

P231: Mathematical Methods in Graduate Physics

Flip Tanedo

flip.tanedo@ucr.edu

Department of Physics & Astronomy, University of California, Riverside, CA 92521

Abstract

This is a crash course on mathematical methods necessary to succeed in the first-year physics graduate curriculum at UC Riverside. The focus is how to solve differential equations using Green's functions. This version is revised for Fall 2021. Last Compiled: September 26, 2021

Contents

1	Introduction: Why mathematical methods?	3
1.1	Green's functions	4
1.2	This is not what I expected from a math methods course	4
1.3	The totally not-mathematical idea of mathematical niceness	5
1.4	Physics versus Mathematics	6
1.5	The most important binary relation	6
1.6	Units	6
2	Dimensional Analysis	7
2.1	Converting Units	7
2.2	Quantifying units	8
2.3	Usage: Sanity Check	8
2.4	Usage: Solving problems	9
2.5	Scaling	9
2.6	Error Estimates	10
2.7	Bonus: Allometry	12
3	Linear Algebra Review	13
3.1	The basics	13
3.2	Linear Transformations and Vector Spaces	15
3.3	A funny vector space: histogram space	15
3.4	Derivative Operators	17
3.5	Derivatives in other function space bases	18
3.6	Locality	19
3.7	Row Vectors and all that	20
3.8	Dual vectors as vector-eaters	21
3.9	Orthonormal Bases	22
3.10	Bra-Ket Notation	22
3.11	Eigenvectors are nice	23
3.12	Linearity of Inverse Operators	24
3.13	The Green's Function Problem	25
3.14	Remark: Implicit assumption of linearity	28

3.15 Metrics on Finite Dimensional Vector Spaces	28
3.16 The dot/inner product, orthonormality	29
3.17 Comment: Why physicists like indices	31
3.18 Adjoints and Hermitian Conjugates	32
4 Function Space	33
4.1 The Green's Function Problem in Function Space	33
4.2 Differential Operators	34
4.3 Inner Product	34
4.4 Dual Vectors	35
4.5 Adjoint operators	35
4.6 Completeness in Function Space	37
4.7 Orthonormality in Function Space	39
4.8 Completeness and Green's Functions	39
4.9 Green's Function by Completeness: why is this helpful?	40
4.10 Patching a Green's function together	42
4.11 Where we're going	44
5 Complex Analysis Review	47
5.1 Teaser: why are we doing this?	47
5.2 Complex Numbers	48
5.3 Complex functions	49
5.4 Analytic complex functions are nice	50
5.5 Analyticity is Differentiability	50
5.6 A geometric point of view (optional)	51
5.7 Analytic and harmonic	53
5.8 Complex functions as maps	53
5.9 Integration on the Complex Plane	58
5.10 Cauchy Integral Theorem	59
5.11 An alternative argument (optional)	62
5.12 Cauchy's Integral Formula	65
5.13 From Taylor to Laurent	68
5.14 The Residue Theorem: a first look	68
5.15 The Residue Theorem: more carefully	69
5.16 Non-simple poles (a case study in being careful)	70
5.17 The Killer App: Real Integrals	72
5.18 The Main Example: follow this carefully	76
5.19 A fancy example	78
6 The Harmonic Oscillator	79
6.1 Fourier Transform	79
6.2 Contour Integral	81
6.3 Going under the poles?	82
6.4 Going over the poles	87
6.5 Other choices? (Feynman Propagator)	88
6.6 Aside: Feynman Green's Function and Convergence of the Path Integral [optional] .	90
6.7 Example: The Damped Harmonic Oscillator	92

7 Green's functions in diverse dimensions	96
7.1 The ‘harmonic oscillator’ in spacetime	96
7.2 Green’s functions for multidimensional spaces	97
7.3 Relativistic Notation	98
7.4 Sketch of the Fourier integral in (3+1)-dimensions	101
8 Coupled Harmonic Oscillators and Fields	104
8.1 A notational interlude	105
8.2 Coupled Harmonic Oscillators	106
8.3 Interpretation	107
8.4 A theoretical digression	108
9 Green’s Functions, Statistics, and Gaussian Integrals	112
9.1 Quantum Mechanics: Green’s Function from the Path Integral	113
9.2 A probability refresher	115
9.3 If this were a different class	117
9.4 A cute trick: the Gaussian integral	117
9.5 Down the Gaussian rabbit hole	118
9.6 Correlation Functions	119
9.7 From Lattice to Field	122
9.8 Interpretation	124
10 Feynman Diagrams	125
11 Closing Thoughts	130
A Method of Variations	131
A.1 First order equation for coefficients	131
A.2 Integrating and boundary conditions	131
B Fourier Conventions	133
C Nuts and Bolts Probability	134
C.1 Were we lucky that the LHC didn’t destroy the world?	136
C.2 The Prosecutor’s Fallacy	136
C.3 Why we are not evidence for God	138
C.4 The End of the World	139
D Legendre Transform	140
D.1 Where you’ve seen this before	140

1 Introduction: Why mathematical methods?

Physics 231: Methods of Theoretical Physics is a course for first-year physics and astronomy graduate students. It is a ‘crash course’ in mathematical methods necessary for graduate courses in electrodynamics, quantum mechanics, and statistical mechanics. It is a *boot camp* rather than a rigorous theorem–proof mathematics class. Where possible, the emphasis is on physical intuition rather than mathematical precision.

1.1 Green's functions

Our primary goal is to solve linear differential equations:

$$\mathcal{O}f(x) = s(x) . \quad (1.1)$$

In this equation, \mathcal{O} is a *differential operator* that encodes some kind of physical dynamics¹, $s(x)$ is the *source* of those dynamics, and $f(x)$ is the system's physical *response* that we would like to determine. The solution to this equation is:

$$f(x) = \mathcal{O}^{-1}s(x) . \quad (1.2)$$

Simply writing that is deeply unsatisfying! In this course, we think carefully about what \mathcal{O}^{-1} actually *means* and how we can calculate it. As you may have guessed, \mathcal{O}^{-1} is the **Green's function** for the differential operator \mathcal{O} .

We approach problem by analogy to linear algebra, where a linear transformation² A acts on a vector to give equations like

$$A\mathbf{v} = \mathbf{w} , \quad (1.3)$$

whose solution is

$$\mathbf{v} = A^{-1}\mathbf{w} . \quad (1.4)$$

We connect the notion of a linear differential operator to a matrix in an infinite dimensional space to give a working definition of \mathcal{O}^{-1} . We then pull out a bag of tricks from complex analysis to actually solve $\mathcal{O}^{-1}s(x)$ given \mathcal{O} and $s(x)$.

1.2 This is not what I expected from a math methods course

This is a course in mathematical methods for *physicists*.³ We will not solve *every* class of differential equation that is likely to pop up in your research careers⁴—that's neither feasible nor particularly enjoyable. This is also not a course in formal proofs—there are plenty of excellent textbooks for you to learn those formal proofs to your heart's content⁵. The goal of this course is to weave together ideas that are not often connected explicitly in undergraduate physics courses in the United States: linear algebra, differential equations, complex analysis.

¹A **differential operator** is just something built out of derivatives that can act on a function. The differential operator may contain coefficients that depend on the variable that we are differentiating with respect to; for example, $\mathcal{O} = (d/dx)^2 + 3x(d/dx)$. Pop quiz: is this operator *linear*? The first term is squared...

²Recall that as physicists, ‘linear transformation’ is a fancy way of saying ‘matrix.’

³By ‘physicists’ I also include astronomers. To quote Prof. Becker during a faculty meeting, “Everything we do in the astronomy group is *astrophysics*.” While I do not quite understand the distinction between astronomy and astrophysics, let it be clear that the methods of theoretical physics we discuss here are equivalently methods of astronomy... er, astrophysics. This is of course all pedantry. What is really exciting is that these are the same methods of emerging fields, like machine learning and quantum information science. Your job as a young scientist is not to let yourself be limited by an artificial label.

⁴That would be a course on mathematical methods for *engineers*.

⁵... and as a graduate student, you should feel well equipped and encouraged to learn all of the necessary material *you* need for *your* research and interests—whether or not they show up in your coursework.

These ideas are not necessarily new—in fact, I *expect* you have seen many them often—but rather we will take a big view of how the interconnection of these ideas come up over and over again in our description of nature.

Do not be surprised if we only mention Bessel functions in passing. Do not think less of our efforts if we do not determine Wronksians or go beyond a single Riemann sheet. As graduate students, it is *your* responsibility to be able to grab your favorite textbook to apply mathematics as needed to your research. *This course* is about the larger narrative that is not often shared explicitly in those books. It is the ‘knack for math’ that physicists are, as a culture, rather proud of. It is what tends to make us employable in Silicon Valley while simultaneously terrible at splitting the bill at a restaurant.

1.3 The totally not-mathematical idea of mathematical niceness

I find it useful to appeal to the notion of a **nice** mathematical situation. This is not a formal idea, and it is one many things mathematicians find ridiculous about me. But as a physicist, the concept of mathematical *niceness* is helpful.

The physical systems that we spend the most time thinking about are all *nice*. While our mathematical cousins may spend years proving every exceptional case to a theorem, we tend to be happy to push onward as long as mathematical results are true for the *nice* cases. Nice mathematical models make tidy predictions. Then we can Taylor expand about these nice predictions to make better predictions. When doing this, we sometimes say *perturbation theory* multiple times in case someone watching us does not think are being rigorous enough.

This is not to say that nature cares at all about our physical models. Every once in a while, we *do* have to worry about the exceptional cases because our models fail to accommodate what is *actually* happening in nature. Those scenarios are the most interesting of all. That’s when our mathematical formalism grabs us by the collar and says, *listen to me—something important is happening and it probably has to do with nature!* This often happens when a calculation tells us that a physical result is infinite.

Exercise 1.1 Consider the potential that an electron feels in the hydrogen atom:

$$V(r) = -\frac{\alpha}{r} . \quad (1.5)$$

As the electron–proton separation goes to zero, $r \rightarrow 0$, the potential goes to infinity. Classical electrodynamics is telling us that something curious is happening. What actually happens? (And why didn’t you ask this question when you were in high school?)

In this course we focus on *nice* functions and *nice* operators and *nice* boundary conditions, etc. For the most part, this is what we need to make progress on our physical models and it’s worth spending our time learning to work with *nice* limits. Leave the degenerate cases to the mathematicians for now. Eventually, though, you may find yourself in a situation where physics demands *not nice* mathematics. In that case—and only when the physics demands it—you will be ready to poke and prod at the mathematical curiosity until the underlying *physics* reason for the not-niceness is apparent.

1.4 Physics versus Mathematics

Let's make one point clear:

$$\text{Physics} \neq \text{Mathematics} . \quad (1.6)$$

This is a truth in many different respects⁶:

- Physicists are rooted in experimental results⁷.
- Physicists Taylor expand to their hearts' content—sometimes even when the expansion is not formally justified⁸.
- Physicists use explicit coordinates, mathematicians abhor this. Even worse, we pick a basis and decorate every tensor with indices⁹.
- Physicists seek to uncover a truth about *this* universe.

1.5 The most important binary relation

When we write equations, the symbol that separates the left-hand side from the right-hand side is a binary relation. We use binary relations like $=$ or \neq . Sometimes to make a point we'll write \cong or \equiv or \doteq to mean something like ‘definition’ or ‘tautologically equivalent to’ or some other variant of *even more equal than equal*.

As physicists the most important binary relation is none of those things¹⁰. Usually what we really care about is in \sim .¹¹ This tells how something *scales*. If I double a quantity on the right-hand side, how does the quantity on the left-hand side scale? Does it depend linearly? Quadratically? Non-linearly? The answer encodes something important about the underlying physics of the system. It's the reason why *imagine the cow is a sphere* is a popular punchline in a joke about physicists.

By the way, implicit in this is the idea that in this class, we will not care about stray factors of 2. As my adviser used to say, if you're worried about a factor of 2, then your additional homework is to figure out that factor of 2.¹²

1.6 Units

There is another way in which physics is different from mathematics. It is far more prosaic. *Quantities in physics have units*. We don't just deal with numbers, we deal with kilograms, electron volts, meters. It turns out that dimensional analysis is a big part of what we do as physicists.

⁶The astronomer Fritz Zwicky would perhaps call this a *spherical truth*; no matter how you look at it, the statement is still true.

⁷Even theorists? *especially* theorists.

⁸<https://johncarlosbaez.wordpress.com/2016/09/21/struggles-with-the-continuum-part-6/>

⁹Those who are not trained may be intimidated by physics because of all the indices we use. Ironically, physicists are often intimidated by mathematics because of the conspicuous absence of any indices.

¹⁰I thank Yuval Grossman for emphasizing this. Unrelated: <https://xkcd.com/2343/>

¹¹I use this the same way as \propto , which is completely different from ‘approximately,’ \approx .

¹²That being said, you're reading these notes and find an error, do let me know about it.

2 Dimensional Analysis

You may be surprised how far you can go in physics by thinking deeply about dimensional analysis. Here we'll only get you started. To go one step further, you may read more about the Buckingham Pi theorem¹³ or dive into neat applications¹⁴.

2.1 Converting Units

Imagine that you have three apples. This is a number (three) and a unit (apple). The meaning of the unit depends on what you're using it to measure. For example, if apples are \$1 each, then you could use an apple as a unit of currency. The way to do this is to simply *multiply by one*:

$$(3 \text{ apples}) \times \left(\frac{\$1}{\text{apple}} \right) = \$3 . \quad (2.1)$$

We have used the fact that the exchange rate is simply the statement that

$$1 \text{ apple} = \$1 \quad \Rightarrow \quad 1 = \frac{\$1}{1 \text{ apple}} . \quad (2.2)$$

You can do a similar thing for [kilo-]calories or any other conversion rate.

All that matters is that the conversion is constant. Indeed, the constants of nature make very good ‘exchange rates.’ For example, in high-energy physics we like to use **natural units**. This is the curious statement that

$$\hbar = c = 1 . \quad (2.3)$$

At face value, this doesn’t make sense. \hbar has units of action, c is a speed, and 1 is dimensionless. However, because nature gives us a *fundamental* unit of action and a *fundamental* unit of speed, we may use them as conversion factors (exchange rates),

$$c = 3 \times 10^{10} \text{ cm/s} . \quad (2.4)$$

If $c = 1$, then this means

$$1 \text{ s} = 3 \times 10^{10} \text{ cm} . \quad (2.5)$$

This, in turn, connects a unit of time to a unit of distance. By measuring time, the constant c automatically gives us an associated distance. The physical relevance of the distance is tied to the nature of the fundamental constant: one second (or ‘light-second’) is the distance that a photon travels in one second. Observe that this only works because c is a constant.

¹³<https://aapt.scitation.org/doi/10.1119/1.1987069>

¹⁴<https://aapt.scitation.org/doi/full/10.1119/1.3535586>, <http://inspirehep.net/record/153032?ln=en>

2.2 Quantifying units

We use the notation that a physical quantity Q has **dimension** $[Q]$ that can be expressed in terms of units of length, mass, and time:

$$[Q] = L^a M^b T^c . \quad (2.6)$$

The dimension is the statement of the powers a , b , and c . You may want to also include units of, say, electric charge. Sticklers may pontificate about whether electric charge formally carries a new unit or not.

Example 2.1 *What are the units of force? We remember that $\mathbf{F} = m\mathbf{a}$, so*

$$[\mathbf{F}] = [m][\mathbf{a}] = M \times LT^{-2} = L^1 M^1 T^{-2} . \quad (2.7)$$

Life is even easier in **natural units**, where $c = 1$ means that units of length and time are ‘the same’ and $\hbar = 1$ means that units of time and energy (mass) are inversely related. In natural units, one typically write $[Q]$ to mean the mass-dimension of a quantity. To revert back to conventional units, one simply multiplies by appropriate factors of $1 = c$ and $1 = \hbar$.

Example 2.2 *What are the units of force in natural units? From (2.7) we multiply by one to convert length and time into mass dimensions:*

$$[\mathbf{F}] = [c^{-3}\hbar\mathbf{F}] = M^2 . \quad (2.8)$$

In natural units we say $[\mathbf{F}] = 2$. Recall that energy and mass have the same dimension, which you may recall from the Einstein relation $E^2 = m^2 c^4 + p^2 c^2$.

2.3 Usage: Sanity Check

The simplest use of dimensional analysis is to check your work. The following expression is obviously wrong:

$$1 + (3 \text{ cm}) . \quad (2.9)$$

This does not make sense. You cannot sum terms with different dimensions. Similarly, $\sin(3 \text{ cm})$ does not make sense. What about $e^{5 \text{ cm}}$? This doesn’t make sense because

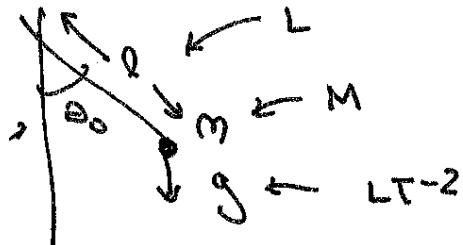
$$e^x = 1 + x + \frac{1}{2!}x^2 + \dots \quad (2.10)$$

Since each term comes with a different power of x , the argument of the exponential must be dimensionless.

Exercise 2.1 *Consider the energy spectrum of light emitted from some constant source—a distant star, the ongoing annihilation of dark matter in the galactic center, a laser in the Hemmerling lab. The spectrum encodes how many photons are emitted per unit time. We can plot this spectrum as a curve on a graph. We can even normalize the curve so that it integrates to one photon. This means we only care about the distribution of energy, not the absolute amount. The horizontal axis of such a plot is the photon energy. What are the units of the vertical axis?*

2.4 Usage: Solving problems

Here's a common problem in introductory physics. Assume you have a pendulum with some [sufficiently small] initial displacement θ_0 . What's the period, τ of the pendulum? We draw a picture like this:



From dimensional analysis, we know that the period has dimensions of time, $[\tau] = T$. The problem gives us a length $[\ell] = L$ and the gravitational acceleration, $[g] = LT^{-2}$. Note that $[\theta_0] = 1$ is dimensionless. This means that the only way to form a quantity with dimensions of time is to use $g^{-1/2}$. This leaves us with a leftover $L^{-1/2}$, which we can fix by inserting a square root of ℓ :

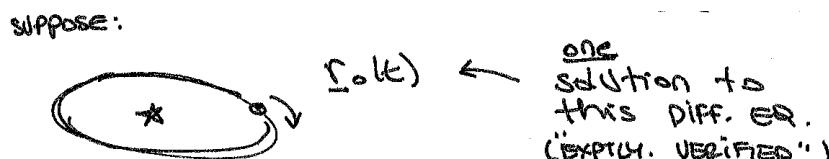
$$\tau \sim g^{-1/2} \ell^{1/2}. \quad (2.11)$$

If we wanted to be fancy, we can make this an equal sign by writing a function of the other dimensionless quantities in the problem:

$$\tau = f(\theta_0) \sqrt{\frac{\ell}{g}}. \quad (2.12)$$

2.5 Scaling

A large part of physics has to do with scaling relations. Here's a somewhat contrived example of how this works¹⁵. Suppose you have some static, central potential $U(\mathbf{r})$. Maybe it's some planet orbiting a star.



The force law gives:

$$m\ddot{\mathbf{r}} = -\frac{\partial U}{\partial \mathbf{r}}. \quad (2.13)$$

¹⁵This is adapted from section 11 of V. I. Arnold's *Mathematical Methods of Classical Mechanics*, one of my favorite differential geometry textbooks because it's disguised as a book on mechanics.

Suppose we are given a solution, $\mathbf{r}_0(t)$. Perhaps this is a trajectory that is experimentally verified. Dimensional analysis gives a way to scale this solution into other solutions. For example, let us scale time by defining a new variable t' :

$$t \equiv \alpha t' . \quad (2.14)$$

If the potential is static, then only the left-hand side of the force law changes. Even though the right-hand side formally has dimensions of time $\sim T^{-2}$, it does not transform because those units are carried in a constant, perhaps G_N , not a $(d/dt)^2$ like the left-hand side. The left-hand side of the force law gives:

$$m \left(\frac{d}{dt} \right)^2 \mathbf{r}_0(t) = m\alpha^{-2} \left(\frac{d}{dt'} \right)^2 \mathbf{r}_0(\alpha t') . \quad (2.15)$$

This begs us to define a new mass $m' = m\alpha^{-2}$. We thus have

$$m' \left(\frac{d}{dt'} \right)^2 \mathbf{r}_0(\alpha t') = -\frac{\partial U}{\partial \mathbf{r}_0} . \quad (2.16)$$

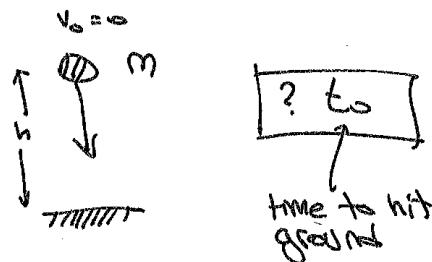
What this tells us is that $\mathbf{r}_1(t') \equiv \mathbf{r}_0(\alpha t')$ is a solution in the same potential that traces the same trajectory but at α times the speed and with mass m' . Changing labels $t' \rightarrow t$ for a direct comparison:

$$m' \left(\frac{d}{dt} \right)^2 \mathbf{r}_1(t) = -\frac{\partial U}{\partial \mathbf{r}_1} , \quad (2.17)$$

which is indeed¹⁶ (2.13) with a new mass m' and a trajectory $\mathbf{r}_1(t') \equiv \mathbf{r}_0(\alpha t')$. For example, if $\alpha = 2$, then $\mathbf{r}_1(t)$ traces the same trajectory at double the velocity with one fourth of the mass.

2.6 Error Estimates

This section is based on a lovely *American Journal of Physics* article by Craig Bohren.¹⁷ Let's go back to another high school physics problem.



¹⁶We were able to swap \mathbf{r}_0 with \mathbf{r}_1 simply because U only depends on the position.

¹⁷<https://doi.org/10.1119/1.1574042>

Suppose you drop a mass m from height h that is initially at rest. How long before this hits the ground? You can integrate the force equation to get

$$t_0 = \sqrt{\frac{2h}{g}} . \quad (2.18)$$

This is the *exact* answer *within our model* of the system. The model made several assumptions. The mass is a point mass, the gravitational acceleration is constant at all positions, there is no air resistance, etc. In fact, we *know* that if we do an experiment, our result will almost certainly *not* be t_0 . All we know is that t_0 is probably a good approximation of what the actual answer is.

How good of an approximation is it?

One way to do this is to do the next-to-leading order ('NLO') calculation, taking into account a more realistic (and hence more complicated) model and then compare to t_0 . But this is stupid. Why do we need to do a *hard* calculation to justify doing an *easy* one? If we're going to do the hard calculation anyway, what's the point of ever doing the easy one?

What we really want is an error estimate. The error is

$$\epsilon = \frac{t_1 - t_0}{t_0} . \quad (2.19)$$

This is a dimensionless quantity that determines how far off t_0 is from a more realistic calculation, t_1 . Ideally we don't actually have to do work to get t_1 .

Let's assume that we're not completely nuts and that we're in a regime where the error is small¹⁸. Then the error is a function of some dimensionless parameters, ξ , in the system. We define these ξ so that as $\xi \rightarrow 0$, $\epsilon(\xi) \rightarrow 0$. In other words, the approximation gets better as the ξ are made smaller. By Taylor expansion:

$$\epsilon(\xi) = \epsilon(0) + \epsilon'(0)\xi + \mathcal{O}(\xi^2) . \quad (2.20)$$

By assumption $\epsilon(0) = 0$ and $\mathcal{O}(\xi^2)$ is small. We can then make a reasonable *assumption* that the dimensionless value $\epsilon'(0)$ is $\mathcal{O}(1)$. This tells us that the error goes like $\epsilon(\xi) \sim \xi$.

By the way $\mathcal{O}(1)$ is read "order one" and is fancy notation for the order of magnitude. Numbers like 0.6, 2, and π are all $\mathcal{O}(1)$. A number like $4\pi^2$, on the other hand, is $\mathcal{O}(10)$. The assumption that a dimensionless number is $\mathcal{O}(1)$ is reasonable. When nature gives you a dimensionless parameter that is both (a) important and (b) very different from $\mathcal{O}(1)$, then there's a good chance that it's trying to tell you something about your model. Good examples of this are the cosmological constant, the strong CP phase, and the electroweak hierarchy problem¹⁹.

¹⁸Note the error has to be dimensionless in order for us to be able to call it 'small,' otherwise it begs the question of 'small with respect to what'.

¹⁹There are also 'bad' examples. The ratio of the angular size of the moon to the angular size of the sun is unity to very good approximation. This is quite certainly a coincidence. Our universe appears to be in an epoch where the density of matter, radiation, and dark energy all happen to be in the same ballpark. Our cosmological models imply that this is purely a coincidence. It would be very curious if this were not the case. As an exercise, you can explore (and critique) the appearance of the anthropic principle in physics.

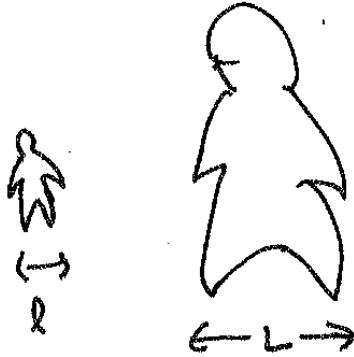
Here's how it works in practice. One effect that we miss in our toy calculation of t_0 is that the earth is round with radius R . This means that assuming a constant g is an approximation. We have two choices for a dimensionless parameter ξ :

$$\xi = \frac{h}{R} \quad \text{or} \quad \xi = \frac{R}{h} . \quad (2.21)$$

There is an obvious choice: $\xi = h/R$, because we know that as h is made smaller (drop the ball closer to the ground) or R becomes bigger (larger radius of Earth) then the constant g approximation gets better. We thus expect that the corrections from the position-dependence of g go like $\mathcal{O}(h/R)$.

2.7 Bonus: Allometry

There's a fun topic called **allometry**. This is basically dimensional analysis applied to biology. A typical example is to consider two people who have roughly the same shape but different characteristic lengths, ℓ and L :



Exercise 2.2 *If both people exercised at the same rate, which one loses more absolute weight? By how much? Let's assume that weight loss is primarily from the conversion of organic molecules into carbon dioxide.*

Exercise 2.3 *David Hu won his first IgNobel prize for determining that mammals take about 21 seconds to urinate, largely independently of their size²⁰. Can you use dimensional analysis to argue why this would be the case? It may be helpful to refer to the paper²¹; as you read this, figure out which terms are negligible (and in what limits), identify the assumptions of the mathematical model (scaling of the bladder and urethra), and prove the approximate scaling relation. Make a note to yourself of which steps were non-trivial and where one may have naively mis-modeled the system. By the way, David Hu won a second IgNobel prize for understanding how wombats poop.*

²⁰I learned about this in his excellent popular science book, *How To Walk on Water and Climb Up Walls*.

²¹<https://doi.org/10.1073/pnas.1402289111>

3 Linear Algebra Review

As physicists, linear algebra is part of our DNA, from the vector calculus in our first electrodynamics course to quantum mechanics. So why should we patronize ourselves with yet another review of linear algebra? We want to understand Green's functions the inverse of a matrix. The ‘matrix’ in question is the differential operator \mathcal{O} in (1.1). This is important:

$$\text{differential operator} = \infty\text{-dimensional matrix} . \quad (3.1)$$

If differential operators are matrices, what vector space do these matrices act on? These matrices act on a space of functions, which turns out to be a vector space:

$$\text{function space} = \infty\text{-dimensional vector space} . \quad (3.2)$$

Don’t be intimidated by terminology like *function space*; this is just an abstract place where functions live. Just recall back to your intuition from 3D Euclidean vector space, \mathbb{R}^3 : any 3-vector \mathbf{v} lives in the vector space \mathbb{R}^3 . If we transform \mathbf{v} by a linear transformation A , you get a new vector $\mathbf{w} = A\mathbf{v} \in \mathbb{R}^3$ that is also in the vector space.

Weird things can happen when we extend our intuition from finite things to infinite things²², but for this course we’ll try to draw as much intuition as we can from finite dimensional linear algebra to apply it to infinite dimensional function spaces.

3.1 The basics

A **linear transformation** A acts on a vector \mathbf{v} as $A\mathbf{v}$. This transformation satisfies

$$A(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha A\mathbf{v} + \beta A\mathbf{w} . \quad (3.3)$$

Here α and β are numbers. This is conventionally matrix multiplication. The result is also a vector. One way that we like to think about vectors is as columns of elements:

$$\begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^N \end{pmatrix} , \quad (3.4)$$

where N is the **dimension** of the vector space. Our notation is that v^i refers to the i^{th} component of \mathbf{v} . Sometimes—as physicists—we refer to v^i as the vector itself, which is a slight abuse of notation that occasionally causes confusion.

In this course we always assume a nice orthonormal basis. In this case, $(\mathbf{v} + \mathbf{w})^i = v^i + w^i$.

Exercise 3.1 *Convince yourself that adding vectors becomes more complicated in polar coordinates. Namely, $(\mathbf{v} + \mathbf{w})^i \neq v^i + w^i$.*

²²For example, the Hilbert Hotel puzzle.

Because the linear transformation of a vector is another vector, we know that the sequential application of linear transformations is itself a linear transformation. This is a bombastic way of saying that you can multiply matrices to produce a matrix. Here's how it works in two dimensions. A transformation that takes vectors into vectors takes the following form:

$$A = \begin{pmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{pmatrix}. \quad (3.5)$$

We have introduced upper and lower indices; for now treat this as a definition. This sometimes causes confusion. So here are some guidelines:

- Treat the upper and lower indices as a definition. The components of the linear transformation A are *defined* by A^i_j , where i is the row number and j is the column number.
- We have not yet explained the significance of the heights, but for now we mandate that the first index is always upper and the lower index is always lower. The following objects do not (yet) make sense: A^1_2 and *not* A_{12} , A^{12} , or A_1^2 .
- We will soon define *additional machinery* to raise and lower indices shortly. This is takes us from a vector space to a metric space.
- The heights of the indices are a convenient shorthand notation that we will elucidate shortly; it is related to the choice of upper indices in (3.4).
- All of this may be familiar from special relativity. Extra credit if you realize that this should also be familiar from quantum mechanics.

If you're squeamish about the indices, don't worry: the elements of A have two indices, the first one is written a little higher than the second one. This notation is neither mathematics nor physics, it's a convention that we use for future convenience.

The action of a linear transformation A on a vector \mathbf{v} is:

$$A\mathbf{v} = \begin{pmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} A^1_1 v^1 + A^1_2 v^2 \\ A^2_1 v^1 + A^2_2 v^2 \end{pmatrix}. \quad (3.6)$$

Look at this carefully. The components of the new vector $(A\mathbf{v})^i$ are sums. In each term, the second/lower index of an A element multiplies the component of \mathbf{v} with the same index. The first/upper index of A tells you whether that term should be in $(A\mathbf{v})^1$ or $(A\mathbf{v})^2$.

A generic component of $(A\mathbf{v})$ is

$$(A\mathbf{v})^i = \sum_j A^i_j v^j = A^i_j v^j \quad (\text{Einstein convention}). \quad (3.7)$$

On the right-hand side we use Einstein notation: *we implicitly sum over repeated upper/lower indices*. We will use this notation from now on. If you are at all in doubt about this, please work out the 2×2 case carefully and compare to the succinct notation above.

Exercise 3.2 Consider three-dimensional Euclidean space, \mathbb{R}^3 . A linear transformation A on this space is a 3×3 matrix with elements of the form A^i_j . Explicitly write out the second component of the vector $A\mathbf{v}$. This is a sum of three terms.

If A and B are linear transformations, then $A + B$ is a linear transformation. The components of $A + B$ are simply the piecewise sum of the corresponding components of A and B :

$$(A + B)^i_j = A^i_j + B^i_j. \quad (3.8)$$

3.2 Linear Transformations and Vector Spaces

Let's be a little more pedantic. We need to move past the idea that a vector \mathbf{v} is some *column of numbers*. A vector space is abstract and we need to start thinking of vector spaces more generally. The layer of abstraction is encoded in the basis vectors, which we write as $\mathbf{e}_{(i)}$. For a space of dimension N , there are N such vectors indexed by the subscript. Let us more formally write the vector \mathbf{v} as

$$\mathbf{v} = v^1 \mathbf{e}_{(1)} + v^2 \mathbf{e}_{(2)} + \cdots + v^i \mathbf{e}_{(i)} . \quad (3.9)$$

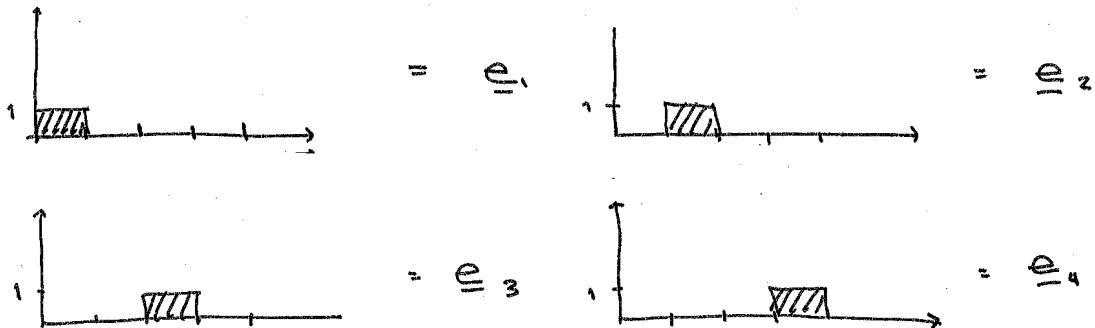
These basis vectors may be unit vectors in space. In the ‘column of numbers’ representation, they can be unit column vectors, e.g.

$$\mathbf{e}_{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{e}_{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \dots \quad (3.10)$$

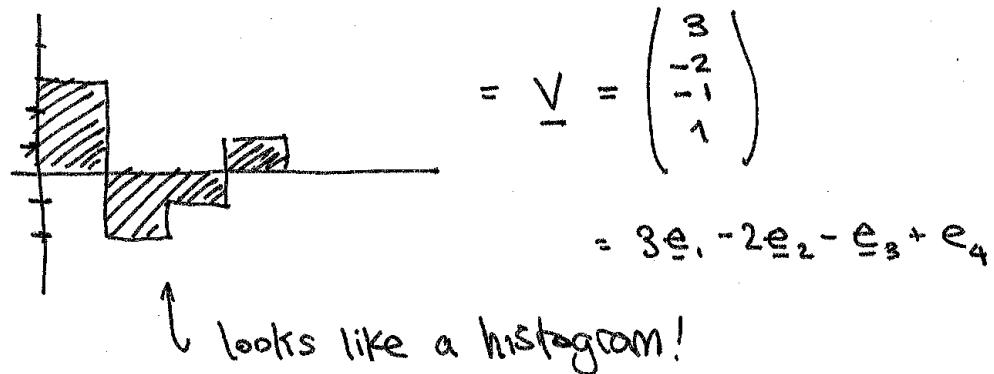
With this basis, (3.9) gives (3.4). But these may be more general objects. For example, you can specify a color of light by specifying the red/green/blue content. We could have $\mathbf{e}_{(1)}$ be a unit amount of red light, $\mathbf{e}_{(2)}$ be a unit amount of green light, and $\mathbf{e}_{(3)}$ be a unit amount of blue light. Then a 3-vector \mathbf{v} would correspond to light of a particular color. This color space is a vector space.

3.3 A funny vector space: histogram space

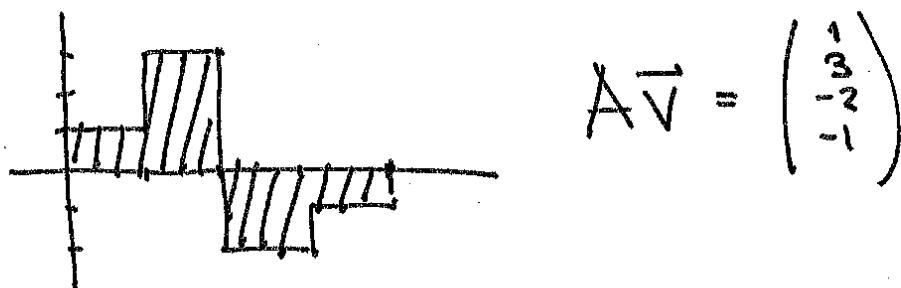
Here's a funny vector space that we're going to use as a pedagogical crutch. Imagine histogram-space. The basis vectors are:



This is a basis for a histogram over unit bins from $x = 0$ to $x = 4$. A vector in this space is, for example:



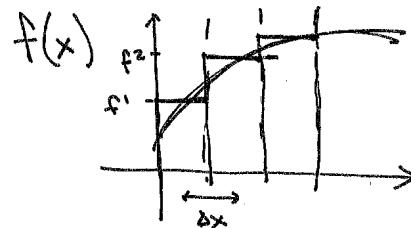
We can perform a linear transformation A on v which outputs another vector. Let's say it's this:



Exercise 3.3 From the image above, can you derive what A is?

The answer to the above exercise is *no*. Please make sure you convince yourself why: there are many different transformations that convert one histogram into the new histogram. If you're not convinced: the matrix A is 4×4 and thus has 16 entries that we need to define. The matrix equation $Av = w$ for known vectors v and w encodes only four equations.

The power of this admittedly strange formalism is that we can think of these histograms as approximations of continuous functions:



Thus a vector in this approximate (discretized) *function* space is

$$\mathbf{f} = \begin{pmatrix} f^1 \\ f^2 \\ \vdots \\ f^N \end{pmatrix}. \quad (3.11)$$

3.4 Derivative Operators

Our discretized function space allows us to define a [forward] derivative²³:

$$\mathbf{f}' = \frac{1}{\Delta x} \begin{pmatrix} f^2 - f^1 \\ f^3 - f^2 \\ \vdots \\ f^{i+1} - f^i \\ \vdots \end{pmatrix}. \quad (3.12)$$

This is familiar if you've ever had to manually program a derivative into a computer program. Note that the right-hand side looks like a linear transformation of \mathbf{f} . In other words, we expect to be able to write a matrix D so that

$$\mathbf{f}' = D\mathbf{f}. \quad (3.13)$$

One problem is apparent: what happens at the ‘bottom’ of the vector? What is the last component of the derivative, \mathbf{f}'^N ? Formally, this is

$$(\mathbf{f}')^N = \frac{1}{\Delta x} (f^{N+1} - f^N) \quad (3.14)$$

but now we have no idea what f^{N+1} is. That was never a component in our vector space. There is no $e_{(N+1)}$ basis vector. This demonstrates an important lesson that we'll need when we move more formally to function spaces:

Boundary conditions are part of the definition of the function space.

That was so important that I put the whole damn sentence in boldface and set it in the middle of the line. The significance of boundary conditions may be a bit surprising—but think of this as part of the definition of which functions we allow into our function space.

Example 3.1 *When one first learns about Fourier series with Dirichlet boundary conditions, one finds that the Fourier expansion only contains sines. The solution to the wave equation in such a system is some function that is zero at each endpoint. So the function space relevant to the system is composed only of functions that are zero at each endpoint.*

For now let's assume **Dirichlet boundary conditions**. A convenient way to impose this is to define what happens to all functions outside the domain of the function space:

$$f^{i>N} = f^{i<1} = 0. \quad (3.15)$$

²³One could have also defined a backward derivative where $(f')^i \sim f^i - f^{i-1}$. Note that you *cannot* try to make this symmetric by defining a ‘centered’ derivative like $(f')^i \sim f^{i+1/2} - f^{i-1/2}$ because there's no such thing as a fractional index. If you tried to write $(f')^i \sim f^{i+1} - f^{i-1}$ you're making a worse approximation. If you're like me, the fact that there's some asymmetry in how we define the first derivative is deeply unsettling. There's something to this intuition!

This solves the problem of the derivative on the last component:

$$(f')^N = \frac{1}{\Delta x} (f^{N+1} - f^N) = \frac{-f^N}{\Delta x} . \quad (3.16)$$

Alternatively, we could have also imposed **periodic boundary conditions**:

$$f^i = f^{i+kN} \quad k \in \mathbb{Z} . \quad (3.17)$$

This would then give

$$(f')^N = \frac{1}{\Delta x} (f^{N+1} - f^N) = \frac{1}{\Delta x} (f^1 - f^N) . \quad (3.18)$$

Periodic boundary conditions amount to wrapping the x -axis into a circle. Older folks sometimes call this *Asteroids* boundary conditions. I'd also accept *Star Control* boundary conditions. Periodic boundary conditions show up *all* the time in physics. Sometimes they show up in obvious places, like the Brillouin zone of a crystal lattice. Other times they show up in not-so-obvious places like the boundary conditions of the known universe. In addition to being crucial for a well-defined function space, the boundary conditions of a system establish its topology²⁴.

Exercise 3.4 *We don't know anything about the universe outside the Hubble radius. Why do you think it would be reasonable in a physical model to assume that it has periodic boundary conditions? Hint: what would happen to the x -momentum of an asteroid in the classic arcade game Asteroids if the game did not have periodic boundary conditions?*

The second derivative may be defined symmetrically:

$$(f'')^i = \frac{(f^{i+1} - f^i) - (f^i - f^{i-1})}{\Delta x^2} . \quad (3.19)$$

You may pontificate about the reason why the first derivative does have a symmetric discretization while the second derivative does.

3.5 Derivatives in other function space bases

There are other ways to write a discrete basis of functions. Here's a natural one for functions that are up to second-order polynomials:

$$\mathbf{e}_{(0)} = 1 \quad \mathbf{e}_{(1)} = x \quad \mathbf{e}_{(2)} = x^2 . \quad (3.20)$$

²⁴I cannot over-emphasize the importance of topology in contemporary physics. Most of the physics you will learn in your first year graduate courses are intrinsically *local* because the laws of physics are causal. Topological quantities are *global*, they are integrals over an entire space. Because winding numbers (and their higher-dimensional cousins) are quantized, they are robust against perturbations. The number of holes in a donut is one, whether or not it's been slightly squished in the box. By the way, the best donuts in Southern California are from *Sidecar Doughnuts* in Costa Mesa. Get the Basil Eggs Benedict donut before 11am; you can thank me later.

Let's sidestep questions about orthonormality for the moment. Clearly linear combinations of these basis functions can produce any quadratic function:

$$f(x) = ax^2 + bx + c \quad \Rightarrow \quad \mathbf{f} = \begin{pmatrix} c \\ b \\ a \end{pmatrix}. \quad (3.21)$$

The derivative operator has an easy representation in this space:

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.22)$$

We can see that

$$D\mathbf{f} = \begin{pmatrix} b \\ 2a \\ 0 \end{pmatrix} \quad D^2\mathbf{f} = \begin{pmatrix} 2a \\ 0 \\ 0 \end{pmatrix} \quad D^3\mathbf{f} = 0. \quad (3.23)$$

The last line is, of course, the realization that the third-derivative of a quadratic function vanishes. Feel free to attach mathy words to this like *kernel*.

There are other bases that we may use for function space. A particularly nice one that we will use over and over is the Fourier basis, which we usually refer to as *momentum space*. The basis vectors are things like sines, cosines, or oscillating exponentials. These do not vanish for any power of D .

3.6 Locality

Notice that in the histogram basis, the derivative matrix D is sparse: it is zero everywhere away from the diagonal. The only non-zero elements on the i^{th} row are around the $(i \pm 1)^{\text{th}}$ column. Higher powers of D sample further away, but the non-zero elements are always clustered near the diagonal.

This is simply a notion of **locality**. Remember the Taylor expansion:

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \dots. \quad (3.24)$$

If we think about the histogram as a discretization of a continuous function, then it is clear what the higher derivatives are doing. Given a function $f(x) = \mathbf{f}$, one might like to know about the function around some point x_0 corresponding to some index i . That is: $f^i = f(x_0)$. If you'd like to learn more about the function around that point, one can express the derivative at x_0 . Thus $D\mathbf{f}$ says something about the slow, $D^2\mathbf{f}$ says something about the curvature, and so on. Because each successive power of D samples terms further away from f^i , you can tell that these higher order terms are learning about the function further and further away from x_0 .

Now think about the types of differential equations that you've encountered in physics. They often include one or two derivatives. You hardly ever see three, four, or more derivatives²⁵. There's a reason for this: at the scales that we can access experimentally, nature appears to be local. Our mathematical models of nature typically have locality built in²⁶.

²⁵With some thought, it may also be clear why spatial derivatives typically appear squared.

²⁶A recent counterexample: <https://www.quantamagazine.org/physicists-discover-geometry-underlying-particle/>

Physics at one spacetime point should not depend on spacetime points that are far away.

This may be familiar from the idea of causality—the idea that A causes B therefore A must have happened *before* B . One of the key results in special relativity is that causality can be tricky if two events do not occur at the same spacetime point. More carefully, A can only cause B if there is a timelike separation of the appropriate sign. If we want to build causal theories of nature, then the dynamics at x_0 should not rely on what is happening at x_1 , a finite distance away.²⁷

3.7 Row Vectors and all that

In high school we did not distinguish between row vectors and column vectors. They both seemed to convey the same information—they were simply one-dimensional arrays of numbers. Row vectors are just ‘tipped over.’ Such a tipping-over is convenient since you could apply the elementary schools rules of ‘matrix multiplication’ have the row vector act on a column vector:

$$(w_1 \ w_2 \ \dots) \begin{pmatrix} v^1 \\ v^2 \\ \vdots \end{pmatrix} = w_1 v^1 + w_2 v^2 + \dots . \quad (3.25)$$

In fact, this is like \mathbf{w}^T is a function that acts *linearly* on its argument, \mathbf{v} :

$$\mathbf{w}^T(\mathbf{v}) = w_1 v^1 + w_2 v^2 + \dots . \quad (3.26)$$

Perhaps you see why we wrote the row vector components with lower indices, w_i so that we may use Einstein summation notation: $\mathbf{w}^T \mathbf{v} = w_i v^i$.

Indeed, let us be a bit more formal about this. This layer of formalism is uncharacteristic of our approach in this course, but this underpins so much of the mathematical structure of our physical theories that it is worth getting right from the beginning. Let V be a vector space. It contains vectors, \mathbf{v} . Sometimes these are called contravariant vectors or kets. They have basis vectors $\mathbf{e}_{(i)}$.

Now introduce a related but *completely distinct* vector space called V^* . This is the space of **dual vectors** to V . A **dual vector** is what you may know as a **row vector**, a **co-vector** a **ket**, a **covariant vector**, or a [differential] **one-form**. These are all words for the *same idea*. A dual vector, say (\mathbf{w}^T) is a *linear function that takes vectors and spits out numbers*:

$$\mathbf{w}^T \in V^* \Rightarrow \mathbf{w}^T : V \rightarrow \mathbb{R} . \quad (3.27)$$

Don’t think about \mathbf{w}^T as some kind of operation on a vector $\mathbf{w} \in V$; at least not yet. For now the ‘ T ’ is just part of the name of \mathbf{w}^T . The two spaces V and V^* are totally different. We haven’t said anything about how to turn elements of V into elements of V^* or vice versa. It should be clear that there is a sense of ‘duality’ here: the vectors V are also linear functions that take a dual vector and spit out a number.

²⁷This is different from saying that information cannot propagate from x_0 to x_1 ; such propagation could come from some causal excitation of the electromagnetic field traveling every infinitesimal distance between the two positions. This is reminiscent of the classical Zeno’s paradox.

