volume V of a cone with radius r and perpendicular height h is given by $V = \frac{1}{3}\pi r^2 h$. Determine the maximum absolute error and the maximum percentage error in calculating V given that $r = 5$ cm and $h = 3$ cm to the nearest millimetre.

max error $\frac{1}{2}$ mm

$$|\Delta V| \leq \left|\frac{\partial V}{\partial r}\Delta r\right| + \left|\frac{\partial V}{\partial h}\Delta h\right|$$

$$|\Delta V| \leq \left|\frac{2}{3}\pi r h\right| |\Delta r| + \left|\frac{1}{3}\pi r^2\right| |\Delta h|$$



$V = \frac{1}{3}\pi r^2 h$

$|\Delta r| \leq 0.05$, $|\Delta h| \leq 0.05$ cm.

$$|\Delta V| \leq \frac{2}{3}\pi(5)(3)(0.05) + \frac{1}{3}\pi(25)(0.05)$$

$\doteqdot 2.88$. = max absolute error in cm^3

when $r = 5$, $h = 3$ $\Rightarrow V = \frac{1}{3}\pi(25)(3) \doteqdot 78.54$

$$\Delta V\% = \frac{2.88}{78.54} \times 100 \doteqdot 3.7\%$$

★2.88, $\frac{2.88}{78.54} \times 100 \approx 3.7\%$★

THE CHAIN RULE

A common situation is that z is a function of x and y with x and y themselves functions of other variables......say u and v. It is then true that z is *ultimately* a function of u and v and thus it makes sense to ask the question "What is $\frac{\partial z}{\partial u}$"? The chain rule enables us to answer this question without actually ever having to produce z as an explicit function of u and v. The chain rule comes in many different flavours, two of which are presented above. However all that needs to be remembered is that you keep on differentiating z with respect to what you can and then always fudge your answer back to what you want.

Note that it is usually more effective to use the chain rule than to explicitly detail the structure of the new function.

Example 2 Suppose that $z = x^2 + 4y$ where $x = u^3 \ln(v)$ and $y = uv^2$. Use a chain rule to find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

z is a function of u & v !!! eventually.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= 2x(3u^2 \ln v) + (4)(v^2)$$

$$= 6xu^2 \ln v + 4v^2$$

$$= 6(u^3 \ln v)(u^2 \ln v) + 4v^2$$

$$= 6u^5 (\ln(v))^2 + 4v^2$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= (2x)\frac{u^3}{v} + 4(2uv)$$

$$= (2)(u^3 \ln v)\frac{u^3}{v} + 8uv = 2u^6 \frac{\ln v}{v} + 8uv$$

$$\bigstar \quad \frac{\partial z}{\partial u} = 6u^5(\ln(v))^2 + 4v^2, \; \frac{\partial z}{\partial v} = \frac{2u^6 \ln(v)}{v} + 8uv \quad \bigstar$$

Example 3 Suppose that $w = a^2 + b^2$ where $a = s^2t^3$ and $b = st$. Use the chain rule to find $\dfrac{\partial w}{\partial t}$.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial a} \cdot \frac{\partial a}{\partial t} + \frac{\partial w}{\partial b}\frac{\partial b}{\partial t}$$

$$= (2a)(3s^2t^2) + (2b)(s)$$

$$= 6as^2t^2 + 2sb$$

$$= 6(s^2t^3)(s^2t^2) + 2s(st)$$

$$= 6s^4t^5 + 2s^2t$$

★ $6s^4t^5 + 2s^2t$ ★

Example 4 Given that $z = x^2y^3$ with $x = 2t$ and $y = \sin(4t)$ use the chain rule to find $\dfrac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= (2xy^3)(2) + (3x^2y^2)\cos(4t)(4)$$

$$= 4xy^3 + 12x^2y^2\cos(4t)$$

$$= 4(2t)\sin^3(4t) + 12(4t^2)\sin^2(4t)\cos(4t)$$

$$= 8t\sin^3(4t) + 48t^2\sin^2(4t)\cos(4t)$$

★ $8t\sin^3(4t) + 48t^2\sin^2(4t)\cos(4t)$ ★

Example 5 If $z = a^2 + b + c^5 + d^7$ where $a = uv,\ b = 2u + 3v,\ c = u^2$ and $d = v^2$ use the chain rule to find $\dfrac{\partial z}{\partial u}$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial a}\frac{\partial a}{\partial u} + \frac{\partial z}{\partial b}\frac{\partial b}{\partial u} + \frac{\partial z}{\partial c}\frac{\partial c}{\partial u} + \frac{\partial z}{\partial d}\frac{\partial d}{\partial u}$$

$$= 2a(v) + (1)(2) + 5c^4(2u) \cancel{+ 7d^6(0)}$$

$$= 2av + 2 + 10c^4 u$$

$$= 2(uv)v + 2 + 10(u^2)^4 u$$

$$= 2uv^2 + 2 + 10u^9$$

★ $2uv^2 + 10u^9 + 2$ ★

Example 6 If $w = a^2 - ab^3$ with $a = e^{uv}$ and $b = 3u + 2v$ use the chain rule to find $\dfrac{\partial w}{\partial u}$ when $u = 0$ and $v = 1$.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial a}\cdot\frac{\partial a}{\partial u} + \frac{\partial w}{\partial b}\cdot\frac{\partial b}{\partial u}$$

$$= (2a - b^3)(e^{uv}(v)) + (-3ab^2)(3)$$

$u = 0,\ v = 1 \Rightarrow$

$$a = e^{uv} = e^{(0)(1)} = e^0 = 1$$

$$b = 3u + 2v = 3(0) + 2(1) = 2$$

$$\therefore \frac{\partial w}{\partial u} = (2 - 8)(e^0(1)) + (-3(1)(4))(3)$$

$$= -6 - 36 = \underline{\underline{-42}}$$

★ -42 ★

Sometimes we need to deal with random undeclared functions and/or variables. This is really no great problem, just carry on in the usual way.

Example 7 If $w = g(u, v)$ with $u = x + y$ and $v = x - y$ show that

$$\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = (\frac{\partial w}{\partial u})^2 - (\frac{\partial w}{\partial v})^2$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x} = \left(\frac{\partial w}{\partial u}\right)(1) + \left(\frac{\partial w}{\partial v}\right)(1)$$

$$= \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial w}{\partial u}(1) + \frac{\partial w}{\partial v}(-1) = \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}$$

$$LHS = \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y} = \left(\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}\right)\left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}\right) = \left(\frac{\partial w}{\partial u}\right)^2 - \left(\frac{\partial w}{\partial v}\right)^2$$

$$= RHS. \quad \star$$

Example 8 Suppose that f is a differentiable function of a single variable and that $F(x, y)$ is defined to be

$$F(x, y) = f(x^2 + 5y).$$

Show that F satisfies the partial differential equation

$$5\frac{\partial F}{\partial x} - 2x\frac{\partial F}{\partial y} = 0$$

Let's start by nominating that f is a function of the single variable t, (any name will do except of course x and y which are used later. Thus $f = f(t)$ and we have a clearer presentation

$$F = f(t) \text{ where } t = x^2 + 5y$$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial t}\cdot\frac{\partial t}{\partial x} = \frac{\partial F}{\partial t}(2x)$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial t}\cdot\frac{\partial t}{\partial y} = \frac{\partial F}{\partial t}(5)$$

$$LHS = 5\frac{\partial F}{\partial x} - 2x\frac{\partial F}{\partial y} = 5(2x)\frac{\partial F}{\partial t} - (2x)(5)\left(\frac{\partial F}{\partial t}\right)$$

$$= 10x\frac{\partial F}{\partial t} - 10x\frac{\partial F}{\partial t} = 0 = RHS \quad \star$$

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 4

TRIGONOMETRIC INTEGRALS AND REDUCTION FORMULAE

Milan Pahor

MATH1231 CALCULUS
TRIGONOMETRIC INTEGRALS AND REDUCTION FORMULAE

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C$$

$$\int \sec(ax)\tan(ax)\,dx = \frac{1}{a}\sec(ax) + C$$

$$\int \mathrm{cosec}(ax)\cot(ax)\,dx = -\frac{1}{a}\mathrm{cosec}(ax) + C$$

$$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + C$$

$$\int \mathrm{cosec}^2(ax)\,dx = -\frac{1}{a}\cot(ax) + C$$

$$\int \sec(ax)\,dx = \frac{1}{a}\ln|\sec(ax) + \tan(ax)| + C$$

$$\int \mathrm{cosec}(ax)\,dx = -\frac{1}{a}\ln|\mathrm{cosec}(ax) + \cot(ax)| + C$$

Note how you can "co" an integral up as long as you pay the price of a minus sign.

Trigonometric Integrals

We have already spent a good deal of time in the MATH1131 course defining the Riemann integral and looking at some specific techniques of integration. We turn now to the special skills required to find certain trigonometric integrals. A frustrating feature of the theory of integration is that it is essentially just the accumulation of a multitude of strange tricks. This is the nature of the beast. Keep in mind that **most** integrals cannot be done, so we should be thankful for the rare occasions where we can actually solve the problem. Applications of the integral will be examined later in this course. For the moment however, we are focused only upon techniques of evaluation.

You should commit the above integrals to memory. A table of integrals will be available for the final examination but NOT for the short quizzes.

$$\int \sin^m(x)\cos^n(x)\,dx \text{ where at least one of } m \text{ or } n \text{ are odd.}$$

Method: Split off the function of odd power and use the substitution $u =$ other one.

Example 1 Find $\int \sin^4(x)\cos^3(x)\,dx$.

Looks like $u = \sin(x)$ might be good!

$$= \int \sin^4(x)\cos^2(x)\,[\cos(x)\,dx]$$

$$= \int \sin^4(x)\{1-\sin^2(x)\}\,[\cos(x)\,dx]$$

Let $u = \sin(x) \Rightarrow du = \cos(x)\,dx$.

$$\int = \int u^4(1-u^2)\,[du]$$

$$= \int u^4 - u^6\,du = \frac{u^5}{5} - \frac{u^7}{7}$$

$$= \frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C$$

$$\star \quad \frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C \quad \star$$

$$\int \sin^m(x)\cos^n(x)\,dx \text{ where both } m \text{ and } n \text{ are even.}$$

Method: Make repeated use of

$$\sin^2(\theta) = \frac{1}{2}\{1-\cos(2\theta)\}$$

$$\cos^2(\theta) = \frac{1}{2}\{1+\cos(2\theta)\}$$

Example 2 Find $\int \sin^4(x)\cos^2(x)\,dx$. Even though Example 2 looks almost identical to Example 1 we need to use a completely different approach since there is no odd power to play with. Thus:

$$\begin{aligned}
&\int \sin^4(x)\cos^2(x)\,dx = \int \sin^2(x)\sin^2(x)\cos^2(x)\,dx\\
&= \int \frac{1}{2}(1-\cos 2x)\frac{1}{2}(1-\cos 2x)\frac{1}{2}(1+\cos 2x)\,dx\\
&= \frac{1}{8}\int (1-\cos 2x)(1-\cos 2x)(1+\cos 2x)\,dx\\
&= \frac{1}{8}\int (1-\cos 2x)(1-\cos^2(2x))\,dx\\
&= \frac{1}{8}\int (1-\cos 2x)\sin^2(2x)\,dx\\
&= \frac{1}{8}\int \sin^2(2x) - \sin^2(2x)\cos 2x\,dx\\
&= \frac{1}{8}\left\{\int \sin^2(2x)\,dx - \int \sin^2(2x)\cos 2x\,dx\right\}\\
&= \frac{1}{8}\{I_1 - I_2\}
\end{aligned}$$

$$I_1 = \int \sin^2(2x)\,dx$$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\sin^2 2x = \frac{1}{2}(1-\cos(4x))$

$\int \sin^2(2x)dx$

$= \frac{1}{2}\int 1-\cos(4x)dx$

$= \frac{1}{2}\{x - \frac{1}{4}\sin 4x\}$

$$I_2 = \int \sin^2(2x)\cos 2x\,dx$$

Let $u = \sin(2x)$

$du = 2\cos(2x)dx$

$I_2 = \int u^2 \frac{du}{2}$

$= \frac{1}{2}\{u^3/3\} = \frac{1}{6}u^3$

$= \frac{1}{6}\sin^3(2x)$

$$\star \quad \frac{x}{16} - \frac{1}{64}\sin(4x) - \frac{1}{48}\sin^3(2x) + C \quad \star$$

$\therefore I = \frac{1}{8}\{\frac{1}{2}x - \frac{1}{8}\sin 4x - \frac{1}{6}\sin^3(2x)\} + C$

$$\int \sin(mx)\cos(nx)\,dx \qquad \int \sin(mx)\sin(nx)\,dx \qquad \int \cos(mx)\cos(nx)\,dx$$

Method: Using

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

OR

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

convert the products to sums and then integrate.

Example 3 Find $\int \sin(7x)\cos(3x)\,dx$.

$$\mathrm{I} \quad \sin(7x+3x) = \sin(7x)\cos(3x) + \cos(\cancel{7x})\sin(3x)$$

$$\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B.$$

$$\mathrm{II} \quad \sin(7x-3x) = \sin(7x)\cos(3x) - \cos(\cancel{7x})\sin(3x)$$

$$\mathrm{I}+\mathrm{II}: \quad \sin(10x) + \sin(4x) = 2\sin(7x)\cos(3x)$$

$$\therefore \int \sin(7x)\cos(3x) = \frac{1}{2}\int \{\sin(10x) + \sin(4x)\}$$

$$= \frac{1}{2}\left\{-\frac{1}{10}\cos(10x) - \frac{1}{4}\cos(4x)\right\} + C$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B$$

$\star \quad -\frac{1}{20}\cos(10x) - \frac{1}{8}\cos(4x) + C \quad \star$

Integrals involving $\tan(x)$ and $\sec(x)$.

Method: Make use of the fact that $\dfrac{d}{dx}\sec(x) = \sec(x)\tan(x)$ and/or $\dfrac{d}{dx}\tan(x) = \sec^2(x)$.

Example 4 Find $\displaystyle\int_0^{\frac{\pi}{3}} \tan(x)\sec^4(x)\,dx$

Option 1:

$u = \sec(x)$!!

$$\int_0^{\frac{\pi}{3}} \tan(x)\sec^4(x)\,dx = \int_0^{\frac{\pi}{3}} \sec^3(x)\boxed{\sec(x)\tan(x)\,dx}$$

Let $u = \sec(x)$: $du = \sec(x)\tan(x)dx$.

$x = 0 \to u = \sec(0) = \frac{1}{\cos(0)} = \frac{1}{1} = 1$

$x = \pi/3 \to u = \sec(\pi/3) = \frac{1}{\cos(\pi/3)} = \frac{1}{\frac{1}{2}} = 2$

$\int = \int_1^2 u^3\,du = \left[\frac{u^4}{4}\right]_1^2 = \frac{2^4}{4} - \frac{1}{4}$

$= 4 - \frac{1}{4} = 15/4$

Option2:

!! $u = \tan(x)$!!

$$\int_0^{\frac{\pi}{3}} \tan(x)\sec^4(x)\,dx = \int_0^{\frac{\pi}{3}} \tan(x)\sec^2(x)\boxed{\sec^2(x)\,dx} = \int_0^{\frac{\pi}{3}} \tan(x)(1+\tan^2(x))\underbrace{\sec^2(x)\,dx}_{du}$$

Let $u = \tan(x)$: $du = \sec^2(x)dx$.

$x = 0 \to u = 0$, $x = \pi/3 \to u = \tan(\pi/3) = \sqrt{3}$

$\int = \int_0^{\sqrt{3}} u(1+u^2)du = \int_0^{\sqrt{3}} u + u^3\,du$

$= \left[\frac{u^2}{2} + \frac{u^4}{4}\right]_0^{\sqrt{3}} = \left(\frac{3}{2} + \frac{(\sqrt{3})^4}{4}\right) - (0)$

$= \frac{3}{2} + \frac{9}{4} = 15/4$

$\star\ \frac{15}{4}\ \star$

Reduction Formulae

Sometimes an integral is too difficult to solve in one simple attempt. Instead we produce a recursive recipe through which the answer may be found. These recipes are called reduction formulae or recurrence relations. Reduction formulae should not be memorised. If you are expected to use one in an exam it will be supplied. Note that reduction formulae are almost always verified using integration by parts

$$\int u\,dv = uv - \int v\,du$$

Example 5 Find $\int 3xe^{4x}\,dx$. (This is just revision of integration by parts)

$u = 3x \rightarrow du = 3$

$dv = e^{4x} \rightarrow v = \frac{1}{4}e^{4x}$

$\int u\,dv = (3x)(\frac{1}{4}e^{4x}) - \int(\frac{1}{4}e^{4x})(3)$

$= \frac{3}{4}xe^{4x} - \frac{3}{4}\int e^{4x}\,dx.$

$= \frac{3}{4}xe^{4x} - \frac{3}{16}e^{4x} + C$

★ $\frac{3}{4}xe^{4x} - \frac{3}{16}e^{4x} + C$ ★

Example 6 Let $I_n = \int x^n e^x \, dx$. Show that

$$I_n = x^n e^x - nI_{n-1}$$

Hence find $\int x^3 e^x \, dx$.

$$I_n = \int x^n e^x \, dx.$$

$$u = x^n \to du = nx^{n-1}$$

$$dv = e^x \to v = e^x$$

$$\int u\,dv = uv - \int v\,du$$

$$= x^n e^x - \int e^x n x^{n-1} dx.$$

$$= x^n e^x - n \int x^{n-1} e^x dx.$$

$$\therefore I_n = x^n e^x - n I_{n-1} \quad \#$$

$$I_3 = x^3 e^x - 3 I_2$$

$$= x^3 e^x - 3\{x^2 e^x - 2 I_1\}$$

$$= x^3 e^x - 3x^2 e^x + 6 I_1$$

$$= x^3 e^x - 3x^2 e^x + 6\{x e^x - 1\, I_0\}$$

$$= x^3 e^x - 3x^2 e^x + 6x e^x - 6 I_0$$

$$I_0 = \int x^0 e^x dx = \int e^x dx = e^x.$$

$$\therefore I_3 = x^3 e^x - 3x^2 e^x + 6x e^x - 6 e^x + C$$

★ $x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$ ★

Example 7 Let

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n(x)\,dx.$$

Show that $\quad I_n = \dfrac{1}{n-1} - I_{n-2}\quad$ and hence evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan^4(x)\,dx.$

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n(x)\,dx = \int_0^{\frac{\pi}{4}} \tan^{n-2}(x)\tan^2(x)\,dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2}(x)\{\sec^2(x) - 1\}$$

$$= \underbrace{\int_0^{\pi/4} \tan^{n-2}(x)\,\underbrace{\sec^2(x)\,dx}_{du}} - \underbrace{\int_0^{\frac{\pi}{4}} \tan^{n-2}(x)\,dx}_{I_{n-2}\,!!}$$

Let $u = \tan(x)$
$du = \sec^2(x)\,dx.$

$x = 0 \to u = 0$
$x = \frac{\pi}{4} \to u = \tan\frac{\pi}{4} = 1$

$$\int = \int_0^1 u^{n-2}\,du = \left[\frac{u^{n-1}}{n-1}\right]_0^1 = \frac{1^{n-1}}{n-1} = \frac{1}{n-1}$$

$$\therefore\ I_n = \frac{1}{n-1} - I_{n-2} \quad \text{as required.}$$

So $I_4 = \frac{1}{3} - I_2 = \frac{1}{3} - \{\frac{1}{1} - I_0\}$

$$= \frac{1}{3} - 1 + I_0$$

$$I_0 = \int_0^{\pi/4} \tan^0(x)\,dx = \int_0^{\frac{\pi}{4}} 1\,dx = [x]_0^{\frac{\pi}{4}} = \frac{\pi}{4}$$

$$\therefore\ I_4 = -2/3 + \frac{\pi}{4}$$

$\star\ \frac{\pi}{4} - \frac{2}{3}\ \star$

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 5

TRIGONOMETRIC AND HYPERBOLIC TRIG. SUBSTITUTIONS

Milan Pahor

MATH1231 CALCULUS
TRIGONOMETRIC AND HYPERBOLIC TRIG. SUBSTITUTIONS

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\frac{d}{dx}\sinh(x) = \cosh(x)$$

$$\frac{d}{dx}\cosh(x) = \sinh(x)$$

$$\frac{d}{dx}\tanh(x) = \text{sech}^2(x)$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$1 - \tanh^2(x) = \text{sech}^2(x)$$

$$\sinh(2x) = 2\sinh(x)\cosh(x)$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x)$$

$$\tanh(2x) = \frac{2\tanh(x)}{1+\tanh^2(x)}$$

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$$

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$$

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$\int$ involving $\sqrt{a^2 - x^2}$ use the substitution $x = a\sin(\theta)$ or $x = a\tanh(\theta)$

$\int$ involving $\sqrt{a^2 + x^2}$ use the substitution $x = a\tan(\theta)$ or $x = a\sinh(\theta)$

$\int$ involving $\sqrt{x^2 - a^2}$ use the substitution $x = a\sec(\theta)$ or $x = a\cosh(\theta)$

Trigonometric and Hyperbolic Trigonometric Substitutions

We have already seen in the Math1131 course that a substitution can often clarify the essential nature of an integral. For a little revision consider:

Example 1 Evaluate $\int_0^{\sqrt{\pi}} x \sin(x^2)\, dx$.

Let $u = x^2 \Rightarrow du = 2x\,dx. \Rightarrow x\,dx = \frac{du}{2}$

$x = 0 \to u = 0$

$x = \sqrt{\pi} \to u = \pi$

$$\int = \int_0^{\pi} \sin(u) \frac{du}{2}$$

$$= \frac{1}{2}\left[-\cos(u)\right]_0^{\pi}$$

$$= \frac{1}{2}\left\{(-\cos\pi) - (-\cos 0)\right\}$$

$$= \frac{1}{2}\left\{(--1) - (-1)\right\}$$

$$= \frac{1}{2}\left\{1+1\right\} = \underline{\underline{1}}$$

★ 1 ★

When implementing a substitution make sure that you always change:

- The function.
- The increment dx.
- The limits (if there are limits).

In this lecture we will focus on trigonometric and hyperbolic trigonometric substitutions. First a little more revision on the hyperbolic trigonometric functions.

The hyperbolic trig. functions $y = \sinh(x)$, $y = \cosh(x)$ and $y = \tanh(x)$ are "fake" trigonometric functions constructed from the exponential function. They mimic (but do not exactly replicate) most of the trigonometric properties. Their definitions are:

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

and

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

The extra h stands for hyperbolic and their unique connections with the trig. functions are as follows:

a) $\frac{d}{dx}\sinh(x) = \cosh(x)$ $\quad\star\ \frac{d}{dx}\sin(x) = \cos(x)\ \star$

b) $\frac{d}{dx}\cosh(x) = \sinh(x)$ $\quad\star\ \frac{d}{dx}\cos(x) = -\sin(x)\ \star$

c) $\frac{d}{dx}\tanh(x) = \text{sech}^2(x)$ $\quad\star\ \frac{d}{dx}\tan(x) = \sec^2(x)\ \star$

d) $\cosh^2(x) - \sinh^2(x) = 1$ $\quad\star\ \cos^2(x) + \sin^2(x) = 1\ \star$

e) $1 - \tanh^2(x) = \text{sech}^2(x)$ $\quad\star\ 1 + \tan^2(x) = \sec^2(x)\ \star$

f) $\sinh(2x) = 2\sinh(x)\cosh(x)$ $\quad\star\ \sin(2x) = 2\sin(x)\cos(x)\ \star$

g) $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$ $\quad\star\ \cos(2x) = \cos^2(x) - \sin^2(x)\ \star$

It is clear from the above properties that the hyperbolic trig. functions do indeed behave very much like the real trig. functions! But keep your eye on those negatives!

When it comes to inverting the hyberbolic trig. functions we also have:

h) $\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$

i) $\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$

j) $\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Note that since the hyperbolic trigonometric functions are defined in terms of the exponential function it should come as no surprise that the inverses are expressed in terms of the natural log function!

Let's prove some of the results. The proofs will involve the exponential definitions and a little algebra.

Proof b): $\dfrac{d}{dx}\cosh(x) = \sinh(x)$

$$\frac{d}{dx}\cosh(x) = \frac{d}{dx}\left\{\frac{e^x + e^{-x}}{2}\right\} = \frac{e^x - e^{-x}}{2}$$

$$= \sinh(x)$$

★

Proof d): $\cosh^2(x) - \sinh^2(x) = 1$

$$LHS = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$\frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4}$$

$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - (\cancel{e^{2x}} - 2 + \cancel{e^{-2x}})}{4}$$

$$= \frac{4}{4}$$

$$= 1 = RHS$$

★

Proof j): $\tanh^{-1}(x) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$

$$y = \tanh(x)$$

$$= \frac{\sinh(x)}{\cosh(x)}$$

$$= \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

swap: $$x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$xe^y + xe^{-y} = e^y - e^{-y}$$

$$xe^y - e^y = -e^{-y} - xe^{-y}$$

$$e^y - xe^y = e^{-y} + xe^{-y}$$

$$e^y(1-x) = e^{-y}(1+x)$$

$$e^{2y}(1-x) = 1+x$$

$$e^{2y} = \frac{1+x}{1-x}$$

$$2y = \ln\left(\frac{1+x}{1-x}\right)$$

★

$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\therefore \tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

When faced with integral involving square roots, trigonometric substitutions or hyperbolic trigonometric substitutions are often effective. As a general rule:

$\int$ involving $\sqrt{a^2 - x^2}$	use the substitution $x = a\sin(\theta)$ or $x = a\tanh(\theta)$
$\int$ involving $\sqrt{a^2 + x^2}$	use the substitution $x = a\tan(\theta)$ or $x = a\sinh(\theta)$
$\int$ involving $\sqrt{x^2 - a^2}$	use the substitution $x = a\sec(\theta)$ or $x = a\cosh(\theta)$

Whether you use trig. or hyperbolic trig. subs is up to you, unless of course the examiner forces a particular approach.

Example 2 Find $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\,dx$:

a) using a non-trig substitution.

b) using a trigonometric substitution.

c) using a hyperbolic trigonometric substitution.

a) $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+4}}$ Let $u = x^2+4$, $du = 2x\,dx.$

$= \displaystyle\int \frac{1}{\sqrt{u}}\frac{du}{2} = \frac{1}{2}\int u^{-\frac{1}{2}}\,du$

$= \left(\frac{1}{2}\right)\dfrac{u^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} = u^{\frac{1}{2}} = (x^2+4)^{\frac{1}{2}} + c$

$= \underline{\sqrt{x^2+4} + C}$

b) Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$

$\displaystyle\int = \int \frac{2\tan\theta\; 2\sec^2\theta\,d\theta}{\sqrt{4\tan^2\theta + 4}}$

$= \displaystyle\int \frac{4\tan\theta\,\sec^2\theta\,d\theta}{\sqrt{4(\tan^2\theta + 1)}}$

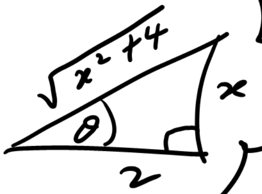
$$= \int \frac{4 \tan\theta \sec^2\theta \, d\theta}{2\sqrt{1+\tan^2\theta}} = 2\int \frac{\tan\theta \sec^2\theta \, d\theta}{\sqrt{\sec^2\theta}}$$

$$= 2\int \frac{\tan\theta \sec^2\theta}{\sec\theta} = 2\int \sec\theta \tan\theta \, d\theta$$

$$= 2\sec\theta + C.$$

$x = 2\tan\theta$

$\tan\theta = \frac{x}{2}$



$$= 2 \cdot \frac{1}{\cos\theta} + C$$

$$= 2 \frac{1}{\frac{2}{\sqrt{x^2+4}}} = 2 \times \frac{\sqrt{x^2+4}}{2}$$

$$= \underline{\sqrt{x^2+4} + C}$$

c) $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+4}}$

Let $x = 2\sinh\theta \Rightarrow dx = 2\cosh\theta \, d\theta$

$$\int = \int \frac{2\sinh\theta \, 2\cosh\theta \, d\theta}{\sqrt{4\sinh^2\theta + 4}} = \int \frac{4\sinh\theta \cosh\theta \, d\theta}{\sqrt{4}\sqrt{1+\sinh^2\theta}}$$

Recall $\cosh^2\theta - \sinh^2\theta = 1 \Rightarrow \underline{\cosh^2\theta = 1+\sinh^2\theta}$

$$\int = 2\int \frac{\sinh\theta \cancel{\cosh\theta} \, d\theta}{\cancel{\sqrt{\cosh^2\theta}}} = 2\int \sinh(\theta)\, d\theta$$

$$= 2\cosh\theta + C$$

$$= 2\sqrt{1+\sinh^2\theta} = 2\sqrt{1+\left(\frac{x}{2}\right)^2}$$

$$= 2\sqrt{\frac{4+x^2}{4}} = \frac{\cancel{2}\sqrt{4+x^2}}{\cancel{\sqrt{4}}} = \underline{\sqrt{4+x^2} + C}$$

★ $\sqrt{4+x^2}$ ★

A

B

Example 3 Find $\int_0^3 \frac{75}{(25-x^2)^{\frac{3}{2}}}\,dx$:

a) using a trigonometric substitution.

b) using a hyperbolic trigonometric substitution.

a) Let $x = 5\sin\theta$.
$dx = 5\cos\theta\,d\theta$

$x=0 \Rightarrow 5\sin\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$x=3 \Rightarrow 5\sin\theta = 3 \Rightarrow \sin\theta = 3/5 \Rightarrow \theta = \sin^{-1}(3/5)$

$$\int = \int_0^{\sin^{-1}(3/5)} \frac{75.(5\cos\theta\,d\theta)}{(25-25\sin^2\theta)^{3/2}}$$

$$= \int_0^{\sin^{-1}(3/5)} \frac{5\times 75\cos\theta\,d\theta}{(25)^{3/2}(1-\sin^2\theta)^{3/2}}$$

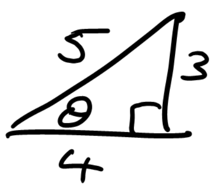
$$\frac{5\times 75}{125}\int_0^{\sin^{-1}(3/5)} \frac{\cos\theta\,d\theta}{(\cos^2\theta)^{3/2}}$$

$$3\int_0^{\sin^{-1}(3/5)} \frac{\cos\theta\,d\theta}{\cos^3\theta\,d\theta} = 3\int_0^{\sin^{-1}(3/5)} \frac{1}{\cos^2\theta}\,d\theta$$

$$= 3\int_0^{\sin^{-1}(3/5)} \sec^2\theta\,d\theta = 3\left[\tan\theta\right]_0^{\sin^{-1}(3/5)}$$

$$= 3\left(\tan\sin^{-1}(3/5) - 0\right) = 3\tan(\underbrace{\sin^{-1}(3/5)}_{\theta})$$

Let $\theta = \sin^{-1}(3/5) \Rightarrow \sin\theta = 3/5 \Rightarrow$



$= 3\tan\theta = 3 \cdot \frac{3}{4} = \frac{9}{4}$ //

$\operatorname{sech}\theta = \frac{1}{\cosh\theta}$

b) $x = 5\tanh(\theta)$

$dx = 5\operatorname{sech}^2\theta \, d\theta$

$$\int_0^3 \frac{75 \, dx}{(25-x^2)^{3/2}}$$

(continued after page 8)

With a little luck in b) we have reached an answer of $3\sinh(\tanh^{-1}(\frac{3}{5}))$.

We have three different ways of simplifying this:

Method 1: Calculator! Quick and Nasty.

Method 2: $\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ (this fact is in our table of integrals!) so

$$\tanh^{-1}(\frac{3}{5}) = \frac{1}{2}\ln\left(\frac{1+\frac{3}{5}}{1-\frac{3}{5}}\right) = \frac{1}{2}\ln\left(\frac{\frac{8}{5}}{\frac{2}{5}}\right) = \frac{1}{2}\ln(4) = \frac{1}{2}\ln(2^2) = \frac{1}{2} \times 2\ln(2) = \ln(2).$$

Thus $3\sinh(\tanh^{-1}(\frac{3}{5})) = 3\sinh(\ln(2)) =$ $3\sinh(\ln(2))$

Recall $\sinh x = \frac{e^x - e^{-x}}{2}$ $\Rightarrow$

$$= 3\left\{\frac{e^{\ln(2)} - e^{-\ln(2)}}{2}\right\}$$

$$= 3\left\{\frac{2 - e^{\ln(2^{-1})}}{2}\right\} = 3\left\{\frac{2 - \frac{1}{2}}{2}\right\}$$

$$= 3\left\{\frac{4-1}{4}\right\} = \frac{9}{4} \quad //$$

Method 3: Let $\alpha = \tanh^{-1}(\frac{3}{5})$.

