Minimal Sets, Union-Closed Families, and Frankl's Conjecture

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- A *set system* or *set family* on a set X is a pair $(X, \mathcal{F})$ where $\mathcal{F}$ is a collection of sets which are subsets of X. Letting $\mathcal{P}(X)$ denote the power set of X, we can write $\mathcal{F} \subseteq \mathcal{P}(X)$.

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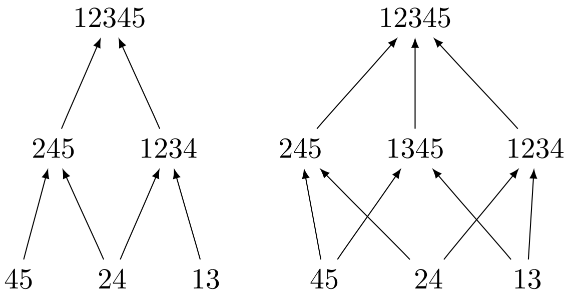
- The set X is called the *ground set*. If the ground set is clear from context, we can refer to $\mathcal{F}$ as a set system.
- We say that set family $\mathcal{F}$ is *union-closed* if the union of any two sets in the family is contained in the family, i.e.

$$\text{if} \quad S, T \in \mathcal{F} \quad \text{then} \quad S \cup T \in \mathcal{F}$$



- The family on the left is not union-closed because it does not contain $\{1,3\} \cup \{4,5\}$.
- The family on the right is union-closed.

- For $x \in X$, the *subfamily* of $\mathcal{F}$ containing x is the collection of sets in $\mathcal{F}$ which have x as an element:

$$\mathcal{F}_x = \{S : x \in S \in \mathcal{F}\}.$$

- The *frequency* of x is the size of $\mathcal{F}_x$.

$$\text{If} \quad \frac{|\mathcal{F}_x|}{|\mathcal{F}|} \geq \frac{1}{2}, \quad \text{then } x \text{ is } \textit{abundant}.$$

$$\text{If} \quad \frac{|\mathcal{F}_x|}{|\mathcal{F}|} < \frac{1}{2}, \quad \text{then } x \text{ is } \textit{rare}.$$

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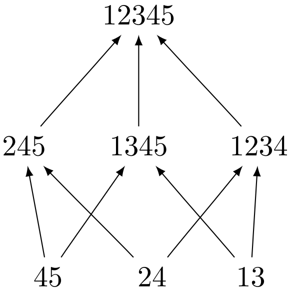
- Thus, $\mathcal{F}$ contains an abundant element if there exists some $x \in X$ which appears in at least half the sets of $\mathcal{F}$.



- In this family $\mathcal{F}$, we have $|\mathcal{F}| = 7$, $|\mathcal{F}_1| = 4$, $|\mathcal{F}_2| = 4$, $|\mathcal{F}_3| = 4$, $|\mathcal{F}_4| = 6$, $|\mathcal{F}_5| = 4$.
- Each element in this family is abundant since each element has frequency at least $7/2 = 3.5$.

Conjecture (Frankl, 1979)

Every union-closed family with non-empty ground set contains an abundant element.

- If a union-closed family $\mathcal{F}$ satisfies Frankl's conjecture (has an abundant element), we say $\mathcal{F}$ *is Frankl*.

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- If a union-closed family $\mathcal{F}$ satisfies Frankl's conjecture (has an abundant element), we say $\mathcal{F}$ *is Frankl*.
- Péter Frankl thought of the conjecture in late 1979 when researching traces of finite sets.
- Frankl told the conjecture to Ron Graham in early 1980.
- Dwight Duffus learned about the problem in 1981 and presented on it in 1984.

D. DUFFUS: This is a problem of Frankl. Let $\mathcal{F}$ denote a finite collection of finite sets containing both $\emptyset$ and $\bigcup\mathcal{F}$.

