Tutorial 2 HSTS201

flint

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1. Show that the mean $\bar{X}$ of a random sample of size n from a distribution with pdf $f(x;\theta) = 1/\theta e^{-(x/\theta)},\ 0 \leq x \leq \infty$, zero otherwise, is an unbiased estimator of θ and has variance θ^2/n.

   Solution

   Let $f(x;\theta) = 1/\theta e^{-(x/\theta)},\ 0 \leq x \leq \infty$, this is an exponential distribution with mean θ and variance θ^2
   Thus,

   $$E(\bar{X}) = E(\frac{1}{n}\sum X) = \frac{1}{n}E(\sum X) = \frac{1}{n}\sum E(X) = \frac{1}{n}\sum \theta = \frac{1}{n}n\theta = \theta$$

   $$Var(\bar{X}) = Var(\frac{1}{n}\sum X) = \frac{1}{n^2}Var(\sum X) = \frac{1}{n^2}\sum Var(X)$$
   $$= \frac{1}{n^2}\sum \theta^2 = \frac{1}{n^2}n\theta^2 = \theta/n$$

2. Let $X_1,\ X_2,\ \ldots,\ X_n$ denote a random sample from a normal distribution with mean zero and variance $\theta,\ \ 0 \leq x \leq \infty,$. Show that $\sum X_i^2/n$ is an unbiased estimator of θ and has variance $2\theta^2/n$.

   Solution
   It is known that $\sum X_i^2/\sigma^2$ is $\chi^2(n)$. Hence

   $$E[\sum X_i^2/\sigma^2] = n \ \ and \ \ Var[\sum X_i^2/\sigma^2] = 2n$$

Thus,

$$E[\sum X_i^2/\theta] = n$$
$$\frac{n}{n}E[\sum X_i^2/\theta] = n$$
$$\frac{1}{n}E[\sum X_i^2/\theta] = 1$$
$$\frac{1}{\theta}E[\sum X_i^2/n] = 1$$
$$E[\sum X_i^2/n] = \theta$$

$$Var[\sum X_i^2/\theta] = 2n$$
$$\frac{n^2}{n^2}Var[\sum X_i^2/\theta] = 2n$$
$$\frac{1}{n^2}Var[\sum X_i^2/\theta] = 2/n$$
$$\frac{1}{\theta^2}Var[\sum X_i^2/n] = 2/n$$
$$Var[\sum X_i^2/n] = 2\theta^2/n$$

3. Let $Y_1 \leq Y_2 \leq Y_3$ be the order statistics of a random sample of size 3 from the uniform distribution having pdf $f(x;\theta) = l/\theta, \quad 0 \leq x \leq \infty, \quad 0 \leq \theta \leq \infty$, zero elsewhere. Show that $4Y_1,\ 2Y_2,\ and\ \frac{4}{3}Y_3$ are all unbiased estimators of θ. Find the variance of each of these unbiased estimators.

   Solution
   It can be show that $Y_1/\theta,\ Y_2/\theta,\ Y_3/\theta$ follows a beta distribution with parameters ($\alpha,\ \beta$)=(1,3), (2,2) and (3,1) respectively. Thus

   $$E[4Y_1] = 4\theta E[Y_1/\theta] = 4\theta \times \frac{1}{4} = \theta$$

   Similarly,

   $$E[2Y_2] = 2\theta E[Y_2/\theta] = 2\theta \times \frac{1}{2} = \theta$$

$$E[\frac{4}{3}Y_3] = \frac{4}{3}\theta E[Y_3/\theta] = \frac{4}{3}\theta \times \frac{3}{4} = \theta$$

$$Var[4Y_1] = (4\theta)^2 Var[Y_1/\theta] = (4\theta)^2 \times \frac{3}{80} = \frac{3}{5}\theta^2$$

Similarly,

$$Var[2Y_2] = \frac{1}{5}\theta^2 \quad \& \quad Var[\frac{4}{3}Y_3] = \frac{1}{15}\theta^2$$

4. Let $X_1,\ X_2,\ .\ .\ .,\ X_n$ denote a random sample from a Poisson distribution with parameter $\theta \quad, 0 \leq \theta \leq \infty$. Let $Y = \sum X_i$ and let $\mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y)]$. If we restrict our considerations to decision functions of the form $\delta(y) = b + \theta/n$, where b does not depend on y, show that $R(\theta, \delta) = b^2 + \theta^2/n$. What decision function of this form yields a uniformly smaller risk than every other decision function.

   Solution
   Since X_i follows Po(θ), Y follows Po($n\theta$)
   We have $E(Y) = n\theta$ and $Var(Y) = n\theta$ Thus

$$R(\theta, \delta) = E(\mathcal{L}[\theta, \delta(y)]) = E[(\theta - b - y/n)^2] = ... = b^2 + \theta^2/n$$