Then $\tanh(\alpha) = \frac{3}{5} \implies \tanh^2(\alpha) = \frac{9}{25} \implies 1 - \operatorname{sech}^2(\alpha) = \frac{9}{25} \implies \operatorname{sech}^2(\alpha) = \frac{16}{25}$

$\implies \cosh^2(\alpha) = \frac{25}{16} \implies 1 + \sinh^2(\alpha) = \frac{25}{16} \implies \sinh^2(\alpha) = \frac{9}{16} \implies \sinh(\alpha) = \frac{3}{4}$. $(\alpha > 0)$

Thus $3\sinh(\tanh^{-1}(\frac{3}{5})) = 3\sinh(\alpha) = 3 \times \frac{3}{4} = \frac{9}{4}$.

The reason Method 3 is so gruesome is that we cannot appeal to a little triangle to sort out all the ratios. But in reality the hyperbolic trigonometric functions are **NOT** trigonometric functions (even though they pretend to be) so triangles will never save us when dealing with hyperbolics.

★ $\frac{9}{4}$ ★

$$\int_0^3 \frac{75\,dx}{(25-x^2)^{3/2}}$$

$x = 5\tanh\theta$
$dx = 5\,\text{sech}^2\theta\,d\theta$

$x=0 \Rightarrow 0 = 5\tanh\theta \Rightarrow \tanh\theta = 0$
$\Rightarrow \theta = 0.$

$x=3 \Rightarrow 3 = 5\tanh\theta \Rightarrow \tanh\theta = 3/5$
$\Rightarrow \theta = \tanh^{-1}(3/5)$

$$\int = \int_0^{\tanh^{-1}(3/5)} \frac{75.(5)\,\text{sech}^2\theta\,d\theta}{(25 - 25\tanh^2\theta)^{3/2}}$$

$$\int_0^{\tanh^{-1}(3/5)} \frac{(5)(75)\,\text{sech}^2\theta\,d\theta}{(25)^{3/2}\,(1-\tanh^2\theta)^{3/2}}$$

$$= 3\int_0^{\tanh^{-1}(3/5)} \frac{\text{sech}^2\theta\,d\theta}{(\text{sech}^2\theta)^{3/2}}$$

$$= 3\int_0^{\tanh^{-1}(3/5)} \frac{\text{sech}^2\theta\,d\theta}{\text{sech}^3\theta\,d\theta}$$

$$= 3\int_0^{\tanh^{-1}(3/5)} \frac{1}{\text{sech}\theta}\,d\theta$$

$$= 3\int_0^{\tanh^{-1}(3/5)} \cosh\theta\,d\theta = 3\Big[\sinh\theta\Big]_0^{\tanh^{-1}(3/5)}$$

$$= 3\sinh(\tanh^{-1}(3/5)) - 0$$

$$= \underline{3\sinh(\tanh^{-1}(3/5))} = \frac{9}{4} \checkmark$$

(calulator)

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 6

INTEGRATION BY PARTIAL FRACTIONS

Milan Pahor

MATH1231 CALCULUS
INTEGRATION BY PARTIAL FRACTIONS

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(\frac{x}{a}) + C = \ln(x+\sqrt{x^2-a^2}) + D$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}(\frac{x}{a}) + C = \ln(x+\sqrt{x^2+a^2}) + D$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$$

$$\int \frac{f'}{f}\,dx = \ln|f| + C$$

Before moving on to integration by partial fractions lets extend your standard table of integrals a little.

Recall that $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C$. But what about $\int \frac{dx}{\sqrt{x^2-1}}$?

Claim:

$$\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}(x) + C = \ln(x+\sqrt{x^2-1}) + C$$

Proof:

$\cosh^2\theta - \sinh^2\theta = 1$

$\cosh^2\theta - 1 = \sinh^2\theta$

Let $x = \cosh(\theta)$

$dx = \sinh(\theta)\,d\theta$

$\int = \int \frac{\sinh\theta\,d\theta}{\sqrt{\cosh^2\theta - 1}}$

$\int = \int \frac{\sinh\theta\,d\theta}{\sqrt{\sinh^2\theta}} = \int 1\,d\theta$

$= \theta = \cosh^{-1}(x) + C$

Let $x = \sec(\theta)$ **(Homework)**

$dx = \sec\theta\tan\theta\,d\theta$

$\int = \int \frac{\sec\theta\tan\theta\,d\theta}{\sqrt{\sec^2\theta - 1}}$

$1 + \tan^2\theta = \sec^2\theta$

$\int = \int \frac{\sec\theta\,\cancel{\tan\theta}\,d\theta}{\cancel{\sqrt{\tan^2\theta}}}$

$= \int \sec\theta\,d\theta$

$= \ln|\sec\theta + \tan\theta|$

$= \ln|x + \sqrt{x^2-1}| + C$

★

Two new table entries you will find in the university table of integrals are:

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C = \ln(x + \sqrt{x^2 - a^2}) + D$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C = \ln(x + \sqrt{x^2 + a^2}) + D$$

Note that the constants of integration C and D are not necessarily the same.

Example 1 Find

$$\int \frac{dx}{\sqrt{x^2 - 25}}$$

$a = 5$

$$\therefore \int = \cosh^{-1}\left(\frac{x}{5}\right) + C$$

$$= \ln(x + \sqrt{x^2 - 25}) + D$$

★ $\cosh^{-1}(\frac{x}{5}) + C$ or $\ln(x + \sqrt{x^2 - 25}) + D$ ★

Example 2 Find

$$\int \frac{dx}{\sqrt{x^2 - 6x + 13}}$$

$$\int \frac{dx}{\sqrt{x^2 - 6x + 9 + 4}} = \int \frac{dx}{\sqrt{(x-3)^2 + 4}}$$

Let $u = x - 3$, $du = dx$

$$= \int \frac{du}{\sqrt{u^2 + 4}}$$

$$= \sinh^{-1}\left(\frac{u}{2}\right) = \sinh^{-1}\left(\frac{x-3}{2}\right) + C$$

★ $\sinh^{-1}(\frac{x-3}{2}) + C$ or $\ln(x - 3 + \sqrt{x^2 - 6x + 13}) + D$ ★

Next a little further revision in preparation for partial fractions.

Recall

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$$

$$\int \frac{f'}{f}\, dx = \ln|f| + C$$

Example 3 Find each of the following integrals:

a) $\displaystyle\int \frac{7}{2x-5}\, dx$

b) $\displaystyle\int \frac{9}{x^2+36}\, dx$

c) $\displaystyle\int \frac{4}{25x^2+36}\, dx$

d) $\displaystyle\int \frac{1}{(3x+5)^7}\, dx$

e) $\displaystyle\int \frac{10x-4}{x^2+6x+13}\, dx$

a) $\displaystyle\int \frac{7}{2x-5}\, dx = \frac{7}{2}\int \frac{2\,dx}{2x-5}$

$= 7/2 \ln|2x-5| + c$

b) $\displaystyle\int \frac{9}{x^2+36}\, dx. = 9\int \frac{dx}{x^2+36}$

$= 9\left(\frac{1}{6}\right)\tan^{-1}\left(\frac{x}{6}\right) = 3/2 \tan^{-1}\left(\frac{x}{6}\right) + c$

c) $\displaystyle\int \frac{4\,dx}{25x^2+36} = 4\int \frac{dx}{25x^2+36}$

$\displaystyle= \frac{4}{25}\int \frac{dx}{x^2+\left(\frac{36}{25}\right)}$ $\quad a^2 \quad \therefore a = \frac{6}{5}.$

$\displaystyle= \frac{4}{25}\left(\frac{1}{\frac{6}{5}}\right)\tan^{-1}\left(\frac{x}{\left(\frac{6}{5}\right)}\right) = \frac{4}{25}\cdot\frac{5}{6}\cdot\tan^{-1}\left(\frac{5x}{6}\right)$

$$= \frac{4}{30}\tan^{-1}\left(\frac{5x}{6}\right) = \frac{2}{15}\tan^{-1}\left(\frac{5x}{6}\right) + c$$

d) $\int \frac{dx}{(3x+5)^7}$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)(a)} + c$$

$$\int (3x+5)^{-7} dx = \frac{(3x+5)^{-6}}{(-6)(3)}$$

$$= -\frac{1}{18}(3x+5)^{-6}$$

$$= -\frac{1}{18}\frac{1}{(3x+5)^6} + c$$

e) $\int \frac{10x-4}{x^2+6x+13} dx$

$$= \int \frac{10x-4 \, dx}{x^2+6x+9+4} = \int \frac{10x-4 \, dx}{(x+3)^2+4}$$

Let $u = x+3$
$du = dx$

$$= \int \frac{10(u-3)-4 \, du}{u^2+4}$$

$$= \int \frac{10u-34}{u^2+4} du = \int \frac{10u \, du}{u^2+4} - 34\int \frac{du}{u^2+4}$$

$$= 5\int \frac{2u \, du}{u^2+4} - 34 \cdot \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) = 5\ln(u^2+4) - 17\tan^{-1}\left(\frac{u}{2}\right)$$

Recall $u = x+3$

★ $a)$ $\frac{7}{2}\ln|2x-5|$ $b)$ $\frac{3}{2}\tan^{-1}(\frac{x}{6})$ $c)$ $\frac{2}{15}\tan^{-1}(\frac{5x}{6})$ $d)$ $-\frac{1}{18}\frac{1}{(3x+5)^6}$ ★

★ $e)$ $5\ln(x^2+6x+13) - 17\tan^{-1}(\frac{x+3}{2})$ $+c.$ ★

A

B

Integration by Partial Fractions (Part 1)

Integrals of the form

$$\int \frac{\text{little poly}}{\text{big factored poly}}\,dx$$

can be knocked off by using partial fraction decompositions. Typical examples of parfrac integrals are:

$$\int \frac{10x-41}{(x-5)(x-2)}\,dx$$

$$\int \frac{5x^2+9x+19}{(x-8)(x^2+6x+25)}\,dx$$

Observe that:

- Only polynomials are involved.
- The poly on the bottom is factorisable.
- The poly on the top is of a smaller degree than the poly on bottom. It doesn't have to be exactly one degree smaller....just smaller.

All of these things need to happen for an integral to be solvable via partial fractions. Our method of attack is to rewrite the function so that it involves sums rather than products. This opens the door so that integration becomes possible.

Note that the quotient of two polynomials is often called a **rational function**. If also, the degree of the numerator is strictly smaller than the degree of the denominator it is call a **proper rational function**....just like a proper fraction!

Example 4

a) Show that

$$\frac{5x-29}{x^2-12x+35} = \frac{3}{x-7} + \frac{2}{x-5}$$

b) Hence find

$$\int \frac{5x-29}{x^2-12x+35}\,dx$$

(a) RHS $= \frac{3}{x-7} + \frac{2}{x-5} = \frac{3(x-5)+2(x-7)}{(x-7)(x-5)}$

$= \frac{3x-15+2x-14}{x^2-5x-7x+35}$

$= \frac{5x-29}{x^2-12x+35} = \text{LHS}$

b) $\displaystyle\int \frac{5x-29}{x^2-12x+35}\,dx$

$$= \int \frac{3}{x-7} + \frac{2}{x-5}\,dx.$$

$$= 3\ln|x-7| + 2\ln|x-5| + C$$

Note that even though we calculated two integrals we need only one constant.

$\bigstar \quad 2\ln|x-5| + 3\ln|x-7| + C \quad \bigstar$

Observe how easy it was to integrate once we had rewritten the function as a sum of two simpler objects. But how do we do this in general?

Example 5

a) Write $\dfrac{17x-52}{(x-2)(x-8)}$ in the form $\dfrac{A}{(x-2)} + \dfrac{B}{(x-8)}$.

b) Hence evaluate $\displaystyle\int \frac{17x-52}{(x-2)(x-8)}\,dx$.

Observe how we have decomposed the function as a sum over its factors. We now need to find the A and B which do the job. Generally when setting up a partial fraction decomposition, the top of the new components is exactly one degree less than the bottom and arbitrary. We have linear bottoms in the decomposition here so the tops are arbitrary constants. If the bottom was quadratic the top would need to be an arbitrary linear function (we will see this later).

After decomposing we recompose and compare numerators:

$$\frac{17x-52}{(x-2)(x-8)} = \frac{A}{(x-2)} + \frac{B}{(x-8)} = \frac{A(x-8)+B(x-2)}{(x-2)(x-8)}$$

Thus $A(x-8) + B(x-2) = 17x - 52$

$x=8$: $0 + 6B = 17 \times 8 - 52$

$6B = 84 \Rightarrow B = 14$

$x=2$: $A(-6) + 0 = 17 \times 2 - 52 = 34 - 52 = -18$

$\therefore -6A = -18 \Rightarrow A = 3$

$\int = \int \frac{3}{x-2} + \frac{14}{x-8}\, dx \Longrightarrow$ ★ $3\ln|x-2| + 14\ln|x-8| + C$ ★

So a typical partial fraction integral is...........a little poly over a bigger factored poly. We decompose the proper rational function as a sum over its denominator's factors and then integrate the simpler components, usually to log and/or inverse tan.

Example 6 Find $\displaystyle\int \frac{9}{(x-5)(2x-7)}\, dx$

$$\frac{9}{(x-5)(2x-7)} = \frac{A}{x-5} + \frac{B}{2x-7}$$

$$= \frac{A(2x-7)+B(x-5)}{(x-5)(2x-7)}$$

$\therefore A(2x-7) + B(x-5) = 9$

$x=5$: $A(3) = 9 \Rightarrow A = 3$

$x = 7/2$: $B(\frac{7}{2} - 5) = 9$

$B(-3/2) = 9$

$B = \frac{18}{-3} = -6$

$\int \frac{3}{x-5} + \frac{-6}{2x-7}\, dx$

$3\ln|x-5| - \frac{6}{2}\int \frac{2\,dx}{2x-7}$

$3\ln|x-5| - 3\ln|2x-7| + C$

★ $3\ln|x-5| - 3\ln|2x-7| + C$ ★

Example 7 Find $\displaystyle\int \frac{4x^3 - 24x^2 - 15x + 184}{(x-5)(2x-7)}\,dx.$

We have a major problem here! The degree of the numerator is larger than the degree of the denominator. Fortunately a little polynomial division will fix this up. You must never start a partial fraction decomposition unless the degree of the numerator is strictly less than the degree of the denominator. Even equal degrees is a problem.

$$\begin{array}{r|l} & 2x+5 \\ 2x^2-17x+35 & 4x^3-24x^2-15x+184 \\ & 4x^3-34x^2+70x \\ \hline & 10x^2-85x+184 \\ & 10x^2-85x+175 \\ \hline & 9 \end{array}$$

$$\therefore \frac{4x^3-24x^2-15x+184}{2x^2-17x+35} = 2x+5 + \frac{9}{2x^2-17x+35}$$

So/

$$\int \frac{4x^3-24x^2-15x+184}{(x-5)(2x-7)} = \int 2x+5 + \int \frac{9}{(x-5)(2x-7)}$$

$$= x^2+5x+\underbrace{3\ln|x-5|-3\ln|2x-7|}_{\text{from previous example!!}}+C$$

★ $x^2 + 5x + 3\ln|x-5| - 3\ln|2x-7| + C$ ★

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 7

INTEGRATION BY PARTIAL FRACTIONS PART 2

Milan Pahor

MATH1231 CALCULUS
INTEGRATION BY PARTIAL FRACTIONS PART 2

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$$

$$\int \frac{f'}{f}\,dx = \ln|f| + C$$

$$\frac{\text{up to linear}}{(x-3)(x-5)} \longleftrightarrow \frac{A}{x-3} + \frac{B}{x-5}$$

$$\frac{\text{up to quadratic}}{(x-3)(x^2+5)} \longleftrightarrow \frac{A}{x-3} + \frac{Bx+C}{x^2+5}$$

$$\frac{\text{up to cubic}}{(x-3)^2(x^2+5)} \longleftrightarrow \frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{Cx+D}{x^2+5}$$

In this lecture we will continue the theory of integration by partial fractions and have a look at all the different situations which can and do arise.

Example 1 Find $\displaystyle\int \frac{8x^2-16x+28}{(1-x)(x^2+9)}\,dx.$

This looks OK, noting that deg(bottom)=3>2=deg(top) and we have some nice factors to play with in the denominator. Remember that in the decomposition we usually choose a random function on top which is exactly one degree less than the factor in the bottom. Thus

$$\frac{8x^2-16x+28}{(1-x)(x^2+9)} = \frac{A}{1-x} + \frac{Bx+C}{x^2+9}$$

$$= \frac{A(x^2+9)+(Bx+C)(1-x)}{(1-x)(x^2+9)}$$

$$A(x^2+9)+(Bx+C)(1-x) = 8x^2-16x+28$$

$$A(x^2+9)+(Bx+C)(1-x) = 8x^2-16x+28$$

Method 1 (usually the quickest): If you run out of sneaky values of x just use nice values of x.

$$A(x^2+9)+(Bx+C)(1-x)=8x^2-16x+28$$

$\underline{x=1}$: $A(1^2+9)+0=8-16+28$

$10A=20 \Rightarrow \underline{A=2}$

$\underline{x=0}$: $9A+(0+c)(1-0)=0-0+28$

$9A+c=28$

$18+c=28 \Rightarrow c=\underline{10}$

$x=2$: $A(4+9)+(2B+c)(-1)=32-32+28$

$13A-2B-c=28 \Rightarrow 26-2B-10=28$

$-2B=12 \Rightarrow \underline{B=-6}$

Method 2: (This is very weird):

$$A(x^2+9)+(Bx+C)(1-x)=8x^2-16x+28$$

$x=1$: $10A=20 \Rightarrow A=2$

$\underline{x=3i}$: $0A+(3iB+c)(1-3i)=8(-9)-48i+28$

$(3Bi+c)(1-3i)=-72-48i+28=-44-48i$

$c+3Bi=\dfrac{-44-48i}{1-3i}\cdot\dfrac{1+3i}{1+3i}=\dfrac{-44-132i-48i+144}{10}$

$=10-18i$

$\therefore$ $c+3Bi=10-18i$ equate real & imag parts

$\underline{c=10}$, $3B=-18$

$\underline{B=-6}$

Method 3:(Homework) This is usually the longest approach of the three.

We just grind everything into dust and compare powers of x:

$A(x^2+9)+(Bx+C)(1-x)=8x^2-16x+28$

$Ax^2+9A+Bx+C-Bx^2-Cx)=8x^2-16x+28$

$(A-B)x^2+(B-C)x+(9A+C)=8x^2-16x+28$

Hence $A-B=8$, $B-C=-16$, $9A+C=28$

Setting up a system and solving vis Gaussian elimination yields:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 8 \\ 0 & 1 & -1 & -16 \\ 9 & 0 & 1 & 28 \end{array}\right) \Longrightarrow R_3 = R_3 - 9R_1$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 8 \\ 0 & 1 & -1 & -16 \\ 0 & 9 & 1 & -44 \end{array}\right) \longrightarrow$$

$$R_3 = R_3 - 9R_2 \quad \left(\begin{array}{ccc|c} 1 & -1 & 0 & 8 \\ 0 & 1 & -1 & -16 \\ 0 & 0 & 10 & 100 \end{array}\right)$$

echelon form! Now back substitute

$\therefore$ $10C = 100 \Rightarrow \underline{C=10}$

$B - C = -16 \Rightarrow B - 10 = -16 \Rightarrow \underline{B=-6}$

$A - B = 8 \Rightarrow A + 6 = 8 \Rightarrow \underline{\underline{A=2}}$

★ Using any of the three methods: $A=2$ $B=-6$ $C=10$ ★

Thus

$$\frac{8x^2-16x+28}{(1-x)(x^2+9)} = \frac{2}{1-x} + \frac{-6x+10}{x^2+9}$$

and hence

$$\int \frac{8x^2-16x+28}{(1-x)(x^2+9)}\,dx = \int \frac{2}{1-x} + \frac{-6x+10}{x^2+9}$$

$$\int \frac{2}{1-x}\,dx = (-2)\int \frac{-1}{1-x}\,dx. = -2\ln|1-x|$$

$$\int \frac{-6x}{x^2+9}\,dx = (-3)\int \frac{2x}{x^2+9}\,dx = -3\ln|x^2+9|$$

$$\int \frac{10}{x^2+9}\,dx = 10\int \frac{dx}{x^2+9} = 10\,\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$$

$$\bigstar \quad -2\ln|1-x| + \frac{10}{3}\tan^{-1}\left(\frac{x}{3}\right) - 3\ln(x^2+9) + K \quad \bigstar$$

Below are some proper rational functions together with the appropriate partial fraction template. We will NOT be carrying out the integrals:

$$\frac{7x-2}{(x-2)(x-8)} \longleftrightarrow \frac{A}{x-2} + \frac{B}{x-8}$$

$$\frac{4x}{(x^2+9)(x-2)(x-5)} \longleftrightarrow \frac{Ax+B}{x^2+9} + \frac{C}{x-2} + \frac{D}{x-5}$$

$$\frac{7x^5+13x^4-12x^3-8x-11}{(x^3+9x^2+11x-3)(x^2+5x+2)(2x+25)} \longleftrightarrow \frac{Ax^2+Bx+C}{x^3+9x^2+11x-3} + \frac{Dx+E}{x^2+5x+2} + \frac{F}{2x+25}$$

In each case above observe how the factors in the denominator drive the decomposition.

$$\frac{7x^6+13x^4-12x^3-8x-11}{(x^3+9x^2+11x-3)(x^2+5x+2)(2x+25)} \longleftrightarrow \text{No good! Must divide first!!}$$

We will finish with one last tricky situation which deals with the case where there are factors of multiplicity greater than 1 in the denominator.

Example 2 Find

$$\int \frac{10x^2 + 10x - 25}{(x+1)^2(x-4)}\,dx$$

The problem here is "how do we perceive $(x+1)^2$ "? Is it linear or quadratic?

The answer is that we view it in the simplest possibly way and thus it is linear (raised to a power) and thus we only need constants on the top in the decomposition. But care needs to be taken to include all possibilities by stepping through the powers when establishing the template. So:

$$\frac{10x^2 + 10x - 25}{(x+1)^2(x-4)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-4)}$$

We are still treating all three denominators as linear and thus we need only constants on top! We also need to recompose with care. You can't just bang the bottoms together since we eventually need to run a comparison with

$$\frac{10x^2 + 10x - 25}{(x+1)^2(x-4)}$$

$$= \frac{A(x+1)(x-4) + B(x-4) + C(x+1)^2}{(x+1)^2(x-4)}$$

$$\therefore A(x+1)(x-4) + B(x-4) + C(x+1)^2 = 10x^2 + 10x - 25$$

$\underline{x=4}$: $C(4+1)^2 = 10(16) + 10(4) - 25$

$25C = 175 \Rightarrow \underline{C = 7}$

$\underline{x=-1}$: $-5B = 10(-1)^2 + 10(-1) - 25$

$-5B = -25 \Rightarrow \underline{B = 5}$

$\underline{x=0}$: $A(1)(-4) + B(-4) + C(1) = -25$

$-4A - 4B + C = -25 \Rightarrow -4A - 20 + 7 = -25$

$$-4A = -25+13 = -12$$

$$\underline{\underline{A = 3}}$$

$$\int = \int \frac{3}{(x+1)} + \frac{5}{(x+1)^2} + \frac{7}{(x-4)} \, dx.$$

$$\int \frac{3\,dx}{x+1} = 3\ln|x+1|$$

$$\int \frac{7}{x-4}\,dx = 7\ln|x-4|$$

$$\int \frac{5}{(x+1)^2}\,dx = 5\int (x+1)^{-2}\,dx$$

$$= 5\,\frac{(x+1)^{-1}}{(-1)(1)} = \frac{-5}{x+1}$$

$$\therefore \int = 3\ln|x+1| + 7\ln|x-7| - \frac{5}{x+1} + K$$

$$\bigstar \qquad A = 3, B = 5, C = 7 \longrightarrow 3\ln|x+1| - \frac{5}{x+1} + 7\ln|x-4| + K \quad \bigstar$$

Below are some more proper rational functions **with factors of multiplicity in the denominator** together with the appropriate partial fraction template. We will NOT be carrying out the integrals:

$$\frac{7x-2}{(x-2)(x-8)^2} \longleftrightarrow \frac{A}{x-2} + \frac{B}{(x-8)} + \frac{C}{(x-8)^2}$$

$$\frac{4x}{(x^2+9)^3(x-2)(x-5)^2} \longleftrightarrow \frac{Ax+B}{(x^2+9)} + \frac{Cx+D}{(x^2+9)^2} + \frac{Ex+F}{(x^2+9)^3} + \frac{G}{(x-2)} + \frac{H}{(x-5)} + \frac{I}{(x-5)^2}$$

One last comment on this topic. For simplicity we have not used limits in any of the above examples but they are not a problem. The presence of limits does not affect the decomposition process and all you really need to do is carry on as usual and just plonk the limits in at the end. Nothing changes.

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 8

INTRODUCTION TO DIFFERENTIAL EQUATIONS

Milan Pahor

MATH1231 CALCULUS
INTRODUCTION TO DIFFERENTIAL EQUATIONS

A differential equation is an equation relating a function to its derivatives.

The order of a differential equation is the biggest derivative that appears in the equation.

The order of a differential equation determines the number of constants in solution.

A differential equation is said to be linear if all terms involving y and its derivatives appear linearly.

First order separable differential equations are solved by separating the variables and completing two integrals.

Arbitrary constants are found by implementing initial conditions (if they exist)

WHAT IS A DIFFERENTIAL EQUATION ?

We have already seen in high school and Math1131 that if you are supplied with a rate of change $R(t)$ then the accumulation under that rate is given by $\int R(t)\,dt$.

But sometimes you do not actually start with a rate but rather with a relationship connecting quantities of interest with their rates of change. This is a differential equation (which we often abbreviate to DE).

Differential equations are incredibly important mathematical objects. As we will soon see, differential equations are the fundamental tools used by applied mathematicians to model reality. Pure mathematicians play with vector spaces. Applied mathematicians fool around with differential equations.

Definition 1: A differential equation is an equation relating a function to its derivatives.

For example

$$\frac{dy}{dx} = y^2x^3$$

is a differential equation.

It is important to note that a solution to a differential equation is not a number......it's a function. What the DE above is asking you is "can you find a function $y = f(x)$ with the property that differentiating the function is the same as squaring it and then multiplying by x^2. We will shortly examine exactly how we would go about finding such a special function (and whether or not such a function is unique). But first some classifying definitions.

Definition 2: The order of a differential equation is the biggest derivative that appears in the equation.

Definition 3: A differential equation is said to be linear if all terms involving y and its derivatives appear linearly. The x's can do as they please.

Example 1 Find the order of each of the following differential equations and state whether each equation is linear or not.

a) $\frac{dy}{dx} = y^2x^3$. first order, non-linear

b) $\frac{dy}{dx} + y = x^3$. first order, linear

c) $y'' + 3y' - 6y = \frac{1}{x}$. second order, linear

★ a) First order, non-linear b) First order linear c) Second order linear ★

What does a solution to a differential equation look like?

Consider the first order linear differential equation

$$\frac{dy}{dx} - \frac{2}{x}y = 1$$

A solution to this equation is $y = 3x^2 - x$. This is easily checked by simply plugging the function into the DE and showing that it does the job!

LHS= $\frac{dy}{dx} - \frac{2}{x}y = (6x-1) - \frac{2}{x}(3x^2 - x)$

$= \cancel{6x} - 1 - \cancel{6x} + 2 = 1 = \underline{RHS}$

Note that we start with the LHS of the differential equation, **NOT** the LHS of the solution!!

Unfortunately there are other solutions!

It is easily checked that $y = 7x^2 - x$ is also a solution!

LHS= $\frac{dy}{dx} - \frac{2}{x}y = (14x-1) - \frac{2}{x}(7x^2 - x)$

$= 14x - 1 - 14x + 2 = 1 = \underline{\underline{RHS}}$

The most general solution is $y = Cx^2 - x$ where C is an arbitrary constant.

!!Differential equations have infinitely many solutions!!

You have already seen that integration always involve arbitrary constants,...the same is true for differential equations. In fact solving a DE can be viewed as an elaborate form of integration and thus the constants of integration must be included. A solution which accommodates all possible constants is called a **general solution.**

Fact: The order of a DE determines the number of constants in solution.

Thus the solution of a first order DE must have one arbitrary constant and a second order DE must have two arbitrary constants in solution. The constants play a central role in the theory of differential equations and you must take care of them!

The question still remains. How do we find the solution to a differential equation?

This depends entirely upon the type of DE we are faced with! Many DE's can't be solved. First order DE's are solved in entirely different ways to second order ones. Let's begin with the simplest possible first order situation, first order separable.

FIRST ORDER SEPARABLE DIFFERENTIAL EQUATIONS

These DE's have only first derivatives in them and they have the property that you can drag all the x's to one side of the equation and all the y's to the other side of the equation. We then solve the DE by completing a pair of integrals. It is important to note that not all first order DE's are separable.

Example 2 Find the general solution to the first order separable DE

$$\frac{dy}{dx} = y(2x+3)$$

Separating the variables we have

$$\frac{dy}{dx} = y(2x+3)$$

$$\frac{dy}{y} = (2x+3)dx$$

$$\int \frac{dy}{y} = \int (2x+3)dx$$

$$\ln(y) = x^2+3x+C$$

$$e^{\ln y} = e^{x^2+3x+C}$$

$$y = e^C e^{x^2+3x}.$$

Let $A = e^C$

$$y = Ae^{x^2+3x}$$

check

$$\frac{dy}{dx} = Ae^{x^2+3x}(2x+3)$$

$$= y(2x+3)$$

$$\star \quad y = Ae^{x^2+3x} \quad \star$$

Note that we only put in one constant even though we've evaluated two integrals.

Note also that the constant **must** be introduced at the point of integration and then manipulated from there. You cannot not just $+C$ at the end!

We will see in the next example that sometimes the differential equation offers a little extra information in the shape of an initial condition to determine the value of the constant. Without such an initial condition we can't do anything about the constant.

Example 3

a) Solve the differential equation

$$\frac{dy}{dx} = \frac{2\cos(x)}{3y^2} \quad ; \qquad y(\frac{\pi}{2}) = 3.$$

b) Check that your solution is correct, first by using implicit differentiation and then by differentiating explicitly. Check also that the i.c. is satisfied.

You can always check your solutions to DE's, but it is not necessary, as it burns up a lot of time in exams.

Because this differential equation has an initial condition we refer to the DE as an initial value problem. Initial conditions are often referred to as i.c.'s.

Keep in mind that the i.c. is only there to help you find the constant C. The initial condition $y(\frac{\pi}{2}) = 3$ is telling you that when $x = \frac{\pi}{2}$ we have $y = 3$.