1.9 *If $\mathcal{F}$ is closed under unions, does there exist an element x such that $|\{F \in \mathcal{F} : x \in F\}| \geq \frac{1}{2}|\mathcal{F}|$?*

The Boolean lattice shows that this, if true, is sharp. You can get another formulation by dualizing:

1.9' *If $\mathcal{F}$ is closed under intersections, does there exist an element x such that $|\{F \in \mathcal{F} : x \in F\}| \leq \frac{1}{2}|\mathcal{F}|$?*

You can also translate the problem into one involving finite lattices, which on the surface seems less likely:

1.9" *If L is a finite lattice, does there exist a join-irreducible element a of L such that $|\{x \in L : x \geq a\}| \leq \frac{1}{2}|L|$?*

This version can be proved for distributive lattices, and Mike Saks showed me an argument that works for geometric lattices.

From the Proceedings of the NATO Advanced Study Institute on Graphs and Order in Banff Canada (published 1985)

- Another formulation of the conjecture introduced by J.C. Renaud in 1990 is an extremal formulation.
- For union-closed family $\mathcal{F}$, let $\phi(\mathcal{F})$ be the frequency of the most frequent element in $\mathcal{F}$, so

$$\phi(\mathcal{F}) = \max\{|\mathcal{F}_x| : x \in X\}.$$

- For positive integer m, let $\phi(m)$ be the smallest number such that all union-closed families with m sets contain an element of frequency at least $\phi(m)$, so

$$\phi(m) = \min\{\phi(\mathcal{F}) : |\mathcal{F}| = m\}.$$

- Showing that $\phi(m) \geq m/2$ for all $m > 0$ is equivalent to Frankl's conjecture.

- One of the first research articles on the conjecture was published in 1989 by D. G. Sarvate and J. C. Renaud with some ideas for how to attack the conjecture.

Idea

Let $\mathcal{F}$ be a union-closed family and S be a non-empty set in $\mathcal{F}$ with minimal cardinality. Can we prove Frankl's conjecture for small values of $|S|$?

Theorem (Sarvate and Renaud, 1989)

If $\mathcal{F}$ is a union-closed family containing a singleton or a pair, then $\mathcal{F}$ satisfies Frankl's conjecture.

Proof.

(Singleton case): let $\mathcal{F}$ be a union-closed family containing a singleton $S = \{x\}$. Write $\mathcal{F}$ as $\mathcal{F} = \{S, S_1, S_2, \dots, S_{\ell-1}\}$. For each $S_i \in \mathcal{F}$, with $x \notin S_i$, the set $S_i \cup \{x\}$ is in $\mathcal{F}$. Each set not containing x has a unique corresponding set containing x, so the number of sets not containing x is at most as large as the number of sets containing x. Thus x is in at least half the sets of $\mathcal{F}$. □

Proof.

(Pair case): let $\mathcal{F}$ be a union-closed family containing a pair $S = \{x, y\}$ and write $\mathcal{F}$ as $\mathcal{F} = \{S, S_1, S_2, \ldots, S_{\ell-1}\}$. Suppose s_0 sets of $\mathcal{F}$ contain neither x nor y, s_{xy} sets contain both x and y, s_x sets contain x but not y, and s_y sets contain y but not x. These values partition $\mathcal{F}$, so

$$s_0 + s_{xy} + s_x + s_y = \ell$$

For every set S_i containing neither x nor y, the set $S_i \cup S$ will be in $\mathcal{F}$, meaning $s_0 \leq s_{xy}$. Thus

$$2s_{xy} + s_x + s_y \geq \ell$$

and so either $s_{xy} + s_x \geq \ell/2$ or $s_{xy} + s_y \geq \ell/2$. $\square$

