Frankl's Conjecture

Christopher Flippen

Virginia Commonwealth University

flippenc@vcu.edu

April 12, 2023

Definitions

Definition

A set family is a collection of sets which are all subsets of a set G which we call the ground set.

- This means a set family $\mathcal{F}$ is a collection of sets $\{S_1, S_2, S_3, \ldots, S_n\}$ where $\bigcup_{i \in [n]} S_i = G$.
- Here, we will always use $[k] = \{1, 2, 3, \ldots, k\}$ as the ground set and n as the number of sets in the family. The *elements* of the family are the elements of $[k]$.

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Definition

A set family $\mathcal{F}$ is *union-closed* (*abbreviated as UC*) if the union of any two sets in the family is in the family.

- This means for any $S_1, S_2 \in \mathcal{F}$, $S_1 \cup S_2 \in \mathcal{F}$.

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For element $x \in [k]$, the subfamily of $\mathcal{F}$ containing x is $\mathcal{F}_x = \{S \in \mathcal{F} : x \in S\}$. The *frequency* of x is $|\mathcal{F}_x|$.

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Lemma

If $\mathcal{F}$ is a UC family, then $\mathcal{F}_x$ is a UC family for all $x \in [k]$.

- Let $\mathcal{F}$ be a union-closed family, $x \in [k]$, and $S_1, S_2 \in \mathcal{F}_x$.
- Since $\mathcal{F}$ is union-closed, then $S_1 \cup S_2 \in \mathcal{F}$ and since $x \in S_1 \cup S_2$, then $S_1 \cup S_2 \in \mathcal{F}_x$.

The Conjecture

Definition

The *abundance* of an element $x \in [k]$ is $|\mathcal{F}_x|/|\mathcal{F}|$. We say x *abundant* if $|\mathcal{F}_x|/|\mathcal{F}| \geq 1/2$. If an element is not abundant, it is *rare*.

Conjecture (Frankl (1979))

Every union-closed family has an abundant element.

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- Péter Frankl thought of the conjecture in late 1979 when researching traces of finite sets.
- Frankl told the conjecture to Ron Graham in early 1980.
- Dwight Duffus learned about the problem in 1981 and presented on it in 1984.

D. DUFFUS: This is a problem of Frankl. Let $\mathcal{F}$ denote a finite collection of finite sets containing both $\emptyset$ and $\bigcup\mathcal{F}$.

1.9 *If $\mathcal{F}$ is closed under unions, does there exist an element x such that $|\{F \in \mathcal{F} : x \in F\}| \geq \frac{1}{2}|\mathcal{F}|$?*

The Boolean lattice shows that this, if true, is sharp. You can get another formulation by dualizing:

1.9' *If $\mathcal{F}$ is closed under intersections, does there exist an element x such that $|\{F \in \mathcal{F} : x \in F\}| \leq \frac{1}{2}|\mathcal{F}|$?*

You can also translate the problem into one involving finite lattices, which on the surface seems less likely:

1.9" *If L is a finite lattice, does there exist a join-irreducible element a of L such that $|\{x \in L : x \geq a\}| \leq \frac{1}{2}|L|$?*

This version can be proved for distributive lattices, and Mike Saks showed me an argument that works for geometric lattices.

From the Proceedings of the NATO Advanced Study Institute on Graphs and Order in Banff Canada (published 1985)

A MUCH-TRAVELLED CONJECTURE

At the recent Annual Meeting of the Australian Mathematical Society, Jamie Simpson publicised the following conjecture, passed on to him by Franz Salzborn (University of Adelaide) who heard it in Holland.

Given a finite collection A of n finite sets which is closed under union (i.e., $S \in A,\ T \in A \Longrightarrow S \cup T \in A$), there exists an element which belongs to at least $n/2$ of the sets.

Has this conjecture been published elsewhere? Has it been proved? Can anyone prove it? These are Jamie's questions. Reply to him at the South Australian Institute of Technology, Whyalla.

From the Report of the 1987 Australian Applied Mathematics Conference

UNION-CLOSED SETS CONJECTURE

Regarding the conjecture given in the June *GAZETTE* (page 63), Peter Winkler (Department of Mathematics, Emory University, Atlanta, Georgia) has written the following.

The "union-closed sets conjecture" is well known indeed, except for (1) its origin and (2) its answer! I first encountered it ten years ago as "Grinstead's Conjecture" but Charles Grinstead, the only Grinstead I have ever heard of, denies originating it. Some of my colleagues thought it was Erdős's but Paul disclaimed it at a Boca talk five years ago, saying that as far as he knows it came from Peter Frankl. I doubt it, although I confess I haven't asked Peter yet. The best claim, in my opinion, comes from an Oxford student who wrote some sort of thesis on the problem and who, I am told, insists vociferously that the problem is originally from Ehrenfeucht.