Let us call the basis of dual vectors $\tilde{\mathbf{e}}^{(i)}$. This notation is cumbersome, so we'll change to something different soon. The upper index is deliberate. The defining property of $\tilde{\mathbf{e}}^{(i)}$ is:

$$\tilde{\mathbf{e}}^{(i)} (\mathbf{e}_{(j)}) \equiv \mathbf{e}_{(j)} (\tilde{\mathbf{e}}^{(i)}) \equiv \delta_j^i. \quad (3.28)$$

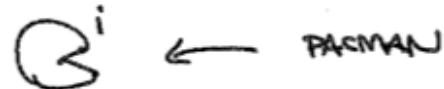
One may check that this gives

$$(w_i \tilde{\mathbf{e}}^{(i)}) (v^j \mathbf{e}_{(j)}) = w_i v^j \delta_j^i = w_i v^i = w_1 v^1 + w_2 v^2 + \dots . \quad (3.29)$$

All that we've done here is defined basis vectors that carry the intrinsic *vector-ness* or *dual-vector-ness* through their relations (3.28). We have ‘derived’ the contraction of a lower-index object with an upper-index object, and hence our summation convention, in terms of these basis vectors.

3.8 Dual vectors as vector-eaters

It is perhaps useful to use a slightly different notation based on Pac-Man. Rather than writing $\tilde{\mathbf{e}}^{(i)}$, lets write the basis dual vectors as



In this notation, the action of a basis dual vector on a basis vector is simply Pac-Man eating the basis vectors:

$$\begin{aligned} \textcircled{B}^i \underline{\mathbf{e}}_{(1)} &= \delta_1^i; \\ \textcircled{B}^1 \underline{\mathbf{e}}_{(2)} &= 0 \\ \textcircled{B}^s \underline{\mathbf{e}}_{(s)} &= 1 \end{aligned}$$

So we can write (3.26) and (3.29) as

$$\begin{aligned}
 \underline{\underline{w}}^T \underline{\underline{v}} &= (w_1 \underline{\underline{e}}^1 + w_2 \underline{\underline{e}}^2 + \dots)(v^1 \underline{\underline{e}}_1 + \dots) \\
 (\text{no dot!} \quad \text{this is } \underline{\underline{w}} \cdot \underline{\underline{v}}) \\
 &= w_1 v^1 \underbrace{\underline{\underline{e}}^1 \underline{\underline{e}}_1}_{=1} + w_2 v^2 \underbrace{\underline{\underline{e}}^2 \underline{\underline{e}}_2}_{=1} + \dots \\
 &\quad + w_1 v^1 \underbrace{\underline{\underline{e}}^1 \underline{\underline{e}}_1}_{=0} + w_2 v^2 \underbrace{\underline{\underline{e}}^2 \underline{\underline{e}}_2}_{=1} + \dots \\
 &= w_1 v^1 + w_2 v^2 + \dots
 \end{aligned}$$

$$\text{so we get } \underline{\underline{w}}^T \underline{\underline{v}} = \sum_i w_i v_i$$

$\uparrow \quad \uparrow$
 "w_i" "v_i"

3.9 Orthonormal Bases

At this point we should take a deep breath and state explicitly that we've been assuming an orthonormal basis. In this course we will continue to use an orthonormal basis. You may object to this and say that you used to believe in orthonormal bases until you were forced to write down the gradient (or worse, the Laplacian) in spherical coordinates. In other words, in principle one could imagine a basis where (3.28) does not hold. There are many things to be said about this, none of them are particularly edifying without a full discussion. With no apologies, I'll make the following [perhaps perplexing] remarks:

1. There is no such thing as a 'position vector.' Positions refer to some base space, whereas vectors (like differential operators) act on the tangent space at a point of that base space.
2. A given tangent space is 'nice' and has a nice orthonormal basis.
3. That basis may not be the same for neighboring tangent spaces (perhaps due to coordinates, perhaps due to intrinsic curvature).

In this course these nuances will not come up. In the rest of your life you'll still have to deal with curvilinear coordinates. But suffice it to say that our study of function space will be nice an orthonormal. We haven't yet given an adequate definition of 'orthonormality,' so let's take (3.28) as a working definition.

3.10 Bra-Ket Notation

There is neither any physics nor mathematics contained in a choice of notation. However, a convenient notation does simplify our lives. Let us introduce bra-ket notation. In this

notation, we denote vectors by kets:

$$|v\rangle = v^i|i\rangle , \quad (3.30)$$

where $|i\rangle = \mathbf{e}_{(i)}$ is the basis of vectors that span the vector space V . There is nothing new or different about this object, $\mathbf{v} = |v\rangle$.

We denote dual vectors (row-vectors, one-forms) as bras:

$$\langle w| = w_i\langle i| , \quad (3.31)$$

where $\langle i| = \tilde{\mathbf{e}}^{(i)}$. The orthonormality of this basis is encoded in

$$\langle i|j\rangle = \delta_j^i . \quad (3.32)$$

In bra-ket notation a linear transformation A has a basis

$$A = A^i_j|i\rangle\langle j| . \quad (3.33)$$

The notation $|i\rangle\langle j|$ is shorthand for the **tensor product** $|i\rangle \otimes \langle j|$. If the \otimes doesn't mean anything to you, that's fine. It doesn't mean much to me either. Maybe you can replace it with the word '*and*' so that $|i\rangle \otimes \langle j|$ means you have a basis ket $|i\rangle$ *and* an basis bra $\langle j|$ that are somehow stuck together but aren't acting on each other. Matrix multiplication proceeds as before:

$$A\mathbf{v} = A|v\rangle = A^i_j|i\rangle\langle j|v^k|k\rangle = A^i_jv^k|i\rangle\langle j|k\rangle = A^i_jv^k|i\rangle\delta_k^j = A^i_jv^j|i\rangle . \quad (3.34)$$

Observe that the power of the notation is clear: the object with the index v^i is just a number. It commutes with everything. All of the vector-ness is carried in the basis objects: the bras, kets, and ket-bras. Those do not commute. But they have a well defined way in which kets act on bras (or vice versa).²⁸

3.11 Eigenvectors are nice

Given a sufficiently *nice* linear transformation, A , there is a particularly convenient basis: the eigenvectors of A . These are kets $|\lambda\rangle$ such that

$$A|\lambda\rangle = \lambda|\lambda\rangle . \quad (3.35)$$

In other words, A acts on the eigenvector by rescaling. The rescaling coefficient is the eigenvalues. For *nice* transformations (see Section 1.3), there is a complete set of such vectors to span the vector space.

If you write a general vector $|v\rangle$ in terms of this eigenbasis,

$$|v\rangle = v^i|\lambda_{(i)}\rangle , \quad (3.36)$$

²⁸This is where the \oplus notation is handy. It keeps track of which kets/bras might hit which other bras/kets. This falls under the name of multi-linear algebra.

Then the action of A on this vector is easy:

$$A|v\rangle = \sum_i \lambda_{(i)} v^i |\lambda_{(i)}\rangle . \quad (3.37)$$

In fact, assuming that all of the eigenvalues are non-zero, even the matrix inverse is easy:

$$A^{-1}|v\rangle = \sum_i \lambda_{(i)}^{-1} v^i |\lambda_{(i)}\rangle . \quad (3.38)$$

The first time you see this should have brought a deep joy to your life: if you can decompose a matrix (linear transformation) into its eigenvectors and eigenvalues, then taking the inverse transformation is simple.

3.12 Linearity of Inverse Operators

Given a linear operator A , the inverse operator A^{-1} is defined by

$$A^{-1}A = \mathbb{1}_{N \times N} . \quad (3.39)$$

For an N -dimensional vector space, this represents N^2 different equations: one for each element. The inverse operator, by the way, is also linear. Let's remind ourselves of what this means. A linear transformation, when written as a matrix, is simply stating what that linear transformation does to your basis vectors. If a matrix B has elements

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = B_j^i |i\rangle\langle j| , \quad (3.40)$$

then this simply means that acting on basis vectors $\mathbf{e}_{(1)} = |1\rangle$ and $\mathbf{e}_{(2)} = |2\rangle$ gives

$$B|1\rangle = a|1\rangle + c|2\rangle \quad (3.41)$$

$$B|2\rangle = b|1\rangle + d|2\rangle . \quad (3.42)$$

In column vector notation:

$$B \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \quad B \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} . \quad (3.43)$$

So knowing the action on basis vectors is the same as knowing the transformation itself. Suppose I told you the action of the inverse transformation A^{-1} on your basis vectors, vis-a-vis (3.42):

$$A^{-1}|1\rangle = x|1\rangle + y|2\rangle \quad (3.44)$$

$$A^{-1}|2\rangle = z|1\rangle + w|2\rangle . \quad (3.45)$$

Then you know exactly how A^{-1} acts on a general vector $|s\rangle = s^1|1\rangle + s^2|2\rangle$:

$$A^{-1}|s\rangle = A^{-1}(s^1|1\rangle + s^2|2\rangle) \quad (3.46)$$

$$= s^1 A^{-1}|1\rangle + s^2 A^{-1}|2\rangle \quad (3.47)$$

$$= (s^1 x + s^2 z)|1\rangle + (s^1 y + s^2 w)|2\rangle . \quad (3.48)$$

You can now keep this in mind when we say we want to solve $A|\psi\rangle = |s\rangle$. If we knew the action of A^{-1} on some basis of the space, then the problem is simple:

$$|\psi\rangle = \psi^i|i\rangle = (A^{-1})_j^i|i\rangle\langle j|s^k|k\rangle \quad (3.49)$$

$$= (A^{-1})_j^i s^k|i\rangle\langle j||k\rangle \quad (3.50)$$

$$= (A^{-1})_j^i s^j|i\rangle. \quad (3.51)$$

We can write this as an equation for each component:

$$\psi^i = \sum_j (A^{-1})_j^i s^j. \quad (3.52)$$

We've restored the explicit sum over j as a convenient reminder. The quantity $(A^{-1})_j^i$ is what we would like to identify with a Green's function.

3.13 The Green's Function Problem

Going back to the big picture: recall that we want to solve differential equations of the form $\mathcal{O}f(x) = s(x)$. If we had a sense of the *eigenfunctions* of \mathcal{O} , then we could expand $s(x)$ in a basis of those eigenfunctions and then apply \mathcal{O}^{-1} to both sides.

The analog is this:

"GREEN'S FUNCTIONS IN LINEAR ALGEBRA"

$$A|\psi\rangle = |s\rangle \quad \xrightarrow{\text{source}} \quad Ax = f$$

↑
 Dynamics
 ↗ State
 ↙
EFFECT
causes

given a "disturbance" $|s\rangle$, what happens to $|\psi\rangle$?

The operator A encodes the *physics* of the system, the underlying dynamics. This is presumably local: it is a near-diagonal matrix coming from one or two powers of derivatives. The ket $|s\rangle$ is the source. This is the thing that *causes* the dynamics. The ket $|\psi\rangle$ is some state that we would like to determine. (3.38) is telling us that to invert a differential operator \mathcal{O} , it may be useful to decompose it into **eigenfunctions**.

By the way, *this* is where all of the 'special functions' in your physics education show up. The reason why you would ever care about Bessel functions (of various kinds) or Legendre polynomials is simply that they are the eigenfunctions of differential operators that we care about²⁹. Sometimes we confuse mathematical physics with 'properties of special functions.'

²⁹In fact, they're mostly the eigenfunctions of the same differential operator in different coordinate systems. Do you know which differential operator? I'll give you a guess. It's starts with a 'har-' and ends with a 'monic oscillator.'

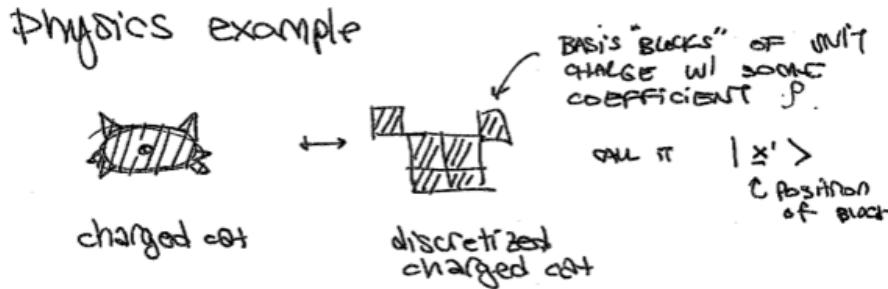
I do not care about special functions; perhaps with the notable exception of the Γ -function. Seriously, to screw special functions. The real intuition for what we're doing is evident in the few-dimensional harmonic oscillator. All the Bessel-schmessel function-ology that will pain you in your Jackson E&M course are just *technical details*.

Example 3.2 Consider a differential operator $A \rightarrow \mathcal{O} = (d/dx)^2$. There's a basis of nice eigenvectors $\xi_{(k)}$:

$$\xi_{(k)} = \sin(kx) \quad \mathcal{O}\xi_{(k)} = -k^2\xi_{(k)}. \quad (3.53)$$

I've deliberately avoided normalizing for now. From this you can see that writing a function as a **Fourier Series** is simply a change of basis to eigenfunctions of $(d/dx)^2$. Can you see how you would 'invert' this operator acting on a general function $f(x)$ with a set of Fourier coefficients $f(x) = \sum_k c_k \sin(kx)$?

Example 3.3 A common example in electrostatics is the triboelectric effect. Go ahead, take a moment to look it up on Wikipedia. The image is very relevant. If you pet your cat with hard rubber³⁰, the cat builds up some electrostatic charge. For simplicity, let's model this system as a bunch of Lego blocks arranged in a cat-shape where each block has a constant electric charge. Let's say that each block has volume dV and constant charge density ρ_0 so that each block has charge $\rho_0 dV$. The entire cat is described by a charge density $\rho(x)$ where $\rho(x) = 0$ if you're outside the cat and $\rho(x) = \rho_0$ if you're inside the cat.



You know how this problem works. You want to solve for the electrostatic potential, $\Phi(x)$, given the charge density $\rho(x)$. The relevant equation is

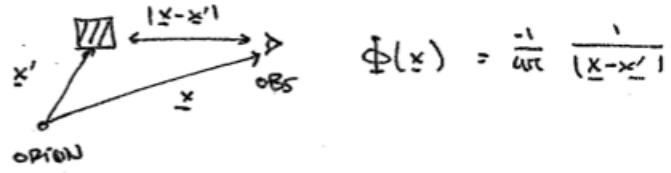
$$\nabla^2\Phi(\mathbf{x}) = -\rho(\mathbf{x}), \quad (3.54)$$

where $\epsilon_0 = 1$ in convenient units. This looks like a tricky differential equation to solve, but the first week of our undergraduate electrodynamics course taught us that the potential from a unit point charge is

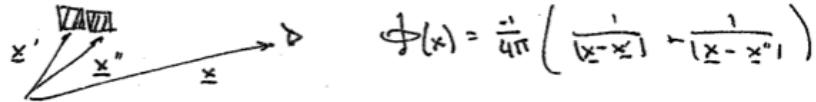
$$\Phi(\mathbf{x}) = \frac{-1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}'|}. \quad (3.55)$$

The relevant diagram is

³⁰I do not recommend doing this.



If we know the potential from a single point charge, then we can invoke linearity—specifically, the linearity of ∇^2)—to write down the potential for two unit point charges:



At this point, you should start to see how this looks just like (3.48). The result is

$$\Phi(\mathbf{x}) = \sum_{\mathbf{x}'} \frac{-1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \rho(\mathbf{x}') dV \rightarrow \int d^3 \mathbf{x}' \frac{-1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \rho(\mathbf{x}') . \quad (3.56)$$

This last example is really useful. You've seen this calculation before, but please review it carefully from the perspective of the action of $(\nabla^2)^{-1}$. Just as we are able to build a ‘Lego cat’ out of unit blocks, each of those unit blocks comes with an electrostatic potential. The solution to $\nabla^2 \Phi(x) = -\rho(x)$ is to simply sum together those point-source solutions in the same way that one sums together the point sources (assembles the blocks) to model the finite source.

Exercise 3.5 In the example above, what plays the role of the basis vectors?

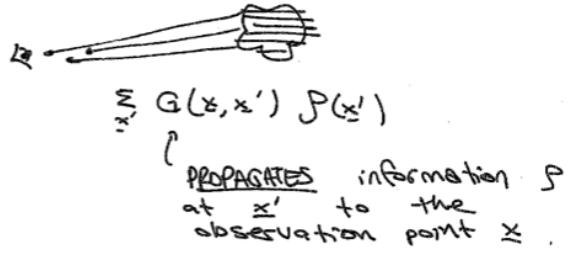
Compare (3.56) to (3.52). The sum over positions \mathbf{x}' that becomes an integral over $d^3 \mathbf{x}'$ is completely analogous to the sum over the index j . $\rho(\mathbf{x}')$ plays the role of a component of the source, s^j . On the left-hand side, $\Phi(\mathbf{x})$ plays the role of ψ^i . We note that the index i is replaced by the position \mathbf{x} . Evidently, the inverse of ∇^2 is simply

$$(\nabla^2)^{-1} \equiv G(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}'|} . \quad (3.57)$$

Here we've written out $G(\mathbf{x}, \mathbf{x}')$, the Green's function for ∇^2 . Observe that the Green's function has two arguments in the same way that the inverse matrix $(A^{-1})_j^i$ has two indices. The \mathbf{x}' ‘index’ is integrated/summed over—it scans each ‘building block’ of the source in the same way that we sum over j in the index contraction $(A^{-1})_j^i s^j$. In other words, we may heuristically write (3.57) as

$$\Phi^i \sim \sum_j G_j^i \rho^j . \quad (3.58)$$

At this point, it's useful to point out that the Green's function is often called a **propagator**. As a function, $G(x, x')$ propagates the information of the source at x' to the observer at x assuming some dynamics—the operator for which G is a Green's function.



Please, please, please make sure you understand this example. This will be the key starting point from which we will generalize our study of Green's functions in physics.

3.14 Remark: Implicit assumption of linearity

In everything we're building we're assuming that the dynamics that relates the source $|s\rangle$ and the state $|\psi\rangle$ is *linear*. That's why we can think of the differential operator \mathcal{O} as a matrix. If $|\psi_0\rangle$ is the effect of source $|s_0\rangle$, then you know by linearity that $2|\psi_0\rangle$ is the effect of a source that is twice as strong, $2|s_0\rangle$.

Linearity is *not* a truth of nature, yet we're spending all of this course developing techniques for dealing with linear dynamics! The fact of the matter is that a good chunk of physics *is* linear—and that's good because those are the parts that we can solve using our standard toolkit. Most of the frontier of physics has to do with how to deal with the *non-linearities*. There are a few options here: numerical solution, perturbation expansions about a linear solution (Feynman diagrams), and looking for topological invariants.

3.15 Metrics on Finite Dimensional Vector Spaces

Thus far we have introduced vector spaces. The dual vector space is a set of linear functions that act on elements of a vector space; these are bras/row-vectors/one-forms. Let us now introduce a new piece of machinery: a **metric**. This is also known as an **inner product** or a **dot product**. A space with a metric is called a metric space. We only state this fact to emphasize that we are *adding this structure by hand*. Vector spaces don't come with metrics—someone makes up a metric and slaps it onto the vector space.

The **metric** is a function that takes two vectors and spits out a number. It is linear in each argument. In other words, a metric g is:

$$g : V \times V \rightarrow \mathbb{R} . \quad (3.59)$$

Occasionally one may want a metric defined such that the output is a complex number. We thus have:

$$g(\alpha\mathbf{v} + \beta\mathbf{w}, \delta\mathbf{x} + \gamma\mathbf{y}) = \alpha\delta g(\mathbf{v}, \mathbf{x}) + \alpha\gamma g(\mathbf{v}, \mathbf{y}) + \beta\delta g(\mathbf{w}, \mathbf{x}) + \beta\gamma g(\mathbf{w}, \mathbf{y}) . \quad (3.60)$$

One more special assumption about the metric is that it is **symmetric**³¹:

$$g(\mathbf{v}, \mathbf{w}) = g(\mathbf{w}, \mathbf{v}) . \quad (3.61)$$

³¹Actually it is conjugate symmetric, $g(\mathbf{v}, \mathbf{w}) = g(\mathbf{w}, \mathbf{v})^*$. This distinction will be important for function spaces.

In indices one may write

$$g = g_{ij} \langle i | \otimes \langle j | \quad (3.62)$$

so that

$$g(\mathbf{v}, \mathbf{w}) = g_{ij} v^i w^j . \quad (3.63)$$

Here we see the usefulness of the \otimes notation. It tells us that the bras and kets resolve as follows:

$$g_{ij} \langle i | \otimes \langle j | (v^k | k \rangle) (w^\ell | \ell \rangle) = g_{ij} v^k w^\ell \langle i | k \rangle \langle j | \ell \rangle = g_{ij} v^k w^\ell \delta_k^i \delta_\ell^j = g_{ij} v^i w^j . \quad (3.64)$$

If this is your first time seeing it, please re-read (3.64) carefully to see exactly how the bras and kets resolve themselves. For ordinary Euclidean space in flat coordinates, the metric is simply the unit matrix: $g_{ij} = \text{diag}(1, \dots, 1)$. In Minkowski space there's a relative minus sign between space and time. In curvilinear coordinates things get ugly.

Here's the neat thing about metrics. We can take a metric g and pre-load it with a vector \mathbf{v} :

$$g(\mathbf{v}, \quad) . \quad (3.65)$$

In fact, we may then define a function with respect to this pre-loaded metric:

$$f(\mathbf{w}) = g(\mathbf{v}, \mathbf{w}) \quad (3.66)$$

Observe that $f(\mathbf{w})$ is a linear function that takes elements of V and returns a number. In other words, this is a *dual vector* (row-vector, one-form, element of V^*). The metric has allowed us to *convert vectors into dual vectors*:

$$g(\mathbf{v}, \quad) = g_{ij} v^i \langle j | . \quad (3.67)$$

Similarly, one may define an inverse metric g^{-1} such that $g^{-1}g = \mathbb{1}$. In a slight abuse of notation, the inverse metric is written with two upper indices: g^{ij} . Note that we do not write the ' -1 '. The inverse metric will *raise* the index on a lower-index object, while the metric *lowers* the index of an upper-index object.³²

3.16 The dot/inner product, orthonormality

You are already familiar with the ‘obvious’ way to take two vectors and spit out a number. This is the dot product of two vectors, also known as the inner product. They are equivalent to each other and equivalent to the action of the metric on a pair of vectors;

$$\mathbf{v} \cdot \mathbf{w} = \langle \mathbf{v}, \mathbf{w} \rangle = g(\mathbf{v}, \mathbf{w}) . \quad (3.68)$$

³²Of course: what’s really happening is that the metric has a basis $\langle i | \otimes \langle j |$ while the inverse metric has a basis $| i \rangle \otimes | j \rangle$.

The norm of a vector \mathbf{v} is simply the square root of $g(\mathbf{v}, \mathbf{v})$. By the linearity of the metric, we can decompose the dot product into the action of the metric on basis vectors:

$$g(\mathbf{v}, \mathbf{w}) = g(v^1 \mathbf{e}_{(1)} + v^2 \mathbf{e}_{(2)} + \cdots, w^1 \mathbf{e}_{(1)} + w^2 \mathbf{e}_{(2)} + \cdots) \quad (3.69)$$

$$= v^1 w^1 g(\mathbf{e}_{(1)}, \mathbf{e}_{(1)}) + v^1 w^2 g(\mathbf{e}_{(1)}, \mathbf{e}_{(2)}) + \cdots, \quad (3.70)$$

where our assumption of an *orthonormal basis*³³ is the statement that

$$g(\mathbf{e}_{(i)}, \mathbf{e}_{(j)}) = \pm \delta_j^i. \quad (3.71)$$

The \pm depends on the signature of the metric. As physicists, this is the distinction between ‘space’ and ‘time.’ For Euclidean space the sign is always plus. For Minkowski spacetime, there’s some choice of sign for the timelike direction and the opposite sign for the spacelike direction. The choice is a convention, there’s no ‘right’ choice³⁴. When (3.71) is not true the basis is not orthonormal. Sometimes this happens when your space is curvy (general relativity); other times you just have curvilinear coordinates for a flat space (e.g. polar coordinates). I’m glossing over several subtleties here, among them is the idea that position vectors do not exist³⁵.

Exercise 3.6 *Speaking of the notion of curved space: Consider a sheet of paper as an idealized two-dimensional surface. If you tape opposite edges of the paper together to make a cylinder, is the two-dimensional space curved or flat? For example, if two-dimensional beings lived on the paper like in Edwin Abbot–Abbot’s novella Flatland, would they say that their space is curved?*

Exercise 3.7 *The cylinder from the previous exercise is an example of periodic boundary conditions. For those familiar with special relativity, this entire section has probably been a big review. Here’s a fun puzzle to think about while you skim. What happens to the twin paradox if the universe were periodic in some spatial direction? The usual resolution to the twin paradox is that the twin that ‘turns around’ must change inertial frame. However, if there were a periodic direction, neither twin has to ‘turn around’ for them to meet once again and compare clocks³⁶.*

Example 3.4 *Given two vectors \mathbf{v} and \mathbf{w} , there are two equivalent ways of thinking about how they may be combined into a number. First, one can lower the index of one of the vectors with the metric $v_i \equiv g_{ij}v^i$ and then contract the ‘row vector’ v_i with the ‘column vector’ w^j : $v_i w^i$. I am of course abusing notation by conflating the components v_i and w^j with the row/column vectors. The second way of doing this is taking the two vectors and contracting them directly with the metric: $v^i w^j g_{ij}$. Please make sure you are comfortable that these ‘two ways’ are exactly the same thing.*

³³Not an ‘orphan normal’ basis like Zoom’s auto-transcription claims I said.

³⁴The right choice is + for timelike and – for spacelike.

³⁵Formally, vectors live in the tangent space of a manifold. They are ‘intrinsically’ tangent vectors (velocities) of some trajectory on the manifold with some ‘time’ parameter. The position on the manifold is a coordinate, but it is not a *vector*. For more on what I mean by this, I recommend hep-th/0611201, Arnold’s *Mathematical Methods of Classical Mechanics*, or a graduate general relativity textbook.

³⁶For a nice solution, see: <https://doi.org/10.1080/00029890.2001.11919789>

3.17 Comment: Why physicists like indices

Einstein's summation convention tells us that we should see repeated upper/lower indices as contracted pairs—we should treat them differently. In contrast, the un-paired indices tell us something about the physical quantity to which those indices are attached. This is why—despite what our mathematician colleagues tell us—we love indices³⁷ Vectors are objects that transform in a well defined way with respect to rotations:

$$v^i \rightarrow v'^i = R^i_j v^j ,$$

where R is a rotation matrix. Similarly, one-forms/column vectors transform ‘oppositely’:

$$w_k \rightarrow w'_k = (R^T)^\ell_k w_\ell . \quad (3.72)$$

Observe that the rotation matrix is *transposed* when acting on an *lower* index object. This is consistent with the idea that the dot product is invariant under rotations:

$$v^i w_i \rightarrow v'^i w'_i = R^i_j v^j (R^T)^\ell_i w_\ell = [(R^T)^\ell_i R^i_j] v^j w_\ell = v^j w_j , \quad (3.73)$$

where we've used the fact that the matrix in the square brackets is simply δ_j^ℓ which is evident from $R^T R = \mathbb{1}$.

Example 3.5 We exploited something neat about the index notation in (3.73). Note that there aren't any ‘vectorial’ objects in (3.73)—all there is are a bunch of numbers (components of vectorial objects). We used the fact that we could re-arrange these products to make it clear how they contract. That's why we were able to move the $(R^T)^\ell_i$ term all the way to the left.

A neat lesson here is that

Indices tell us how an object transforms under ‘rotations’.

Here ‘rotations’ mean whatever the appropriate symmetry is. In quantum mechanics, for example, the rotations are unitary transformations. If an object has indices, then it transforms. If an object has no indices (or is only composed of contracted indices) then it is a scalar with respect to those transformations. This ‘indexology’ is the backbone of how lazy physicists apply representations of symmetry groups³⁸. It's especially useful in particle physics³⁹.

The generalization to tensors is hopefully clear. An object with some number of upper indices and some number of lower indices transforms with several factors of the rotation matrix and its transpose:

$$T^{i_1 i_2 \dots i_n}_{j_1 j_2 \dots j_m} \rightarrow R^{i_1}_{k_1} R^{i_2}_{k_2} \dots R^{i_n}_{k_n} (R^T)^{\ell_1}_{j_1} (R^T)^{\ell_2}_{j_2} \dots (R^T)^{\ell_m}_{j_m} T^{k_1 k_2 \dots k_n}_{\ell_1 \ell_2 \dots \ell_m} . \quad (3.74)$$

Yes, we are so wealthy with indices that our indices have indices.

Exercise 3.8 How does the moment of inertia tensor transform under a rotation R ?

³⁷Recent Nobel laureate Roger Penrose has a clever notation that replaces indices with lines; https://en.wikipedia.org/wiki/Penrose_graphical_notation.

³⁸<https://github.com/Tanedo/Physics262-2019>

³⁹<https://sites.google.com/ucr.edu/p165>

3.18 Adjoints and Hermitian Conjugates

Because the metric lowers the index of a vector and produces a dual vector, you may want to think about this as ‘tipping over’ a column vector—almost like a transpose, right? We should be careful with this notion. Suppose we have a vector $\mathbf{w} = A\mathbf{v}$. How do we express the components of the *dual vector* w_i with respect to the elements of the dual vector v_i ?

$$w_i = g_{ij}w^j = g_{ij}A^j{}_k v^k = A_{jk}v^k = A_j{}^k v_k \equiv v_k (A^T)_j{}^k . \quad (3.75)$$

In the second-to-last step we have used $g^{ij}g_{jk} = \delta_j^i$. In the last step we have *defined* the transpose of a real matrix in the usual way: swap the first and second indices. These happen to have heights that come along for the ride. For convenience we moved the v_k to the left-side of the expression—it should be clear that this does not affect the expression at all (just write out the sum explicitly). We’ve gone through these gymnastics simply to motivate the notion that

$$\mathbf{w}^T = \mathbf{v}^T A^T . \quad (3.76)$$

Notice that the T *acts* on the matrix A . This is in contrast to our notation where the T in \mathbf{w}^T is simply a label to remind us that the object is a dual (row) vector.

For complex vector spaces—for example, the space of wavefunctions in quantum mechanics—we need to be a little bit more careful. We won’t say much about complex vector spaces in general, but us simply state that the ‘transpose’ is generalized to the Hermitian conjugate. You may be most familiar with this in bra-ket notation from quantum mechanics:

$$\langle x | = |x\rangle^\dagger = \langle x, \quad | , \quad (3.77)$$

where $\langle x, \quad |$ is simply the inner product on the complex space with one component pre-loaded⁴⁰. You should now see why the Hermitian conjugate isn’t just a ‘transpose’ but also a complex conjugation. The **norm** of a vector is $\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle$. We would like the **norm** to be positive definite⁴¹, even when \mathbf{v} is complex. If \mathbf{v} has a complex phase, then there needs to be a ‘complex conjugation’ built into the notion of an inner product.

Example 3.6 While this may sound like hand-waving, you already know this from quantum mechanics where the wavefunction $\psi(x)$ is complex. You know that a nice wavefunction is normalized, $\langle \psi, \psi \rangle = 1$, which we interpret as:

$$\int dx \psi^*(x)\psi(x) = 1 . \quad (3.79)$$

The $\psi^*(x)$ in the integrand is related to the complex conjugation implicit when going from a ket $|\psi\rangle$ to a bra $\langle\psi|$.

⁴⁰By the way, this motivates the bra-ket notation for vectors and dual-vectors:

$$\langle x|y \rangle = \langle x, y \rangle . \quad (3.78)$$

In the above equation, the left-hand side is a bra (dual vector) acting on a ket (vector), while the right-hand side is the inner product acting on two kets. This is simply repeating the statement in Example 3.4.

⁴¹The notable exception is when you have a space with nontrivial signature like Minkowski space where the relative sign between space and time is relevant.

We will work this out more carefully when we re-introduce function spaces systematically—but all of this should feel familiar.

Let us write **adjoint** to mean transpose for real spaces and Hermitian conjugate for complex spaces. The importance of the adjoint is not converting vectors into dual vectors, but rather the action on operators. Armed with a metric/inner product, the adjoint A^\dagger of an operator A satisfies

$$\langle A^\dagger w, v \rangle = \langle w, Av \rangle . \quad (3.80)$$

In a given inner product, the adjoint shifts the action from one vector to the other.

Example 3.7 *From this definition, it should be clear that*

$$\langle w | Av \rangle = \langle A^\dagger w | v \rangle . \quad (3.81)$$

This equation does not invoke the metric in contrast to (3.80). Of course, the dual vector $\langle w |$ is related to the vector $|w\rangle$ by the metric.

Exercise 3.9 *If you are mathematically inclined, prove the existence, uniqueness, and linearity of the adjoint A^\dagger of a linear operator A . Hint: see Byron & Fuller chapter 4.4.*

The adjoint/Hermitian conjugate is important because of the result that **self-adjoint operators**, $A^\dagger = A$, have real eigenvalues and a complete set of orthogonal eigenvectors. Then we can use the strategy of Section 3.11 to use eigenvectors of A to simplify the solution of $A\mathbf{v} = \mathbf{w}$.

4 Function Space

Function spaces are vector spaces where the vectors are functions. We introduced ‘histogram space’ in Section 3.3 as a crude model of a finite dimensional function spaces. An alternative finite dimensional model is the space of polynomials up to some degree, which we introduced in Section 3.5. Our models of nature are typically continuous⁴². This requires *infinite* dimensional function spaces, or Hilbert spaces.

4.1 The Green’s Function Problem in Function Space

The Green’s function can be defined by analogy to the finite-dimensional inverse transformation. The finite-dimensional linear system $A\mathbf{v} = \mathbf{w}$ can be solved by applying the inverse transformation A^{-1} in the same way that the continuum (infinite-dimensional) system $\mathcal{O}\psi(x) = s(x)$ can be solved with the Green’s function $G(x, y)$ of the operator \mathcal{O} :

$$v^i = \sum_i (A^{-1})_j^i w^j \quad \Rightarrow \quad \psi(x) = \int dy G(x, y) s(y) . \quad (4.1)$$

We’ve explicitly written out the sum over the dummy index i to emphasize the analogy to the integration over the dummy variable y . The arguments of the functions play the role of ‘continuum indices.’

⁴²This does not mean that nature is fundamentally continuous. There is a deep sense in which our models are valid whether or not nature is continuous so long as any granularity is smaller than our experimental probes.

4.2 Differential Operators

Linear transformations on function space are differential operators. In principle you can imagine linear transformations that are not differential operators, for example a finite translation. However, because our models of nature are typically *local* and *causal*, the linear transformations that we obtain from physical models are differential operators⁴³.

Let's write a general differential operator as:

$$\mathcal{O} = p_0(x) + p_1(x) \frac{d}{dx} + p_2(x) \left(\frac{d}{dx} \right)^2 + \dots \quad (4.2)$$

where the $p_i(x)$ are polynomials. Sometimes we will write this as \mathcal{O}_x to make it clear that the argument of the polynomials is x and the variable with which we are differentiating is x .

Exercise 4.1 Explain why (4.2) is a linear operator acting on function spaces.

Exercise 4.2 A confused colleague argues to you that (4.2) cannot possibly be ‘linear.’ Just look at it, your colleague says: the functions $p_i(x)$ are polynomials—those aren’t linear! There are also powers of derivatives—how is that possibly linear? Explain to your colleague why the $p_i(x)$ does not have to be linear nor is one restricted to finite powers of derivatives for the operator \mathcal{O} to be a linear operator acting on function space.

Technically (4.2) is called a **formal operator** because we haven’t specified the boundary conditions of the function space. Recall in our discretized ‘histogram space’ in Section 3.3 that we had to be careful about how to define the derivative acting on the boundaries of the space. A differential operator along with boundary conditions is called a **concrete operator**.

4.3 Inner Product

There’s a convenient inner product that you may be familiar with from quantum mechanics. For two functions $f(x)$ and $g(x)$ in your function space, define the inner product to be

$$\langle f, g \rangle = \int dx f^*(x)g(x) . \quad (4.3)$$

Example 4.1 Wave functions in 1D quantum mechanics obey this norm. For an infinite domain, we typically restrict to square-integrable functions meaning that $|f|^2$ goes to zero fast enough at $\pm\infty$ so that the integral $\langle f, f \rangle$ is finite.

⁴³This is not to say that finite transformations are somehow not permitted. The dynamics that govern our models of nature, however, only dictate how information is transmitted infinitesimally in space and time. Propagation forward in time by some finite interval is described by the exponentiation of infinitesimal forward time translations. This is, of course, why the time-translation operator in quantum mechanics is $e^{i\hat{H}t}$, where the Hamiltonian H is described as a local function with perhaps one or two derivative operators.

Sometimes the inner product is defined with respect to a **weight** function $w(x)$:

$$\langle f, g \rangle_w = \int dx w(x) f^*(x)g(x) . \quad (4.4)$$

There's nothing mysterious about inner products with weights. They typically boil down to the fact that one is not using Cartesian coordinates.

Example 4.2 *Have you met the Bessel functions? If not, you're in for a treat in your electrodynamics course. The Bessel functions satisfy a funny orthogonality relation with weight $w(x) \sim x$ because they show up as the radial part of a solution when using polar coordinates. When you separate variables, $d^2x = rdr d\theta$, we see that the measure over the radial coordinate r carries a weight r .*

We will assume unit weight until we go to higher spatial dimensions⁴⁴.

4.4 Dual Vectors

What are the ‘dual functions’ (dual vectors, bras) in function space? These are linear functions on act on functions and spit out numbers. Taking inspiration from (3.67), these are integrals that are pre-loaded with some factors. Assuming unit weight:

$$\langle f | = \langle f, \quad \rangle = \int dx f^*(x) [\text{insert ket here}] . \quad (4.5)$$

4.5 Adjoint operators

What is the adjoint of a differential operator? The definition of the adjoint (3.80) and the function space inner product (4.3) give us a hint. We define \mathcal{O}^\dagger by the property

$$\int dx [\mathcal{O}f(x)]^* g(x) = \int dx f^*(x) [\mathcal{O}^\dagger g(x)] . \quad (4.6)$$

The strategy is: given an inner product (integral) over f^* and g where there is some stuff (\mathcal{O}) acting on g , can we re-write this as an integral with no stuff acting on g and some *other* stuff acting on f^* ? If so, then the ‘other stuff’ is the adjoint \mathcal{O}^\dagger .

Example 4.3 *What is the adjoint of the derivative operator, $\mathcal{O} = d/dx$? Assume an interval $x \in [a, b]$ and Dirichlet boundary conditions, $f(a) = f(b) = 0$. There's a simple way to do this: integrate by parts.*

$$\int dx \left[\frac{d}{dx} f(x) \right]^* g(x) = - \int dx f^*(x) \left[\frac{d}{dx} g(x) \right] + [f^*(x)g(x)]_a^b = - \int dx f^*(x) \left[\frac{d}{dx} g(x) \right] . \quad (4.7)$$

From this we deduce that

$$\left(\frac{d}{dx} \right)^\dagger = - \frac{d}{dx} . \quad (4.8)$$

⁴⁴My dissertation focused on theories of extra dimensions. I also noticed that my weight increased in my final year of graduate school as I spent most of my time writing about extra dimensions and eating cafe pastries.

We will be especially interested in **self-adjoint** (Hermitian) operators for which

$$\mathcal{O}^\dagger = \mathcal{O} . \quad (4.9)$$

This is, as we mentioned for the finite-dimensional case, because self-adjoint operators are *nice*: they have real eigenvalues and orthogonal eigenvectors. Since most physical values are real eigenvalues of some operator, one may expect that the differential operators that show up in physics are typically self-adjoint.

Exercise 4.3 *We saw above that the derivative operator is not self-adjoint. What is an appropriate self-adjoint version of the derivative operator? Hint: what is the momentum operator in quantum mechanics?*⁴⁵

Example 4.4 Consider $\mathcal{O} = -\partial_x^2$ defined on the domain $x \in [0, 1]$ with the boundary conditions $f(0) = f(1) = 0$. Is this operator self-adjoint? We want to check if $\langle f, \mathcal{O}g \rangle = \langle \mathcal{O}f, g \rangle$. We have one trick: integration by parts. Let's see how this works.

$$\langle f, \mathcal{O}g \rangle = - \int dx f^*(x) \partial^2 g(x) . \quad (4.10)$$

This is compared to

$$\langle \mathcal{O}f, g \rangle = - \int_0^1 dx [\partial^2 f(x)] * g(x) \quad (4.11)$$

$$= - (\partial f(x))^* g(x)|_0^1 + \int_0^1 dx [\partial f(x)]^* \partial g(x) \quad (4.12)$$

$$= f^*(x) \partial^2 g(x)|_0^1 - \int_0^1 dx f^*(x) \partial^2 g(x) \quad (4.13)$$

$$= - \int_0^1 dx f^*(x) \partial^2 g(x) . \quad (4.14)$$

And so we see that indeed $(-\partial^2)^\dagger = -\partial^2$.

Exercise 4.4 In the previous example, what is the significance of the overall sign of the operator? Hint: the sign doesn't matter, it's because we typically think of $-\partial^2$ and its higher-dimensional derivatives as the square of the momentum operator.

Example 4.5 The **eigenfunctions** f_n of $-\partial^2$ defined on $x \in [0, 1]$ with Dirichlet boundary conditions are simply

$$f_n(x) = A_n \sin(n\pi x) \quad \lambda_n = -n^2\pi^2 , \quad (4.15)$$

where λ_n is the associated eigenvalue and A_n is some normalization that. These eigenfunctions are orthonormal in the following sense:

$$\langle f_n, f_m \rangle = \int_0^1 dx \sin(n\pi x) \sin(m\pi x) = \frac{A_n A_m}{2} \delta_{nm} , \quad (4.16)$$

from which we deduce that the normalization is $A_n = \sqrt{2}$. That's basically all there is to know about Fourier series.

⁴⁵<https://aapt.scitation.org/doi/abs/10.1119/1.9932>

Exercise 4.5 A function $g(x)$ defined on an interval $x \in [0, 1]$ with Dirichlet boundary conditions can be written with respect to the Fourier basis (4.15). In ket notation, the n^{th} component of g with respect to this basis is

$$g^n = \langle f_n | g \rangle . \quad (4.17)$$

Confirm that this is precisely what you know from Fourier series. In other words, we can decompose $g(x)$ as

$$g(x) = \sum_n \langle f_n | g \rangle f_n(x) . \quad (4.18)$$

4.6 Completeness in Function Space

We rarely have much to say about the unit matrix in linear algebra. However, much like when we discussed units, we can squeeze a lot out of inserting the identity in our mathematical machinations. In order to help with translate this to function space, let's review how it works in finite dimensional vector spaces. The unit matrix is $\mathbb{1}$ and may be written:

$$\mathbb{1} = \sum_i |i\rangle\langle i| , \quad (4.19)$$

where $|i\rangle$ and $\langle j|$ are basis (dual-)vectors.

Exercise 4.6 Take a moment and convince yourself that (4.19) is true and obvious. It may be helpful to explicitly write out $|i\rangle\langle j|$ as a matrix.

Exercise 4.7 Suppose you have a two-dimensional Euclidean vector space. Show that (4.19) is true for the basis

$$|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (4.20)$$

$$\langle 1| = \frac{1}{\sqrt{2}} (1 \ 1) \quad \langle 2| = \frac{1}{\sqrt{2}} (1 \ -1) . \quad (4.21)$$

In fact, (4.19) defines what it means that a set of basis vectors is **complete**. You can write any vector $|v\rangle$ with respect to the basis $|i\rangle$ —the components are simply

$$v^i = \langle i | v \rangle \quad (4.22)$$

so that

$$|v\rangle = \sum_i |i\rangle\langle i | v \rangle , \quad (4.23)$$

which we recognize as nothing more than ‘multiplying by the identity.’

What does completeness look like in function space?

Let $e_{(n)}(x)$ be a set of basis functions. The basis is **complete** if

$$\sum_n [e_{(n)}(x)]^* e_{(n)}(y) = \delta(x - y) . \quad (4.24)$$

Compare this *very carefully* with the completeness relation (4.19). The sum over i in the finite-dimensional case has been relabeled into a sum over n in the function space—this is just my preference⁴⁶. The $\mathbb{1}$ has been replaced by a Dirac δ -function, $\delta(x - y)$. Let’s confirm that this makes sense. The *multiply by one* completeness relation (4.23) in function space is

$$|g\rangle = \sum_n |e_{(n)}\rangle \langle e_{(n)}|g\rangle \quad \langle e_{(n)}|g\rangle = \int dy [e_{(n)}(y)]^* g(y) . \quad (4.25)$$

We have deliberately changed the name of the integration variable to y to avoid confusion; since this variable is integrated over it’s simply a *dummy variable* and it doesn’t matter what we name it—the quantity $\langle e_{(n)}|g\rangle$ is independent of y because y is integrated over⁴⁷. Writing this out explicitly as functions:

$$g(x) = \sum_n \left[\int dy e_{(n)}^*(y) g(y) \right] e_{(n)}(x) . \quad (4.26)$$

The factor in the square brackets is simply $\langle e_{(n)}|g\rangle$, which is just a *number*—it has no functional dependence on x . If this seems unusual, please refer back to Example 4.5 and Exercise 4.5.

By the way, you’ll often hear people (perhaps even me) say that the Dirac δ function is not strictly an *function* but rather a **distribution**—this means that it only makes sense when it is integrated over. As physicists we’ll sometimes be sloppy and talk about physical quantities that could be Dirac δ -functions. There is *never* an appropriate, measurable physical quantity that is described by a $\delta(x)$. Anything with a $\delta(x)$ is an object that was meant to be integrated over. When you imagine that a point charge density is a δ -function, this is only because you will eventually integrate over it to determine the total charge. This is precisely what we saw in the charged cat in Example 3.3. If you ever calculate a *measurable* quantity to be $\delta(x)$ check your work. If you ever find $\delta(x)^2$, then go home, it’s past your bed time.

Example 4.6 One can vaguely motivate the δ -function as the unit matrix by appealing to the ‘histogram basis’ of discretized function space. In an ordinary finite-dimensional vector space, unit matrix can be written as

$$\mathbb{1} = |1\rangle\langle 1| + |2\rangle\langle 2| + \dots = \sum_{i,j} \delta_i^j |i\rangle\langle j| . \quad (4.27)$$

⁴⁶I think this is because we will deal with complex functions and I want to avoid using i as an index. But if we’re being honest, it’s just become a habit.

⁴⁷By the way, this should ring a bell from our summation convention. When an upper and lower tensor index are contracted, the resulting object behaves as if it didn’t have those indices: $A^i{}_j v^j$ behaves as a vector with one upper index.

The Dirac δ -function in histogram space is analogous to

$$\delta(x - x') \rightarrow \delta_x^{x'} |x\rangle\langle x'| , \quad (4.28)$$

where x and x' are discrete bins on the right-hand side. Thus for a discretized function $f = f(x_1)|x_1\rangle + f(x_2)|x_2\rangle + \dots$, one has

$$\int dy \delta(x - y)f(y) = f(x) \longrightarrow \sum_j \delta_{x_j}^{x_i} |x_i\rangle\langle x_j|f\rangle = f(x_i)|x_i\rangle . \quad (4.29)$$

4.7 Orthonormality in Function Space

One should contrast the notion of completeness of a basis this with that of **orthonormality** of the basis. Orthonormality is the statement that

$$\langle i|j\rangle = \delta_i^j . \quad (4.30)$$

Completeness has to do with the ‘outer product’ $|i\rangle\langle i|$ while orthonormality has to do with the ‘inner product’ $\langle i|i\rangle = \langle i, i\rangle$. The function space generalization of orthonormality is⁴⁸

$$\langle e_{(n)}|e_{(m)}\rangle = \int dx e_{(n)}^*(x)e_{(m)}(x) = \delta_{nm} . \quad (4.31)$$

Exercise 4.8 Why does (4.31) have a Kronecker δ with discrete indices when (4.24) has a Dirac δ ? Please make sure you can answer this; it establishes the conceptual foundation of the analogy between finite- and infinite-dimensional vector spaces.

