Methods in Artificial Intelligence Homework 2

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1 Hidden Markov Model for the Umbrella World

In this section, we describe the umbrella world as a Hidden Markov Model (HMM).

1.1 Unobserved and Observed Variables

For a given time-slice t, we define:

- Our **unobserved variable** (represented as X_t in the book) is whether or not it is raining. We can call this variable R_t, and therefore have $X_t = R_t$, for a given day t.
- Our **observable variable** (represented as E_t in the book) is whether or not the director has an umbrella. We can therefore say $E_t = U_t$, where U_t represents whether the director carries an umbrella on day t.

1.2 Dynamic Model and Observation Model

The transition probabilities in the dynamic model $P(X_t \mid X_{t-1})$ are given as:

$$P(R_t \mid R_{t-1}) = \begin{bmatrix} P(R_t = \text{true} \mid R_{t-1} = \text{true}) & P(R_t = \text{false} \mid R_{t-1} = \text{true}) \\ P(R_t = \text{true} \mid R_{t-1} = \text{false}) & P(R_t = \text{false} \mid R_{t-1} = \text{false}) \end{bmatrix} = \begin{bmatrix} p & 1-p \\ q & 1-q \end{bmatrix} \tag{1}$$

where p and q represent the probabilities of rain persistence and rain onset, respectively. In page 482 of the book, these are defined as 0.7 and 0.3, so we can therefore use these and get:

$$P(R_t \mid R_{t-1}) = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} \tag{2}$$

The observation model $P(E_t \mid X_t)$ is given as:

$$P(U_t \mid R_t) = \begin{bmatrix} P(U_t = \text{true} \mid R_t = \text{true}) & P(U_t = \text{false} \mid R_t = \text{true}) \\ P(U_t = \text{true} \mid R_t = \text{false}) & P(U_t = \text{false} \mid R_t = \text{false}) \end{bmatrix} = \begin{bmatrix} r & 1-r \\ s & 1-s \end{bmatrix} \tag{3}$$

where r and s are the probabilities of the director carrying an umbrella given that it is raining or not raining, respectively. This is given in the book as 0.9 and 0.2. We therefore get:

$$P(U_t \mid R_t) = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \tag{4}$$

1.3 Assumptions in the Model

With the model, we have the following assumptions:

- The Markov assumption: The probability of rain on day t depends only on whether it rained on day $t-1$.
- The observation independence assumption: The presence of the umbrella on day t depends only on whether it is raining on that day and not on previous days.
- Stationarity: The transition probabilities (p and q) and observation probabilities (r and s) do not change over time.

2 Forward Algorithm for Hidden Markov Model

We consider a Hidden Markov Model (HMM) with the following parameters:

2.1 Transition Probabilities

The state transition probabilities are given as:

$$P(R_t = \text{true} \mid R_{t-1} = \text{true}) = 0.7$$
$$P(R_t = \text{false} \mid R_{t-1} = \text{true}) = 0.3$$
$$P(R_t = \text{true} \mid R_{t-1} = \text{false}) = 0.3$$
$$P(R_t = \text{false} \mid R_{t-1} = \text{false}) = 0.7$$

2.2 Observation Probabilities

The observation probabilities are defined as:

$$P(U_t = \text{true} \mid R_t = \text{true}) = 0.9$$
$$P(U_t = \text{false} \mid R_t = \text{true}) = 0.1$$
$$P(U_t = \text{true} \mid R_t = \text{false}) = 0.2$$
$$P(U_t = \text{false} \mid R_t = \text{false}) = 0.8$$

2.3 Observations

The observed values for U_t over five time steps are:

$$U_1 = \text{true}, U_2 = \text{true}, U_3 = \text{false}, U_4 = \text{true}, U_5 = \text{true}$$

3 Forward Algorithm Computation

We define the forward probabilities:

$$f_t(\text{true}) = P(U_1, U_2, \ldots, U_t, R_t = \text{true})$$
$$f_t(\text{false}) = P(U_1, U_2, \ldots, U_t, R_t = \text{false})$$

3.1 Initialization (Day 1)

Using uniform priors:

$$f_1(\text{true}) = P(R_1 = \text{true})P(U_1 = \text{true} \mid R_1 = \text{true}) = 0.5 \times 0.9 = 0.45$$
$$f_1(\text{false}) = P(R_1 = \text{false})P(U_1 = \text{true} \mid R_1 = \text{false}) = 0.5 \times 0.2 = 0.1$$

Normalization factor:

$$\alpha = f_1(\text{true}) + f_1(\text{false}) = 0.55$$
$$f_1(\text{true}) = \frac{0.45}{0.55} \approx 0.8182, \quad f_1(\text{false}) = \frac{0.1}{0.55} \approx 0.1818$$

3.2 Induction for Days 2 to 5

For each subsequent day t:

$$f_t(\text{true}) = [f_{t-1}(\text{true})P(R_t = \text{true} \mid R_{t-1} = \text{true}) + f_{t-1}(\text{false})P(R_t = \text{true} \mid R_{t-1} = \text{false})]\, P(U_t = \text{true} \mid R_t = \text{true})$$
$$f_t(\text{false}) = [f_{t-1}(\text{true})P(R_t = \text{false} \mid R_{t-1} = \text{true}) + f_{t-1}(\text{false})P(R_t = \text{false} \mid R_{t-1} = \text{false})]\, P(U_t = \text{true} \mid R_t = \text{false})$$

3.3 Automated Computation

I wrote a python script to compute these values, which gave the following results:

$$
\begin{aligned}
P(R_1 = \text{true}) &= 0.8182, \quad P(R_1 = \text{false}) = 0.1818 \\
P(R_2 = \text{true}) &= 0.8834, \quad P(R_2 = \text{false}) = 0.1166 \\
P(R_3 = \text{true}) &= 0.1907, \quad P(R_3 = \text{false}) = 0.8093 \\
P(R_4 = \text{true}) &= 0.7308, \quad P(R_4 = \text{false}) = 0.2692 \\
P(R_5 = \text{true}) &= 0.8673, \quad P(R_5 = \text{false}) = 0.1327
\end{aligned}
$$

3.4 Final Result

The final probability of rain on day 5 given the observations is:

$$P(R_5 = \text{true} \mid U_1, \ldots, U_5) = f_5(\text{true}) \approx 0.8673$$