Initial conditions are used absolutely last when solving a DE. If there is no initial condition supplied with the equation then the arbitrary constant will be a part of the solution. First order problems have only one constant in solution and thus need only one initial condition.

$$\frac{dy}{dx} = \frac{2\cos(x)}{3y^2}$$

$$\int 3y^2\,dy = \int 2\cos(x)\,dx$$

$$y^3 = 2\sin(x) + C$$

$$y = (2\sin(x) + C)^{\frac{1}{3}}$$

But $y(\pi/2) = 3$

i.e. $x = \frac{\pi}{2} \Leftrightarrow y = 3$

sub into $y^3 = 2\sin(x) + C$

$$27 = 2\sin\frac{\pi}{2} + C$$

$$27 = 2 + C \Rightarrow C = 25$$

$$y^3 = 2\sin(x) + 25$$

$$y = \sqrt[3]{2\sin(x) + 25}$$

check $y^3 = 2\sin(x) + 25$

i.c. $x = \pi/2 : y^3 = 2\sin\frac{\pi}{2} + 25$

$$y^3 = 27 \Rightarrow y = 3$$

D.E. $y^3 = 2\sin(x) + 25$

$$\frac{d}{dx}(y^3) = 2\cos(x) + 0$$

$$\frac{d}{dy}(y^3)\frac{dy}{dx} = 2\cos(x)$$

$$3y^2\frac{dy}{dx} = 2\cos(x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\cos(x)}{3y^2}$$

★ $y^3 = 2\sin(x) + 25$ or $y = (2\sin(x) + 25)^{\frac{1}{3}}$ ★

A

B ↓

Example 4 Solve the initial value problem

$$\frac{dy}{dt} = \frac{2t}{y\cos(y)} \quad ; \quad y(0) = \frac{\pi}{2}.$$

Do not be put off by the fact that we are using t rather than x as the independent variable. This is of no concern. The variable names do not matter.

$$\frac{dy}{dt} = \frac{2t}{y\cos(y)} \Rightarrow y\cos(y)\,dy = 2t\,dt$$

$$\Rightarrow \int \overset{u}{y}\,\overset{dv}{\cos(y)\,dy} = \int 2t\,dt$$

parts

Let $u = y$ $dv = \cos(y)$

$du = 1$ $v = \sin(y)$

$$\int u\,dv = uv - \int v\,du$$

$$= y\sin(y) - \int \sin(y)(1)\,dy$$

$$= y\sin(y) + \cos(y)$$

$$\therefore \quad y\sin(y) + \cos(y) = t^2 + C$$

i.c. $t = 0$, $y = \frac{\pi}{2}$

$$\frac{\pi}{2}\sin(\pi/2) + \cancel{\cos\left(\frac{\pi}{2}\right)} = 0^2 + C$$

$$\frac{\pi}{2} = C \Rightarrow C = \frac{\pi}{2}$$

$$\therefore \quad y\sin(y) + \cos(y) = t^2 + \frac{\pi}{2}$$

$$\bigstar \quad y\sin(y) + \cos(y) = t^2 + \frac{\pi}{2} \quad \bigstar$$

Example 5 Find the general solution of

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{x^2-4}}$$

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{\sqrt{x^2-4}} \quad \text{— } \underline{\underline{\text{tables}}}$$

$$\sin^{-1}(y) = \cosh^{-1}\left(\frac{x}{2}\right) + C$$

$$y = \sin\left(\cosh^{-1}\left(\frac{x}{2}\right) + C\right)$$

★ $y = \sin(\cosh^{-1}(\frac{x}{2}) + C)$ ★

When modelling using differential equations the variables have specific meanings and the solution predicts future behaviour. We close with a modelling question. There will be a whole lecture of modelling questions next week. The harder modelling questions will ask you to actually create an appropriate differential equation and then solve. For the moment we will just give you the DE.

Example 6 The population P of a small colony is initially 200 and at any subsequent time t (in years) satisfies the differential equation

$$\frac{dP}{dt} = -\frac{P^2}{400}$$

a) By referencing the differential equation explain why the population is decreasing.

b) Solve the differential equation to find a formula for the population P in terms of time t.

c) What will the population be when $t = 3$ years?

d) When will the population be $P = 50$?

e) What is the eventual fate of the population?

f) Sketch P vs t.

a) Clearly $\frac{dP}{dt} < 0$

b) $\frac{dP}{dt} = \frac{-P^2}{400}$

$\frac{dP}{P^2} = -\frac{1}{400}\, dt$

$\int \frac{dP}{P^2} = -\frac{1}{400} \int dt$

$\int P^{-2} dP = -\frac{1}{400} \{t + c\}$

$-\frac{1}{P} = -\frac{1}{400} \{t + c\}$

$= \frac{t+c}{-400}$

$P = \frac{400}{t+c}$

Given $t=0$, $P = 200$

$200 = \frac{400}{0+c}$

$c = 2$

$\therefore P = \frac{400}{t+2}$

c) $P = \frac{400}{3+2} = 80$

d) $50 = \frac{400}{t+2}$

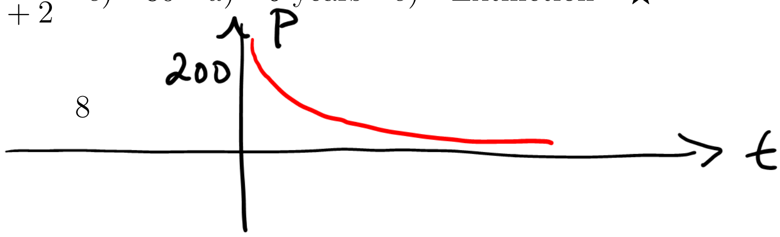
$t+2 = \frac{400}{50} = 8$

$\therefore t = 6$ yrs

e) $\lim_{t \to \infty} P$

$= \lim_{t \to \infty} \frac{400}{t+2} = 0$

★ *a)* $P' < 0$ *b)* $P = \frac{400}{t+2}$ *c)* 80 *d)* 6 years *e)* Extinction ★



Please observe what happened in the above example. We moved from a situation where we had vague information about the rate of change of a population

$$\frac{dP}{dt} = -\frac{P^2}{400}$$

over to a clear explicit formula for the population P:

$$P = \frac{400}{t+2}$$

This is what solving a DE does for you! It gives you a solution in the form of a formula, which you can wave around and use to answer subsequent explicit questions.

So that's first order separable done. Are all first order DEs separable? Definitely not! But if you can separate the variables in a first order differential equation there is a good chance that evaluating a couple of integrals will knock the DE off. The problem of course is that the integrals may be difficult or even impossible to calculate.

In the next lecture we will look at two other types of first order DEs, linear and exact. These are solved using **completely** different methods. Once we have mastered techniques of solution we will examine in the following lecture how DEs are used in real life.

Australia's Global University

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 9

FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

Milan Pahor

MATH1231 CALCULUS
FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

$$\boxed{\frac{dy}{dx} + p(x)y = q(x) \longrightarrow R(x) = e^{\int p(x)\,dx} \longrightarrow y = \frac{1}{R(x)}\int R(x)q(x)\,dx}$$

The separable techniques of the previous lecture are easy to implement, but the reality is that many first order differential equations are simply not separable. That is, it is algebraically impossible to drag all the $x's$ to one side and all the $y's$ to the other. In this lecture we will examine the method of solution for another type of first order differential equation, first order linear.

FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

These are DEs where the y variable and all the derivatives y', y'' etc appear in a simple linear fashion.... the $x's$ can do as they please. Every first order linear DE can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x)$$

where $p(x)$ and $q(x)$ are functions of x only. These differential equations can be solved by simply evaluating two special integrals. The first produces what is called an integrating factor $R(x)$ and the integrating factor is then used to find the general solution. We are still expecting exactly one constant in solution since the DE is of first order.

Claim: Suppose that

$$\frac{dy}{dx} + p(x)y = q(x)$$

is a first order linear differential equation. If

$$R(x) = e^{\int p(x)\,dx} \qquad \text{(the integrating factor)}$$

then the general solution of the differential equation is given by

$$y = \frac{1}{R(x)}\int R(x)q(x)\,dx$$

Proof: (A little later)

★

There are many different approaches to solving first order linear differential equations. Some tutors may spend quite some time essentially reformulating the above proof every time they solve a problem. Other will quickly evaluate the two integrals and move on. Either approach is OK in the final examination however you should chat with your tutor regarding what is expected in the quizzes.

Example 1 Find the general solution of

$$\frac{dy}{dx} + \frac{2}{x}y = 4x$$

and check that your solution is correct.

Observe that this DE is NOT separable. It is algebraically impossible to drag all the x's to one side.

It **is** however first order linear with $p(x) = \dfrac{2}{x}$ and $q(x) = 4x$. Thus

$R(x) = e^{\int p(x)\,dx}$

$$R(x) = e^{\int \frac{2}{x}} = e^{2\ln(x)} = e^{\ln(x^2)} = x^2$$

$$y = \frac{1}{R}\int Rq = \frac{1}{x^2}\int x^2(4x)\,dx.$$

$$= \frac{1}{x^2}\int 4x^3\,dx = \frac{1}{x^2}\{x^4 + C\}$$

$$y = x^2 + C/x^2$$

Check:

$$y = x^2 + Cx^{-2}$$

$$\frac{dy}{dx} = 2x - 2Cx^{-3} = 2x - \frac{2C}{x^3}$$

$$LHS = \frac{dy}{dx} + \frac{2}{x}y = \overbrace{2x - \frac{2C}{x^3}} + \underbrace{\frac{2}{x}\left\{x^2 + \frac{C}{x^2}\right\}}$$

$$= 2x - \cancel{\frac{2C}{x^3}} + 2x + \cancel{\frac{2C}{x^3}} = 4x = RHS$$

$\star \quad y = x^2 + \dfrac{C}{x^2} \quad \star$

Note that we do NOT include an arbitrary constant when finding the integrating factor $R(x)$ but DO include an arbitrary constant when finding the solution y. Note also that the constant is in the end multiplied by $\dfrac{1}{R(x)}$. Don't let it run around by itself.

Example 2 Solve the initial value problem

$$x\frac{dy}{dx} - y = x^3\cos(x) \quad ; \quad y(\pi) = 0.$$

We MUST rewrite the equation:

$$\frac{dy}{dx} - \frac{1}{x}y = x^2\cos(x)$$

$$p(x) = -\frac{1}{x}, \quad q(x) = x^2\cos(x).$$

$$R(x) = e^{\int p(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln(x)} = e^{\ln(x^{-1})} = x^{-1} = \frac{1}{x}$$

$$y = \frac{1}{R}\int Rq = \frac{1}{\left(\frac{1}{x}\right)}\int \frac{1}{x}\cdot(x^2\cos x)\,dx.$$

$$y = x\int x\cos(x)\,dx.$$

$$u = x \to du = 1$$
$$dv = \cos x \to v = \sin x$$

$$\int u\,dv = uv - \int v\,du$$

$$\int u\,dv = x\sin(x) - \int \sin(x)(1)$$
$$= x\sin x - -\cos x = x\sin(x) + \cos(x) + c$$

$$y = x\{x\sin(x) + \cos(x)\} = x^2\sin(x) + x\cos(x) + cx$$

$$y = x^2\sin(x) + x\cos(x) + cx.$$

$$y(\pi) = 0 \implies x = \pi \iff y = 0$$
$$0 = \pi^2\cancel{\sin(\pi)} + \pi\cos(\pi) + c\pi$$
$$0 = -\pi + c\pi \implies c\pi = \pi \implies c = 1$$

$$y = x^2\sin(x) + x\cos(x) + x$$

★ $y = x^2\sin(x) + x\cos(x) + x$ ★

Example 3 Find the general solution of the differential equation

$$\frac{dy}{dx} + xy = x$$

a) By treating the DE as linear.

b) By treating the DE as separable.

c) Check that the two solutions are equivalent.

a) $p(x) = x, \; q(x) = x$ $\quad R(x) = e^{\int p(x)\,dx} = e^{\int x} = e^{\frac{1}{2}x^2}$

$$y = \frac{1}{R}\int Rq = e^{-\frac{1}{2}x^2}\int e^{\frac{1}{2}x^2}\underline{x\,dx}$$

$\int$: Let $\quad u = \frac{1}{2}x^2 \implies du = \underline{x\,dx}.$

$$\int = \int e^u\,du = e^u = e^{\frac{1}{2}x^2} \underline{\underline{+c}}$$

$$y = e^{-\frac{1}{2}x^2}\left\{e^{\frac{1}{2}x^2} + c\right\} = 1 + ce^{-\frac{1}{2}x^2}$$

b) $\dfrac{dy}{dx} = x - xy = x(1-y) \implies \dfrac{dy}{1-y} = x\,dx$

$$\implies \int \frac{dy}{1-y} = \int x\,dx \implies \int \frac{-dy}{1-y} = \int -x\,dx$$

$$\ln|1-y| = -\frac{1}{2}x^2 + C \implies 1-y = e^{-\frac{1}{2}x^2 + c} = e^c e^{-\frac{1}{2}x^2} = De^{-\frac{1}{2}x^2} \quad (D = e^c)$$

$$\implies y = 1 - De^{-\frac{1}{2}x^2}$$

$$\implies y = 1 + Ce^{-\frac{1}{2}x^2} \quad (C = -D)$$

★ $y = 1 + De^{-\frac{x^2}{2}}$ ★

It is quite rare for a differential equation to be of two completely different types!

Example 4 The population N (in millions) of a small town at time t (in years) satisfies the first order linear differential equation

$$\frac{dN}{dt} + \frac{1}{1+t}N = \frac{3t^2}{1+t}$$

Given that the population is initially 6 million solve the DE to determine what it will be in 4 years time.

$p = \frac{1}{1+t}$, $q = \frac{3t^2}{1+t}$

$R = e^{\int p(t)dt} = e^{\int \frac{1}{1+t}dt}$

$= e^{\ln(1+t)} = \underline{1+t}$

$N = \frac{1}{R}\int Rq$

$= \frac{1}{1+t}\int \cancel{(1+t)}\left(\frac{3t^2}{\cancel{1+t}}\right)dt$

$= \frac{1}{1+t}\int 3t^2 dt$

$N = \frac{1}{1+t}\{t^3 + c\}$

$t = 0$, $N = 6$

$6 = \frac{1}{1+0}\{0^3 + c\} \Rightarrow c = 6$

$N = \frac{1}{1+t}\{t^3 + 6\}$

$N(4) = \frac{1}{1+4}\{64 + 6\}$

$= 14$ $\underline{\text{million}}$

★ 14 ★

Example 5 Bruce has had a little too much to drink and is worried about his blood alcohol level L(in mg alcohol per litre of blood). The legal limit is $L = 50$. His level is currently $L = 70$ and it varies over the next 3 hours according to the differential equation

$$\frac{dL}{dt} + L = 20; \qquad 0 \leq t \leq 3.$$

a) Solve the DE to find Bruce's blood alcohol level L in terms of time t.

b) What is his alcohol level at the beginning and end of the trial?

c) How long should he wait before he drives?

a) $p = 1$, $q = 20$

$R = e^{\int 1} = e^{\int 1} = e^t$

$L = \frac{1}{R}\int Rq = e^{-t}\int e^t(20)$

$= e^{-t}\{20e^t + c\}$

$L = 20 + ce^{-t}$

$t = 0$, $L = 70$

$70 = 20 + ce^{-0} \Rightarrow c = 50$

$L = 20 + 50e^{-t}$

b) $L(0) = 20 + 50 = 70$

$L(3) = 20 + 50e^{-3} \doteq 22.49$

c) $50 = 20 + 50e^{-t}$

$30 = 50e^{-t} \Rightarrow e^{-t} = 3/5$

$-t = \ln(3/5) \Rightarrow t \doteq \frac{1}{2}$

$\doteq 30$ minutes

★ *a*) $L = 20 + 50e^{-t}$ *b*) 70 22.49 *c*) 30 minutes ★

Let's now prove that these formulae for first order linear DEs work! Recall:

Claim: Suppose that

$$\frac{dy}{dx} + p(x)y = q(x)$$

is a first order linear differential equation. If

$$R(x) = e^{\int p(x)\, dx} \qquad \text{(the integrating factor)}$$

then the general solution of the differential equation is given by

$$y = \frac{1}{R(x)} \int R(x)q(x)\, dx$$

Proof: This is quite an interesting and very subtle argument.

$$\frac{dy}{dx} + p(x)y = q(x)$$

$$\boxed{R(x)\frac{dy}{dx} + R(x)p(x)y} = R(x)q(x)$$

$$(uv)' = u'v + v'u$$

Consider $\frac{d}{dx}(R(x)y)$ prod. rule

$$= \boxed{R(x)\frac{dy}{dx} + y\frac{dR}{dx}}.$$

To align the boxes

$$R(x)p(x) \equiv \frac{dR}{dx}$$

$$\int p(x)dx = \int \frac{dR}{R}$$

$$= \ln(R)$$

$$\therefore \ln(R) = \int p(x)dx.$$

$$R = e^{\int p(x)dx}.$$

DE becomes

$$\frac{d}{dx}(R(x)y) = R(x)q(x)$$

$$R(x)y = \int R(x)q(x)dx$$

$$y = \frac{1}{R(x)}\int R(x)q(x)dx$$

★

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 10

FIRST ORDER EXACT DIFFERENTIAL EQUATIONS

Milan Pahor

MATH1231 CALCULUS
EXACT DIFFERENTIAL EQUATIONS

The differential equation $Mdx + Ndy = 0$ is said to be **exact** if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Simultaneously integrate $\dfrac{\partial f}{\partial x} = M$ and $\dfrac{\partial f}{\partial y} = N$ to find the solution $f(x, y) = C$

We now move on to our third class of first order differential equations, Exact DEs. These are very different from separable and linear DEs and the technique of solution is quite strange. Hopefully we will finish a little early today to give you an opportunity to clear out any final questions you may have regarding the first order theory. We need to settle the material because in the next lecture we will be applying these techniques to real life modelling problems.

FIRST ORDER EXACT DIFFERENTIAL EQUATIONS

We have already looked at first order linear and separable differential equations. Our final first order technique is **first order exact**. This is again a very different type of DE.

Consider the situation where we have $f(x, y) = C$ and y is a function of x. Differentiating both sides with respect to x and using the chain rule we have

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} = 0$$

Hence

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

which we usually express as

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

The quantity

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

is often referred to as the **total differential**.

This means that the differential equation $\dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy = 0$ has solution $f(x, y) = C$.

More generally the differential equation $Mdx + Ndy = 0$ has solution $f(x, y) = C$ provided that M can be identified with $\dfrac{\partial f}{\partial x}$ and N can be identified with $\dfrac{\partial f}{\partial y}$.

Noting that $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$ this will happen exactly when $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Under these circumstances we say that the DE is **exact** and the solution may be found through careful simultaneous partial integration.

In summary:

The differential equation $Mdx + Ndy = 0$ is said to be **exact** if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

That is, the DE is exact if: "the derivative of the coefficient of dx with respect to y equals the derivative of the coefficient of dy with respect to x".

The solution $f(x,y) = C$ of an exact DE may then be found by noting that $\dfrac{\partial f}{\partial x} = M$ and that $\dfrac{\partial f}{\partial y} = N$ and simultaneously integrating partially.

Example 1 Consider the differential equation

$$(2xy^5 + 3)dx + (5x^2y^4 + 2y)dy = 0$$

a) Show that the DE is exact.

b) Find the general solution.

c) Check your solution.

---

a) We need to show that the derivative of the coefficient of dx with respect to y equals the derivative of the coefficient of dy with respect to x.

We have $M = 2xy^5 + 3$ and $N = 5x^2y^4 + 2y$. So

$$\frac{\partial M}{\partial y} = 10xy^4 + 0$$
$$\frac{\partial N}{\partial x} = 10xy^4 + 0$$

) equal $\therefore$ exact

Thus $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and hence the DE is exact. Thus:

$\dfrac{\partial f}{\partial x}$ = coefficient of dx= $2xy^5 + 3$

$\dfrac{\partial f}{\partial y}$ = coefficient of dy=$5x^2y^4 + 2y$.

Now watch this carefully:

$$\frac{\partial f}{\partial x} = \underline{2x}y^5 + 3 \Rightarrow f = x^2y^5 + 3x + g_1(y)$$

$$\frac{\partial f}{\partial y} = 5x^2y^4 + 2y \Rightarrow f = 5x^2\frac{y^5}{5} + y^2 + g_2(x)$$

$$f = x^2y^5 + y^2 + g_2(x)$$

$$\Rightarrow f(x,y) = x^2y^5 + 3x + y^2 = C$$

c) $\frac{d}{dx}(x^2y^5) + 3 + \frac{d}{dx}(y^2) = 0$

check

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}.$$

$$\frac{d}{dx}(x^2y^5) = 2xy^5 + \frac{d}{dx}(y^5)(x^2)$$

$$= 2xy^5 + \frac{d}{dy}(y^5)\left(\frac{dy}{dx}\right)(x^2)$$

$$= 2xy^5 + (5y^4)(x^2)\frac{dy}{dx}.$$

$$\Rightarrow 2xy^5 + 5x^2y^4\frac{dy}{dx} + 3 + 2y\frac{dy}{dx} = 0$$

Check Solution:

$$(2xy^5 + 3) + (5x^2y^4 + 2y)\frac{dy}{dx} = 0$$

$$(2xy^5 + 3)dx + (5x^2y^4 + 2y)dy = 0$$

Note that we cannot rewrite this general solution in the form of $y = f(x)$ but that's OK. Don't forget the =C !! You must have a constant in your general solution.

$\bigstar \quad x^2y^5 + 3x + y^2 = C \quad \bigstar$

Example 2 Solve the initial value problem

$$(y + 2xy^2 - \frac{1}{x^2})dx + (x + 2x^2y + \frac{1}{y})dy = 0 \quad ; \qquad y(3) = 1$$

The shape of the problem $Mdx + Ndy = 0$ leads us to believe that it may be an exact DE. But we still need to check for exactness!

$$M = y + 2xy^2 - x^{-2} \Rightarrow \frac{\partial M}{\partial y} = 1 + (2x)(2y) - 0$$
$$= 1 + 4xy$$

$$N = x + 2x^2y + \frac{1}{y} \Rightarrow \frac{\partial N}{\partial x} = 1 + 4xy + 0$$
$$= 1 + 4xy$$

$\therefore$ exact

$$\frac{\partial f}{\partial x} = y + 2xy^2 - x^{-2} \Rightarrow f = xy + x^2y^2 + x^{-1} + g_1(y)$$

$$\frac{\partial f}{\partial y} = x + 2x^2y + \frac{1}{y} \Rightarrow f = xy + x^2y^2 + \ln|y| + g_2(x)$$

$\therefore$ $xy + x^2y^2 + \frac{1}{x} + \ln|y| = C$

$\boxed{\begin{matrix} x=3 \\ y=1 \end{matrix}}$ I.C.

$$3(1) + (9)(1) + \frac{1}{3} + \cancel{\ln(1)} = C$$

$$12\tfrac{1}{3} = C \Rightarrow C = 37/3$$

$\therefore$ $xy + x^2y^2 + \frac{1}{x} + \ln y = 37/3$

Observe that all terms involving both x and y appear in both of the simultaneous equations. Take care not to double them up. There are not two of them. Just one of them found twice.

$$\bigstar \quad xy + x^2y^2 + \frac{1}{x} + \ln(y) = \frac{37}{3} \quad \bigstar$$

Example 3 Find the general solution of the differential equation

$$2x\sin(y)dx + x^2\cos(y)dy = 0$$

a) By treating the DE as exact.

b) By treating the DE as separable.

c) Verify that the two solutions are both valid.

a) $M = 2x\sin(y)$ $\quad N = x^2\cos(y)$

check: $\frac{\partial M}{\partial y} = 2x\cos(y)$

$\qquad \| \longrightarrow \therefore$ exact

$\frac{\partial N}{\partial x} = 2x\cos(y)$

$\therefore \frac{\partial f}{\partial x} = 2x\sin(y) \Rightarrow f = x^2\sin y + g_1(y)$

$\frac{\partial f}{\partial y} = x^2\cos(y) \Rightarrow f = x^2\sin y + g_2(x)$

$\therefore \boxed{x^2\sin y = c}$

b) $2x\sin(y)dx + x^2\cos y\,dy = 0$

$x^2\cos y\,dy = -2x\sin y\,dx.$

$\frac{dy}{dx} = -\frac{2x\sin y}{x^2\cos y} = \frac{-2}{x}\frac{\sin y}{\cos y}$

$dy\frac{\cos(y)}{\sin(y)} = -\frac{2}{x}dx \Rightarrow \int\frac{\cos y}{\sin y}dy = \int -\frac{2}{x}dx$

$\ln(\sin y) = -2\ln x + c \Rightarrow \ln(\sin y) = \ln(x^{-2}) + c$

$$\bigstar \qquad x^2\sin(y) = D \qquad \bigstar$$

Note again that it is quite rare for a differential equation to be of two completely different types!

$e^{\ln(\sin y)} = e^{\ln(x^{-2})+c} = e^{\ln(x^{-2})}\underbrace{e^{c}}_{\text{rename as } D}$

$\Rightarrow \sin y = x^{-2}D \Rightarrow x^2\sin y = D$

Example 4 Solve the first order initial value problem

$$\frac{dy}{dx} = \frac{-2x - e^{2y}}{3y^2 + 2xe^{2y}} \quad ; \quad y(5) = 0.$$

$$(3y^2 + 2xe^{2y})\,dy = (-2x - e^{2y})\,dx.$$

$$\Rightarrow \underbrace{(2x + e^{2y})}_{}dx + \underbrace{(3y^2 + 2xe^{2y})}_{\frac{\partial f}{\partial y}}dy = 0$$

<u>check</u>: $\frac{\partial}{\partial y}(2x + e^{2y}) = 2e^{2y}$ $\rightarrow$ exact de!

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \shortparallel$

$\frac{\partial}{\partial x}(3y^2 + 2xe^{2y}) = 2e^{2y}$

$$\frac{\partial f}{\partial x} = 2x + e^{2y} \Rightarrow f = x^2 + e^{2y}x + g_1(y)$$

$$\frac{\partial f}{\partial y} = 3y^2 + 2xe^{2y} \Rightarrow f = y^3 + 2x\left(\tfrac{1}{2}e^{2y}\right) + g_2(x)$$

$$= y^3 + xe^{2y} + g_2(x)$$

$$\therefore \quad x^2 + xe^{2y} + y^3 = C$$

I.C. $y(5) = 0$: $x = 5,\ y = 0$

$$25 + 5e^0 + 0^3 = C \Rightarrow C = 30$$

$$\therefore \quad x^2 + xe^{2y} + y^3 = 30$$

★ $x^2 + xe^{2y} + y^3 = 30$ ★

Example 5 (HOMEWORK) Consider the differential equation

$$\frac{dy}{dx} = \frac{(-3y - \frac{y^2}{x})}{x+y}$$

Show that it is not exact as it stands but that it becomes exact after multiplication of both sides by the integrating factor x^2. Hence solve.

$$(x+y)dy = (-3y - \frac{y^2}{x})dx.$$

$$(3y + y^2/x)dx + (x+y)dy = 0$$

$$M = \frac{\partial f}{\partial x} \qquad N = \frac{\partial f}{\partial y}$$

$$\underbrace{\frac{\partial M}{\partial y} = 3 + \frac{2y}{x}, \quad \frac{\partial N}{\partial x} = 1 + 0 = 1}_{\neq}$$

$\therefore$ Not exact.

$$(\times x^2) \quad (3x^2y + xy^2)dx + (x^3 + x^2y)dy = 0$$

integrating factor.

★ $x^3y + \frac{1}{2}x^2y^2 = C$ ★

Note that you will not need to find your own integrating factors in this course.

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 11

APPLICATIONS OF FIRST ORDER DEs

Milan Pahor

LECTURE 11 MATH1231 CALCULUS
APPLICATIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS

When setting up a differential equation keep in mind that the rate of change of a quantity is the rate of increase minus the rate of decrease. Keep your eye out for initial conditions buried in the text.

Modelling with First Order Differential Equations

Sometimes you are not actually given a differential equation but rather are told various details about a situation and it is left up to you to construct the DE yourself. Keep in mind that DEs are all about rates of change and thus your first task is to establish equations which govern the rate of change. The rate of change of any quantity is always equal to the rate at which it increases minus the rate at which it decreases. Thus for example the rate of change of a population will be the rate of increase (births, immigration etc) minus the rate at which the population decreases (deaths etc). Also there is usually some information given as to initial conditions. Once the differential equation is formed there is still of course the problem of solving it to obtain formulae for the quantities of interest.

Almost all applications of applied mathematics to real world problems is enacted through the prism of differential equations.

Example 1 A population P (in millions) evolves over time t (in years) according to the logistic differential equation

$$\frac{dP}{dt} = P(1-P)$$

a) Given that the population is initially $\frac{1}{4}$ (million) solve the differential equation to find a formula for P in terms of t.

b) What is the population after 2 years?

c) To what limit does the population tend?

d) To what limit does the rate of change of population tend?

a) $\int \frac{dP}{P(1-P)} = \int dt = t + c.$

$$\frac{1}{P(1-P)} = \frac{A}{P} + \frac{B}{1-P} = \frac{A(1-P)+BP}{P(1-P)}$$

$A(1-P)+BP = 1 \qquad P=0: A=1, \; P=1: B=1$

$$\int \frac{1}{P(1-P)} = \int \frac{1}{P} + \frac{1}{1-P} = \ln P + \ln(1-P)$$

$$= \ln\left(\frac{P}{1-P}\right)$$

$\therefore \ln\left(\frac{P}{1-P}\right) = t+c \Rightarrow \frac{P}{1-P} = e^{t+c} = e^c e^t = Ae^t$

$\frac{P}{1-P} = Ae^t \Rightarrow P = Ae^t - PAe^t$

$\Rightarrow P(1+Ae^t) = Ae^t \Rightarrow P = \frac{Ae^t}{1+Ae^t}$

$t=0 \Rightarrow P = \frac{1}{4} \Rightarrow \frac{1}{4} = \frac{A}{1+A} \Rightarrow 1+A = 4A$

$3A = 1 \Rightarrow A = \frac{1}{3}$.

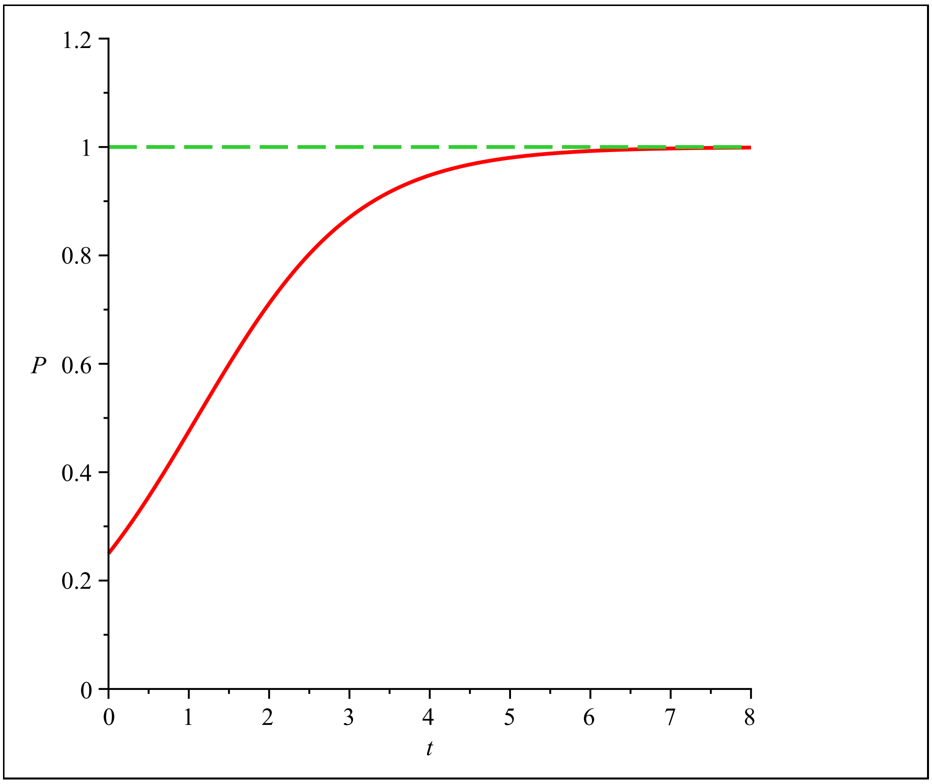
$P = \frac{\frac{1}{3}e^t}{1+\frac{1}{3}e^t} = \frac{e^t}{3+e^t}$

$P(2) = \frac{e^2}{3+e^2} = \text{calc.}$

c) $\lim_{t\to\infty} \frac{e^t}{3+e^t} = 1$

d) $\lim_{t\to\infty} \frac{dP}{dt} = 0$.

The logistic curve



★ a) $P = \frac{e^t}{e^t+3}$ b) 711235 c) 1 million d) 0 ★

In the last example we were fortunate that the differential equation was supplied, ready to be used. Sometimes we have to work hard just to produce the DE.

