TP5 for Sub-pixel

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1 Ex 13

$u : \mathbb{Z} \to \mathbb{R}, v : \mathbb{Z} \to \mathbb{R}$

v is zoom of u by factor 2:

$$\forall k \in \mathbb{Z},\ v[2k] = u[k], v[2k+1] = (u[k] + u[k+1])/2$$

$$\begin{aligned}
\forall \xi \in \mathbb{R},\ \hat{v}(\xi) &= \sum_{k \in \mathbb{Z}} v[k] e^{-ik\xi} = \sum_{l \in \mathbb{Z}} v[2l] e^{-i2l\xi} + \sum_{l \in \mathbb{Z}} v[2l+1] e^{-i2l\xi - i\xi} \\
&= \sum_{l \in \mathbb{Z}} u[l] e^{-il2\xi} + \frac{1}{2} e^{-i\xi} \sum_{l \in \mathbb{Z}} (u[l] + u[l+1]) e^{-il2\xi} \\
&= \hat{u}(2\xi) + \frac{1}{2} e^{-i\xi} \hat{u}(2\xi) + \frac{1}{2} e^{-i\xi} \sum_{l \in \mathbb{Z}} u[l+1] e^{-il2\xi} \\
&= \hat{u}(2\xi) + \frac{1}{2} e^{-i\xi} \hat{u}(2\xi) + \frac{1}{2} e^{-i\xi} \sum_{l \in \mathbb{Z}} u[l] e^{-il2\xi + i2\xi} \\
&= \hat{u}(2\xi) + \frac{1}{2} e^{-i\xi} \hat{u}(2\xi) + \frac{1}{2} e^{i\xi} \hat{u}(2\xi) \\
&= (1 + \cos(\xi)) \hat{u}(2\xi)
\end{aligned}$$

Set $U(x),\ x \in \mathbb{R}$ as the Shannon interpolation of $u[k], k \in \mathbb{Z}$

$$U(x) = \sum_{k \in \mathbb{Z}} u[k] sinc(x - k)$$

Then set $W(x),\ x \in \mathbb{R}$ as the zoom of U, $W(x) = U(x/2)$

$$\begin{aligned}
\forall \xi \in \mathbb{R},\ \hat{W}(\xi) &= \int_{\mathbb{R}} U(x/2) e^{-ix\xi} dx \\
&= \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} u[k] sinc(x/2 - k) e^{-ix\xi} dx \\
&= \sum_{k\in\mathbb{Z}} u[k] \int_{\mathbb{R}} sinc(x/2 - k) e^{-ix\xi} dx \\
&= \sum_{k\in\mathbb{Z}} u[k] \int_{\mathbb{R}} sinc(y) e^{-i2(y+k)\xi} 2dy \\
&= 2 \sum_{k\in\mathbb{Z}} u[k] e^{-ik2\xi} \int_{\mathbb{R}} sinc(y) e^{-iy2\xi} dy \\
&= 2\hat{u}(2\xi)\widehat{sinc}(2\xi) \\
&= 2\hat{u}(2\xi)\mathbb{1}_{[-\pi/2,\pi/2]}
\end{aligned}$$

w is discrete W of interval 1, so $\hat{w}$ is the 2π-periodization of $\hat{W}$

$$\hat{w}(\xi) = \begin{cases} 2\hat{u}(2\xi) & \xi \in [-\pi/2, \pi/2] \\ 0 & \xi \in [-\pi, -\pi/2] \cup [\pi/2, \pi] \\ 2\pi\text{-periodic} & \text{else} \end{cases}$$

Compare with $\hat{v}(\xi) = (1+\cos(\xi))\hat{u}(2\xi)$: Both $\hat{v}(\xi)$ and $\hat{w}(\xi)$ are 2π periodic, so we only need to compare the interval $[-\pi, \pi]$.

