VCU Math—Summer 2022 Research
Graphs & Knots!

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2106 Harris Hall
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MTWThF, 10:00-1:00

An *independent set* in a graph is a set of vertices which are pairwise non-adjacent. For any graph we can find *all* of the independent sets C of the graph and compute:

$$\sum_{C \text{ independent}} (-1)^{|C|}.$$

This is a graph **invariant**. This means that it may be related to other graph invariants—and these connections can be studied. Understanding this number might lead to new insights about graphs—or knots.

Initial Research Questions

Let's call this the *Bae-Morton number* β. This number is *hard* to calculate (NP-hard). A simple formula for this number would be amazing—but probably unlikely. So one thing that researchers do is investigate **bounds** for hard-to-compute numbers. Bounds can be both theoretically and computationally useful.

1. Can we find upper bounds for the Bae-Morton number?

   An *upper bound* is a number that is always at least as large as the Bae-Morton number. If a graph has n vertices then $\beta \leq 2^n$, so 2^n is an upper bound.

2. Can we find lower bounds for the Bae-Morton number?

   A *lower bound* is a number that is always no more than the Bae-Morton number. If a graph has n vertices then $-2^n \leq \beta$, so -2^n is an upper bound.

3. Can we find a formula for the Bae-Morton number for special classes of graphs?

Research Goal

The **goal** of mathematical research is the creation of new mathematical knowledge, particularly knowledge related to or that advances pre-existing mathematical questions, or pre-existing issues. Can we find any knew theorems, or make any new connections?

Concepts & Notes

As we go we'll learn lots of new concepts. It will be useful to you to write out your own definitions and examples. **We assume that no one knows anything and that we'll build up our knowledge as we go. So ask lots of questions!** Here are some beginning concepts.

1. What is a *graph*?

2. What are examples of graph *classes*? What is a *cycle* graph? What is a *path* graph? What is a *complete* graph?

Research Practices

1. Can you find a conjecture for *all* graphs (or maybe a special class of graphs)? Can you prove it, or find a *counterexample*? If you found a counterexample, can you find a revised conjecture that is true for this counterexample (in particular)?
2. What is known (from the literature)? Do known theorems imply any conjecture you are investigating?

New Research Questions

Our initial investigation is narrowly focused. But the invariant we're studying arises from knot theory. What is the connection? What can the Bae-Morton number tell us about knots? How can our Bae-Morton number investigation advance any knot theory question?

1. What is a *knot*?
2. What does the Bae-Morton number tell us about knots?

Tools

Mathematical research involves learning relevant concepts, diving into the mathematical literature and finding out what is already known, computing lots of examples, making conjectures, proving or disproving them, revising them, writing up the results, discussing article drafts, and submitting an article to a journal.

1. `Sage` (and `CoCalc`) are useful for computations.
2. Google Scholar and MathSciNet are useful for literature review.
3. GT is a file with lots of predefined graphs, invariants and properties.
4. Conjecturing is a program can be used to generate conjectures upper and lower bounds for a graph invariant, or conjectures sufficient or necessary conditions for a graph property.
5. nauty is a program that can exhaustively generate all graphs of any order (there are more than 10 million graphs of order no more than 10!)
6. LaTeX is a mathematical typesetting program that all mathematicians use to write their research papers (and `overleaf.com` is a LaTeX-based website for math paper writing collaboration).
7. compile is a VCU computer cluster for large-scale console-based computation. Any VCU-affiliated person can get an account at: `https://ts.vcu.edu/askit/research-math-s technology-services-research-servers/compile-server---linux/`

Known Results & Literature Review

What is already known? What theorems exist? What might be related that can give us knew ideas?

1. Keep a log or record of concepts and theorems that might be useful.
2. Maintain a list of *references* where you found these ideas—we will need and want to report these in our paper (or other research product).

Research Techniques

1. Calculate lots of small examples. Make sure that you *really* understand the concepts involved. Test conjectures with lots of small examples.
2. **Be bold**. False conjectures are OK. Scientists are wrong most of the time. By producing false conjectures and learning *why* they are wrong we learn obstacles that future conjectures will need to overcome.
3. In many cases counterexamples can be searched for in various systematic ways, either by exhaustive (computer) search of small examples, by the generation of random examples, or by careful construction of examples with special properties.
4. Dive in to the literature. This is something to do when you are stuck and not feeling creative—when you are reading you can't be creating—so going back and forth is useful.

Computing Questions

1. How can we represent a graph in Sage/CoCalc?
2. How can we produce upper or lower bound conjectures for the Bae-Morton invariant?
3. How can we evaluate a conjecture for a specific graph of interest?
4. How can we evaluate a conjecture for all graphs of some specific order?
5. How can we represent a knot in Sage/CoCalc?
6. What knot invariants are built-in to CoCalc? What can we compute about a knot?