Example 2 The population N (in millions) of a small country at time t (in years) is initially 5 (million) and subsequently varies according to

$$\frac{dN}{dt} = \underline{B(t)} - \underline{D(t)}$$

where $B(t) = 2te^{-\frac{t}{100}}$ is the birth rate and the death rate $D(t)$ is equal to $\frac{1}{100}$th of the current population.

a) Show that $\frac{dN}{dt} + \frac{1}{100}N = 2te^{-\frac{t}{100}}$

b) Solve this differential equation to find N in terms of t.

c) What will the population be in 3 years time? $(t=3)$

d) What will the eventual population be?

a) $\frac{dN}{dt} = 2te^{-\frac{1}{100}t} - \frac{1}{100}N$

$\underline{\frac{dN}{dt}} + \underbrace{\frac{1}{100}}_{P}\underline{N} = \underbrace{2te^{-\frac{1}{100}t}}_{q}$ $\#$

b)

$R(t) = e^{\int P} = e^{\int \frac{1}{100}} = e^{\frac{1}{100}t}$ no $+c$

$N = \frac{1}{R}\int Rq = e^{-\frac{1}{100}t}\int e^{\frac{1}{100}t}\, 2te^{-\frac{1}{100}t}\,dt$

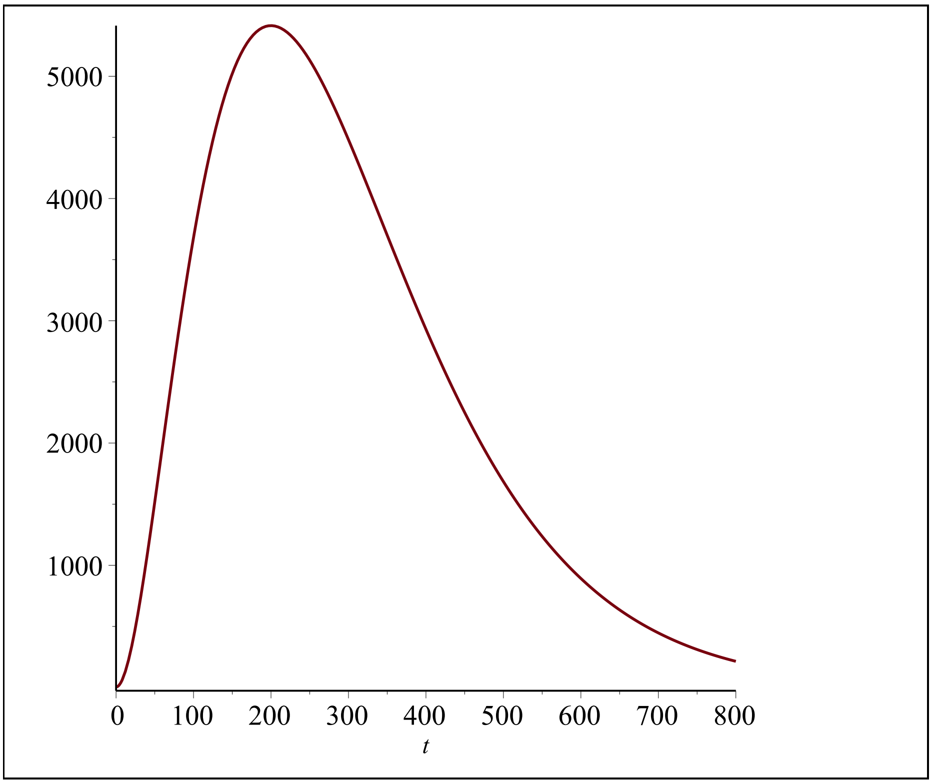
$= e^{-\frac{1}{100}t}\int 2t\,dt = e^{-\frac{1}{100}t}\{t^2+c\}$

$t=0, N=5 \Rightarrow 5 = e^0(0+c) \Rightarrow c=5.$

$\therefore \quad N = e^{-\frac{1}{100}t}\{t^2+5\}$

$\uparrow t=3$

d) $\lim_{t\to\infty} \frac{t^2+5}{e^{\frac{1}{100}t}} = 0$



★ $a)$ $Proof$ $b)$ $N = (t^2+5)e^{-\frac{t}{100}}$ $c)$ $N(3) = 13.59$ million $d)$ 0 ★

Example 3 Suppose that the rate at which an enzyme reaction takes place is governed by the Michaelis-Menton equation

$$\frac{dy}{dt} = -\frac{y}{1+y}$$

where y is the amount (in grams) of substrate present at time (in hours).

a) Solve the differential equation to obtain an implicit relation between y and t.

b) Assuming that there is initially one gram of substrate present, find the value of the constant in your solution.

c) Determine when there will be only $\frac{1}{2}$ gram of substrate remaining?

a) $\int \frac{1+y}{y} dy = \int dt = -t + c.$

$\int \frac{1}{y} + 1 \, dy = \ln y + y$

$\ln y + y = -t + c.$

$t = 0, y = 1 \Rightarrow \cancel{\ln}(1) + 1 = -0 + c$

$c = 1$

$\therefore \ln y + y = -t + 1$

c) $y = \frac{1}{2}$

$\ln(\frac{1}{2}) + \frac{1}{2} = -t + 1$

$t = \frac{1}{2} - \ln(\frac{1}{2}) =$

★ *a)* $\ln(y) + y = -t + C$ *b)* $C = 1$ *c)* 1.19 hours ★

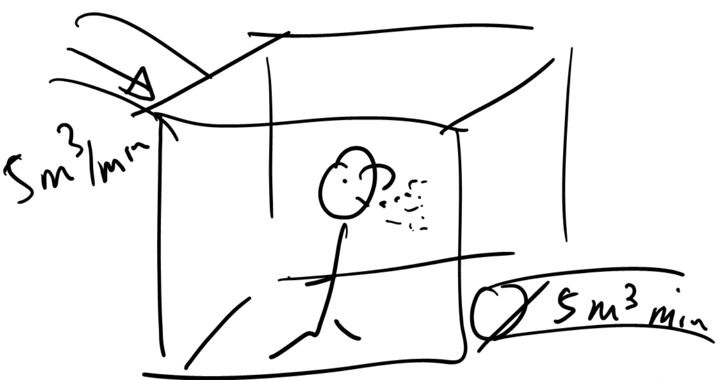
Example 4 The air in a 50 cubic metre room is initially clean. Chris lights up a cigarette introducing smoke into the room's air at a rate of 2 mg/minute. An air conditioning system exchanges the mixture of air and smoke with clean air at a rate of 5 cubic metres per minute. Assume that the pollutants are mixed uniformly throughout the room and that Chris's cigarette lasts 4 minutes. Let $S(t)$ be the amount of smoke in the room at time t (in minutes).

a) Explain why $\dfrac{dS}{dt} = 2 - \dfrac{S}{10}$.

b) Hence show that $S(t) = 20 - 20e^{-\frac{t}{10}}$

c) What is the level of pollution after 4 minutes?

d) What is the level of pollution after 14 minutes? (Hint: You will need a new clock and a new DE after $t = 4$.



a) $\frac{dS}{dt} = S_{in} - S_{out}$

$= \cancel{2} - \frac{S}{50} \cdot 5$

smoke per m³

$\frac{dS}{dt} = 2 - \frac{S}{10}$.

b) $\frac{dS}{dt} + \frac{1}{10}S = 2$ p q

$\int e^{ax} = \frac{1}{a}e^{ax}$

$R = e^{\int p} = e^{\int \frac{1}{10} dt} = e^{\frac{1}{10}t}$

$S = \frac{1}{R}\int Rq = e^{-t/10} \int 2e^{t/10} dt$

$= e^{-t/10}\left\{ 2 \frac{1}{(\frac{1}{10})} e^{t/10} + c \right\}$

$= 20 + ce^{-t/10}$

$t=0, S=0$ $0 = 20 + ce^{0} \Rightarrow c = -20$

$\therefore S = 20 - 20e^{-t/10}$

$S(4) = 20 - 20e^{-4/10} \doteq 6.6$ mg

d) $\frac{dS}{dt} = S_{in} - S_{out} = 0 - \frac{S}{10}$

$\frac{dS}{dt} = -\frac{1}{10} S$ (after smoking stopped)

$\int \frac{dS}{S} = \int -\frac{1}{10}\,dt$

$\ln S' = -\frac{1}{10}t + C$

$S = e^{-\frac{1}{10}t + c} = e^{c} e^{-t/10} = A e^{-t/10}$

$t=0 \to S \doteq 6.6$ $\qquad$ $6.6 = Ae^{0} = A.$

$S' = 6.6\, e^{-t/10}$

$S'(10) = 6.6\, e^{-1} \doteq \underline{\underline{2.43 \text{ mg.}}}$

★ $c)$ $6.60mg$ $d)$ $2.43mg$ ★

Example 5 A tank can hold 100 litres of fluid. Initially it holds 50 litres of pure water and salt water containing $2g$ of salt per litre is poured into the tank at 3 litres per minute. The mixture is stirred continuously and drained from the bottom of the tank at 2 litres per minute. Let $y(t)$ be the number of grams of salt in the tank at time t.

a) How much fluid is in the tank at time t minutes?

b) Explain why y satisfies the differential equation

$$\frac{dy}{dt} + \frac{2y}{50+t} = 6 \quad ; \qquad y(0) = 0.$$

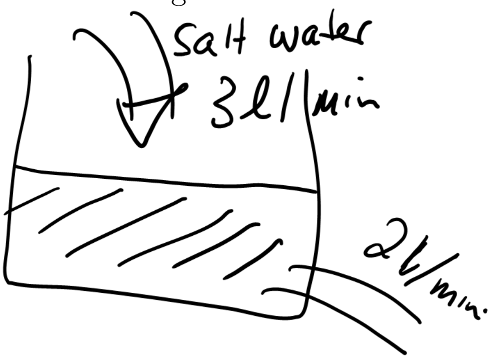
linear

$p = \frac{2}{50+t}$, $q = 6$

c) (HOMEWORK) Show that $y = 2t + 100 - \dfrac{250000}{(50+t)^2}$.

$t = 50$

d) (HOMEWORK) Determine how much salt is in the tank when it is at the point of overflowing.



a) $50+t$ litres

b) $\frac{dy}{dt} = \text{Salt}_{in} - \text{Salt}_{out}$

$= 6 - \frac{y}{50+t} \times 2$

Salt per litre

$= 6 - \frac{2y}{50+t}$

$\therefore \frac{dy}{dt} = 6 - \frac{2y}{50+t}$

$t=0, y=0$

★ $a)$ (50+t) litres $d)$ $175g$ ★

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 12

SECOND ORDER CONSTANT COEFFICIENT HOMOGENEOUS DE's

Milan Pahor

LECTURE 12 MATH1231 CALCULUS
SECOND ORDER CONSTANT COEFFICIENT HOMOGENOUS DEs

To solve the homogeneous second order constant coefficient differential equation

$$ay'' + by' + cy = 0$$

first form the auxiliary (also called characteristic) equation

$$a\lambda^2 + b\lambda + c = 0.$$

The auxiliary equation is not a DE, it is just a quadratic equation with two roots λ_1 and λ_2. Remarkably the solution to the DE is completely determined by the nature of λ_1 and λ_2!

- $\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 \neq \lambda_2 \to y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
- $\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 = \lambda_2 \to y = (Ax + B)e^{\lambda_1 x}$
- $\lambda_1, \lambda_2 \in \mathbb{C},\ \lambda_1, \lambda_2 = r \pm is \to y = e^{rx}\{A\cos(sx) + B\sin(sx)\}$

We turn now to the theory of second order differential equations. These are DEs where the second derivative also makes an appearance. Second order problems are in general more difficult to solve than their first order cousins. In fact the theory is so tangled that we restrict our attention to only the very special case of linear DEs with constant coefficients. These take the form

$$ay'' + by' + cy = RHS \qquad \text{where } a, b, c \in \mathbb{R}.$$

In this lecture we will deal with the homogeneous case (RHS=0). In the next lecture we will broaden our scope a little and allow some simple functions to appear on the RHS.

Keep in mind that all second order DEs must have two independent arbitrary constants in solution and hence require **a pair** of i.c.'s to deal with the constants. The method of solution for second order problems is *totally* different from first order techniques. As a relief you will find that in most cases no integration is involved.

Proof of first bullet point above: $ay'' + by' + cy = 0$

Assume sol^n of form $y = e^{\lambda x}$

$\Rightarrow y' = \lambda e^{\lambda x}$ & $y'' = \lambda^2 e^{\lambda x}$. $\underline{\text{Sub in DE}}$

$a\lambda^2 \cancel{e^{\lambda x}} + b\lambda \cancel{e^{\lambda x}} + c\cancel{e^{\lambda x}} = 0 \Rightarrow \underbrace{a\lambda^2 + b\lambda + c = 0}_{\text{Aux equ}^n}$

Example 1 Solve each of the following second order D.E.s:

a) $y'' - 3y' - 10y = 0$

b) $y'' - 8y' + 16y = 0$

c) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 58y = 0$

Aux equ^n

a) $\lambda^2 - 3\lambda - 10 = 0$

$(\lambda - 5)(\lambda + 2) = 0$

$\lambda = -2, 5$

$y = Ae^{-2x} + Be^{5x}$

b) Aux equ^n $\lambda^2 - 8\lambda + 16 = 0$

$(\lambda - 4)^2 = 0 \Rightarrow \lambda = 4, 4$

$y = Ae^{4x} + Be^{4x}$

$= (A+B)e^{4x} = Ce^{4x}$

$y = (Ax+B)e^{4x}$

c) Aux equ^n: $\lambda^2 - 6\lambda + 58 = 0 \Rightarrow \lambda = \dfrac{6 \pm \sqrt{36 - 232}}{2}$

$= \dfrac{6 \pm \sqrt{-196}}{2} = \dfrac{6 \pm 14i}{2}$

$\therefore \lambda = 3 \pm 7i$

$y = e^{3x}\{A\cos 7x + B\sin 7x\}$

★ *a)* $y = Ae^{-2x} + Be^{5x}$ *b)* $y = (Ax + B)e^{4x}$ *c)* $y = e^{3x}(A\cos(7x) + B\sin(7x))$ ★

Note that is some texts, $y = e^{3x}(A\cos(7x)+B\sin(7x))$ will be written in the equivalent form $Re^{3x}\cos(7x - \delta)$. We will not do so here.

Example 2 Solve

$$y'' + 3y' - 10y = 0; \quad y(0) = 7, y'(0) = -14$$

Check that your solution satisfies both the differential equation and the initial conditions.

Observe that we now have two initial conditions since we have two constants in the solution.

Aux equ^n: $\lambda^2 + 3\lambda - 10 = 0 \Rightarrow (\lambda+5)(\lambda-2) = 0$

$\lambda = 2, -5.$

$\therefore \quad y = Ae^{2x} + Be^{-5x}.$

$y(0)=7$: $\quad 7 = Ae^0 + Be^0 \Rightarrow A + B = 7$

$y' = 2Ae^{2x} - 5Be^{-5x}$

$y'(0) = -14$: $\quad -14 = 2A - 5B.$

$$\begin{bmatrix} A+B=7 \\ 2A-5B=-14 \end{bmatrix} \Rightarrow \underline{A=3, B=4.}$$

$$\therefore \quad \underline{y = 3e^{2x} + 4e^{-5x}}$$

$\underline{\text{Check}}$: $y' = 6e^{2x} - 20e^{-5x}$, $y'' = 12e^{2x} + 100e^{-5x}$

$LHS = y'' + 3y' - 10y = \cancel{12e^{2x}} + \cancel{100e^{-5x}} + \cancel{18e^{2x}} - \cancel{60e^{-5x}} - \cancel{30e^{2x}} - \cancel{40e^{-5x}}$

$= \underline{\underline{0}} = RHS$

$$\bigstar \quad y = 3e^{2x} + 4e^{-5x} \quad \bigstar$$

It is easily checked by direct substitution that the above solution

$$y = 3e^{2x} + 4e^{-5x}$$

does actually satisfy the DE

$$y'' + 3y' - 10y = 0$$

together with the i.c.'s $y(0) = 7, y'(0) = -14$. But since the actual process of solution is so elementary, there is usually no need to check.

APPLICATION: SIMPLE HARMONIC MOTION

Recall from Extension 1 Mathematics that an object oscillates along the x axis in Simple Harmonic Motion if

$$a = -n^2 x$$

where $a = x''$ is the acceleration of the particle and x is its displacement. We now have the necessary tools to deal with this theory at a formal level!

Example 3 A particle moves in Simple Harmonic Motion according to

$$a = -4x \qquad a = \ddot{x}$$

It is initially at $x = \sqrt{3}$ and after $\dfrac{\pi}{4}$ seconds is at $x = 1$.

a) By solving an appropriate DE find a formula for x in terms of t.

b) When does the particle first stop?

a) $x'' = -4x \Rightarrow \dfrac{d^2x}{dt^2} = -4x.$

$\Rightarrow x'' + 4x = 0.$ Aux equ^n: $\lambda^2 + 4 = 0 \Rightarrow \lambda^2 = -4$

$\lambda = \pm 2i = 0 \pm 2i$

$x = \cancel{e^{0t}}\{A\cos 2t + B\sin 2t\} = A\cos 2t + B\sin 2t$

$t = 0, x = \sqrt{3} \Rightarrow \sqrt{3} = A(1) + B(0) \Rightarrow A = \sqrt{3}.$

$t = \frac{\pi}{4}, x = 1 \Rightarrow 1 = \cancel{A\cos\frac{\pi}{2}} + B\sin\left(\frac{\pi}{2}\right) \Rightarrow B = 1$

$\therefore x = \sqrt{3}\cos 2t + \sin 2t.$

b) $v = -2\sqrt{3}\sin(2t) + 2\cos(2t) = 0$

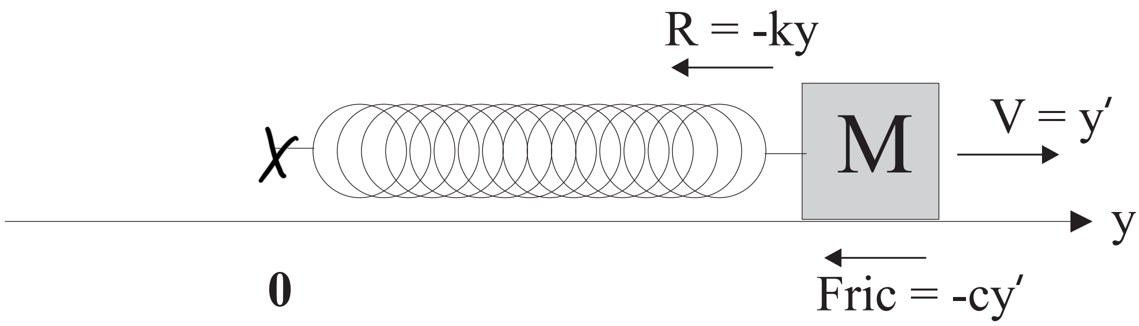
$\cancel{2}\cos 2t = \cancel{2}\sqrt{3}\sin 2t \Rightarrow \tan 2t = \frac{1}{\sqrt{3}}$

$2t = \frac{\pi}{6} \Rightarrow t = \frac{\pi}{12}$ seconds

★ *a)* $x = \sqrt{3}\cos(2t) + \sin(2t)$ *b)* $\dfrac{\pi}{12}$ seconds ★

APPLICATION: FREE OSCILLATIONS

We saw in the previous example that the equations of Simple Harmonic Motion are just constant coefficient second order differential equations. We can even introduce friction into the mix! For simplicity we will now assume that the motion is up and down the y axis.



Consider an object M of mass m attached to a spring oscillating up and down the y axis. At time t its velocity is $v = y' = \dfrac{dy}{dt}$ and its acceleration $a = y'' = \dfrac{d^2y}{dt^2}$.

The total force $F = ma = my''$ acting upon the mass is the sum of two forces; $R = -ky$ the resistive force due to the spring and Fric$= -cy'$ the frictional force. Note that Fric is proportional to v and always points in a direction opposite to the velocity and R is proportional to y and always points opposite to the position.

Thus we have $F = R$+Fric implying that $my'' = -cy' - ky$. This leads to the DE

$$my'' + cy' + ky = 0$$

where m (mass),c (coefficient of friction) and k (spring constant) are all non negative.

Example 4 In the motion above, assume that the mass of the object is $m = 1$ and that the spring constant is $k = 25$. Assume also that there is friction and that the coefficient of friction is $c = 6$.

a) Find the general solution of the differential equation governing the motion.

b) Describe and sketch the motion.

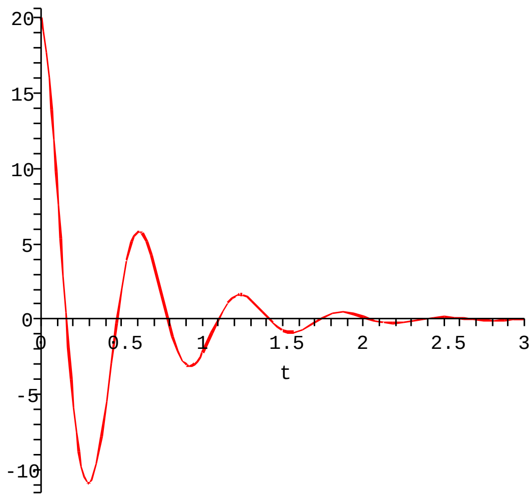
a) $my'' + cy' + ky = 0 \Rightarrow \underline{y'' + 6y' + 25y = 0}$

Aux eqn : $\lambda^2 + 6\lambda + 25 = 0$

$\lambda = \dfrac{-6 \pm \sqrt{36 - 100}}{2} = \dfrac{-6 \pm \sqrt{-64}}{2}$

$= \dfrac{-6 \pm 8i}{2} = \underline{-3 \pm 4i}$

$y = e^{-3t}\underline{\{A\cos 4t + B\sin 4t\}}$



The above sketch certainly looks like Simple Harmonic Motion with friction!

★ *a)* $y = e^{-3t}\{A\cos(4t) + B\sin(4t)\}$ *b)* Damped Oscillation ★

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 13

SECOND ORDER CONSTANT COEFFICIENT NON-HOMOGENEOUS DEs

Milan Pahor

LECTURE 13 MATH1231 CALCULUS
NON-HOMOGENEOUS SECOND ORDER DIFFERENTIAL EQUATIONS

To solve the non-homogeneous second order constant coefficient differential equation

$$ay'' + by' + cy = r(x)$$

we first solve the homogeneous problem $ay'' + by' + cy = 0$ to obtain a homogenous solution y_h. We then find a particular solution y_p by using the method of undetermined coefficients. The final solution is then $y = y_h + y_p$.

To find the long term steady state solution, consider the behaviour of the solution for large x.

We now deal with the constant coefficient second order D.E.'s of the previous lecture except that we will allow "nice" functions to appear on the RHS. This makes the D.E. non-homogeneous. We begin by finding the homogeneous solution y_h by using the methods of the previous lecture. We then find a particular solution y_p by essentially guessing an answer. The final solution is then $y = y_h + y_p$. It should be noted that this technique is prone to fail and will only work when the RHS is uncomplicated.

Example 1 Solve $y'' + 5y' + 6y = 48x + 22$. Describe the long term steady state solution to the differential equation.

$$y'' + 5y' + 6y = 0$$

Aux equ^{n} $\lambda^2 + 5\lambda + 6 = 0$

$(\lambda+2)(\lambda+3) = 0 \Rightarrow \lambda = -2, -3$

$\therefore\ y_h = Ae^{-2x} + Be^{-3x}$

Particular Sol^{n}:

Guess $y = \alpha x + \beta$

$y' = \alpha,\ y'' = 0.$

Sub in DE

$0 + 5(\alpha) + 6(\alpha x + \beta) = 48x + 22$

$6\alpha x + 5\alpha + 6\beta = 48x + 22$

$6\alpha = 48 \Rightarrow \alpha = 8$

$5\alpha + 6\beta = 22 \Rightarrow 40 + 6\beta = 22$

$6\beta = -18 \Rightarrow \beta = -3.$

$\therefore\ y_p = 8x - 3$

General sol^{n}:

$y = y_h + y_p$

$y = Ae^{-2x} + Be^{-3x} + 8x - 3$

$$\lim_{x \to \infty} y = 8x - 3$$

★ $y = Ae^{-2x} + Be^{-3x} + 8x - 3$. Steady State Solution is $y = 8x - 3$ ★

Why does this work? $y = y_h + y_p$

$$0 + 48x + 22$$

$$= 48x + 22$$

The crucial question is of course how do we know what to guess for y_p? The general rule is that you guess an arbitrary representation of the RHS. The following table gives some RHS's and their associated guesses for y_p .

RHS	Choice of y_P
$22e^{9x}$	Ce^{9x}
$x^3 - 7$	$\alpha x^3 + \beta x^2 + \gamma x + \delta$
$3\sin(4x)$	$\alpha \cos(4x) + \beta \sin(4x)$
$5e^{7x}\cos(2x)$	$e^{7x}(\alpha \cos(2x) + \beta \sin(2x))$
$8x^2e^{3x}$	$e^{3x}(\alpha x^2 + \beta x + \gamma)$

Note that if the RHS is too weird then no amount of guessing will save you.

Example 2 Solve the initial value problem

$$y'' + y = 55e^{2x} + 3x^2 + 14$$

where $y(0) = 20$ and $y'(0) = 28$.

This will take a lot of effort!

Aux. Equation $\lambda^2 + 1 = 0 \to \lambda^2 = -1 \to \lambda = 0 \pm i \to y_h = e^{0x}(A\cos(x) + B\sin(x))$

So our homogeneous solution is $y_h = A\cos(x) + B\sin(x)$

Our guess for y_p is

$$y = \alpha e^{2x} + \beta x^2 + \gamma x + \delta \to y' = 2\alpha e^{2x} + 2\beta x + \gamma \to y'' = 4\alpha e^{2x} + 2\beta$$

Sub. into the full differential equation $y'' + y = 55e^{2x} + 3x^2 + 14$:

$$(\cancel{4\alpha e^{2x}} + 2\beta) + \cancel{\alpha e^{2x}} + \beta x^2 + \gamma x + \delta = 55e^{2x} + 3x^2 + 14$$

$$5\alpha e^{2x} + \beta x^2 + \gamma x + (2\beta + \delta) = 55e^{2x} + 3x^2 + 14$$

$$5\alpha = 55 \Rightarrow \alpha = 11, \quad \gamma = 0$$

$$\beta = 3, \quad 2\beta + \delta = 14 \Rightarrow 6 + \delta = 14 \Rightarrow \delta = 8$$

$$\therefore y_p = 11e^{2x} + 3x^2 + 8$$

$$y = y_h + y_p$$
$$= A\cos x + B\sin x + 11e^{2x} + 3x^2 + 8$$

$y(0) = 20 \Rightarrow \quad 20 = A(1) + B(0) + 11 + 8$

$$A = 1$$

$$y = \cos x + B\sin x + 11e^{2x} + 3x^2 + 8$$
$$y' = -\sin x + B\cos x + 22e^{2x} + 6x$$

$y'(0) = 28: \quad 28 = 0 + B + 22 + 0$

$$\therefore B = 6$$

$$\therefore y = \cos x + 6\sin x + 11e^{2x} + 3x^2 + 8$$

This strange and somewhat dodgy technique of guessing a particular solution y_p and then plugging the guess into the DE is called *"the method of undetermined coefficients"*.

$$\bigstar \quad y = \cos(x) + 6\sin(x) + 11e^{2x} + 3x^2 + 8 \quad \bigstar$$

We saw in the previous example that the method of undetermined coefficients only works now and then. Sometimes it fails in a very special way and needs to be carefully patched up.

Example 3 Solve the second order differential equation

$$y'' - 4y' = 12e^{4x}.$$

Let's try the usual approach and see what happens.

Aux. Equation $\lambda^2 - 4\lambda = 0 \to \lambda(\lambda - 4) = 0 \to \lambda = 0, 4 \to y_h = Ae^{0x} + Be^{4x}$.

So our homogeneous solution is $y_h = A + Be^{4x}$. Our natural guess for y_p is

$$y = \alpha e^{4x} \to y' = 4\alpha e^{4x} \to y'' = 16\alpha e^{4x}$$

Sub. into the full differential equation

$y'' - 4y' = 12e^{4x}$:

$$16\alpha e^{4x} - 4(4\alpha e^{4x}) = 12e^{4x}$$

$$0 = 12e^{4x}.$$

We need a fix!

$$y = (\alpha x)e^{4x}$$

$$y' = \alpha e^{4x} + 4e^{4x}(\alpha x)$$

$$= \alpha e^{4x} + 4\alpha x e^{4x}$$

$$= (\alpha + 4\alpha x)e^{4x}$$

$$y'' = 4\alpha e^{4x} + 4e^{4x}(\alpha + 4\alpha x)$$

$$= 4\alpha e^{4x} + 4\alpha e^{4x} + 16\alpha x e^{4x}$$

$$= 8\alpha e^{4x} + 16\alpha x e^{4x}$$

sub into $y'' - 4y' = 12e^{4x}$

$$8\alpha e^{4x} + 16\alpha x e^{4x} - 4(\alpha e^{4x} + 4\alpha x e^{4x}) = 12e^{4x}$$

$$8\alpha e^{4x} + \cancel{16\alpha x e^{4x}} - 4\alpha e^{4x} - \cancel{16\alpha x e^{4x}} = 12e^{4x}$$

$$4\alpha e^{4x} = 12e^{4x}.$$

$$4\alpha = 12 \Rightarrow \alpha = 3$$

$$y_p = 3xe^{4x}$$

$$y = y_h + y_p$$

$$\bigstar \quad y = A + Be^{4x} + 3xe^{4x} \quad \bigstar$$

As a general rule, there must be no common terms between your guess for y_p and the pre-calculated y_h. To guarantee this you will sometimes need to modify the guess for y_p by multiplying by the independent variable, usually x or t. If there is still an overlap you must then multiply by x or t again and keep on doing so until you get something completely new! This multiplication process generates new solutions but we do pay a price. Often the solution becomes unstable and exhibits resonance. More on this later in the Appendix at the end of the lecture.

Example 4 Find the correct guess for the particular solution y_p for each of the following second order DE's. You are not required to solve the equation:

a) $y'' - 6y' + 9y = x^3 + \sin(8x)$ $\Rightarrow y_p = \alpha x^3 + \beta x^2 + \gamma x + \delta + \epsilon \sin 8x + \zeta \cos 8x$

b) $y'' - 6y' + 9y = 12e^{7x}$ $\Rightarrow y_p = \alpha e^{7x}$.

c) $y'' - 6y' + 9y = 12e^{3x}$ $y_p = \cancel{\alpha e^{3x}}$ $\cancel{\alpha x e^{3x}}$ $y_p = \alpha x^2 e^{3x}$

d) $y'' - 6y' + 9y = 12x^2 e^{3x}$ $y_p = (\alpha x^2 + \beta x + \gamma)e^{3x}$ $y_p = (\alpha x^4 + \beta x^3 + \gamma x^2)e^{3x}$

Note that in all the cases above the auxiliary equation is

$$\lambda^2 - 6\lambda + 9 = 0 \longrightarrow (\lambda - 3)^2 = 0 \longrightarrow \lambda = 3, 3.$$

and hence we have a homogeneous solution

$$y_h = (Ax + B)e^{3x} :$$

★ *a)* $y_p = \alpha x^3 + \beta x^2 + \gamma x + \delta + \epsilon \sin(8x) + \zeta \cos(8x)$ ★

Note the greek character ζ is called "zeta" and almost never gets a go.....for obvious reasons.

★ *b)* $y_p = \alpha e^{7x}$ *c)* $y_p = \alpha x^2 e^{3x}$ *d)* $y_p = (\alpha x^4 + \beta x^3 + \gamma x^2)e^{3x}$ ★

Appendix

Forced Oscillations and Resonance.

For Engineering Interest. Not examinable

Studied in detail in second year Math2019

We return now to the physical system of a mass on a spring described earlier.

Recall that this system was governed by the D.E.

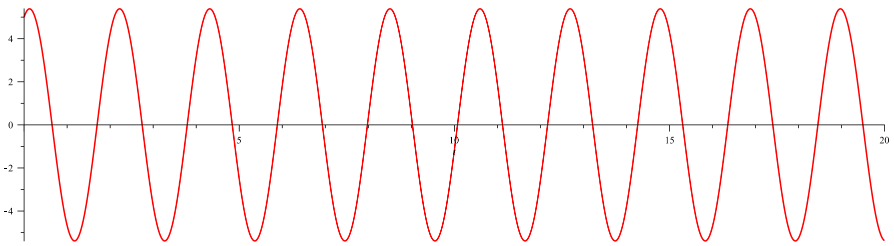
$$my'' + cy' + ky = 0$$

where m (mass), c (damping frictional coefficient) and k (spring constant) are all non negative.