For the completeness relation, we sum over the same eigenfunction label n for a function and its conjugate evaluated at different continuous positions. For the orthonormality relation, we integrate over the positions of two different eigenfunction indices, n and m .

Do not confuse the eigenfunction label with the index of a vector. If this is confusing, please refer back to Exercise 4.7. You may be stuck thinking about basis vectors in the Cartesian basis—this is the analog of thinking about basis functions in the ‘histogram basis’ of Section 3.3. What we want to do is generalize to more convenient bases, like the eigenfunctions of differential operators (e.g. the Fourier basis for $-\partial^2$).

4.8 Completeness and Green’s Functions

The utility of the completeness relation should be clear. If you happen to have a nice (self-adjoint) linear differential operator \mathcal{O} with a nice (complete, orthogonal) eigenfunctions $e_{(n)}$

⁴⁸If you’re a purist, you’ll note that δ_{nm} should really be written as δ_m^n because the dual basis vector has an upper index. While this may be true, I’m making the present notational choice because the object that we would call $\tilde{\mathbf{e}}^{(n)}$ really does contain $e_{(n)}^*(x)$, the complex conjugate of $e_{(n)}(x)$.

and eigenvalues λ_n , then we can expand any function $\psi(x)$ with respect to these eigenfunctions. Then it is easy to invert the differential equation $\mathcal{O}\psi(x) = s(x)$ to determine the response $\psi(x)$ to a source $s(x)$:

$$\psi(x) = \mathcal{O}^{-1} \sum_n \langle e_{(n)} | s \rangle e_{(n)}(x) = \sum_n \frac{\langle e_{(n)} | s \rangle}{\lambda_n} e_{(n)}(x) , \quad (4.32)$$

where we've simply used (3.38). The inner product $\langle e_{(n)} | s \rangle$ is an overlap integral between known functions:

$$\psi(x) = \int dy \sum_n \frac{e_{(n)}^*(y) e_{(n)}(x)}{\lambda_n} s(y) , \quad (4.33)$$

where we have rearranged terms rather suggestively. This is now in the same form as our prototype Green's function example (3.56).

Referring back to (4.1), we see that our completeness relation—that is, our trick of inserting unity—in (4.33) tells us an explicit form for the Green's function of a differential operator \mathcal{O} if you know the eigenfunctions and eigenvalues of that operator:

$$G(x, y) = \sum_n \frac{e_{(n)}^*(y) e_{(n)}(x)}{\lambda_n} . \quad (4.34)$$

This is formally an infinite sum and so is only practically useful if each term is successively smaller.

4.9 Green's Function by Completeness: why is this helpful?

If you look at (4.34) and think about our goals for the class, you may say *hooray! We're done.* After all, given the Green's function $G(x, x')$ for a given differential operator⁴⁹ \mathcal{O}_x , then we know how to *invert* \mathcal{O}_x . So for any source $s(x)$ and differential equation $\mathcal{O}_x\psi(x) = s(x)$, we can find $\psi(x)$ by

$$\psi(x) = \int dx' G(x, x') s(x') . \quad (4.35)$$

The integral is over the domain on which we've defined the function space and subject to functions satisfying the boundary conditions. We interpreted the integral over x' as an integral over the source configuration. Armed with an expression for $G(x, x')$, we can simply perform the overlap integral with $s(x')$ —numerically if needed—and that gives us $\psi(x)$. Easy! What are we missing?

First, all of this *assumed* that you know the eigenfunctions of \mathcal{O} . This is actually a fairly safe assumption. There are only so many differential operators that matter in physics, especially since the physically motivated operators are typically self-adjoint and respect many symmetries. In fact, there are so few of these that their eigenfunctions are all famous—so when you're slogging through electrodynamics dealing with spherical harmonics, Bessel

⁴⁹We've explicitly written the x in \mathcal{O}_x to indicate that derivatives are with respect to that variable, *not* x' .

functions, and Legendre polynomials—you know that these special functions are ‘special’ because they’re eigenfunctions of variations of the Laplacian that show up in physics over and over again. They are so important that ancient graduate students had to use them *before* one could just plug them into *Mathematica*⁵⁰. All that is to say for any differential equation that you will probably ever care about in physics, the eigenfunctions are probably known and their properties are well documented⁵¹.

Okay, so if the identification of eigenfunctions is not a problem, why isn’t (4.34) the end of this course? One reason is that it is an *infinite* series. The differential operator is a ‘matrix’ in infinite-dimensional space, so there are an infinite number of eigenfunctions that space the space. If you’re like me, you really only want to deal with one or two terms—very rarely is it worth it to have to go to many more terms⁵². This means that the infinite sum is only practical if each successive term is a small correction to the previous terms. While this is not always the case, this may start to sound familiar to you. Let’s see it in action with an example.

Example 4.7 *The eigenfunctions for the angular part of the Laplacian, ∇^2 , in spherical coordinates are the **spherical harmonics**, $Y_{\ell m}(\theta, \varphi)$. When you tack on the radial piece, the Green’s function for the Laplacian in spherical coordinates is*

$$G(\mathbf{r}, \mathbf{r}') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} Y_{\ell m}(\theta, \varphi) Y_{\ell m}^*(\theta', \varphi') \frac{r'_<}{r'_>}^{\ell} , \quad (4.36)$$

where $r_> = \max(\mathbf{r}, \mathbf{r}')$ and $r_< = \min(\mathbf{r}, \mathbf{r}')$. To be concrete you can assume that $r > r'$ so that $r_> = r$ and $r_< = r'$. This corresponds to an observer further away from the origin than the source. Remind yourself of where expressions just like this show up in electrodynamics—for example, a charged cat curled up into a small lump near the origin of your coordinate system.

Note that (4.36) has two sums over eigenfunction ‘labels’ m and ℓ . That’s okay—this simply generalizes the case of a single sum. Clearly this expression has the form of a completeness relation with the radial piece tacked on.

The upshot of having this expression is that you can take any source $\rho(\mathbf{r})$, such as that lump of charged cat, and write a closed form expression for the state (e.g. the electrostatic

⁵⁰Have you ever heard of Gradshteyn and Ryzhik? When I was a student there was a story that most of it was written while the authors were bored in Siberia. In Cornell the theoretical physics journal club used to be called the Gradsteyn seminar because it would “integrate the knowledge of the graduate student participants.” (Source: Michael Peskin, private communication.) Anyway, if you’ve made it this far in the footnote: you should consider running a journal club with your lab/classmates. It may be the best preparation you can give yourself for being a young academic.

⁵¹By the way, if you were expecting this class to be about the properties of Bessel functions and all that, then forget it! I find nothing fun about that. We’re going to stick to good old sines and cosines because *all* of the essential intuition is already there. If you deeply understand the orthogonality, completeness, projections onto trigonometric functions, then you can ‘read’ the special functions as generalizations of the trigonometric functions for their respective differential operators. By the way, beware of any young person who seems to know the Bessel function properties *a little too well...* that person has probably been through some shit.

⁵²One notable local exception is Prof. Hai-Bo Yu’s work on self-interacting dark matter calculations. In the resonant regime, some of these numerical results require sums over hundreds of partial waves.

potential):

$$\Phi(\mathbf{r}) = \int d^3\mathbf{r}' \frac{r_<^\ell}{r_>^{\ell+1}} \left[\sum_{\ell,m} \frac{1}{2\ell+1} Y_{\ell m}(\theta, \varphi) Y_{\ell m}^*(\theta', \varphi') \right] \rho(\mathbf{r}') , \quad (4.37)$$

where the expression in the bracket has a special name:

$$P_\ell(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = \sum_{\ell,m} \frac{1}{2\ell+1} Y_{\ell m}(\theta, \varphi) Y_{\ell m}^*(\theta', \varphi') . \quad (4.38)$$

The P_ℓ 's are called Legendre polynomials⁵³ In the limit where $r \gg r'$, the expression takes the following form:

$$\Phi(\mathbf{r}) = \sum_{\ell,m} \frac{1}{2\ell+1} \frac{Y_{\ell m}(\theta, \varphi)}{r^{\ell+1}} \left[\int d^3\mathbf{r}' r'^\ell Y_{\ell m}^*(\theta', \varphi') \rho(\mathbf{r}') \right] , \quad (4.39)$$

where now the term in the brackets is purely a property of the source. Do you recognize what it is? This is simply the **multipole expansion** of the charged, lumpy cat. Observe that each successive term in the sum is suppressed by an additional power of r'/r . As long as $r \gg r'$ —that is, as long as we are far away from the charged, lumpy cat—we can approximate its electrostatic potential as the sum of a monopole term, dipole term, etc.

What we see from the above example is that in the limit where there is a small parameter, the Green's function series expression (4.34) coming from the completeness of eigenfunctions can be seen as a Taylor expansion.

4.10 Patching a Green's function together

There is another clever⁵⁴ way of solving for Green's functions. We'll leave most of this work to your homework, but let's sketch the procedure.

Recall that Green's functions are the analogs to inverse matrices in a finite dimensional vector space. In other words,

$$A(A^{-1}) = A^i{}_j (A^{-1})^j{}_k = \mathbb{1}^i{}_k = \delta^i{}_k . \quad (4.40)$$

The infinite dimensional version of this is

$$\mathcal{O}_x G(x, x') = \delta(x - x') . \quad (4.41)$$

⁵³Once when I was teaching a class of undergraduates in electromagnetism I asked them if they knew what these special functions, P_ℓ were called. One of them enthusiastically shouted, *ooh! Is that a Pessel function?* That's when I learned to appreciate the joy of serendipity in teaching.

⁵⁴'Clever' is not always a positive word. A mathematical technique that is *clever* may have an aesthetic quality that we can appreciate, but it's not practically useful if you have to be *clever* to know to use it. We would rather prefer something that is general and systematic. By the way, this is the reason that high-energy experimentalists all know how to use version control software for their thousand-person publications while theorists have a hard time working simultaneously on a draft between three people.

Example 4.8 Let's do a quick 'sanity' check for why $\delta(x - x')$ could plausibly play the role of an identity element, δ_k^i . When δ_k^i acts on a vector, it acts on each component as (writing the sum explicitly):

$$\sum_k \delta_k^i v^k = v^i . \quad (4.42)$$

Recalling that finite-dimensional indices are arguments in function space, the analog for the $\delta(x - x')$ acting on a function f is

$$\int dx' \delta(x - x') f(x') = f(x) . \quad (4.43)$$

If we compare (4.41) to the class of equations that we wanted to solve, $\mathcal{O}\psi(x) = s(x)$, we realize that the Green's function $G(x, x')$ is simply the state ψ at position x coming from an idealized δ -function source at position x' . Of course, there's no such thing as Dirac δ -function sources in nature, so we emphasize that this interpretation should not be taken literally⁵⁵. Heuristically, the Green's function equation looks like:

$$\mathcal{O}_x G(x, x') = \delta(x - x') . \quad (4.44)$$

Note that the source is zero for everywhere. This means that everywhere to the left of $x = x'$ is described by a homogeneous equation,

$$\mathcal{O}_x G_<(x, x') = 0 . \quad (4.45)$$

Further, everything to the right of $x = x'$ is described by another homogeneous equation,

$$\mathcal{O}_x G_>(x, x') = 0 . \quad (4.46)$$

These are different equations for *different functions*: $G_<$ and $G_>$ are two different functions that obey homogeneous equations *in their respective domains*. $G_<(x, x')$ is *not* defined for $x > x'$. Usually solving homogeneous equations is easier⁵⁶.

The strategy then is to solve for $G_<(x, x')$ and $G_>(x, x')$ as functions of x and *patch them together*:

$$G(x, x') = \begin{cases} G_<(x, x') & \text{if } x < x' \\ G_>(x, x') & \text{if } x > x' \end{cases} . \quad (4.47)$$

⁵⁵In undergraduate electrodynamics we say that the Coulomb potential is the result of a δ -function point source... but you don't actually believe that electrons are δ functions in charge do you? If you do, take some time to think about this. By the way, this is related to Exercise 1.1.

⁵⁶I wouldn't really know, but it seems to take less time on *Mathematica* so there you go.

Here x' is just a spectator variable—we’re keeping it fixed. For simplicity, you may even want to shift your coordinates so that $x' = 0$. When we do this, we usually have a second order differential equation—some variant of the Laplacian because that’s 99% of what we do—so we need to have enough boundary conditions to fix our coefficients. Since we have two functions in a second order differential equation, we need *four* boundary conditions. When we defined the Green’s function problem, presumably we are considering functions over some interval $x, x' \in [a, b]$. This gives boundary conditions at a and b , which may even be at $a = -\infty$ and $b = \infty$. The two additional boundaries are obtained at $x = x'$. These come from requiring the continuity of the solutions

$$G_<(x', x') = G_>(x', x') \quad (4.48)$$

and a ‘jump condition’ between the first derivatives of the solutions:

$$\lim_{\epsilon \rightarrow 0} \int_{x' - \epsilon}^{x' + \epsilon} dx \mathcal{O}_x G(x, x') = 1 , \quad (4.49)$$

where this comes from simply integrating the defining equation $\mathcal{O}_x G(x, x') = \delta(x - x')$ over a sliver around $x = x'$. Since \mathcal{O}_x is assumed to be second order, the jump condition reduces to saying that the first derivatives of $G_<$ and $G_>$ are discontinuous at $x = x'$ by a certain amount. Applying these boundary conditions then gives a piece-wise solution for the Green’s function.

4.11 Where we’re going

Our primary goal in this course is to find the Green’s function $G(x, x')$ given a differential operator \mathcal{O} . There are three primary ways to do this:

1. **Eigenfunctions and completeness**, Section 4.9. Assuming one knows the eigenfunctions of the differential operator, this gives a series solution for the Green’s function. It is practically useful only when the series is convergent.
2. **Patching**, Section 4.10. This method assumes that one can solve the *homogeneous* differential equation $\mathcal{O}_x G(x, x') = 0$ and then produces a piece-wise solution to the *inhomogeneous* differential equation that defines the Green’s function, $\mathcal{O}_x G(x, x') = \delta(x - x')$. This is practically useful in one dimension where the boundary conditions where the pieces are connected are easy to define.
3. **Fourier transform and its cousins**. This will be the topic of the rest of our course. We convert the differential equation into an *algebraic equation* in momentum space. Aspects of the causal structure of the system that are manifested in complex momentum space. Furthermore, one can use contour integrals to do the ‘hard work.’

Recently I filled a hole in my undergraduate education and used a fourth method called the *method of variations* to solve inhomogeneous differential equations. The method is sketched out in Appendix A. We won’t have anything further to say about that here, except that it turns out you can get a faculty job in theoretical particle physics without knowing how to use it.

Example 4.9 This problem is from Matthews & Walker Section 9-4. Consider a unit string with frequency $k = \omega/c$ and Dirichlet boundary conditions at $x = 0, 1$; where we note that we are using units of ‘length of the string.’ The differential operator describing standing waves is

$$\mathcal{O}_x = \frac{d^2}{dx^2} + k^2 . \quad (4.50)$$

Let’s solve this using eigenfunctions. We know the normalized eigenfunctions from Example 4.5:

$$f_n(x) = \sqrt{2} \sin(n\pi x) \quad \mathcal{O}_x f_n(x) = (-n^2\pi^2 + k^2) f_n(x) \equiv \lambda_n f_n(x) . \quad (4.51)$$

By the way, it should have been obvious that these are the eigenfunctions and eigenvalues, even though \mathcal{O}_x is not the same as d^2/dx^2 . Using our completeness relation, the Green’s function is

$$G(x, x') = \sum_n \frac{f^*(x)f(x')}{\lambda_n} = 2 \sum_n \frac{\sin(n\pi x) \sin(n\pi x')}{k^2 - n^2\pi^2} . \quad (4.52)$$

Thus the solution to the system with some inhomogeneous source $s(x)$

$$\left[\frac{d^2}{dx^2} + k^2 \right] f(x) = s(x) \quad (4.53)$$

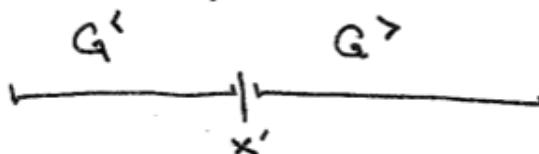
is simply

$$f(x) = \int_0^1 dx' G(x, x') s(x') . \quad (4.54)$$

Example 4.10 Let’s do the same example with the patching method. In this case we start with the equation (the analog of $A(A^{-1}) = \mathbb{1}$):

$$\left[\frac{d^2}{dx^2} + k^2 \right] G(x, x') = \delta(x - x') \quad (4.55)$$

We now separate the domain into $x < x'$ and $x > x'$, with a priori independent solutions $G_<(x, x')$ and $G_>(x, x')$:



Applying the Dirichlet boundary conditions at $x = 0, 1$ gives

$$G(x, x') = \begin{cases} G_<(x, x') = a \sin(kx) & \text{for } x < x' \\ G_>(x, x') = b \sin(k(x - 1)) & \text{for } x > x' \end{cases} . \quad (4.56)$$

Make sure you understand why $G_>(x, x')$ has a factor of $(x - 1)$ and not x ; this is simply the boundary condition at $x = 1$ without setting $b = 0$. The two coefficients a and b must be fixed by the matching at $x = x'$. To do this, integrate the second order differential equation over a sliver around x' :

$$\int_{x'-\varepsilon}^{x'+\varepsilon} dx , \left[\frac{d^2}{dx^2} + k^2 \right] G(x, x') = \int_{x'-\varepsilon}^{x'+\varepsilon} dx \delta(x - x') . \quad (4.57)$$

Note that the $k^2 G$ term in the integrand vanishes since it scales like ε . The integral of the second derivative is simple since it is simply the integral of a derivative, $\int dx d/dx(G') = \int d(G') = G'$ so that

$$\frac{d}{dx} G \Big|_{x'-\varepsilon}^{x'+\varepsilon} = G'_>(x', x') - G'_<(x', x') = 1 . \quad (4.58)$$

This is the jump condition of the first derivative. If we integrate the jump condition once more over a sliver between $x \pm \varepsilon$ gives the continuity condition:

$$G_<(x', x') = G_>(x', x') . \quad (4.59)$$

The jump and continuity conditions give

$$ka \cos(kx') + 1 = kb \cos(k(x' - 1)) \quad (4.60)$$

$$a \sin(kx') = b \sin(k(x' - 1)) . \quad (4.61)$$

One can solve for the coefficients:

$$a = \frac{\sin(k(x' - 1))}{k \sin k} \quad b = \frac{\sin kx'}{k \sin k} . \quad (4.62)$$

Plugging this all in gives

$$G(x, x') = \frac{1}{k \sin k} \begin{cases} \sin kx \sin(k(x' - 1)) & \text{if } x < x' \\ \sin kx' \sin(k(x - 1)) & \text{if } x > x' \end{cases} . \quad (4.63)$$

Observe that you find two *different* expressions for G by these methods. Please confirm—perhaps numerically—that these indeed represent the same function $G(x, x')$.

Exercise 4.9 This example is from Butkov, chapter 12.1. Consider a taut string of length L under the load of a weirdly-shaped rock:



The force equation for the vertical displacement of the string, $u(x)$, is

$$Tu''(x) = F(x), \quad (4.64)$$

where T is the tension and $F(x)$ is the force-per-unit-length. One may write this more simply as $u''(x) = f(x) = F(x)/T$. Show that the Green's function may be written as a piecewise function

$$G(x, x') = \begin{cases} \left(\frac{x'-L}{L}\right)x & \text{if } x < x' \\ \left(\frac{x-L}{L}\right)x' & \text{if } x > x' \end{cases}. \quad (4.65)$$

Suppose you replace the weird rock by a square paper weight of size $x < L$ in the middle of the string. Sketch what the paper weight on the string looks like.

5 Complex Analysis Review

5.1 Teaser: why are we doing this?

We now switch gears to complex analysis. One of the results that I hope you'll come to appreciate in your study of physics is the rich role of complex numbers (and their generalizations) in describing nature. Our study of complex analysis will focus on motivating the residue theorem to calculate integrals in complex space. This, in turn, is useful when we represent functions in *momentum space* by Fourier transforming. For example, a Green's function $G(x, x')$ can be written with respect to the Fourier momentum variable k by

$$G(x, x') = \int dk e^{-ikx} \tilde{G}(k, x'), \quad (5.1)$$

where I've used a convenient notation where $\hat{d} = d/2\pi$. This, in turn is useful because our defining relation for the Green's function is (4.41):

$$\mathcal{O}_x G(x, x') = \delta(x - x'), \quad (5.2)$$

which we may then write (Fourier transforming both sides) as

$$\mathcal{O}_x \int dk e^{-ikx} \tilde{G}(k, x') = \int dk e^{-ikx} e^{ikx'} . \quad (5.3)$$

On the right-hand side we've written the $\delta(x - x')$ in its Fourier representation. We have assumed that \mathcal{O}_x is a polynomial in derivatives with respect to x :

$$\mathcal{O}_x = \sum_{n=0}^{\infty} p_n(x) \left(\frac{d}{dx}\right)^n \equiv P\left(x, \frac{d}{dx}\right) . \quad (5.4)$$

What's powerful about this is that when we apply a differential operator in x onto the Fourier representation of a function, say (5.1), then the derivatives only 'see' the exponential factor e^{-ikx} . This means that derivatives are particularly easy:

$$\left(\frac{d}{dx}\right)^n e^{-ikx} = (-ik)^n . \quad (5.5)$$

This means that we may write the differential operator \mathcal{O}_x in (5.4) when it acts on the Fourier representation as

$$\mathcal{O}_x = \sum_{n=0}^{\infty} p_n(x) (-ik)^n \equiv P(x, -ik) , \quad (5.6)$$

so that the defining relation for G , (5.3), may in turn be simplified:

$$\int dk e^{-ikx} P(x, -ik) \tilde{G}(k, x') = \int dk e^{-ik(x-x')} , \quad (5.7)$$

which suggests a simple expression for the Fourier coefficients:

$$\tilde{G}(k, x') = e^{ikx'} [P(x, -ik)]^{-1} , \quad (5.8)$$

where one may find $G(x, x')$ from simply taking the inverse Fourier transform. There will turn out to be a few neat features of this approach, but before getting there, let's start from the very beginning.

5.2 Complex Numbers

A complex number z may be decomposed into real and imaginary parts

$$z = x + iy , \quad (5.9)$$

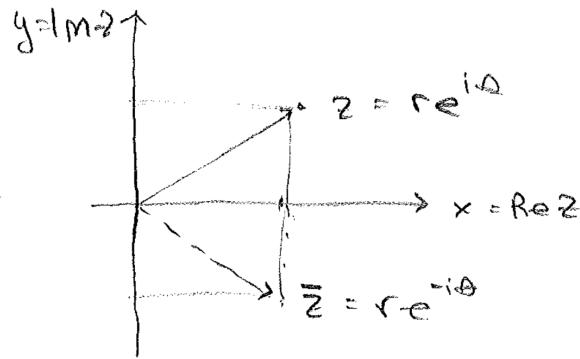
where x and y are both real numbers. We may also define the complex conjugate as

$$\bar{z} = z^* = x - iy . \quad (5.10)$$

One may also write these variables in a polar notation,

$$z = re^{i\theta} \quad \bar{z} = re^{-i\theta} , \quad (5.11)$$

where $r \in [0, \infty]$ and $\theta \in [0, 2\pi]$:



Complex numbers live in the space of complex numbers \mathbb{C} , a *two-dimensional* real space with coordinates (x, y) in \mathbb{R}^2 augmented by an additional rule for multiplication (take two complex numbers and return a complex number) that is called the *complex structure*. This is basically the definition $i^2 = -1$, though one may generalize to higher complex dimensions.

5.3 Complex functions

Complex functions take complex numbers and return complex numbers. Inspired by the two-dimensional-ness of \mathbb{C} , let us write a complex function as $f(z) = f(x, y)$ where $z = x + iy$. Because $f(x, y)$ is, itself, a complex number for a given $z = x + iy$, we can decompose f into two real-valued functions

$$f(z) = f(x, y) = u(x, y) + iv(x, y) . \quad (5.12)$$

This is clearly a map from $\mathbb{C} \rightarrow \mathbb{C}$ and so we can simply “do multivariable calculus” on this space.

If we forget the complex structure, then this *seems* to simply be the calculus of maps between $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ where

$$f(x, y) = u(x, y)\hat{\mathbf{x}} + v(x, y)\hat{\mathbf{y}} . \quad (5.13)$$

In that case, we need to collect the partial derivatives. We know that we can write the variation of f as

$$\delta f(x, y) = \delta u(x, y)\hat{\mathbf{x}} + \delta v(x, y)\hat{\mathbf{y}} = \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y \right) \hat{\mathbf{x}} + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y \right) \hat{\mathbf{y}} . \quad (5.14)$$

Remembering our complex structure, we can write this succinctly as

$$\delta f = \frac{\partial f}{\partial z}\delta z + \frac{\partial f}{\partial \bar{z}}\delta \bar{z} , \quad (5.15)$$

where we’re using

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial(iy)} \right) \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial(-iy)} \right) \quad (5.16)$$

$$= \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) . \quad (5.17)$$

The fact that partial derivatives combine like this may be familiar⁵⁷.

So far we have emphasized that $f(x, y)$ behaves like a function in two real dimensions. Critically, in some ways we want to think about functions $f(z)$ that are functions of ‘one’ [complex] dimension rather than two real dimensions. In two real dimensions, calculus came with a notion of directional derivative. When we wanted the rate of change for a function, we had to ask *in what direction* are we checking the rate of change. At a given function may change by a positive amount in the x -direction, but a negative amount in the y -direction, for example. This is in contrast to one-dimensional calculus where each point had an unambiguous derivative that represented how much the function changed in the positive x direction with respect to a point x_0 :

$$\delta f(x) = \frac{df(x_0)}{dx}(x - x_0) + \dots . \quad (5.18)$$

⁵⁷This comes from the underlying mathematical structure. The partial derivatives are basis vectors for the *tangent space* at some point (x, y) . These partial derivatives ask: how much does the function I’m acting on change if I move by one unit in this direction?

This is in contrast to the change in a general complex function (5.15). In fact, if we explicitly wrote some expansion with respect to a point z_0 , it looks a little funny:

$$\delta f(z) = \frac{\partial f}{\partial z}(z - z_0) + \frac{\partial f}{\partial \bar{z}}(\bar{z} - \bar{z}_0) + \dots . \quad (5.19)$$

What is going on here? Why should $f(z)$ be a function of one complex variable z but the expansion seems to depend not only on $(z - z_0)$ but also $(z - z_0)^*$?

5.4 Analytic complex functions are nice

This leads us to a definition of “nice” complex functions. The following terms are all more-or-less equivalent for this sense of nice-ness: **[complex] differentiable, analytic, holomorphic, regular**. This is simply the restriction that the $\partial f / \partial \bar{z}$ term should vanish so that the variation $f(z)$ only depends on δz and not $\delta \bar{z}$. In other words, an analytic function admits usual single-dimensional definition of the derivative,

$$\frac{df(z_0)}{dz} = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} . \quad (5.20)$$

The key aspect of this is that there is only *one* value of df/dz at z_0 no matter how you approach z_0 . In other words, it doesn’t matter if $\delta z = z - z_0$ is coming from the positive/negative real/imaginary direction. The value of df/dz is the same.

For most of these lectures we will use the term **analytic** to refer this property. Let us see what happens when we impose $df/d\bar{z} = 0$ onto the component functions $u(x, y)$ and $v(x, y)$:

$$\frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) [u(x, y) + iv(v, y)] = \frac{1}{2} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 0 . \quad (5.21)$$

This implies the **Cauchy–Riemann equations**, which are equivalent to the function f being analytic:

$$\frac{\partial u}{\partial x} = + \frac{\partial v}{\partial y} \qquad \qquad \qquad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x} . \quad (5.22)$$

Thus far we’ve talked about functions being analytic or not. It turns out that analytic functions are *so* nice that there really aren’t that many physically relevant functions that are analytic *everywhere*⁵⁸. Instead, we will often talk about functions that are analytic in most places but are not analytic (differentiable) in other places. Indeed, the *non-analyticity* of Green’s functions is a key result in this class.

5.5 Analyticity is Differentiability

We have motivated the Cauchy–Riemann equations from the notion of being independent of z^* . This is a useful crutch and fits our theme of motivating rather than rigorously proving results. Let us be clear, however, that the underlying definition of analyticity is that:

⁵⁸The analog here is people who are nice. Most people are nice most of the time. But very few people are nice *all* of the time. I suspect Fred Rogers may be one of the few who could plausibly be *analytic* *nice* everywhere.

a function that is analytic at a point z_0 is differentiable at z_0 .

A function is differentiable at a point z_0 if its derivative, (5.20), has a well-defined *finite* value. A function that is itself infinite at a point cannot have a well-defined limit (5.20), and thus are *not* analytic at that point. We distinguish this from $df/dz^* = 0$ because of the importance of *singular* functions in physics.

Example 5.1 The following function is non-analytic at $z = 2i$ and $z = 1$:

$$f(z) = \frac{3z}{(z - 2i)(z - 1)} . \quad (5.23)$$

It may seem like $df/dz^ = 0$ at these points because $f(z)$ is independent of z^* . However, one cannot say that $df/dz^* = 0$ because the limit definition analogous to (5.20) is not defined.*

Exercise 5.1 The notion of ‘independent of z^* ’ sometimes causes more confusion than its worth. After all, z and z^* are clearly not independent variables. Given z , you know exactly what z^* is. The notion in which we pretend that z and z^* are independent is that

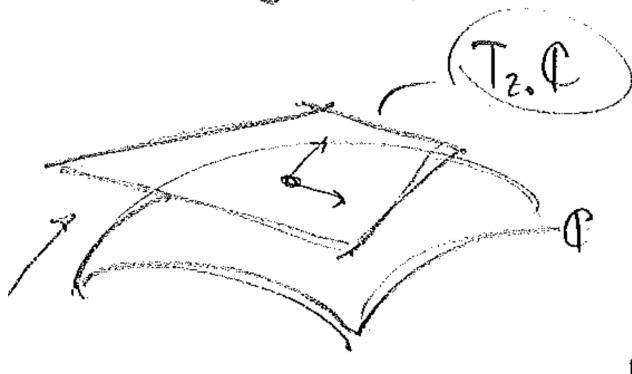
$$\frac{\partial z}{\partial z^*} = \frac{\partial z^*}{\partial z} = 0 , \quad (5.24)$$

which we understand to come from an interpretation of $\mathbb{C} \sim \mathbb{R}^2$. Use (5.17) to prove the above partial derivative relations.

5.6 A geometric point of view (optional)

For culture, let us comment on a geometric perspective of analyticity/complex-differentiability. The differential of a function, df , is a **differential one-form**. If you recall some of our nomenclature for dual vectors, a one-form is a kind of ‘row vector’: it is a linear function that takes in a vector and spits out a number. If df is such a dual vector, what is the vector space upon which it acts?

Let’s specifically consider the differential of f at a specific point z_0 in the complex plane: $df(z_0)$. This is not a number. In order to get a number from this, you have to feed it a vector: I claim this vector is an element of the tangent plane⁵⁹ of the complex plane at z_0 : $T_{z_0}\mathbb{C}$. We can sketch this if you allow me to artificially ‘curve’ the complex plane to help make it easy to distinguish:

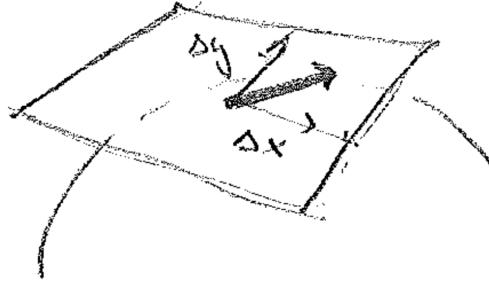


⁵⁹The tangent plane is exactly what it sounds like: if you have a curved surface like a globe, the tangent plane at z_0 and glued a rigid piece of cardboard to one point, z_0 . Now imagine that the piece of paper has graph paper on it.

Using the jargon of differential geometry, we call the complex plane the *base manifold*. The set of all tangent spaces is called a *tangent bundle*⁶⁰. Each tangent space is a vector space ‘attached’ to a single point of the base space.

The object $df(z_0)$ is a dual vector on the vector space⁶¹ $T_{z_0}(\mathbb{C})$. The vectors of this space are ‘velocities’ of trajectories that pass through z_0 . For now let’s think of them as quantities

$$\mathbf{v} = \Delta x + i\Delta y . \quad (5.25)$$



Then we may write the linear action of $df(z_0)$ on $\mathbf{v} \in T_{z_0}(\mathbb{C})$ as

$$df(z_0) [\mathbf{v}] = f'(z_0)\Delta x + f'(z_0)i\Delta y , \quad (5.26)$$

where we assume that f is analytic so that $f'(z_0)$ is unambiguously defined (independent of the direction of δz) as in (5.20).

However, if we think about this complex space as a two-dimensional real space, the following is also true:

$$df(z_0) [\mathbf{v}] = \left. \frac{\partial f}{\partial x} \right|_{z_0} \Delta x + \left. \frac{\partial f}{\partial y} \right|_{z_0} \Delta y \quad (5.27)$$

Comparing (5.26) and (5.27) gives

$$\frac{\partial f}{\partial x} = f'(z_0) = -i \frac{\partial f}{\partial y} . \quad (5.28)$$

Recalling that $f = u + iv$ then gives

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} . \quad (5.29)$$

Setting the real and imaginary parts equal to one another give, as expected, the Cauchy–Riemann equations.

⁶⁰More general types of vector spaces can be attached to each point of the base manifold. These are called fiber bundles. They are the underlying mathematical structure of mechanics and reflect why Hamilton’s equations are so damn special. They are also the mathematical structure that defines gauge theories and so address the question “what is a force?”

⁶¹The standard notation is to write a dual vector of a space V as a vector in the dual space V^* , so one could write $df(z_0) \in T_{z_0}^*(\mathbb{C})$.

5.7 Analytic and harmonic

One more comment is in order. If a function $f(z) = u(x, y) + iv(x, y)$ is analytic, then its real and imaginary parts are independently 2D harmonic with respect to \mathbb{R}^2 :

$$\partial_x^2 u(x, y) + \partial_y^2 u(x, y) = 0 \quad \partial_x^2 v(x, y) + \partial_y^2 v(x, y) = 0 . \quad (5.30)$$

Harmonic functions show up all over the place—though admittedly you probably care mostly about three-dimensional harmonic functions. The above realization may be helpful, however, for 3D systems for which the third dimension is trivial. For example, you may want to consider fluid flow across an airplane wing. One can take the limit where the wing is infinitely long so that the fluid dynamics reduces to the two-dimensional motion around a cross section of the wing. Alternatively, maybe you care about an electrostatic system which are similarly ‘effectively’ 2D. In this case you can ‘promote’ your real harmonic variable (a fluid dynamics potential or electrostatic potential) to an analytic function and then bear the full power of complex analysis to attack the problem. One of the most fascinating—if outdated—methods is called **conformal mapping** by which one can solve for a harmonic function with weird boundary conditions by mapping to much simpler boundary conditions. This topic is beautiful and gives a first, intuitive example of *conformal transformations* which are a pillar of theoretical physics (in any discipline). The technique is how engineers in the pre-personal computer era designed airplane wings. For experimentalists, it is pretty neat that UCR’s own Nathan Gabor has created an analogous micromagnetic ‘solver’ of this very same problem⁶².

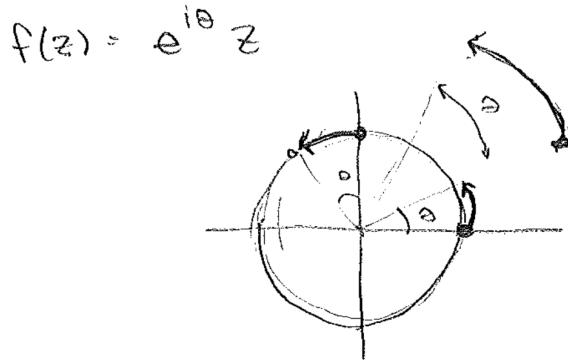
5.8 Complex functions as maps

A complex function $f(z)$ takes in a complex number and returns a complex number. These are maps of the complex plane to itself. It is useful to build an intuition for this rather trivial-sounding statement: complex functions ‘deform’ the complex plane.

Example 5.2 Consider the function that multiplies a complex number by a constant phase,

$$f(z) = e^{i\theta} z . \quad (5.31)$$

This takes each point and rotates it in the counterclockwise direction by θ .



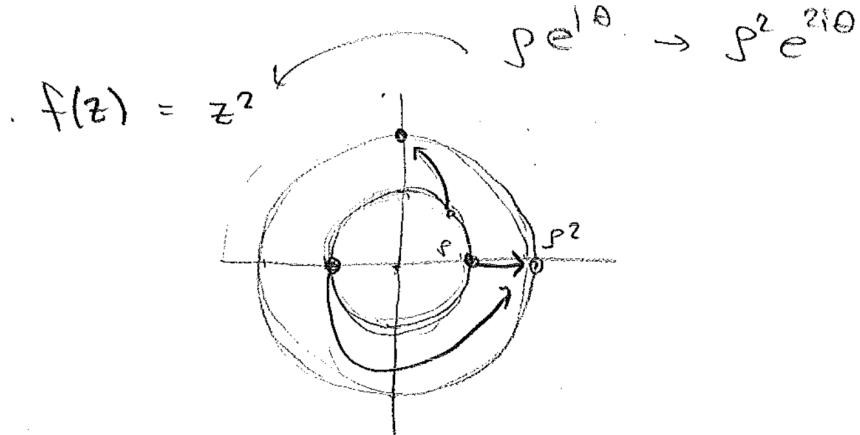
⁶²<https://arxiv.org/abs/2002.07902>

Exercise 5.2 What happens to points under $f(z) = z + z_0$?

Example 5.3 Now consider the simplest non-trivial polynomial, the function that squares its argument:

$$f(z) = z^2 . \quad (5.32)$$

For a point $z = \rho e^{i\theta}$, $f(z) = \rho^2 e^{2i\theta}$. This squares the modulus (length) of each point and rotates according to how much the point is already rotated relative to the positive real axis.



Points that are inside the unit circle get pulled closer to the origin while points outside the unit circle are pushed away from the origin. Points in the first quadrant (positive real and imaginary parts) are sent to points in the upper half plane (positive imaginary part).

Exercise 5.3 Show that under the map $f(z) = z^2$, a vertical line gets mapped onto a parabola:

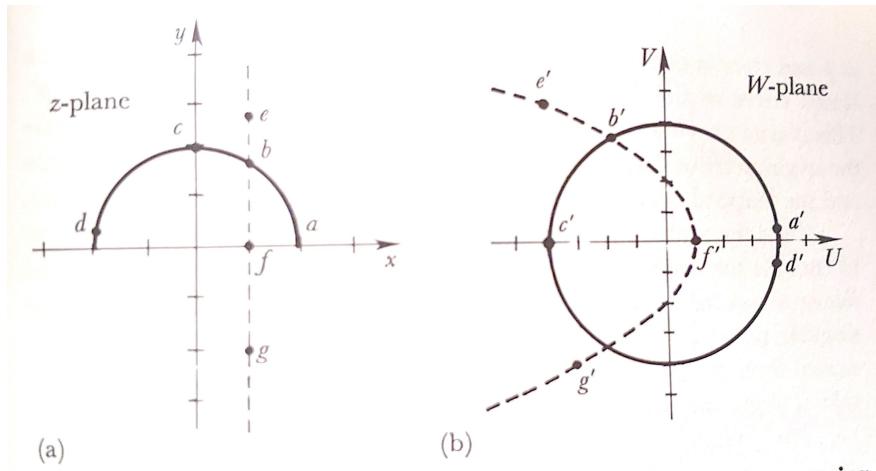
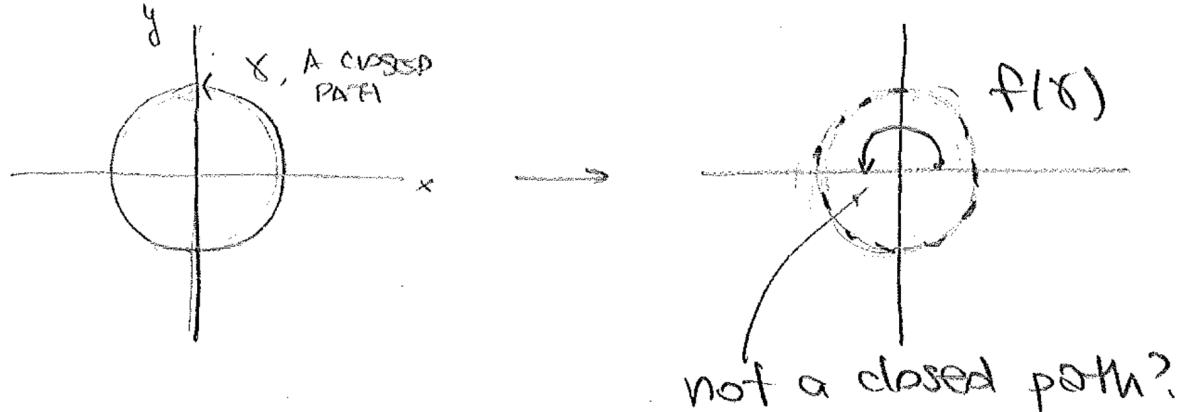


Figure from Matthews and Walker, points a , b , etc. are mapped onto a' , b' etc. The W -plane corresponds to the image of the complex plane under z^2 .

Now let's consider a curious case. What about the square root function?

$$f(z) = \sqrt{z} . \quad (5.33)$$

Something very weird happens here. Consider a closed path $\gamma(t) = e^{it}$ with path parameter $t \in [0, 2\pi]$. This just means consider a bunch of points corresponding to $\gamma(t)$ with a bunch of values of t in the specified range. Clearly this corresponds to a circle. However, when we put those points through the square root function, these only map onto *half* of the circle.

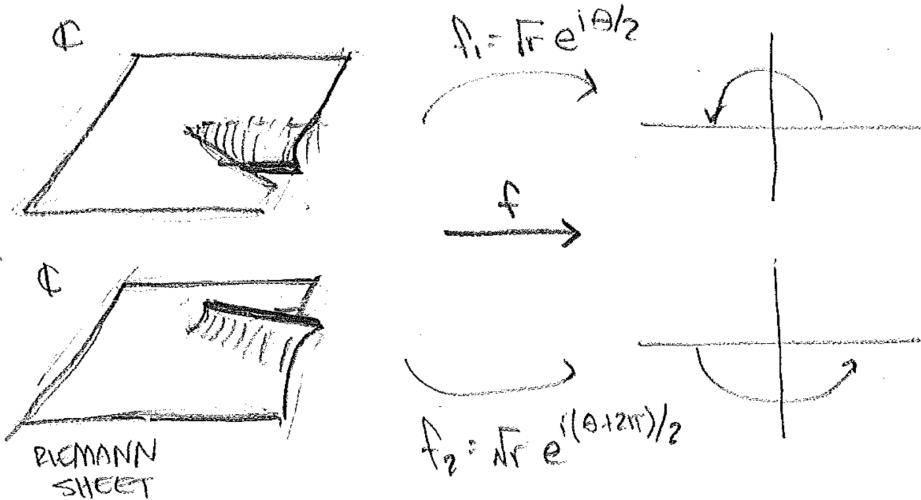


The technical fact that this happens is not surprising at all, you can see that $f(\gamma(t)) = e^{it/2}$ which only spans the upper half circle. But a deeper question bubbles to the surface: if we're thinking about complex functions as maps of the complex plane to itself, what does it mean that $f(z) = \sqrt{z}$ appears to have 'lost' the entire lower half of \mathbb{C} ? Or more strangely, how is it that a *closed path* has been mapped to an *open path* with endpoints?

It's clear that if we wanted to reach the *lower* half plane in the image, we'd need $\theta \in [0, 4\pi]$. Algebraically that means that $f(\gamma(t)) = e^{it/2}$ covers the entire complex plane *only* if we allow angles from 0 to 4π ⁶³. In some sense, this is totally ridiculous, since $\rho e^{i\theta} \equiv \rho e^{i(\theta+2\pi)}$. I put that third bar on the equal sign because they're supposed to be *that* equal.

If, say, $z_1 = e^{i\pi/3}$ and $z_2 = e^{7i\pi/3}$ are supposed to be the *same* point, $z_1 \equiv z_2$, how is it that $f(z_1) = e^{i\pi/6}$ and $f(z_2) = e^{7i\pi/6}$ so that $f(z_1) \neq f(z_2)$? Stated differently, functions are supposed to be single valued. It appears that $f(z) = \sqrt{z}$ is *multivalued* since the same point $z_1 = z_2$ is mapped onto two different points. One way to do this is to *define* additional structure and extend the domain (pre-image) of f . In order to do this, we take *two* copies of the complex plane and stitch them together:

⁶³You may be familiar with another case where 'rotating by 2π ' only takes you halfway—spinors in relativistic quantum mechanics. You should know that the 'double cover' nature of the spinor is (to the best of my view) independent of the present discussion of Riemann sheets. That double cover has to do with the universal cover of the Poincaré group. I refer to the literature on the Wigner theorem, see e.g. volume 1 of Weinberg's *Quantum Theory of Fields*. For an unrelated cute spinor paper, see <https://www.jstor.org/stable/2318771>.



Each copy of the complex plane is called a **Riemann sheet**. We ‘glue’ these sheets together by defining that $\rho e^{i(\theta+2\pi)}$ of one sheet maps onto $\rho e^{i\theta}$ of the other sheet. Implicitly this chooses the positive real axis as a place where we *cut* each sheet and glue them together. We call this a **branch cut**. One could, of course, have chosen the domain of each sheet to be any interval, say $\theta \in [-\pi, \pi]$ and picked a different branch cut ($\theta_{\text{branch}} = \pi$).

Exercise 5.4 Is $f(z) = \sqrt{z}$ analytic on the extended complex plane (two Riemann sheets glued together at, say, $\theta = 2\pi$)?

Take a moment to ponder the above exercise. As you may suspect, the relevant question isn’t whether a function $f(z)$ ‘is analytic,’ but rather *where* is that function analytic? Is it analytic (differentiable) anywhere? Sure—when stay within one Riemann sheet, say $\theta \in (0, 2\pi)$ and staying away from the boundary, $f(z) = \sqrt{z}$ is perfectly differentiable. Further, there’s nothing special about the branch cut—we could have placed it anywhere. Thus it doesn’t seem like there’s *any* place at which $f(z)$ is *not* analytic.

This should make sense: analyticity is about whether a function is differentiable at a given point. This is inherently a *local* notion. The misbehavior of $f(z) = \sqrt{z}$ —the necessity of Riemann sheets and a branch cut—are *global* issues when we try to explore the *whole* space. The interplay between local properties (like derivatives and Taylor expansions) and global properties (like integrals over a closed surface or topology) is a critical theme in mathematical physics. The function $f(z) = \sqrt{z}$ is analytic everywhere, but that doesn’t mean we don’t have to be careful. One of the lessons from this section is that the existence of a branch cut can be critically important if you happen to take trajectories that are *loops* in the complex plane. For those who know where this is going: whenever there are branch cuts, you have to be careful with your contour integrals. This typically happens whenever you have a function that is not some series of integer powers of z .