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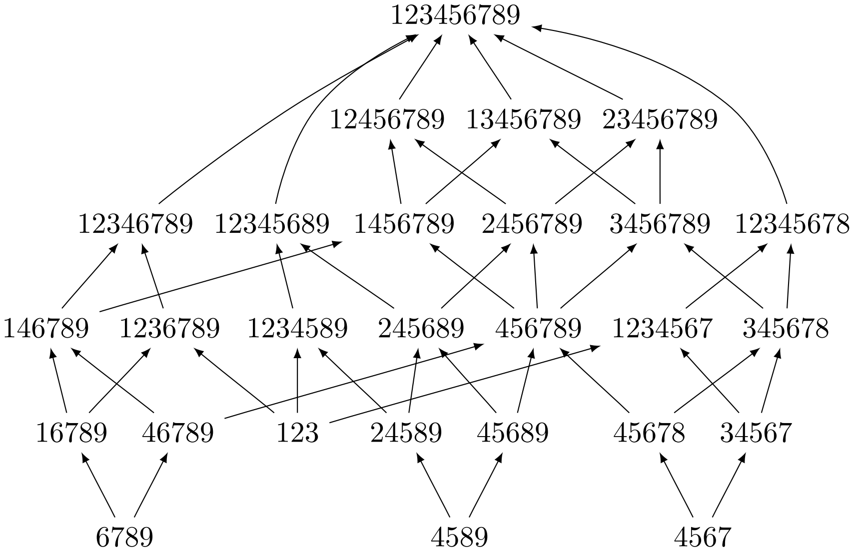
- The previous proofs proved that if a union-closed family contains a singleton or pair, then that singleton is abundant or one of the elements in the pair is abundant.

- Can we use a similar argument for the case where $\mathcal{F}$ contains a triple ($S = \{x, y, z\}$)?
- The previous proofs proved that if a union-closed family contains a singleton or pair, then that singleton is abundant or one of the elements in the pair is abundant.
- However, if a union-closed family contains a triple, it's possible that none of the elements in the triple are abundant. An example of such a family was found by Sarvate and Renaud in 1990.



- $|\mathcal{F}| = 27$, $|\mathcal{F}_1| = 13$, $|\mathcal{F}_2| = 13$, $|\mathcal{F}_3| = 13$, $|\mathcal{F}_4| = 23$

- Instead of just looking at sets with minimum cardinality, can we generalize what we're looking for?

- Instead of just looking at sets with minimum cardinality, can we generalize what we're looking for?
- For union-closed family $\mathcal{F}$, we say that M is a *minimal set* in $\mathcal{F}$ if M is not a superset of any set in $\mathcal{F}$ (besides possibly $\varnothing$).
- So, if M is a minimal set in $\mathcal{F}$, there can't be a set $M' \in \mathcal{F} \setminus \varnothing$ with $M' \subset M$.

Conjecture

In a union-closed family $\mathcal{F}$ with $x \in X$, if $|\mathcal{F}_x| = \phi(\mathcal{F})$, then x is contained some minimal set of $\mathcal{F}$.

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- This conjecture is not true if we only require that x is abundant. Consider the family

$$\mathcal{F} = \{\{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}\}.$$

- Here, $\{1\}$ is the only minimal set.
- However, 2 is abundant but not in $\{1\}$.

- Even if the conjecture is not true in general, we can describe a class of families where it is true.
- For a family of sets $\mathcal{G}$, let the union-closed family *generated by* $\mathcal{G}$ be the family

$$\langle \mathcal{G} \rangle = \left\{ \bigcup_{S \in \mathcal{S}} S : \varnothing \subset \mathcal{S} \subseteq \mathcal{G} \right\}.$$

- We say that a union-closed family is *minimally-generated* if it is generated by its collection of minimal sets. Let $\mathcal{M}$ be the collection of minimal sets in union-closed family $\mathcal{F}$. Then $\mathcal{F}$ is minimally-generated if $\mathcal{F} = \langle M \rangle$ (or $\mathcal{F} = \langle M \rangle \cup \{\varnothing\}$ if $\varnothing \in \mathcal{F}$).

- What kinds of families are minimally-generated?

- What kinds of families are minimally-generated?
- Power-set families $\mathcal{P}(X)$ are.

Proof.