I) Consider the differential equation

$$y'' + 9y = 0.$$

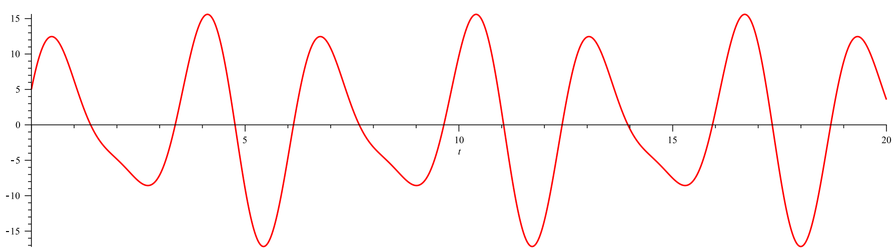
Since $c = 0$ there is no friction. The solution of this DE is $y = A\cos(3t) + B\sin(3t)$ and the differential equation could be viewed as modelling the motion of a child on a frictionless swing, oscillating forever. It's just Simple Harmonic Motion! A sketch of a possible solution is:



II) Now consider the differential equation   a forcing function

$$y'' + 9y = 60\sin(2t).$$

The function $60\sin(2t)$ is called a forcing function and can be interpreted as an external force on the system. This DE describes a child on a swing being pushed occasionally, but the pushing and the swinging are out of phase. The solution of this differential equation is $y = A\cos(3t) + B\sin(3t) + 12\sin(2t)$ and a sketch of a possible solution is:

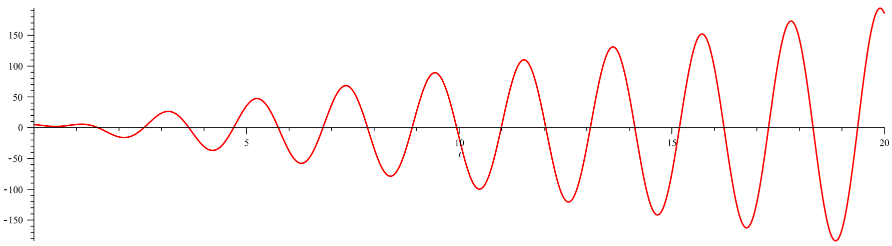


Observe that the pushing is distorting the natural swinging motion.

III) Now consider the differential equation

$$y'' + 9y = 60\sin(3t).$$

This is almost the same as (II)! The only difference is that the forcing function is $60\sin(3t)$ rather than $60\sin(2t)$. Remarkably this causes chaos. The forcing function is now in phase with the natural frequency of the motion and results in an unstable situation called harmonic resonance. **This is the situation where our usual guess for y_p needs to be multiplied by t.** This DE describes a child on a swing being pushed not occasionally, but rather at exactly the right time over and over again! The solution of this differential equation is $y = A\cos(3t) + B\sin(3t) - 10t\cos(3t)$. Note the t on the cosine function generating instability! A sketch of a possible solution is:



Note the scale on the vertical axis!!

What is happening here is that the system when not forced tends to naturally oscillate like a child on a swing. The forcing function in (II) adds a periodic note of oscillation to the system which is different from the natural frequency and hence just produces a minor perturbation. The ride just gets a bit bumpy.

But in (III) something critical happens! The forcing function is in perfect harmony with the system and makes it resonate (it hums!!) The entire system becomes unstable and the amplitude goes through the roof. This is sometimes a good thing since you may want to amplify a waveform...for example the amplification of a distant NASA signal. But in physical structures it is generally bad news and engineers have had to learn the hard way how disastrous harmonic resonance can be.

Have a look at the Tacoma bridge disaster!

$$https://www.youtube.com/watch?v = nkXl8JJBH7E$$

Resonance can be controlled with hydraulic dampers (bringing c into play).

Our second year engineering courses examine resonance in much greater detail.

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 14

MACLAURIN SERIES

Milan Pahor

LECTURE 14 MATH1231 CALCULUS
MACLAURIN SERIES

$$f(x) \approx f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^n(0)}{n!}x^n + \dots$$

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$e^x \approx 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + \dots$$

In this lecture we will begin the analysis of series approximations to functions. But first a little revision on factorials.

Definition: $n! = (n)(n-1)(n-2)\dots(1)$ for any integer $n \geq 0$.

Note that $n!$ is read as n factorial.

Example 1 Evaluate each of the following, first by hand and then via the calculator.

a) $3! = 3 \times 2 \times 1 = 6$

b) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

c) $0! = 1$

d) $100! =$ XX

★

Example 2 Simplify each of the following:

a) $\dfrac{7!}{6!}$. $= \dfrac{7 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = 7$

b) $\dfrac{n!}{(n+2)!}$. $= \dfrac{1}{(n+2)(n+1)}$

★

MACLAURIN SERIES

Have you ever wondered how your calculator evaluates something like $e^{0.2}$? It surely cannot have every possible value of e^x it could be asked stored in memory! What is actually done instead is to approximate e^x with polynomials

$$e^x \approx 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots.$$

We will see in this lecture that every standard function can be easily approximated by polynomials using what are called Maclaurin or Taylor series.

From a calculator's point of view e^x is a mystery but

$$1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

is nothing but addition and multiplication....exactly what a calculator is good at! All that needs to be done is to use a sufficiently precise series so that the approximation is accurate to within the limits of the calculator.

$e^{0.2}$ to 20 decimal places is given by:

$$e^{0.2} = 1.22140275816016983392$$

We can instead evaluate

$$1+\frac{0.2^1}{1!}+\frac{0.2^2}{2!}+\frac{0.2^3}{3!}+\frac{0.2^4}{4!}+\frac{0.2^5}{5!}+\frac{0.2^6}{6!}+\frac{0.2^7}{7!}+\frac{0.2^8}{8!}++\frac{0.2^9}{9!}=1.2214027581601410935$$

In reality no-one would ever find out that we are using a series approximation rather than the actual function. The advantage of polynomial series of course is that they suit calculators and computers perfectly........ it's all just addition and multiplication.

There are many other situations where it is advantageous to replace a complicated function with a polynomial approximation. Polynomials are very simple objects which are easily manipulated under calculus and by moving across to a polynomial approximation impossible questions can sometimes open up a little. The downside is that since polynomials are such basic functions we need to use infinitely many to do the job. Remarkably the actual calculation of the polynomial approximation for any standard function is a relatively straightforward task.

You should be aware of the standard series outlined above but also be capable of calculating a new series from first principles. In this lecture we will focus on Maclaurin series which are series constructed at the origin. In the next lecture we will move away from the origin by considering the more general Taylor Series. Note that a Maclaurin series is often called a Taylor series about $x = 0$.

Both Maclaurin and Taylor series are special cases of power series. We will examine the general theory of power series a little later on.

A Maclaurin series approximates the original function with simple polynomials. To construct the series we essentially force the function and its approximating polynomial series to agree at the origin for all derivatives. This forces the function and its polynomial approximation to look and behave in a similar way.

All you really need to remember when constructing a Maclaurin series is that:

$$\text{The coefficient of } x^n \text{ is } \frac{f^n(0)}{n!}$$

Note finally that Maclaurin series are almost never arithmetic or geometric progressions. Your 2 unit mathematics theory cannot be used for this topic.

There is only one simple formula which needs to be remembered to calculate any MacLaurin series.

Claim: Suppose that $y = f(x)$ is n-times differentiable at $x = 0$. Then the Maclaurin series for f is given by $f(x) \approx f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^n(0)}{n!}x^n + \ldots$

Proof: Assume that

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_n x^n + \ldots$$

$f(0) = a_0$

$f'(x) = a_1 + \cancel{2a_2 x} + \cancel{3a_3 x^2} + \cancel{\cdots} + \cancel{n a_n x^{n-1}}$

$f'(0) = a_1$

$f''(x) = 2a_2 + \cancel{3.2 a_3 x^1} + \cancel{\cdots} \; \cancel{n(n-1) a_n x^{n-2}}$

$f''(0) = 2a_2 \Rightarrow a_2 = \frac{f''(0)}{2\times 1}$

$f'''(x) = 3.2.1\, a_3 + \cancel{\cdots} - \cancel{n(n-1)(n-2) a_n x^{n-3}}$

$f'''(0) = 3!\, a_3$

$a_3 = \frac{f'''(0)}{3!}$ etc

★

Example 3 Find the first three non-zero terms in the Maclaurin series for $f(x) = \cos(x)$ from first principles.

$$f(x) = \cos(x) \Rightarrow f(0) = 1$$

$$f'(x) = -\sin(x) \Rightarrow f'(0) = 0$$

$$f''(x) = -\cos(x) \Rightarrow f''(0) = -1$$

$$f'''(x) = \sin(x) \Rightarrow f'''(0) = 0$$

$$f^{iv}(x) = \cos(x) \Rightarrow f^{iv}(0) = 1$$

$$\cos x = \boxed{1} + \boxed{\frac{0}{1!}}x + \boxed{\frac{-1}{2!}}x^2 + \boxed{\frac{0}{3!}}x^3 + \boxed{\frac{1}{4!}}x^4 + \ldots$$

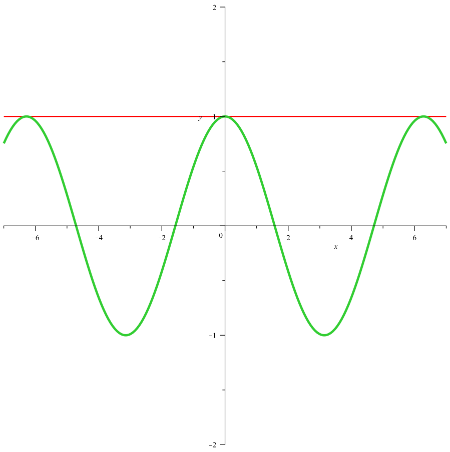
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\bigstar \quad \cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \quad \bigstar$$

Note that you will normally be told how far to take the series. The process of constructing a Maclaurin series begins with a detailed analysis of the function and all of its derivatives at the origin. Not surprisingly this means that the series works best at $x = 0$.

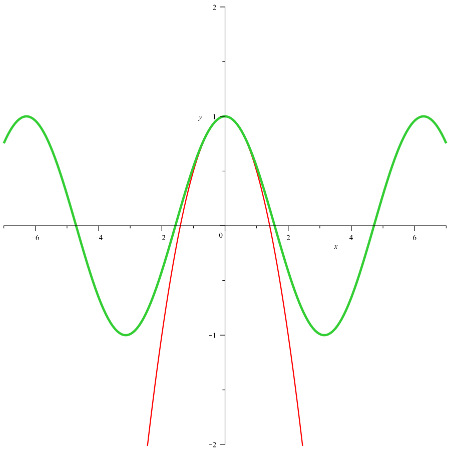
The following sketches show the increasingly accurate approximations using this series:
i)

$$\cos(x) \quad \text{and} \quad 1$$



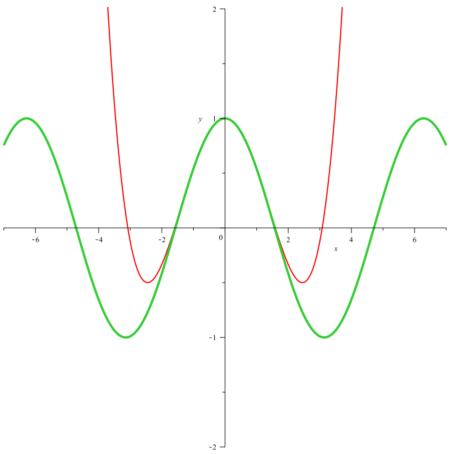
ii)

$$\cos(x) \quad \text{and} \quad 1 - \frac{x^2}{2}$$



iii)

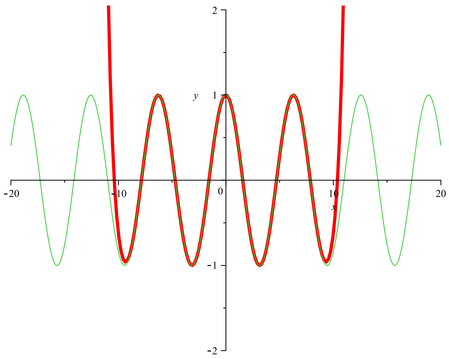
$$\cos(x) \quad \text{and} \quad 1 - \frac{x^2}{2} + \frac{x^4}{24}$$



Observe that the approximation is best at the origin and that the more terms you take the better the approximation is.

Just for a laugh lets take a look at the Maclaurin series up to 24th order terms:

$$\cos(x) \quad \text{and} \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!} + \frac{x^{16}}{16!} - \frac{x^{18}}{18!} + \frac{x^{20}}{20!} - \frac{x^{22}}{22!} + \frac{x^{24}}{24!}$$



It does a great approximating job near the origin but eventually gives up near $x = 10$.

To clear out some language issues let's look at the approximation

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

This is referred to as the Maclaurin series up to and including fourth order terms.

We will also say that $p_4(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$ is the 4th Maclaurin polynomial for $\cos(x)$.

Sometimes we will call this the first three non-zero terms in the series.

It is also possible to use Σ notation to represent a Maclauren or Taylor series in a compact format. Thus we could say that

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$= 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4$$

We can also use standard series to produce more complicated ones.

Example 4 Use the series obtained for $\cos(x)$ above to find the first three non-zero terms in the Maclaurin series of:

a) $f(x) = x^3 \cos(x)$

b) $f(x) = \cos(x^2)$

c) $f(x) = \sin(x)$

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

a) $x^3 \cos(x) \doteqdot x^3 \left\{ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \right\}$

$= x^3 - \frac{x^5}{2!} + \frac{x^7}{4!}$

b) $\cos(x^2) = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} + \ldots = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} + \ldots$

c) $-\sin x = 0 - \frac{2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \ldots = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \ldots$

$\sin x = x - x^3/3! + x^5/5! - \ldots$

★ *a)* $x^3 - \frac{x^5}{2} + \frac{x^7}{24} \ldots$ *b)* $1 - \frac{x^4}{2} + \frac{x^8}{24} \ldots$ *c)* $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ ★

You should be able to create Maclaurin series from scratch. But there are also many well known series with which you should be familiar. These include:

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad (\text{when } -1 < x < 1) \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots
\end{aligned}$$

Note the similarity between the series for $\sin x$, $\cos x$ and e^x.

Example 5

a) Find from first principles the first five terms in the Maclaurin series for e^x.

b) Hence find a series representation for the real number e.

c) Differentiate your series in a). What do you notice?

d) Hence find the first five terms in the Maclaurin series for e^{-x}.

e) Hence find the first three terms in the Maclaurin series for $\cosh(x)$.

f) Hence find the first two terms in the Maclaurin series for $\sinh(x)$.

a) $f(x) = e^x \to f(0) = e^0 = 1$

$f'(x) = e^x \to f'(0) = 1$

$= f''(0) = f'''(0)$

$\dots$

$e^x = \boxed{1} + \boxed{\frac{1}{1!}} x + \boxed{\frac{1}{2!}} x^2 + \boxed{\frac{1}{3!}} x^3 \dots$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$

$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$

$\frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)$

$= 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

e) $\cosh(x) = \frac{e^x + e^{-x}}{2}$

$= \frac{1}{2}\left\{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \dots\right)\right.$

$= \frac{1}{2}\left\{2 + 2\left(\frac{x^2}{2!}\right) + 2\left(\frac{x^4}{4!}\right) \dots\right\}$

$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \dots$

$\frac{d}{dx} = 0 + \frac{2x}{2!} + \frac{4x^3}{4!} + \dots$

$= x + \frac{x^3}{3!} + \dots$

★ $\cosh(x) \approx 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$ $\sinh(x) \approx x + \frac{x^3}{3!} + \dots$ ★

Example 6

a) Find from first principles the first three terms in the Maclaurin series for $\frac{1}{1+t}$.

b) Can you see another way of verifying the series in a)?

c) Hence determine the first three terms in the Maclaurin series for $\ln(1+x)$.

d) Hence find a rational approximation to $\int_0^{\frac{1}{2}} \frac{\ln(1+x)}{x}\,dx$.

Note that the integral in d) is impossible to express in terms of standard functions! Observe also that we are not evaluating the integral too far from zero as the quality of the approximation begins to decay.

e) Let $h(x) = x^4 \ln(1+x)$. Find $h^6(0)$ where h^6 is the 6th derivative of h.

a) $f(t) = \frac{1}{1+t} \Rightarrow f(0) = \frac{1}{1+0} = 1.$

$f'(t) = \frac{0 - 1(1)}{(1+t)^2} \Rightarrow f'(0) = \frac{-1}{1} = -1$

$= -(1+t)^{-2}$

$f''(t) = --2(1+t)^{-3} = \frac{2}{(1+t)^3} \Rightarrow f''(0) = 2$

$\frac{1}{1+t} = \boxed{1} + \boxed{\frac{-1}{1!}}\,t + \boxed{\frac{2}{2!}}\,t^2$

$= 1 - t + t^2$

$\frac{1}{1+t} = \underbrace{1 - t + t^2 - t^3 + t^4 \ldots}_{G.P.}$

$a = 1$

$r = -t$

$S_\infty = \frac{a}{1-r} = \frac{1}{1--t} = \frac{1}{1+t}$

c) $\int_0^x \frac{1}{1+t} = \int_0^x 1 - t + t^2 + \ldots$

$[\ln(1+t)]_0^x = \left[t - \frac{t^2}{2} + \frac{t^3}{3} + \ldots\right]_0^x$

$$\ln(1+x) - \cancel{\ln(1)} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

d) $$\int_0^{\frac{1}{2}} \frac{\ln(1+x)}{x}\,dx = \int_0^{\frac{1}{2}} \frac{x - \frac{x^2}{2} + \frac{x^3}{3}}{x}\,dx$$

$$\int_0^{\frac{1}{2}} 1 - \frac{x}{2} + \frac{x^2}{3}\,dx$$

e) $$h(x) = x^4 \ln(1+x)$$

$$= x^4 \left\{ x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \right\}$$

$$= x^5 - \frac{x^6}{2} + \frac{x^7}{3} + \cdots$$

So

$$-\frac{1}{2} = \frac{h^6(0)}{6!} \Rightarrow h^6(0) = \frac{6!}{-2}$$

$$= -\frac{720}{2}$$

$$= -360$$

© Milan Pahor 2020

★ $a)$ $1 - x + x^2$ $b)$ $x - \frac{x^2}{2} + \frac{x^3}{3}$ $d)$ $\frac{65}{144}$ $e)$ -360 ★

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 15

TAYLOR SERIES

Milan Pahor

LECTURE 15 MATH1231 CALCULUS
TAYLOR SERIES

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_{n+1}(x)$$

$$R_{n+1}(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1} \qquad \text{for some } a \leq c \leq x.$$

In the last lecture we saw that Maclaurin series can be used to approximate functions with polynomials and that the approximation is most accurate near the origin $x = 0$. Suppose that we need a polynomial approximation which works best at $x = a$ instead? We then use a general Taylor series about $x = a$.

Taylor series are almost the same as Maclaurin series except that the analysis is carried out at $x = a$ rather than $x = 0$. The formula for a Taylor series of f about $x = a$ is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_{n+1}(x)$$

$$= p_n(x) + R_{n+1}(x)$$

where the remainder term $R_{n+1}(x)$ measures the error when terminating the series at the nth Taylor polynomial $p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$.

We will examine error estimates a little more carefully in this lecture. The proof for general Taylor series is essentially the same as that for Maclaurin series presented in the last lecture.

To calculate a Taylor series for a function about $x = a$ we need to calculate the derivatives of f at $x = a$. All you really need to remember is that

$$\text{The coefficient of } (x-a)^n \text{ is } \frac{f^n(a)}{n!}$$

Keep in mind that a Maclaurin series is just a Taylor series about $x = 0$ so everything discussed and proven in this lecture applies to Maclaurin series as well.

A Taylor series about $x = a$ works best at $x = a$ and its quality decays the further from a you attempt to use the series.

Please be aware that the printed notes presents the theory of Taylor and Maclaurin series in two sections separated by a few weeks. We will instead do all the theory in one go.

Let's see how easily Taylor series may be constructed. The examiner must always tell you the point about which the series needs to be built and also to how many terms the series needs to be taken.

Example 1 Find the first four non-zero terms in the Taylor series for $f(x) = \cos(x)$ about $x = \pi$.

$f(x) = \cos(x) \Rightarrow f(\pi) = \cos(\pi) = -1$

$f'(x) = -\sin(x) \Rightarrow f'(\pi) = -\sin(\pi) = 0$

$f''(x) = -\cos(x) \Rightarrow f''(\pi) = 1$

$f'''(x) = \sin(x) \Rightarrow f'''(\pi) = 0$

$f^{iv}(x) = \cos(x) \Rightarrow f^{iv}(\pi) = -1$

$f^{v}(x) = -\sin(x) \Rightarrow f^{v}(\pi) = 0$

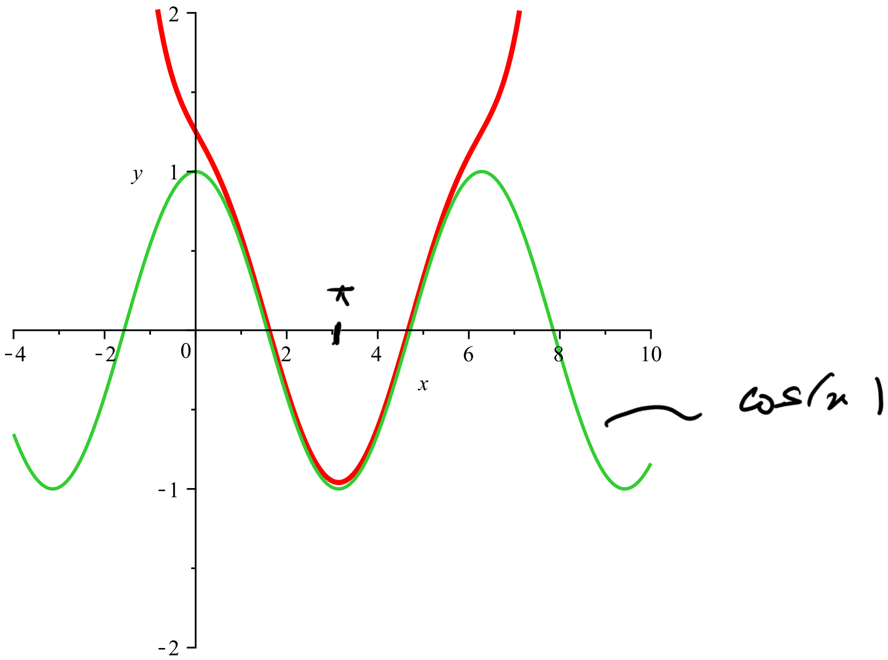
$f^{vi}(x) = -\cos(x) = f^{vi}(\pi) = 1$

$$\cos(x) \cong (-1) + \left(\frac{0}{1!}\right)(x-\pi) + \left(\frac{1}{2!}\right)(x-\pi)^2 + \left(\frac{0}{3!}\right)(x-\pi)^3 + \left(\frac{-1}{4!}\right)(x-\pi)^4 + \left(\frac{0}{5!}\right)(x-\pi)^5 + \left(\frac{1}{6!}\right)(x-\pi)^6$$

$$= -1 + \frac{1}{2!}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 + \frac{1}{6!}(x-\pi)^6 + \ldots$$

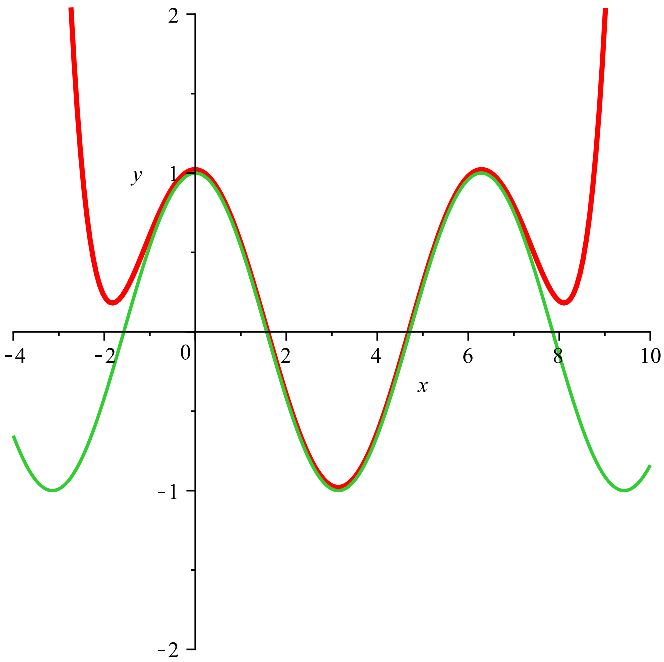
$$\bigstar \quad \cos(x) \approx -1 + \frac{1}{2!}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 + \frac{1}{6!}(x-\pi)^6 + \ldots \quad \bigstar$$

Lets take a look at the graph of $y = \cos(x)$ together with that of its approximating Taylor polynomial $p_6(x) = -1 + \frac{1}{2!}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 + \frac{1}{6!}(x-\pi)^6$.



Observe that the series works best near the point $x = \pi$ about which it is constructed. In other words a Taylor series is most accurate near where you did all the work!

The following graph extends the series to the first 6 non-series terms instead of just 4. Observe that it does a much better job, but the approximation is still best near $x = \pi$.



Example 2

a) Find the first three non-zero terms in the Taylor series for $f(x) = e^{3x}$ about $x = 1$.

b) Let $h(x) = e^{3x}(x-1)^7$. Using your answer in part a) write down the first three non-zero terms in the Taylor series of h about $x = 1$.

c) If you had created the three terms in the series in b) from first principles, how many times would you have needed to differentiate h.

d) Evaluate $h^9(1)$ where h^9 is the ninth derivative of $h(x) = e^{3x}(x-1)^7$.

a) $f(x) = e^{3x} \Rightarrow f(1) = e^3$

$f'(x) = 3e^{3x} \Rightarrow f'(1) = 3e^3$

$f''(x) = 9e^{3x} \Rightarrow f''(1) = 9e^3$

$$e^{3x} \cong e^3 + \left(\frac{3e^3}{1!}\right)(x-1) + \left(\frac{9e^3}{2!}\right)(x-1)^2$$

$$= e^3 + 3e^3(x-1) + \left(\frac{9}{2}e^3\right)(x-1)^2$$

$$\underline{(x-1)^7 e^{3x}} \cong e^3(x-1)^7 + 3e^3(x-1)^8 + \left(\frac{9}{2}e^3\right)(x-1)^9 + \ldots$$

$$\frac{h^9(1)}{9!}$$

d) $$\frac{h^9(1)}{9!} = \frac{9}{2}e^3$$

$$\Rightarrow h^9(1) = \frac{9}{2}e^3(9!)$$

a) $f(x) \approx e^3 + 3e^3(x-1) + \frac{9}{2!}e^3(x-1)^2$ *b)* $h(x) \approx e^3(x-1)^7 + 3e^3(x-1)^8 + \frac{9}{2!}e^3(x-1)^9$ ★

★ *c)* 9 times *d)* $\frac{9 \times 9!}{2}e^3 = 1632960e^3$ ★

ERROR ESTIMATES

When we set up a Taylor series about $x = a$ we have:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_{n+1}(x)$$

The polynomial

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

is called the nth Taylor polynomial of f about $x = a$ and the remainder term $R_{n+1}(x)$ measures the error when terminating the series after n terms. Observe that $R_{n+1}(x)$ depends upon both how far the series has been taken and point x where the series is being evaluated. The error term may be approximated using the Lagrange formula for the remainder.

Fact: Lagrange's form for the Remainder.

Suppose that f has $n+1$ continuous derivatives on an open interval I containing a. Then for each $x \in I$ we have $f(x) = p_n(x) + R_{n+1}(x)$ where

$$R_{n+1}(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$$

for some real number c between a and x.

Proof: See the printed notes.

Lagrange's form for the Remainder essentially tells us that if we terminate the Taylor series after n terms then the error is at worst the next term.

Note that as x moves away from a the value of $(x-a)^{n+1}$ will get larger and larger and hence so too will the error $R_{n+1}(x)$. We have $f^{n+1}(c)$ in the numerator rather than $f^{n+1}(x)$ since the $(n+1)$th derivative may take on its maximal value somewhere in the interval (a, x) rather than at the right hand endpoint x. We are just looking for the worst (biggest) possible error that could arise.

Remember that the error will rise when you reduce the number of terms in the Taylor series OR drift away from the point of expansion $x = a$. Conversely if you wish to reduce the error you should evaluate the Taylor series to a deeper level OR stay close to $x = a$.

In fact the Taylor series about $x = a$ is always **exact** at $x = a$.

Example 3 Let $f(x) = \sqrt{1+x}$. Find the second Taylor polynomial $p_2(x)$ of f about $x = 0$. Estimate $f(0.1)$ and find an error bound for this estimate.

$f(x) = (1+x)^{\frac{1}{2}} \Rightarrow f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \Rightarrow f'(0) = \frac{1}{2}(1) = \frac{1}{2}.$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2} \Rightarrow f''(0) = -\frac{1}{4}$

$\sqrt{1+x} = \boxed{1} + \boxed{\frac{\frac{1}{2}}{1!}}\, x + \boxed{\frac{-\frac{1}{4}}{2!}}\, x^2$

$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} = p_2(x)$

errors in video here!

$p_2(0.1) = 1 + \frac{0.1}{2} - \frac{(0.1)^2}{8} \doteq 1.04875$ ✓

$f'''(x) = \frac{3}{8}(1+x)^{-5/2} = \frac{3}{8} \cdot \frac{1}{(1+x)^{5/2}}$

$R_3(x) = \dfrac{\frac{3}{8}\left(\frac{1}{1+c}\right)^{5/2}}{3!}(x-0)^3$ where $0 \le c \le x$.

$R_3(0.1) = \dfrac{\frac{3}{8}\frac{1}{(1+c)^{5/2}}}{3!}(.1)^3$ $\quad \le 1 \quad 0 \le c \le .1$

$= \frac{3}{8} \cdot \frac{1}{6}(.1)^3 \frac{1}{(1+c)^{5/2}}$

error in video here

$\le \frac{3}{8} \cdot \frac{1}{6}(.1)^3(1)$

$= \frac{3}{8} \cdot \frac{1}{6}(.1)^3$

★ *a)* $p_2(x) = 1 + \frac{x}{2} - \frac{x^2}{8}$ *b)* $p_2(0.1) = 1.04875$ vs $f(0.1) = 1.048808848$ ★

★ *c)* $R_3(0.1) = \frac{1}{3!}\frac{3}{8}\frac{1}{(1+c)^{\frac{5}{2}}}(.1)^3 \le \frac{1}{3!}\frac{3}{8}(.1)^3 = 0.0000625$ ★

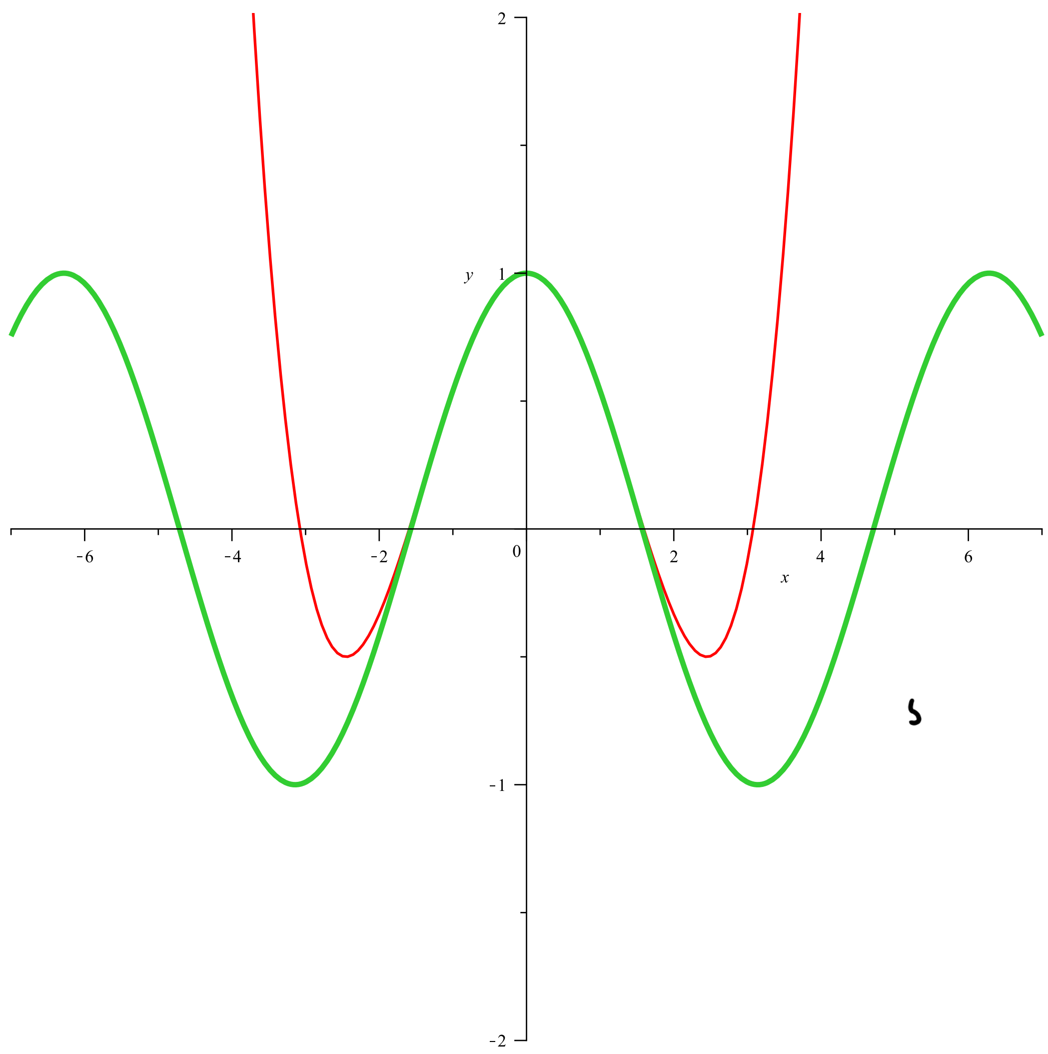
Recall from Example 3 in the previous lecture that the fourth Maclaurin approximating polynomial for $\cos(x)$ is

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + R_5(x) = p_4(x) + R_5(x)$$

where $R_5(x)$ is the error at x when approximating $\cos(x)$ by $p_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$.