As a final example, let’s consider the complex logarithm. We’ll use the notation of Byron and Fuller and write \log to denote the complex natural logarithm and \ln to denote the ‘usual’

natural logarithm acting on real numbers. For $z = re^{i\theta}$, the complex logarithm satisfies

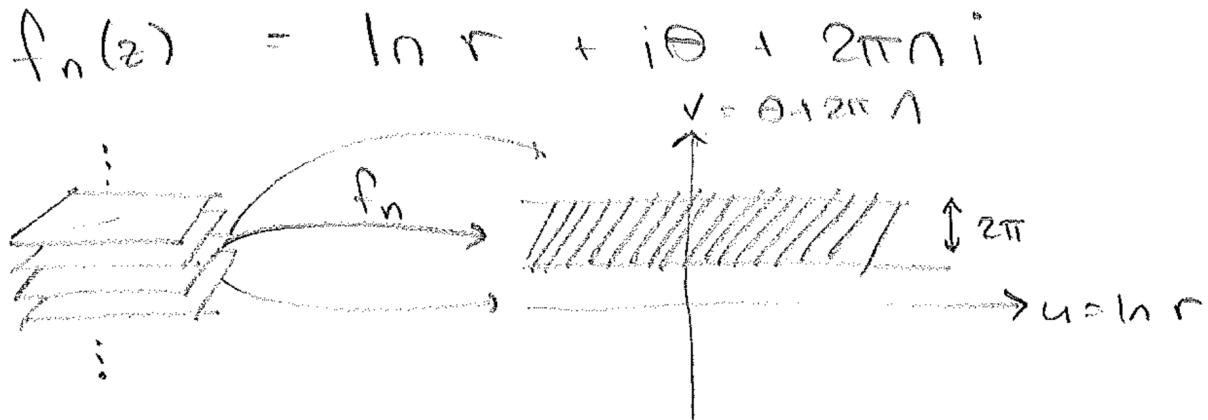
$$\log z = \ln r + i\theta \quad (5.34)$$

$$\log(z_1 z_2) = \ln(r_1 r_2) + i(\theta_1 + \theta_2) = \log z_1 + \log z_2. \quad (5.35)$$

Example 5.4 What does the domain of the complex logarithm look like? As a function, $f(z) = u(z) + iv(z)$, we see that $v(z) = \theta$ and $v(z_1 z_2) = \theta_1 + \theta_2$. This means that if we take a trajectory that keeps going around the origin, $v(z)$ just keeps increasing⁶⁴. Each time you go around you must be on a new Riemann sheet.

Rather than having an infinite number of Riemann sheets, an alternative way of thinking about this is to imagine an infinite number of equally valid ‘complex logarithm’ functions labeled by n :

$$f_n(z) = \ln r + i\theta + 2\pi n i. \quad (5.36)$$



The image of the $f_n(z)$ corresponds to a horizontal strip with $v \in (2\pi n, 2\pi(n_1))$.

Example 5.5 Is the complex logarithm analytic? Everywhere but at $z = 0$.

Let us end by repeating the main point of introducing branch cuts:

Do not integrate across a branch cut!

⁶⁴This is where you can start humming ‘stairway to heaven.’

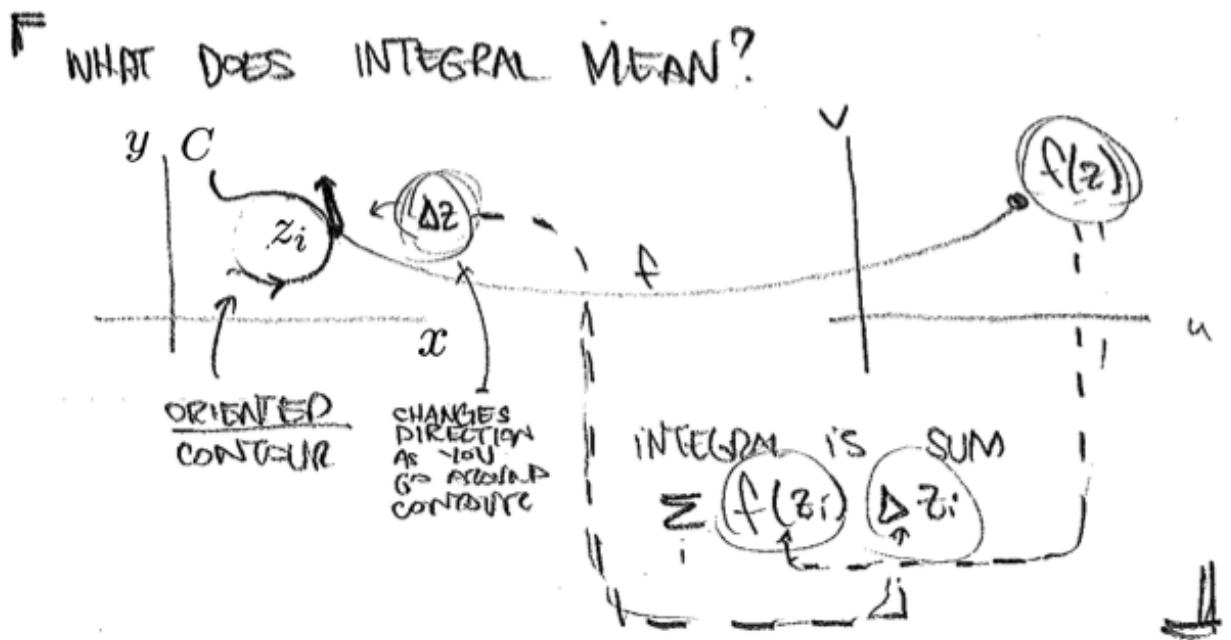
5.9 Integration on the Complex Plane

Let's now focus on integrating analytic functions. Oops, I just mis-spoke a bit there. What I meant to write was that we want to integrate functions in a region of \mathbb{C} where those functions are analytic. Rarely are functions simply ‘analytic’ or ‘not-analytic.’ The functions we care about will be analytic in most places, but non-analytic in others.

When we integrate on the complex plane, \mathbb{C} , we have to define a **contour**—this is just the curve, C , that we are integrating over. This is similar to doing a “line integral” in \mathbb{R}^2 : you are doing a ‘one-dimensional integral’ over a two[ish]-dimensional space. An integral over the contour C is:

$$\int_C dz f(z) = \int_C (dx + idy) , [u(x, y) + iv(x, y)] = \int_C [u(x, y)dx - v(x, y)dy] + i \int_C [u(x, y)dy + v(x, y)dx] . \quad (5.37)$$

Written this way, we are simply doing calculus on \mathbb{R}^2 and keeping track of funny factors of i . It may be helpful to remind ourselves what this means:



The integral along a curve can be thought of as two-dimensional version of a Riemann sum. Recall that the one-dimensional Riemann sum approximated the integral as a sum of terms $f(x_i)\Delta x$. In more than one dimension, the quantity Δx is promoted to a ‘vector’⁶⁵. In other words, we may write

$$\int_C dz f(z) = \sum_i \Delta z_i f(z_i) , \quad (5.38)$$

⁶⁵As I write this I feel sick to my stomach. In fact, Δx is *not* promoted to a ‘vector,’ though it is promoted to something that points in a direction. More properly, it is a [differential] one-form which is more like an *dual vector* in the sense referenced in Section 5.6.

where Δz has a real and imaginary part: it *points* to a direction in the complex plane. In the picture above, the left-hand side shows the curve C on which we are integrating. Consider some point z_i on the curve. That point is mapped by the function f to a point $f(z_i) = u(z_i) + iv(z_i)$. At that point, the curve also has a differential element, Δz_i which is tangent to the curve C at point z_0 and has some fixed infinitesimal length for all i . The *direction* of Δz matters. It contributes to the overall phase of the term $\Delta z_i f(z_i)$.

Example 5.6 Consider the function $f(z) = z^2 + i$. Consider a specific point, $z_0 = 1 + i$. If we consider different curves C_i that pass through z_0 , the integral $\int_{C_i} dz f(z)$ will have different contributions at z_0 depending on the direction of the curve as it passes through z_0 . To see this, consider the specific term in the sum (5.38) corresponding to $z_i = z_0$. If the curve C happens to be moving in the positive vertical (pure imaginary) direction, then the contribution to the sum from this point is

$$\int_C dz f(z) = \dots + i\Delta y [(1+i)^2 + i] + \dots . \quad (5.39)$$

However, if the curve C happens to be moving in the horizontal (pure real) direction, the contribution to the sum from this point is

$$\int_C dz f(z) = \dots + \Delta x [(1+i)^2 + i] + \dots . \quad (5.40)$$

Exercise 5.5 Suppose f is itself a derivative of an analytic function, $f(z) = dF(z)/dz$. Convince yourself that if C is a smooth connected path between two points z_0 and z_N , then the integral of $f(z)$ over C is

$$\int_C dz f(z) = F(z_N) - F(z_0) , \quad (5.41)$$

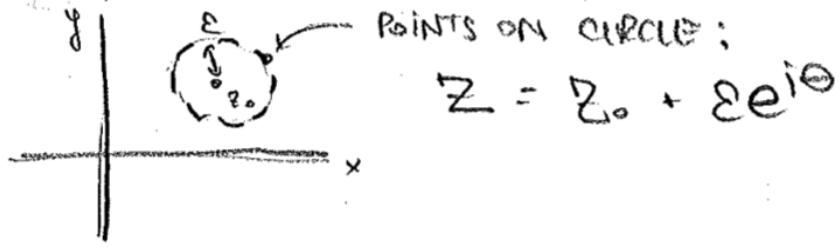
as you would expect from real-number calculus.

5.10 Cauchy Integral Theorem

As we may have referred to earlier—function that are analytic everywhere are too nice. Have you ever read a novel where everyone just got along nicely? Not very interesting. Complex functions are the same—it turns out that integrals of functions around closed curves in domains where they are analytic end up being zero.

5.10.1 Little tiny circles

Let's show this starting from the simplest possible case. Consider a function f that is analytic in some region $R \in \mathbb{C}$. The boundary of this region is a curve $C = \partial R$. First consider the integral of f around a small circle of radius ε around some point z_0 :



In other words, C can be parameterized by the angle θ . We can write points z on the curve as

$$z(\theta) = z_0 + \varepsilon e^{i\theta} \quad dz = i\varepsilon e^{i\theta} d\theta . \quad (5.42)$$

Then the integral around the little circle around z_0 is

$$\oint_C dz f(z) = \int_0^{2\pi} i\varepsilon e^{i\theta} d\theta f(z_0 + \varepsilon e^{i\theta}) . \quad (5.43)$$

Now we observe that since f is analytic, we can differentiate it—which means we can write it as a Taylor expansion:

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \mathcal{O}(\varepsilon^2) , \quad (5.44)$$

where we recognize that $z - z_0 = \varepsilon e^{i\theta}$. Plugging this in gives

$$\oint_C dz f(z) = \int_0^{2\pi} i\varepsilon e^{i\theta} d\theta f(z_0) + \int_0^{2\pi} i\varepsilon^2 e^{2i\theta} d\theta f'(z_0) + \dots . \quad (5.45)$$

Observe that the only θ -dependence in the integrand shows up in factors of $e^{ni\theta}$ for positive integers n . However, we note that

$$\int_0^{2\pi} d\theta e^{in\theta} = \frac{1}{in} (e^{2\pi in} - e^0) = 0 . \quad (5.46)$$

What we find is that

$$\oint_C dz f(z) = 0 \quad (5.47)$$

for the contour C being a small circle around some arbitrary point z_0 inside the region R in which f is analytic.

Exercise 5.6 *Convince yourself that it didn't matter that C is a small circle. It could have been any small shape.*

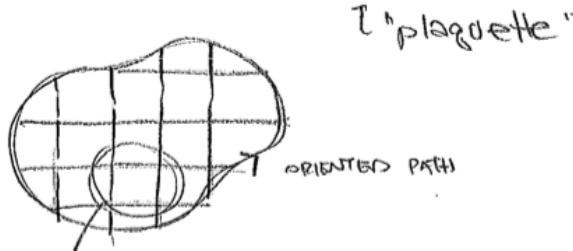
Notice that we did not rely on $\varepsilon \rightarrow 0$ in order to motivate this argument. The circle didn't actually have to be small. We used $\varepsilon \rightarrow 0$ to motivate the idea of truncating the Taylor series. But armed with (5.46), we realize that *every* term in the Taylor series will vanish, no matter how high the power since that simply corresponds to some larger positive integer n . At this point you should start to wonder whether there may be any loopholes in this argument.

5.10.2 Finite regions

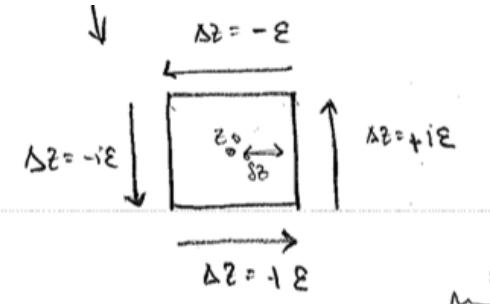
Let's now prove a more general version of this known as the **Cauchy Integral Theorem**. If f is analytic in a connected region $R \in \mathbb{C}$ with some boundary $C = \partial R$, then

$$\oint_{C=\partial R} dz f(z) = 0 , \quad (5.48)$$

even if R is some *finite* region, not just some infinitesimally small circle. From the discussion of the case where C is a little circle, you may have already guessed this. Let's show this more carefully. For any such region R , let's break it up into boxes:



The boundary of a region ∂R has some orientation. By convention a positive orientation corresponds to counterclockwise. Now that we've carved up the region R into a grid like Manhattan⁶⁶, we integrate around the boundary of one of these *plaquettes*:



We've assumed that each plaquette has characteristic size ϵ . Note that for each side of the plaquette the orientation of Δz is different. Let z_0 correspond to the center of this plaquette. Since the function is analytic, we can write the function as a Taylor function around the plaquette:

$$f(z) = f(z_0) + f'(z_0)(z - z_0)^2 + \dots \quad (5.49)$$

This means that we can approximate the integral around the square as

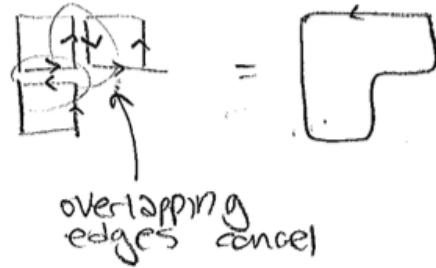
$$\oint_{\text{plaquette}} dz f(z) = f(z_0) (\Delta z_1 + \Delta z_2 + \Delta z_3 + \Delta z_4) + \dots \quad (5.50)$$

where the Δz_i are given in the figure above. It should be clear that the sum of the Δz_i is zero since opposite sides give equal and opposite contributions. We have shown that to leading order the integral of an analytic function f around a little box around it is zero—not too surprising given our previous result for integrating around a little circle.

⁶⁶Search for ‘Manhattanhenge.’ It has nothing to do with this course, but this is a photogenic consequence of the Manhattan grid.

Exercise 5.7 What about the next-to-leading order contribution⁶⁷? Show that the next-to-leading order contribution, which goes like $\sum_i f'(z_0) \delta z_i \Delta z_i$ vanishes. Here the δz_i is the separation from z_0 to the i^{th} side, as shown in the figure above.

The above statement is true about each plaquette. However, note that when we piece the plalettes together, the sum of the integrals around each little boundary corresponds to the integral around the boundary of the combined region:



One can see that neighboring boundaries have opposite orientations so that the integrals along those regions cancel. This means that the integral around a the boundary C of finite region R is equivalent to the sum of the integrals around the little plalettes that tile that region:

$$\oint_C dz f(z) = \sum_i \oint_{\text{plaquette}_i} dz f(z) = 0 . \quad (5.51)$$

Since the integral around the boundary of each plaquette is zero, the integral along C is zero. This proves the Cauchy integral theorem, which may now be stated as follows:

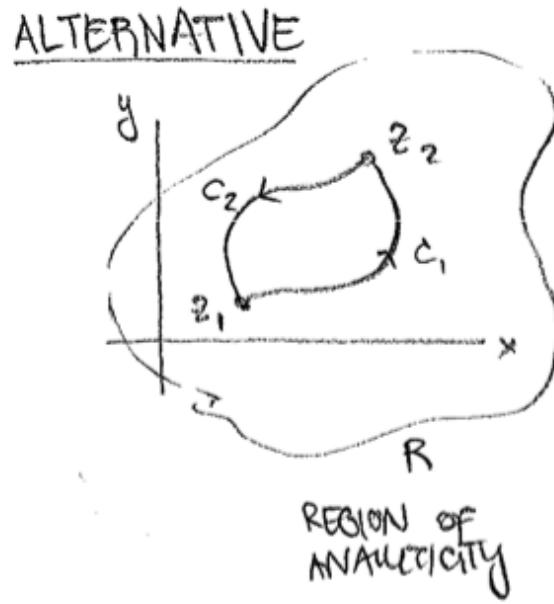
Analytic functions are *so* nice that they're boring.

In other words, if a function f is analytic in a connected region R and you try to integrate over a closed path C that is inside R , then the result is zero.

5.11 An alternative argument (optional)

For those who are mathematically inclined, here's a geometrically-motivated argument for Cauchy's integral theorem. Consider two points z_1 and z_2 in a connected region R where a function f is analytic. Now consider any path C_1 that connects z_1 to z_2 and any other path C_2 that connects z_2 to z_1 . The orientation matters.

⁶⁷Life advice: “What about the next-to-leading order contribution” is a good question to ask whenever you have shown that the leading order contribution is zero.



In the region of analyticity, a function $f(z)$ has an anti-derivative $F(z)$ such that $f(z) = dF(z)/dz$, recall Exercise 5.5. Then we know that

$$\int_{C_1} dz f(z) = F(z_2) - F(z_1) \quad \int_{C_2} dz f(z) = F(z_1) - F(z_2) . \quad (5.52)$$

We deduce that the integral of the combined path $C = C_1 + C_2$ is zero. The above argument is true without having to specify what $F(z)$ is and for *any* paths C_1 and C_2 that share common endpoints. We thus prove Cauchy's Integral Theorem.

Commentary This argument highlights a general idea in differential geometry, which is the generalized Stokes' theorem. In words, one can state this as:

The integral of the derivative of some function f over some domain D is simply the function evaluated on the boundary of the domain, ∂D . The ∂D is notation for ‘boundary of D .’

Technically it is relevant for the domain D to be sufficiently *nice*; this includes the domain being connected and having a reasonable boundary. Note that the boundary ∂D is always oriented—when we integrate over an interval $[a, b] \in \mathbb{R}$, the overall sign depends on which boundary is ‘on top’ of the integral and which is ‘on the bottom.’ In other words, it matters if we integrate over $[a, b]$ or $[b, a]$.

We write formally as follows:

$$\int_D df = \int_{\partial D} f . \quad (5.53)$$

Here the domain D is an n -dimensional space and df is called a *differential n-form*⁶⁸. From the name you should guess that it is related to the idea of a *one-form* as a dual vector; see the

⁶⁸See 2009.10356 for a recent introduction to the utility of differential forms to elucidate physics at an undergraduate level.

optional discussion in Section 5.6. This is a type of tensor with n indices that is contracted with the ‘volume form’ of the n -dimensional space. Formally it looks like this:

$$df = (\partial_{\mu_1} f_{\mu_2 \dots \mu_n}) dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n} . \quad (5.54)$$

The quantity $dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$ is the volume form and is an *oriented* n -dimensional differential volume element. The funny wedge symbols (\wedge) represent an antisymmetric tensor product. This should not be so surprising: recall that the vector triple product $\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})$ is a totally antisymmetric combination of vectors that produces the volume of the parallelogram with edges corresponding to the three vectors \mathbf{v} , \mathbf{w} , \mathbf{u} . This is precisely $\mathbf{v} \wedge \mathbf{w} \wedge \mathbf{u}$. The n -dimensional wedge product generalizes this notion to n -dimensional volumes.

On the right-hand side of (5.54) is an integral over the *boundary* of D . This is an oriented $(n - 1)$ -dimensional space. The integrand is the differential $(n - 1)$ -form. We thus see that the integral of df over D is related to a lower-dimensional integral of its primitive, f , on the boundary of D .

Stokes’ theorem is the underpinning of everything you’ve seen in vector calculus. In one dimension:

$$\int_a^b dx \frac{df(x)}{dx} = \int_{f(a)}^{f(b)} df = f(b) - f(a) . \quad (5.55)$$

In two dimensions, when you have a vector field $\mathbf{f}(x)$, the appropriate derivative (and the one that pops out of differential form notation) is the curl, $\nabla \times \mathbf{f}(x)$. This becomes an integral with respect to the differential surface element $d\hat{S} = \hat{\mathbf{n}} dS$, which may be chosen to be oriented in the z -direction perpendicular to the plane⁶⁹. Then we have the usual Green’s theorem:

$$\int_S dS \hat{\mathbf{n}} \cdot (\nabla \times \mathbf{f}(x)) = \oint_{C=\partial S} \mathbf{f}(x) \cdot d\mathbf{x} , \quad (5.56)$$

where $d\mathbf{x}$ is a differential line element along the oriented curve C that is the boundary of the integration region S . Finally, the divergence theorem in 3D relates the divergence of a vector field to its value on the surface:

$$\int_V dV \nabla \cdot \mathbf{f}(x) = \int_{S=\partial V} dS \hat{\mathbf{n}} \cdot \mathbf{f}(x) , \quad (5.57)$$

where $\hat{\mathbf{n}}$ is the unit normal vector at each point on the surface S bounding the volume V . These three versions of the ‘fundamental theorem of calculus’ all look tantalizingly similar—and here we see that in fact, they’re all manifestations of the general Stokes’ theorem.

Observe that the ‘boundary of a boundary’ is nothing. If you have a disc, the boundary is a circle. The circle itself has no endpoints—in contrast to an interval. We can start to see some of the neat features of differential geometry when we look at Stokes’ theorem, (5.54), from this perspective. Indeed, one can read Stokes’ theorem as a relation between the differential operator d acting on an integrand and the ‘boundary operator’ ∂ acting on the space. The ‘boundary of a boundary = 0’ mantra can be loosely translated into ‘derivative

⁶⁹Alternatively, $\hat{\mathbf{n}}$ is the unit normal of the 2D curved surface in a 3D space.

of a derivative' is zero; where we are not being technically rigorous at all. You've already seen variants of this in vector calculus:

$$\nabla \times \nabla f(x) = 0 \quad \nabla \cdot \nabla \times \mathbf{f}(x) = 0 , \quad (5.58)$$

and so forth. I used to think vector calculus was very challenging because there seemed to be so many different rules for how to differentiate and integrate. It turns out that differential geometry unifies this nicely into a general mathematical structure that, when applied to specific dimensions of space, produce all of the funny things you learn in undergraduate electrodynamics. It should not surprise you that this mathematical structure has a lot to say about the structure of theories like electrodynamics and gravity⁷⁰.

Exercise 5.8 *To see how this structure shows up in a simple physical system, look up Montgomery's treatment of the Falling Cat Problem. Similarly, Shapere and Wilczek's treatment of swimming animals at low Reynolds number. Wilczek famously won the Nobel prize for his contributions to the gauge theory of the strong nuclear force; a theory based on precisely this type of geometric structure we've discussed here. A final reference is to look up the Parallel Parking Problem, which was a topic of discussion in the first version of this P231 class that I taught in 2016. In that problem, one asks how a car can move transversely given that its only degrees of freedom are to move forward/backward and to turn the steering wheel. For the next two years students raised concerns that this class was hopelessly mathematical—so now here we are at the tail end of an exercise with no specific directions in an optional section of the lecture notes.*

5.12 Cauchy's Integral Formula

Cauchy's Theorem tells us that integrating functions over domains where they are analytic is boring. Cauchy's Integral Formula⁷¹ is the first step to something that is decidedly *not* boring. The formula applies to a function f that is analytic in some connected domain and a path C entirely contained in that domain:

$$f(z_0) = \frac{1}{2\pi i} \oint_C dz \frac{f(z)}{(z - z_0)} . \quad (5.59)$$

This is the statement that the value of some function at some point z_0 is related to an integral of the function around the point. In fact, maybe this shouldn't be so surprising: the right-hand side looks like some kind of average of the function in the neighborhood of the function. Given the relation between analytic functions and harmonic functions, this sounds plausible. On the right-hand side there's a factor of $[2\pi(z - z_0)]^{-1}$ which indeed looks like one is averaging over the circumference around the point z_0 . The factor of i is curious. Those who are familiar with all of this will notice the 'famous' combination $(2\pi i)$.

Let us highlight that the integrand on the right-hand side is

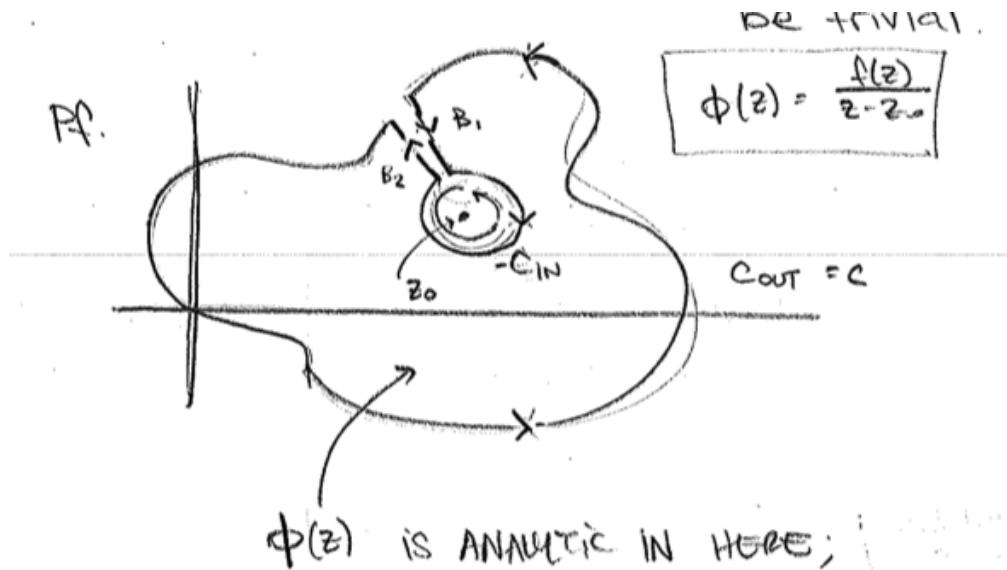
$$g(z) = \frac{f(z)}{z - z_0} . \quad (5.60)$$

⁷⁰One of my favorite introductions: <https://arxiv.org/abs/hep-th/0611201v1>.

⁷¹At this point you wonder: how is it that Cauchy got his name on everything in this business?

Unlike $f(z)$, $g(z)$ is absolutely *not* analytic ‘everywhere’ in the region that we’re looking at. It is not analytic at $z = z_0$. What’s the derivative of $g(z)$ at the singularity? Heck if I know⁷². This is important because it is our first example of a function that is analytic in a region up to a single point⁷³; we say that the function $g(z)$ has a **pole** at $z = z_0$.

Since the integrand $g(z)$ has a pole, we probably shouldn’t integrate over it. No problem, our integration contour C away from z_0 . However, z_0 still punctures our domain over which the integrand is analytic⁷⁴. To see what this does, let’s consider the integral on the right-hand side.



Let us call $C_{\text{out}} = C$, the original contour over which we’re integrating. It is oriented counter clockwise by assumption. Because we’re curious about the singularity at $z = z_0$, let’s deform the contour by building a little bridge B_1 that heads towards z_0 , then a little circle $-C_{\text{in}}$ that goes around the pole⁷⁵, and then a little bridge B_2 that returns to C_{out} where we originally left it. Clearly the integral over B_1 and B_2 cancel because

$$B_1 = -B_2 \quad (5.61)$$

⁷²Is it infinity? I suppose, but the notion of infinity in complex space can be a little tricky. At any rate, usually ‘infinity’ is not the kind of answer that inspires much confidence.

⁷³One observant student said that it is not obvious that $g(z)$ is non-analytic at the singularity $z = z_0$. If we define analytic to mean that a function is independent of z^* , then it is indeed not clear why $g(z)$ is non-analytic at z_0 . This ends up being a failure of that definition: the more constructive definition of analyticity is having a unique and well-defined derivative as defined by the limiting procedure,

$$\lim_{\Delta z \rightarrow 0} [g(z + \Delta z) - g(z)]/\Delta z .$$

⁷⁴We’ve given our domain a topology. This, by the way, perhaps the one of the lamest things you can give someone. Once when I was a child I was sad when my parents got me a pair of socks for my birthday. I’d have been even more disappointed with topology. At least the socks kept my feet warm.

⁷⁵Note the minus sign! We define C_{in} to have positive/counter-clockwise orientation. In the picture above, we see that we traverse this little circle in the clockwise direction, so we put a minus sign on C_{in} .

as paths. Note, however, that the region enclosed by the total curve $C_{\text{out}} + B_1 - C_{\text{in}} + B_2$ is a region in which $g(x)$ is totally analytic. The $-C_{\text{in}}$ boundary separates the pole from the region enclosed⁷⁶. This means that Cauchy's Theorem holds. Because the bridge integrals over B_1 and B_2 cancel, the theorem tells us that

$$\oint_{C_{\text{out}}} dz g(z) - \oint_{C_{\text{in}}} dz g(z) = 0 , \quad (5.62)$$

where the minus sign came from $\oint_{-C_{\text{in}}} = -\oint_{C_{\text{in}}}$. The first term is precisely the right-hand side of (5.59). Apparently the second term is supposed to be $f(z_0)$. We can evaluate the second term along the contour by parameterizing C_{in} as

$$C_{\text{in}} : \quad z(\theta) = z_0 + \varepsilon e^{i\theta} , \quad (5.63)$$

which goes around C_{in} for $\theta \in [0, 2\pi]$. Now watch carefully. The relevant quantities in our integrand are:

$$dz = i\varepsilon e^{i\theta} d\theta \quad z - z_0 = \varepsilon e^{i\theta} . \quad (5.64)$$

Now watch carefully:

$$\oint_{C_{\text{in}}} dz g(z) = \oint_{C_{\text{in}}} dz \frac{f(z)}{z - z_0} = i \int_0^{2\pi} d\theta f(z) = 2\pi i f(z_0) . \quad (5.65)$$

Note that once we wrote the integral with respect to $d\theta$, this is just an ordinary ‘real’ integral where the integrand happens to have complex numbers in it. What is critical is that the powers of $e^{i\theta}$ canceled. Compare this to what happened in (5.46), which was the analogous critical step for showing that the integral of analytic functions vanishes. In (5.65) we ended up with *no* factors of $e^{i\theta}$ so that the $d\theta$ integral ended up being non-zero.

Exercise 5.9 Derive the last equality of (5.65):

$$i \int_0^{2\pi} d\theta f(z) = 2\pi i f(z_0) . \quad (5.66)$$

Why are we able to insert $f(z_0)$ in place of $f(z)$? Does this depend on the smallness of ε ? (Answer: no.) Hint: $f(z)$ is analytic, which means it admits a Taylor expansion. Show that only the zeroth order term contributes, independently of how small ε may be.

Putting this all together gives the desired result,

$$f(z_0) = \frac{1}{2\pi i} \oint_C dz \frac{f(z)}{z - z_0} . \quad (5.67)$$

⁷⁶An engineer, a physicist, and a mathematician are tasked to optimize the amount of space surrounded by a finite length of fencing. The engineer builds a square pen since that makes it simple to construct. The physicist mumbles something about variational principles and builds a circle, stating that it optimizes the area enclosed for fixed perimeter. The mathematician takes the fence, throws away most of it, and then makes a tiny enclosure. The mathematician then carefully steps inside and says, “I declare myself to be on the outside.”

The star of this discussion is not the analytic function $f(z)$; rather it is the analytic-up-to-a-pole function $g(z)$. Unlike our *boring* scenario of functions that are analytic in a given domain, interesting stuff happens when our functions have singularities. We get to live dangerously and dance around these singularities. In general, we will refer to singularities that go like $(z - z_0)^{-n}$ to be poles at $z = z_0$. The positive integer n is called the order of the pole. The case $n = 1$ is called a **simple pole**.

5.13 From Taylor to Laurent

Functions with poles are clearly not analytic *everywhere*. However, they're pretty close to being analytic. They're analytic except for isolated poles. A function that is analytic up to poles is called **meromorphic**. If you're like me, you should classify these as *nice, but not too nice* functions. They're just not-nice enough to be interesting. If you're keeping up with the ‘big picture,’ you’ll recall that the Fourier transform of a differential operator’s Green’s function (5.8) appears to be in this class.

In a region where f is analytic, differentiability meant that one could write a Taylor expansion: a series of terms that go like $(z - z_0)^n$ for positive integers n . When f is merely *meromorphic*, the Taylor expansion is generalized to a **Laurent expansion**:

$$f(z) = \sum_{n=-N}^{\infty} a_n(z_0)(z - z_0)^n , \quad (5.68)$$

where N is the order of the pole at z_0 (if there is one).

5.14 The Residue Theorem: a first look

Now we arrive at our main tool. Suppose $f(z)$ is meromorphic in some region of the complex plane. In fact, suppose $f(z)$ has a simple pole at z_0 ; the function $g(z)$ in (5.60) is a function precisely of this type. Consider the integral of $f(z)$ around a closed contour that goes around the pole once. For example, the pole is in some connected region R and the contour is the boundary of this region, $C = \partial R$. Applying the Laurent expansion about z_0 for this meromorphic function f gives

$$\oint_C dz f(z) = \oint_C dz \left[\sum_{n<0} a_n(z - z_0)^n + \sum_{n \geq 0} a_n(z - z_0)^n \right] . \quad (5.69)$$

All we have done is separated the positive-power terms of the Laurent expansion from the negative power terms. We know that the positive power terms integrate to zero because those terms are analytic in R . Thanks, Cauchy’s Theorem.

What about the term with negative powers? Since we assumed that z_0 is a simple pole, we know that only the a_{-1} term is non-zero in this Laurent expansion about z_0 . That means that we can write the integral as

$$\oint_C dz f(z) = \oint_C dz \frac{a_{-1}}{z - z_0} . \quad (5.70)$$

The coefficient a_{-1} of the Laurent expansion about a pole is called the **residue** of the function f at the pole z_0 . We will use the notation $\text{Res}_f(z_0)$.

Now recall Cauchy's Integral Theorem. For the sake of clarity (I have used f too many times), let us write the integral theorem with respect to an analytic-in-this-neighborhood function h :

$$\oint_C \frac{h(z)}{z - z_0} = 2\pi i h(z_0) , \quad (5.71)$$

we've gone ahead and re-arranged some terms. Comparing this to (5.70), we may take $h(z) = a_{-1} = \text{Res}_f(z_0)$. This means that $h(z_0) = \text{Res}_f(z_0)$ so that we ultimately have

$$\oint_C dz f(z) = 2\pi i \text{Res}_f(z_0) . \quad (5.72)$$

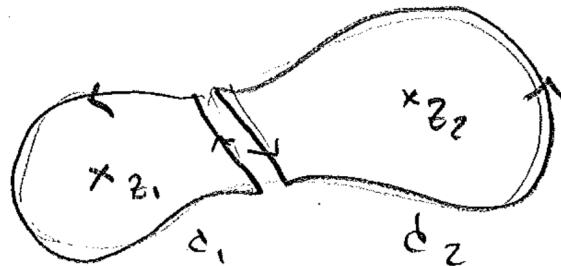
We will generalize this shortly, but the main idea is in this simple example. If you can identify the a_{-1} coefficient of a meromorphic function at its pole, then you can easily integrate the function around the pole. As long as there aren't any other poles in the neighborhood, it doesn't matter what your contour is as long as you are going around counter-clockwise and you go around exactly once.

5.15 The Residue Theorem: more carefully

The residue theorem is our primary tool for calculating integrals on the complex plane. Let's see how it generalizes. Consider a meromorphic function $g(z)$ with two simple poles:

$$g(z) = \frac{h(z)}{(z - z_1)(z - z_2)} \quad h(z) \text{ analytic over } \mathbb{C} . \quad (5.73)$$

The simple poles are located at $z = z_1$ and one at $z = z_2$. Consider a contour C that encloses both poles. The integral of $g(z)$ along C is simply $2\pi i$ times the sum of the residues of the two simple poles. This is easy to see from dividing C into the sum of two paths:



Here C_1 is a closed contour that encloses z_1 but not z_2 . Similarly, C_2 is a closed contour that encloses z_2 but not z_1 . Observe that C_1 and C_2 overlap along a line that separates the two poles. Because both C_1 and C_2 are positively oriented, the integrals along this line cancel. From this it is clear that

$$\oint_C dz g(z) = \oint_{C_1} dz g(z) + \oint_{C_2} dz g(z) = 2\pi i (\text{Res}_f(z_1) + \text{Res}_f(z_2)) . \quad (5.74)$$

This gives a more general form of the residue theorem. The integral of a meromorphic function $f(z)$ around a closed contour C that is the boundary of some region of the complex plane⁷⁷ is

$$\oint_C dz f(z) = 2\pi i \sum_{i \in \text{poles}} \text{Res}_f(z_i) , \quad (5.75)$$

where the sum is over the poles of f at z_i enclosed inside C .

Exercise 5.10 How would the residue theorem change if the contour C were oriented in the opposite direction? What if the contour circled the poles multiple times? What if the contour circled some poles some number of times, and other poles a different number of times?

5.16 Non-simple poles (a case study in being careful)

So far we've focused only on meromorphic functions that have *simple* poles. What about higher order singularities? Here's a tangible example:

$$\oint_C dz f(z) = ? \quad f(z) = \frac{1}{(z - 2i)^2} , \quad (5.76)$$

where C is a positively oriented contour that circles the second-order pole $z_0 = 2i$. Because $f(z)$ has no simple pole at z_0 , it looks like there's no contribution to the integral. To check this, we recall Cauchy's integral formula, (5.65). The reason why simple poles contributed to the contour integral is the observation that for an integer, n , the integral of $e^{in\theta} d\theta$ behaves as follows:

$$\int_0^{2\pi} d\theta e^{in\theta} = \begin{cases} 0 & \text{if } n \neq 0 \\ 2\pi & \text{if } n = 0 \end{cases} . \quad (5.77)$$

Following the same in the previous sub-sections, the integral in (5.76) about a contour C that circles $z_0 = 2i$ once is equivalent to the integral over smaller contour C' that is a small circle that surrounds z_0 . Parameterize C' by the angular variable:

$$z(\theta) = z_0 + \varepsilon e^{i\theta} \quad dz = i\varepsilon e^{i\theta} d\theta . \quad (5.78)$$

Then the integral is:

$$\oint_C dz f(z) = \oint_{C'} dz f(z) = \int_0^{2\pi} i\varepsilon e^{i\theta} d\theta \frac{1}{\varepsilon^2 e^{2i\theta}} = \frac{i}{\varepsilon} \int_0^{2\pi} d\theta e^{-i\theta} = 0 . \quad (5.79)$$

So this is all consistent with the mantra of *find the a_{-1} coefficient of the Laurent expansion, that's the residue*.

We should be careful, though. If we're too slick we can convince ourselves of wrong things. For example, suppose we wanted to generalize (5.76) by changing the numerator of $f(x)$:

$$\oint_C dz f(z) = ? \quad f(z) = \frac{h(z)}{(z - 2i)^2} , \quad (5.80)$$

⁷⁷Note that this means that C is positively oriented and only circles around the region once.

where $h(z)$ is an analytic function. For the sake of argument, let's assume that $h(z)/(z - 2i)^2$ has been simplified so that there are no common factors between the numerator and denominator⁷⁸. One might think that when we shrink the contour from C to C' , we can approximate $h(z) = h(z_0)$ so that

$$\oint_{C'} dz f(z) = \oint_{C'} i\varepsilon e^{i\theta} d\theta \frac{h(z_0)}{(z - 2i)^2} = \frac{i}{\varepsilon} h(z_0) \int_0^{2\pi} d\theta e^{-i\theta} = 0 ? \quad (5.81)$$

In general, this argument is *wrong*. It is wrong even though the (5.79) happens to be true.

Exercise 5.11 *Can you spot the incorrect assumption?*

The incorrect assumption is that we could approximate $h(z) = h(z_0)$ because the little circle C' is always close to z_0 . The intuition is fine, but we weren't careful enough: even though $h(z)$ is analytic everywhere, we should remember write a Taylor expansion:

$$h(z) = h(z_0) + h'(z_0)(z - z_0) + \dots \quad (5.82)$$

Even if our intent is to only keep the first term, it's a good habit to remember that the Taylor expansion contains many terms. Let's see what happens:

$$\oint_{C'} dz f(z) = \oint_{C'} i\varepsilon e^{i\theta} d\theta \frac{h(z_0) + h'(z_0)(z - z_0) + \dots}{(z - 2i)^2} \quad (5.83)$$

$$= \frac{ih(z_0)}{\varepsilon} \int_0^{2\pi} d\theta e^{-i\theta} + ih'(z_0) \int_0^{2\pi} d\theta + \dots \quad (5.84)$$

While the first term vanishes, the second term is clearly non-zero since it's a contour integral. Indeed, one ends up with

$$\oint_{C'} dz \frac{h(z)}{(z - 2i)^2} = 2\pi i h'(z_0), \quad (5.85)$$

where evidently $\text{Res}_f(2i) = h'(2i)$ is the residue of $f(z) = h(z)/(z - 2i)^2$ at $z_0 = 2i$.

Exercise 5.12 *Why don't higher order terms in (5.84) do not contribute to the contour integral.*

The purpose of this example was to show that it can be a little tricky to identify the residue of a function by simply 'looking at the denominator.' You will derive an explicit formula in your homework. Fortunately, we will rarely consider non-simple poles in this course.

Let's go through a few examples. These are from the third edition of Boas, chapter 14.6.

Example 5.7 *Let $f(z) = \cot z$. Find the residue of f at $z = 0$.*

To solve this, we can write out the numerator and denominator of the cotangent:

$$\cot z = \frac{\cos z}{\sin z} = \frac{1 - z^2/2 + \dots}{z - z^3/3!}. \quad (5.86)$$

From this we see that $z \rightarrow 0$ is indeed a simple pole, and one may write the residue as

$$\text{Res}_f(0) = \lim_{z \rightarrow 0} z \frac{1}{z} = 1. \quad (5.87)$$

⁷⁸For example, if $h(z) = (z - 2i)$ then clearly $f(z)$ has a simple pole at $z_0 = 2i$ so the integral picks up a non-zero residue.

Example 5.8 Let $f(z) = \cot^2 z$. Find the residue of f at $z = 0$.

Writing out the numerator and denominator again:

$$\cot^2 z = \frac{1 - z^2 + \dots}{z^2 - 2z^3/3! + \dots} \sim \frac{1}{z^2} + \mathcal{O}(1) . \quad (5.88)$$

From this we deduce that $\text{Res}_f(0) = 0$.

Exercise 5.13 Why were we able to be slick in Example (5.8) after making a big deal about being careful in (5.84)?

Exercise 5.14 Let $f(z) = z \cot^2 z$. Find the residue of f at $z = 0$.

Exercise 5.15 Let $f(z)$ be

$$f(z) = \left(\sum_{n=0}^{\infty} c_n z^n \right) \cot^2 z \quad (5.89)$$

Show that the residue of f at $z = 0$ is c_1 .

5.17 The Killer App: Real Integrals

The *killer app*⁷⁹ for complex contour integrals is to solve integrals along the *real line*. It is often the case that some of the real functions that we would like to integrate happen to have poles in the complex plane. In this case, the integral along the real line can be completed into a closed contour C in the complex plane. If we can separate the contribution of the real line from the rest of the contour, then we can use the residue theorem to do the integral without any of the hard work of, uh, integrating.

Example 5.9 Consider the real function

$$f(x) = \frac{1}{x^2 + 1} . \quad (5.90)$$

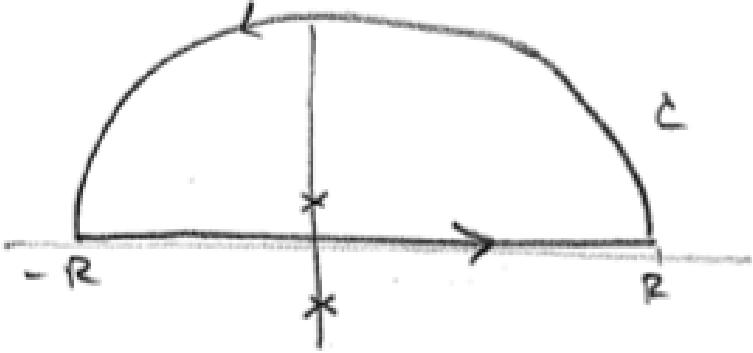
Typically we care about real functions with real arguments. However, let's analytically continue $f(x)$ into a meromorphic function $f(z)$

$$f(z) = \frac{1}{z^2 + 1} . \quad (5.91)$$

This is the (unique⁸⁰) complex function that is analytic and agrees with the original function on the real line. We immediately notice that $z^2 + 1 = (z + i)(z - i)$ so that $f(z)$ has simple poles at $z = \pm i$. Consider the following contour which includes the real interval $x \in [-R, R]$ as part of it:

⁷⁹This is an old phrase from the first dot-com bubble.

⁸⁰The proof is left to you to derive or look up.



We will care about the limit $R \rightarrow \infty$ where the integral includes the entire real line. Observe that the contour only encloses the pole at $z = i$. Let us integrate the function around C . From the residue theorem, we have

$$\oint_C dz \frac{1}{(z+i)(z-i)} = 2\pi i \text{Res}_f(i) = \pi , \quad (5.92)$$

where we've used $\text{Res}_f(i) = 1/2i$. However, we can also write the integral as a sum of the real integral plus a counter-clockwise arc of radius R :

$$\oint_C dz f(z) = \int_{-R}^R dx f(x) + \int_0^\pi iRe^{i\theta} d\theta f(Re^{i\theta}) . \quad (5.93)$$

where in the second term we've used the parameterization $z(\theta) = Re^{i\theta}$. Note that the first term on the right-hand side is the ‘ordinary’ purely real integral. The second term behaves as follows:

$$\int_0^\pi iRe^{i\theta} d\theta f(Re^{i\theta}) = \frac{Re^{i\theta}}{R^2 e^{2i\theta} + 1} d\theta . \quad (5.94)$$

As we take the limit $R \rightarrow \infty$, the integral goes like $\sim 1/R \rightarrow 0$. Thus as long as we actually care about the integral along the entire real line, $R \rightarrow \infty$, we have

$$\oint_C dz \frac{1}{(z+i)(z-i)} = \pi = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{x^2 + 1} . \quad (5.95)$$

And so we find

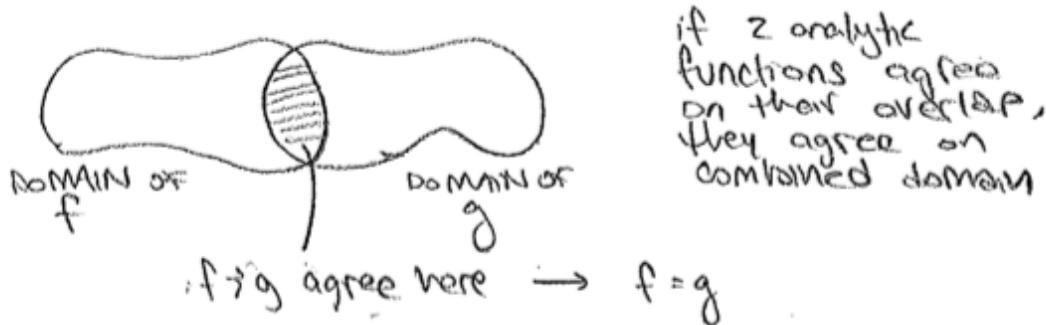
$$\int_{-\infty}^{\infty} dx \frac{dx}{x^2 + 1} = \pi . \quad (5.96)$$

This is an absolutely correct result about a real integral that we sneakily derived without actually doing the integral. I assume one may check this by the appropriate trigonometric substitution⁸¹. Observe that the critical step was that the curved part of the contour (the one that actually had imaginary parts) went to zero in a well defined limit.