Let $\mathcal{F} = \mathcal{P}(X)$. The collection of minimal sets of $\mathcal{F}$, $\mathcal{M}$ is the collection of singletons $\{\{x\} : x \in X\}$. Any non-empty set $S = \{s_1, s_2, \ldots, s_\ell\}$ in $\mathcal{F}$ can be represented as the union

$$S = \{s_1\} \cup \{s_2\} \cup \cdots \cup \{s_\ell\}$$

and is thus generated by a subset of $\mathcal{M}$. $\square$

- The family $\mathcal{S}(n,k)$ where $k < n$ created by starting with $\mathcal{P}(X)$ (where $|X| = n$) and removing sets of size k or smaller is minimally-generated.

Proof.

The collection of minimal sets of $\mathcal{S}(n,k)$ is $\mathcal{M} = X^{(k+1)}$ (the collection of size $k+1$ subsets of X). Select $S = \{s_1, s_2, \ldots, s_{k+2}\}$ of size $k+2$. We can write

$$S = \{s_1, s_2, \ldots, s_{k+1}\} \cup \{s_2, s_3, \ldots, s_{k+2}\},$$

so S is a union of sets in $\mathcal{M}$. Thus we can write any set in $X^{(k+2)}$ as a union of sets in $\mathcal{M}$. Using the same logic, we can write any set in $X^{(k+3)}$ as a union of sets in $X^{(k+1)}$ (and thus as a union of sets in $\mathcal{M}$). Continuing in this way allows us to generate any set in $\mathcal{S}(n,k)$ using $\mathcal{M}$. □

- All union-closed families contain minimally-generated subfamilies since for union-closed family $\mathcal{F}$, $\langle \mathcal{M} \rangle$ is union-closed and $\langle \mathcal{M} \rangle \subseteq \mathcal{F}$.

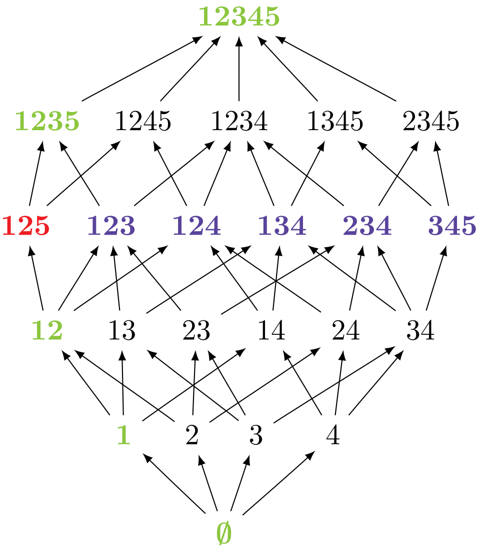
Question

What other interesting minimally-generated families exist?

- We can say more about minimal sets by representing set systems as partially ordered sets.

- We can say more about minimal sets by representing set systems as partially ordered sets.
- Given a set A, a *partial order* on A, usually denoted $\leq$ or $\subseteq$ is a relation which for any $a, b, c \in A$ is
  - reflexive, $a \leq a$
  - antisymmetric, so $a \leq b$ and $b \leq a$ imply that $a = b$
  - transitive, so $a \leq b$ and $b \leq c$ imply that $a \leq c$
- A *poset* is a pair $(A, \leq)$ consisting of a set and a partial order on it.
- We can view a union-closed family $\mathcal{F}$ as a poset $(\mathcal{F}, \subseteq)$ where $\subseteq$ is the usual subset relation.

- Let $(\mathcal{F}, \subseteq)$ be a poset.
- A *chain* is a subset $\mathcal{C} \subseteq \mathcal{F}$ where for any two sets $C_1, C_2 \in \mathcal{C}$, either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$. The *height* of a poset, denoted $\mathrm{h}(\mathcal{F})$, is the size of its largest chain.
- An *antichain* is a subset $\mathcal{A} \subseteq \mathcal{F}$ where for any two sets $A_1, A_2 \in \mathcal{A}$, $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$. The *width* of a poset, denoted $\mathrm{w}(\mathcal{F})$, is the size of its largest antichain.