$\hat{w}(\xi)$ shrinks all frequency in $\hat{u}(\xi)$ by factor 2. Then the rest part is left zero. So there is no high frequency in w. But $\hat{w}(\xi)$ not only shrink frequency in $\hat{u}(\xi)$, but also copied shrunken $\hat{u}(\xi)$ into interval $[-\pi, -\pi/2] \cup [\pi/2, \pi]$, then multiply by $(1 + \cos(\xi))$ which is a soft hat shaped function in $[-\pi, \pi]$. So there will be extra high frequency in v.

2 Ex 14

Set

$$\Omega = \{0, \dots, M-1\} \times \{0, \dots, N-1\}$$

and set

$$\Omega_2 = \{0, \dots, 2M-1\} \times \{0, \dots, 2N-1\}$$

.

$u : \Omega \to \mathbb{R}, v : \Omega_2 \to \mathbb{R}$

v is zoom of u by factor 2:

$$\begin{aligned}
\forall (k,l) \in \Omega,\ v[2k, 2l] &= u[k,l] \\
v[2k+1, 2l+1] &= (u[k,l] + u[k+1,l] + u[k,l+1] + u[k+1,l+1])/4 \\
v[2k+1, 2l] &= (u[k,l] + u[k+1,l])/2 \\
v[2k, 2l+1] &= (u[k,l] + u[k,l+1])/2
\end{aligned}$$

$$
\begin{aligned}
\forall (p,q) \in \Omega_2,\ \hat{v}[p,q] &= \sum_{(k,l)\in\Omega_2} v[k,l] e^{-i2\pi kp/2M} e^{-i2\pi lq/2N} \\
= & \sum_{(k,l)\in\Omega} v[2k,2l] e^{-i2\pi 2kp/2M} e^{-i2\pi 2lq/2N} \\
& + \sum_{(k,l)\in\Omega} v[2k+1,2l] e^{-i2\pi (2k+1)p/2M} e^{-i2\pi 2lq/2N} \\
& + \sum_{(k,l)\in\Omega} v[2k,2l+1] e^{-i2\pi 2kp/2M} e^{-i2\pi (2l+1)q/2N} \\
& + \sum_{(k,l)\in\Omega} v[2k+1,2l+1] e^{-i2\pi (2k+1)p/2M} e^{-i2\pi (2l+1)q/2N} \\
= & \sum_{(k,l)\in\Omega} u[k,l] e^{-i2\pi kp/M} e^{-i2\pi lq/N} \\
& + \frac{1}{2} e^{-i2\pi p/2M} \sum_{(k,l)\in\Omega} (u[k,l] + u[k+1,l]) e^{-i2\pi kp/M} e^{-i2\pi lq/N} \\
& + \frac{1}{2} e^{-i2\pi q/2M} \sum_{(k,l)\in\Omega} (u[k,l] + u[k,l+1]) e^{-i2\pi kp/M} e^{-i2\pi lq/N} \\
& + \frac{1}{4} e^{-i2\pi p/2M} e^{-i2\pi q/2M} \sum_{(k,l)\in\Omega} (u[k,l] + u[k+1,l] + u[k,l+1] + u[k+1,l+1]) e^{-i2\pi kp/M} e^{-i2\pi lq/N} \\
= & \ \hat{u}[p,q] \\
& + \frac{1}{2} e^{-i2\pi p/M} \hat{u}[p,q] + \frac{1}{2} e^{+i2\pi p/M} \hat{u}[p,q] \\
& + \frac{1}{2} e^{-i2\pi q/M} \hat{u}[p,q] + \frac{1}{2} e^{+i2\pi q/M} \hat{u}[p,q] \\
& + \frac{1}{4} e^{-i2\pi p/M} e^{-i2\pi q/M} \hat{u}[p,q] + \frac{1}{4} e^{+i2\pi p/M} e^{-i2\pi q/M} \hat{u}[p,q] \\
& + \frac{1}{4} e^{-i2\pi p/M} e^{+i2\pi q/M} \hat{u}[p,q] + \frac{1}{4} e^{+i2\pi p/M} e^{+i2\pi q/M} \hat{u}[p,q] \\
= & \ \dot{\hat{u}}[p,q] (cos(2\pi p/M) + cos(2\pi q/M) + cos(2\pi p/M) cos(2\pi q/M))
\end{aligned}
$$