No-one has made any real progress on the problem; Ron Graham is rumoured to have claimed a counterexample (which later fell through) some years ago. Along with the Erdős-Faber-Lovasz problem, this is considered one of the most embarrassing gaps in combinatorial knowledge.

A curious observation, which suggests that the problem had a single source: intersection-closed set systems (e.g., subalgebras) occur with great frequency in mathematics; thus one might expect to hear the equivalent dual formulation of the problem more often (is there an element belonging to at most half the sets of an intersection-closed system?). In fact, I have heard the union version 25 times out of 25.

From the Fall 1987 Australian Mathematical Society Gazette

Graph Theory Formulation

In 2015, Bruhn, Charbit, Schaudt, and Telle found equivalent formulations of Frankl's conjecture using graph theory.

Conjecture (Frankl (1979))

Every union-closed family has an abundant element.

Conjecture (Bruhn et al. (2015))

Let G be a graph with at least one edge. There will be two adjacent vertices each belonging to at most half of the maximal independent sets.

Conjecture (Bruhn et al. (2015))

Let G be a bipartite graph with at least one edge. Each of the independent sets contains a vertex belonging to at most half of the maximal independent sets.

Graph Theory Results

Theorem (Bruhn et al. (2015))

Chordal bipartite graphs satisfy Frankl's conjecture.

Theorem (Bruhn et al. (2015))

Subcubic bipartite graphs satisfy Frankl's conjecture.

Theorem (Bruhn et al. (2015))

Bipartite series-parallel graphs satisfy Frankl's conjecture.

Recent Results

Theorem (Gilmer (2022))

Every union-closed family contains an element x with $|\mathcal{F}_x| \geq 0.01|\mathcal{F}|$.

Theorem (Several Authors (2022))

Every union-closed family contains an element x with

$$|\mathcal{F}_x| \geq \frac{3-\sqrt{5}}{2}|\mathcal{F}| \approx 0.38197|\mathcal{F}|.$$

- Later in 2022, this bound was improved to $|\mathcal{F}_x| \geq 0.38234|\mathcal{F}|$.

Some Attempts at the Conjecture

Question

Let $|\mathcal{F}| = n$. *Can we try to prove Frankl's conjecture for small values of* n *and then find a pattern or use induction on* n *to prove the whole conjecture?*

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For integer m, if the conjecture is true for families of size $2m + 1$, then the conjecture is true for families of size $2m + 2$.

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- Let $\mathcal{F}$ be a UC family of size $2m+2$, written as $\mathcal{F} = \{S_1, S_2, S_3, \ldots, S_{2m+2}\}$ where $|S_i| \leq |S_{i+1}|$.

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- Let $\mathcal{G} = \{S_2, S_3, \ldots, S_{2m+2}\}$.
- If $\mathcal{G}$ satisfies Frankl, there exists an $x \in [k]$ where $|\mathcal{G}_x| \geq \left\lceil \frac{2m+1}{2} \right\rceil = \left\lceil \frac{2m+2}{2} \right\rceil$.

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- Let $\mathcal{G} = \{S_2, S_3, \dots, S_{2m+2}\}$.
- If $\mathcal{G}$ satisfies Frankl, there exists an $x \in [k]$ where $|\mathcal{G}_x| \geq \lceil \frac{2m+1}{2} \rceil = \lceil \frac{2m+2}{2} \rceil$.
- Since $\mathcal{G} \subseteq \mathcal{F}$, then $|\mathcal{F}_x| \geq |\mathcal{G}_x| \geq \lceil \frac{2m+2}{2} \rceil = |\mathcal{F}|/2$.

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- Gao and Weidong proved in 1998 that Frankl is true for $|\mathcal{F}| \leq 32$ and if $\mathcal{F}$ has a ground set of size up to 8.
- Ivica Bošnjak and Peter Marković proved in 2008 that Frankl is true if $\mathcal{F}$ has a ground set of size up to 11.
- Roberts and Simpson in 2010 used the 2008 result to show that Frankl is true for $|\mathcal{F}| < 46$.

First Results

Observation

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Theorem (Sarvate and Renaud (1989))

If for some $x \in [k]$, the singleton $\{x\}$ is in $\mathcal{F}$, then x is abundant.

