TP3 for Sub-pixel

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November 8, 2017

1 Ex 8

1.1 Ex 8.1

Let's firstly calculate Fourier transform of Dirac comb in 2D. Define Dirac comb with period (T_1, T_2) as:

$$\Pi_{(T_1,T_2)} = \sum_{k,j\in\mathbb{Z}^2} \delta(x - jT_1, y - kT_2) \tag{1}$$

Use Fourier series in 2d, $\Pi_{(T_1,T_2)}$ can be decomposed as:

$$\Pi_{(T_1,T_2)} = \sum_{k,j\in\mathbb{Z}^2} c_{j,k} \exp(i\frac{2j\pi}{T_1}x)\exp(i\frac{2k\pi}{T_2}y) \tag{2}$$

where

$$\begin{aligned} c_{j,k} &= \frac{1}{T_1T_2}\int_{-T_1/2}^{T_1/2}\int_{-T_2/2}^{T_2/2} \Pi_{(T_1,T_2)} \exp(-i\frac{2j\pi}{T_1}x)\exp(-i\frac{2k\pi}{T_2}y) \\ &= \frac{1}{T_1T_2}\langle\delta, \exp(-i\frac{2j\pi}{T_1}x)\exp(-i\frac{2k\pi}{T_2}y)\rangle \\ &= \frac{1}{T_1T_2} \end{aligned} \tag{3}$$

So we have

$$\Pi_{(T_1,T_2)} = \frac{1}{T_1T_2}\sum_{k,j\in\mathbb{Z}^2} \exp(i\frac{2j\pi}{T_1}x)\exp(i\frac{2k\pi}{T_2}y) \tag{4}$$

Then we calculate $\hat{\Pi}_{(T_1,T_2)}$

$$
\begin{aligned}
\hat{\Pi}_{(T_1,T_2)} &= \iint \Pi_{(T_1,T_2)} \exp(-i\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \rangle) dxdy \\
&= \langle \sum_{k,j\in\mathbb{Z}^2} \delta(x - jT_1, y - kT_2), \exp(-i\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \rangle)\rangle \\
&= \sum_{k,j\in\mathbb{Z}^2} \exp(-ijT_1\xi_1)\exp(-ikT_2\xi_2) \\
&= \sum_{k,j\in\mathbb{Z}^2} \exp(-i\frac{2j\pi}{s_1}\xi_1)\exp(-i\frac{2k\pi}{s_2}\xi_2)
\end{aligned} \tag{5}
$$

where $s_1 = \frac{2\pi}{T_1}, s_2 = \frac{2\pi}{T_2}$, then we can rewrite

$$
\hat{\Pi}_{(T_1,T_2)} = \frac{4\pi^2}{T_1T_2}\Pi_{(\frac{2\pi}{T_1},\frac{2\pi}{T_2})} \tag{6}
$$

Now define translation operator $\mathscr{T}^t$ in 2D:

$$
\mathscr{T}^t(f)(x) = f(x-t) \tag{7}
$$

Fourier transform has property:

$$
(\mathscr{F} \circ \mathscr{T}^t \circ f)(x) = \exp(-i\langle t, \xi\rangle)\hat{f}(\xi) \tag{8}
$$

$$
\begin{aligned}
U &= \sum_{k\in[\![0,M-1]\!],l\in[\![0,N-1]\!]} u[k,l] \sum_{m,n\in\mathbb{Z}^2} \delta_{(k+mM,l+nN)} \\
&= \sum_{k,l} u[k,l]\mathscr{T}^{(k,l)} \circ \Pi_{(M,N)} \\
\hat{U} &= \sum_{k,l} u[k,l]\mathscr{F} \circ \mathscr{T}^{(k,l)}\Pi_{(M,N)} \\
\hat{U} &= \sum_{k,l} u[k,l]\exp(-i\langle (k,l), \xi\rangle)\hat{\Pi}_{(M,N)} \\
\hat{U} &= \sum_{k,l} u[k,l]\exp(-ik\xi_1)\exp(-il\xi_2)\frac{4\pi^2}{MN}\Pi_{(\frac{2\pi}{M},\frac{2\pi}{N})}
\end{aligned} \tag{9}
$$

