TP2 for Optim Image

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Gradient Projeté

1.1

$\nabla J = 2A^T(Ax - b)$

1.2

Prove J is strongly convex:

$$J(x+h) = J(x) + \nabla J(x)^T h + h^T A^T A h$$

Notice that

$$h^T A^T A h =< h, A^T A h >\geq \lambda_{min} \|h\|^2$$

where λ_{min} is the minimum eigenvalue of matrix $A^T A$. So

$$J(x+h) \geq J(x) + \nabla J(x)^T h + \frac{2\lambda_{min}}{2} \|h\|^2$$

the strongly convex parameter is $\mu = 2\lambda_{min}$.

$$\|\nabla J(x_1) - \nabla J(x_2)\| = 2\|A^T A(x_1 - x_2)\| \leq 2\|A^T A\| \|x_1 - x_2\|$$

So the Lipshitzien constant M is $2\|A^T A\|$. Because $A^T A$ is symmetric, so $\|A^T A\| = \lambda_{max}$, the max eigenvalue of $A^T A$.

1.3

Data: x_0, τ, ϵ
$x \leftarrow x_0$
while x *not converged* **do**
  $x \leftarrow x - \tau \nabla J$
  $x \leftarrow \max(0, x)$
end

According to Theorem 31 in poly, $\mu = 2\lambda_{min}$ and $M = 2\lambda_{max}$, so the step τ should be smaller than $\frac{2\mu}{M^2} = \frac{4\lambda_{min}}{4\lambda_{max}^2} = \frac{\lambda_{min}}{\lambda_{max}^2}$

1.5

Solution is $[0, 0.4615]$

1.6

We want to find a vector $\mathbf{b}$ such that at point $[5, 0]$ constrain $x_1 \geq 0$ is not actived and constrain $x_2 \geq 0$ is actived.

Kuhn Tucker relation gives:

$$\nabla J(\mathbf{x}) + \lambda[0, -1]^T = 0, \lambda > 0$$

$$\Rightarrow [10 + b_2 - b_1, -5 - 2b_1 - 3b_2] = [0, \lambda/2]$$

$$\Rightarrow b_1 = 10 + b_2,\ b_2 < -5$$

We can take $\mathbf{b} = [0, -10]^T$

Gradient projete bis

1

a

C is a cylinder with infinite height. C is convex because:

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2$$

is a convex function, so $f \leq 1$ defines a convex set.

b

$$P_C(\mathbf{x}) = \begin{cases} [x_1/\sqrt{x_1^2 + x_2^2}, x_2/\sqrt{x_1^2 + x_2^2}, x_3] & x_1^2 + x_2^2 > 1 \\ [x_1, x_2, x_3] & else \end{cases}$$

2

a

There exists a unique solution because C is a convex set and J is strongly convex.

b

Data: x_0, τ, ϵ
$x \leftarrow x_0$
while x *not converged* **do**
  $x \leftarrow x - \tau \nabla J$
  $x \leftarrow P_C x$
end

Step should be smaller than $\frac{2\mu}{M^2} = \frac{\lambda_{min}}{\lambda_{max}^2} = 0.0093$

d

$x^* = [0.9911, -0.1334, 0.8854], J = 1.609$

e

When $\tau > 0.089$, the algorithm diverges.

3 Uzawa

1

$\forall \mathbf{x} \neq 0$

$$\begin{aligned}\mathbf{x}^T A\mathbf{x} &= 2x_1^2 - x_1x_2 + 2x_2^2 - x_2x_3 + \cdots + x_n^2 \\ &= x_1^2 + (x_1 - x_2)^2 + \cdots + (x_{n-1} + x_n)^2 + x_n^2 \\ &> 0\end{aligned}$$

$x^T Ax$ correspond to $\int |f'|$

2

$$L = \frac{1}{2}x^T Ax - b^T x + \lambda^T Cx - \lambda^T d$$

3

Use the result from Question 1, function $\frac{1}{2}x^T Ax - b^x$ is strongly convex with parameter $\mu = \lambda_{\min}(A)$, the step ρ should verify $0 < \rho < \frac{2\lambda_{\min}(A)}{\|C\|^2}$

4

Algorithm converged to

$$\lambda = [0.3661, 0.8542, 1.70171], x = [0.4915, 0.2271, 1.0373, 1.7017, 2.7017, 1.0000]$$

All λ_i is positive, so all constraints are actived.
$\nabla J = Ax - b = [-0.2441, -2.0746, -0.8542, -1.3356, 1.7017, -1.7017]$
$\lambda \nabla (Cx - d) = [0.2441, 2.0746, 0.8542, 1.3356, -1.7017, 1.7017]$
Clearly we have $\nabla J + \lambda \nabla G = 0$

5

When replace b by $-b$, $\lambda = [0, 0, 0]$, which means all constraints are not actived.