- The sets highlighted in **green** and **red** form a chain.
- The sets highlighted in **blue** and **red** form an antichain.

- We can prove that union-closed families with small height and width values are Frankl.

Proposition

If $\mathcal{F}$ is a union-closed family with height two, then for any $x \in X$, $|\mathcal{F}_x| \geq |\mathcal{F}| - 1$.

Proof.

Select $x \in X$ and suppose for contradiction that there exist two distinct sets $A, B \in \mathcal{F}$ which do not contain x. Then $A \subset A \cup B \subset X$, so $\{A, A \cup B, X\}$ is a chain of size three, contradicting that $\mathcal{F}$ is height two. Thus at most one set of $\mathcal{F}$ does not contain x. □

- We can also show that Frankl's conjecture is true for families with width up to 3. For the $\mathrm{w}(\mathcal{F}) = 3$ case, we use Dilworth's Theorem

Theorem (Dilworth's Theorem)

Let $\mathcal{F}$ be a set family and the size of its largest antichain be w. Then $\mathcal{F}$ can be partitioned into w chains and no fewer.

Proof sketch for Frankl being true for $\mathrm{w}(\mathcal{F}) = 3$.

- Via Dilworth, we can partition $\mathcal{F}$ into chains $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$. Let M_1, M_2, and M_3 be the minimal sets within these chains. We consider the cases when M_1, M_2, and M_3 are pairwise disjoint and when they are not.
- Case 1: suppose there exists $x \in M_1 \cap M_2$.
- Since M_1 and M_2 are minimal, then x is in every set of $\mathcal{C}_1$ and $\mathcal{C}_2$.
- Depending on the size of $|\mathcal{C}_1 \cup \mathcal{C}_2|$, we show either that x is abundant or that $\mathcal{C}_3$ contains an abundant element.

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- Depending on the size of $|\mathcal{C}_1 \cup \mathcal{C}_2|$, we show either that x is abundant or that $\mathcal{C}_3$ contains an abundant element.
- Case 2: M_1, M_2, M_3 mutually disjoint
- Suppose $|\mathcal{C}_1| \geq |\mathcal{C}_2| \geq |\mathcal{C}_3|$ and notice that $|\mathcal{C}_1 \cup \mathcal{C}_2| \geq 2|\mathcal{F}|/3$.
- Notice that since the M_i's are disjoint, $M_1 \cup M_2$ is in either $\mathcal{C}_1$ or $\mathcal{C}_2$. In either case, we end up with some $x \in M_1$ or $x \in M_2$ such that x is in almost all of the sets of $\mathcal{C}_1$ and $\mathcal{C}_2$

$\square$

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For any union-closed family $\mathcal{F}$ *with collection of minimal sets* $\mathcal{M}$*,* $|\mathcal{M}| \leq \phi(\mathcal{F})$*.*

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- Using the fact that $\mathcal{M}$ is an antichain, we can prove that these two conjectures are actually equivalent.

Proof that $\mathrm{w}(\mathcal{F}) \leq \phi(\mathcal{F})$ implies $|\mathcal{M}| \leq \phi(\mathcal{F})$.

Since $\mathcal{M}$ is an antichain, then $|\mathcal{M}| \leq \mathrm{w}(\mathcal{F})$. Thus if $\mathrm{w}(\mathcal{F}) \leq \phi(\mathcal{F})$, then

$$|\mathcal{M}| \leq \mathrm{w}(\mathcal{F}) \leq \phi(\mathcal{F})$$

and so $|\mathcal{M}| \leq \phi(\mathcal{F})$. □

Proof that $|\mathcal{M}| \leq \phi(\mathcal{F})$ implies $\mathrm{w}(\mathcal{F}) \leq \phi(\mathcal{F})$.

- Suppose that $|\mathcal{M}| \leq \phi(\mathcal{F})$ is true.

Proof that $|\mathcal{M}| \leq \phi(\mathcal{F})$ implies $\mathrm{w}(\mathcal{F}) \leq \phi(\mathcal{F})$.

