TP4 for Sub-pixel

Leman FENG
Email: flm8620@gmail.com
Website: lemanfeng.com

November 15, 2017

1 Ex 11

see code in main.mlx

2 Ex 12

2.1 Ex 12.1

$$\widehat{u - per}(u) = \frac{\hat{v}}{\hat{k}}$$

where k is a convolutional kernel and v only depend on boundary values of u. So $u - per(u)$ only depend on restriction of u on $\partial\Omega$

2.2 Ex 12.2

$s = u - per(u)$ and we have:

$$v(z) = (k * s)(z)$$

where $v(z) = 0$ for $z \in \mathring{\Omega}$. And convolution with k is the discret Laplacien operator: $k * s = \Delta s$. So $\Delta(u - per(u))(z) = v(z) = 0$ for $z \in \mathring{\Omega}$.

2.3 Ex 12.3

$per(u) = u - s = u - (Q_1 + Q_2)^{-1}Q_1 u = [(Q_1 + Q_2)^{-1}(Q_1 + Q_2) - (Q_1 + Q_2)^{-1}Q_1]u$
$= (Q_1 + Q_2)^{-1}Q_2 u$

$(Q_1 + Q_2)^{-1}Q_2$ is invertible as we see in course. So per is a injection from $\mathbb{R}^\Omega$ to $\mathbb{R}^\Omega$.

per is also a linear map, and $\mathbb{R}^\Omega$ has finite dimension. That means $dim(Ker(per)) = 0$. According to Rank–nullity theorem, $dim(Im(per)) = dim(\mathbb{R}^\Omega)$. So per is bijective.

2.4 Ex 12.4

Say $\lambda, u \neq 0$ is a pair of eigenvalue and eigenvector. Assume they are complex.

$$per(u) = \lambda u \Leftrightarrow (Q_1 + Q_2)^{-1} Q_2 u = \lambda u \Leftrightarrow Q_2 u = \lambda (Q_1 + Q_2) u$$

Multiply at left by conjugate transpose u^* of u

$$\lambda(u^* Q_1 u + u^* Q_2 u) = u^* Q_2 u$$

and Q_1 is positive semi-definite, Q_2 positive definite. So

$$\lambda = \frac{u^* Q_2 u}{u^* Q_1 u + u^* Q_2 u} \in]0, 1]$$

2.5 Ex 12.5

Let u be the fix point of per. So $per(u) = u$, $s = u - per(u) = 0$. Which means $(Q_1 + Q_2)^{-1} Q_1 u = 0$. While $(Q_1 + Q_2)^{-1}$ is invertible, so we must have $Q_1 u = 0$. Multiply at left by u^T:

$$u^T Q_1 u = 0$$

By definition, $u^T Q_1 u$ means sum of squared difference around across periodic boundary. So $u^T Q_1 u = 0$ iff boundary pixels equal to their external periodic neighbours, which means $u \in \mathbf{P}$

2.6 Ex 12.6

Use the indication, $\exists P$ invertible, $\exists D$ diagonal such that $P^T Q_1 P = D$ and $P^T Q_2 P = I$. Then

$$\begin{aligned} per(u) &= (Q_1 + Q_2)^{-1} Q_2 u = (P^{-T} D P^{-1} + P^{-T} I P^{-1})^{-1} P^{-T} I P^{-1} u \\ &= P(D + I)^{-1} P^T P^{-T} I P^{-1} u = P(D + I)^{-1} P^{-1} u \end{aligned}$$

where $(D + I)^{-1}$ is obviously diagonal. So we just diagonalized the matrix applied to u, so per is diagonalizable.

2.7 Ex 12.7

Use the result of Ex 12.6. per is diagonalizable. Ex 12.4 tells us that all eigenvalues are real and $\in]0, 1]$. We decompose one image $\mathbf{u}$ into basis of per.

$$\mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + \cdots + u_m \mathbf{e}_m$$

Apply per n times:

$$per^n(\mathbf{u}) = \lambda_1^n u_1 \mathbf{e}_1 + \lambda_2^n u_2 \mathbf{e}_2 + \cdots + \lambda_m^n u_m \mathbf{e}_m$$

All λ_i strictly smaller than one will vanish when $n \to \infty$. For $i s.t. \lambda_i = 1$, $\mathbf{e}_i$ must be a fix point of per. Ex 12.5 tells us fix point of per is in $\mathbf{P}$. So $per^\infty(\mathbf{u})$ is composed of elements in $\mathbf{P}$, and $\mathbf{P}$ is a subspace, so $per^\infty(\mathbf{u}) \in \mathbf{P}$