⁸¹Just kidding, you can just check it on Mathematica. You can even tell people that you did the trig substitution if it makes you feel better.

Exercise 5.16 What result do you get when you solve for the same integral of function $f(x) = (x^2 + 1)^{-1}$ over the real line, but this time you use the contour C' that encircles the lower half of the complex plane?

Let's address the question of **analytic continuation**⁸². We started with a purely real function $f(x)$ and 'generalized it' to a complex function $f(z)$ that happened to have poles in the complex plane. Because we used the residue theorem, the details of these poles (residue and whether they're enclosed by C) are critical for performing the integral. So a good question is: is the *complex function* $f(z)$ even uniquely defined given the *real function* $f(x)$?⁸³



There is a handy theorem that states that when two analytic functions agree on their domain of overlap, then they agree on their common domain. For *meromorphic* functions, we can just imagine the domain where those singularities are not included. In our example above, $f(x)$ is analytic along the real line and $f(z)$ is a function that trivially (by construction) is analytic and matches $f(x)$ on the real line. Then the theorem says that $f(z)$ and $f(x)$ 'agree' in the entire domain of analyticity; so one can *analytically continue* $f(x)$ to the complex plane as long as one avoids any singularities.

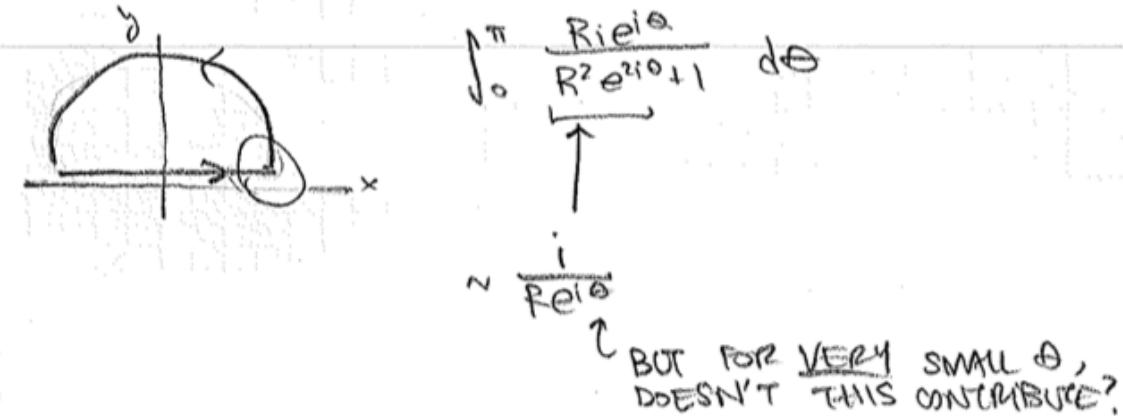
The gist of the proof is that if two functions f and g agree in some overlapping domain of analyticity, then their difference $h(z) \equiv f(z) - g(z)$ is also an analytic function in this domain. Analyticity means one has finite radius of convergence because it has a well defined Taylor expansion. One can then use this to argue that $h(z)$ can be defined beyond the overlapping domain. Continuity requires that $h(z)$ remains zero even outside the overlapping domain, which ensures that $f(z) = g(z)$. In other words, there is a unique analytic continuation of a function to a maximal domain of analyticity.

There's another effect you may wonder about. Are there some 'edge' effects from the corner of the contour?

⁸²See Appel section 5.3 for a more careful discussion.

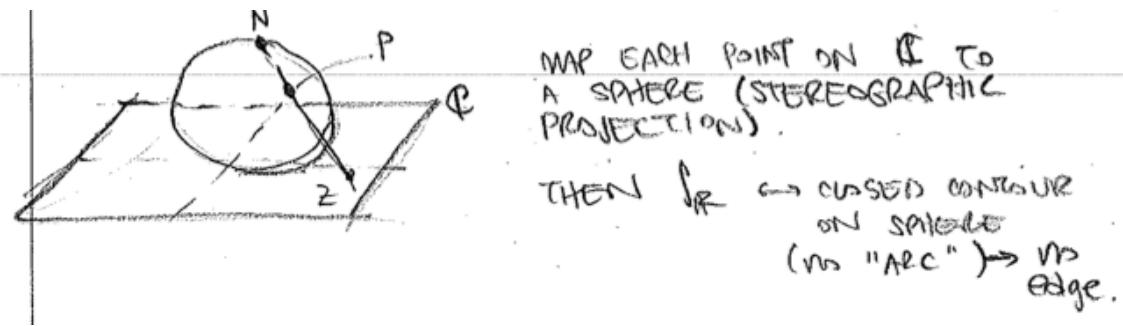
⁸³My notation is a little glib and assumes the answer. If you want, you can write $f_{\mathbb{C}}$ and $f_{\mathbb{R}}$ to differentiate the two functions that are different in principle.

ANOTHER CONCERN : EDGE EFFECTS?



A hand-waving answer is to say that we take the $R \rightarrow \infty$ limit ‘first’ and so that corner is pushed off into infinity. This should be totally unsatisfying since mathematical physics *rarely* depends on the order in which limits are taken.

A better answer is that the complex plane *with infinity* is a slightly different object from the complex plane without infinity. One way to see this is to note that the complex plane can be mapped one-to-one onto a sphere. This is called *stereographic projection*. The picture is this:



The idea is to place a unit sphere at the origin of \mathbb{C} . The south pole of the sphere is touching $z = 0$. Then for any point $z \in \mathbb{C}$, one can draw a line from z to the north pole of the sphere. There is a single unique point on the sphere that intersects that line. This maps the entire complex plane onto the sphere—except for the north pole of the sphere. We identify the north pole with complex infinity. That’s right: all infinities are the same: $\infty, i\infty, \infty e^{2\pi/3}, -\infty$, etc. From this point of view, the real line maps onto the prime meridian of the sphere and there are no ‘corners’ in the contour. In fact, there’s no arc component, either! The integral along the real line is a closed contour with respect to the stereographic projection. Note that the orientation of the contour depends on which hemisphere you are figuratively standing in.

5.18 The Main Example: follow this carefully

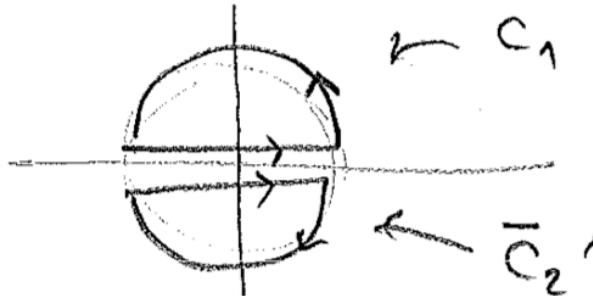
Let's try another instructive example. Here's an real integral that we would like to solve using contour integral techniques:

$$\int dx f(x) = \int_{-\infty}^{\infty} dx \frac{2 \cos x}{x^2 + 1}. \quad (5.97)$$

We can analytically continue this into a complex function by simply replacing the real variable with a complex variable, $x \rightarrow z$. This complex function $f(z)$ is analytic on the real line. The presence of simple poles at $z = \pm i$ mean that it is meromorphic with respect to the whole complex plane. It is convenient—for reasons that will be clear momentarily—to write the cosine as a sum of exponentials:

$$f(z) = \frac{e^{iz} + e^{-iz}}{(z+i)(z-i)} = \frac{e^{iz}}{(z+i)(z-i)} + \frac{e^{-iz}}{(z+i)(z-i)} \equiv f_+(z) + f_-(z). \quad (5.98)$$

With some foresight, we have separated $f(z) = f_+(z) + f_-(z)$ into two different functions. There are now two obvious choices for convenient integration contours that include the real line with the appropriate orientation:



We label the two contours are C_1 and \bar{C}_2 . The bar over \bar{C}_2 is to remind us of the clockwise orientation; alternatively one could⁸⁴ have written $\bar{C}_2 = -C_2$. Does it matter which contour we pick?

Our criteria for a convenient contour are that (1) the contour includes the real line with the appropriate orientation and (2) the rest of the contour integrates to zero. That way we can use the residue theorem:

$$\int_C f(z) dz = \int_{-\infty}^{\infty} dx f(x) + \int_{\text{arc}} dz f(z) = \int_{-\infty}^{\infty} dx f(x) = \sum_{i=\text{poles}} \text{Res}_f(z_i), \quad (5.99)$$

where we recall that i is runs over the poles enclosed by the contour C . The condition that the integral over the arc vanishes determines which contour we take. We may conveniently parameterize the arcs as follows:

$$z(\theta) = R \cos \theta + iR \sin \theta \quad (5.100)$$

⁸⁴This all boils down to notation. You are free to be creative with notation, but the underlying quest is *clarity*. Notation doesn't change the underlying mathematics or physics, but a good notation make your ideas clearer—perhaps even to yourself.

where for C_1 the arc is given by $\theta \in [0, \pi]$ while for \bar{C}_2 the arc is $\theta \in [0, -\pi]$. We assume the limit $R \rightarrow \infty$. The arc integrals are then:

$$\int_{\text{arc}} dz f_{\pm}(z) = \int_{\text{arc}} d\theta \frac{e^{\pm iz}}{z^2 + 1}. \quad (5.101)$$

When we plug in the parameterization (5.100), we see that the denominator of f_{\pm} scales like R^2 . There's an additional factor of R in the numerator coming from dz , and so it may appear that either C_1 or \bar{C}_2 could be used. However, recall that $f_{\pm}(z) \sim e^{\pm iz}$ and exponentials beat polynomials. Plugging in $z(\theta)$ into the exponentials gives

$$e^{\pm iz} = e^{\pm i(R \cos \theta + iR \sin \theta)}, \quad (5.102)$$

where the difference between C_1 and \bar{C}_2 is the range (and direction) of θ .

- C_1 lives in the upper half plane so the imaginary part of z is positive: $R \sin \theta > 0$.
- C_2 lives in the lower half plane so the imaginary part of z is negative: $R \sin \theta < 0$.

This makes it easy to read off the convergence properties of the exponential

$$e^{\pm iz} = (\text{oscillating}) e^{\mp R \sin \theta}. \quad (5.103)$$

The oscillating part is not relevant: since $|e^{i\varphi}| = 1$, this factor doesn't actually affect the magnitude of the contribution of the integrand, just the phase. The $\exp(\mp R \sin \theta)$ factor, on the other hand, is everything.

- $e^{-R \sin \theta}$ is exponentially suppressed when $\sin \theta > 0$, this corresponds to the upper half plane and so we use the C_1 arc to complete the integration contour of the $f_+(z) \sim \exp(-R \sin \theta)$ integrand.
- $e^{+R \sin \theta}$ is exponentially suppressed when $\sin \theta < 0$, this corresponds to the lower half plane and so we use the \bar{C}_2 arc to complete the integration contour of the $f_-(z) \sim \exp(+R \sin \theta)$ integrand.

Because the e^{-R} exponential suppression defeats the polynomial R^{-1} scaling, the vanishing of the integral along the arc is determined only by this exponential factor: $\lim_{R \rightarrow \infty} e^{-R}/R \rightarrow 0$. We thus have the following approach to the original real integral:

$$\int f(x) = \int_{-\infty}^{\infty} dx \frac{2 \cos x}{x^2 + 1} = \int_{-\infty}^{\infty} dx \frac{e^{+ix}}{x^2 + 1} + \int_{-\infty}^{\infty} dx \frac{e^{-ix}}{x^2 + 1} \quad (5.104)$$

$$= \int_{C_1} dz \frac{e^{+iz}}{z^2 + 1} + \int_{\bar{C}_2} dz \frac{e^{-iz}}{z^2 + 1} \quad (5.105)$$

$$= 2\pi i \text{Res}_{f_+}(i) - 2\pi i \text{Res}_{f_-}(-i). \quad (5.106)$$

In the second step we have used the fact that the integral along the real line can be extended by large arcs along C_1 or \bar{C}_2 as long as the integral along those arcs vanishes. The residues are straightforward:

$$\text{Res}_{f_{\pm}}(\pm i) = \pm \frac{1}{2ie} \quad (5.107)$$

so that the final result is

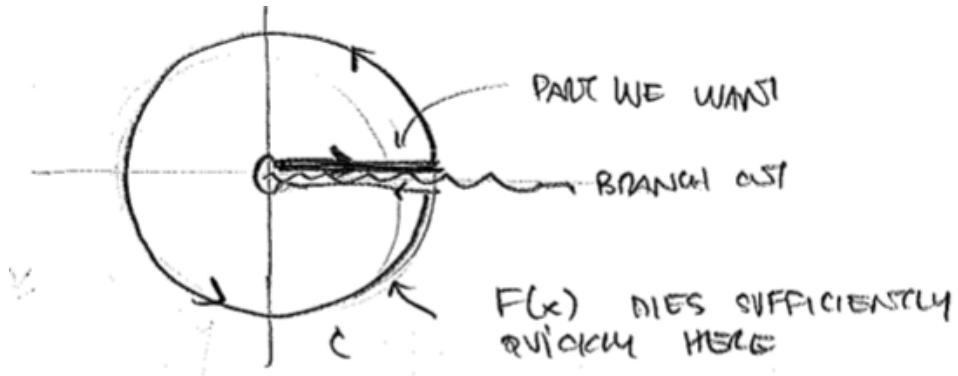
$$\int_{-\infty}^{\infty} dx f(x) = \frac{2\pi}{e}. \quad (5.108)$$

5.19 A fancy example

Here's an example of the residue theorem used to calculate an integral along just the positive real line,

$$\int_0^\infty dx x^{1/3} F(x) . \quad (5.109)$$

We can trivially analytically continue the integrand by replacing $x \rightarrow z$. Let us assume that $F(z)$ is a meromorphic function with poles away from the positive real line. Let us further assume further that $F(z)$ is exponentially suppressed for $|z| = R \gg 1$, say $F(z) \sim e^{-R}$. Note that the $x^{1/3} \rightarrow z^{1/3}$ factor is problematic. It tells us that there's an *branch cut* in our integrand: if we start at some point and went a full circle around the origin, say $z(\theta) = e^{i\theta}$ for $\theta \in [0, 2\pi]$, then $z^{1/3}$ doesn't return to the origin. This means you probably need to extend the complex plane into Riemann sheets. Practically what this means is that we have to pick a branch cut that prevents us from taking contours that are 'full circles' around the origin. We have the freedom to pick the orientation of this branch cut, but in this case it is convenient to pick it along the positive real line⁸⁵. Let's take a peculiar contour that includes the integration region we want, $x \in [0, \infty)$, then circles the entire complex plane, and returns back to zero along $x \in (\infty, 0]$.



In the past, we'd have said that the integral along $x \in [0, \infty)$ should cancel that from $x \in (\infty, 0]$. However, we now notice that there's a branch cut separating those two regions. Indeed, you may already see that the *integrand* is not identical along those two lines.

Construct the following contour integral:

$$\oint_C dz z^{1/3} F(z) = \int_0^\infty dx x^{1/3} F(x) + \int_{\text{arc}} dz z^{1/3} F(z) + \int_\infty^0 dx (xe^{2\pi i})^{1/3} F(e^{2\pi i}) . \quad (5.110)$$

The left-hand side is simple to evaluate with the residue theorem and will depend on the poles of the unspecified function $F(z)$. On the right-hand side, the integral along the arc vanishes by the assumptions we've made about $F(z)$ becoming exponentially small for large arguments, $|z| = R \gg 1$. Finally, we notice that the third term on the right-hand side we've written

⁸⁵If you want to be careful to make sure that you're not interfering with the original integral, you can put the branch cut just below the real line by some vanishingly small amount and take that amount to zero at the end of the calculation.

$z = xe^{2\pi i}$; this is critical since it includes the phase that z picks up when traversing the large circular path. Since $F(z)$ is meromorphic, we may simply replace $F(x \exp(2\pi i)) \rightarrow F(x)$. However, the $z^{1/3}$ needs some care. We find:

$$2\pi i \sum_{j=\text{poles}} z_j^{1/3} \text{Res}_F(z_j) = (1 - e^{2\pi i/3}) \int_0^\infty dx x^{1/3} F(x) . \quad (5.111)$$

The $-e^{2\pi i/3}$ came from the $1/3$ -power of the phase. The minus sign comes from $\int_\infty^0 dx = -\int_0^\infty dx$. Converting the difference of exponentials into a sine gives

$$(1 - e^{2\pi i/3}) = 2e^{i\pi/3} \cdot \frac{1}{2} (e^{-i\pi/3} - e^{i\pi/3}) = -2e^{i\pi/3} \sin(\pi/3) . \quad (5.112)$$

We thus find an expression for the original integral:

$$\int_0^\infty dx x^{1/3} F(x) = \sum_{j=\text{poles}} \frac{\pi i e^{i\pi/3} z_j^{1/3} \text{Res}_F(z_j)}{\sin(\pi/3)} . \quad (5.113)$$

We won't worry too much about branch cuts in this course. However, they do occasionally show up in physical manifestations, such as in dispersion relations⁸⁶.

6 The Harmonic Oscillator

Let's return to the problem of solving for Green's functions. Let's focus on our favorite example—arguably, the *only* example⁸⁷—is the harmonic oscillator. The differential operator is

$$\mathcal{O} = \left(\frac{d}{dt} \right)^2 + \omega_0^2 . \quad (6.1)$$

The Green's function equation tells us the response $G(t, t_0)$ at time t from a ‘unit displacement’ at t_0 :

$$G''(t, t_0) + \omega_0^2 G(t, t_0) = \delta(t - t_0) . \quad (6.2)$$

Recall that the arguments t and t_0 are analogous to the indices of a finite-dimensional matrix. For notational convenience, we will set $t_0 = 0$ and not list it explicitly. We are primarily concerned about the t -dependence of $G(t, t_0)$.

6.1 Fourier Transform

The first thing we're going to do is write $G(t, t_0)$ as a Fourier transform with respect to t . Please refer to Appendix B for our set of Fourier transform conventions. We can write $G(t)$

⁸⁶See, e.g., <https://arxiv.org/abs/1610.06090> for a peek at how it can show up in quantum field theory.

⁸⁷Upon generalizing to higher dimensions, curvilinear coordinates. Physicists have Harmonic Oscillators in different area codes.

as an integral over Fourier modes with frequency ω and weight (Fourier transform) $\tilde{G}(\omega)$:

$$G(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \tilde{G}(\omega) \quad d = \frac{d}{2\pi}. \quad (6.3)$$

We say that $\tilde{G}(\omega)$ is the Fourier transform of $G(t)$. The key point is that the t -dependence of $G(t)$ has been sequestered into the $e^{-i\omega t}$ plane waves. This is convenient since these plane waves are eigenfunctions of the derivative operator:

$$\frac{d}{dt} e^{-i\omega t} = -i\omega e^{-i\omega t}. \quad (6.4)$$

The left-hand side of the Green's function equation (6.2) is

$$\mathcal{O}_t G(t, t_0) = - \int_{-\infty}^{\infty} d\omega (\omega^2 - \omega_0^2) e^{-i\omega t} \tilde{G}(\omega, t_0). \quad (6.5)$$

The right-hand side is simply the Fourier transform of $\delta(t - t_0)$:

$$\delta(t - t_0) = \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t_0)}. \quad (6.6)$$

Exercise 6.1 Use our conventions for the Fourier transform (6.3) (see also Appendix B) to confirm the Fourier representation of $\delta(t - t_0)$ in (6.6). In our notation, the Fourier coefficients $\tilde{f}(\omega)$ of a function $f(t)$ is

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} f(t). \quad (6.7)$$

So the Green's function equation for the 1D harmonic oscillator, (6.2), tells us

$$- \int_{-\infty}^{\infty} d\omega (\omega^2 - \omega_0^2) e^{-i\omega t} \tilde{G}(\omega) = \int_{-\infty}^{\infty} d\omega e^{i\omega t}. \quad (6.8)$$

For simplicity we have set $t_0 = 0$ and don't write it explicitly. There's a rather unscrupulous⁸⁸ way to solve this equation for $\tilde{G}(\omega)$. Since the two sides of this expression are equal, they have the same Fourier expansion. This implies that the Fourier coefficients are equal. Since both sides are already written as Fourier expansions, we can just match the coefficients of the basis functions, $e^{-i\omega t}$. This gives us:

$$\tilde{G}(\omega) = \frac{-1}{\omega^2 - \omega_0^2} \quad (6.9)$$

⁸⁸I don't think this is rigorously valid, but the result is true. The ends don't justify the means, but let's take this morally ambiguous shortcut to make the big picture clear. I encourage you to live the rest of your lives with virtue.

Exercise 6.2 Prove (6.9) honestly. Hint: Start with (6.8) and project out the Fourier coefficients. Recall that you do this by taking the inner product with one of the basis functions and then using the orthogonality of the eigenbasis. You may need to be careful with the normalization.

Exercise 6.3 In (6.8) we had already set $t_0 = 0$. What is the expression for $\tilde{G}(\omega)$ if we kept t_0 explicit?

That was the critical step: we have successfully solved for the Green's function Fourier coefficient. This means that we have a closed form expression for the Green's function by plugging $\tilde{G}(\omega)$ into (6.3):

$$G(t) = \int_{-\infty}^{\infty} d\omega \frac{-e^{-i\omega t}}{\omega^2 - \omega_0^2}. \quad (6.10)$$

All that's left is for us to actually *do* this integral. Fortunately, this integral should look very similar. It seems to beg for us to solve using the residue theorem.

Exercise 6.4 What are the poles of the integrand in (6.10)? What are their associated residues? What is the residue if $t_0 \neq 0$?

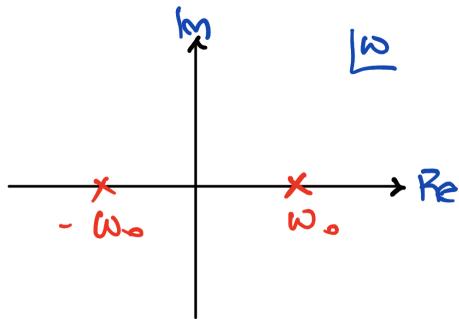
6.2 Contour Integral

The integral (6.10) looks like it's perfect for contour integration. There's an exponential factor on top that will determine the convergence, and the denominator can be factored to see where the poles are. Except we notice something troubling:

$$\frac{-e^{-i\omega t}}{\omega^2 - \omega_0^2} = \frac{-e^{-i\omega t}}{(\omega - \omega_0)(\omega + \omega_0)}. \quad (6.11)$$

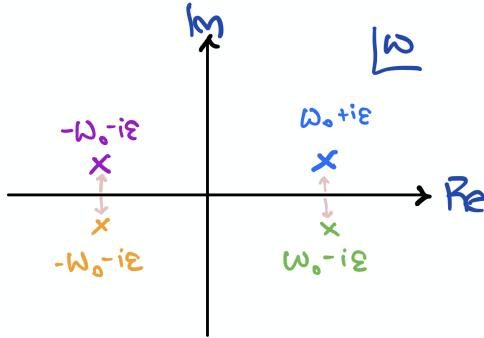
The poles are located at $\omega = \pm\omega_0$. These are *on the real axis*, precisely along the integration contour! How annoying!

Now you may want to have an existential moment. Remind yourself that all we're doing is solving for the behavior of the one-dimensional *harmonic oscillator*. This is an eminently *physical* system. We could have written this system as $\mathcal{O}f(t) = s(t)$ where $f(t)$ is the displacement of a harmonic oscillator and $s(t)$ is some driving function. The Green's function, $G(t, t_0)$ gives the response of the system to a 'unit' driving function, $s(t) = \delta(t - t_0)$. The response of the system should be perfectly physical. And yet—and yet—we now face an integral (6.10) that seems to run right into not only one, but *two* singularities along the integration contour!



Please do take a brief moment to ponder at this existential crisis. This is an apparent failure of our theory—our calculation hits a pole, you get an infinity, and it's not clear what's going on. Rather than anguish, though, we should see if our theory is trying to tell us something.

In order to figure out what's going on, let's do something wild. The poles on the real axis are in the way. Let's just move them out of the way by giving them small imaginary parts⁸⁹. Each pole can either be pushed above or below the real line:



To be clear: are *changing* the integral. We are exploring what happens when the expression (6.10) is changed, as if the poles were slightly complex the whole time. In other words, we are *changing* the original differential operator, (6.1). We're not yet motivating *why* we want to do this; we'll justify it post-facto. For now, let's just be happy that our contours no longer bump into the poles.

6.3 Going under the poles?

Let's try going under the poles, that is: let's push the poles into the upper half-plane with small imaginary parts:



This means that we have deformed the integrand (6.11):

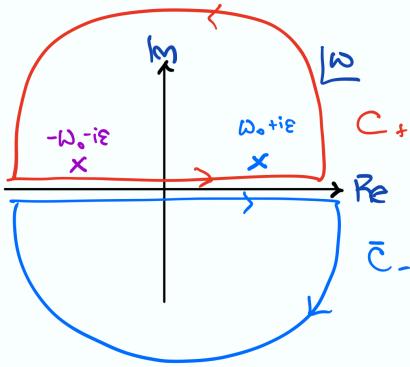
$$\frac{-e^{-i\omega t}}{(\omega - \omega_0)(\omega + \omega_0)} \rightarrow \frac{-e^{-i\omega t}}{(\omega - \omega_0 - i\varepsilon)(\omega + \omega_0 - i\varepsilon)}. \quad (6.12)$$

To make it clear that we are treating ω as a complexified variable, let's write it as z :

$$\omega \rightarrow z. \quad (6.13)$$

Now let's go through the exercise of performing the integral along the real line by using the residue theorem. To do this have to choose a contour that includes the real line. We have two options, C_+ and \bar{C}_- , where the bar reminds us of the orientation:

⁸⁹I have altered the poles. Pray that I do not alter them further.



Which contour do we pick? Does it matter? (Yes!) The contours differ by whether the ‘large arc’ has positive or negative real part. We can parameterize these arcs as we did in (5.100):

$$z(\theta) = R \cos \theta + iR \sin \theta , \quad (6.14)$$

where for C_+ the arc is given by $\theta \in [0, \pi]$ while for \bar{C}_- the arc is $\theta \in [\pi, 2\pi]$. In other words,

$$\text{for } C_+: \sin \theta > 0 \quad \text{for } \bar{C}_-: \sin \theta < 0 . \quad (6.15)$$

How do we pick? The criteria for completing the real line into a contour is that the integral along the contour vanishes, that way

$$\oint_C dz f(z) = \int_{-\infty}^{\infty} dx f(x) + \lim_{R \rightarrow \infty} R \int_{\text{arc}} d\theta f(Re^{i\theta}) = \int_{-\infty}^{\infty} dx f(x) . \quad (6.16)$$

And hence one can use the residue theorem to solve the contour integral on the left-hand side and relate it to the real integral on the right-hand side⁹⁰. So which contour satisfies

$$\lim_{R \rightarrow \infty} R \int_{\text{arc}} d\theta \frac{-e^{-i(R \cos \theta + iR \sin \theta)t}}{(Re^{i\theta} - \omega_0 - i\varepsilon)(Re^{i\theta} + \omega_0 - i\varepsilon)} \rightarrow 0 ? \quad (6.17)$$

I was a little slick and chose to write $z = R \cos \theta + iR \sin \theta$ in the numerator while writing the more compact $z = Re^{i\theta}$ in the denominator. The convergence⁹¹ of the integrand depends on the exponential factor in the numerator:

$$e^{-i(R \cos \theta + iR \sin \theta)t} = e^{-Rt \cos \theta} e^{Rt \sin \theta} . \quad (6.18)$$

We can see that the convergence depends specifically on the second factor, $e^{-Rt \sin \theta}$. Since $R > 0$, this depends on two things: the sign of $\sin \theta$ and the sign of t . The integral along the arc vanishes in the $R \rightarrow \infty$ limit when:

$$t > 0 \quad \text{and} \quad \sin \theta < 0 \quad \text{or} \quad t < 0 \quad \text{and} \quad \sin \theta > 0 . \quad (6.19)$$

⁹⁰I say ‘real integral’ but of course the integrand (6.12) is now a complex quantity, but the point is that we’re only integrating along the real line.

⁹¹I tend to say convergence, what I really mean is ‘vanishes as $R \rightarrow \infty$.’

Or in other words:

$$t > 0 \rightarrow \bar{C}_- \quad t < 0 \rightarrow C_+ . \quad (6.20)$$

Recall that we have set $t_0 = 0$; if you did Exercise 6.3, you would know that restoring t_0 corresponds to $t \rightarrow t - t_0$ in the conditions above.

The Green's function equation (4.41) is simply that

$$\mathcal{O}_t G(t, t') = \delta(t - t') .$$

This tells us that the Green's function $G(t, t')$ is the *response* measured at time t to a δ -function source at t_0 . Thus the sign of $t - t_0$ corresponds to whether we are measuring the response *before* or *after* the cause. If you are like me, your eyes just lit up with joy. Something very exciting has happened: in this odd discussion about pushing around the poles of \tilde{G} , we have found that we were secretly talking about *causality*.

Before we progress, let's be clear: we expect that effects happen *after* causes. This means that physical dynamics obey $t - t_0 > 0$. A nice theory would tell us that you cannot have an effect before a cause⁹².

Causal propagation. For the specific case of $t > 0$ (or $t - t_0 > 0$ if we restore t_0), we observe from (6.19) that the integral (6.12) converges when $\sin \theta < 0$. In other words, for times *after* the initial cause at $t_0 = 0$, the appropriate contour is \bar{C}_- .

Exercise 6.5 *What is the value of the contour integral*

$$\oint_{\bar{C}_-} dz \frac{-e^{-itz}}{(z - \omega_0 - i\varepsilon)(z + \omega_0 - i\varepsilon)} ? \quad (6.21)$$

Hint: you shouldn't have to do any work.

The contour integral above vanishes along the arc when $t > 0$, so it is the correct contour when doing the Fourier transform to get $G(t)$ from $\tilde{G}(\omega)$. However, the contour encloses no poles. No poles means no residues. This means that the contour integral vanishes and the Fourier transform vanishes. In other words,

$$G(t > 0) = 0 . \quad (6.22)$$

Example 6.1 *The statement that $G(t > 0) = 0$ should make you sad and possibly upset. This means that there is no effect that happens after some cause at $t_0 = 0$.*

⁹²At the beginning of this course, we talked about locality as one of the motivating factors for why physical dynamics are described by differential operators. This was closely tied to the idea of causality, since non-local effects with spacelike separation can be non-causal in some reference frame. Perhaps amusingly—but beyond the scope of this course or my expertise—the notion of causality and locality in fundamental theories of nature continues to be a theme in cutting edge theoretical research.

Acausal propagation. By now you see the writing on the wall. Something perverse has happened: after a δ -function disturbance, information is not traveling forward in time because $G(t) = 0$ for $t > 0$. This, in turn, was self-evident in because for $t > 0$, we were required to take the \bar{C}_- contour in order that the contribution of the arc vanishes. At this point, you must feel the dread of what's coming next. If we take the $t < 0$ case in (6.19), we are forced to take the C_+ contour. This contour indeed picks up poles and so we expect it to be non-zero. Thus we find that $G(t < 0) \neq 0$, which is awful because this $t < 0$ corresponds to an *effect that happens before the cause*.

Exercise 6.6 Taking $t_0 = 0$ for simplicity, show that the exact result for the acausal Green's function is

$$G(t < 0) = \frac{-1}{\omega_0} \sin \omega_0 t . \quad (6.23)$$

Hint: the relevant residues are $\pm e^{\pm i\omega_0 t} / 2\omega_0$.

You should do the above exercise as practice for the main contour integral of this entire course. If you are tempted to just keep reading these notes⁹³

To help you along, let me even set up the integral for you. We start with $G(t)$ in (6.11), where we rewrote the integrand in (6.10) to make the pole structure clear. We then said that we wanted to push the poles off the real line. This changes the physical problem, but we'll worry about what that means later. In this case, we move the poles into the upper half-plane so that each pole has a small imaginary part. The poles are then located at $\pm\omega_0 + i\varepsilon$. The resulting integral is:

$$G(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{-e^{-i\omega t}}{(\omega - \omega_0 - i\varepsilon)(\omega + \omega_0 - i\varepsilon)} . \quad (6.24)$$

Complete the real integral into a contour integral; we'll re-label our variables to remind us that we're in the complex plane, $\omega \rightarrow z$. The contour that we select, C_+ , came from the $e^{-i\omega t}$ factor. The result is:

$$\oint_{C_+} \frac{dz}{2\pi} \frac{-e^{-izt}}{(z - \omega_0 - i\varepsilon)(z + \omega_0 - i\varepsilon)} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{-e^{-i\omega t}}{(\omega - \omega_0 - i\varepsilon)(\omega + \omega_0 - i\varepsilon)} + \lim_{R \rightarrow \infty} \int_0^\pi \frac{R d\theta}{2\pi} \frac{-e^{-i(R \cos \theta + iR \sin \theta)t}}{(Re^{i\theta} - \omega_0 - i\varepsilon)(Re^{i\theta} + \omega_0 - i\varepsilon)} \quad (6.25)$$

$$= 2\pi i \sum_{z_\pm = \pm\omega_0 + i\varepsilon} \text{Res}_f(z) , \quad (6.26)$$

where f is understood to be the integrand. From here you should be able to read off the result. Confirm that we can take the $\varepsilon \rightarrow 0$ limit; this tiny shift was only there to tell us that the poles live in the upper half-plane.

⁹³By now you should know that learning physics requires *doing* physics. Reading/watching someone else do physics is one of the least efficient ways of improving yourself as a physicist.

Advanced propagator. The causal ($t > 0$) and acausal ($t < 0$) cases in (6.23) and (6.22) respectively may be combined into a single expression:

$$G_{\text{adv}}(t) = \frac{-1}{\omega_0} \sin(\omega_0 t) \Theta(-t) , \quad (6.27)$$

where we use the Heaviside step function

$$\Theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 . \end{cases} \quad (6.28)$$

This is the **advanced** propagator. Even though ‘advanced’ is often a positive adjective when applied to your progress⁹⁴ In this context, however, ‘advanced’ is a *bad* word. It means *acausal*: the cause comes *after* the effect. This does not make any sense, and we reject this as non-sense.

After an appropriate amount of indignant disdain, we can reflect on how the heck we got here⁹⁵. We make two observations:

1. After checking all of our work, the main conclusion from this non-sense result is that our approach of pushing the poles *up* into the upper half-plane must have been wrong. We see that the position of the poles tells us about the causality of the system. By pushing the poles off the real axis, we have imposed a sense of causality in time. We should see what happens when we push the poles *down* into the lower half-plane.
2. Even though the *physics* of advanced propagator is utter rubbish because it breaks causality, we also appreciate that it is a perfectly reasonable *mathematical* solution. The harmonic oscillator dynamics encoded in our differential operator \mathcal{O}_t is symmetric with respect to time-reversal symmetry⁹⁶. The ‘physical’ interpretation is analogous to seeing a bunch of ripples starting at the edges of a pond that move inward until they reach a singularity (δ -function!?) at which point the pond spits out a small pebble.

Exercise 6.7 At this point, you should be able to solve the case of the causal Green’s function without any additional guidance. In fact, without doing any work, you should see that by pushing the poles down into the lower half plane, the propagator will be causal. With a little bit of thought, you may be able to write the analog of (6.27), the retarded Green’s function⁹⁷,

⁹⁴I once read a story about a particle physicist who started a conversation with a stranger on the plane. The stranger explains, “My son is really quite brilliant. He’s studying physics at an Ivy league university.” The physicist says, “Oh, how nice. You know, I’m a physicist.” The stranger then asks what kind of physics, to which the physicist says “I study elementary particles.” The stranger responds with, “Ah yes. Well, my son studies advanced particles.” I no longer remember where I read this story, but I can attest to meeting people like this.

⁹⁵If your response is *Christmas! Christmas eve last year*, then I applaud your taste in musical theater.

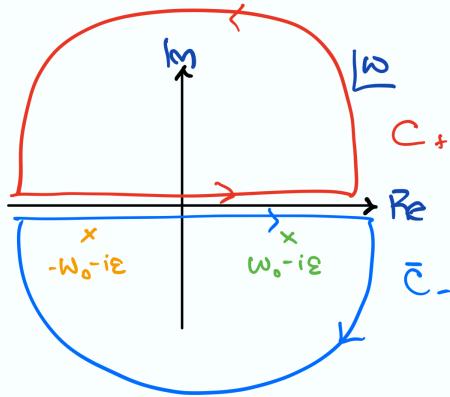
⁹⁶The behavior of our models under $t \rightarrow -t$ is a powerful diagnostic for understanding our physical theories. One of my friends was in a pub the night before lecture and some of the pub regulars, who knew he was a physicists, asked him what he was lecturing on the next day. He said he was going to talk about reversing the direction of time, to which they cautiously responded: ‘... we can do that??’

⁹⁷In the same way that ‘advanced’ means something different colloquially versus for physics, we use this word very carefully. One of my friends in graduate school was an international student who liked to make physics puns. At some point she playfully admonished late arriving undergraduates for being ‘retarded,’ which required quite a bit of care to clarify the situation to everyone involved.

without doing any work. You should still proceed to calculate $G_{\text{ret}}(t)$ the ‘honest way.’ We walk through it together in the next subsection.

6.4 Going over the poles

Assuming that you have completed Exercise 6.7, you have figured out the main result of this course⁹⁸. However, because this example is so important, let’s go ahead and walk through it together. We push the poles down into the lower half-plane by giving them a small imaginary part, $-i\varepsilon$.



The integral is exactly the same, except the sign of ε has changed.

Exercise 6.8 *Pop quiz! Unlike the advanced Green’s function where we gave the poles a small positive imaginary part, we have now given the poles a small **negative** imaginary part. How do we assign the contours C_+ and \bar{C}_- to the cases $t > 0$ and $t < 0$?*

Did I fool you? The poles may have moved, but the way that we pick contours does not change: this is fixed by the $e^{-i\omega t}$ factor in the integrand. Thus we *still* have the contour choices in (6.20):

$$t > 0 \rightarrow \bar{C}_- \quad t < 0 \rightarrow C_+ . \quad (6.29)$$

What has changed is that now the \bar{C}_- contour encloses poles, whereas the C_+ contour encloses no poles. This clearly tells us that the integral over ω (or its complexification z) is zero for $t < 0$, which means we do not have acausal propagation:

$$G(t < 0) = 0 . \quad (6.30)$$

⁹⁸Do the exercises. I’ve come to appreciate how annoying it is to come up with meaningful exercises, rather than rote calculations. At some point there’s a phase transition in education where exercises are no longer just ‘practice’ but are pedagogical tools to make sure your brain knows how to *think* physics rather than just *read* physics. To say it differently, as my undergraduate advisor once told me: you should do every single exercise problem you can. If you’re short on time, just focus on the ones that you cannot do.

What about the causal propagation? Remember that \bar{C}_- has a negative orientation. To keep our signs clear, let's write the integral as negative a contour C_- which has the positive orientation:

$$\oint_{\bar{C}_-} = - \oint_{C_-} . \quad (6.31)$$

We can go ahead and do the integral over C_- using the residue theorem. We're simply re-doing (6.25) and (6.26) with $\varepsilon \rightarrow -\varepsilon$. Writing $f(z)$ to be the integrand, we have:

$$G(t > 0) = - \frac{1}{2\pi} \oint_{C_-} dz f(z) \quad (6.32)$$

$$= 2\pi i \times -\frac{1}{2\pi} (\text{Res}_f(\omega_0) + \text{Res}_f(-\omega_0)) \quad (6.33)$$

$$= \frac{1}{\omega_0} \sin(\omega_0 t) . \quad (6.34)$$

Combining this with (6.30) using the Heaviside Θ -function (6.28) gives the **retarded Green's function**:

$$G_{\text{ret}}(t) = \frac{1}{\omega_0} \sin(\omega_0 t) \Theta(t) . \quad (6.35)$$

Exercise 6.9 Restore $t_0 \neq 0$ in the above equation for the retarded Green's function.

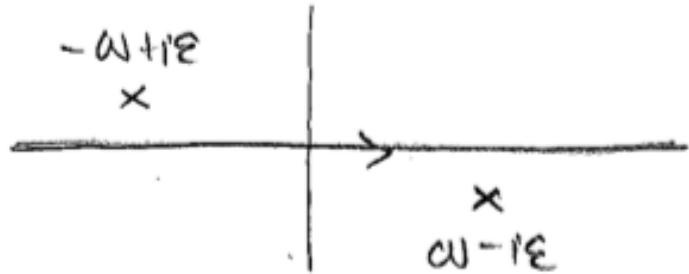
By the way, we will often use the word **propagator** in place of Green's function. This is because the Green's function propagates information from the source to the observation point⁹⁹.

6.5 Other choices? (Feynman Propagator)

We've introduced the advanced G_{adv} and retarded G_{ret} Green's functions. These are both Green's functions for the harmonic oscillator, but for technically different versions of the harmonic oscillator where the differential operator \mathcal{O}_t has been tweaked to make it slightly complex. We always take the complex part to zero at the end, but we have nudged the poles of the Fourier integrand over a bit and discovered that these have implications on the causality of the Green's function. Clearly the Green's function that we want for most classical applications is the retarded Green's function; this one reflects an effect that happens after the cause.

However, we may as well also explore alternative pole-pushing options. Consider the following mixed prescription:

⁹⁹This is especially clear when solving linear differential equations for which the Green's function represents an exact solution. For non-linear problems, one typically resorts to a perturbation expansion. This is what Feynman diagrams are: lines represent propagators, which are Green's functions for the linear part of the dynamical equation. The vertices represent non-linearities which must be treated as a Taylor series about a partition function.



Here one pole is pushed up and the other is pushed down. We'll only focus on the case of $+\omega_0$ being pushed into the lower half-plane and $-\omega_0$ being pushed into the upper half-plane. I'll leave it to you to figure out the implications of the converse choice.

Example 6.2 Even before we do any of the work, you should identify why this is strange. The exponential factor of the Fourier integrand determines which contour we select. This, in turn, is based on the sign of t and connects to our notion of causality. By splitting the poles like this, we see that there is a non-zero contribution to both causal and acausal propagation. What's even more curious, we no longer get the combination of two complex exponentials that gives a (real) sine. Thus the Green's function is manifestly complex. Curious!

We call this the Feynman propagator, G_F , which should make you suspect that this choice is not just some mathematical curiosity but actually has physical significance. The result is:

$$G_F(t > 0) = \frac{i}{2\omega_0} e^{-i\omega_0 t} \quad G_F(t < 0) = \frac{-i}{2\omega_0} e^{i\omega_0 t}. \quad (6.36)$$

Exercise 6.10 Prove (6.36) using the techniques demonstrated in this section.

This Green's function turns out to be the correct choice for quantum field theory. Even if you don't know what quantum field theory is, you can guess from the name that this makes sense:

1. The ‘quantum’ implies that maybe complex numbers aren’t so odd. In fact, the $e^{\pm i\omega_0 t}$ looks a lot like $e^{i\hat{H}t}$, the time-evolution operator. Indeed, frequencies and energies have the same dimension once you throw in the appropriate factors of $\hbar = 1$.
2. The word ‘field’ implies something like a continuum. Continua are represented by a bunch of nearest-neighbor connected springs—in other words, they are naturally represented as coupled harmonic oscillators¹⁰⁰.

What the Feynman propagator/Green’s function represents two things, depending on the sign of t :

1. $t > 0$: a plane wave moving forward in time with positive energy, $G_F \sim e^{-i\omega t}$. This seems totally fine. The fact that G_F is imaginary is not a big deal, since it now represents something like a wavefunction in quantum mechanics.

¹⁰⁰This fact should be better presented in the standard undergraduate curriculum.

2. $t < 0$: a plane wave moving *backward in time* with negative energy, $G_F \sim e^{-i(-\omega)t}$.

This is the suspicious case: note the ‘double negative’ of moving backward in time and having negative energy.

You may know the resolution of the $t < 0$ solution since it’s a popular story in physics. Dirac found that the solution to the dynamical equation for relativistic electrons contained negative energy electrons moving backward in time. These corresponded to currents of negative charge moving backward in time, which you can hand-wave into understanding as a *positive* charge moving *forward* in time. In other words, rather than describing negative energy electrons moving backward in time, these states correspond to positive-energy positrons moving forward in time. Thus we have a theoretical invitation to the existence of anti-particles¹⁰¹.

6.6 Aside: Feynman Green’s Function and Convergence of the Path Integral [optional]

This section is purely ‘for culture’. We present a qualitative discussion of how one could have anticipated the Feynman propagator from the perspective of the path integral formulation of quantum mechanics. While path integrals are outside the scope of our course, the overall story here is helpful to see how a theory predicts an equation of motion. The Fourier components of the Feynman Green’s function are:

$$\tilde{G}(k) = \frac{-1}{[k - (\omega_0 - i\varepsilon)][k - (\omega_0 + i\varepsilon)]} = \frac{-1}{k^2 - \omega_0^2 + \frac{i}{2}\delta}, \quad (6.37)$$

where $2i\epsilon\omega_0 \equiv \delta = \frac{i}{2}\delta$ is just shorthand for a small, positive quantity. This is what we would expect from a Lagrangian:

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}(\omega_0^2 + i\delta)x^2. \quad (6.38)$$

Exercise 6.11 Check that the equation of motion from this Lagrangian indeed gives a differential operator whose Green’s function has Fourier components (6.37).

Since the the Lagrangian is related to the action, S , by $S = \int dt \mathcal{L}$, one can integrate the \dot{x}^2 term by parts to write:

$$S = - \int dt \frac{1}{2}x \left[\left(\frac{d}{dt} \right)^2 + \omega_0^2 - i\delta \right] x = - \int dt \frac{1}{2}x [\mathcal{O}_{HO} - i\delta] x, \quad (6.39)$$

¹⁰¹For those interested in particle or nuclear physics, you may also want to notice that the particle and the anti-particle propagators differ by complex conjugation. This is related to the idea that charges reflect some complex nature of the particles. The geometric description of electrodynamics is a $U(1)$ fibration of spacetime. What this means is that every field is allowed to have a complex phase: the $U(1)$ refers to rephasing symmetry, $z \rightarrow e^{i\phi}z$. The other familiar forces of fundamental physics are described by the generalization to what is called $SU(N)$ where an N -vector of particles Ψ^i transforms under the $N \times N$ special unitary matrix $U_j^i = \exp(i\phi^a T^a)_j^i$ so that $\Psi^i \rightarrow U_j^i \Psi^j$. The matrices T^a are called the ‘generators’ of these N -dimensional rephasings. For the case $SU(2)$, the T^a are simply the three Pauli matrices, σ^a and the N -vectors are essentially two-component spinors. The anti-particles are Hermitian conjugates of the particles Ψ .

where \mathcal{O}_{HO} is the original harmonic oscillator differential operator that we started with in (6.1). This shows us that the Feynman prescription for pushing poles around corresponds to adding a small imaginary part to the harmonic oscillator Lagrangian. This is a little weird since the resulting operator is clearly not Hermitian.