- Suppose that $|\mathcal{M}| \leq \phi(\mathcal{F})$ is true.
- Let $\mathcal{A}$ be a maximum antichain in $\mathcal{F}$ (so $|\mathcal{A}| = \mathrm{w}(\mathcal{F})$) and let $\mathcal{F}' = \langle \mathcal{A} \rangle$.

Proof that $|\mathcal{M}| \leq \phi(\mathcal{F})$ implies $\mathrm{w}(\mathcal{F}) \leq \phi(\mathcal{F})$.

- Suppose that $|\mathcal{M}| \leq \phi(\mathcal{F})$ is true.
- Let $\mathcal{A}$ be a maximum antichain in $\mathcal{F}$ (so $|\mathcal{A}| = \mathrm{w}(\mathcal{F})$) and let $\mathcal{F}' = \langle \mathcal{A} \rangle$.
- Since $\mathcal{A}$ forms the collection of minimal sets in $\mathcal{F}'$, then our assumption gives $|\mathcal{A}| \leq \phi(\mathcal{F}')$. Since $\mathcal{F}' \subseteq \mathcal{F}$, then $\phi(\mathcal{F}') \leq \phi(\mathcal{F})$. Combining our results gives

$$\mathrm{w}(\mathcal{F}) \leq |\mathcal{A}| \leq \phi(\mathcal{F}') \leq \phi(\mathcal{F})$$

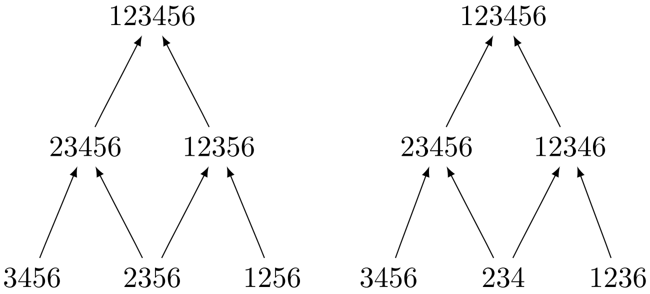
and so $\mathrm{w}(\mathcal{F}) \leq \phi(\mathcal{F})$.

$\square$

- Let $\mathcal{F}$ and $\mathcal{F}'$ be union-closed families and $\varphi : \mathcal{F} \to \mathcal{F}'$ be a bijective function such that for all $S, T \in \mathcal{F}$,
  - $\varphi(S \cup T) = \varphi(S) \cup \varphi(T)$ and
  - $S \subseteq T$ if and only if $\varphi(S) \subseteq \varphi(T)$
- We call such a φ a *structure isomorphism* and say that $\mathcal{F}$ and $\mathcal{F}'$ are *structure isomorphic*.
- Note that if $\varphi : \mathcal{F} \to \mathcal{F}'$ is a structure isomorphism, then so is $\varphi^{-1} : \mathcal{F}' \to \mathcal{F}$.

$$
\begin{tikzcd}
 & & 1234 & & \\
 & 234 \arrow[ur] & & 123 \arrow[ul] & \\
34 \arrow[ur] & & 23 \arrow[ul] \arrow[ur] & & 12 \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & 12345 & & \\
 & 2345 \arrow[ur] & & 1234 \arrow[ul] & \\
345 \arrow[ur] & & 234 \arrow[ul] \arrow[ur] & & 123 \arrow[ul]
\end{tikzcd}
$$

- These two families have ground sets of different sizes, but are still structure isomorphic.



- These two families have the same ground set and are structure isomorphic, but their elements have different frequencies.
- Element 3 appears 5 times in the left family and 6 times in the right one.

Proposition

Structure isomorphisms preserve the height of union-closed families.

Proof.