Notice that distribution can be multiplied by infinitely differentiable functions, so $\exp(-ik\xi_1)\exp(-il\xi_2)\frac{4\pi^2}{MN}\Pi_{(\frac{2\pi}{M},\frac{2\pi}{N})}$ make sense

$$
\begin{aligned}
\hat{U} &= \sum_{k,l} u[k,l] \exp(-ik\xi_1)\exp(-il\xi_2)\frac{4\pi^2}{MN}\sum_{m,n\in\mathbb{Z}^2}\delta_{(\frac{2\pi}{M}m,\frac{2\pi}{N}n)} \\
\hat{U} &= \frac{4\pi^2}{MN}\sum_{k,l} u[k,l]\sum_{m,n\in\mathbb{Z}^2}\delta_{(\frac{2\pi}{M}m,\frac{2\pi}{N}n)}\exp(-i\frac{2\pi m}{M}k)\exp(-i\frac{2\pi n}{N}l) \\
\hat{U} &= \frac{4\pi^2}{MN}\sum_{m,n\in\mathbb{Z}^2}\delta_{(\frac{2\pi}{M}m,\frac{2\pi}{N}n)}\sum_{k,l} u[k,l]\exp(-i\frac{2\pi k}{M}m)\exp(-i\frac{2\pi l}{N}n)
\end{aligned} \tag{10}
$$

The DFT in 2D is defined as:

$$
\hat{u}[p,q] = \sum_{k=0}^{M-1}\sum_{l=0}^{N-1} u[k,l]\exp(-i\frac{2\pi k}{M}p)\exp(-i\frac{2\pi l}{N}q), 0 \le p < M, 0 \le q < N \tag{11}
$$

So

$$
\begin{aligned}
\hat{U} &= \frac{4\pi^2}{MN}\sum_{m,n\in\mathbb{Z}^2}\delta_{(\frac{2\pi}{M}m,\frac{2\pi}{N}n)}\hat{u}[m \mod M, n \mod N] \\
\hat{U} &= \frac{4\pi^2}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}\hat{u}[k,l]\sum_{m,n\in\mathbb{Z}^2}\delta_{(\frac{2\pi k}{M}+2\pi m,\frac{2\pi l}{N}+2\pi n)}
\end{aligned} \tag{12}
$$

1.2 Ex 8.2

$$
\begin{aligned}
\hat{u}'[p,q] &= \sum_{k=0}^{M-1}\sum_{l=0}^{N-1} u'[k,l]\exp(-i\frac{2\pi k}{M}p)\exp(-i\frac{2\pi l}{N}q) \\
\hat{u}'[p,q] &= \sum_{k=0}^{M-1}\sum_{l=0}^{N-1} \dot{u}[k+k_0,l+l_0]\exp(-i\frac{2\pi k}{M}p)\exp(-i\frac{2\pi l}{N}q)
\end{aligned} \tag{13}
$$

Take $a = k + k_0, b = l + l_0$:

$$
\begin{aligned}
\hat{u}'[p,q] &= \sum_{a=k_0}^{M-1+k_0}\sum_{b=l_0}^{N-1+l_0} \dot{u}[a,b]\exp(-i\frac{2\pi(a-k_0)}{M}p)\exp(-i\frac{2\pi(b-l_0)}{N}q) \\
\hat{u}'[p,q] &= \exp(i\frac{2\pi k_0}{M}p)\exp(i\frac{2\pi l_0}{M}q)\sum_{a=k_0}^{M-1+k_0}\sum_{b=l_0}^{N-1+l_0} \dot{u}[a,b]\exp(-i\frac{2\pi a}{M}p)\exp(-i\frac{2\pi b}{N}q)
\end{aligned} \tag{14}
$$

$\dot{u}$ and $\exp(-i\frac{2\pi a}{M}p)\exp(-i\frac{2\pi b}{N}q)$ are both (M,N) periodic in 2D. So

$$
\begin{aligned}
\hat{u}'[p,q] &= \exp(i\frac{2\pi k_0}{M}p)\exp(i\frac{2\pi l_0}{M}q)\sum_{a=0}^{M-1}\sum_{b=0}^{N-1} u[a,b]\exp(-i\frac{2\pi a}{M}p)\exp(-i\frac{2\pi b}{N}q) \\
\hat{u}'[p,q] &= \exp(i\frac{2\pi k_0}{M}p)\exp(i\frac{2\pi l_0}{M}q)\hat{u}[p,q]
\end{aligned} \tag{15}
$$