If you're familiar with statistical mechanics or quantum field theory, you may know that the fundamental object is neither the Lagrangian nor the action¹⁰², but actually the path integral, Z which is defined as an integral over all possible functions $x(t)$:

$$Z = \int \mathcal{D}x e^{iS[x]} = \int \mathcal{D}x e^{iS_{\text{HO}}[x] - \frac{\delta}{2}x^2}. \quad (6.40)$$

Here $\mathcal{D}x$ is just the measure of the path integral. For discretized space, it corresponds to $\mathcal{D}x \sim dx(t_0)dx(t_1)dx(t_2)\dots$. The harmonic oscillator action S_{HO} is real and thus $e^{iS_{\text{HO}}}$ oscillates and the path integral doesn't converge. By adding the $-\frac{\delta}{2}x^2$ term to the action we are imposing a small perturbation that forces the integrand to converge for very large values of $x(t)$. With this factor the path integral is well defined. If we approached the harmonic oscillator assuming that path integral is sufficiently convergent¹⁰³ and plugging in some factor to force this convergence, then we can run this argument backward and 'deduce' the Feynman prescription for the Green's function.

If everything in this subsection is a bit boring, then here's what you should know. This discussion gives us a hint of where all of our physical differential equations come from. The equations of motion that define a theory come from a variational principle with respect to some action.

Exercise 6.12 (optional) *What is the action (Lagrangian, Hamiltonian, whatever...) that yields Maxwell's equations? Note that curiously, you only get half of Maxwell's equations from an action principle. The other half comes from the geometry alluded to in Footnote 101 and the ideas in Section 5.11. For a discussion by a well-known UCR mathematical physicist, see Gauge Fields, Knots And Gravity by Prof. John Baez.*

We then recall that our actions (Lagrangians, Hamiltonians, ...) tend to be *local* function(al)s, so it makes sense that they can be written as a derivative expansion. The actions depend on particles/fields, say¹⁰⁴ $q(t)$. If we write out the leading terms of the action with respect to $q(t)$ you get something like

$$S[q] = \int dt (\text{const.}) + \mathcal{O}_1 q(t) + \mathcal{O}_2 q^2(t) + \mathcal{O}_3 q^3(t) + \dots \quad (6.41)$$

where the \mathcal{O}_i are differential operators. We can ignore the constant. Usually the \mathcal{O}_1 term doesn't show up, so let's ignore that too; you can remove it by redefining the particle¹⁰⁵. The

¹⁰²... and is definitely not the Hamiltonian; sorry condensed matter folks and Lin-Manuel Miranda fans.

¹⁰³Physically this is saying that these extreme path configurations should not contribute since the path that minimizes the action should be far away from these cases; so we're just helping the theory focus on the part of the path integral that matters.

¹⁰⁴I'm deliberately changing notation from $x(t) \rightarrow q(t)$. This is important because we will eventually want to treat x and t as spacetime coordinates rather than $x(t)$ as 'the position of the particle at time t '. This unfortunately notational confusion tends to be one of the conceptual hurdles in field theory.

¹⁰⁵The linear term tells you that you're probably not defining the 'zero' of your field sensibly. Amusingly, this is closely related to the Higgs phenomenon that gives known fundamental particles most of their mass.

$\mathcal{O}_2 q^2$ term can be integrated by parts. The details of the integration by parts depend on the details of the derivatives in \mathcal{O}_2 , but one can write typically write this term as

$$\mathcal{O}_2 q^2(t) \equiv -q(t) \mathcal{O}_{\text{quad}} q(t) , \quad (6.42)$$

which is in the form that we used in (6.39). When we derive the equation of motion by varying the action, we end up with

$$-\mathcal{O}_{\text{quad}} q(t) = 0 , \quad (6.43)$$

where I'm probably assuming that $\mathcal{O}_{\text{quad}}$ is Hermitian if we want to be technical. The main observation here is that this is a *linear* differential equation. Linear differential equations in physics come from the quadratic pieces of actions. Since we argued that the linear piece is usually not there and we know that the constant piece corresponds to a shift in the vacuum energy (that's *usually* not physical¹⁰⁶), then this quadratic piece of the action is the *lowest order* contribution to the action in an expansion of the particle $q(t)$ with respect to the vacuum ($q(t) = 0$). The higher-order terms correspond to differential equations that are *not* linear since the variation of that term contains $q^2(t)$. So yes, even though the mathematical structures here are a bit more fancy, let us state out loud that all of our main differential equations in physics basically boil down to taking the leading term in a Taylor expansion. Typical for physicists, eh?

6.7 Example: The Damped Harmonic Oscillator

At this point we are happy that the physical solution to the classical harmonic oscillator is the retarded Green's function, $G_{\text{ret}}(t)$. There is an unphysical solution, the advanced Green's function, $G_{\text{adv}}(t)$ that reflects a mathematical symmetry of the theory (time-reversal invariance) but is otherwise not of physical interest. Finally, there is a solution that makes sense for quantum physics, the Feynman Green's function, $G_F(t)$, which is complex but reflects the existence of anti-particles. Each of the above cases came from *almost* the same harmonic oscillator differential operator, \mathcal{O} , except the operators were slightly deformed so that its poles were off of the real line in the complex momentum plane. We took the limit where this deformation goes to zero, but not until after we've exploited those deformations to do the Fourier integral using the residue theorem. We came to appreciate that the infinitesimal nudge in the imaginary direction was telling us something about the causality of the harmonic oscillator.

We can understand the nature of this imaginary push by looking at the *damped harmonic oscillator*. The dynamical equation for the damped harmonic oscillator is

$$\mathcal{O}_{\text{DHO}} x(t) = \ddot{x}(t) + 2\gamma \dot{x}(t) + \omega_0^2 x(t) = F(t) , \quad (6.44)$$

where γ is the **damping coefficient** and $F(t)$ is a driving force.

Exercise 6.13 *What are the dimensions of the damping coefficient, γ ?*

¹⁰⁶Damn you, dark energy.

Back of the envelope version. Before doing anything quantitative, we can guess the behavior. A non-zero γ suppresses the response $x(t)$ to the source $F(t)$. Unlike an idealized instrument, a guitar string has non-zero damping and the vibrations get smaller the longer you wait after plucking it. If we imagine a plane wave¹⁰⁷ $x(t) \sim e^{-i\omega t}$ and ask what this wave looks like far away from the source, $F(t) \rightarrow 0$, then the differential equation reads:

$$-(\omega^2 + 2i\gamma\omega - \omega_0^2) e^{-i\omega t} = 0 . \quad (6.45)$$

The solution for the plane wave's frequency ω is

$$\omega = \omega_R - i\Gamma , \quad (6.46)$$

where for $\gamma^2 \ll \omega_0^2$, $\omega_R \approx \omega_0$ is the frequency. Plugging ω back into our plane wave gives

$$x(t) = e^{-i\omega_R t} e^{-\gamma t} . \quad (6.47)$$

The first factor is the usual plane wave. We recognize the $\exp(-\gamma t)$ as an exponential suppression that causes the plane wave to get rapidly smaller for times larger than $1/\gamma$. There are a few lessons:

1. In (6.46) we have implicitly but decisively bought into the idea of a complex frequency. The interpretation of this complex frequency is that the real part is the usual oscillation and the imaginary part corresponds to damping.
2. The solution (6.46) corresponds to finding the poles of the Fourier integrand; recall, for example, (6.8). So damping has something to do with the imaginary part of the poles in the complex plane.
3. Because the position of the pole in the imaginary direction is related to the *causality* of the theory, we have now connected ‘damping’ to causality in a round-about way.
4. When we derived the retarded Green’s function $G_{\text{ret}}(t)$ that describes the classical, ideal harmonic oscillator, we took $i\varepsilon \rightarrow 0$. For the damped harmonic oscillator the corresponding value is $i\gamma$, which we keep finite¹⁰⁸; it has some physical significance after all.
5. The exponential notation for the Fourier transform made all of this very easy: we just let ω be complex in $e^{-i\omega t}$ and we are able to capture the physics of oscillations and damping. If we had used trigonometric functions then we’d have a bunch of sinh or cosh floating around and the convergence would not be so clear.

More carefully. Let’s go through the steps to derive the Green’s function for the damped harmonic oscillator properly. The Green’s function equation is

$$\left[\left(\frac{d}{dt} \right)^2 + 2\gamma \frac{d}{dt} + \omega_0^2 \right] G(t) = \delta(t) , \quad (6.48)$$

¹⁰⁷I keep calling $e^{-i\omega t}$ a plane wave because I’m thinking of the generalization to spacetime. For now there are no transverse directions, so of course $e^{-i\omega t}$ is really just a wave.

¹⁰⁸Here’s a funny bit of linguistics in our discipline. Any physical quantity is naturally not-infinite, so it seems silly when physicists make a point to call something ‘finite.’ You would think that this is in contrast to that quantity being infinite. In fact, when physicists say *finite* what they really mean is *not zero*, which is quite different from ‘not infinite.’

where we've again set $t_0 = 0$. It is easy at this stage to restore¹⁰⁹ t_0 by taking $t \rightarrow (t - t_0)$. Plugging in the Fourier representations for the t -dependence,

$$G(t) = \int d\omega e^{-i\omega t} \tilde{G}(\omega) \quad \delta(t) = \int d\omega e^{-i\omega t}, \quad (6.49)$$

then gives:

$$\tilde{G}(\omega) = \frac{-1}{\omega^2 + 2i\gamma\omega - \omega_0^2}. \quad (6.50)$$

The poles occur at

$$\omega_{\pm} = \frac{-2i\gamma \pm \sqrt{-4\gamma + 4\omega_0^2}}{2} = \pm\sqrt{\omega_0^2 - \gamma^2} - i\gamma \equiv \pm\omega_R - i\gamma. \quad (6.51)$$

Example 6.3 *The poles in (6.51) have a real part as long as $\omega_0^2 > \gamma^2$. What does it mean when $\omega_0^2 < \gamma^2$? This is the observation that the damping is stronger than the characteristic oscillation frequency, ω_0 . Thus the system is properly described by an exponentially suppressed because it never has a chance to oscillate.*¹¹⁰

Assuming $\omega_0^2 > \gamma^2$, then we may identify $\omega_R = +\sqrt{\omega_0^2 - \gamma^2}$ and the poles may be written as $\pm\omega_R - i\gamma$. This looks a *just* like the pole prescription for the retarded Green's function of the ideal harmonic oscillator with $\omega_0 \rightarrow \omega_R$ and $i\varepsilon \rightarrow i\gamma$. In fact, we can just follow all of the same steps that we did for the retarded Green's function:

1. The poles are in the lower half-plane of ω . This means that the propagator is automatically causal (it's the retarded, not the advanced Green's function).
2. The exponential factor $e^{-i\omega t}$ in the Fourier integral determines the convergence of the integrand and hence how we complete the contour. For $t > 0$ we close the contour on the lower half-plane and pick up these poles in the residue theorem. For $t < 0$ we close the contour on the upper half-plane; since there are no poles here, the Fourier integral vanishes.
3. We end up with our own version of (6.35) with $\omega_0 \rightarrow \omega_R$ except we need to account for the finite γ .

The only substantial difference between the damped harmonic oscillator and the ordinary harmonic oscillator is that ω_{\pm} contains a non-negligible imaginary part. The contour integral

¹⁰⁹There is a theorem that $G(t, t')$ only depends on $|t - t'|$ under reasonable assumptions. You should be able to argue this on physical grounds.

¹¹⁰This is one of the magical things about B mesons. B mesons are really nice systems whose characteristic oscillation frequency (between B and anti- B mesons) and characteristic decay width are roughly of the same order of magnitude. When we produce B mesons at, say, the LHCb experiment, we get to observe the decays (imaginary part) of these mesons after ~ 1 oscillation which gives a powerful tool for studying matter-anti-matter asymmetry.

takes the form:

$$\oint_C dz \frac{-e^{-izt}}{(z - \omega_+)(z - \omega_-)} = 2\pi i (\text{Res}_f \omega_+ + \text{Res}_f \omega_-) \quad (6.52)$$

$$= -2\pi i \left[\frac{e^{i\omega_+ t}}{\omega_+ - \omega_-} + \frac{e^{i\omega_- t}}{\omega_- - \omega_+} \right] \quad (6.53)$$

$$= -2\pi i \frac{e^{i\omega_+ t} - e^{i\omega_- t}}{\omega_+ - \omega_-} \quad (6.54)$$

$$= -2\pi i \frac{e^{-i\gamma t}}{2\omega_R} (e^{i\omega_R t} - e^{-i\omega_R t}) \quad (6.55)$$

$$= \pi \frac{e^{-i\gamma t}}{2\omega_R} \sin(\omega t) . \quad (6.56)$$

Plugging this into the full solution—which involves keeping track of which contour integral is non-zero and where the factors of 2π belong—we find:

$$G(t) = \frac{e^{-\gamma t}}{\sqrt{\omega_0 - \gamma^2}} \sin\left(\sqrt{\omega_0^2 - \gamma^2} t\right) \Theta(t) . \quad (6.57)$$

Exercise 6.14 Compare and contrast (6.57) to (6.35). Why is the general form the same? How do you explain the differences? What is the behavior for $\omega \gg \gamma$, $\omega \ll \gamma$? What does it look like if you restored t_0 ?

The key observation that we have from this discussion is that the choice of pushing the poles into the lower half plane—which we found was critical for a causal Green’s function—is actually physically motivated. Any *physical* oscillating system actually has some small amount of friction (or the appropriate generalization) that causes it to not only oscillate, but also decay.

Exercise 6.15 In quantum mechanics the time evolution operator is $e^{i\hat{H}t}$. Maybe there’s a minus sign in there, I don’t remember and I don’t really care. It is important that this operator is unitary, or else probabilities are not preserved in time. However, there are plenty of physical systems (say, the B meson in footnote 110) that are described by non-Hermitian Hamiltonians, i.e. non-unitary time evolution operators. Reason tells us that the form of the time-evolution operator should be $e^{i\hat{H}t}e^{-\gamma t}$, since probabilities less than one make more sense than probabilities bigger than one. What is the physical interpretation of this non-unitary time evolution? Hint: this has to do with the notion that our theory (our Hamiltonian) models something in nature, but it not model all of the relevant pieces nature for a given problem.

At this point in my note I have the following written:

I took 7 weeks of physics 231 and all I have is this one stupid Green’s function.

7 Green's functions in diverse dimensions

Our usual approach is to take the simplest example and then focus on understanding the physics, while leaving each of you to generalize to the horrible real-world scenarios that you'll encounter in your other graduate courses and research. There's one generalization that we should take time to do properly: the shift from one dimension to multiple dimensions. This is the shift from treating time as a dependent variable to working on *spacetime*. Equivalently, the harmonic oscillator converts into the wave equation. Keep in mind that we're not doing any special relativity, even though we'll borrow notation from special relativity¹¹¹.

7.1 The ‘harmonic oscillator’ in spacetime

The second derivative, d^2/dt^2 , is generalized in Euclidean space to the Laplacian, ∇^2 . But when you combine space and time (Minkowski space), there's the famous relative minus sign¹¹²:

$$\frac{d^2}{dt^2} \rightarrow \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \mathbf{x}^2}, \quad (7.1)$$

where $\partial/\partial \mathbf{x} = \nabla$ is the gradient. Against all of my hard-developed instincts for natural units, I have replaced the explicit value of c so that the operator makes sense dimensionally. Each term has powers of inverse length squared. In what follows I'm likely to make mistakes¹¹³ with the factors of c .

In (1+1)-dimensions of spacetime¹¹⁴ this is $\partial^2 = c^{-2} \partial_t^2 - \partial_x^2$. That looks familiar, doesn't it? The minus sign tells us that if we move forward in time a little bit, but look ‘backward’ in space, then the function doesn't change. The description of the state at time $t - \delta t$ and position $x - \delta x$, subject to $\delta x = c\delta t$, is the same as the state at time t and position x . This, in turn, means that the information was propagated *forward* in space as time also moves forward. This is exactly what we expect from a wave. We recall that the solution to second derivative differential equations are usually trigonometric functions or their exponential counterparts. We further recall that ‘plane waves’ are described with the same funny minus sign:

$$f(t) \sim \sin(\omega t - kx). \quad (7.2)$$

From this you can read off that the plane wave velocity is ω/k . Of course, you already knew that from dimensional analysis.

¹¹¹There was once a professor teaching electrodynamics who used notation informed by relativity. The condensed matter students complained the class is about electricity and magnetism, not relativity. The professor replied: just where do you think magnetism comes from?

¹¹²At this level the minus sign is a convenient convention, but we know that the second derivative should really be relativistically invariant: $\partial^2 = \partial_\mu \partial^\mu$.

¹¹³My adviser used to say: if you think there may be a sign error or a factor of two error, then your homework is to fix those errors. In this case, if you think there's a missing factor of $c = 1$, I'm not even sure if I'd even acknowledge that it's actually an error.

¹¹⁴The (1+1) notation means one dimension of space, one dimension of time.

7.2 Green's functions for multidimensional spaces

Our ‘harmonic oscillator’ equation becomes:

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right] \varphi(\mathbf{x}, t) = \rho(\mathbf{x}, t) . \quad (7.3)$$

Huh, that looks familiar, doesn’t it? We’ve chosen variables so that this looks just like the wave equation for the scalar potential φ subject to a charged source ρ in electrodynamics! In fact, you know how this works. There are three more equations that correspond to the vector potential:

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right] \mathbf{A}(\mathbf{x}, t) = \mathbf{j}(\mathbf{x}, t) . \quad (7.4)$$

Now let’s go through a series of questions about (7.3) and (7.4) to make sure we’re on the same page.

1. **Are these differential equations still linear?** Yes. Remember what it means to be linear! $\mathcal{O}(f + g) = \mathcal{O}f + \mathcal{O}g$, for example.
2. **Are we worried that there are multiple arguments?** Not really. The functions now depend on t and $\mathbf{x} = (x, y, z)$. You went from an infinite dimensional ‘vector space’ to a still-infinite dimensional vector space¹¹⁵. You can also mumble reassuring words to yourself, perhaps recalling how in prior encounters with the wave equation you’ve perhaps separated your functions into single-argument factors, $\Psi(t, x) = T(t)X(x)$, or something like that.
3. **How many equations are there?** There are *four*¹¹⁶ equations. There is an equation for φ and one equation for each component of \mathbf{A} . In fact, we typically bundle this all up into a four-vector $A_\mu = (\varphi, \mathbf{A})$.
4. **How many differential operators are there?** Just one. There is only *one* differential operator, ∂^2 . It acts on different components of A_μ , but it’s the same wave operator acting on each component.
5. **... so how many equations are there, really?** All four equations are essentially the same equation with different state functions and different sources. So it’s really just one class of differential equation.

Okay, here’s the really important one. I’m going to put it in a box to make sure you pay really close attention. Please answer the following question before reading on:

How many Green’s functions are there?

¹¹⁵You can pontificate about whether or not this infinity has gotten ‘bigger.’ It doesn’t really make a difference.

¹¹⁶Obligatory TNG reference: <https://www.youtube.com/watch?v=wjKQQpPVifY>

Do we need a Green's function for each component of A_μ ? Do we need a Green's function for each dependent variable, $x^\mu = (ct, x, y, z)$? *No!* There is only *one* differential equation, and thus there is only *one* Green's function.

Let's see how this works. Suppose you have a differential equation in multiple variables,

$$\mathcal{O}_x f(\mathbf{x}) = s(\mathbf{x}) . \quad (7.5)$$

Let's say that $\mathbf{x} = (x_1, \dots, x_N)$, so that we're working in N -dimensional space. Suppose you are given the Green's function $G(\mathbf{x}, \mathbf{x}')$ for \mathcal{O}_x . This means that G satisfies the Green's function equation

$$\mathcal{O}_x G(\mathbf{x}, \mathbf{x}') = \delta^{(N)}(\mathbf{x} - \mathbf{x}') , \quad (7.6)$$

where the N -dimensional δ -function is simply the product of one-dimensional δ -functions in the expected way:

$$\delta^{(N)}(\mathbf{x} - \mathbf{x}') = \delta(x_1 - x'_1)\delta(x_2 - x'_2) \cdots \delta(x_N - x'_N) . \quad (7.7)$$

The solution to the above differential equation is

$$f(\mathbf{x}) = \int d^N \mathbf{x}' s(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') , \quad (7.8)$$

which you can check by acting on both sides with \mathcal{O}_x .

7.3 Relativistic Notation

Let's borrow some four-vector notation from special relativity. Nothing that we're doing is based on special relativity, but the fact that it pops out for free is telling. Our indices μ run over four values which we take to be $\mu = 0, 1, 2, 3$ corresponding to timelike (0) and spacelike (1,2,3) directions. Some objects are naturally defined with an upper index, like $x^\mu = (ct, x, y, z)$. You can lower these indices using the metric, $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. This means that for some vector v^μ ,

$$v^2 \equiv v^\mu v_\mu = c^2(v^0)^2 - (v^1)^2 - (v^2)^2 - (v^3)^2 . \quad (7.9)$$

More importantly, this means that the four-vector of partial derivatives, which is naturally a lower-index object, squares to:

$$\partial^2 \equiv \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} . \quad (7.10)$$

The placement of the c^2 factors should be clear from dimensional analysis. We see that indeed the wave operator is the natural generalization of the harmonic oscillator in spacetime. We won't actually be raising and lowering indices, but we want to be clear that there's a natural 'dot product' that includes the relative minus sign between spatial and temporal components.

The states and sources in our system are the (four-)vector potential A_μ and the current j_μ :

$$A_\mu = (\varphi, \mathbf{A}) \qquad j_\mu = (\rho, \mathbf{j}) . \quad (7.11)$$

The four equations for the vector potential are

$$\partial^2 A_\mu = j_\mu . \quad (7.12)$$

That's one equation for each value of μ , but each equation has the *same* differential operator ∂^2 . Our task is to find the Green's function for this operator in spacetime. The Green's function equation is

$$\partial^2 G(x, x') = \delta^{(4)}(x - x') , \quad (7.13)$$

where I'm writing x and x' to mean *four*-vectors with *components* x^μ and $(x')^\mu$. Given a current $j_\mu(x)$, the vector potential is the solution to the wave equation which integrates over all of the current (source) positions:

$$A_\mu(x) = \int d^4x' G(x, x') j_\mu(x') . \quad (7.14)$$

Observe that G has two arguments, x and x' , but no indices. The two arguments correspond to the observation point x and the source point x' . The integral over d^4x' means we're accounting for sources from all positions and all times¹¹⁷. We haven't specified our choice of coordinates. For now you can assume that we are using Cartesian coordinates, but you may recall that the Green's function for the Poisson equation¹¹⁸ is most cleanly expressed in spherical coordinates, $G(r) \sim (4\pi r)^{-1}$. You should expect that the natural way to write $G(x, x')$ in (3+1)-dimensional spacetime is some hybrid of spherical and Cartesian coordinates.

Exercise 7.1 At this point you may want to start going over Homework #4 where you solve the (3+1)- and (2+1)-dimensional Green's functions in detail. I'm quite proud of that homework assignment, there's a lot of neat observations. One observation is that with some physical insight, you could take the Poisson equation's Green function and guess the Green's function for the wave equation simply by using the finite speed of light. This is discussed nicely in Tony Zee's latest textbook, Fly By Night Physics.

We solve (7.13) in what is now the ‘usual’ way. We Fourier decompose both sides. Making deliberate choices for our Fourier transform conventions, let us write

$$f(t, \mathbf{x}) = \int d^4k e^{ik \cdot x} \tilde{f}(\omega, \mathbf{k}) \quad d^4k = \frac{dE d^3\mathbf{k}}{(2\pi)^4} . \quad (7.15)$$

For simplicity we'll just write $f(x)$ and $\tilde{f}(k)$ to refer to the four-component arguments¹¹⁹. We have used the Minkowski spacetime inner product,

$$k_\mu x^\mu \equiv k \cdot x = Et - \mathbf{k} \cdot \mathbf{x} . \quad (7.16)$$

¹¹⁷This should remind you of starlight that is emitted a long time ago (t') in a galaxy far, far away (\mathbf{x}' that we observe today (t, \mathbf{x}) .

¹¹⁸Which we can now call the ‘harmonic oscillator’ in (3+0)-dimensional spacetime.

¹¹⁹Later on we'll use $k = |\mathbf{k}|$ to mean the spatial length of the 3-momentum; there should be no ambiguity because the context should be clear. This is an example of ‘read what I mean, not what I [literally] write.’ This is a common ‘chalkboard physics’ peccadillo; usually it’s a pain to write *literally* what you mean. Sometimes it’s better to write the ‘easy to interpret’ equation, then tell people to interpret it slightly differently. Mathematicians should be incensed. That’s okay. This is mathematics for physicists, not physics for mathematicians. The latter course is also quite interesting and theorists should take it if it’s ever offered from John Baez.

From the perspective of the Fourier transform, this is just a sign convention for each Cartesian direction that we are free to make. From the perspective of physics, we know that it is convenient to pick a notation that is manifestly Lorentz invariant¹²⁰ and automatically looks like a traveling wave. By the way, if you're keeping track of factors of c :

$$x^\mu = (ct, \mathbf{x}) \quad k^\mu = (E/c, \mathbf{p}) \quad k_\mu = (E/c, -\mathbf{p}) \quad x \cdot k = Et - \mathbf{p} \cdot \mathbf{x} . \quad (7.17)$$

I hate keeping track of factors of c .

Here's what the Fourier transform of the spacetime δ -function looks like:

$$\delta^{(4)}(x - x') = \int dE e^{-iE(t-t')} \int dk_x e^{+ik_x(x-x')} \int dk_y e^{+ik_y(y-y')} \int dk_z e^{+ik_z(z-z')} . \quad (7.18)$$

Again, we have *chosen* the sign of the Fourier transform so that the time-like component has a minus sign, $e^{-iE(t-t')}$, while the space-like component has a plus sign, $e^{+ik_x(x-x')}$. This is convenient since we could package all of these exponential together with respect to $k_\mu = (E, -k_x, -k_y, -k_z)$,

$$\delta^{(4)}(x - x') = \int d^4k e^{-ik \cdot (x-x')} . \quad (7.19)$$

Conveniently, this looks just like a—*wait for it*—plane wave:

$$e^{-ik \cdot x} = e^{-i(Et - \mathbf{k} \cdot \mathbf{x})} . \quad (7.20)$$

Compare this to (7.2). By the way, if you're really picky you can bring up factors of c again because you argue that $\delta(t - t')$ should really be $\delta(ct - ct')$. This is worth checking, but I'd rather relegate it to yet another parenthetical footnote¹²¹.

The left-hand side of the Green's function equation (7.13) is

$$\partial^2 \int d^4k e^{-ik \cdot x} \tilde{G}(k, x') = \int d^4k (-\bar{E}^2 + \mathbf{k}^2) e^{-ik \cdot x} \tilde{G}(k, x') \quad \bar{E} \equiv E/c . \quad (7.21)$$

We have introduced the \bar{E} notation to keep track of the factors of c coming from $\partial^2 \sim c^{-2}\partial_t^2 - \dots$. If you'd had set $c = 1$, you could have guessed this because you want to make sure that both terms in the denominator have the same dimensions. Equating the Fourier coefficients of (7.19) with (7.21) gives us the Fourier modes:

$$\tilde{G}(k, x') = \frac{-e^{ikx'}}{\bar{E}^2 - \mathbf{k}^2} . \quad (7.22)$$

¹²⁰If we had chosen a different notation, the theory would still be Lorentz invariant—it's built into the operator—but it would not be *obviously* Lorentz invariant. This would be like using cylindrical coordinates and losing track about whether or not space is rotationally symmetric.

¹²¹If we keep factors of c explicit, then we recall that $t \rightarrow ct$ and $E \rightarrow E/c$. The expression $\delta(t - t')$ is replaced by $\delta(ct - ct') = \delta(t - t')/c$. The factor of $1/c$ shows up in the corresponding replacement $dE \rightarrow dE/c$. To solve for the Fourier transform \tilde{G} , we compare Fourier coefficients of the δ function with the Fourier coefficients of the Green's function. In that comparison, these factors of c cancel out. Thus we are free to simply write $\delta(t - t')$ as long as our integration measure is dE and not dE/c . If you're ever lost, you should just set $c = 1$ and then replace factors of c at the end to make sure the dimensions of G work out.

Thus the wave equation Green's function in (3+1)-dimensions is:

$$G(x, x') = \int d^4k \frac{-e^{ik \cdot (x-x')}}{\bar{E}^2 - \mathbf{k}^2} . \quad (7.23)$$

This looks almost trivial—you've integrated something with this *exact* integrand before when we did the harmonic oscillator. The only issue is that now the $d\omega$ has been replaced with a much more¹²² ominous d^4k .

7.4 Sketch of the Fourier integral in (3+1)-dimensions

Rather than solve this in gory detail, we'll just provide the framework and then we'll let it to you to dot all the *is* and cross all the *ts*. It'll be a little tedious the first time you do do this integral, but don't worry. If you drink alcohol, I recommend a glass of wine¹²³ before starting.

The solution is most clear if we use **hypercylindrical** coordinates that correspond to spherical coordinates over the spatial directions and a Cartesian coordinate in the time direction. The integration measure is

$$d^4k = \frac{1}{(2\pi)^4} dE |\mathbf{k}|^2 d|\mathbf{k}| d\cos\theta d\varphi . \quad (7.24)$$

The polar angle θ is defined by

$$(\mathbf{x} - \mathbf{x}') \cdot \mathbf{k} = |\mathbf{x} - \mathbf{x}'| |\mathbf{k}| \cos\theta . \quad (7.25)$$

The azimuthal integral. The azimuthal angle φ is defined with respect to the azimuthal direction set by $\mathbf{x} - \mathbf{x}'$. Observe that the integrand of the Fourier integral (7.23) is completely independent of φ so we might as well do the $d\varphi$ integral first:

$$\int_0^{2\pi} d\varphi = 2\pi . \quad (7.26)$$

This conveniently cancels a factor of (2π) in the denominator of (7.24).

The polar integral. We can tidy up our remaining expressions by defining $c = \cos\theta$ and $k = |\mathbf{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2}$ for the radial direction in momentum space; hopefully this doesn't cause any ambiguity with $k = (E/c, k_x, k_y, k_z)$ the four-vector¹²⁴. Similarly, let's write r to be the separation distance between the observer and the source, $r = |\mathbf{x} - \mathbf{x}'|$. Thus we have

$$\int_{-1}^1 d\cos\theta e^{i\mathbf{k} \cdot (\mathbf{x}-\mathbf{x}')} = \int_{-1}^1 dc e^{ikrc} = \frac{1}{ikr} (e^{ikr} - e^{-ikr}) . \quad (7.27)$$

¹²² *The more integrals we come across, the more problems we see*, with apologies to the Notorious BIG.

¹²³ Or the appropriate substance relevant for you personally.

¹²⁴ It will not cause any ambiguity because you'll notice that all of the factors of c have been removed in the Fourier integrand (7.23).

Plugging this into (7.23) gives

$$G(x, x') = \frac{1}{(2\pi)^3} \int dk dE k^2 \frac{1}{ikr} (e^{ikr} - e^{-ikr}) \frac{-e^{-iE(t-t')}}{\bar{E}^2 - k^2} \quad (7.28)$$

$$= \frac{1}{8\pi^3 ir} \int_0^\infty dk k (e^{ikr} - e^{-ikr}) \int_0^\infty dE \frac{-e^{-iE(t-t')}}{\bar{E}^2 - k^2}. \quad (7.29)$$

The energy integral. Ah! The dE integral is an old friend: you remember it when it used to go by the name $d\omega$ for the harmonic oscillator. You know how to do this one: just follow what we did to derive the retarded Green's function for the harmonic oscillator. Nudge the poles infinitesimally into the lower-half plane¹²⁵. The solution is

$$\frac{1}{2\pi} \int_0^\infty dE \frac{-e^{-iE\Delta t}}{\bar{E}^2 - k^2} = \frac{-i}{2k} (e^{ikc\Delta t} - e^{-ikc\Delta t}) \Theta(\Delta t). \quad (7.30)$$

Note that we have been careful to replace the factors of the speed c when integrating since we remember $\bar{E} \equiv E/c$. This is straightforward from writing $E = c\bar{E}$ and then changing integration variables $dE = cd\bar{E}$. You could have also guessed this because you know the right-hand side will have exponentials with arguments that go like k and Δt . In order for these to make sense dimensionally, there must be a factor of c in there¹²⁶.

Compare this to (6.35). We have chosen not to write the difference of exponentials into sines; we will see why shortly. Otherwise, the three-momentum magnitude k plays the role of the frequency ω_0 . We went ahead and wrote $\Delta t \equiv t - t'$; in the harmonic oscillator we equivalently said¹²⁷ “set $t' = 0$ ”.

The [magnitude of the] three-momentum integral. The remaining integral over $k = |\mathbf{k}|$ is

$$G(x, x') = -\frac{\Theta(\Delta t)}{8\pi^2 r} \int_0^\infty dk [e^{ik(r+c\Delta t)} - e^{ik(r-c\Delta t)} - e^{ik(-r+c\Delta t)} + e^{ik(-r-c\Delta t)}] \quad (7.31)$$

$$= -\frac{\Theta(\Delta t)}{8\pi^2 r} \int_0^\infty dk [e^{ik(r+c\Delta t)} - e^{ik(r-c\Delta t)} - e^{i(-k)(r-c\Delta t)} + e^{i(-k)(r+c\Delta t)}] \quad (7.32)$$

$$= -\frac{\Theta(\Delta t)}{8\pi^2 r} \int_{-\infty}^\infty dk [e^{ik(r+c\Delta t)} - e^{ik(r-c\Delta t)}]. \quad (7.33)$$

In the second line we were slick and realized that we could convert the last two terms into the first two terms if we extended the region of integration to include negative values of

¹²⁵This is as good a time as any to point out that the rule of nudging the poles into the *lower* half plane for the causal propagator is directly related to our Fourier integral convention. Some references use the opposite convention and thus nudge the poles into the upper half plane.

¹²⁶If you're being strict about units, recall that the ‘energies’ are really frequencies and the ‘three-momenta’ are really wave numbers. Those who like to think in natural units will be confused why this footnote is necessary.

¹²⁷Technically we said $t_0 = 0$ because that subscript made sense for an ‘initial time,’ but now we use a more standard Green’s function convention that x' is the spacetime position of the source.

k. Pat yourself on the back if you caught this simplification. The integrands are now pure exponentials. We know what these are: they're δ -functions!

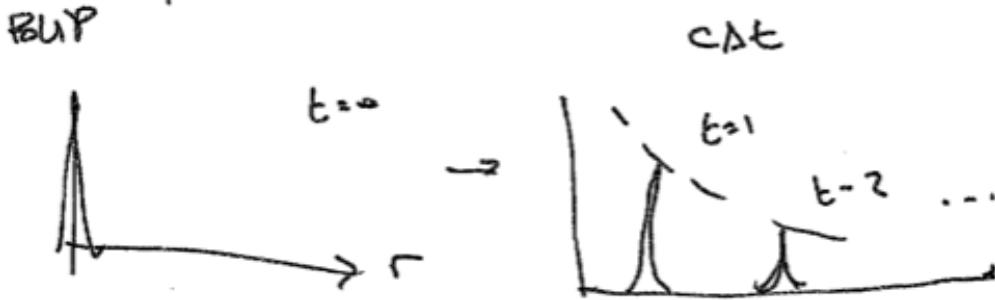
$$G(x, x') = \frac{\Theta(\Delta t)}{4\pi r} [\delta(r - c\Delta t) - \delta(r + c\Delta t)] . \quad (7.34)$$

At this point a bit of either physical or mathematical insight is required. The second term is problematic: $\delta(r + c\Delta t)$. Recall that r is a radial coordinate, which means that $r > 0$. However, we also have this $\Theta(\Delta t)$ sitting in the function that imposes that $\Delta t > 0$. That means that $\delta(r + c\Delta t) = 0$ for any interesting values of r and t away from the original ' δ -function' source¹²⁸. The non-physical term apparently encodes acausal propagation, so we'll go ahead and drop it like its hot.

The result for the retarded Green's function in spacetime is remarkably simple

$$G_{\text{ret}} \frac{1}{4\pi r} \Theta(\Delta t) \delta(r - c\Delta t) . \quad (7.35)$$

Take a moment to appreciate this. The first factor is *exactly* the Green's function for the Poisson operator, ∇^2 , in three spatial dimensions. The Θ function imposes forward-propagation in time (retarded versus advanced). Then there's a δ -function whose sole purpose seems to impose that $r = c\Delta t$. A moment's thought reveals the interpretation. Recall that the Green's function is the solution to an idealized δ -function source at spacetime point x' : $\partial^2 G = \delta^4(x - x')$. The $\delta(r - c\Delta t)$ is telling me that the response from my initial idealized δ^4 -function source propagates outward (increasing r) at speed c along a *wave front*. At time t the wave front reaches radial distance $c(t - t')$. In other words, this is just reminding us about propagation speed in the wave equation. In other words, the speed of electrodynamic waves (light) is constant¹²⁹. Here's what the propagation of a 'blip' of light looks like (this is no surprise to you):



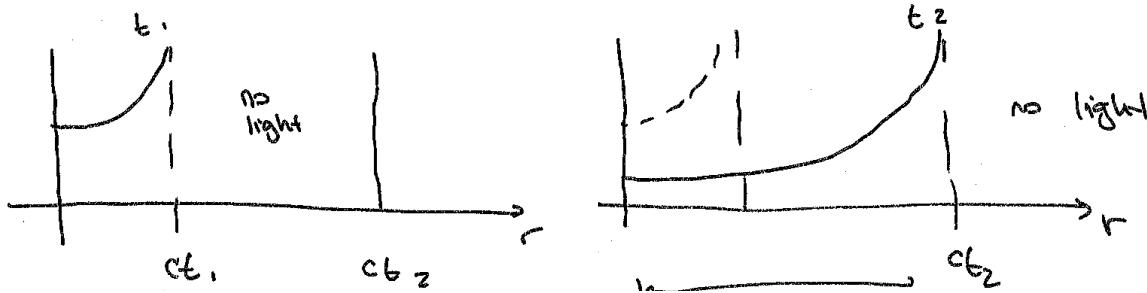
¹²⁸Such a source is not at all physical and is just a model for a physical source whose details are not resolved by our model. For example: in Newtonian mechanics, when we say that an ideal particle has an ideal elastic collision with an ideal wall, we say that the momentum changes from \mathbf{p} to $-\mathbf{p}$ and that this happens instantaneously at the moment the particle hits the wall. This is a discontinuous jump in the momentum, which means that the force, $\mathbf{F} = \dot{\mathbf{p}}$ must be infinite. We don't actually believe that there is an infinite force, but the model doesn't care about the details of how the ideal wall is actually a little squishy. The details of that *microphysics* are only a negligible contribution to the model of, say, a basketball hitting the backboard. This is the idea that physical models as effective theories with domains of validity.

¹²⁹We can be snotty and say "you mean, the speed of light... *in vacuum*," but this is really beside the point. We started with Maxwell's equations. We didn't even specify the medium. Maybe it's uniform water. The point is that the *wave equation* told us that the speed of propagation is constant.

With a little bit of thought¹³⁰, you could have *guessed* down this Green's function for the wave equation based on the Green's function from the Poisson equation. All you have to do is to remember that the speed of light is constant and you probably want to mathematically enforce causality. In this way, the Poisson equation, the harmonic oscillator, and the wave equation are all *the same damn thing*. They're the appropriate second derivative in a given (3+0)-, (0+1)-, and (3+1)-dimensional spacetime.

To pontificate further on this: all of the uglier versions of the Poisson equation and the wave equation in cylindrical and spherical coordinates are just cousins of what we've done here. Sure, you may have to replace your Fourier expansion over nice plane waves with spherical harmonics or Bessel functions. But you know the deal: we never cared much about the basis functions, we just cared about using the *orthonormality* of these *nice* basis functions to read off the coefficients (\tilde{G}) so that it's easy to represent the solution. Keep this in mind and you won't get lost when you get to the mathematically more tedious examples in your electrodynamics course—you're doing nothing more and nothing less than the harmonic oscillator in some number of dimensions and in some choice of basis.

Exercise 7.2 Solve for the Green's function of the wave equation in (2+1)-dimensional spacetime. Notice that the behavior is quite different from the one in (3+1)-dimensions. Comment on the physics of waves in Flatland. Answer:



8 Coupled Harmonic Oscillators and Fields

Let's take a break from Green's functions. In the previous section, we called the wave equation the *harmonic oscillator in spacetime*. We proceeded to solve the wave equation for the electromagnetic field, $\partial^2 A_\mu = j_\mu$. We were very hand-wavy in arguing that the appropriate generalization of the harmonic oscillator operator $(d/dt)^2$ is the spacetime second derivative, $\partial^2 = c^{-2} \partial_t^2 - \nabla^2$. Furthermore, we noticed that our state functions went from being functions of only one variable, $\psi(t)$, to functions of space and time, $A_\mu(\mathbf{x}, t)$. In other words, our states became *fields*. It's instructive to see why a field is naturally understood as the continuum limit of a lattice of harmonic oscillators¹³¹.

¹³⁰This addresses the point in Exercise 7.1.

¹³¹I'm saying something 'deep' here. Actual systems in condensed matter physics are atomic lattices with complicated electronic potentials. These are approximated by harmonic oscillators to leading order. At long wavelengths, the physics of this lattice often has a *continuum limit* described by a field theory. The field theory is a model that has no underlying lattice, but whose long-wavelength predictions are designed to match that of a lattice with small spacing. In particle physics one usually assumes that nature is

8.1 A notational interlude

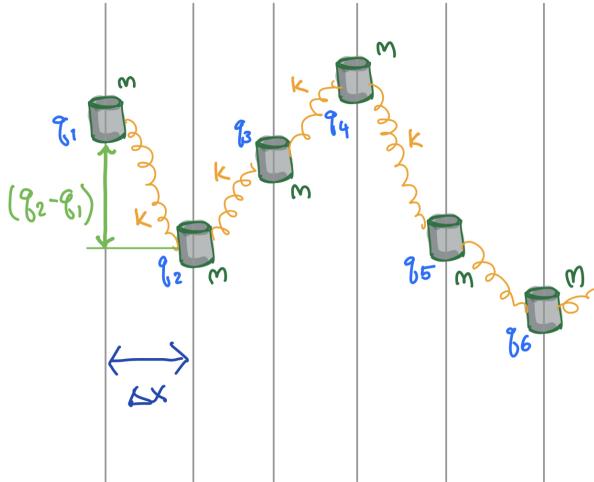
Here's a bad habit: I often use the shorthand $x(t)$ to describe the position of a particle at time t . The particle doesn't need to literally be a particle, it could be the displacement of a spring from equilibrium. If the particle takes a position in three-space then we write $\mathbf{x}(t)$. Note that this is very different from a *field*, where the spatial coordinates are a *dependent* variable analogous to time. Just as $x(t)$ has a value for all t , a field $\varphi(x, t)$ takes some value for all x and t .

This is a potential source of confusion because we want to show the transition from a single harmonic oscillator to a field. The intermediate state is a lattice of coupled harmonic oscillators. To do this, let us break the bad habit of writing $x(t)$ and write the displacement of a harmonic oscillator to be $q(t)$. The displacement can be abstract, it doesn't have to be a distance separation; for example, it could be the value of a wavefunction at a point in space. If we have a bunch of harmonic oscillators evenly spaced on a line, we can index them $q_i(t)$.

If you imagine that you have *many* harmonic oscillators that are spaced very closely together, then rather than specifying an index you could just specify their position along the line, x . Note that x is playing the role of an index, not as the state variable. In other words:

$$q_i(t) \rightarrow q(x, t). \quad (8.1)$$

If I pick some point along the line, x , then the displacement of the harmonic oscillator at x observed at time t is $q(x, t)$. Congrats, $q(x, t)$ is now a field. You can see why we didn't want to use $x(t)$: it confuses the position along the line with the displacement of the harmonic oscillator at that position. The picture is as follows:



The generalization to a dense three-dimensional lattice is straightforward, you end up with $q(x, y, z, t)$ or $q(\mathbf{x}, t)$.

continuous so that the fundamental objects are actually fields. Whether or not this is literally true doesn't matter: it is sufficient that the range of phenomena that your theory hopes to describe are agnostic to whether or not there is a lattice. Sometimes particle physicists go the opposite way and use lattice techniques to make predictions where the continuum limit is difficult. The notion (and meaning) of a continuum limit is tied closely to the idea of renormalization group flow, which is one of the most elegant ideas in theoretical physics.

8.2 Coupled Harmonic Oscillators

The Lagrangian for a single harmonic oscillator $q(t)$ is

$$L[q(t)] = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 . \quad (8.2)$$

This assumes some reference equilibrium value $q = 0$. Consider instead a series of identical beads of mass m along a line that can move up and down freely, but are connected to their nearest neighbors by springs of uniform spring constant k . The Lagrangian for this system is

$$L[q_1(t), q_2(t), \dots] = \sum_i \frac{1}{2}m\dot{q}_i^2 - \sum_i \frac{1}{2}k(q_i - q_{i-1})^2 . \quad (8.3)$$

Because life is short, I'm not going to worry about the boundary conditions for the index i . You can assume that the system is periodic if you're nervous. Now let us pass into the continuum limit and consider some small separation Δx . We make the following replacement to continuum variables:

$$i \rightarrow x \quad (8.4)$$

$$q_i(t) \rightarrow q(x, t) \quad (8.5)$$

$$(q_i - q_{i-1})^2 \rightarrow \Delta x^2 \left(\frac{\partial q}{\partial x} \right)^2 \quad (8.6)$$

$$\sum_i \rightarrow \int \frac{dx}{\Delta x} \quad (8.7)$$

$$m \rightarrow \rho \Delta x . \quad (8.8)$$

The factors of Δx should be clear from dimensional analysis. You can think about them as some characteristic small length scale analogous to the lattice spacing. Observe that the mass of a single harmonic oscillator is replaced by the mass density ρ multiplied by a ‘volume’ (Δx in one dimension). Once we write out the Lagrangian the Δx factors almost cancel:

$$L = \int \frac{dx}{\Delta x} \left[\frac{1}{2}\rho \Delta x \left(\frac{\partial q}{\partial t} \right)^2 - \frac{1}{2}k \Delta x^2 \left(\frac{\partial q}{\partial x} \right)^2 \right] \quad (8.9)$$

$$= \int dx \left[\frac{1}{2}\rho \left(\frac{\partial q}{\partial t} \right)^2 - \frac{1}{2}(k \Delta x) \left(\frac{\partial q}{\partial x} \right)^2 \right] . \quad (8.10)$$

If we had a lattice in D spatial dimensions, then this result generalizes to

$$L = \int d^D x \left[\frac{1}{2}\rho \left(\frac{\partial q}{\partial t} \right)^2 - \frac{1}{2}(k \Delta x^{2-D}) \left(\frac{\partial q}{\partial \mathbf{x}} \right)^2 \right] . \quad (8.11)$$

Finally, let us rescale the state variable by defining a ‘better’ state variable

$$\varphi(x, t) \equiv \frac{q(x, t)}{\sqrt{\rho}} . \quad (8.12)$$

Further, let's go ahead and write the action $S = \int dt L$ with the idea of writing one big *spacetime* integral over a Lagrangian *density* $S = \int dt d^Dx \mathcal{L}$:

$$S = \int dt d^Dx \frac{1}{2} \left[\left(\frac{\partial \varphi}{\partial t} \right)^2 - c^2 \left(\frac{\partial \varphi}{\partial \mathbf{x}} \right)^2 \right]. \quad (8.13)$$

Observe that we have defined a speed

$$c^2 = k \Delta x^{2-D} \rho. \quad (8.14)$$

If we take one more step and integrate by parts, we may write the action in the form:

$$S = \int dt d^Dx \frac{-c^2}{2} \varphi \left[\frac{1}{c^2} \left(\frac{\partial}{\partial t} \right)^2 - \left(\frac{\partial}{\partial \mathbf{x}} \right)^2 \right] \varphi = -c^2 \int d^{D+1}x \frac{1}{2} \varphi \partial^2 \varphi. \quad (8.15)$$

Aha! Check it out, we have recovered the *wave operator* ∂^2 . Now if you permit me some sloppy calculus, then if I squint at the action S and vary with respect to φ , I get

$$\delta S = 0 \quad \Rightarrow \quad \partial^2 \varphi = 0. \quad (8.16)$$

Let's be completely honest: we should do this variation more carefully... and we will, but just not right now; see Example 9.6. But this result is rather compelling: $\partial^2 \varphi(x, t) = 0$ is simply the wave equation.