- Consider UC family $\mathcal{F} = \{S_1, S_2, \ldots, S_k\}$ with $S_1 = \{x\}$. For each $S_i \in \mathcal{F}$ such that $x \notin S_i$, there exists the set $S_i \cup \{x\}$ in $\mathcal{F}$ and for such sets $S_i \neq S_j$, $S_i \cup \{x\} \neq S_j \cup \{x\}$ and so x is abundant in $\mathcal{F}$.

First Results

Theorem (Sarvate and Renaud (1989))

If for some $x, y \in [k]$, the pair $\{x, y\}$ is in $\mathcal{F}$, then x or y is abundant.

- Consider UC family $\mathcal{F} = \{S_1, S_2, \ldots, S_k\}$ with $S_1 = \{x, y\}$. Suppose s_0 sets of $\mathcal{F}$ contain neither x nor y, s_{xy} sets contain both, s_x contain x but not y, and s_y sets contain y but not x. So $n = s_0 + s_{xy} + s_x + s_y$.

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- Since for every set $S_i \in \mathcal{F}$ containing neither x nor y, there is a set $S_i \cup S_j$ containing both x and y, we have $s_0 \leq s_{xy}$ and so $2s_{xy} + s_x + s_y \geq n$, giving

$$s_{xy} + s_x \geq n/2 \quad \text{or} \quad s_{xy} + s_y \geq n/2.$$

Thus at least one of x or y is abundant in $\mathcal{F}$.

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If the triple $\{x, y, z\}$ is in $\mathcal{F}$, must one of x, y, or z be abundant?

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- Sarvate and Renaud found in 1990 that the answer is no.

A Disappointing Result

Consider $\mathcal{F} = \{S_1, \ldots, S_{27}\}$ with

$S_1 = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ $\quad S_{10} = \{\mathbf{4}, 6, 7, 8, 9\}$ $\quad S_{19} = \{\mathbf{2}, \mathbf{4}, 5, 6, 7, 8, 9\}$

$S_2 = \{\mathbf{4}, 5, 6, 7\}$ $\quad S_{11} = \{\mathbf{1}, \mathbf{4}, 6, 7, 8, 9\}$ $\quad S_{20} = \{\mathbf{3}, \mathbf{4}, 5, 6, 7, 8, 9\}$

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$S_4 = \{6, 7, 8, 9\}$ $\quad S_{13} = \{\mathbf{3}, \mathbf{4}, 5, 6, 7, 8\}$ $\quad S_{22} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, 5, 6, 8, 9\}$

$S_5 = \{\mathbf{1}, 6, 7, 8, 9\}$ $\quad S_{14} = \{\mathbf{4}, 5, 6, 7, 8, 9\}$ $\quad S_{23} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, 6, 7, 8, 9\}$

$S_6 = \{\mathbf{2}, \mathbf{4}, 5, 8, 9\}$ $\quad S_{15} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, 5, 6, 7\}$ $\quad S_{24} = \{\mathbf{1}, \mathbf{2}, \mathbf{4}, 5, 6, 7, 8, 9\}$

$S_7 = \{\mathbf{3}, \mathbf{4}, 5, 6, 7\}$ $\quad S_{16} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, 5, 8, 9\}$ $\quad S_{25} = \{\mathbf{1}, \mathbf{3}, \mathbf{4}, 5, 6, 7, 8, 9\}$

$S_8 = \{\mathbf{4}, 5, 6, 7, 8\}$ $\quad S_{17} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, 6, 7, 8, 9\}$ $\quad S_{26} = \{\mathbf{2}, \mathbf{3}, \mathbf{4}, 5, 6, 7, 8, 9\}$

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This is a family with $|\mathcal{F}| = 27$, but $|\mathcal{F}_1| = 13$, $|\mathcal{F}_2| = 13$, $|\mathcal{F}_3| = 13$, and $|\mathcal{F}_4| = 23$. This means none of the elements in the set of size 3 are abundant, but the family is still Frankl.

A Slightly Weaker Result

Definition

We say that a set M in a family $\mathcal{F}$ is *minimal* if no subset of M (besides the empty set) is a member of $\mathcal{F}$. If every set in $\mathcal{F}$ (besides the empty set) can be represented as a union of the family's minimal sets, we say the family is *minimally-generated*.

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Theorem

Any element of maximum frequency in a UC family will appear in a minimal set.

Proof of Theorem

- Let $\mathcal{F}$ be a UC family and $x \in [k]$ be a maximum frequency element of $\mathcal{F}$. We can show this via induction. The base case is when $|\mathcal{F}| = 1$ and is trivial.

