Complexity Classes

DTM - Deterministic Turing Machine
NTM - Non-deterministic Turing Machine
PT - Probabilistic Turing Machine
PDTM - Deterministic Turing Machine *running in polynomial time*
PNTM - Non-deterministic Turing Machine *running in polynomial time*
PPT - Probabilistic Turing Machine *running in polynomial time*
n - Input length

Time Complexity Classes

$\mathsf{DTIME}(f) = \{A \in \Sigma^* \mid \exists \text{ DTM } T : T \text{ decides } A \wedge T \text{ runs in } \mathcal{O}(f(n))\ \}$
$\mathsf{NTIME}(f) = \{A \in \Sigma^* \mid \exists \text{ NTM } T : T \text{ decides } A \wedge T \text{ runs in } \mathcal{O}(f(n))\ \}$

$$P = \bigcup_{k \geq 1} \mathsf{DTIME}(n^k) \qquad NP = \bigcup_{k \geq 1} \mathsf{NTIME}(n^k)$$
$$E = \bigcup_{k \geq 1} \mathsf{DTIME}(2^{kn}) \qquad NE = \bigcup_{k \geq 1} \mathsf{NTIME}(2^{kn})$$
$$EXP = \bigcup_{k \geq 1} \mathsf{DTIME}(2^{n^k}) \qquad NEXP = \bigcup_{k \geq 1} \mathsf{NTIME}(2^{n^k})$$

Space Complexity Classes

$\text{DSPACE}(s) = \{A \in \Sigma^* \mid \exists \text{ DTM } T : T \text{ decides } A \wedge T \text{ uses } \mathcal{O}(s(n)) \text{ space }\}$
$\text{NSPACE}(s) = \{A \in \Sigma^* \mid \exists \text{ NTM } T : T \text{ decides } A \wedge T \text{ uses } \mathcal{O}(f(n)) \text{ space }\}$

$$
\begin{aligned}
LOGSPACE = L = \quad & \text{DSPACE}(\log n) \\
NLOGSPACE = NL = \quad & \text{NSPACE}(\log n) \\
LINSPACE = \quad & \text{DSPACE}(n)
\end{aligned}
$$

$$
\begin{aligned}
PSPACE = \bigcup_{k \geq 1} \text{DSPACE}(n^k) \quad &= \quad NPSPACE = \bigcup_{k \geq 1} \text{NSPACE}(n^k) \\
EXPSPACE = \bigcup_{k \geq 1} \text{DSPACE}(2^{n^k}) \quad &= \quad NEXPSPACE = \bigcup_{k \geq 1} \text{NSPACE}(2^{n^k})
\end{aligned}
$$

Polynomial Hierarchy

$$
\begin{aligned}
&\Delta_0^P = \Sigma_0^P = \Pi_0^P = P \\
&\qquad \Delta_{k+1}^P = P^{\Sigma_k^P} \qquad \Delta_1^P = P \qquad \Delta_2^P = P^{NP} \\
&\qquad \Sigma_{k+1}^P = NP^{\Sigma_k^P} \qquad \Sigma_1^P = NP \qquad \Sigma_2^P = NP^{NP} \\
&\qquad \Pi_{k+1}^P = co\text{–}NP^{\Sigma_k^P} \qquad \Pi_1^P = co\text{–}NP \qquad \Pi_2^P = co\text{–}NP^{NP}
\end{aligned}
$$

$PH = \bigcup_{k \geq 0} \Delta_k^P = \bigcup_{k \geq 0} \Sigma_k^P = \bigcup_{k \geq 0} \Pi_k^P$

Probabilistic Complexity Classes

$$
\begin{aligned}
P = \{A \in \Sigma^* \mid \exists \text{ PDTM } T: \quad & \forall x \in A : \mathsf{P}(T(x) = 1) = 1 \land \\
& \forall x \notin A : \mathsf{P}(T(x) = 1) = 0 \ \} \\
NP = \{A \in \Sigma^* \mid \exists \text{ PNTM } T: \quad & \forall x \in A : \mathsf{P}(T(x) = 1) > 0 \land \\
& \forall x \notin A : \mathsf{P}(T(x) = 1) = 0 \ \} \\
PP = \{A \in \Sigma^* \mid \exists \text{ PPT } T: \quad & \forall x \in A : \mathsf{P}(T(x) = 1) > 0.5 \land \\
& \forall x \notin A : \mathsf{P}(T(x) = 1) \leq 0.5 \ \} \\
BPP = \{A \in \Sigma^* \mid \exists \text{ PPT } T: \quad & \forall x \in A : \mathsf{P}(T(x) = 1) > 0.5 + \epsilon \land \\
& \forall x \notin A : \mathsf{P}(T(x) = 1) \leq 0.5 - \epsilon \ \} \\
RP = \{A \in \Sigma^* \mid \exists \text{ PPT } T: \quad & \forall x \in A : \mathsf{P}(T(x) = 1) > 0.5 \land \\
& \forall x \notin A : \mathsf{P}(T(x) = 1) = 0 \ \} \\
ZPP = \{A \in \Sigma^* \mid \exists \text{ PPT } T: \quad & \forall x \in A : \mathsf{P}(T(x) = 1) > 0.5 \land \\
& \qquad\quad\ \mathsf{P}(T(x) = ?\,) < 0.5 \land \\
& \forall x \notin A : \mathsf{P}(T(x) = 1) = 0 \land \\
& \qquad\quad\ \mathsf{P}(T(x) = ?\,) < 0.5 \ \}
\end{aligned}
$$

Relations between Complexity Classes

Time Hierarchy Theorem

$P \subset E \subset EXP$

Borodin-Trakhtenbrot Gap-Theorem

$\forall$ total computable function g with $g(n) \geq n$:
$\exists$ total computable function $f : DTIME(f) = DTIME(g \circ f)$

Space Hierarchy Theorem

$L \subseteq NL \subset PSPACE \subset EXPSPACE$

$L \subseteq NL \subseteq P \subseteq NP \subseteq PH \subseteq PSPACE \subseteq EXP$

Savitch

$NSPACE(s(n)) \subseteq DSPACE(s(n)^2)$
(compare to $NTIME(f(n)) \subseteq DTIME(2^{\mathcal{O}(f(n))})$)

Immerman-Szelepcsnyi

$s(n) \geq \log(n) \Rightarrow NSPACE(s(n)) = co\text{–}NSPACE(s(n))$

Probabilistic World

$$\begin{array}{rcl} P \subseteq \; ZPP \subseteq & RP \subseteq \; BPP & \subseteq PP \subseteq \; PSPACE \\ P \subseteq \; ZPP \subseteq & RP \subseteq \; NP & \subseteq PP \subseteq \; PSPACE \\ P \subseteq \; ZPP \subseteq & co\text{–}RP \subseteq \; co\text{–}NP & \subseteq PP \subseteq \; PSPACE \end{array}$$

$RP \cap co\text{–}RP = ZPP$

Misc

$DIP = PV = NP \subseteq PH \subseteq PSPACE$

$P = NP \iff PH = P$