Exercise 8.1 Confirm that (8.11) is the appropriate generalization to D spatial dimensions.

Exercise 8.2 Confirm by dimensional analysis that c in (8.14) is a speed.

8.3 Interpretation

What we have described here is a lattice of harmonic oscillator states. The harmonic oscillator potential comes from each lattice point being connected ‘by a spring’ to its nearest neighbor lattice points in each spatial direction. The propagation of a wave comes from a perturbation of single harmonic oscillator *locally* affecting the harmonic oscillators nearby. Those oscillators, in turn, cause perturbations to *their* neighbors. The speed c is a characteristic speed of this propagation. In the continuum theory it’s just some prefactor that is required by dimensional analysis. In the lattice theory it is related to the mass of each harmonic oscillator and the spring constant.

Now that we’ve gone from the lattice to the continuum, you can just look at the Lagrangian for a field and see that it can be understood as each local piece of the field tugging at nearby parts of the field.

Example 8.1 The relative minus sign between the time derivatives and the space derivatives in $\partial^2 = c^{-2} \partial_t^2 - \nabla^2$ is understood as the relative minus sign between the kinetic term and the potential term in the Lagrangian¹³²

¹³²You could, in turn, ask where that relative minus sign comes from. There are a few different ways to answer this, but I think the way that makes the most sense to me the one that is based on the path integral formulation of quantum mechanics.

Exercise 8.3 We assumed that each harmonic oscillator in the lattice has an identical mass m and an identical spring constant k . What is the physical significance of this choice in the continuum representation?

Exercise 8.4 We assumed that each harmonic oscillator on the lattice only couples to its nearest neighbor. What would interactions with next-to nearest neighbors look like in the continuum representation? What is a good physical reason why you wouldn't have couplings to lattice points that are very far away¹³³? HINT: we've already discussed this when we presented discretized function space... the only difference was when we did that, we started with a continuum and motivated a discrete representation. Now we're going in the other direction.

8.4 A theoretical digression

If you'll permit yet another digression, let me remind you that the reason why we spend so much time solving for the second derivative is that our models of physics tend to be local. Consider the action, $S = \int dt L$. If we have some state φ that propagates in spacetime, then φ is a *field*. The natural form of the action is an integral with respect to a Lagrangian density,

$$S = \int d^4x \mathcal{L} = \int d^4x (\text{const.} + a\varphi + b\partial_\mu\varphi + \dots) , \quad (8.17)$$

where on the right-hand side we just started writing out *all* possible polynomials of $\varphi(x, t)$ and the four-derivative ∂_μ with respect to coefficients a, b, \dots . At this point we're not writing *the* theory of the field φ , we are parameterizing *all* theories of the field φ . Different choices for the infinite number of coefficients (typically called *couplings*) correspond to different theories of the field φ .

We have already seen that the wave operator ∂^2 emerges from the kinetic term and the nearest-neighbor harmonic oscillator coupling on a lattice of spatial points. As we imagine the infinite sum of all possible terms in the action S , we write ∂_μ instead of d/dt or ∇ because we expect the theory to be Lorentz invariant¹³⁴. In fact, we probably don't want to have any terms that have any free indices like $\partial_\mu\varphi$ because that means it transforms like a Lorentz vector and thus the term is not Lorentz invariant. Furthermore, we can use field redefinitions to remove linear terms¹³⁵. Subject to symmetries, the action looks like:

$$S = \int d^4x \left[\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}\omega_0^2\varphi^2 + c\varphi^3 + d\varphi^4 + e\partial^2\varphi^2 + \dots \right] . \quad (8.18)$$

¹³³You can think about this problem the other way around and consider that the choice of the nearest neighbor couplings is *defining* a sense of spacetime locality. Some theorists thinking about the nature of quantum mechanics propose that an analogous idea with quantum entanglement may be responsible for the macroscopic emergence of spacetime. See, e.g. arXiv:1606.08444.

¹³⁴If your theory is not Lorentz invariant, then replace ‘Lorentz’ with whatever symmetries your system does have. If it has no appreciable symmetries, then may Boltzmann have mercy on your soul.

¹³⁵This is not an obvious statement in the action, but the variation of the action usually comes from a path integral, where x varies over $\varphi(x, t)$. Shifting $\varphi(x, t)$ is like integrating dx versus $d(x+3) = dx$.

You recognize that the term in parenthesis is simply the wave operator. It is usually convenient to lump together all of the quadratic terms

$$S_{\text{quad.}} = \int d^4x (\partial\varphi)^2 - \frac{1}{2}\omega_0^2\varphi^2 = -\frac{1}{2} \int d^4x \varphi \mathcal{O}_{\text{quad}} \varphi . \quad (8.19)$$

When you vary this part of action and ignore the other terms, you end up with the appropriate *linear* differential equation, $\mathcal{O}\varphi = 0$. This should make sense from basic calculus: if you have a quadratic function $f(x) = \frac{1}{2}ax^2$, then the first derivative is linear: $f'(x) = ax$ and the extrema satisfy $ax = 0$. The wave equation comes from the same observation when you replace $x \rightarrow \varphi$ and $a \rightarrow \mathcal{O}_{\text{quad}}$. Of course you're really going from a function $f(x)$ to a functional (function of functions) $S[\varphi]$, but this is a technical detail¹³⁶.

The operator $\mathcal{O}_{\text{quad}}$ includes the usual wave equation that represented a lattice of coupled harmonic oscillators, as well as the term $\omega_0^2\varphi(x)^2$.

Exercise 8.5 *The ω_0^2 term looks like the potential for a harmonic oscillator. How is it different from the ∇^2 harmonic oscillator potential? How would you describe a lattice of harmonic oscillators with both the ∇^2 and the ω_0^2 potentials? Hint: what is the equilibrium value of the harmonic oscillators?*

What about the other terms? First we should do a bit of dimensional analysis. For my own sanity, let's use natural units where we set $c = \hbar = 1$ and all units are measure in mass dimension:

$$[x] = [t] = -1 . \quad (8.20)$$

This means that $[d^4x] = -4$. Since the action is dimensionless in natural units—after all, it shows up in a e^{iS} —then we know the Lagrangian density has dimension $[\mathcal{L}] = +4$. Since $[\partial] = +1$, we deduce that the variable q has mass dimension $[\varphi] = +1$. With that in mind, we can look at the ‘higher order’ terms. The coefficients of these terms must have some *negative* mass dimension:

$$c, d, e, \dots = \left(\frac{1}{M}\right)^n , \quad (8.21)$$

where n is a positive integer. We've written the mass scale as M . Any mass scale in the theory must have physical significance. We could question whether the mass scale M is big or small compared to either ω_0 or the characteristic energies at which we are studying the system. If M is small, then these coefficients are large, and their dynamics are important. However, if we include these terms in our dynamics, then the equations of motion become very nonlinear:

$$\mathcal{O}_{\text{quad.}} q + 3c\varphi^2 + 4d\varphi^3 + \dots = 0 . \quad (8.22)$$

That would mean that the wave equation approximation is quite bad. Further, these additional terms are also clearly non-linear. If we see physics that is approximately described by

¹³⁶You may want to check that this intuition is true by again imagining a discretized version of φ .

the wave equation, then these coefficients must be small. We thus expect M to be large—it is an ultraviolet scale.

What we intuit is that the scale M is some energy scale where our theory is breaking down. If, for example, we were to *experimentally*¹³⁷ study this system at energies $E \sim M$, we expect that the effect of these non-linear terms are no longer small and become significant. In other words, the scale M plays the role of a *cutoff* at which our theory described by the linear operator $\mathcal{O}_{\text{quad.}}$ breaks down. For energies well below M , we can study the linear system and do *perturbation theory* to study the first few non-linear coefficients that have coefficients that go like $(1/M)$ to a relatively small power. In the regime where our experiment has characteristic energy $E \ll M$, the in-principle infinite number of higher-order coefficients are negligible because we expect those effects to go like $(E/M)^n$; so keeping the first few should be an accurate description of the system.

This is the notion of an **effective theory**. All physical models are effective theories. We're simply parameterizing our ignorance about the universe and working in a regime where we are predictive. The effective theory philosophy explains why we are so obsessed with Green's functions of second-derivative operators:

- Constant operators are trivial, they're not even differential.
- Operators with a single derivative are not Lorentz invariant. (The exceptions, like the diffusion operator, are only valid in the non-relativistic regime.)
- Operators with three, five, or any odd number of derivatives are not Lorentz invariant. They are also suppressed by inverse powers of the cutoff, M .
- Operators that are nonlinear, i.e. terms in the Lagrangian density that have more than four powers of $q(x, t)$, are also suppressed by inverse powers of the cutoff M .

This tells you that in (3+1)-dimensions, the most interesting non-linear terms are $\mathcal{L} \supset c\varphi^3 + d\varphi^4$. If you assume that your theory has some $q \rightarrow -q$ symmetry, then you can even remove the c term. By the way, this is how you build theories: you start with the most general \mathcal{L} with an infinite number of terms. Then you argue based on symmetries and the relevance of high-dimensional operators which terms you can throw out. The result is that any reasonable theory in (3+1)-dimensions should be some perturbation of the harmonic oscillator/wave/Poisson system.

Let me layer onto this a bit more: implicit in writing out $S = \int d^4x \mathcal{L}$ is the idea that our theory should be manifestly *local*. When we write terms like q^3 , we really mean $\varphi(x, t)\varphi(x, t)\varphi(x, t)$ at the *same* spacetime point. If this were not true, then the theory would be non-local and the causal structure of the theory would depend on the reference frame. This, in turn would put causality and Lorentz invariance at odds with one another and you'd have to pick one but not both. Some recent alternative formulation of quantum physics based on amplitudes (and not Lagrangians) have proposed that by giving up on *manifest* locality, there may be more elegant descriptions of a theory. Those descriptions are local, they're just not obviously so.

Exercise 8.6 *If we lived in Flatland, then the action would take the form $S = \int d^3x \mathcal{L}$. How does the power counting change for a state q ? How does it change for a general number of*

¹³⁷In the *gedanken* sense.

spatial dimensions, D ? Note: quantum effects can change the story quite a bit in $D = 1$ and $D > 3$. That's different story

Exercise 8.7 *The Lagrangian density for electrodynamics is*

$$\mathcal{L} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \quad (8.23)$$

where $F_{\mu\nu}$ is the field strength tensor. Write out the corresponding equations of motion. Notice that you only get half of Maxwell's equations. Where does the other half come from?
ANSWER (partial): *the other half of Maxwell's equations come from the geometry of that system. It's most clear in the language of differential forms, but then one has to build up that mathematical machinery.*

Finally, let me end this theoretical digression by re-emphasizing the effective theory philosophy. For the most part, a theory for the field φ is an action with respect to this field. The equation of motion for the field come from $\delta S = 0$. The quadratic terms in φ give a linear differential equation $\mathcal{O}_{\text{quad}}\varphi = 0$. In the presence of a source¹³⁸ there's some non-trivial right-hand side. This linear equation can be solved using Green's functions. The terms coming from higher powers of φ are *non-linear*, but they are also small as long as the theory is being studied at energies that are small compared to the dimensionful scale M that suppresses those terms. This is the regime where the theory is under control (perturbative) and where it makes sense to talk about excitations of the φ field. At much higher energies, this description may break down—for example, because it may become important to include the infinite number of non-linear terms. Or perhaps the underlying theory is a completely different theory where φ were effective degrees of freedom¹³⁹.

Example 8.2 *Our theories of protons, neutrons, and electrons make sense until you study them at energies $E > \text{GeV}$ at which point the substructure of the nucleons becomes important. The protons and neutrons are no longer fundamental degrees of freedom and are replaced by a new description, quantum chromodynamics, that looks nothing like a theory of protons and neutrons. To say it differently, a chef doesn't need to know plasma physics when boiling a soup at 373 Kelvin. However, if you want to describe the phase transitions of early universe at 10^{13} Kelvin, then you really do need to know about subnuclear physics. The relevant description of nature depends on the energy scale. Often times, the idea of a relevant description is far more important than getting the right description¹⁴⁰.*

¹³⁸One way to write the source is to put in a linear term in the Lagrangian density with the source $j(x)$ as a coefficient, $\Delta\mathcal{L} = -j(x)\varphi(x)$. The resulting differential equation is $\mathcal{O}_{\text{quad}}\varphi = j(x)$.

¹³⁹There is a lot of ongoing research in ‘pure’ theoretical physics along these lines. One of the most interesting papers in the past 15 years has been [hep-th/0602178](#), which pulls in many advanced ideas in quantum field theory, but is written in a way that anyone in this course can hopefully appreciate.

¹⁴⁰One of the most intriguing ideas in theoretical physics is the existence of dualities that relate the mathematical description of two completely different theories. One of the implications of these observations is that there may not be a single ‘right’ description of nature at a given energy scale. It may be that many completely different-looking theories make equivalent predictions about the same phenomena. Typically, though, only one of these descriptions is perturbative and easily calculable. Turning this observation upside down, one may use these dualities as a way to understand what happens to one description of nature when it breaks down—perhaps a dual description that was non-perturbative becomes perturbative in that break-down region. This turns out to be one of the main motivations to study extra dimensions.

If we only took the quadratic part of the action, then the theory is completely linear and every field looks like it's described by the wave equation. We call that case a *free* field theory because it has no interactions and is completely solvable. Theories are distinguished by the kinds of interactions that *perturb* the linear theory¹⁴¹. Suppose, for example, we included a $-\lambda\varphi^4$ term in the action. Now we have a theory that is not *just* the wave equation—in fact, it is not even linear. Since our theory is non-linear, what good is any of the Green's function garbage we've learned in this class?! There are two standard options:

- Numerical approach: simulate the system on a lattice and solve for the extrema of the action. This is called lattice QCD in the high-energy physics community. In the astronomy community this is just called ‘simulation.’
- Feynman diagrams: do perturbation theory with respect to the couplings as long as they are small.

I have little to say about the numerical approach other than that it is limited by computing power. There's always a handful of open questions in physics that one wonders: if we focused all of our supercomputing power on this problem for a week, maybe humanity would just know the answer.

The Feynman diagram approach is limited by perturbativity. The idea, though, is that you can use the Green's function from the linear (quadratic) part of the theory, but you have to *perturb* about this Green's function by inserting powers of the non-linearity. The Green's function encodes the propagation of information from a source point in spacetime to the observation point in spacetime, hence the alternative name of ‘propagator’. When you include the non-linearities, you systematically deform the Green's function at intermediate points in spacetime. This is most easily represented graphically by diagrams where straight lines represent linear propagation through spacetime and vertices represent non-linear perturbations. For example, the φ^4 term is a vertex connecting four lines: this is because it's like four ‘harmonic oscillators’ that are chained together; when one of them wiggles (because a wave hits it), it causes the other three to wiggle and leads to *three* outgoing waves. We leave a careful exploration of this technique to a course on statistical field theory or quantum field theory.

Example 8.3 *If you have two fields, φ and ψ , then the effective theory approach follows. You imagine writing down not only all possible polynomials in each field alone, but mixed polynomials like $\varphi^2\psi^2$ subject to the symmetries of your theory. The more symmetric your theory, the fewer terms are allowed. Higher-order terms that are suppressed by large powers of the heavy scale M are negligible compared to terms with smaller M -suppression.*

9 Green's Functions, Statistics, and Gaussian Integrals

In Section 8 we showed how fields emerge from a system of nearest-neighbor coupled harmonic oscillators. To leading order, fields are described by the wave operator. We remarked that the differential equation that governs a physical theory comes from a variation of the action

¹⁴¹The observation that all theories basically look alike up to a handful of parameters is called **universality**. This idea is formalized in theoretical physics in what is called the **renormalization group**.

functional $\delta S[\varphi] = 0$. We even motivated how one builds theories: you start by imagining the most general action S and then you remove terms because they violate symmetries or because they're suppressed by an energy scale M much larger than the energy E or physical phenomena you're examining. The latter point amounts to a $(E/M)^n$ suppression on physical observables.

At this point, you may ask: well, where does $\delta S = 0$ come from? Curious, aren't we? It turns out that this connects to a few different ideas which we'll map out in this section. In case you haven't noticed, the latter part of these notes are much less technical than the first half. The point is no longer to teach you techniques. We are stepping back and seeing how these ideas interconnect with different theoretical structure that you may [soon] know. Don't worry if there are more results that are left to you to prove or look up, the goal isn't the technical derivation. The goal is to see the interconnection of these tools.

In this section, we'll make a connection to a bit of probability theory and a cute trick for solving Gaussian integrals. First, though, a quantum mechanical interlude.

9.1 Quantum Mechanics: Green's Function from the Path Integral

Time evolution by some amount of time Δt in quantum mechanics is enacted by the Hamiltonian (energy) as an operator: $e^{i\hat{H}\Delta t}$. This time evolution may be described as a Green's function that takes an initial state $\Psi(q_0, t_0)$ and evolves it into a final state $\Psi(q, t)$. The wavefunctions are defined by a projection of a state $|\psi(t)\rangle$ onto the position basis q :

$$\Psi(q, t) = \langle q | \psi(t) \rangle . \quad (9.1)$$

Since $|\psi(t)\rangle$ is the time evolution of $|\psi(t_0)\rangle$ by a time $\Delta t = t - t_0$, we may continue:

$$\Psi(q, t) = \langle q | e^{-i\hat{H}\Delta t} | \psi(t_0) \rangle \quad (9.2)$$

$$= \int dq_0 \langle q | e^{-i\hat{H}\Delta t} | q_0 \rangle \langle q_0 | \psi(t_0) \rangle \quad (9.3)$$

$$= \int dq_0 \langle q | e^{-i\hat{H}\Delta t} | q_0 \rangle \Psi(q_0, t_0) , \quad (9.4)$$

where we have simply inserted a complete set of states (*multiplied by one*) $\mathbb{1} = \int dq_0 |q_0\rangle\langle q_0|$; this is simply the completeness relation that we recall from our review of linear algebra. We identify the quantity $\langle q | e^{-i\hat{H}\Delta t} | q_0 \rangle$ as the Green's function $G(q, t; q_0, t_0)$ since it now manifestly plays the role of a Green's function to solve for the state $\Psi(q, t)$ given the initial state $\Psi(q_0, t_0)$:

$$\Psi(q, t) = \int dq_0 G(q, t; q_0, t_0) \Psi(q_0, t_0) . \quad (9.5)$$

We didn't have to explicitly write the Schrödinger equation because that's already implicit in our starting point that $e^{-i\hat{H}\Delta t}$ is the time-evolution operator.

If you go over the path integral formulation of quantum mechanics, one finds that by repeating this trick and inserting a complete set of states over *many* time slices between t_0

and t you end up with¹⁴² a closed form expression for the Green's function:

$$G(q, t; q_0, t_0) = \int \mathcal{D}q(t) e^{iS[q]/\hbar}. \quad (9.6)$$

Exercise 9.1 For those with some familiarity with quantum mechanics, derive the above result. Be sure to keep track of how $e^{i\hat{H}t}$ turns into $e^{iS[q]}$. I personally find this to be the most compelling motivation for defining the action S as the difference of the kinetic and the potential energies. Hint: you can slice up the finite time interval t into a bunch of tiny time slices εt . You can use the following matrix/operator relation: $e^{\varepsilon(A+B)} = e^{\varepsilon A}e^{\varepsilon B} + \mathcal{O}(\varepsilon^2)$ to separate out the kinetic and potential terms. Note that the kinetic terms depend only on momentum while the potential terms depend only on position. You can insert complete sets of momentum states, $\mathbb{1} = \int dp|p\rangle\langle p|$ (check the normalization) and use $\langle q|p\rangle \sim e^{ipq}$, subject to conventions.

I've broken my usual convention of natural units and explicitly written out the \hbar required to make $S[q]$ dimensionless. Recall that $[L]$ is energy and $S = \int dt L$ so that $[S] = E \times t$, which just happens to be the units of \hbar . The curious integration measure $\mathcal{D}q(t)$ is an integral over different functions $q(t)$. The meaning is clear if we discretize in time:

$$\mathcal{D}q(t) = dq(t_0) dq(t_1) dq(t_2) \cdots dq(t_{N-1}) dq(t_N = t). \quad (9.7)$$

In other words, for each discrete time t_i , we vary the position $q(t_i)$ independently of the other positions. In this way, the integral over $\mathcal{D}q(t)$ is an integral over all possible functions $q(t)$ subject to the initial and final states.

You should interpret this expression as the famous quantum mechanical *sum over histories*. The amplitude for a state to go from $\Psi(q_0, t_0)$ to $\Psi(q, t)$ includes a sum over the amplitude for each possible way to transition from the initial to the final state. At this point, you should recall the double slit experiment¹⁴³. Evidently the quantity $e^{iS/\hbar}$ is the weight of each path. It corresponds to the amplitude of each path we're summing together. The total amplitude is given by the sum of these complex numbers with unit magnitude¹⁴⁴.

\hbar is also a measure of ‘quantumness’. Recall that it shows up in statistical mechanics due to the Gibbs paradox. The ‘quantum’ of quantum mechanics refers to the fact that $\hbar \neq 0$ so that $[\hat{p}, \hat{q}] \neq 0$. As $\hbar \rightarrow 0$ one recovers the classical limit. We see that in the classical limit, the argument of the exponential $e^{iS/\hbar}$ becomes large and highly sensitive to variations of S . At this point one can make the *saddle point approximation*. This is the observation that for $\hbar \rightarrow 0$, changes in $S[q]$ coming from nearby paths will rapidly change the phase

¹⁴²You may refer to the first couple of lectures here: <https://sites.google.com/ucr.edu/p230b/>

¹⁴³The argument goes like this: imagine a double slit experiment. Now poke a third slit through the board; you sum over three paths. Now insert another board with two slits; you sum over $3 \times 2 = 6$ paths. Now poke more holes, add more boards until you have an infinite number of boards each with an infinite number of holes. This corresponds to summing over all possible continuum paths from the initial to the final positions.

¹⁴⁴You may want to look up ‘phasor’ diagrams to see how to interpret this sum. Feynman describes this well in his for-the-public book *QED: The Strange Theory of Light and Matter*, or you can see a more recent pop physics summary from PBS Spacetime: <https://youtu.be/vSFRN-ymfge>.

of $e^{iS/\hbar}$ and cause the contributions of these nearby paths to cancel as one integrates over paths $q(t)$. There is one exception: when $S[q]$ is near an extremum (say, a minimum) then the contribution from nearby paths will change more slowly—this is obvious, you’re at an extremum so the functional $S[q]$ is flat—and those paths will dominate the integral. The result is that in the classical limit, the paths $q(t)$ that give the dominant contribution are simply those which realize an extremum of the action:

$$\delta S[q] = 0 . \quad (9.8)$$

And there you go, we have ‘derived’ the principle of least action as the classical limit of time evolution in quantum mechanics.

Exercise 9.2 *How do you expect the Green’s function (9.6) to change when we go from a particle at $q(t)$ at time t to a field $\varphi(x, t)$ that takes in position x and time t as variables? Congratulations, you’re on your way to quantum field theory.*

9.2 A probability refresher

Now you may have a bit of mental whiplash as I change gears completely, but humor me a moment, we’ll get back to the quantum mechanical picture shortly.

A probability distribution function is an integrand $p(x)dx$ that tells you about the probability for some quantity x to take some value. $p(x)$ is an infinitesimal probability defined by the statement that the probability for x to take values within x and $(x + dx)$ is $p(x)dx$. The probability for x to take a value within some finite range $x \in [a, b]$ is

$$P(x \in [a, b]) = \int_a^b dx p(x) . \quad (9.9)$$

As you can expect, distributions are normalized so that

$$\int_{-\infty}^{\infty} dx p(x) = 1 . \quad (9.10)$$

A surprising amount of physics boils down to probability distributions. This shows up because we often deal with thermal/statistical uncertainty in condensed matter systems, data uncertainty in observational and experimental fields, and quantum uncertainty in anything ‘small.’ Often you get multiple kinds of uncertainty¹⁴⁵. Physics trains us to be skeptics. This is also part of what makes research-level training in physics so valuable in many other fields like machine learning or finance¹⁴⁶.

Example 9.1 *The most famous distribution is the Gaussian:*

$$g(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} . \quad (9.11)$$

¹⁴⁵There’s also the notorious ‘theory’ uncertainty which is a statement about us not knowing what we don’t know.

¹⁴⁶It’s almost surprising that this hasn’t translated into fields like government and policy, though I suspect that this may be due to aspects that are lacking from a traditional physics training.

This is a distribution with mean μ and standard deviation σ . It is relevant for us in this course because up to pre-factors, it is an exponential of a quadratic polynomial—this looks similar to e^{iS} , where we argued that the quadratic parts of S correspond to terms that give linear differential equations.

A probability distribution can be decomposed into moments. This is the same way that you can decompose a finite mass distribution into *moments* of inertia or the way that we can decompose a charge distribution into monopoles, dipoles, quadrupoles, etc. The n^{th} moment of a distribution $p(x)$ is

$$M_n = \int_{-\infty}^{\infty} dx x^n p(x) \equiv \langle x^n \rangle . \quad (9.12)$$

The notation should remind you of a ‘correlation function’ or an ‘expectation value.’ The moments are an expansion that encode the whole distribution. There is, in turn, a fancy way to encode the moments in a **generating function**:

$$Z(J) \equiv \int dx e^{Jx} p(x) = \langle e^{Jx} \rangle = \langle 1 \rangle + J \langle x \rangle + \frac{1}{2} J^2 \langle x^2 \rangle + \dots , \quad (9.13)$$

where J is the argument of the generating function. The moments of the distribution are derivatives of the generating function:

$$M_n = \left. \frac{\partial^n Z(J)}{\partial J^n} \right|_{J=0} . \quad (9.14)$$

The generating function is a *Laplace transform* of the distribution $p(x)dx$. Like the Fourier transform, this is an integral transform that can be used to solve differential equations. To the best of my limited experience, engineers learn about Laplace transforms and physicists learn about Fourier transforms¹⁴⁷.

Example 9.2 One could also write the Fourier transform of the distribution $p(x)dx$. This is called the **characteristic function** and is defined by

$$\tilde{p}(k) = \int_{-\infty}^{\infty} dx e^{ikx} p(x) = 1 + ikM_1 + \dots . \quad (9.15)$$

Example 9.3 Observe that $\tilde{p}(k) = Z(ik)$.

Example 9.4 Note that one can write the distribution in a Fourier representation:

$$p(x) = \int dk \left(1 + ikM_1 + \frac{(ik)^2}{2!} M_2 + \dots \right) e^{-ikx} . \quad (9.16)$$

¹⁴⁷I suspect this is because the momentum basis has physical significance for us. Maybe engineers are a little more skeptical of imaginary numbers than we are. Who knows. ... okay, I suspect someone with a mathematical background knows and will say that both physicists and engineers are being silly. <https://xkcd.com/435/>

The e^{-ikx} looks like a δ -function in momentum space¹⁴⁸. In fact, we can think of this as a derivative expansion:

$$p(x) = \int dk \left(1 + (-\partial_x) M_1 + \frac{(-\partial_x)^2}{2!} M_2 + \dots \right) e^{-ikx}. \quad (9.17)$$

Recalling that $\int dk e^{-ikx} = \delta(x)$, we see that this is indeed an expansion in derivatives of the δ -function:

$$p(x) = \left[1 + (-\partial_x) M_1 + \frac{(-\partial_x)^2}{2!} M_2 + \dots \right] \delta(x). \quad (9.18)$$

This means that if we would like to determine the expectation value of a test function, $\langle f(x) \rangle$, with respect to the distribution $p(x)$, we have

$$\langle f(x) \rangle = \int dx f(x) p(x) = f(0) + m_1 f'(0) + \dots. \quad (9.19)$$

9.3 If this were a different class

If this were a different class, we would take a moment and talk about probabilities in physics. Rather than doing that, let me refer to some fun ideas in Appendix C.

9.4 A cute trick: the Gaussian integral

One of the neat things that you can show off to any young family members is the Gaussian integral. Removing annoying prefactors, one could imagine asking a high school student to integrate the Gaussian:

$$G = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2}. \quad (9.20)$$

The exponent is supposed to be easy to solve, but the argument is quadratic. A change of variables doesn't help. What can we do?

It turns out we can simplify the problem by solving it in two dimensions. We may write G^2 as a two dimensional integral over independent variables:

$$G^2 = \int dx dy e^{-\frac{1}{2}(x^2+y^2)}. \quad (9.21)$$

The trick is to pass to polar coordinates:

$$G^2 = 2\pi \int_0^{\infty} r dr e^{-\frac{1}{2}r^2} \quad (9.22)$$

$$= 2\pi \int_0^{\infty} dw e^{-w} \quad (9.23)$$

$$= 2\pi. \quad (9.24)$$

¹⁴⁸This is a toy version of the partition function in quantum field theory and the δ -function imposes overall momentum conservation on any scattering process.

From this we deduce that $G = \sqrt{G^2} = \sqrt{2\pi}$. This is a cute parlor trick that may earn you a tip of the hat in the right company. But we're not done yet; let's squeeze this parlor trick for all that it's worth.

9.5 Down the Gaussian rabbit hole

The action appears in our quantum Green's function (9.6). We've identified something special about the quadratic part of the action: it gives linear differential equations. What can we learn about the nature of the Green's function from the Gaussian integral?

Gaussian with a prefactor. The following generalization of the Gaussian integral should be ‘obvious’ for you:

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}. \quad (9.25)$$

This is ‘obvious’ from dimensional analysis. Imagine giving x some dimension (x -ness). In order for the argument of the exponential to be dimensionless, a must carry dimension $[a] = -2$ of x -ness. The total integral has x -ness dimension $[G] = 1$, from which we deduce that $[G] \sim 1/\sqrt{a}$. However, since the result must match (9.24), we know the proportionality.

Gaussian with a source. Now consider generalizing further by adding a linear term. In field theory we call this a *source* of x because terms like this will give the right-hand sides (the source terms) of our differential equations. The result is:

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+Jx} = \sqrt{\frac{2\pi}{a}} e^{J^2/2a}. \quad (9.26)$$

Exercise 9.3 Prove (9.26) by completing the square and changing variables. Use:

$$-\frac{1}{2}ax^2 + Jx = -\frac{1}{2}a \left(x - \frac{J}{a} \right)^2 + \frac{J^2}{2a}. \quad (9.27)$$

Note that the source term amounts to multiplying by e^{Jx} so that the integral is now the Laplace transform with respect to the (not yet normalized) Gaussian. Up to normalization, this is the *generating function* of the Gaussian distribution (9.13). You can write the moments of the Gaussian distribution simply by differentiating with respect to the source and then setting it to zero:

$$M_n = \int_{-\infty}^{\infty} dx x^n e^{-\frac{1}{2}ax^2} = \left(\frac{\partial}{\partial J} \right)^n \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+Jx} \Big|_{J=0}. \quad (9.28)$$

We can plug in the integrated expression (9.26) to get, for example:

$$\langle x^2 \rangle = \left(\frac{\partial}{\partial J} \right)^2 \left[\sqrt{\frac{2\pi}{a}} e^{J^2/2a} \right]_{J=0}. \quad (9.29)$$

Exercise 9.4 Argue in a few words why $\langle x \rangle = 0$.

Gaussians in many dimensions. Here's the fun one. If the Gaussian integral is a parlor trick, then doing this one on a bar napkin should earn you a drink or two. Let's consider 3-dimensional space with x, y, z directions. No—*even better!* (You have to say it this way if you really want to impress the crowd.) Let's go to N -dimensional space described by N -vectors \mathbf{q} with components q^i . Furthermore, consider a symmetric $N \times N$ matrix A with components A^i_j . Now consider the N -dimensional generalization of (9.25):

$$\int \cdots \int dq^1 \cdots dq^n e^{-\frac{1}{2} q_i A^i_j q^j} = \sqrt{\frac{(2\pi)^n}{\det A}} . \quad (9.30)$$

After some inspection the trick should be clear: you can diagonalize a symmetric matrix A by a rotation $\mathbf{q} \rightarrow \mathbf{q}' = R\mathbf{q}$. Since this rotation does not change the integration measure, one can freely change variables to $(q')^i$, in which case A is diagonal. Then the matrix multiplication turns into a product of factors of the form

$$\int dq'^i e^{-\frac{1}{2} a_i (q'_i)^2} = \sqrt{\frac{2\pi}{a_i}} , \quad (9.31)$$

where a_i is just the i^{th} diagonal entry of A , otherwise known as the i^{th} eigenvalue. By multiplying all N of these factors together, the denominator inside the square root is simply $\det A = a_1 \cdots a_N$.

Gaussians in many dimensions with sources. Ok, you know where we're going, right? If we add a source to the multidimensional Gaussian with a symmetric matrix A , the generalization of (9.26) is

$$\int \cdots \int dq^1 \cdots dq^n e^{-\frac{1}{2} \mathbf{q} \cdot \mathbf{A} \cdot \mathbf{q} + \mathbf{J} \cdot \mathbf{q}} = \sqrt{\frac{(2\pi)^n}{\det A}} e^{\frac{1}{2} \mathbf{J} \cdot \mathbf{A}^{-1} \cdot \mathbf{J}} . \quad (9.32)$$

The implied matrix multiplication should be clear.

Exercise 9.5 *Prove (9.32). Hint: you can complete the square once again. Hint: the extremum of $-\frac{1}{2} \mathbf{q} \mathbf{A} \mathbf{q} + \mathbf{J} \mathbf{q}$ occurs at $\mathbf{q} = \mathbf{A}^{-1} \mathbf{J}$. Thus you can complete the square by writing*

$$-\frac{1}{2} \mathbf{q} \mathbf{A} \mathbf{q} + \mathbf{J} \mathbf{q} = -\frac{1}{2} (\mathbf{q} - \mathbf{A}^{-1} \mathbf{J}) \mathbf{A} (\mathbf{q} - \mathbf{A}^{-1} \mathbf{J}) + \frac{1}{2} \mathbf{J} \mathbf{A}^{-1} \mathbf{J} . \quad (9.33)$$

9.6 Correlation Functions

We're done with cute parlor tricks and now return to physics. Let's take stock of what we've achieved. We now have the expression for the multi-dimensional Gaussian integral with a source (9.32). We recall that the one-dimensional Gaussian with a source was not-so-secretly the generating function (9.28). The same observation holds for the multi-dimensional case, except now that there are many variables (q^1, \dots, q^N) we ask about more general objects than moments $\langle x^n \rangle$ of a one-dimensional distribution. The particular generalization of interest is called a **correlation function**. The simplest one is the two-point correlation function:

$$\langle q_i q_j \rangle \equiv \frac{\int d^N q q_i q_j e^{-\frac{1}{2} \mathbf{q} \mathbf{A} \mathbf{x}}}{\int d^N q e^{-\frac{1}{2} \mathbf{q} \mathbf{A} \mathbf{x}}} = A_{ij}^{-1} . \quad (9.34)$$

Very interesting! Let's make a few hot takes:

1. I'm changing the notation to lower all indices, e.g. q_i , to avoid confusing indices with powers. This shouldn't introduce any ambiguity.
2. On the right-hand side, we have A^{-1} showing up. This entire course has been about taking the inverse of infinite-dimensional matrices (operators on function space). You can bet that we're going to go to the $N \rightarrow \infty$ limit and identify A with a differential operator. In that case, the right-hand side is the Green's function.
3. If we think about each 'dimension' i, \dots, N as the displacement of some harmonic oscillator q_i at position i , then the numerator is telling us something about the mean value of $q_i q_j$ over all i and j .
4. When we were playing our Gaussian parlor tricks, we didn't bother normalizing the distribution. The denominator is simply accounting for this distribution. Let's go ahead and write this normalization as

$$\mathcal{N} = \int d^N q e^{-\frac{1}{2}\mathbf{q} \cdot \mathbf{A} \cdot \mathbf{q}} = \sqrt{\frac{(2\pi)^n}{\det A}}. \quad (9.35)$$

The two-point correlation function can, in turn, be written with respect to the multidimensional generating functional, which is simply our multi-dimensional Gaussian with a source:

$$Z[\mathbf{J}] \equiv \mathcal{N}^{-1} \int \dots \int dq^1 \dots dq^n e^{-\frac{1}{2}\mathbf{q} \cdot \mathbf{A} \cdot \mathbf{q} + \mathbf{J} \cdot \mathbf{q}} = e^{\frac{1}{2}\mathbf{J} \cdot \mathbf{A}^{-1} \cdot \mathbf{J}}. \quad (9.36)$$

Then it is clear that the two-point correlation function (or *correlator*) is indeed a second derivative of $Z[q]$ evaluated at $\mathbf{J} = 0$:

$$\langle q_i q_j \rangle = \left. \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} Z[q] \right|_{\mathbf{J}=0}. \quad (9.37)$$

Exercise 9.6 Argue in one sentence that the one-point correlation function is zero¹⁴⁹.

You can also ask for the three-point correlator $\langle q_i q_j q_k \rangle$, the four-point correlator $\langle q_i q_j q_k q_\ell \rangle$ and so forth.

Exercise 9.7 What is the seven-point correlator of the multi-dimensional Gaussian? You shouldn't have to do any work.

Example 9.5 The two-point function corresponds to the Green's functions we've been studying in this class. It is the solution to a linear differential equation. In this section, we've seen that it's straightforward to define higher-point correlation functions like $\langle q_i q_j q_k \rangle$. These correspond to non-linear differential equations (do you see why?). While the Green's function method does not work for non-linear differential equations, people will sometimes refer to $\langle q_i q_j q_k \rangle$ as the three-point Green's function by analogy to the ordinary Green's function¹⁵⁰.

¹⁴⁹Side comment: the funny thing about the Higgs boson is that its one-point correlation function is non-zero. This is related to its identity as the order parameter of electroweak symmetry.

¹⁵⁰If you've been doing Gaussian integrals for too long, everything is a Green's function. This is the analog of doing particle physics for too long and then seeing everything as a quantum field. If you do astronomy for too long every non-hydrogen-or-helium element is 'metal.'

Contextual interlude. At this point let's remind ourselves of the basic canon of this course. We have differential operators \mathcal{O} acting on states q . The equations we'd like to solve take the form

$$\mathcal{O}q = J , \quad (9.38)$$

where J is the source (we used to call it s). We can solve for q by inverting the operator \mathcal{O}

$$q = \mathcal{O}^{-1}J , \quad (9.39)$$

where on the right-hand side what we really mean is $\int dx' G(x, x')J(x')$, where G is the Green's function of the operator \mathcal{O} . This is what we mean by the Green's function being the inverse of the differential operator. The differential equation $\mathcal{O}q = J$ comes from an variational principle which we may write as

$$L = q\mathcal{O}q - Jq . \quad (9.40)$$

This shows up in an action $S = \int dt L$, which in turn shows up in an exponential $e^{iS[q]/\hbar}$ that gets integrated over the states q .

Example 9.6 If we interpret the argument of the exponential in $Z[q]$ to be an action, then one can imagine varying the action with respect to q_i to determine the condition $\delta S = 0$. Here one can see that the variation of the ‘action’ gives

$$-\mathbf{A}\mathbf{q} + \mathbf{J} = 0 . \quad (9.41)$$

Observe that when $J = 0$, this is simply (8.16). In other words, from the discrete function space picture, the variation of the action indeed gives the differential equation $\mathcal{O}q = 0$. Restoring J gives a source term on the right-hand side.

Probabilistic Interpretation Mathematically, you can interpret the 2-point correlation function $\langle q_i q_j \rangle$ as a conditional probability. It is called a *correlation* function if the a priori independent variables q_i and q_j have some correlation. Look at the numerator of (9.34):

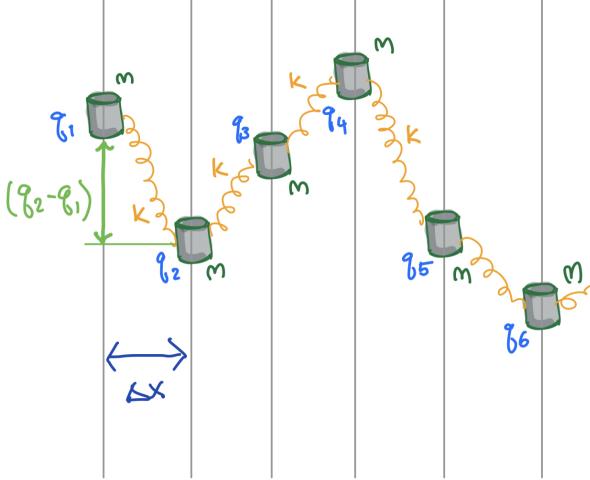
$$\int d^N q q_i q_j e^{-\frac{1}{2}\mathbf{q}\mathbf{A}\mathbf{x}} . \quad (9.42)$$

The integrals over $q_{k \neq i,j}$ don't matter: they just give you Gaussian prefactors that are canceled out by the normalization \mathcal{N}^{-1} . The two integrals $dq_i dq_j$ tell us that we are summing over all possible q_i and q_j configurations independently. If the distribution A treats q_i and q_j as totally independent (we'll clarify this momentarily), then the $dq_i dq_j$ integrals should integrate to zero in the same way the one-point correlation function integrates to zero. For example, if A were a diagonal matrix, then for every value of q_i and q_j in the sum, there's another part of the sum where, say q_j has the opposite sign and the two contributions cancel exactly.

In general, A is *not* diagonal, and so we end up with $\langle q_i q_j \rangle = A_{ij}$ in (9.34). Probabilistically, this tells us that q_i and q_j are correlated. If q_i has some positive value, then q_j is likely to be positive. If q_i has some negative value, then q_j is likely to be negative.

9.7 From Lattice to Field

Thus far the lattice sites i and j have been abstract. We have used language analogous to the coupled harmonic oscillator in Section 8.2. Let's draw the picture again:



In the above picture we treated each value of i as some point on a discretized spatial dimension. One could have equivalently treated i as an index of a discretized time direction. In that way,

$$q_i \rightarrow q(t) , \quad (9.43)$$

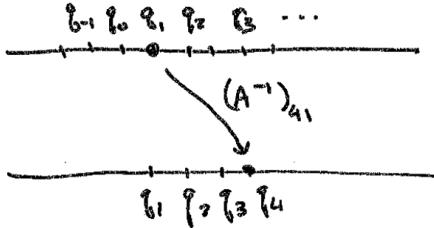
and the matrix/operator A encodes something about propagation in time. For example, the harmonic oscillator would be $A \sim -(d/dt)^2$, where we take the appropriate discretized version of the second derivative:

$$A \sim \begin{pmatrix} 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & -2 & 1 & \\ & & & & & \ddots \end{pmatrix} \quad (9.44)$$

One can see clearly that A couples nearby lattice points in time. If I displace the oscillator at time i , then at some later¹⁵¹ time j the oscillator will have some correlated displacement. In other words, the disturbance at time i has *propagated* to time j . The object that enabled that propagation is $A^{-1} \sim [-(d/dt)^2]^{-1} \sim G(t, t')$.

Observe that even though A is local—the discretized second derivative connects nearest-neighbors and next-to-nearest neighbor sites—the Green's function is *non-local*: it allows the information to propagate over long distances. The physical mechanism is clear: if we think about this as a bunch of coupled oscillators (in time), then each oscillator causes its neighbor to move. You should imagine ‘the wave’ in a stadium when the crowd cheers. As a matrix, the indices of the Green's function (A_{ij}^{-1}) gives the value of q_j given a value at q_i :

¹⁵¹One still has to impose causality by hand since the operator is symmetric under time reversal.



When we pass to the continuum limit, the product over $dq_1 \cdots dq_N$ becomes a *path integral* over all configurations in time:

$$\mathcal{D}q(t) = dq_1 \cdots dq_N . \quad (9.45)$$

The matrix multiplication turns into integrals

$$\mathbf{q} \mathbf{A} \mathbf{q} \rightarrow \int dt q(t) \left(-\frac{d^2}{dt^2} + \omega_0^2 \right) q(t) , \quad (9.46)$$

where you may want to take a moment to convince yourself that there's only one integral and not two. And, up to a factor of i , it looks like the

$$Z[J = 0] \sim \int \mathcal{D}q(t) e^{i \int dt L} , \quad (9.47)$$

where L is the harmonic oscillator Lagrangian.

In fact, we can generalize even further and go from a particle propagating in time $q(t)$ to a field propagating in spacetime $q(x, t) = \varphi(x, t)$. We just have to imagine there being more indices. We can imagine A becoming a tensor that acts as a ‘matrix’ for each direction. I say ‘imagine’ because it’s a pain in the ass to actually write out, and it’s better that we just imagine that one could do this: the point is that it is conceptually simple. In continuum language, A is simply the quadratic part of the Lagrangian density:

$$A \rightarrow \partial^2 = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 , \quad (9.48)$$

and the generating function (which I usually just call a *partition function* inspired by statistical physics) is

$$Z[J = 0] = \int \mathcal{D}\varphi(x, t) e^{iS[\varphi]} \quad S[\varphi] = \int d^4x \mathcal{L} , \quad (9.49)$$

with \mathcal{L} the Lagrangian *density* for the theory. At this point, we can remind ourselves of (9.6), throw in factors of \hbar for dimensional consistency, and pat ourselves on the back. By the way, I don’t think there’s anything too deep to read into the fact that (9.6) is a quantum mechanical Green’s function while Z is a partition function. There’s a slight nuance related to what is called second quantization (which is related to what we’ve been doing), but that is now beyond the focus of this course¹⁵².

¹⁵²See, e.g. <https://sites.google.com/ucr.edu/p230b/>

9.8 Interpretation

What's the lesson of all of this? The ‘*fields are coupled harmonic oscillators in spacetime*’ picture gives us a lot of intuition for solving [linear] differential equations. It brings us to the *origin* of these differential equations, the action. It connects the solution to the linear differential equations to a Gaussian integral. For the non-linear cases, we still need to do some work, but this is where life gets difficult. The coupled harmonic oscillator picture also brings us back to the notion of a *discretized function space* as a crutch to understand the continuum¹⁵³. That discrete function space picture helps reinforces the idea that the Green’s function is the inverse of some quadratic form¹⁵⁴ in the action.

We develop some intuition for *field theory* by placing *sources* $J(x, t)$ on the lattice and taking functional derivatives with respect to them. This corresponds to imagining that there’s some microscopic trouble-maker jumping up and down on one the harmonic oscillator at $\varphi(x, t)$ to create excitations of the field. The Green’s function transmits these excitations to the rest of the lattice, according to the dynamics $\mathcal{O} \sim A$. Even though the dynamics are manifestly *local* (derivatives \sim nearest neighbor interactions), the Green’s function is a *propagator* that ‘exponentiates’¹⁵⁵ microscopic interactions into long-range interactions.