Let $\mathcal{F}$ and $\mathcal{F}'$ be structure isomorphic union-closed families and $\varphi : \mathcal{F} \to \mathcal{F}'$ be a structure isomorphism. Consider a maximum chain in $\mathcal{F}$, with sets $S_1, S_2, \ldots, S_k$ where $S_i \subseteq S_{i+1}$. Then by definition of structure isomorphism,

$$\varphi(S_1) \subseteq \varphi(S_2) \subseteq \ldots \subseteq \varphi(S_k)$$

so $\mathrm{h}(\mathcal{F}) \leq \mathrm{h}(\mathcal{F}')$. Now consider a maximum chain in $\mathcal{F}'$ with sets $T_1, T_2, \ldots, T_\ell$ where $T_i \subseteq T_{i+1}$. Then again by definition of structure isomorphism,

$$\varphi^{-1}(T_1) \subseteq \varphi^{-1}(T_2) \subseteq \ldots \subseteq \varphi^{-1}(T_\ell)$$

meaning that $\mathrm{h}(\mathcal{F}') \leq \mathrm{h}(\mathcal{F})$ and so $\mathrm{h}(\mathcal{F}) = \mathrm{h}(\mathcal{F}')$. □

Lemma

Structure isomorphisms map minimal sets to minimal sets.

Proof.

Let $\mathcal{F}$ and $\mathcal{F}'$ be structure isomorphic and $\varphi : \mathcal{F} \to \mathcal{F}'$ be a structure isomorphism. Suppose for contradiction that M is minimal in $\mathcal{F}$, but $\varphi(M)$ is not minimal in $\mathcal{F}'$. This means there exists $S \in \mathcal{F}'$ with $S \subset \varphi(M)$ and so $\varphi^{-1}(S) \subset M$, contradicting that M was minimal in $\mathcal{F}$. □

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- Using this result, we can show that structure isomorphisms preserve being minimally-generated.

Theorem

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Proof.

Let $\mathcal{F}$ and $\mathcal{F}'$ be structure isomorphic and $\varphi : \mathcal{F} \to \mathcal{F}'$ be a structure isomorphism. Suppose for contradiction that $\mathcal{F}$ is minimally-generated, but $\mathcal{F}'$ is not, meaning $\mathcal{F}'$ contains a set S not generated by its minimal sets. Let $\mathcal{M}_{\mathcal{F}}$ be the collection of minimal sets of $\mathcal{F}$. Since $\mathcal{F}$ is minimally-generated, we can write

$$\varphi^{-1}(S) = M_1 \cup M_2 \cup \ldots \cup M_k$$

for some $M_1, M_2, \ldots, M_k \in \mathcal{M}_{\mathcal{F}}$. However, this means

$$S = \varphi(M_1) \cup \varphi(M_2) \cup \ldots \cup \varphi(M_k)$$

and our previous result gives that each $\varphi(M_i)$ is minimal in $\mathcal{F}'$, contradicting that S is not generated by minimal sets in $\mathcal{F}'$. $\square$

Conjecture

Structure isomorphisms preserve being Frankl.

Theorem

This conjecture is equivalent to Frankl's conjecture.

Proof.

- ($\Rightarrow$) Take any union-closed family $\mathcal{F}$. Select element $y \notin X$ and construct $\mathcal{F}' = \{S \cup \{y\} : S \in \mathcal{F}\}$. It is easy to see that the function $\varphi : \mathcal{F}' \to \mathcal{F}$ given by $\varphi(S) = S \setminus \{y\}$ is a structure isomorphism. Since y is in every set of $\mathcal{F}'$, $\mathcal{F}'$ satisfies Frankl. Thus if structure isomorphisms preserve being Frankl, $\mathcal{F}$ is Frankl.

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- ($\Leftarrow$) Structure isomorphisms preserve being union-closed, so if Frankl's conjecture is true, then any structure isomorphism will map from a Frankl family to another Frankl family.