Consider the following graph of $\cos(x)$ (green) together with its fourth Maclaurin approximating polynomial (red)

$$p_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$



Observe that the error $R_5(x)$ (that is the vertical difference between the two graphs) is small at $x = 1$, a little larger at $x = 2$ and quite significant at $x = 3$. How do we estimate these errors?

Example 4 For the series above use the Lagrange form of the remainder to estimate all three absolute errors $|R_5(1)|$, $|R_5(2)|$ and $|R_5(3)|$.

As we are interested here in just the distance between the curves rather than which one is actually bigger we consider absolute error $|R_5(x)|$ rather than error $R_5(x)$.

Since the series is about $x = 0$ we have

$$R_5(x) = \frac{f^5(c)}{5!}(x-0)^5 \text{ for some } c \text{ where } 0 < c < x.$$

That is

$$|R_5(x)| = \frac{|f^5(c)|}{5!}(x)^5$$

Note now that $f(x) = \cos(x) \to f'(x) = -\sin(x) \to f''(x) = -\cos(x)$

$\to f^3(x) = \sin(x) \to f^4(x) = \cos(x) \to f^5(x) = -\sin(x) \to f^5(c) = -\sin(c)$.

We need to choose c so that $|f^5(c)|$ is as large as possible. We want the worst possible case. Fortunately we can simply argue here that $|-\sin(c)| \leq 1$. Thus

$$R_5(x) \leq \frac{x^5}{5!}.$$

So $R_5(1) \leq \frac{1^5}{5!} \to R_5(1) \leq \frac{1}{120} \approx 0.0083$. (the error at $x = 1$ when using p_4).

So $R_5(2) \leq \frac{2^5}{5!} \to R_5(2) \leq \frac{4}{15} \approx 0.267$. (the error at $x = 2$ when using p_4).

So $R_5(3) \leq \frac{3^5}{5!} \to R_5(3) \leq \frac{243}{120} \approx 2.025$. (the error at $x = 3$ when using p_4).

★

Stationary Points via Taylor Series

Example 5 For each of the following curves $y = f(x)$:

a) Sketch the curve.

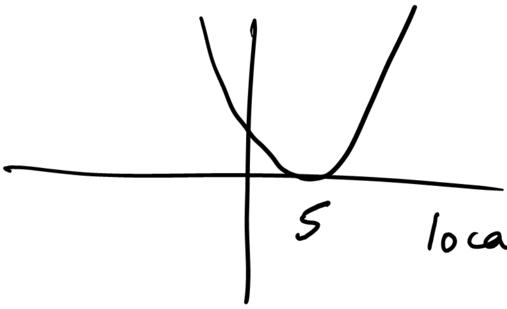
b) Evaluate $f''(5)$.

(I) $f(x) = 7(x-5)^4$.

$f'(x) = 28(x-5)^3$

$f''(x) = 84(x-5)^2$

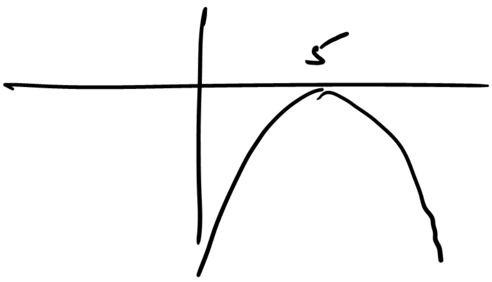
$f''(5) = 0$!!



local min

(II) $f(x) = -7(x-5)^4$.

$f''(5) = 0$



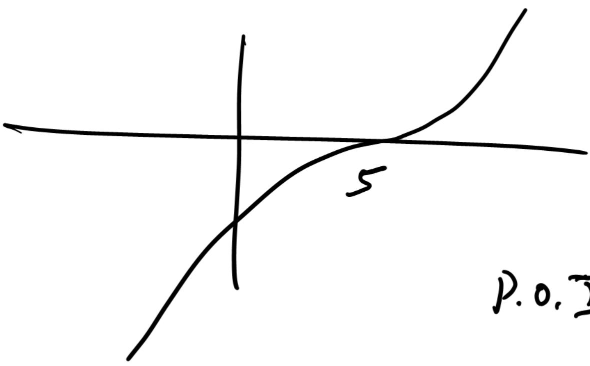
local max

(III) $f(x) = 7(x-5)^3$.

$f'(x) = 21(x-5)^2$

$f''(x) = 42(x-5)$

$f''(5) = 0.$



P.O.I.

★

There are three critical observations to make from the above example:

- $f''(\alpha) = 0$ does **NOT** imply a point of inflection at $x = \alpha$.

- If k is even then the graph of $(x-\alpha)^k$ is essentially the shape a parabola.

- If k is odd then the graph of $(x-\alpha)^k$ is essentially the shape of a cubic.

This will give us options in the situation where the standard second derivative test fails to classify a stationary point.

Example 6 Let $f(x) = x^6 - 4x^5 + 7x^4 - 8x^3 + 7x^2 - 4x + 9$. You are given that the graph has a stationary point when $x = 1$. Determine its nature.

$f(1) = 1 - 4 + 7 - 8 + 7 - 4 + 9 = 8$. So the point in question is $(1, 8)$.

$f'(x) = 6x^5 - 20x^4 + 28x^3 - 24x^2 + 14x - 4 \rightarrow f'(1) = 0 \rightarrow (1, 8)$ is a stationary point.

$f''(x) = 30x^4 - 80x^3 + 84x^2 - 48x + 14 \rightarrow f''(1) = 0$.

This is bad news! The second derivative test has failed! DO NOT conclude that $(1, 8)$ is a point of inflection! Having a zero second derivative does not imply point of inflection. Anything could still happen ..... max, min or P.O.I. Lets continue:

$f'''(x) = 120x^3 - 240x^2 + 168x - 48 \rightarrow f'''(1) = 0$.

$f^4(x) = 360x^2 - 480x + 168 \rightarrow f^4(1) = 48$. Finally a non-zero derivative!

The fourth Taylor polynomial f about $x = 1$ is therefore

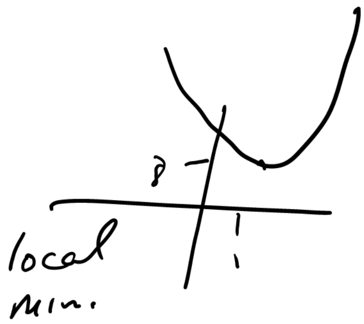
$$f(x) = 8 + \frac{48}{4!}(x - 1)^4 = 8 + 2(x - 1)^4$$

We can now consider the simpler function

$$y = 8 + 2(x - 1)^4$$

instead of

$$y = x^6 - 4x^5 + 7x^4 - 8x^3 + 7x^2 - 4x + 9.$$



Since $(x - 1)^4$ is an even power we have either $\bigcup$ or $\bigcap$. Since $2 > 0$ it must be $\bigcup$.

Thus the point $(1, 8)$ is a **local minimum** of $f(x) = x^6 - 4x^5 + 7x^4 - 8x^3 + 7x^2 - 4x + 9$.

★

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 16

SEQUENCES

Milan Pahor

LECTURE 16 MATH1231 CALCULUS
SEQUENCES

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers and that $L \in \mathbb{R}$. We then write

$$\lim_{n \longrightarrow \infty} a_n = L$$

if, for every positive number ϵ (no matter how small) there exists an integer M (usually large) such that $|a_n - L| < \epsilon$ whenever $n > M$.

That is, as $n \to \infty$ the terms of the sequence get close and stay close to L.

We turn now to the theory of infinite sequences. A sequence is simply a list (usually infinite) of real numbers formed according to some rule. For example $\{1, 4, 9, 16, 25, \dots\}$ is an infinite sequence.

We have many different ways of writing sequences down. The sequence $\{1, 4, 9, 16, 25, \dots\}$ could also be expressed as

$$a_n = n^2 \quad n = 1, 2, 3 \dots \qquad \text{or simply} \qquad \left\{n^2\right\}_{n=1}^{\infty}$$

Example 1 Write down the first three terms in the sequence $\left\{\dfrac{1}{n+1}\right\}_{n=3}^{\infty}$

$\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots$

★

Observe that the indexing of a sequence does not **have** to start at $n = 0$ or $n = 1$, though it usually does.

There are two important points to be made before continuing:

• Sequences **DO NOT** have to be arithmetic progressions or geometric progressions. In high school almost all of your sequences were either AP's or GP's. In Math1231 almost **none** of the sequences will be AP's or GP's. Observe already that none of the sequences on this page are arithmetic or geometric progressions.

• Sequences and series, although closely related, are **completely** different objects! The sequence $1, 4, 9, 16, 25$ has an associated series $1 + 4 + 9 + 16 + 25$. That is, a series is just the sum of the terms of a sequence. But these two concepts need to be very carefully separated in your mind. In this lecture we will look at nothing but **sequences**. In the next two lectures we will move on to **series**.

Sequences can be defined recursively rather than in closed form.

Example 2 Define a sequence by $a_n = \dfrac{a_{n-1}}{n}$; $a_0 = 1$.

Find the first 5 terms of the sequence and express the sequence in closed form.

$a_1 = \frac{a_0}{1} = \frac{1}{1} = 1$ $\Longrightarrow$ $a_n = \frac{1}{n!}$

$a_2 = \frac{a_1}{2} = \frac{1}{2!}$ $n = 0, 1, 2, \ldots$

$a_3 = \frac{a_2}{3} = \frac{\frac{1}{2}}{3} = \frac{1}{3 \times 2} = \frac{1}{3!}$

$$\bigstar \quad a_n = \frac{1}{n!} \quad n = 0, 1, 2 \ldots \quad \bigstar$$

Sequences fall into two classes, those that converge and those that do not converge (we say then that they diverge).

Convergence and Divergence

A converging sequence $\{a_n\}_{n=1}^{\infty}$ is (roughly speaking), one that tends towards a limit L. That is, as $n \to \infty$ the terms of the sequence get close and stay close to a some number L. We then write

$$\lim_{n \longrightarrow \infty} a_n = L$$

We will present a formal definition of convergence later in the lecture.

When attempting to find the limit of a sequence we use either common sense or, if desperate, L'Hopitals rule from Math1131.

In reality finding the limit of a sequence is not much different from finding the limit of a function as $x \to \infty$. Limits of functions were fully investigated in Math1131. All the techniques come across for free.

Example 3 Evaluate $\displaystyle\lim_{n \longrightarrow \infty} \frac{1}{n+1}$. $= 0$

This is trivial. All we need to do is think about what happens when n is large!

$$\bigstar \quad 0 \quad \bigstar$$

Example 4 Evaluate $\lim_{n\longrightarrow\infty} \frac{2n^2+6n-4}{3n^2+n-7}$.

Method 1

$$= \lim_{n\to\infty} \frac{2+\cancel{6/n}-\cancel{4/n^2}}{3+\cancel{1/n}-\cancel{7/n^2}} = 2/3.$$

$$= \text{“}\frac{\infty}{\infty}\text{”}$$

Method 2

$$\overset{l'H}{=} \lim_{n\to\infty} \frac{4n+6}{6n+1} = \text{“}\frac{\infty}{\infty}\text{”}$$

$$\overset{l'H}{=} \lim_{n\to\infty} \frac{4}{6} = 2/3.$$

★ $\frac{2}{3}$ ★

Example 5 Find $\lim_{n\longrightarrow\infty} 3 + \frac{(-1)^n}{n^2}$ decays to 0

$$= 3$$

★ 3 ★

Example 6 Find $\lim_{n\longrightarrow\infty} \sin(\frac{\pi}{4} - \frac{1}{n})$ goes to 0

$= \frac{1}{\sqrt{2}}$

$\star \quad \frac{1}{\sqrt{2}} \quad \star$

Example 7 Find $\lim_{n\longrightarrow\infty} \frac{\ln(n)}{n}$ " $= \frac{\infty}{\infty}$ "

$\overset{l'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$

$\star \quad 0 \quad \star$

Example 8 Find $\lim_{n\longrightarrow\infty} \frac{n!}{n^n}$

Hint $\quad \frac{4!}{4^4} = \frac{4\times 3\times 2\times 1}{4\times 4\times 4\times 4} < \frac{\cancel{4}\times \cancel{4}\times \cancel{4}\times 1}{\cancel{4}\times \cancel{4}\times \cancel{4}\times 4} = 1\times 1\times 1\times \frac{1}{4} = \frac{1}{4}.$

Simlarly $\quad \frac{n!}{n^n} < \frac{1}{n}.$

$\lim_{n\to\infty} \frac{n!}{n^n} \leq \lim_{n\to\infty} \frac{1}{n} = 0.$

$\star \quad 0 \quad \star$

Example 9 Find $\lim_{n \longrightarrow \infty} (\sqrt{n^2+n} - n)$

This is a strange one! Keep in mind that if you get desperate you can simply place a large integer in the formula and take a guess!

For example when $n = 1000$ we have $(\sqrt{n^2+n} - n) \approx 0.499875$. A guess of 0.5 for the limit seems reasonable! How do we get there mathematically?

$$\lim_{n \to \infty} (\sqrt{n^2+n} - n)\left(\frac{\sqrt{n^2+n} + n}{\sqrt{n^2+n} + n}\right)$$

$$= \lim_{n \to \infty} \frac{(n^2+n) - n^2}{\sqrt{n^2+n} + n} \qquad (a+b)(a-b) = a^2 - b^2$$

$$= \lim_{n \to \infty} \frac{n}{\sqrt{n^2+n} + n} = \lim_{n \to \infty} \frac{1}{\frac{\sqrt{n^2+n}}{\sqrt{n^2}} + 1}$$

$$= \lim_{n \to \infty} \frac{1}{\sqrt{\frac{n^2+n}{n^2}} + 1} = \lim_{n \to \infty} \frac{1}{\sqrt{\frac{1+\cancel{\frac{1}{n}}}{1}} + 1}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

$$\star \quad \frac{1}{2} \quad \star$$

We have been evaluating limits intuitively. What is the formal definition of a limit of a sequence as $n \to \infty$? It is almost the same as the definition of the limit of a function as $x \to \infty$:

Definition: Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers and that $L \in \mathbb{R}$. We then write

$$\lim_{n \longrightarrow \infty} a_n = L$$

if, for every positive number ϵ (no matter how small) there exists an integer M (usually large) such that $|a_n - L| < \epsilon$ whenever $n > M$. In other words $\lim_{n \longrightarrow \infty} a_n = L$ if we can make a_n as close to L as we like by simply choosing n to be sufficiently large.

$$\lim_{n \to \infty} \left(2 + \frac{1}{n}\right) = 2$$

$$\left\{2 + \frac{1}{n}\right\}_{n=1}^{\infty} = 3, 2\tfrac{1}{2}, 2\tfrac{1}{3}, 2\tfrac{1}{4}, 2\tfrac{1}{5}, \ldots$$

Example 10 Prove that

$$\lim_{n\longrightarrow\infty} \frac{8n+5}{2n+1} = 4$$

a) Informally $= \lim_{n\to\infty} \frac{8+\frac{5}{n}}{2+\frac{1}{n}} = 8/2 = 4.$

b) Formally Claim $\lim_{n\to\infty} \frac{8n+5}{2n+1} = 4.$

$\underline{\underline{\text{Pf}}}$ Let $\varepsilon > 0$ be given.

$$\underbrace{\left| \frac{8n+5}{2n+1} - 4 \right| < \varepsilon}$$

$$\Leftrightarrow \left| \frac{8n+5-4(2n+1)}{2n+1} \right| < \varepsilon \Rightarrow \left| \frac{1}{2n+1} \right| < \varepsilon$$

$$\frac{1}{2n+1} < \varepsilon \Rightarrow 2n+1 > \frac{1}{\varepsilon} \Rightarrow 2n > \frac{1}{\varepsilon} - 1$$

$$n > \frac{1}{2\varepsilon} - \frac{1}{2} \qquad \therefore \ M = \frac{1}{2\varepsilon} - \frac{1}{2}$$

c) Using b) determine how far down the sequence we would need to go until the terms in the sequence $\left\{\frac{8n+5}{2n+1}\right\}_{n=1}^{\infty}$ are within $\frac{1}{1000} = 0.001$ of the limit $L = 4$.

$$M = \frac{1}{2(.001)} - \frac{1}{2} = \underline{\underline{499.5}}$$

$$\therefore \boxed{M = 500}$$

★ *c*) At $n = 499$, $\frac{8n+5}{2n+1} = 4.001001$. Not quite! But at $n = 500$, $\frac{8n+5}{2n+1} = 4.0009$. ★

It is worthwhile at this stage to consider the relative strength of various sequences for large n. We have

$$\ln(n) << n^2 << e^n << n! << n^n$$

That is $\ln(n)$ gets to infinity very slowly whereas n^n is super fast.

So we would expect $\displaystyle\lim_{n\longrightarrow\infty} \frac{\ln(n)}{n!}$ to be 0, since $n!$ dominates $\ln(n)$ but

$\displaystyle\lim_{n\longrightarrow\infty} \frac{e^n}{n^2}$ to be infinite since e^n dominates n^2.

Divergence

If a sequence does not converge we say that it diverges. Sequences may diverge in a number of different ways.

For example the sequence $\{1, 4, 9, 16, \dots\}$ diverges to infinity. The terms just keep on getting bigger.

But $\{1, -1, 1, -1, 1, -1 \dots\}$ is boundedly divergent. The terms are not getting large. They are just wobbling all over the place and never settling down.

Most of this lecture is intuitively reasonable. In the next few lectures however, we will look at the convergence of **series** rather than sequences. This theory is very surprising and you may find that many things that you thought were true for the real number system are not quite as you imagined them.

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 17

INFINITE SERIES OF POSITIVE TERMS (PART 1)

Milan Pahor

LECTURE 17 MATH1231 CALCULUS
INFINITE SERIES OF POSITIVE TERMS (PART I)

If an infinite series $\sum_{n=1}^{\infty} a_n$ sums to a finite number S we say that the series converges to S. Else we say that the series diverges.

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

nth term Test for divergence If $\lim_{n \longrightarrow \infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

The integral Test $\sum_{n=1}^{\infty} a_n$ converges/diverges iff the associated improper integral

$$\int_1^{\infty} a_n(x)\, dx \text{ converges/diverges.}$$

p-Series Test The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

In the previous lecture we examined the convergence theory of infinite **sequences**. We will now devote two lectures to the convergence of infinite **series** of **positive** numbers. Note that initially, we will only consider series of positive numbers in order to simplify the analysis.

Many of the results here are highly counter-intuitive, so be alert to the possibility that infinite series may not behave as expected! Come down at the end of the lecture for a chat if you have general concerns regarding what is happening.

If an infinite series $\sum_{n=1}^{\infty} a_n$ sums to a finite number S we say that the series converges to S.

Else we say that the series diverges. Our focus in this course is NOT the question "What is the sum of this infinite series ?" since this question is in general way too hard to answer. Rather, we simply want to know "Does this series converge?"

Note that we are not interested in finite series as all finite series converge. You just add up the numbers. Note also that the series we will be considering are much more general than AP's and GP's. Indeed AP's and GP's will rarely show up.

The first task is to convince ourselves that it is actually possible for the sum of an infinite set of positive numbers to be finite!

Many students are mathematically troubled by the results of this lecture. We will finish a little early today, so please feel free to come down and have a chat re any concerns.

Consider the sequence $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$. That is $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$. Clearly the limit of this sequence is 0, that is $\lim_{n \longrightarrow \infty} \frac{1}{2^n} = 0$.

The associated **series** is $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

You will no doubt recognise this as the limiting sum of a geometric progression with $a = \frac{1}{2}$ and $r = \frac{1}{2}$. Thus $S_\infty = \frac{a}{1-r} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$.

So this infinite **sequence** converges to 0 but the associated infinite **series** converges to 1. We say that the series is summable and that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Does the sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ ever actually get to 1?

The answer is **NO**. How close to 1 does it get? As close as you like! It gets arbitrarily close. This is what is meant by a series converging to a limit. The sum gets (and stays) arbitrarily close to 1 even though it may never actually get there!

Observe that convergence of sequences and convergence of series are completely different theories.

Harmonic Sequence

Now consider the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. That is $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$.

Clearly the limit of this sequence is also 0. In other words $\lim_{n \longrightarrow \infty} \frac{1}{n} = 0$.

The associate series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is called the Harmonic series. It is one of the most mysterious objects in mathematics.

Fact: The Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ diverges!

This is a little surprising. The terms are getting smaller and smaller and it feels as if the sum should be finite. Instead the series becomes arbitrarily large and approaches infinity.

Proof:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \dots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right) + \dots$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right) + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

$= \infty$

It follows that when summing the Harmonic series it is larger than adding up infinitely many copies of $\frac{1}{2}$. Thus the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. It just blows up!

★

Suppose we terminate the infinite series $\sum_{n=1}^{\infty} a_n$ after N terms and calculate the sum to N rather than the sum to infinity. This is called the Nth partial sum and is denoted by S_N. So

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \cdots + a_N$$

We say that

$$\sum_{n=1}^{\infty} a_n = L \quad \longleftrightarrow \quad \lim_{N \longrightarrow \infty} S_N = L$$

In other words a series converges if and only if the sequence of partial sums converges.

Let's take a look at some partial sums for $\sum_{n=1}^{\infty} \frac{1}{n}$ calculated via Maple:

$$S_{1000} = \sum_{n=1}^{1000} \frac{1}{n} = 7.485$$

$$S_{10000} = \sum_{n=1}^{10000} \frac{1}{n} = 9.788$$

$$S_{100000} = \sum_{n=1}^{100000} \frac{1}{n} = 12.090$$

$$S_{1000000} = \sum_{n=1}^{1000000} \frac{1}{n} = 14.393$$

$$S_{10000000} = \sum_{n=1}^{10000000} \frac{1}{n} = 16.695$$

Observe that the S_N's are not settling down. The Harmonic series is diverging pretty slowly....but still it's diverging not converging. For contrast let's take a look at the partial sums for $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$S_{10} = \sum_{n=1}^{10} \frac{1}{n^2} = 1.54977$$

$$S_{100} = \sum_{n=1}^{100} \frac{1}{n^2} = 1.63498$$

$$S_{1000} = \sum_{n=1}^{1000} \frac{1}{n^2} = 1.64393$$

$$S_{10000} = \sum_{n=1}^{10000} \frac{1}{n^2} = 1.64483$$

$$S_{100000} = \sum_{n=1}^{100000} \frac{1}{n^2} = 1.64492$$

$$S_{1000000} = \sum_{n=1}^{1000000} \frac{1}{n^2} = 1.64493$$

This is clearly a totally different story. The partial sums are approaching a definite limit L, although it is still unclear at this stage what the exact value of L is.

We say that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges to approximately 1.64493. An exact value for L will be given later in the lecture.

What is clear from the Harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ above, is that for a series to converge, it is NOT enough for the terms to go to 0. They need to approach zero with a little bit of urgency or else the sum is overwhelmed by the number of terms.

In this lecture and the next we will develop a batch of tests which can be used to determine whether an infinite series of positive terms converges or diverges. Keep in mind that we do not in general ask for the value to which a converging series sums. That is too hard a question. All we wish to know is does it converge (that is sums to a finite number) or doesn't it?

TESTS FOR CONVERGENCE OF $\sum_{n=1}^{\infty} a_n$ where $a_n \geq 0$

(Proofs for these tests may be found in your printed notes)

Test 1: nth term test for Divergence.

If $\lim_{n\longrightarrow\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 1 Prove that $\sum_{n=1}^{\infty} \frac{6n+1}{2n+4}$ diverges.

$\lim_{n\to\infty} \frac{6n+1}{2n+4} = \underline{\underline{3}} \neq 0.$

$\therefore$ Series diverges by nth term test. ★

There is no chance of the series above converging since in the end you will just be adding up infintely many 3's! Note carefully in this example that the **sequence** converges to 3 but the **series** diverges!

Discussion Question: What does the nth term test tell you about the Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}?$$

$\lim_{n\to\infty} \frac{1}{n} = 0$ test says nothing.

★

The nth term test cannot be used to prove convergence....only divergence! The Harmonic series clearly shows you that $\lim_{n\longrightarrow\infty} a_n = 0$ does NOT imply that $\sum_{n=1}^{\infty} a_n$ converges.

Discussion Question: The first million terms in the series $\sum_{n=1}^{\infty} \frac{6n+1}{2n+4}$ were lost in an office fire and replaced with numbers which were randomly chosen. How could this effect the convergence or divergence of the series.

<u>It won't</u>

★

Convergence happens in the tail of the series (at ∞) NOT the head! Altering a finite collection of terms will never impact upon the convergence or divergence of the series.

Test 2: The Integral Test

$$\sum_{n=1}^{\infty} a_n \text{ converges/diverges iff the associated improper integral } \int_1^{\infty} a_n(x)\,dx \text{ converges/diverges.}$$

The integral test is a very handy little test. It allows us to transfer the problem of convergence of series over to the issue of convergence of improper integrals (from Math1131).

Example 2 Use the integral test to determine whether $\sum_{k=0}^{\infty}\left(\frac{1}{k^2+1}\right)$ converges or diverges.

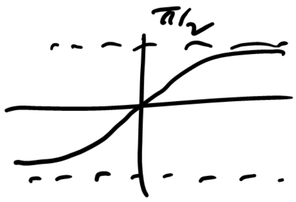
A few small points first.

• Do not be confused by the use of k rather than n. Both are equivalent. The index is usually either k or n but it doesn't matter which.

• The fact that we are starting at 0 rather than 1 is also of no consequence. In fact the first M terms where M is any integer are of no consequence when it comes to considering convergence/divergence of series OR sequences.

Consider $\int_0^{\infty} \frac{1}{x^2+1}\,dx. = \lim_{n \to \infty} \int_0^n \frac{1}{x^2+1}\,dx.$

$= \lim_{n \to \infty} \left[\tan^{-1}(x)\right]_0^n$



$= \lim_{n \to \infty} \tan^{-1}(n) - \cancel{\tan^{-1}(0)}$

$= \frac{\pi}{2}.$ $\therefore \int$ converges.

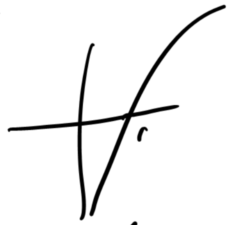
$\therefore$ By integral test series $\sum_{k=0}^{\infty} \frac{1}{k^2+1}$ also converges.

★ Converges ★

Example 3 Use the integral test to prove that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Consider $\int_1^{\infty} \frac{1}{x}\,dx = \infty$ ↑

$= \lim_{n\to\infty} \int_1^{n} \frac{1}{x}\,dx = \lim_{n\to\infty} \left[\ln x\right]_1^{n}$



$= \lim_{n\to\infty} \ln n - \cancel{\ln(1)}$ $\therefore$ integral diverges ★

$= \infty$ $\therefore$ By integral test, series diverges.

Test 3: p-Series Test

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

The p-series are a nice class of series where the convergence/divergence can be trivially determined. They will be very important in the next lecture where we will need to make comparisons between series. The p-series test follows as a consequence of the p-integral theory in the Math1131 course.

Note that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ sits right at the convergence boundary for p-series.

Example 4 Determine whether the following series converge or diverge:

a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Converges, p-series with $p = 2 > 1$

b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. $= \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$,

Diverges, p-series with $p = \frac{1}{2} \leq 1$.

★ $a)$ Converges $b)$ Diverges ★

Example 5 Determine whether the following series converge or diverge:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.0000000001}}$ – converge

p-series

$p>1$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.9999999999}}$.

diverge

p-series

$p\leq 1$

Looking at the partial sums to a million is fascinating. We have

$a)$ $\displaystyle\sum_{n=1}^{1000000} \frac{1}{n^{1.0000000001}} = 14.39272295$ b) $\displaystyle\sum_{n=1}^{1000000} \frac{1}{n^{0.9999999999}} = 14.39272584$ → diverge

↑ converge

This took Maple 15 minutes!

The sums of the first million terms are almost identical! Yet a) converges while b) diverges ★

conv. div

In conclusion lets take a close look at $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ versus $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

They do not look all that different yet $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (converging p-series $p = 2 > 1$) while $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (diverging p-series $p = 1 \leq 1$). Well

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

– finite

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

– not finite

The terms in both series are converging to 0.

What saves $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is that the terms are getting to zero **fast**. The terms in the Harmonic series are dawdling in too slowly and eventually get swamped. Roughly speaking a series will converge if the terms get to zero with a little bit or urgency.

Does that finish off the tests for convergence.........NO we have another full lecture of tests to consider!

One last comment. It is quite rare to be able to actually evaluate the sum of a converging series. By taking partial sums we saw earlier in this lecture that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1.64493$.

Remarkably you will be able to show in second year mathematics that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.$$

$\dfrac{\pi^2}{6} = 1.644934067$ to 9 decimal places so it looks right!

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 18

INFINITE SERIES OF POSITIVE TERMS (PART II)

Milan Pahor

LECTURE 18 MATH1231 CALCULUS
INFINITE SERIES OF POSITIVE TERMS (PART II)

p-Series Test The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Comparison Tests:

If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges and $a_n \leq b_n$ for all n then $\displaystyle\sum_{n=1}^{\infty} a_n$ also converges. (pushed down)

If $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges and $a_n \geq b_n$ for all n then $\displaystyle\sum_{n=1}^{\infty} a_n$ also diverges. (pushed out)

$$-1 \leq \sin(n) \leq 1$$

$$\ln(n) < n$$

For n sufficiently large it is true that $\ln(n) < n^c$ for any $c > 0$

Ratio Test:

Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ is an infinte series of positive terms and let $\displaystyle\rho = \lim_{n \longrightarrow \infty} \frac{a_{n+1}}{a_n}$.

If $\rho < 1$ the series converges.

If $\rho > 1$ the series diverges.

If $\rho = 1$ the test fails.

In this lecture we will continue our study of infinite series of positive terms:

$$\sum_{n=1}^{\infty} a_n \quad \text{where} \quad a_n \geq 0.$$

Our main concern is still whether the series adds up to a finite number (convergence) or whether it doesn't (divergence). In the last lecture we presented some tests for convergence and divergence. Today will will expand upon these tests. Proofs justifiying the tests can be found in your printed notes.

Keep in mind we are examining convergence of series here NOT sequences! Note also the blanket assumption in this lecture that the series do not have negative terms. We will look at series with positive and negative terms in the next lecture.

Test 4: The Comparison Test.

We used comparison tests when dealing with the convergence of improper integrals in Math1131. We can also use comparison tests on series in an almost identical fashion.

