Complexity Classes

DTM - Deterministic Turing Machine
NTM - Non-deterministic Turing Machine
PT - Probabilistic Turing Machine
PDTM - Deterministic Turing Machine *running in polynomial time*
PNTM - Non-deterministic Turing Machine *running in polynomial time*
PPT - Probabilistic Turing Machine *running in polynomial time*
n - Input length

Time Complexity Classes

$\text{DTIME}(f) = \{A \in \Sigma^* \mid \exists \text{ DTM } T : T \text{ decides } A \wedge T \text{ runs in } \mathcal{O}(f(n))\ \}$
$\text{NTIME}(f) = \{A \in \Sigma^* \mid \exists \text{ NTM } T : T \text{ decides } A \wedge T \text{ runs in } \mathcal{O}(f(n))\ \}$

$$P = \bigcup_{k \geq 1} \text{DTIME}(n^k) \qquad NP = \bigcup_{k \geq 1} \text{NTIME}(n^k)$$

$$E = \bigcup_{k \geq 1} \text{DTIME}(2^{kn}) \qquad NE = \bigcup_{k \geq 1} \text{NTIME}(2^{kn})$$

$$EXP = \bigcup_{k \geq 1} \text{DTIME}(2^{n^k}) \qquad NEXP = \bigcup_{k \geq 1} \text{NTIME}(2^{n^k})$$

Space Complexity Classes

$\mathsf{DSPACE}(s) = \{A \in \Sigma^* \mid \exists \text{ DTM } T : T \text{ decides } A \wedge T \text{ uses } \mathcal{O}(s(n)) \text{ space } \}$
$\mathsf{NSPACE}(s) = \{A \in \Sigma^* \mid \exists \text{ NTM } T : T \text{ decides } A \wedge T \text{ uses } \mathcal{O}(f(n)) \text{ space } \}$

$$\begin{aligned} LOGSPACE = L = &\quad \mathsf{DSPACE}(\log n) \\ NLOGSPACE = NL = &\quad \mathsf{NSPACE}(\log n) \\ LINSPACE = &\quad \mathsf{DSPACE}(n) \end{aligned}$$

$$PSPACE = \bigcup_{k \geq 1} \mathsf{DSPACE}(n^k) \quad = \quad NPSPACE = \bigcup_{k \geq 1} \mathsf{NSPACE}(n^k)$$
$$EXPSPACE = \bigcup_{k \geq 1} \mathsf{DSPACE}(2^{n^k}) \quad = \quad NEXPSPACE = \bigcup_{k \geq 1} \mathsf{NSPACE}(2^{n^k})$$

Polynomial Hierarchy

$$\Delta_0^P = \Sigma_0^P = \Pi_0^P = P$$
$$\begin{array}{lll} \Delta_{k+1}^P = P^{\Sigma_k^P} & \Delta_1^P = P & \Delta_2^P = P^{NP} \\ \Sigma_{k+1}^P = NP^{\Sigma_k^P} & \Sigma_1^P = NP & \Sigma_2^P = NP^{NP} \\ \Pi_{k+1}^P = co\text{–}NP^{\Sigma_k^P} & \Pi_1^P = co\text{–}NP & \Pi_2^P = co\text{–}NP^{NP} \end{array}$$

$$PH = \bigcup_{k \geq 0} \Delta_k^P = \bigcup_{k \geq 0} \Sigma_k^P = \bigcup_{k \geq 0} \Pi_k^P$$

Probabilistic Complexity Classes

$$
\begin{aligned}
P = \{A \in \Sigma^* \mid \exists \text{ PDTM } T: \quad & \forall x \in A : \mathsf{P}(T(x) = 1) = 1 \wedge \\
& \forall x \notin A : \mathsf{P}(T(x) = 1) = 0 \ \} \\
NP = \{A \in \Sigma^* \mid \exists \text{ PNTM } T: \quad & \forall x \in A : \mathsf{P}(T(x) = 1) > 0 \wedge \\
& \forall x \notin A : \mathsf{P}(T(x) = 1) = 0 \ \} \\
PP = \{A \in \Sigma^* \mid \exists \text{ PPT } T: \quad & \forall x \in A : \mathsf{P}(T(x) = 1) > 0.5 \wedge \\
& \forall x \notin A : \mathsf{P}(T(x) = 1) \leq 0.5 \ \} \\
BPP = \{A \in \Sigma^* \mid \exists \text{ PPT } T: \quad & \forall x \in A : \mathsf{P}(T(x) = 1) > 0.5 + \epsilon \wedge \\
& \forall x \notin A : \mathsf{P}(T(x) = 1) \leq 0.5 - \epsilon \ \} \\
RP = \{A \in \Sigma^* \mid \exists \text{ PPT } T: \quad & \forall x \in A : \mathsf{P}(T(x) = 1) > 0.5 \wedge \\
& \forall x \notin A : \mathsf{P}(T(x) = 1) = 0 \ \} \\
ZPP = \{A \in \Sigma^* \mid \exists \text{ PPT } T: \quad & \forall x \in A : \mathsf{P}(T(x) = 1) > 0.5 \wedge \\
& \qquad\qquad \mathsf{P}(T(x) = 0) = 0 \wedge \\
& \qquad\qquad \mathsf{P}(T(x) = ?\,) \leq 0.5 \wedge \\
& \forall x \notin A : \mathsf{P}(T(x) = 1) = 0 \wedge \\
& \qquad\qquad \mathsf{P}(T(x) = 0) > 0.5 \wedge \\
& \qquad\qquad \mathsf{P}(T(x) = ?\,) \leq 0.5 \ \}
\end{aligned}
$$

Complexity Classes Relations

Time Hierarchy Theorem

$P \subset E \subset EXP$

Borodin-Trakhtenbrot Gap-Theorem

$\forall$ total computable function g with $g(n) \geq n$:
$\exists$ total computable function f : $DTIME(f) = DTIME(g \circ f)$

Space Hierarchy Theorem

$L \subseteq NL \subset PSPACE \subset EXPSPACE$

$L \subseteq NL \subseteq P \subseteq NP \subseteq PH \subseteq PSPACE \subseteq EXP$

Savitch

$NSPACE(s(n)) \subseteq DSPACE(s(n)^2)$
(compare to $NTIME(f(n)) \subseteq DTIME(2^{\mathcal{O}(f(n))})$)

Immerman-Szelepcsnyi

$$s(n) \geq \log(n) \Rightarrow NSPACE(s(n)) = co\text{–}NSPACE(s(n))$$

Probabilistic World

$$\begin{array}{rcl}
P \subseteq ZPP \subseteq & RP \subseteq BPP & \subseteq PP \subseteq PSPACE \\
P \subseteq ZPP \subseteq & RP \subseteq NP & \subseteq PP \subseteq PSPACE \\
P \subseteq ZPP \subseteq & co\text{–}RP \subseteq co\text{–}NP & \subseteq PP \subseteq PSPACE
\end{array}$$

$$RP \cap co\text{–}RP = ZPP$$

Misc

$$DIP = PV = NP \subseteq PH \subseteq PSPACE$$

$$P = NP \iff PH = P$$

$$NP \neq co\text{–}NP \Rightarrow P \neq NP$$