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- Our induction hypothesis is that the claim is true for families with less than n sets. Let $\mathcal{F}$ be a UC family with n sets. If any minimal set contains x, then we're done.

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- Otherwise, *no* minimal set contains x. Let M be any minimal set of $\mathcal{F}$. Consider $\mathcal{F}' = \mathcal{F} - M$. The resulting family is still UC. Since $x \notin M$, x is still a maximum frequency element in $\mathcal{F}'$.

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- The induction hypothesis gives that there exists at least at least one minimal set in $\mathcal{F}'$ containing x. If one or more of these minimal sets in $\mathcal{F}'$ is not minimal in $\mathcal{F}$, then M is a subset of each of them.

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- The induction hypothesis gives that there exists at least at least one minimal set in $\mathcal{F}'$ containing x. If one or more of these minimal sets in $\mathcal{F}'$ is not minimal in $\mathcal{F}$, then M is a subset of each of them.
- Let $y \in M$. Then y is in every set of $\mathcal{F}$ containing x. Since $x \notin M$, then this gives $\mathcal{F}_x \subset \mathcal{F}_y$, so x is not of maximum frequency, a contradiction.

A Non-Minimally-Generated Family

$$\{1,2,3,4,5,6,7,8\}$$
$$\{1,2,3,4,5,6,7\}$$
$$\{1,2,3,4,5,6\}$$
$$\{1,2,3,4,5\}$$
$$\{1,2,3,4\}$$
$$\{1,2,3\}$$
$$\{1,2\}$$
$$\{1\}$$

- The only minimal set in this family is $\{1\}$. Since we only have one minimal set, we can't represent any of the other sets of the family as unions of the minimal sets.

A Minimally-Generated Family

$$\{1,2,3,4\}$$
$$\{1,2,3\} \qquad \{1,2,4\}$$
$$\{1,2\} \qquad \{2,3\} \qquad \{1,4\}$$

- We have minimal sets $M_1 = \{1,2\}$, $M_2 = \{2,3\}$, and $M_3 = \{1,4\}$. Every set in the family is either minimal, or a union of minimal sets.

Blocks

Definition

Given a UC family $\mathcal{F}$, a *block* B is a collection of elements of $[k]$ such that $\mathcal{F}_x = \mathcal{F}_y$ for any $x, y \in B$

Lemma (Poonen (1992))

It is sufficient to consider UC families where all the blocks are singletons.

- Let $\mathcal{F}$ be a UC family. Let $\mathcal{F}'$ be the family obtained from $\mathcal{F}$ by replacing every non-singleton block of $\mathcal{F}$ with a representative element. If an element of $\mathcal{F}'$ is abundant, then every element of its corresponding block in $\mathcal{F}$ was abundant and vise versa.

Normal Families

Definition

If all blocks in a family are singletons, we say that the family is *Poonen-normal*.

Definition

If a family is both minimally-generated and Poonen-normal, we say that it is *normal*.

Normal Families

Conjecture

All normal families are Frankl.

Question

What can we use to study normal families?

Poset Definitions

Definition

A *partially ordered set*, or *poset*, is a set with a partial order defined on it. A partial order $\leq$ on a set P is a relation satisfying for all $a, b, c \in P$:

- Reflexivity: $a \leq a$
- Antisymmetry: if $a \leq b$ and $b \leq a$, then $a = b$
- Transitivity: if $a \leq b$ and $b \leq c$, then $a \leq c$

Viewing a set family as a poset ordered with set inclusion ($A \leq B \Leftrightarrow A \subseteq B$ and $A \nleq B \Leftrightarrow A \nsubseteq B$) allows us to define several invariants on the family.

Poset Definitions

Definition

A *chain* in a UC family is a collection of sets $\{S_1, S_2, S_3, \ldots, S_h\}$ satisfying $S_1 \subseteq S_2 \subseteq S_3 \subseteq \ldots \subseteq S_h$. The maximum size of a chain is called the *height* of the family $\mathrm{h}(\mathcal{F})$.

Definition

An *antichain* in a UC family is a collection of sets $\{S_1, S_2, S_3, \ldots, S_w\}$ satisfying $S_i \not\subseteq S_j$ for all $i \neq j$. The maximum size of an antichain is called the *width* of the family $\mathrm{w}(\mathcal{F})$.

Height and Width Results

Lemma

If $\mathrm{h}(\mathcal{F}) \leq 2$, *then every element in* $[k]$ *is abundant*

- If $\mathrm{h}(\mathcal{F}) = 1$, then $\mathcal{F}$ has $[k]$ as its only set. Thus every element in $[k]$ has $|\mathcal{F}_x| = 1 = |\mathcal{F}|$ and is abundant.