(i) If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges and $a_n \le b_n$ for all n then $\displaystyle\sum_{n=1}^{\infty} a_n$ also converges. (pushed down)

(ii) If $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges and $a_n \ge b_n$ for all n then $\displaystyle\sum_{n=1}^{\infty} a_n$ also diverges. (pushed out)

Note that

If $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges and $a_n \le b_n$ for all n then NOTHING.

Being less than infinite tells you nothing. It could be finite it could be infinite.

Also

If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges and $a_n \ge b_n$ for all n then NOTHING.

Being bigger than finite tells you nothing. It could be finite it could be infinite.

Example 1 Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^3+1}$ converges or diverges.

When using the comparison test we usually compare against a p-series. Recall the p-series test:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1 \text{ and diverges if } p \le 1.$$

Remember that bigger bottoms make smaller fractions. For example $\frac{8}{13} < \frac{8}{9}$

Also smaller bottoms make bigger fractions. For example $\frac{11}{5} > \frac{11}{7}$

Now

$$\frac{1}{2n^3+1} < \frac{1}{2n^3}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{2n^3+1} < \sum_{n=1}^{\infty} \frac{1}{2n^3} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges by p-series $p=3>1$

$\therefore$ By comparison test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^3+1}$ also converges

★ Converges ★

Example 2 Examine the convergence of $\sum_{k=7}^{\infty} \frac{1}{\sqrt{k-2}}$.

$\frac{1}{\sqrt{k-2}} > \frac{1}{\sqrt{k}}$. and $\sum_{k=7}^{\infty} \frac{1}{\sqrt{k}} = \sum \frac{1}{k^{\frac{1}{2}}}$

is a diverging p-series $p = \frac{1}{2} < 1$.

$\therefore$ By comparison test

$\sum_{k=7}^{\infty} \frac{1}{\sqrt{k-2}}$ also diverge.

★ Diverges ★

Some Useful Comparisons

Example 3 Discuss the convergence/divergence of $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{2^n}$.

$$\boxed{\sin^2(n) \leq 1}$$

$\frac{\sin^2(n)}{2^n} < \frac{1}{2^n}$. Consider $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

converging G.P.

$\underbrace{-1 < r = \frac{1}{2} < 1}$

$\therefore$ By comparison test.

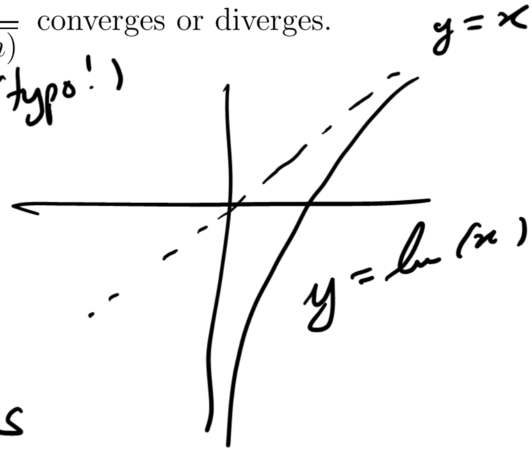
$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{2^n}$ also converges

★ Converges ★

Example 4 Determine whether the infinite series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{\ln(n)}}$ converges or diverges.

$n=2$ (typo!)

$$\boxed{\ln(n) < n}$$



$\ln(n) < n$

$\ln(n)^{\frac{1}{2}} < n^{\frac{1}{2}}$

$\frac{1}{\ln(n)^{\frac{1}{2}}} > \frac{1}{n^{\frac{1}{2}}}$

$\Rightarrow \frac{1}{\sqrt{\ln n}} > \frac{1}{\sqrt{n}}$. ✓

$\sum \frac{1}{\sqrt{n}}$ is a diverging p-series. $p = \frac{1}{2} < 1$

★ Diverges ★

$\therefore \sum_{n=2}^{\infty} \frac{1}{\sqrt{\ln(n)}}$ also diverges via comparison test.

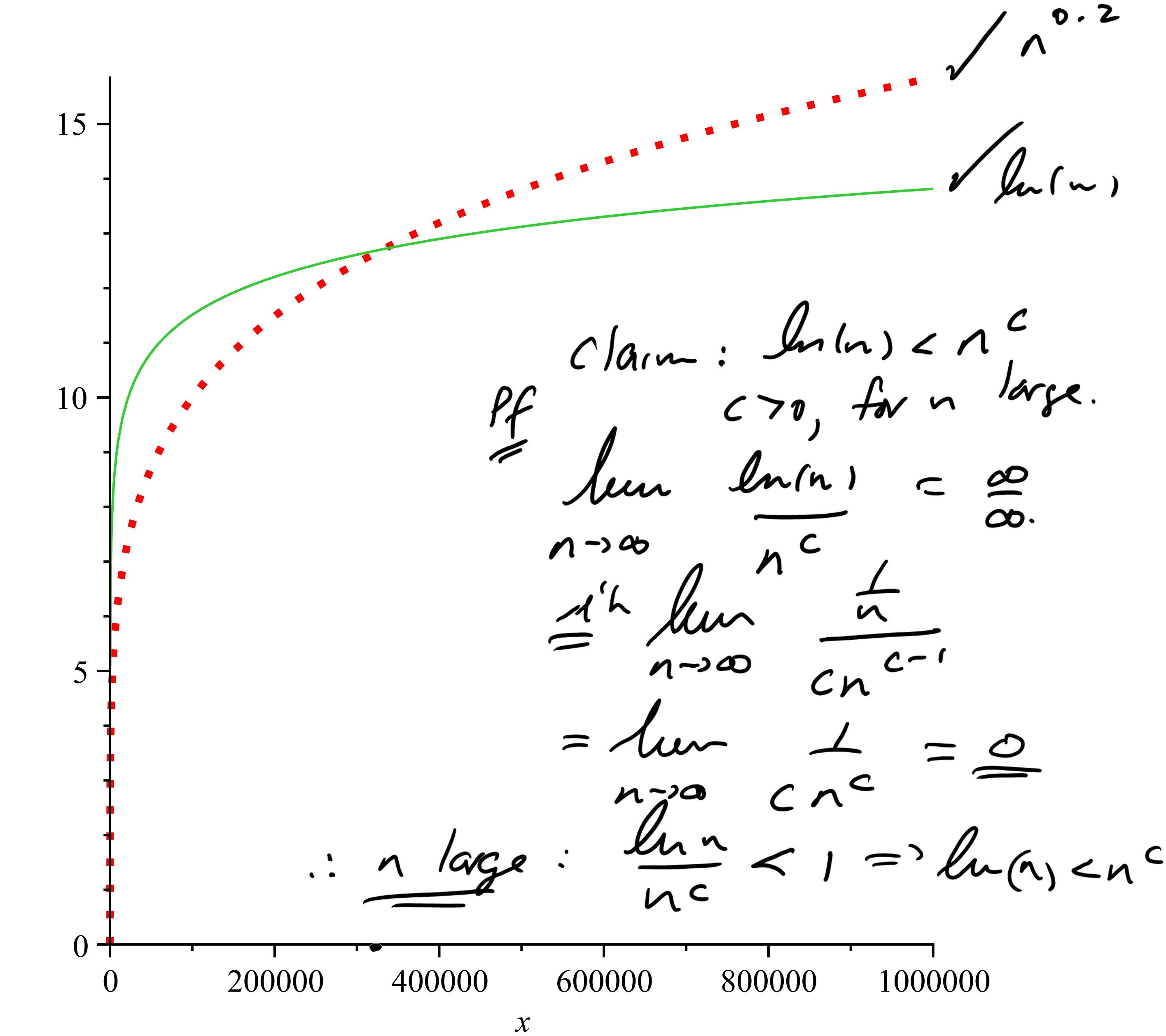
We saw in the previous example that $\ln(n) < n$ for $n = 1, 2, 3 \ldots$. There exists a slightly sharper result than this:

Fact: Let $c > 0$. Then $\ln(n) < n^c$ provided that n is sufficiently large.

Proof of Fact: See printed notes.

This means that $\ln(n)$ is soooooo weak that it is eventually dominated by any power of n. So for example $\ln(n)$ is eventually smaller than $n^{0.0000000000000001}$.

In the following diagram the solid graph is that of $\ln(n)$ and the dotted graph is $n^{0.2}$. That is $c = 0.2$. Observe that *eventually* $\ln(n) < n^{0.2}$. That is, *eventually* the solid line sits underneath the dotted line. You have to wait until $n \approx 300000$ but it does happen. Remember that for series all the action is in the tail so waiting until $n \approx 300000$ is of no concern at all. Series have all the time in the world to wait for something to occur.



Claim: $\ln(n) < n^c$, $c > 0$, for n large.

Pf

$$\lim_{n\to\infty} \frac{\ln(n)}{n^c} = \frac{\infty}{\infty}.$$

$$\stackrel{l'h}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{cn^{c-1}}$$

$$= \lim_{n\to\infty} \frac{1}{cn^c} = 0$$

$\therefore$ n large: $\frac{\ln n}{n^c} < 1 \Rightarrow \ln(n) < n^c$

Example 5 Examine the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^{\frac{3}{2}}}$

$$\boxed{\text{For any } c > 0, \quad \ln(n) < n^c, \text{ provided } n \text{ is sufficiently large}}$$

We can argue that since $\ln(n) < n$

$$\frac{\ln(n)}{n^{\frac{3}{2}}} < \frac{n}{n^{\frac{3}{2}}} = \frac{1}{n^{\frac{1}{2}}}$$

But this doesn't quite work! We have proven that our series is smaller than a **diverging** p-series. We need a *sharper* inequality.

For n sufficiently large it is also true that $\ln(n) < n^{\frac{1}{4}}$ say. Note that we have just made the $\frac{1}{4}$ up! We use any c that is sufficiently small to do the job. Then

$$\frac{\ln(n)}{n^{\frac{3}{2}}} < \frac{n^{\frac{1}{4}}}{n^{\frac{3}{2}}} = \frac{n^{\frac{1}{4}}}{n^{\frac{6}{4}}} = \frac{1}{n^{\frac{5}{4}}}$$

$$\frac{\ln(n)}{n^{3/2}} < \frac{1}{n^{5/4}}$$

Now $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{4}}}$ is a converging p-series with $p = \dfrac{5}{4} > 1$. Hence by the comparison test $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^{\frac{3}{2}}}$ also converges.

$\sum \frac{1}{n^{5/4}}$ is a convergig p-series $p = 5/4 > 1$

★

Test 5: The Ratio Test

Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ is an infinte series of positive terms and let $\rho = \displaystyle\lim_{n \longrightarrow \infty} \frac{a_{n+1}}{a_n}$. Then

If $\rho < 1$ the series converges.

If $\rho > 1$ the series diverges.

If $\rho = 1$ the test fails. You will need to use a different test.

Proof: See your printed notes.

Intuitively, the reason the ratio test works is that if $\rho < 1$, the terms are getting smaller and approaching zero quite quickly, guaranteeing convergence.

The ratio test is very handy for sequences involving factorials and powers.

Observe carefully that when using the ratio test we are analysing limit properties of the associated **sequence** in order to determine whether the **series** converges or diverges. This is similar to the nth term test for divergence.

Example 6

a) Show that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

b) To what value does $\sum_{n=0}^{\infty} \frac{1}{n!}$ converge?

a) $a_n = \frac{1}{n!}$ $\quad a_{n+1} = \frac{1}{(n+1)!}$

$$\rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}$$

$$= \lim_{n \to \infty} \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{1}{n+1} = 0$$

$\underline{\rho < 1}$ $\quad \therefore$ series converges

b) $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \longrightarrow e$

★ *b)* e ★

Example 7 Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2}{5^n}$ converges or diverges.

$a_n = \frac{n^2}{5^n}$, $a_{n+1} = \frac{(n+1)^2}{5^{n+1}}$

$$\rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)^2}{\cancel{5^{n+1}}} \times \frac{\cancel{5^n}}{n^2}$$

$$= \lim_{n \to \infty} \frac{1}{5} \frac{n^2 + 2n + 1}{n^2} = \frac{1}{5} \lim_{n \to \infty} \frac{n^2 + 2n + 1}{n^2}$$

$$= \frac{1}{5} \lim_{n \to \infty} \frac{1 + \cancel{\frac{2}{n}} + \cancel{\frac{1}{n^2}}}{1} = \frac{1}{5} \times 1 = \underline{\underline{\frac{1}{5} < 1}}$$

$\therefore$ By ratio test $\sum_{n=1}^{\infty} \frac{n^2}{5^n}$ $\underline{\underline{\text{converges.}}}$

★ Converges ★

Example 8 Determine whether the series $\sum_{k=1}^{\infty} \frac{3^k}{k}$ converges or diverges.

$a_k = \frac{3^k}{k}$, $a_{k+1} = \frac{3^{k+1}}{k+1}$

$\rho = \lim_{k \to \infty} \frac{3^{k+1}}{k+1} \Big/ \frac{3^k}{k} = \lim_{k \to \infty} \frac{\cancel{3^{k+1}}}{k+1} \times \frac{k}{\cancel{3^k}}$

$= \lim_{k \to \infty} 3 . \frac{k}{k+1} = 3 \times 1 = 3 > 1.$

$\therefore \sum_{k=1}^{\infty} \frac{3^k}{k}$ diverges by <u>ratio test</u>

★ Diverges ★

Example 9 Apply the ratio test to $\sum_{n=1}^{\infty} \frac{1}{n}$. Does this series converge or diverge?

$a_n = \frac{1}{n}$, $a_{n+1} = \frac{1}{n+1}$

$\rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{1}{(n+1)} \times \left(\frac{n}{1}\right)$

$= \lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{1}{1+\frac{1}{n}}$

$= \underline{\underline{1}}$

$\therefore$ Ratio <u>test fails</u>

★ Diverges ★

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 19

TELESCOPIC AND ALTERNATING SERIES

Milan Pahor

LECTURE 19 MATH1231 CALCULUS
TELESCOPIC SERIES AND ALTERNATING SERIES

Consider the alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$ where $a_n > 0$.

- If $\sum_{n=0}^{\infty} a_n$ converges (that is the associated positive series converges) then $\sum_{n=0}^{\infty}(-1)^n a_n$ also converges. This is called absolute convergence.
- If $\sum_{n=0}^{\infty} a_n$ diverges it is **still** possible for for the original series $\sum_{n=0}^{\infty}(-1)^n a_n$ to converge. This is called **conditional convergence** and is guaranteed when:

(i) $a_{n+1} \leq a_n$ for all n (that is the terms are getting smaller) and

(ii) $\lim_{n\longrightarrow\infty} a_n = 0$ (that is the terms are approaching zero)

TELESCOPIC SERIES

We very rarely have the ability to actually find the sum of a convergent series. The best we can do is prove that a sum exists. A particular and very special case however, where we can add all of the numbers up to infinity and actually find the sum is with telescopic series.

Example 1

a) Show that $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$ converges.

b) By implementing a partial fraction decomposition of $\frac{1}{n^2+n}$ find the sum of the finite Nth partial sum $S_n = \sum_{n=1}^{N}\frac{1}{n^2+n}$. (this is called a telescopic series).

c) Hence find the sum of the infinite series $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$.

a) $\frac{1}{n^2+n} \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a converging p-series $\underline{p=2>1}$

more in bottom.

$\therefore$ By comparison test $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$ <u>also</u> converges.

b) $\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1)+Bn}{n(n+1)}$

$\therefore \quad A(n+1)+Bn = 1$

$\underline{n=0}: \quad A = 1 \quad , \quad n=-1 \Rightarrow -B = 1 \Rightarrow B = -1$

$\therefore \quad \frac{1}{n^2+n} = \frac{1}{n} - \frac{1}{n+1} \quad //$

$\sum_{n=1}^{N} \frac{1}{n^2+n} = \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1}$

$S_N = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \cancel{\left(\frac{1}{2} - \frac{1}{3}\right)} + \cancel{\left(\frac{1}{3} - \frac{1}{4}\right)} + \cdots + \left(\cancel{\frac{1}{N}} - \frac{1}{N+1}\right)$

$S_N = 1 - \frac{1}{N+1}$

c) $S_\infty = \sum_{n=1}^{\infty} \frac{1}{n^2+n} = \lim_{N \to \infty} 1 - \frac{1}{N+1}$

$= 1.$

$\bigstar \quad b) \quad 1 - \frac{1}{N+1} \quad c) \quad 1 \quad \bigstar$

Telescopic series are very special and quite rare. They collapse in on themselves so that only the first term and the last term survive. This makes them easily summable.

ALTERNATING SERIES

All of our tests in the previous two lectures are valid only for infinite series of positive terms. We will now let negative numbers into the mix. To simplify the analysis we will assume that the terms alternate from being positive to being negative. These series are called alternating series. In the next lecture we will allow a more random mix of positives and negatives.

Definition: An alternating series takes the form $\sum_{n=0}^{\infty}(-1)^n a_n$ where $a_n > 0$.

Example 2 Write down the first 5 terms in the alternating series $\sum_{n=4}^{\infty}(-1)^n \frac{1}{n^2}$

$$\frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{8^2} \cdots\cdots$$

★

Observe how the signs strictly alternate.

Test 6: The Alternating Series Test (The Leibnitz Test)

Consider the alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$ where $a_n > 0$.

• If $\sum_{n=0}^{\infty} a_n$ converges (that is the associated positive series converges) then $\sum_{n=0}^{\infty}(-1)^n a_n$ also converges. This is called absolute convergence.

If we have absolute convergence we can stop. Nothing further needs to be done.

However. . .

• If $\sum_{n=0}^{\infty} a_n$ diverges it is **still** possible for for the original series $\sum_{n=0}^{\infty}(-1)^n a_n$ to converge. This is called **conditional convergence** and is guaranteed when:

(i) $a_{n+1} \leq a_n$ for all n (that is the terms are getting smaller) and

(ii) $\lim_{n\longrightarrow\infty} a_n = 0$ (that is the terms are approaching zero)

Proof: See printed notes.

Always check for absolute convergence first. If you don't get absolute convergence then go on to check for conditional convergence.

Example 3

Explore the convergence of the alternating series $\sum_{n=4}^{\infty}(-1)^n \frac{1}{n^2}$ in Example 1.

Consider $\sum_{n=4}^{\infty} \frac{1}{n^2}$ (the positive series)

This is a converging p-series $p = 2 > 1$

$\therefore$ By Leibnitz test $\sum_{n=4}^{\infty} (-1)^n \frac{1}{n^2}$

converges absolutely

★ Converges Absolutely ★

Example 4 Discuss the convergence of the alternating series $\sum_{n=2}^{\infty}(-1)^n \frac{n}{n^2-1}$.

(Hint: You may need to use a little calculus to prove condition (i) of the Leibnitz test).

First consider

$\sum_{n=2}^{\infty} \frac{n}{n^2-1}$

Now $\frac{n}{n^2-1} > \frac{n}{n^2} = \frac{1}{n}$

and $\sum_{n=2}^{\infty} \frac{1}{n}$ is a

diverging harmonic series

$\therefore$ No absolute convergence

Conditional Convergence ??

Claim $\frac{n}{n^2-1}$ is decreasing.

Pf Consider $y = \frac{x}{x^2-1}$

$$y' = \frac{(x^2-1)(1) - x(2x)}{(x^2-1)^2}$$

$$y' = \frac{x^2-1-2x^2}{(x^2-1)^2}$$

$$= \frac{-1-x^2}{(x^2-1)^2} < 0$$

$x > 2$.

$\therefore$ y decreasing function

$\therefore$ $a_{n+1} < a_n$.

Also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2-1} = 0$

$\therefore$ $\sum_{n=2}^{\infty} (-1)^n \frac{n}{n^2-1}$

converges conditionally

★ Converges Conditionally ★

Example 5 Consider the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots$$

a) Prove that the series converges conditionally.

I) Consider $\sum_{n=1}^{\infty} \frac{1}{n}$. This is a divergent p-series (Harmonic series)

II) Let $a_n = \frac{1}{n}$. ie series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$.

i) $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$.

(ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = \underline{\underline{0}}$

$\therefore \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges conditionally

★

b) We showed earlier in the course that the Maclaurin series for $\ln(1 + x)$ is

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots$$

Use this series to prove that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots = \ln(2).$$

$\underline{\underline{x=1}} \Rightarrow \ln(2) = 1 - \frac{1^2}{2} + \frac{1^3}{3} - \frac{1^4}{4} \dots$

$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

★

c) Suppose that the infinite series

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots$$

is terminated at the 8th partial sum

$$S_8 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}.$$

Find an error bound for the absolute difference between S_8 and $\ln(2)$.

Just as in Maclaurin series the error when terminating a convergent alternating series is bounded by the absolute value of the next term following the point of termination.

$$\therefore \quad \text{error} \leq \frac{1}{9}.$$

$$\star \quad \frac{1}{9} \quad \star$$

d) How far does the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots$$

need to be taken in order for the difference between the series and $\ln(2)$ to be less than or equal to 0.001?

$$\frac{1}{n+1} \leq \frac{1}{1000}$$

$$\Rightarrow n+1 \geq 1000$$

$$\Rightarrow n \geq 999$$

$$\star \quad n = 999 \quad \star$$

Using Maple we have $\ln(2) \approx 0.6931471$ and $\displaystyle\sum_{n=1}^{999}(-1)^{n+1}\frac{1}{n} \approx 0.6936474$ which agree to the third decimal place.

And now for some very disturbing observations.

Let

$$U = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots$$

Observe that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots = \frac{1}{2}(1 + \frac{1}{2} + \frac{1}{3} + \dots)$$

The series in the brackets is the diverging harmonic series. Thus

$$U = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots$$

is unbounded. It can be made as large as we wish.

Similarly let

$$V = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$$

Observe that

$$(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots) > (\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots) = \frac{1}{2}(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots)$$

The series in the brackets is again the diverging harmonic series. Thus

$$V = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$$

is unbounded. It can also be made as large as we wish.

Recall that the conditionally convergent alternating series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots$$

sums to $\ln(2)$ which is approximately 0.69.

Example 6 Add up all the terms of

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots$$

in a different order to get a sum of 127 instead of ln(2).

Please be clear about what is about to happen. We are going to add up exactly the same numbers to get a different answer.

$\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots\dots\right)$ until $> \underline{127}$.

$\left(-\frac{1}{2} - \frac{1}{4} - \frac{1}{8} \dots\dots\right)$ until < 127

(select from remainder of $\underline{V}$) until > 127.

(select from remainder of V) until < 127.

$> 127 \quad < 127 \quad > 127 \quad < 127 \quad > 127$ ....

final sum $\longrightarrow 127$.

★

What is special about 127? Nothing! The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots$$

can be rearranged so as to add to up **any** real number, positive or negative or even zero!

Fact: The terms of any conditionally convergent alternating series may be rearranged so that the sum converges to **any** real number. The terms may also be rearranged so that the series becomes divergent.

Absolute convergence is very powerful and nothing strange happens for absolutely convergent alternating series. But conditional convergence of alternating series is so weak that simply rearranging the order in which you add the numbers has the potential to effect your answer!

By rearranging the terms, a conditionally convergent alternating series can be manipulated to add up to whatever you want.

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 20

POWER SERIES

Milan Pahor

LECTURE 20 MATH1231 CALCULUS
POWER SERIES

If the series $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges then the original series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutlely.

A series of the form $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ is called a power series in powers of x.

A series of the form $\displaystyle\sum_{n=0}^{\infty} a_n (x-\beta)^n$ is called a power series in powers of $(x-\beta)$.

To find an open interval upon which a power series converges we simply apply the ratio test to the absolute series. That is, we force

$$\lim_{n\longrightarrow\infty} \left| \frac{a_{n+1}(x-\beta)^{n+1}}{a_n(x-\beta)^n} \right| < 1.$$

The radius of convergence is simply half the length of the interval of convergence.

ARBITRARY SERIES

We have been very careful with negatives in series up until this point, as the presence of negatives tends to make a mess of the summation process. We started by only considering series of positive terms and then made the concession to allow alternating series. We will now allow the terms to be of any sign at any time. An immediate consequence is that many of our previous tests no longer apply.

Definition: An infinite series of arbitrary real numbers $\displaystyle\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if the infinite positive series $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges.

Fact: If a series is absolutely convergent then it converges.

Proof: See your printed notes.

What this means is that if you have a series which is all over place in sign, just consider the associated positive series of absolute values. If this converges then the original series converges (absolutely).

If the associated positive series diverges then we could still have conditional convergence though we have no real test for this in general. For example the bizarre series we constructed at the end of the last lecture using the terms of the alternating Harmonic series to converge to 127 is not absolutely convergent but **is** conditionally convergent.

Example 1 Examine the convergence of the infinite series

$$1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\frac{1}{49}+\frac{1}{64}-\frac{1}{81}+\frac{1}{100}+\dots$$

Every third term is negative here so this is NOT an alternating series. However taking absolute values, the associated positive series is

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\frac{1}{49}+\frac{1}{64}+\frac{1}{81}+\frac{1}{100}+\dots$$

which is just $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ a well known converging p-series. Thus the original series converges absolutely and hence converges.

★

POWER SERIES

Definition:

A series of the form $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ is called a power series in powers of x.

A series of the form $\displaystyle\sum_{n=0}^{\infty} a_n (x-\beta)^n$ is called a power series in powers of $(x-\beta)$.

Our final topic on series and sequences is power series. These are series where there is a variable in the series. You have seen these before with Maclaurin and Taylor series. For example:

$$1+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$$

and

$$-1+\frac{1}{2!}(x-\pi)^2-\frac{1}{4!}(x-\pi)^4+\frac{1}{6!}(x-\pi)^6-\dots$$

The first power series above would be called a power series in powers of x while the second is a power series in powers of $(x-\pi)$.

Power series are similar to, but a little more general that Taylor or Maclaurin series since the coefficients are not restricted to being derivatives evaluated at a point. They behave in much the same manner.

Please note carefully that we have already covered the entire Maclaurin and Taylor series section of the course in Lectures 14 and 15. Nothing further will be done here.

The problem with power series is that we may have convergence for some values of x and divergence for other values of x. So the questions we see are not "Does it converge"? but rather "for which values of x does it converge"?

You have already seen this with your high school G.P.'s where

$$1 + x + x^2 + x^3 + \ldots$$

converges to $\dfrac{1}{1-x}$ if $-1 < x < 1$ BUT diverges if $x \geq 1$ or $x \leq -1$.

To find an open interval upon which a power series converges we simply apply the ratio test to the absolute series. That is, we force

$$\lim_{n \longrightarrow \infty} \left| \frac{a_{n+1}(x-\beta)^{n+1}}{a_n(x-\beta)^n} \right| < 1$$

The radius of convergence is simply half the length of the interval of convergence.

Example 2 Determine an open interval I of convergence and the radius R of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n+5}$$

Observe that the series converges trivially at $x = 2$ since then we are just adding zeros. The question is how far from $x = 2$ can we wander before the trouble starts?

$\lim_{n\to\infty} \left|\frac{T_{n+1}}{T_n}\right| < 1$ for convergence

Let $T_n = \dfrac{(x-2)^n}{n+5}$ and $T_{n+1} = \dfrac{(x-2)^{n+1}}{n+6}$. Then

$$\lim_{n \longrightarrow \infty} \left| \frac{T_{n+1}}{T_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{n+6}}{\frac{(x-2)^n}{n+5}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{n+6} \times \frac{n+5}{(x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| (x-2) \frac{n+5}{n+6} \right| = |x-2| \lim_{n\to\infty} \left(\frac{n+5}{n+6} \right) < 1$$

$= 1$

$\Rightarrow |x-2| < 1.$

$x - 2 < 1$ OR $x - 2 > -1$

$x < 3$ OR $x > 1$

$1 < x < 3$

$\bigstar$ $I = (1, 3)$ $R = 1$ $\bigstar$

This means that the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n+5}$ only converges absolutely for $1 < x < 3$.

Example 3 Determine an open interval I of convergence and the radius R of convergence for the power series

$$\sum_{k=0}^{\infty} \frac{(-5)^k (x-3)^k}{k^2+1}$$

Once again we clearly have convergence at the point $x = 3$ about which the power series is based. But how far from $x = 3$ can we go before divergence kicks in?

$$T_k = \frac{(-5)^k (x-3)^k}{k^2+1}, \quad T_{k+1} = \frac{(-5)^{k+1}(x-3)^{k+1}}{(k+1)^2+1} = \frac{(-5)^{k+1}(x-3)^{k+1}}{k^2+2k+2}$$

$$\lim_{k\to\infty} \left| \frac{T_{k+1}}{T_k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{(-5)^{k+1}(x-3)^{k+1}}{k^2+2k+2} \times \frac{k^2+1}{(-5)^k (x-3)^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{-5(x-3)(k^2+1)}{k^2+2k+2} \right|$$

$$= 5|x-3| \left(\lim_{k\to\infty} \frac{k^2+1}{k^2+2k+2} \right) \qquad = 1.$$

$$\therefore \quad 5|x-3| \times (1) < 1 \implies |x-3| < \frac{1}{5}.$$

$$x-3 < \frac{1}{5} \qquad x-3 > -\frac{1}{5}$$

$$x < 16/5 \qquad x > 14/5$$

$$\therefore \frac{14}{5} < x < 16/5$$

$$\bigstar \quad I = \left(\frac{14}{5}, \frac{16}{5}\right) \qquad R = \frac{1}{5} \quad \bigstar$$

This means that the power series $\displaystyle\sum_{k=0}^{\infty} \frac{(-5)^k (x-3)^k}{k^2+1}$ only converges absolutely for $\frac{14}{5} < x < \frac{16}{5}$.

Remember that all Taylor and Maclaurin series are power series, so their convergence may also be analysed using the above methods.

Example 4 Determine the interval I of convergence and the radius R of convergence for the Maclaurin series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$T_k = \frac{x^k}{k!}, \quad T_{k+1} = \frac{x^{k+1}}{(k+1)!}$$

$$\lim_{k\to\infty}\left|\frac{T_{k+1}}{T_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{(k+1)!}\cdot\frac{k!}{x^k}\right|$$

$$\Rightarrow |x| \underbrace{\lim_{k\to\infty}\frac{1}{k+1}}_{=0.} < 1$$

$$\Rightarrow |x| \times 0 < 1$$

True for <u>all</u> x.

Converges absolutely for <u>all</u> x.

$\bigstar$ $I = \mathbb{R}$ $\quad R = \infty$ $\bigstar$

The Maclaurin series for e^x converges for all real x and hence the radius of convergence is infinite.

This concludes the theory of series and sequences. This last lecture has been left a little light to give you the opportunity to ask any final questions regarding the topic. Many of our results have been highly counter-intuitive and sometimes almost paradoxical.

Feel free to come down and have a chat if you have any questions or concerns from lectures or tutorials.

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 21

PARAMETRICALLY DEFINED CURVES, VELOCITY AND AVERAGES

Milan Pahor

LECTURE 21 MATH1231 CALCULUS
PARAMETRICALLY DEFINED CURVES AND AVERAGES

For a parameterised curve $x = x(t)$ and $y = y(t)$ we have

$$\mathbf{d} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \qquad \text{speed } = |\mathbf{v}| = \sqrt{(\dot{x})^2 + (\dot{y})^2} \qquad \mathbf{a} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$$

$$\frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)}$$

The average value $\bar{f}$ of $y = f(x)$ over an interval $[a, b]$ is given by

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Parametric Curves

In most cases curves are defined in terms of a Cartesian equation. Thus for example $y = x^2 + 1$ is a parabola in $\mathbb{R}^2$, while $x^2 + y^2 = 25$ is a circle. Sometimes however it is preferable to define a curve by expressing the x and y variables in terms of a third party t, called a parameter. You have already seen this approach in Math1131 where lines in space were defined in parametric vector form.

We will see in this lecture that parametrically defined curves enjoy several advantages over their Cartesian counterparts.