Big Picture. You now have the big picture the most common differential operators in physics. You’ll have to slog through all sorts of second-order differential operators that are versions of the ∂^2 in different coordinate systems. Sometimes you’ll have differential equations that are only single derivatives in time, and you’ll know that this corresponds to a non-relativistic limit. You’ll know that you won’t find differential operators with only one power of ∇ unless the system somehow breaks translation invariance. You’ll have to solve these differential equations, usually by solving for the operators’ Green’s functions. You can do this in multiple ways; we reviewed the most popular methods in Section 4.11:

1. **Eigenfunctions and completeness**, Section 4.9. Assuming one knows the eigenfunctions of the differential operator, this gives a series solution for the Green’s function. It is practically useful only when the series is convergent.
2. **Patching**, Section 4.10. This method assume that one can solve the *homogeneous* differential equation $\mathcal{O}_x G(x, x') = 0$ and then produces a piece-wise solution to the *inhomogeneous* differential equation that defines the Green’s function, $\mathcal{O}_x G(x, x') = \delta(x - x')$. This is practically useful in one dimension where the boundary conditions where the pieces are connected are easy to define.
3. **Fourier transform and its cousins**. This will be the topic of the rest of our course. We convert the differential equation into an *algebraic equation* in momentum space.

¹⁵³If you’re a condensed matter physicist, then the lattice holds some physical significance. If you’re a particle physicist, then you assume that the lattice is just a coarse model of the continuum. In either case, you gain a lot from working with a handful of fields rather than $N \rightarrow \infty$ coupled harmonic oscillators.

¹⁵⁴This is the fancy way of saying matrix.

¹⁵⁵At the level of this discussion, the word ‘exponentiates’ is not a good choice. In quantum mechanics and the representation theory of groups, however, this is literally taking infinitesimal translations in spacetime, generated by e.g. \hat{H} , and then creating a finite transformation by exponentiating it, $e^{-i\hat{H}t}$. In fact, in all of what we have done, the one picture that we have ‘missed’ is the geometric one, which I keep referring to obliquely in footnotes like this. Sometimes the real story is in the footnotes (e.g. the novel *Pale Fire*).

Aspects of the causal structure of the system that are manifested in complex momentum space. Furthermore, one can use contour integrals to do the ‘hard work.’

When solving the Laplacian in different coordinates, you’ll have to use different basis functions in place of the Fourier transform. You’ll become good friends with the Bessel functions and spherical harmonics. They may look intimidating, but you’ll remember that they’re just a different basis for function space and they obey all the same nice features: they are orthonormal, they have real eigenvalues, and you can project functions onto that basis. The technical work you’ll have to do will be cumbersome, but you will never be lost conceptually. You can always refer back to the simpler examples in this course.

The one thing that we didn’t stress in this course is actually solving physical problems. We’ve brought you to the point where the solution $\psi(x)$ to a differential equation $\mathcal{O}_x\psi(x) = s(x)$ is

$$\psi(x) = \int dx' G(x, x') s(x') . \quad (9.50)$$

Actually *doing* this integral is not interesting to me. It’ll be interesting to you when the differential equation has some significance in your life (e.g. a homework problem in electrodynamics). But this is just technical work. As far as I’m concerned, you could put that integral into *Mathematica* or *NumPy* and get the answer. Most of the physical intuition went into writing down that integral in the first place.

Great! We’re now experts in solving linear differential equations. These correspond to the quadratic parts of the action and tend to be variants of the wave operator. However, you know from Section 8.4 that there are typically *more* terms in the action than the quadratic part. For example, terms that go like φ^3 or φ^4 in the action give *non-linear* differential equations, $\mathcal{O}_{\text{quad}}\varphi + \lambda\varphi^3 = s$, for example. We even argued that it is precisely these non-linear terms that distinguish theories from one another: otherwise they all tend to be the *same* second-derivative theory of non-interacting¹⁵⁶ waves. The two approaches are (1) put everything on a computer and (2) appeal to perturbation theory with respect to the non-linear terms.

10 Feynman Diagrams

The idea of Feynman diagrams is to do perturbation theory with respect to the quadratic piece of the action. Rather than jumping into the field theory language, let’s keep using the more familiar N -site lattice language. Let the matrix A represent the quadratic part of the action which, at the very least, includes the kinetic term and the nearest-neighbor interactions. There is also the linear term, $\mathbf{J}\mathbf{q}$ that we interpret as a source for \mathbf{q} . Let us add to this a quartic term which we write as

$$-V(\mathbf{q}) = -\lambda \sum_i q_i^4 . \quad (10.1)$$

¹⁵⁶We say non-interacting because the plane waves that pop out of the wave equation add linearly. This is obvious, right? The whole system was linear. The result is a theory which is completely solved, but rather uninteresting.

The minus sign is to remind us that this is a potential term and such terms appear with a minus sign in the Lagrangian. λ is a number that represents the four-point coupling or interaction strength. When q_i is a discrete harmonic oscillator λ has some dimension, but upon passing to the continuum field φ using the normalization (8.12) one can check that λ is dimensionless in natural units. By the way, it is important that we wrote $\sum q_i^4$ and not $(\sum_i q_i)^4$. The latter mixes terms at different sites, whereas q_i^4 is a local term that is evaluated at each site i .

The partition function/path integral/generating function for \mathbf{q} in the presence of $V(\mathbf{q})$ is

$$Z_V[\mathbf{J}] = \mathcal{N}^{-1} \int d^N \mathbf{q} e^{-\frac{1}{2}\mathbf{q} \mathbf{A} \mathbf{q} + \mathbf{J} \mathbf{q} - V(\mathbf{q})} \quad (10.2)$$

$$= \mathcal{N}^{-1} \int d^N \mathbf{q} e^{-V(\frac{\partial}{\partial \mathbf{J}})} e^{-\frac{1}{2}\mathbf{q} \mathbf{A} \mathbf{q} + \mathbf{J} \mathbf{q}} . \quad (10.3)$$

In the second line we have used the fact that because

$$\mathcal{N}^{-1} \int d^N \mathbf{q} q_i^n e^{-\frac{1}{2}\mathbf{q} \mathbf{A} \mathbf{q} + \mathbf{J} \mathbf{q}} = \mathcal{N}^{-1} \int d^N \mathbf{q} \left(\frac{\partial}{\partial J_i} \right)^n e^{-\frac{1}{2}\mathbf{q} \mathbf{A} \mathbf{q} + \mathbf{J} \mathbf{q}} , \quad (10.4)$$

one may write any function of q_i , including $e^{-V(\mathbf{q})}$ as a function of $(\partial/\partial J_i)$ acting on $Z[J]$, the generating function *without* $V(\mathbf{q})$. Thus

$$Z_V[\mathbf{J}] = e^{-V(\frac{\partial}{\partial \mathbf{J}})} Z[\mathbf{J}] . \quad (10.5)$$

It is conventional to write the generating function(al) to be a function of the source \mathbf{J} since it is formally an integral transform of the distribution. When we take correlation functions, we take derivatives of $Z_V[\mathbf{J}]$ with respect to \mathbf{J} , recall (9.37).

As long as λ is small, one may Taylor expand the exponential in $V(\mathbf{q})$,

$$Z_V[\mathbf{J}] = Z[\mathbf{J}] + \lambda \left[- \sum_i \left(\frac{\partial}{\partial J_i} \right)^4 \right] Z[\mathbf{J}] + \dots . \quad (10.6)$$

Let's put this to work. Suppose I'd like to ask about the four-point correlation function between four oscillators, which I'll call $q_{1,2,3,4}$. I'm labeling them 1–4 for notational simplicity, not because they have to be sequential oscillators. You could have picked any four numbers, say q_4 , q_9 , q_8 , and q_{24} . Mathematically, we know what this object corresponds to:

$$\langle q_1 q_2 q_3 q_4 \rangle = \frac{1}{\mathcal{N}} \int d^N \mathbf{q} q_1 q_2 q_3 q_4 e^{-\frac{1}{2}\mathbf{q} \mathbf{A} \mathbf{q} - V(\mathbf{q})} \quad (10.7)$$

$$= \frac{\partial^4 Z_V[\mathbf{J}]}{\partial J_1 \partial J_2 \partial J_3 \partial J_4} \Big|_{\mathbf{J}=0} \quad (10.8)$$

$$= \partial_{J_1} \partial_{J_2} \partial_{J_3} \partial_{J_4} \left(1 - \lambda \sum_i \partial_{J_i}^4 + \dots \right) Z[\mathbf{J}] \Big|_{\mathbf{J}=0} . \quad (10.9)$$

At this point, let us recall the closed form expression for $Z[\mathbf{J}]$ from performing the Gaussian integral, (9.36):

$$Z[\mathbf{J}] = e^{\frac{1}{2} \mathbf{J} \mathbf{A}^{-1} \mathbf{J}} . \quad (10.10)$$

This means that

$$\partial_{J_k} Z[\mathbf{J}]|_{\mathbf{J}=0} = (\mathbf{A}^{-1} \mathbf{J})_k = 0 \quad \partial_{J_k} \partial_{J_\ell} Z[\mathbf{J}]|_{\mathbf{J}=0} = A_{k\ell}^{-1}. \quad (10.11)$$

This means that every *pair* of derivatives with respect to source lattice sites J_k and J_ℓ produces a propagator (Green's function) $A_{k\ell}^{-1}$. Any single derivative ∂_{J_k} of Z is proportional to $A^{-1} J$, which vanishes when we set the sources to zero. The $\mathcal{O}(\lambda^0)$ term of (10.9) may be written as:

$$\partial_{J_1} \partial_{J_2} \partial_{J_3} \partial_{J_4} Z[\mathbf{J}]|_{\mathbf{J}=0} = A_{12}^{-1} A_{34}^{-1} + A_{13}^{-1} A_{24}^{-1} + A_{14}^{-1} A_{23}^{-1}. \quad (10.12)$$

The indices refer to particular lattice points where the sources are turned on. These, in turn, correspond to the specific oscillators in the correlation function $\langle q_1 q_2 q_3 q_4 \rangle$.

Exercise 10.1 Check that (10.12) is correct. You may need to do this the hard way and write out a bunch of terms to see how it works out. It is worth your time doing this to see how (10.12) pops out.

There is a graphical way of drawing this. Assign each point $i = 1, 2, 3, 4$ a point on the page and draw $A_{k\ell}^{-1}$ as a line that connects point k to point ℓ . The result is

$$A_{12}^{-1} A_{34}^{-1} + A_{13}^{-1} A_{24}^{-1} + A_{14}^{-1} A_{23}^{-1} = \overline{\text{---}} + | | + \times. \quad (10.13)$$

The $\mathcal{O}(\lambda)$ term now has *eight* derivatives with respect to \mathbf{J} ; the four new ones are at some *new* point J_i , which is then summed over:

$$-\lambda \sum_i \partial_{J_1} \partial_{J_2} \partial_{J_3} \partial_{J_4} \partial_{J_i}^4 Z[\mathbf{J}] \Big|_{\mathbf{J}=0}. \quad (10.14)$$

Following our observation above that pairs of $\partial_{J_k} \partial_{J_\ell}$ bring down a propagator $A_{k\ell}^{-1}$, we can write this diagrammatically in terms of a general intermediate point i that is summed over:

$$-\lambda \sum_i \partial_{J_1} \partial_{J_2} \partial_{J_3} \partial_{J_4} \partial_{J_i}^4 Z[\mathbf{J}] \Big|_{\mathbf{J}=0} = -4! \lambda \sum_i A_{1i}^{-1} A_{2i}^{-1} A_{3i}^{-1} A_{4i}^{-1}. \quad (10.15)$$

The $4!$ is a combinatorial factor that counts the number of different ways in which you can pair up J_i derivatives with $J_{1,2,3,4}$ derivatives.

Exercise 10.2 Confirm the factor of $4!$ by explicitly deriving the above result.

Finally, we may write this diagrammatically as well:

$$-4! \lambda \sum_i A_{1i}^{-1} A_{2i}^{-1} A_{3i}^{-1} A_{4i}^{-1} = \begin{array}{c} 1 \\ \diagup \diagdown \\ 3 \quad 2 \\ \diagup \diagdown \\ 4 \end{array}. \quad (10.16)$$

What we end up with is a rule for describing the Taylor expansion of $e^{-V[\partial \mathbf{J}]}$ in (10.5) when calculating n -point correlation functions $\langle q_1 \cdots q_n \rangle$:

1. For each of the n points, draw a point on the page. Your goal will be the connect these points into a graph.
2. To leading order, connect all the points using lines that do not intersect. This represents free propagation and is the $\mathcal{O}(\lambda^0)$ contribution.
3. The $\mathcal{O}(\lambda)$ graphs correspond to the same set up, but with one additional point that accepts four lines. Draw all graphs where each point is connected, with the reminder that the new point must have four lines connecting to it.
4. You can read this graph as the product of propagators $A_{k\ell}^{-1}$ corresponding to each line. Each four-point vertex picks up a factor λ (up to other combinatorial factors that one may determine).
5. For higher-order terms, you insert more vertices that accept four lines. One such vertex for each order in λ . You interpret more complicated diagrams the same way: each line corresponds to a propagator between those two nodes, each vertex comes with a prefactor of λ up to combinatorial factors.

As long as λ is small, one may truncate the sum at a small number of insertions to approximate the correlation function. These are called Feynman diagrams.

Interpretation. Note that each line is a Green's function from the *quadratic* operator \mathbf{qAq} : each line represents the solution to the *linear* differential equation. Recall that this Green's function is precisely what 'propagates' over long distances as a free wave. The vertices are the non-linearities. You can see this because rather than one q coming in and one q coming out, you have many q 's interacting at the same point. For example, you could read this as one q turning into three lower-energy qs . This non-conservation of particle number for an interacting theory is part of what makes a quantum theory of fields different from quantum mechanics.

By the way, the diagrams represent the *dynamics* of a system. A rough definition of this is the physical content of the Lagrangian—how the states interact. This is in contrast to the *kinematics* of the system, which has to do with things like the conservation of momentum and dispersion-like relations ('on-shell-ness') that connect the energy of a state to its momentum. The diagrams tell you about possible [quantum] interactions, but you have to impose the phase space constraints before converting the diagrams into, say, a probability or a cross section.

The sum over all of these Feynman diagrams is completely analogous to the sum over histories in quantum mechanics. You can imagine that q_1 and q_2 are some harmonic oscillators in the asymptotic past that have excitations. We would like to know the amplitude for those excitations to show up at harmonic oscillators q_3 and q_4 in the asymptotic future. In order to figure that out, we sum over all the ways in which those excitations could propagate. The leading terms include no interactions; this term is usually zero since you can't exchange momentum. The next-to-leading terms include an interaction where the initial particles overlap at a point q_i before spitting out two new particles. In this way, you can 'read off' a story of what the excitations are doing in spacetime.

Connected versus disconnected. The types of diagrams that we draw fall into two classes. There are disconnected diagrams like (10.13) and connected diagrams like (10.16). The disconnected diagrams represent propagation in spacetime, but not necessarily causal or kinematically allowed propagation. This is clear since each line is a propagator (Green's function, A^{-1} , \dots). For a dynamical field, the propagator carries some well defined momentum. You know this because we know the propagator is the solution to a wave equation and so it's something like a sine wave with some well defined angular frequency (energy) and wave number (momentum). If your initial and final states conserve total energy and momentum but are not otherwise identical, then the diagrams in (10.13) don't actually contribute. On the other hand, the four-point interaction in (10.16) allows the states to exchange momenta.

There's some good physics here. The standard introduction to field theory in statistical mechanics is the Ising model for magnetism: a lattice of spins that take binary values (spin-up or spin-down) with nearest-neighbor interactions. You may have already met this system: at high temperatures, the spins each have a lot of thermal energy and end up randomly oriented. At low temperatures, the dipole interactions between neighboring spins cause them to align. The typical length scale of these correlations are small—if you wiggle a spin at one lattice site, its neighbor will respond, but the neighbor's neighbor may already be largely insensitive¹⁵⁷. However, if one tunes the temperature to be near a *critical temperature*, then the typical number of sites that are correlated becomes large: as if a wave is propagating across the lattice¹⁵⁸. I'll leave way in which this happens to your statistical mechanics/field theory courses, but suffice it to say that this is an example where a microphysical lattice has a continuum limit that is best described by a quantum field theory¹⁵⁹.

The connected and disconnected diagrams have a physical meaning in this system. Correlators like $\langle q_i q_j \rangle$ tell you how likely a spin at site i is likely to be aligned with a spin at site j . If the temperature is near absolute zero, then this correlation is large no matter how far away i and j are: the thermal energy is so low that all of the spins want to line up. However, that's not really ‘propagation’ of information: that's just the ground state of the system, reflected in the fact that the single-spin expectation value (“one-point correlation function,” which sounds silly) is non-zero, $\langle q_i \rangle \approx \pm 1$. What we really care about is what happens when we flip one of the spins relative to that background. Thus the figure of merit is the *connected* two-point function:

$$\langle q_i q_j \rangle_{\text{connected}} = \langle q_i q_j \rangle - \langle q_i \rangle \langle q_j \rangle , \quad (10.17)$$

where the second term removes the effect of the ground state being aligned. As you can expect from its name, the connected correlation function only includes diagrams that are connected. In a particle-physics-based field theory course, one usually sees the removal of disconnected diagrams when carefully normalizing the partition function¹⁶⁰. Here it's nice to

¹⁵⁷This is just like in social situations: if you are blasting Rammstein at 3am in the morning, your next-door neighbors will get upset, but the people one apartment away will barely hear it.

¹⁵⁸This as if you played Rammstein at 3am in the morning, causing your neighbors to say ‘oh hell yes’ and then turn on all of *their* speakers up to also play Rammstein, and then *their* neighbors do the same thing.

¹⁵⁹What more, the quantum field theory exhibits universal behavior. It turns out to capture precisely the same physics as the water–ice transition.

¹⁶⁰By the way, in the discussion above I did *not* carefully normalize the partition function. The value of \mathcal{N} should be with respect to the full theory, not just the free theory. However, we didn't actually use this normalization so it was convenient to lie a bit for pedagogical efficiency.

have a concrete physical interpretation.

Onward and upward. Now you know everything you need to know to play with Feynman diagrams¹⁶¹.

11 Closing Thoughts

What I hope you've come to appreciate is that the mathematical machinery that you face in your first year of graduate school may appear daunting at first, but that you're never too far off from some variation of the harmonic oscillator. Green's functions are a powerful tool for solving differential equations, but we found that their *analytic structure* tells us about the underlying physics of our theory. We waxed poetic about going from a harmonic oscillator to a field and how this is connected to the idea of extending time to spacetime. We danced carefully with the picture of functions as infinite-dimensional function spaces that could be understood in their discretized form as large-but-finite-dimensional vector spaces. We said a few words about the notion of effective theory and action principles. After arguing that everything really does reduce to something like a harmonic oscillator, we showed one way (perturbation theory) to deal with the *non-linearities* that make *actual* physics really interesting. There are any number of directions for you to go from here. Experimentalists and observers may want to dig into statistics and probability, condensed matter folks may flock to statistical mechanics while particle folks head to quantum field theory (only to discover that at the hearts of their respective fields they are speaking dialects of the same language¹⁶²). From here on out, *you* define your path through mathematical physics: what you need for your work, what tickles your fancy, and what gets a slice of your precious attention as you set forth in your scientific careers.

Acknowledgments

I thank the students of Physics 231 (2016–2020) for their patience and feedback on this course. Especially notable are those poor souls who suffered the first iteration of this course in 2016. I appreciate the advice of my more experienced colleagues when shaping this course and the many fantastic physics references out there that each take their own twists through mathematical physics. I am especially appreciative to the textbooks by Stone and Goldbart as well as that by Cahill. Those two references not only inspired big portions of these notes, but also led me down hours of delightful rabbit holes of enlightenment that did not necessarily end up in my course.

¹⁶¹See, e.g. <https://sites.google.com/ucr.edu/p165/>. For a teaser, you can try this exercise that I give to the undergraduate particle physics course: https://github.com/Tanedo/Physics165-2020/blob/master/P165_L02.pdf.

¹⁶²... that differ by an i

A Method of Variations

The method of variations is a way to solve inhomogeneous differential equations starting from a basis of solutions to the homogeneous differential equation. We specialize to the case of second-order differential operators, though the technique is valid for higher-order operators.

A.1 First order equation for coefficients

Consider the second-order inhomogeneous linear differential equation

$$\mathcal{O}f(x) = a_2(x)f''(x) + a_1(x)f'(x) + a_0(x) = g(x) . \quad (\text{A.1})$$

One can determine the solution to this equation from varying with respect to the solutions to the homogeneous differential equation, $u(x)$ and $v(x)$. Write the solution to the inhomogeneous equation as

$$f(x) = c(x)u(x) + d(x)v(x) \quad (\text{A.2})$$

Make the ansatz that

$$c'(x)u(x) + d'(x)v(x) = 0 . \quad (\text{A.3})$$

Then the inhomogeneous differential equation reduces to

$$\mathcal{O}f = a_2(x)(c'(x)u'(x) + d'(x)v'(x)) = g(x) . \quad (\text{A.4})$$

This gives expressions for c' and d' :

$$c'(x) = \frac{-v(x)g(x)/a_2(x)}{u(x)v'(x) - u'(x)v(x)} \quad d'(x) = \frac{u(x)g(x)/a_2(x)}{u(x)v'(x) - u'(x)v(x)} , \quad (\text{A.5})$$

where one may recognize the denominators to be the Wronksian, $W = uv' - u'v$. For simplicity, write $\tilde{g}(x) \equiv g(x)/a_2(x)$.

A.2 Integrating and boundary conditions

Suppose that the inhomogeneous differential equation has boundary conditions at x_1 and x_2 given by differential operators $\mathcal{B}_{1,2}$:

$$\mathcal{B}_1 f(x) = \alpha_1 f'(x_1) + \beta_1 f(x_1) = 0 \quad \mathcal{B}_2 f(x) = \alpha_2 f'(x_2) + \beta_2 f(x_2) = 0 . \quad (\text{A.6})$$

These boundary conditions are, in general, first order for a second order inhomogeneous differential equation. For example, these may come from integrating a second-order equation of motion by parts, leaving a first-order operator on either boundary. The ansatz (A.3) simplifies these boundary conditions:

$$\mathcal{B}_i f(x) = c(x_i)\mathcal{B}_i u(x) + d(x_i)\mathcal{B}_i v(x) = 0 . \quad (\text{A.7})$$

For simplicity, let us write $(\mathcal{B}_i u) = \alpha_i u'(x_i) + \beta_i f(x_i)$ as a constant that depends on u and u' evaluated at the boundary x_i . We define $(\mathcal{B}_i v)$ similarly.

Integrate the first-order differential equations for the coefficients (A.5) using these boundary conditions. The simplest approach is to integrate a convenient linear combinations:

$$\mathcal{I}_1 = \int_{x_1}^x dy c'(y)(\mathcal{B}_1 u) + d'(y)(\mathcal{B}_1 v) = c(x)(\mathcal{B}_1 u) + d(x)(\mathcal{B}_1 v) , \quad (\text{A.8})$$

where the lower limit vanishes because $(\mathcal{B}_i f) \equiv 0$. Similarly, one finds

$$\mathcal{I}_2 = \int_x^{x_2} dy c'(y)(\mathcal{B}_2 u) + d'(y)(\mathcal{B}_2 v) = -c(x)(\mathcal{B}_2 u) - d(x)(\mathcal{B}_2 v) . \quad (\text{A.9})$$

The integrals in (A.8) and (A.9) are equivalently expressed using the expressions for $c'(x)$ and $d'(x)$ in (A.5):

$$\mathcal{I}_1 = \int_{x_1}^x dy [-(\mathcal{B}_1 u)v(y) + (\mathcal{B}_1 v)u(y)] \frac{\tilde{g}(y)}{W(y)} \quad (\text{A.10})$$

$$\mathcal{I}_2 = \int_x^{x_2} dy [-(\mathcal{B}_2 u)v(y) + (\mathcal{B}_2 v)u(y)] \frac{\tilde{g}(y)}{W(y)} . \quad (\text{A.11})$$

Combining this with the right-hand sides of (A.8) and (A.9) then gives the desired integral solutions for the coefficient functions:

$$[(\mathcal{B}_1 u)(\mathcal{B}_2 v) - (\mathcal{B}_2 u)(\mathcal{B}_1 v)] c(x) = (\mathcal{B}_2 v)\mathcal{I}_1 + (\mathcal{B}_1 v)\mathcal{I}_2 \quad (\text{A.12})$$

$$[(\mathcal{B}_2 u)(\mathcal{B}_1 v) - (\mathcal{B}_1 u)(\mathcal{B}_2 v)] d(x) = (\mathcal{B}_2 u)\mathcal{I}_1 + (\mathcal{B}_1 u)\mathcal{I}_2 . \quad (\text{A.13})$$

Define the left-hand side prefactors:

$$(\text{den.}) \equiv (\mathcal{B}_1 u)(\mathcal{B}_2 v) - (\mathcal{B}_2 u)(\mathcal{B}_1 v) , \quad (\text{A.14})$$

so that we have

$$f(x) = \frac{(\mathcal{B}_2 v)\mathcal{I}_1 + (\mathcal{B}_1 v)\mathcal{I}_2}{(\text{den.})} u(x) - \frac{(\mathcal{B}_2 u)\mathcal{I}_1 + (\mathcal{B}_1 u)\mathcal{I}_2}{(\text{den.})} v(x) \quad (\text{A.15})$$

Collecting the terms by the integration range gives:

$$(\text{den.})f(x) = \int_{x_1}^x dy \frac{\tilde{g}(y)}{W(y)} [-(\mathcal{B}_1 u)(\mathcal{B}_2 u)v(x)v(y) + (\mathcal{B}_1 v)(\mathcal{B}_2 v)u(x)u(y) \quad (\text{A.16})$$

$$- (\mathcal{B}_1 u)(\mathcal{B}_2 v)u(x)v(y) + (\mathcal{B}_1 v)(\mathcal{B}_2 u)v(x)u(y)] + \dots \quad (\text{A.17})$$

$$= \int_{x_1}^x dy \frac{\tilde{g}(y)}{W(y)} [(\mathcal{B}_1 u)v(y) - (\mathcal{B}_1 v)u(y)][(\mathcal{B}_2 u)v(x) - (\mathcal{B}_2 v)u(x)] + \dots . \quad (\text{A.18})$$

The \dots represent the \mathcal{I}_2 terms. Performing the same grouping for the \mathcal{I}_2 terms then gives:

$$(\text{den.})f(x) = \int_{x_1}^x dy \frac{\tilde{g}(y)}{W(y)} [(\mathcal{B}_1 u)v(y) - (\mathcal{B}_1 v)u(y)][(\mathcal{B}_2 u)v(x) - (\mathcal{B}_2 v)u(x)] + \quad (\text{A.19})$$

$$\int_x^{x_2} dy \frac{\tilde{g}(y)}{W(y)} [(\mathcal{B}_1 v)u(x) - (\mathcal{B}_1 u)v(x)][(\mathcal{B}_2 v)u(y) - (\mathcal{B}_2 u)v(y)] . \quad (\text{A.20})$$

These two terms can be combined rather nicely:

$$f(x) = \frac{1}{(\text{den.})} \int_{x_1}^{x_2} dy \frac{\tilde{g}(y)}{W(y)} [(\mathcal{B}_1 u)v(x_<) - (\mathcal{B}_1 v)u(x_<)] [(\mathcal{B}_2 u)v(x_>) - (\mathcal{B}_2 v)u(x_>)] \quad (\text{A.21})$$

where we introduce the convenient notation that

$$x_< = \min(x, y) \quad x_> = \max(x, y) . \quad (\text{A.22})$$

B Fourier Conventions

This section is inspired by an excellent post on [physics.stackexchange](#)¹⁶³. There are many different conventions for Fourier transforms. The danger is that you accidentally use one convention to do the Fourier transform and a different convention for the inverse transform.

There are two choices one can make when defining a Fourier transform convention; we parameterize these choices by real numbers a and b . The Fourier transform $\tilde{f}(\omega)$ of a function $f(t)$ is

$$\tilde{f}(\omega) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{\infty} dt e^{ib\omega t} f(t) . \quad (\text{B.1})$$

We see that a tells us about the (2π) factors and b tells us about the argument of the basis function $e^{ib\omega t}$. With this basis, the inverse Fourier transform is

$$f(t) = \sqrt{\frac{|b|}{(2\pi)^{1+a}}} \int_{-\infty}^{\infty} d\omega e^{-ib\omega t} \tilde{f}(\omega) . \quad (\text{B.2})$$

One may check that the inverse Fourier transform of a Fourier transform gives the original function:

$$\tilde{\tilde{f}} = \frac{|b|}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-ib\omega t} \int_{-\infty}^{\infty} ds e^{ib\omega s} f(s) \quad (\text{B.3})$$

$$= \frac{|b|}{2\pi} \int ds f(z) \int d\omega e^{ib\omega(s-t)} \quad (\text{B.4})$$

$$= \int ds \delta(s-t) f(s) , \quad (\text{B.5})$$

where we have used $\int d\xi \exp(2\pi i x \xi) = \delta(x)$.

The convention that we will choose for the *time-frequency* [inverse] Fourier transform is

$$f(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \tilde{f}(\omega) \quad d\omega \equiv \frac{d\omega}{2\pi} . \quad (\text{B.6})$$

¹⁶³<https://physics.stackexchange.com/a/308248>

This corresponds to $a = b = 1$. The corresponding transform for the frequency-domain function is

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} f(t) . \quad (\text{B.7})$$

All of this generalizes to higher dimensions: you simply Fourier transform each dimension. In fact, one is free to use a different Fourier transform convention for each direction. We can use this freedom to pick a convention that ‘automatically’ fits our conventions for spacetime. In particular, given a four-vector $x = (t, \mathbf{x})$ and its conjugate four-momentum $p = (\omega, \mathbf{k})$, one may choose to Fourier transform as follows:

$$f(x) = \int d\omega d^3\mathbf{k} e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} \tilde{f}(p) . \quad (\text{B.8})$$

With this convention, the basis function is simply

$$e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} = e^{-ip\cdot x} , \quad (\text{B.9})$$

where $p \cdot x$ is the usual Minkowski dot product, $p_\mu x^\mu$. This makes it clear that the basis function is Lorentz invariant. The Fourier transform would still respect the spacetime symmetries even if we had not chosen a convenient notation—it just wouldn’t be as simple to see.

C Nuts and Bolts Probability

Let’s forget Green’s functions now and follow up on something to do with probability: Bayes’ theorem. This is fully outside the scope of the class, but it is significant enough that it deserves to be mentioned in a ‘math methods’ course.

You are already familiar with *conditional probability*. If you roll two 6-sided dice, the probability of each coming up with one is simply the product of each probability:

$$P(\text{both dice are 1}) = P(\text{one die is 1})^2 . \quad (\text{C.1})$$

In this case we’re asking about two events, A (first die rolls 1) and B (second die rolls 1). In general, though, these two events may not be independent. In this case, one has to deal with *conditional probability*. We write $P(A|B)$ to mean the probability of A assuming that B is true. This, in turn, is related to the probability that A and B are true divided by the probability that B is true:

$$P(A|B) = \frac{P(A \& B)}{P(B)} . \quad (\text{C.2})$$

Stop to make sure this makes sense: if you write the probabilities as

$$P(X) = \frac{\text{number of times } X \text{ happens}}{\text{total number of samples}} , \quad (\text{C.3})$$

then the right-hand side of (C.2) is simply

$$P(A|B) = \frac{\text{number of times } A \text{ and } B \text{ happen}}{\text{number of times } B \text{ happens}} , \quad (\text{C.4})$$

which is precisely what we want from $P(A|B)$: what is the probability that A is true if we already know B is true. Once we say B is true, we can throw away all the samples where B is not true.

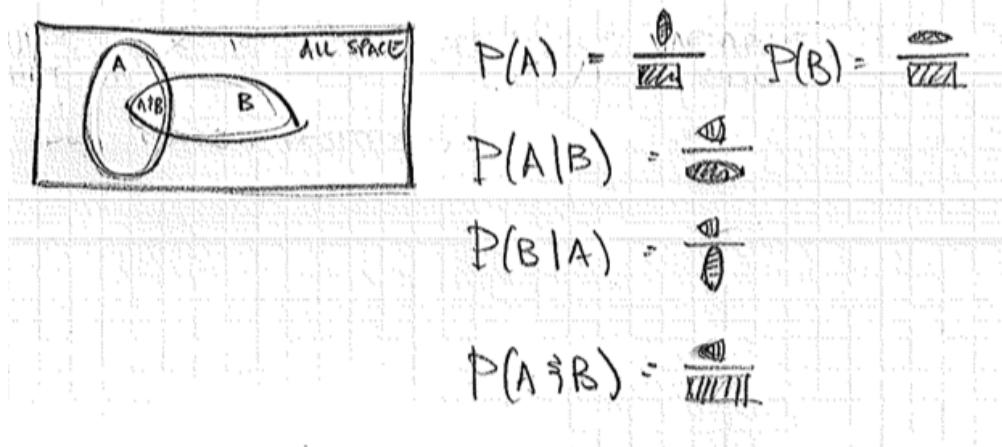
Here's the key insight: $P(A|B)P(B) = P(A \& B)$. On the right hand side, $A \& B$ is completely symmetric in A and B . That means we could swap them on the left-hand side as well:

$$P(A \& B) = P(B \& A) = P(B|A)P(A) . \quad (\text{C.5})$$

We thus have $P(A|B)P(B) = P(B|A)P(A)$, which gives us the famous **Bayes theorem** that relates the conditional probabilities $P(A|B)$ and $P(B|A)$:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} . \quad (\text{C.6})$$

Here's a useful graphical representation from Bob Cousins:



What is important here is that $P(A|B)$ and $P(B|A)$ can differ significantly if $P(A)/P(B)$ is very big or very small. This can be problematic because human beings can be sloppy when distinguishing $A|B$ versus $B|A$.

Example C.1 *The probability that someone knows how to use ROOT¹⁶⁴ given that they are a particle physicist is roughly $P(\text{ROOT}|\text{particle person}) \approx 50\%$. To rough approximation, particle experimentalists use ROOT and theorists avoid it like the plague. However, the probability that someone is a particle physicist given that they use ROOT is $P(\text{particle person}|\text{ROOT}) \approx 100\%$. If someone uses ROOT, then you can be pretty sure that they do particle physics.*

¹⁶⁴<https://root.cern/>

There's a simple reason why this is important: usually A is something that we can actually see and B is something that we want to *infer* but cannot directly test. Specifically, in physics we typically have the following case:

$$A = \text{data} \quad B = \text{theory .} \quad (\text{C.7})$$

A is the particular data that an experimentalist or an observer actually records. B roughly a statement about whether a theory is correct. We've already made may caveats in Section 8.4 about the idea of effective theories and what it means for a theory to be 'correct.' To summarize, usually B is asking: is a given theory a good description of the underlying phenomenon related to the data A ?

At this point, I'll leave it to you to dig deeper into the formalism for probabilities¹⁶⁵. In summary, you should be precise about your hypothesis tests and be able to clearly articulate the meaning of whatever figure of merit you're using to interpret your data. In what follows, let's just play with some silly manifestation of these ideas.

C.1 Were we lucky that the LHC didn't destroy the world?

One of the favorite discussion topics when I was a graduate student was a funny clip from *The Daily Show* about the Large Hadron Collider in 2009¹⁶⁶. At the time, there was speculation that the LHC could destroy the world. There were many fun and silly ideas for how this could come about, though there were many very good reasons why these ideas were more 'silly science fiction' rather than environmental concern¹⁶⁷. During the clip, John Oliver—then a correspondent for John Stewart's *The Daily Show*—interviews some dude concerned about the possibility that the LHC may destroy the world. The interview went something like this:

OLIVER: What are the chances that the world is going to be destroyed?

DUDE: It's about a one in two chance.

OLIVER: 50–50?

DUDE: It's either going to happen or not happen. So the best guess is 50–50.

Exercise C.1 *What's wrong with that dude's argument?*

C.2 The Prosecutor's Fallacy

This comes from Bergstrom and West's book *Calling Bullshit*. Imagine some high-profile *Ocean's Eleven* or *Carmen Sandiego* style heist. The FBI is able to recover a set of fingerprints which they run through their database of 50 million people and guess what: *you're* a match. Against everyone's advice, you decide to represent yourself in court. When the FBI agent

¹⁶⁵Some of my favorite references: *Statistics: A Guide to the Use of Statistical Methods in the Physical Sciences*, the lectures by Bob Cousins 1807.05996 (and his 1995 *American Journal of Physics* article), and Kyle Cranmer's statistics and data science textbook <https://cranmer.github.io/stats-ds-book/>.

¹⁶⁶<http://www.cc.com/video-clips/hzqmb9/the-daily-show-with-jon-stewart-large-hadron-collider>

¹⁶⁷<https://home.cern/science/accelerators/large-hadron-collider/safety-lhc>

explains that your fingerprint match makes is an open-and-shut case, you politely ask: *what is the chance that the wrong fingerprint would be matched with a print in the database?*

“Hah!” the prosecutor scoffs at you. “The chance of this happening happens to be *one in ten million*. That’s beyond any reasonable doubt.” Fortunately you know a thing or two about probability and there’s a chalkboard in the courtroom. You draw the following table:

	Match	No Match
Guilty		
Innocent		

You start by stating the obvious assumptions. There is *one* guilty person—not you—whose fingerprints match the prints lifted from the crime scene. While it’s neither here nor there, you also point out that there are no people who are both guilty but whose prints do not match those at the crime scene. You start to fill in the table:

	Match	No Match
Guilty	1 person	0 people
Innocent		

Next you say that there are approximately 50 million people in the fingerprint database. Almost all of them are innocent and almost all of them are completely distinct from the evidence from the crime scene. You fill in the table a bit more:

	Match	No Match
Guilty	1 person	0 people
Innocent		50,000,000 people

Finally, you point out: given a false-match rate of one in 10 million, you can estimate that around *five* innocent people will have fingerprints that match those at the crime scene.

	Match	No Match
Guilty	1 person	0 people
Innocent	5 person	50,000,000 people

The prosecutor starts to sweat, they can see what’s coming. You point out that the *data* in the case is that you are one person whose fingerprints match those at the crime scene. That means we should focus only on the population of people whose fingerprints match those at the crime scene: there are approximately six: five innocent and one guilty. You point to the numbers and say, “I know that my *Math Methods* class in graduate school wasn’t *especially* rigorous, but clearly there is a five in six chance that even though my fingerprint matches the one from the crime scene, I am in fact innocent.”

Exercise C.2 *What was the mistake that the prosecutor made?*

Bergstrom and West made up the above story to explain the fallacy, but they point out that this scenario indeed played out in 2018 in the highly publicized case that led to the capture of the Golden State Killer using DNA samples from a genealogy company. What was not shared

widely in the press coverage was that the genetic screening alone initially led to the *wrong* suspect. The true culprit was identified by combining the DNA match with other evidence from more traditional detective work. Timely applications to medicine are all over¹⁶⁸.

C.3 Why we are not evidence for God

Mathematician Jordan Ellenberg takes the above example a bit further to highlight another fallacy. This comes from his pop mathematics book *How Not To Be Wrong*, which you’re probably a bit advanced for, but is none-the-less a fun read and a lesson in effective technical communication to a lay audience.

There is an argument that the universe we observe is evidence for God. After all, look how incredibly engineered and fine-tuned life is¹⁶⁹. Indeed all this *stuff* that we see around us does seem rather complicated—why isn’t the universe more like the spherical cow¹⁷⁰ theories that we study in grad school?

The argument is something like the one in Appendix C.1. Let’s write down the *likelihood* that we see this complicated universe. Ellenberg makes some very rough estimates that will be sufficient for our point. The existence of an omnipotent creator could seem to be far more likely to have created such a finely-tuned, complicated universe.

	God Exists	No God
Simple universe		
Complicated universe	10^{-6}	10^{-18}

We don’t bother to write out the first row—after all, that’s not the case we care about. There you have it: it seems like the observation that our universe is complicated gives overwhelming evidence that God exists.

Of course, the problem is that we have *assumed* that God exists and God doesn’t exist are the only possible outcomes. This is analogous to assuming that you have a theory for an exotic new particle and the results of an experiment either rejects existence of the exotic new particle in favor of the null hypothesis (no new particles) or confirms the existence of the exotic new particle (alternate hypothesis). You know this is too naïve. Sometimes there are other reasons why the data of your experiment looks funny—maybe you didn’t plug in your cables properly¹⁷¹. So in our ontological argument, Ellenberg says that we should be sure to consider all possibilities. For example, often times complicated things come about by the dreaded ‘design by committee.’ So maybe it’s not that God exists, but that there’s a whole pantheon of gods who designed the universe through a complex process of peer review and feedback. The result is *this* ugly mess. Our table now looks like

	God Exists	No God	Many Gods
Simple universe			
Complicated universe	10^{-6}	10^{-18}	10^{-5}

¹⁶⁸<https://youtu.be/1G4VkPoG3ko>

¹⁶⁹You can also point to more physical arguments, like the apparent fine tuning of the cosmological constant.

¹⁷⁰https://en.wikipedia.org/wiki/Spherical_cow

¹⁷¹https://en.wikipedia.org/wiki/Faster-than-light_neutrino_anomaly

The number we threw in here is fairly arbitrary, but the point is that you could make the argument that *many gods* may be more likely than *one god*. Fine, but for a physics class this is starting to look a little bit too much like religious studies, no? Well, as we ponder the origin of our universe, we go to knock on the door of our cosmology colleagues, only to find that they're busy running their cosmological simulation. Aha! You remember playing *The Sims*, which is an odd knock-off experience compared to actual real life. However, it seems plausible that a sufficiently advanced civilization would have created a computer game to simulate their existence. Perhaps when that simulated reality is sufficiently advanced, it too will create a computer program that simulates *their* reality. And so forth. You start to think that this, too, is a rather plausible origin for a strangely complicated universe. Ellenberg updates our table as follows

	God Exists	No God	Many Gods	Simulated reality
Simple universe				
Complicated universe	10^{-6}	10^{-18}	10^{-5}	10^{-1}

At the level of this crude example, it seems that the most likely possibility is that not only does any god *not* exist, but it's unlikely that we exist.

C.4 The End of the World

There's a cute example of Bayesian statistics that I read about in William Poundstone's *The Doomsday Calculation*. I had the pleasure of meeting William Poundstone once and he seems like a nice and intelligent person. Unfortunately, I did not much enjoy the book. You can read the main argument in Poundstone's article for *Vox*¹⁷².

The example comes from Richard Gott, a renowned astrophysicist. Gott visited the Berlin wall in the summer of 1969. The Berlin wall was built in 1961, so by then the wall was $t = 8$ years old. Gott claims to have wondered how long the Berlin wall would stand and came up with the following argument.

Gott figured that there was nothing special about him visiting the Berlin wall in 1969. Assuming that the Berlin wall would be torn down, it had a finite lifetime T . He just happened to sample the existence of the Berlin wall at some moment of time within that lifetime. He assumed was no reason for him to be visiting the wall particularly early or particularly late in its lifetime; thus, there's a uniform 'prior' probability that he'd visit the wall at any time during its lifetime. With this assumption, he said that there's a 50% chance that his visit in 1969 happens to fall within the middle 50% interval of T . In other words, there's a 50% chance that $t = 8$ is somewhere between $.25 \times T$ and $.75 \times T$.

- If $t = 8$ corresponds to $0.25 \times T$ —that is, he happened to show up a little *early* in the wall's lifetime—then $T = 32$ and the wall would stand for another 24 years.
- Alternatively, if $t = 8$ to $0.75 \times T$ —that is, he happened to show up a little *late* in the wall's lifetime—then $T = 10.6$ and the wall would stand for only another 2.6 years.

¹⁷²<https://www.vox.com/the-highlight/2019/6/28/18760585/doomsday-argument-calculation-prediction-j-richard-gott>

This brackets an anticipated lifetime for the Berlin wall to be between $T = 10.6$ and $T = 32$ years, or a range of dates 1972–1993. Spoiler alert: the Berlin wall was torn down in 1989, which is indeed in this range of years.

Gott went on to speculate about the implications on how long humanity would survive¹⁷³.

Exercise C.3 *Using Gott's technique for estimating the lifespan of the Berlin wall and the approximation that humanity has existed for around 200,000 years, what is a plausible range of lifetimes for human existence?*

D Legendre Transform

The **Legendre transform** is a mathematical tool that shows up often in physics. Like many of the ideas in mathematical physics, it one can get lost going down a rabbit hole of the mathematical significance and different ways in which it pops up in physics. As a result, physics textbooks often avoid talking about the Legendre transform—I think they assume that you must have learned it ‘properly’ in some other textbook.

D.1 Where you've seen this before

The Legendre transform converts between Lagrangian and Hamiltonian mechanics. Depending on the sophistication of your last mechanics course, this may seem trivial: *isn't the Hamiltonian just the Lagrangian with a positive sign on the potential?* The answer is *yes, but...*

The answer is *yes* because the statement is true. The ‘but...’ is because the statement completely misses the significance of the passage between the Lagrangian and Hamiltonian formalisms. This is usually described toward the back of graduate mechanics textbooks under the title of Hamilton–Jacobi relations or symplectic manifolds. Borrowing some of the language that we have used in this course: the Lagrangian formalism describes particle motion using its position, q , and its velocity, \dot{q} . The Hamiltonian formalism describes particle motion using the position, q , and *momentum*, p . Formally, we have

$$H(q, p) = qp - L(q, \dot{q}) . \quad (\text{D.1})$$

Since $p = m\dot{q}$, this seems like a trivial change. In fact, we recognize that $qp = m\dot{q}^2 = 2(\text{kinetic energy})$; so the above ‘formal’ definition seems like an arbitrary way to get $H = \text{KE} + V$. On the other hand, geometrically this goes from describing the dynamics using the tangent space (where vectors live) versus the co-tangent space (where one-forms live)¹⁷⁴. The equation of motion for the Lagrangian formalism is, in general, second order in derivatives. On the other hand, the equation of motion for the Hamiltonian formalism is first order¹⁷⁵.

¹⁷³<https://www.nature.com/articles/363315a0.epdf>

¹⁷⁴Technically, since the dynamics are imposed on $\{q, \dot{q}\}$ and $\{q, p\}$, these formalisms describe the tangent *bundle* and co-tangent *bundle*.

¹⁷⁵If you argue that this is only a superficial change since $p = m\dot{q}$, then you're missing the power of the Hamiltonian formalism. For most purposes—and certainly for most applications in a first year graduate course—the second-order Lagrangian formalism is fine. The mathematically inclined may want to look up

Legendre transforms show up again in thermodynamics and statistical mechanics. Unsurprisingly, they show up most robustly in the language of differential forms, even if we don't typically write it as such¹⁷⁶. Here there is a useful notion of a generalization of the energy–force–displacement relation

$$dE = F dx \quad \Rightarrow \quad dE = \sum F_i ds_i , \quad (\text{D.2})$$

where the F_i are *generalized forces* with respect to *generalized displacements* s_i . In the thermodynamic context, the generalized force is typically an *intensive* quantity while the displacement is *extensive*. Recall that intensive means that the quantity doesn't scale with volume (for example, local quantities like pressure) while extensive means that it scales with the volume of the system. Usually the generalized force is written as the derivative of some type of energy with respect to the generalized displacement: $F_i = \partial U_i / \partial s_i$. The pair $\{F_i, s_i\}$ are called conjugate variables with respect to energy and are related by a Legendre transform.

An excellent example of the Legendre transform in thermodynamics is the **Helmholtz free energy** which describes the amount of energy available to do work at constant volume.

jet bundles in physics. For the more physically inclined, read the section of your favorite mechanics book about the Hamilton–Jacobi equation and canonical transformations. If you can't find that chapter, then it's time to find a new favorite mechanics book. (By the way: never get relationship advice from people who talk like this.)

¹⁷⁶Though see 0711.4319 and 1908.07583.