# TP6 for Sub-pixel

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## Ex 15

### 1

Notice that $\frac{dx_+^{n+1}}{dx} = (n+1)x_+^n, x \neq 0$. So inversely,

$$\int_a^b x_+^n = \frac{1}{n+1}[x_+^{n+1}]_a^b$$

### 2

It's easy to verify this is true for $n = 1$. Assume it's true for $n$.

$$\begin{aligned}
\beta^{n+1}(y) = \beta^n \star \beta^0 &= \frac{1}{n!}\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k \int (x - k + \frac{n+1}{2})_+^n \cdot \mathbb{1}_{[-1/2,1/2[}(y-x)dx \\
&= \frac{1}{n!}\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k \int_{y-1/2}^{y+1/2} (x - k + \frac{n+1}{2})_+^n dx \\
&= \frac{1}{n!}\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k \frac{1}{n+1}\int_{y-k+n/2}^{y-k+n/2+1} x_+^{n+1} dx \\
&= \frac{1}{(n+1)!}\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k [x_+^{n+1}]_{y-k+n/2}^{y-k+n/2+1} \\
&= \frac{1}{(n+1)!}\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k ((y-k+n/2+1)_+^{n+1} - (y-k+n/2)_+^{n+1}) \\
&= \frac{1}{(n+1)!}(\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k (y-k+(n+2)/2)_+^{n+1} + \sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^{k+1}(y-(k+1)+(n+2)/2) \\
&= \frac{1}{(n+1)!}(\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k (y-k+(n+2)/2)_+^{n+1} + \sum_{k=1}^{n+2}\binom{n+1}{k-1}(-1)^k (y-k+(n+2)/2)_+^{n+1})
\end{aligned}$$

We know that

$$\begin{cases} \binom{n+2}{k} = \binom{n+1}{k} + \binom{n+1}{k-1} & 1 \leq k \leq n+1 \\ \binom{n+2}{k} = \binom{n+1}{k-1} & \leq k = n+2 \\ \binom{n+2}{k} = \binom{n+1}{k} & \leq k = 0 \end{cases}$$

So we have

$$\beta^{n+1}(y) = \frac{1}{(n+1)!} \sum_{k=0}^{n+2} \binom{n+2}{k} (-1)^k (y - k + (n+2)/2)_+^{n+1}$$

so this is also true for $n+1$. By recurrence, this is true for all $n \geq 1$

### 3

$$\begin{aligned}
\forall p \in \mathbb{Z},\ \hat{u}[p] &= \sum_{l=0}^{N-1} u[l] e^{-i\frac{2\pi l}{N}p} \\
&= \sum_{l=0}^{N-1} e^{-i\frac{2\pi l}{N}p} \sum_{k\in\mathbb{Z}} v[k]w[l-k] \\
&= \sum_{l=0}^{N-1} e^{-i\frac{2\pi l}{N}p} \sum_{k\in\mathbb{Z}} v[l-k]w[k] \\
&= \sum_{k\in\mathbb{Z}} w[k] e^{-i\frac{2\pi k}{N}p} \sum_{l=0}^{N-1} v[l-k] e^{-i\frac{2\pi(l-k)}{N}p} \\
&= \sum_{k\in\mathbb{Z}} w[k] e^{-i\frac{2\pi k}{N}p} \sum_{l=0}^{N-1} v[(l-k) \mod N] e^{-i\frac{2\pi(l-k) \mod N}{N}p} \\
&= \sum_{k\in\mathbb{Z}} w[k] e^{-i\frac{2\pi k}{N}p} \hat{v}[p] \\
&= \hat{w}(\frac{2\pi k}{N}) \hat{v}[p]
\end{aligned}$$

### 4

Assume there exist a $c$ such that $U(x)$ interpolate exactly $u$

$$\forall l \in \mathbb{Z},\ u[l] = U(l) = \sum_{k\in\mathbb{Z}} c[k]\beta^n(l-k) = \sum_{k\in\mathbb{Z}} c[k]\beta_d^n[l-k]$$

use the result from 3:

$$\begin{aligned}
\forall p \in \mathbb{Z},\ \hat{u}[p] &= \hat{c}[p]\hat{\beta_d^n}(\frac{2\pi p}{N}) \\
\hat{c}[p] &= \frac{\hat{u}[p]}{\hat{\beta_d^n}(\frac{2\pi p}{N})}
\end{aligned}$$

where $\hat{\beta^n_d}$ is the $2\pi$-periodic of $\hat{\beta^n}$.