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- Let $\mathcal{F}$ be a UC family with $\mathrm{h}(\mathcal{F}) = 2$ and $|\mathcal{F}| = n \geq 3$. Suppose for contradiction that $\mathcal{F}$ has an element $x \in [k]$ not present in sets $S_1, S_2 \in \mathcal{F}$.

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- Let $\mathcal{F}$ be a UC family with $\mathrm{h}(\mathcal{F}) = 2$ and $|\mathcal{F}| = n \geq 3$. Suppose for contradiction that $\mathcal{F}$ has an element $x \in [k]$ not present in sets $S_1, S_2 \in \mathcal{F}$.
- Since $\mathcal{F}$ is UC, then $S_1 \cup S_2 \in \mathcal{F}$. However, since $x \notin S_1 \cup S_2$, then $S_1 \cup S_2 \neq [k]$. This means we have a chain $S_1 \subseteq S_1 \cup S_2 \subseteq [k]$. This chain is length three, a contradiction. This means that $|\mathcal{F}_x| \geq |\mathcal{F}| - 1$, so x is abundant.

Height and Width Results

Theorem (Dilworth's Theorem)

Any poset of width w can be partitioned into w disjoint chains.

Lemma

Frankl's conjecture is true for UC families with $\mathrm{w}(\mathcal{F}) \leq 3$.

- Let $\mathcal{F}$ be a UC family with $\mathrm{w}(\mathcal{F}) = 1$. Every element in $[k]$ must be part of the same chain, so there exists some element present in every non-empty set of $\mathcal{F}$, so $\mathcal{F}$ satisfies Frankl.

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- Let $\mathcal{F}$ be a UC family with $\mathrm{w}(\mathcal{F}) = 2$. By Dilworth's theorem, $\mathcal{F}$ can be partitioned into two chains C_1 and C_2 with $|C_1| \geq |C_2|$. The chain C_1 will contain some element in all of its sets and is size at least $|\mathcal{F}|/2$, so $\mathcal{F}$ satisfies Frankl.

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- The $\mathrm{w}(\mathcal{F}) = 3$ case uses Dilworth's theorem and multiple cases.

The Width Conjecture

Conjecture (Width Conjecture)

For any UC family $\mathcal{F}$, $\max_{x \in [k]} |\mathcal{F}_x| \geq \mathrm{w}(\mathcal{F})$.

Conjecture (Minimal Sets Conjecture)

For any UC family $\mathcal{F}$ with $\mathcal{M}$ as its collection of minimal sets, $\max_{x \in [k]} |\mathcal{F}_x| \geq |\mathcal{M}|$.

Theorem

The Minimal Sets conjecture and the Width Conjecture are equivalent.

Structure Isomorphism

Definition

We say that two UC families $\mathcal{F}$ and $\mathcal{G}$ are *structure-isomorphic* if there exists some bijective function $\varphi : \mathcal{F} \to \mathcal{G}$ such that for all $S, T \in \mathcal{F}$, $\varphi(S \cup T) = \varphi(S) \cup \varphi(T)$ and $S \subseteq T$ in T if and only if $\varphi(S) \subseteq \varphi(T)$ in $\mathcal{G}$. We call such a φ a structure isomorphism.

Lemma

Structure isomorphisms preserve height and width.

- Let $\mathcal{F}$ and $\mathcal{F}'$ be structure isomorphic UC families with structure isomorphism φ. Suppose for contradiction that $\mathcal{F}$ has $\mathrm{w}(\mathcal{F}') < \mathrm{w}(\mathcal{F}) = \ell$ and $\{S_i : i \in [\ell]\}$ is a maximum anti-chain.

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- Since φ is a structure isomorphism, then $\varphi(S_i) \nsubseteq \varphi(S_j)$ for $i \neq j$ and $i, j \in [\ell]$. This means that $\{\varphi(S_i) : i \in [\ell]\}$ is an antichain in $\mathcal{F}'$, so $\mathrm{w}(\mathcal{F}') \geq \ell$, a contradiction. The proof is identical for height being preserved

Structure Isomorphism

Question

Do structure isomorphisms between UC families preserve being Frankl?

Question

Is it possible to uniquely identify (up to structure isomorphism) normal families with a small set of invariants? They aren't uniquely defined by just height, width, and the number of minimal sets. What else can we use to identify them?

Conclusion

- Thank you for coming to the seminar.
- Are there any questions?