$\square$

Some Interesting Family Constructions

- Earlier, we stated that if we can show $\phi(m) \geq m/2$ for all $m > 0$, then Frankl's conjecture is true.
- It is natural to ask if there is a way to construct families $\mathcal{F}_m$ where $\phi(\mathcal{F}_m) = \phi(m)$. It is possible to compute the values of $\phi(m)$ for small m and obtain the sequence (starting at $\phi(1)$):

$$1, 1, 2, 2, 3, 4, 4, 4, 5, 6, 7, 7, 8, 8, 8, 8, 9, 10, \ldots$$

- These first few terms match the first terms of Conway's challenge sequence, defined by $a(1) = a(2) = 1$ and

$$a(m) = a(a(m-1)) + a(m - a(m-1)).$$

- Additionally, it is known that $a(m) \geq m/2$, so it is natural to ask if $\phi(m) = a(m)$ and if we can construct $\mathcal{F}_m$ such that $\phi(\mathcal{F}_m) = a(m)$.

- The Renaud-Fitina families, denoted $\mathcal{R}(m)$ satisfy $\phi(\mathcal{R}(m)) = a(m)$.
- These families are constructed by taking the initial segment of length m of $\mathbb{N}^{(<\omega)}$ ordered by the relation $<_R$ defined by $A <_R B$ if:
  - $\max A < \max B$ or
  - $\max A = \max B$ and $|A| > |B|$ or
  - $\max A = \max B$ and $|A| = |B|$ and $\max(A \Delta B) \in B$

  where $A\Delta B$ represents the symmetric difference of A and B.
- The initial segment of $<_R$ is

$$\begin{array}{llllllll} <_R\colon & \varnothing, & 1, & 12, & 2, & 123, & 13, & 23, \quad \ldots \\ & 3, & 1234, & 124, & 134, & 234, & 14, & 24, \quad \ldots \\ & 34, & 4, & 12345, & 1235, & 1245, & 1345, & 2345, \quad \ldots \end{array}$$

- Another desirable property of union-closed families is seeing how the height, width, and number of sets in the family are related.
- For positive integers h, w, ℓ with $h > 1$ and $h + w - 1 \leq \ell \leq w(h-1) + 1$, we can construct the family $\mathcal{F}(h, w, \ell)$ which has height h, width w, and ℓ sets.

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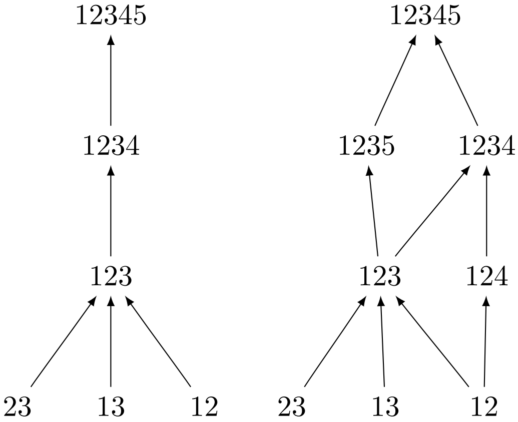
- Start with a union-closed family consisting of a chain of size h and an antichain of size w which intersect in one set.
- Create the chain and antichain such that the union of any two sets in the antichain are included in the chain. The result is a family with size $h + w - 1$.

$$
\begin{tikzcd}
 & & {[w+h-2]} & & \\
 & & {[w+h-3]} \arrow[u] & & \\
 & & \vdots \arrow[u] & & \\
 & & {[w+1]} \arrow[u] & & \\
 & & {[w]} \arrow[u] & & \\
{[w] \setminus \{1\}} \arrow[urr] & {[w] \setminus \{2\}} \arrow[ur] & \cdots \arrow[u] & {[w] \setminus \{w-1\}} \arrow[ul] & {[w-1]} \arrow[ull]
\end{tikzcd}
$$

- Here, $[m] = \{1, 2, 3 \ldots, m\}$.
- Add sets of sizes $w, w+1, \ldots, w+h-3$ while keeping the family union-closed. This works until the family has size $w(h-1)+1$.



- The family on the left has $h = 4$, $w = 3$, and $\ell = 6$.
- The family on the right has $h = 4$, $w = 3$, and $\ell = 8$.

Conclusion

- Thank you for coming to the seminar.
- Are there any questions?