Example 1 Suppose that a curve C in $\mathbb{R}^2$ is defined as:

$$\begin{cases} x = 3t + 7 \\ y = t^2 \end{cases}$$

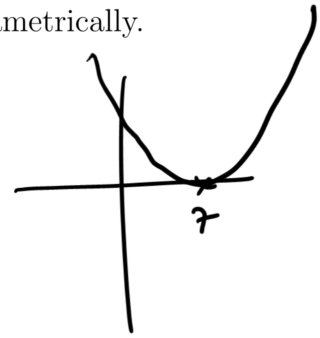
a) Find the Cartesian equation of C.

b) Which point (x, y) on C corresponds to $t = 2$?

c) Which value of the parameter t corresponds to the point $(4, 1)$ on C.

d) Find the gradient of the curve at the point in b) by differentiating parametrically.

e) Check your answer in d) by using standard calculus.



a) $3t = x - 7 \Rightarrow t = \frac{1}{3}(x-7)$

$y = \left[\frac{1}{3}(x-7)\right]^2 = \frac{1}{9}(x-7)^2$

b) $x = 3(2) + 7 = 13$ $\Rightarrow$ $(13, 4) \Longleftrightarrow t = 2$

$y = 2^2 = 4$ $\quad (x, y)$

c) $x = 3t + 7 = 4 \Rightarrow 3t = -3 \Rightarrow \underline{t = -1}$

$y = t^2 = 1 \Rightarrow t = \pm 1$ $\quad \cancel{t = +1}$ ~ doesn't work for $\underline{x}$.

d) $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \boxed{\frac{2t}{3}}$

$\underline{t = 2}$: $\frac{dy}{dx} = \frac{2(2)}{3} = \frac{4}{3}$

e) $y = \frac{1}{9}(x-7)^2 \Rightarrow \frac{dy}{dx} = \frac{2}{9}(x-7)^1(1)$

at $x = 13$: $\frac{2}{9}(13-7) = \frac{2}{9}(6) = \frac{12}{9} = \frac{4}{3}$

★ $\quad a) \quad y = \frac{(x-7)^2}{9} \quad b) \quad (13, 4) \quad c) \quad t = -1 \quad d) \quad \frac{4}{3}$ ★

SPEED

A really nice feature of using parametrically defined curves is that we can examine the **motion** of a particle travelling along the curve by interpreting the parameter t as time. It is then a simple task to find the position vector $\mathbf{d}$ and the velocity vector $\mathbf{v}$ for the motion, the speed $|\mathbf{v}|$ of the particle as well as the acceleration vector $\mathbf{a}$. This will put to good use all of your vector theory from Math1131 algebra.

Please note that a dot over a variable **always** denotes a time derivative. This notation goes all the way back to Newton.

So $\dot{y}$ is most definitely $\dfrac{dy}{dt}$ and **NOT** $\dfrac{dy}{dx}$.

For a parametric curve $x = x(t)$ and $y = y(t)$ we have

$$\mathbf{d} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \qquad \text{speed } = |\mathbf{v}| = \sqrt{(\dot{x})^2 + (\dot{y})^2} \qquad \mathbf{a} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$$

Example 2 Suppose that the motion of a particle on a curve C in $\mathbb{R}^2$ is defined as:

$$\begin{cases} x = t^2 + 7 \\ y = 5t^3 \end{cases}$$

At time $t = 1$ second find:

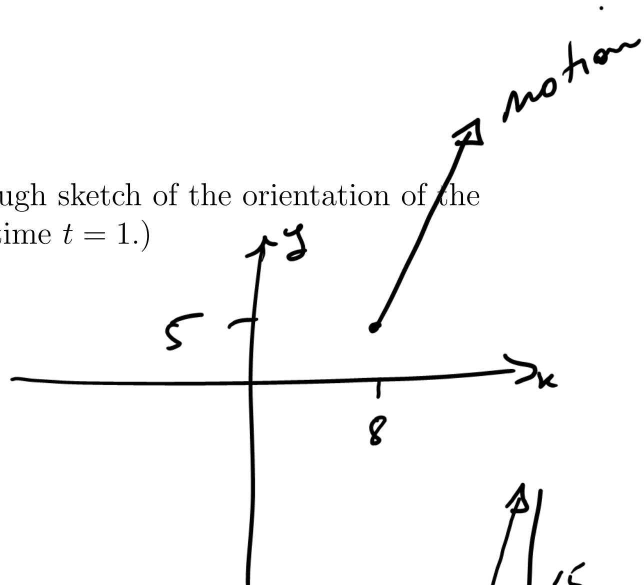
a) The position vector of the particle.

b) The velocity vector of the particle. (Draw a rough sketch of the orientation of the curve, the position vector and the velocity vector at time $t = 1$.)

c) The speed of the particle.

d) The acceleration vector.

e) The magnitude of the acceleration.



a) $\underset{\sim}{d} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$

b) $\underset{\sim}{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2t \\ 15t^2 \end{pmatrix} = \begin{pmatrix} 2 \\ 15 \end{pmatrix}$ @ $t = 1$

c) speed $= |\underset{\sim}{v}| = \sqrt{2^2 + 15^2} = \sqrt{229}$

d) $\underset{\sim}{a} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 2 \\ 30t \end{pmatrix} = \begin{pmatrix} 2 \\ 30 \end{pmatrix}$ @ $t=1$.

$|\underset{\sim}{a}| = \sqrt{30^2 + 2^2} = \sqrt{904}$

★ *a)* $\mathbf{d} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$ *b)* $\mathbf{v} = \begin{pmatrix} 2 \\ 15 \end{pmatrix}$ *c)* speed $= \sqrt{229}$ *d)* $\mathbf{a} = \begin{pmatrix} 2 \\ 30 \end{pmatrix}$ *e)* $|\mathbf{a}| = \sqrt{904}$ ★

Example 3 Suppose that the motion of a particle on a curve C_1 in $\mathbb{R}^2$ is defined as:

$$\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$$

a) Find the Cartesian equation of C_1.

$x^2 + y^2 = \cos^2 t + \sin^2(t) = 1$

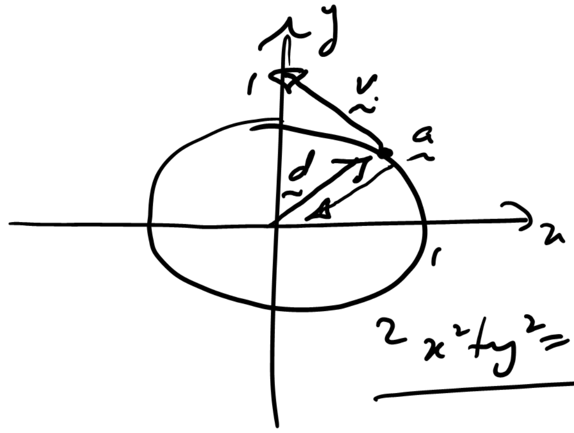
b) Find the position vector $\mathbf{d}$ of the particle.

c) Find the velocity vector $\mathbf{v}$ of the particle.

d) Find the acceleration vector $\mathbf{a}$ of the particle.

e) Show that $\mathbf{v} \perp \mathbf{d}$. Comment on this result.

f) Show that $\mathbf{a} = -\mathbf{d}$. Comment on this result.



$\therefore \; x^2 + y^2 = 1$

b) $\underset{\sim}{d} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

c) $\underset{\sim}{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$

d) $\underset{\sim}{a} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix}$

e) Claim $\underset{\sim}{v} \perp \underset{\sim}{d} \Leftrightarrow \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$= -\sin t \cos t + \cos t \sin t = 0$

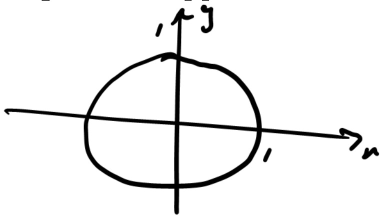
$\therefore \underset{\sim}{v} \perp \underset{\sim}{d}$. (in a circle tangent is $\perp$ radius)

f) $\underset{\sim}{a} = \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix} = -\underset{\sim}{d}$

$\therefore$ a particle moving in uniform circular motion accelerates to the centre.

What we see above, is that for uniform circular motion, the velocity vector is always tangential to the circle and the acceleration vector points back in towards the centre. That is the particle accelerates towards the centre!

Example 4 Suppose now that the motion of a particle on a curve C_2 in $\mathbb{R}^2$ is defined as:



$$\begin{cases} x = \cos(3t) \\ y = \sin(3t) \end{cases}$$

$x^2+y^2 = \cos^2(3t)+\sin^2(3t)$
$= 1$!!!

same curve

$\underset{\sim}{v} = \begin{pmatrix} -3\sin 3t \\ 3\cos 3t \end{pmatrix}$

How is this motion different from that of the previous example?

Same curve
But motion is 3x as fast

★

Example 5 Suppose that the motion of a particle on a curve C_3 in $\mathbb{R}^3$ is defined as:

$$\begin{cases} x = \cos(t) \\ y = \sin(t) \\ z = t \end{cases}$$

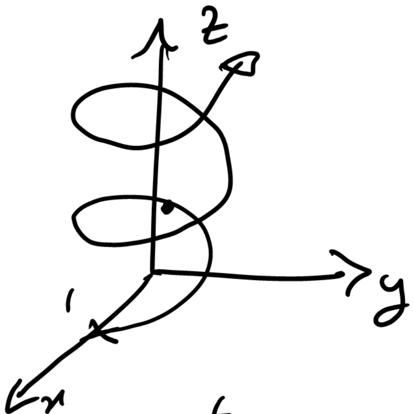
$t=0$

a) Where is the particle initially? $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

b) Where is the particle when $t = 2\pi$ seconds. $= \begin{pmatrix} 1 \\ 0 \\ 2\pi \end{pmatrix}$

c) Describe the path C_3 in space.

d) Find the speed of the particle after π seconds.



Helix

$\underset{\sim}{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

This is $\underset{\sim}{v}$ not $\underset{\sim}{d}$ or $\underset{\sim}{a}$.

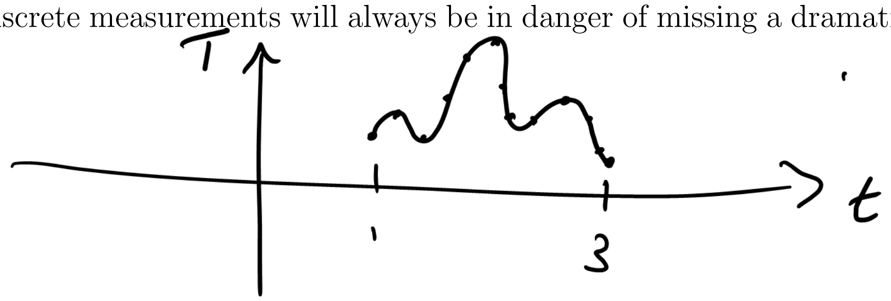
speed $= |\underset{\sim}{v}| = \sqrt{0^2+(-1)^2+(1)^2}$
$= \sqrt{2}$

★ *d)* $\sqrt{2}$ ★

Observe how easily we have analysed motion along a strange curve in space in Example 5. Parametric representations are ideal for this job. This is why we defined lines in space in Math1131 in parametric vector form.

Averages

If a quantity is varying continuously, the calculation of an average over an interval is tricky since discrete measurements will always be in danger of missing a dramatic change.



By using integration we can measure the quantity at **every** point!

FACT: The average value $\bar{f}$ of $y = f(x)$ over an interval $[a, b]$ is given by

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Proof: See your printed notes.

Example 6 Suppose that the temperature T (in °C) varies over 5 hours according to

$$T = t^2 + 15 \qquad 0 \le t \le 5.$$

a) Find the temperature at $t = 0, 1, 2, 3, 4, 5$ and the average of these temperatures?

b) What is the average temperature over the **entire** five hours?

c) Sketch both T and its average on the same set of axes.

---

a) $T(0) = 15, \quad T(1) = 16, \quad T(2) = 19, \quad T(3) = 24, \quad T(4) = 31, \quad T(5) = 40.$

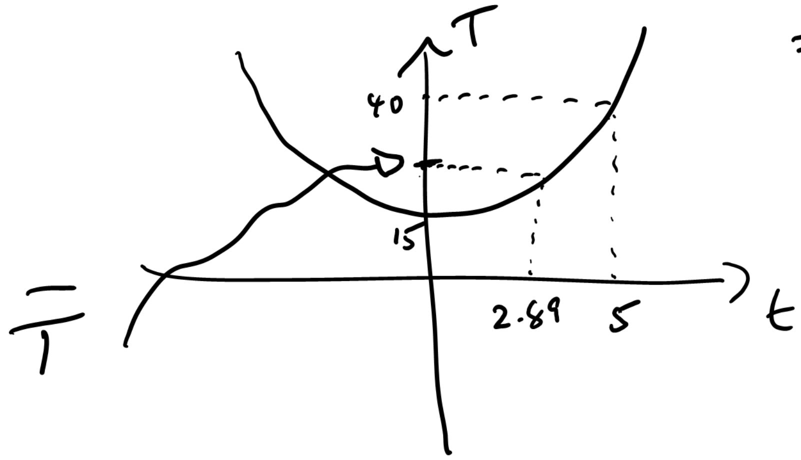
So the discrete average is $\dfrac{15+16+19+24+31+40}{6} = \dfrac{145}{6} \approx 24.17.$

Now the **true average** is

$$\overline{T} = \frac{1}{b-a}\int_a^b T(t)dt = \frac{1}{5-0}\int_0^5 t^2+15\,dt$$

$$\overline{T} = \frac{1}{5}\left[t^3/3 + 15t \right]_0^5 = \frac{1}{5}\left\{ \frac{5^3}{3} + 15 \times 5 \right\}$$

$$= \frac{70}{3} = 23.\dot{3}$$



★ *a)* $\dfrac{127}{6} \approx 24.1\dot{6}$ *b)* $\dfrac{70}{3} \approx 23.\dot{3}$ *c) Sketch* ★

The Mean Value theorem for Integrals:

Suppose that f is a continuous function on $[a, b]$. Then there exists $c \in (a, b)$ such that $\bar{f} = f(c)$.

In other words a continuous function over a closed interval always takes on its average value at some point in the interval.

Note differentiability of f is NOT required here, as there are no tangents involved, just areas.

Proof: See your printed notes.

Example 7 Verify the "mean value theorem for integrals" for the previous example.

$$T = \frac{70}{3} \qquad 0 \le t \le 5$$

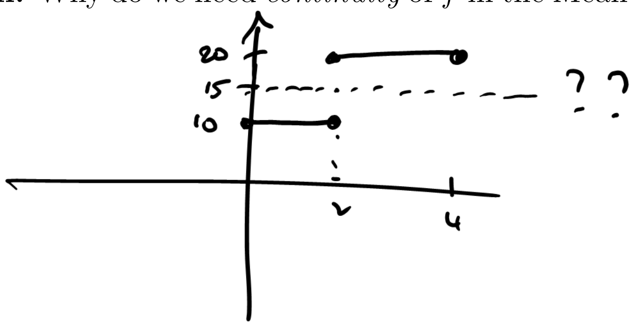
$$t^2 + 15 = \frac{70}{3}$$

$$t^2 = \frac{70}{3} - \frac{45}{3} = \frac{25}{3}$$

$$\therefore \quad t = \frac{5}{\sqrt{3}} \simeq \underline{\underline{2.89}}.$$

$$\bigstar \qquad c = \frac{5}{\sqrt{3}} \approx 2.89 \qquad \bigstar$$

Discussion: Why do we need *continuity* of f in the Mean Value theorem for integrals?



 $\bigstar$

School of Mathematics and Statistics

Math1231 Mathematics 1B

CALCULUS LECTURE 22

ARC LENGTH AND SURFACE AREA

Milan Pahor

LECTURE 22 MATH1231 CALCULUS
ARC LENGTH AND SURFACE AREA

The arc Length of a curve C is given by:

$$L_1 = \int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

OR

$$L_2 = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

OR

$$L_3 = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

depending upon whether the curve has been presented parametrically $[L_1]$, as a function $y = f(x)$ $[L_2]$ or in polar form $[L_3]$.

Assume that a curve C sitting above the x axis is rotated about the x axis to produce a solid of revolution. Then the surface area S of this solid is given by

$$S_1 = \int_{t_a}^{t_b} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

OR

$$S_2 = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

OR

$$S_3 = \int_{\theta_0}^{\theta_1} 2\pi r \sin(\theta) \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

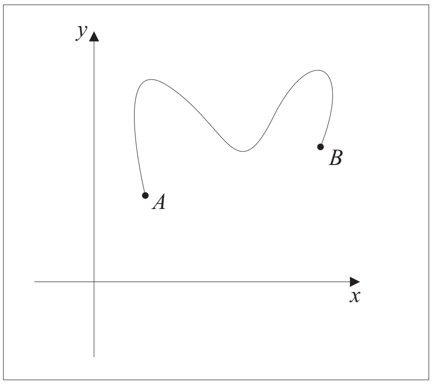
depending upon whether the curve has been presented parametrically $[S_1]$, as a function $y = f(x)$ $[S_2]$ or in polar form $[S_3]$.

In high school you used integrals to find volumes and areas and in the previous lecture, the integral was used to find the average value of a function. We close the course with a few other applications of the integration process. Note that all proofs may be found in your printed notes.

Arc Length

We now have three very different ways of presenting a curve, Cartesians, Parametrics or Polars. We close the lecture by establishing arc length formulae for each of the three forms.

Consider a curve C in the $x-y$ plane and imagine walking along C from A to B.



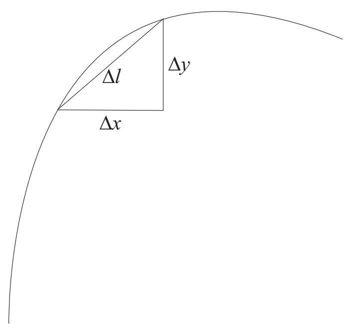
The arc length of the path is quite simply the distance walked. The path is not straight however (indeed it could be completely wild) so arc length is not an easy thing to calculate!

Fact: For a curve define parametrically by $x = x(t)$ and $y = y(t)$ the arc length l from $t = t_a$ to $t = t_b$ is given by

$$l = \int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Proof:

$$\Delta l^2 = \Delta x^2 + \Delta y^2 = \frac{\Delta x^2 + \Delta y^2}{\Delta t^2}\Delta t^2 = \left(\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2\right)\Delta t^2.$$



So

$$\Delta l = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\ \Delta t$$

Summing and letting $\Delta t \to 0$ yields

$$l = \int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

★

Example 1 Suppose that a curve C_1 in $\mathbb{R}^2$ is defined parametrically as:

$$\begin{cases} x = 1 + 3t^2 & \Rightarrow \dfrac{dx}{dt} = 6t \\ y = 4 + 2t^3 & \Rightarrow \dfrac{dx}{dt} = 6t^2 \end{cases}$$

Find the arc length along C_1 from the point $(1, 4)$ to the point $(4, 6)$

$\begin{pmatrix}1\\4\end{pmatrix}$: $4 = 4 + 2t^3$

$0 = 2t^3$

$t = 0$

$\begin{pmatrix}4\\6\end{pmatrix}$: $6 = 4 + 2t^3$

$2 = 2t^3$

$t^3 = 1 \Rightarrow t = 1$

Let $u = 1 + t^2$. Then $du = 2t\,dt$ $\Rightarrow 3du = 6t\,dt$

$t = 0 \Rightarrow u = 1$

$t = 1 \Rightarrow u = 2$

$$l = \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\,dt$$

$$= \int_0^1 \sqrt{36t^2 + 36t^4}\,dt$$

$$= \int_0^1 \sqrt{36t^2(1+t^2)}\,dt$$

$$= \int_0^1 6t\sqrt{1+t^2}\,dt$$ group

$$= \int_1^2 \sqrt{u}\; 3\,du$$

$$= 3\int_1^2 u^{\frac{1}{2}}\,du = 3\left[\frac{u^{3/2}}{3/2}\right]_1^2$$

$$= 2\{2^{3/2} - 1\}$$

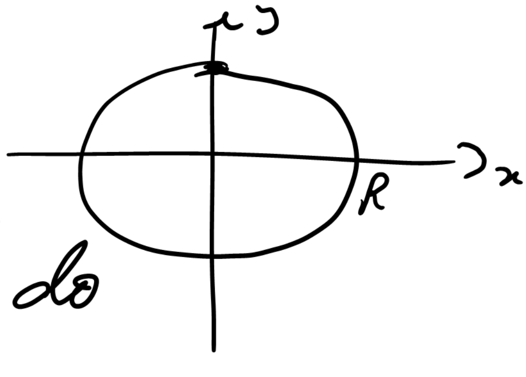
$3 \times \frac{2}{3} = 2$

★ $2(2^{\frac{3}{2}} - 1)$ ★

Example 2 By describing the circle in parametric form prove that the circumference C of a circle or radius R is given by $C = 2\pi R$.

$x = R\cos\theta$

$y = R\sin\theta$: $\theta : 0 \to 2\pi$



$$L = \int_0^{2\pi} \sqrt{(-R\sin\theta)^2 + (R\cos\theta)^2}\,d\theta$$

$$= \int_0^{2\pi} \sqrt{R^2\{\sin^2\theta + \cos^2\theta\}}\,d\theta$$

$$= \int_0^{2\pi} \sqrt{R^2} = \int_0^{2\pi} R = [R\theta]_0^{2\pi} = \underline{\underline{2\pi R}}$$ (nice!)

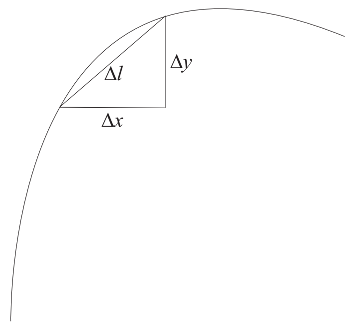
★

Arc Length in Cartesian Form

Fact: For a curve $y = f(x)$ the arc length l from $x = a$ to $x = b$ is given by

$$l = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Proof:



$$\Delta l^2 = \Delta x^2 + \Delta y^2 = \left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right) \Delta x^2$$

So

$$\Delta l = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

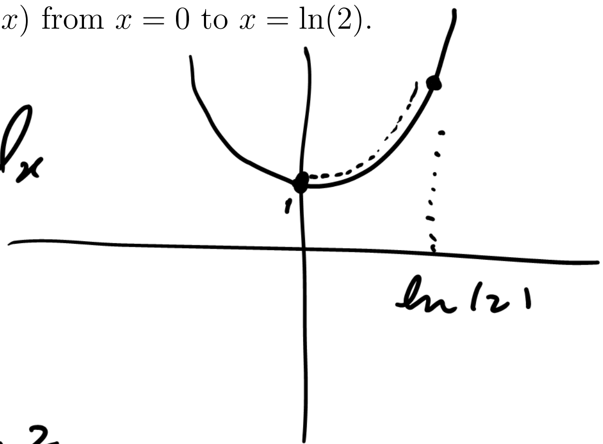
Summing and letting $\Delta x \to 0$ we have

$$l = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

★

Although the above two formulae for arc length look handy, the sad truth is that most of the time the integral is impossible to calculate. Sometimes you get lucky.

Example 3 Find the exact arc length of $y = \cosh(x)$ from $x = 0$ to $x = \ln(2)$.



$$l = \int_0^{\ln(2)} \sqrt{1 + (\sinh x)^2}\ dx$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh^2 x = 1 + \sinh^2 x.$$

$$= \int_0^{\ln 2} \sqrt{\cosh^2(x)}\ dx = \int_0^{\ln 2} \cosh(x)\,dx.$$

$$= \left[\sinh(x)\right]_0^{\ln(2)} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$= \sinh(\ln(2))$$

$$= \frac{e^{\ln 2} - e^{-\ln(2)}}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{4-1}{4} = \underline{\underline{3/4}}$$

★ $\frac{3}{4}$ ★

Note $e^{-\ln(2)} = e^{\ln 2^{-1}} = 2^{-1} = \frac{1}{2}.$

Arc Length for a Polar Curve

Fact: Suppose that a curve is described using polar coordinates by

$$r = f(\theta) \quad \text{where} \quad \theta_0 \le \theta \le \theta_1.$$

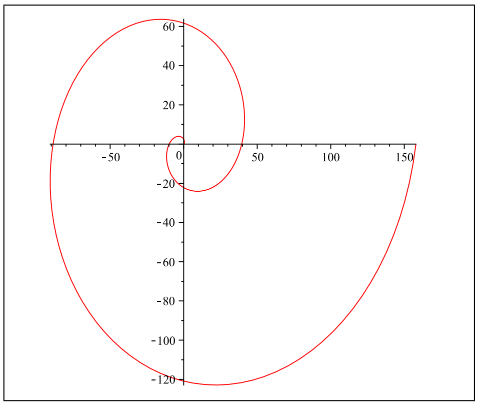
Then the arc length l is given by the integral

$$l = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$r = \theta^2$$
$$\frac{dr}{d\theta} = 2\theta.$$

Proof: See your printed notes.

Example 4 Calculate the length of $r = \theta^2$ for $0 \le \theta \le 4\pi$.



$$L = \int_0^{4\pi} \sqrt{\theta^4 + (2\theta)^2}\, d\theta$$
$$= \int_0^{4\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$$
$$= \int_0^{4\pi} \sqrt{\theta^2(\theta^2+4)}\, d\theta$$
$$= \int_0^{4\pi} \theta\sqrt{\theta^2 + 4}\, d\theta$$

Let $u = \theta^2 + 4$

$du = 2\theta\, d\theta.$

$\theta = 0 \to u = 4$

$\theta = 4\pi \to u = 16\pi^2 + 4$

$$\int_4^{4+16\pi^2} \sqrt{u}\, \frac{du}{2}$$
$$\frac{1}{2}\int_4^{4+16\pi^2} u^{\frac{1}{2}}\, du$$
$$= \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_4^{4+16\pi^2}$$
$$= \frac{1}{3}\left[u^{3/2}\right]_4^{4+16\pi^2}$$
$$= \frac{1}{3}\left\{\left((4+16\pi^2)^{\frac{3}{2}}\right) - \left(4^{\frac{3}{2}}\right)\right\}$$
$$= \frac{1}{3}\left\{4^{3/2}(1+4\pi^2)^{3/2} - 4^{3/2}\right\}$$
$$= \frac{1}{3}\left\{8(1+4\pi^2)^{3/2} - 8\right\}$$

$$\star \quad \frac{8}{3}\left((4\pi^2+1)^{\frac{3}{2}} - 1\right) \quad \star$$

Surface Areas

In high school you found volumes of solids of revolution using integration. We will now find the **surface area** of solids of revolution once again by implementing an appropriate integral.

Assume that a curve C sitting above the x axis is rotated about the x axis to produce a solid of revolution. Then the curved surface area S of this solid (without the end-caps) is given by

$$S_1 = \int_{t_a}^{t_b} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

OR

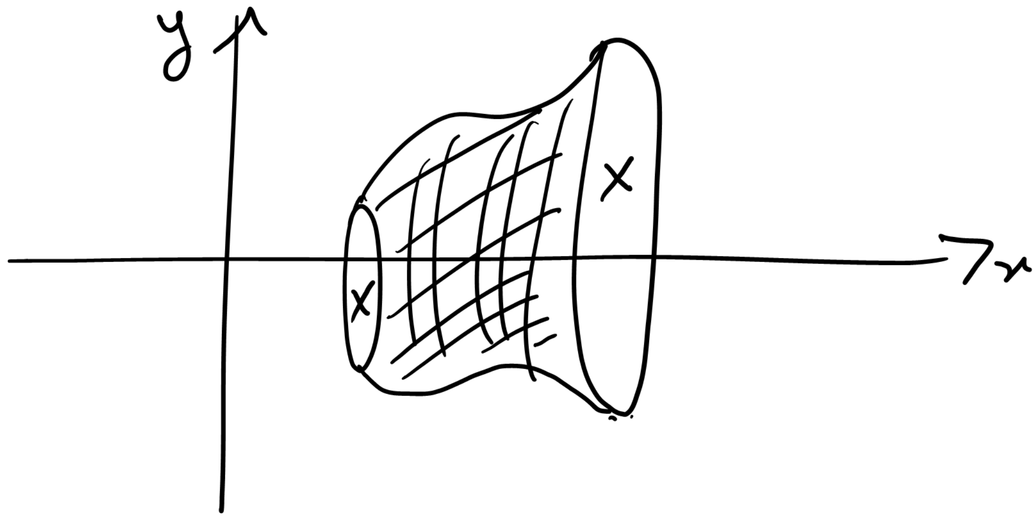
$$S_2 = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

OR

$$S_3 = \int_{\theta_0}^{\theta_1} 2\pi r \sin(\theta) \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

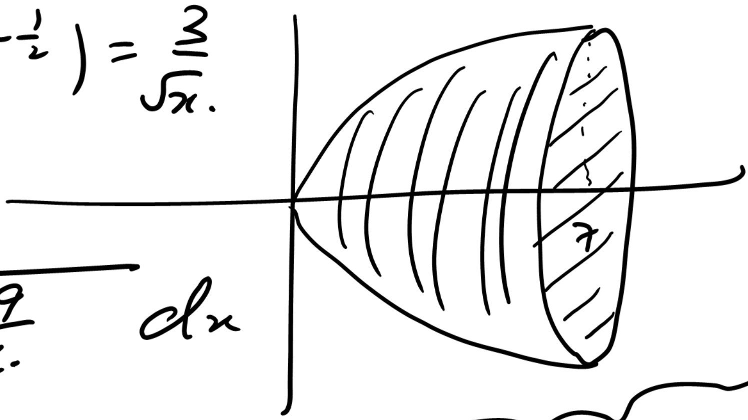
depending upon whether the curve has been presented parametrically [S_1], as a function $y = f(x)$ [S_2] or in polar form [S_3].

Observe in all three cases that the arc length formula has simply been modified by a "circumference" term $2\pi y$. This makes the surface area equations easy to remember once you have the arc length formulae committed to memory.



Example 5 Determine the **total** surface area of revolution (including any end caps) obtained when the curve $y = 6\sqrt{x}\ ;\ 0 \le x \le 7$ is rotated about the x axis.

$$y = 6x^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = 6\left(\tfrac{1}{2}x^{-\frac{1}{2}}\right) = \frac{3}{\sqrt{x}}.$$



$$S = \int_0^7 2\pi\left(6x^{\frac{1}{2}}\right)\sqrt{1 + \frac{9}{x}}\ dx$$

$$= 12\pi \int_0^7 \sqrt{x}\ \sqrt{1 + 9/x}\ dx.$$

$$\int (ax+b)^n = \frac{(ax+b)^{n+1}}{(n+1)(a)}$$

$$= 12\pi \int_0^7 \sqrt{x+9}\ dx.$$

$$= 12\pi \int_0^7 (x+9)^{\frac{1}{2}} dx = 12\pi\left[\frac{(x+9)^{3/2}}{(3/2)(1)}\right]_0^7$$

$$= 12\pi\left(\tfrac{2}{3}\right)\left[(x+9)^{3/2}\right]_0^7 = 8\pi\left\{\left(16^{3/2}\right) - \left(9^{3/2}\right)\right\}$$

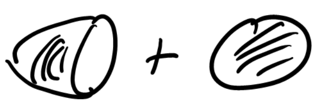
$$= 8\pi\{64 - 27\} = 8\pi(37) = 296\pi\ u^2$$

End cap: $x = 7 \rightarrow y = 6\sqrt{7} = r$

Area $= \pi r^2 = \pi(6\sqrt{7})^2 = \pi(36)(7)$

$= 252\pi\ u^2$

★ $(296 + 252)\pi = 548\pi\, u^2$ ★



SUMMARY

(I) For a curve defined parametrically by $x = x(t)$ and $y = y(t)$.

$$\mathrm{d} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathrm{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \qquad \text{speed } = |\mathrm{v}| = \sqrt{(\dot{x})^2 + (\dot{y})^2} \qquad \mathrm{a} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$\text{Arc Length } = \int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\text{Surface Area of Revolution } = \int_{t_a}^{t_b} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

(II) For a Cartesian curve defined by $y = f(x)$.

$$\text{Average} = \bar{f} = \frac{1}{b-a} \int_a^b f(x)dx$$

$$\text{Arc Length } = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\text{Surface Area of Revolution } = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

(III) For a curve defined defined in polar coordinates $r = f(\theta)$.

$$\text{Arc Length} = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\text{Surface Area of Revolution} = \int_{\theta_0}^{\theta_1} 2\pi r \sin(\theta) \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

SOME FINAL INFORMATION

1. Please check online that all your marks are recorded correctly.

2. Read the school pages on additional assessment/special consideration so that you are fully aware of the rules that apply.

3. Past papers are on Moodle.

4. Make sure that you are aware of the format and date of the final exam. If in any doubt please consult the Moodle page or contact the first year office.

8. Please take the time to complete all online surveys regarding the administration and teaching of the course.

9. Check Moodle for consultation options during stuvac.

Good Luck!

Milan Pahor