Set $U(x,y),\ (x,y) \in \mathbb{R}^2$ as the Shannon (M,N)-periodic interpolation of $u[k,l], (k,l) \in \Omega$ in the sense periodic.

$$
U(x,y) = \sum_{(k,l)\in\Omega} u[k,l] sincd_M(x-k) sincd_N(y-l)
$$

Then set $W(x,y),\ (x,y) \in \mathbb{R}^2$ as the zoom of U, $W(x,y) = U(x/2, y/2)$

$$
\forall (k,l) \in \Omega,\ w(k,l) = U(k/2, l/2)
$$

U can also be written as

$$U(x,y) = \frac{1}{MN} \sum_{|p|\le M/2, |q|\le N/2} \hat{u}[p,q] e^{2i\pi(px/M)} e^{2i\pi(qy/N)}$$

Then we have

$$\forall (k,l) \in \Omega,\ w[k,l] = U(k/2, l/2) = \frac{1}{MN} \sum_{|p|\le M/2, |q|\le N/2} \hat{u}[p,q] e^{2i\pi(pk/2M)} e^{2i\pi(ql/2N)}$$

Meanwhile, w can be represented by its inverse transform of Fourier:

$$\forall (k,l) \in \Omega,\ w[k,l] = \frac{1}{4MN} \sum_{|p|\le M, |q|\le N} \hat{w}[p,q] e^{2i\pi(pk/2M)} e^{2i\pi(ql/2N)}$$

The representation should be unique. So we have:

$$\forall (p,q) \in \Omega,\ \hat{w}[p,q] = \begin{cases} 4\hat{u}[p,q] & |p| \le M, |q| \le N \\ 0 & else \end{cases}$$

which is the zero padding.

Similar to the first question, v introduces extra high frequency in the zero padding zone of $\hat{w}$. Then multiplied by 2D soft hat $cos(2\pi p/M) + cos(2\pi q/M) + cos(2\pi p/M)cos(2\pi q/M)$.

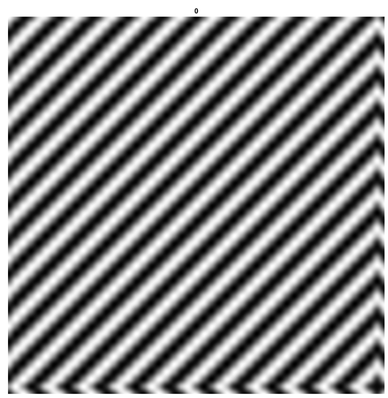
2.1 Ex 17

For image "crop_bouc.pgm", splines with order $n \ge 3$ are very good. Keys Bicubic has a bad performance, it shows weird strips in image. For image "crop_cameraman.pgm", Keys Bicubic is the best, while high order splines will show ripples around sharp edges.

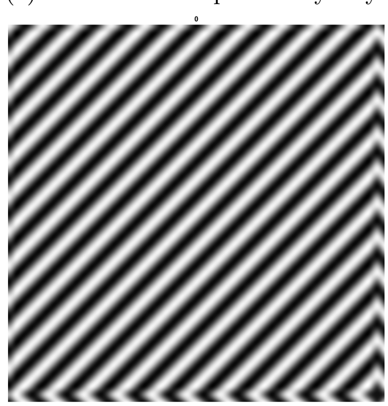
By checking their Fourier transform, we can see that "bouc.pgm" has mainly low frequencies, and many of them are along one direction, because the regular stripes in image. While "cameraman.pmg" has many high frequencies, due to sharp edges around cameraman.

When doing a high order spline interpolation, it's like a Shannon interpolation, high frequencies will be lost in "cameraman.pmg", so we see ripples. While Keys Bicubic is a direct method with a kernel with small support. Thus sharp edges cannot influence far pixels.

The problem of Keys Bicubic is that it is axis aligned. The formular is hand-crafted. Once we have a inclined strip in image, there will be artifacts. See Figure 1a:



(a) Pure wave interpolated by Keys



(b) Pure wave interpolated by spline order 3