So we proved that there is one way to construct such $c$, and for all $c$ that satisfy this property, its fourier transform must be in the above form, so $c$ exist and is unique.

## 5

$$\forall 0 \leq l \leq 2N-1,\ u_2[l] = U(l/2) = \sum_{k\in\mathbb{Z}} c[k]\beta^n(l/2-k)$$

$$\begin{aligned}
\forall 0 \leq p \leq 2N-1,\ \hat{u_2}[p] &= \sum_{l=0}^{2N-1} u_2[l]e^{-i\frac{2\pi l}{2N}p} \\
&= \sum_{l=0}^{2N-1}\sum_{k\in\mathbb{Z}} c[k]\beta^n(l/2-k)e^{-i\frac{2\pi l}{2N}p} \\
&= \sum_{a=0}^{N-1}\sum_{k\in\mathbb{Z}} c[k]\beta^n(a-k)e^{-i\frac{2\pi a}{N}p} + \sum_{b=0}^{N-1}\sum_{k\in\mathbb{Z}} c[k]\beta^n(b-k+1/2)e^{-i\frac{2\pi b}{N}p}e^{-i\frac{\pi}{N}p}
\end{aligned}$$

According to question 4, first term is simplified:

$$\forall 0 \leq p \leq 2N-1,\ \hat{u_2}[p] = \hat{u}[p] + \sum_{b=0}^{N-1}\sum_{k\in\mathbb{Z}} c[k]\beta^n(b-k+1/2)e^{-i\frac{2\pi b}{N}p}e^{-i\frac{\pi}{N}p}$$

Note $\beta^n_T(x) = \beta^n(x-T)$

$$\begin{aligned}
\forall 0 \leq p \leq 2N-1,\ \hat{u_2}[p] &= \hat{u}[p] + e^{-i\frac{\pi}{N}p}\sum_{b=0}^{N-1} e^{-i\frac{2\pi b}{N}p}\sum_{k\in\mathbb{Z}} c[k]\beta^n_{-\frac{1}{2}}(b-k) \\
&= \hat{u}[p] + e^{-i\frac{\pi}{N}p}\sum_{b=0}^{N-1} e^{-i\frac{2\pi b}{N}p}\sum_{k\in\mathbb{Z}} c[b-k]\beta^n_{-\frac{1}{2}}(k) \\
&= \hat{u}[p] + e^{-i\frac{\pi}{N}p}\sum_{k\in\mathbb{Z}} \beta^n_{-\frac{1}{2}}(k)e^{-i\frac{2\pi k}{N}p}\sum_{b=0}^{N-1} c[b-k]e^{-i\frac{2\pi(b-k)}{N}p} \\
&= \hat{u}[p] + e^{-i\frac{\pi}{N}p}\widehat{\beta^n_{-\frac{1}{2}d}}(\frac{2\pi p}{N})\hat{c}[p]
\end{aligned}$$

By properties of Fourier transform, we have $\hat{\beta^n_T}(\xi) = -\hat{\beta^n}(\xi)e^{i\xi T}$.

$\widehat{\beta^n_{Td}}$ is $2\pi$-periodic of $\hat{\beta^n_T}(\xi)$

$$\begin{aligned}
\widehat{\beta^n_{-\frac{1}{2}d}}(\xi) &= \sum_{a\in\mathbb{Z}} \hat{\beta^n_{-\frac{1}{2}}}(\xi+2\pi a) = e^{i\xi\frac{1}{2}}\sum_{a\in\mathbb{Z}}\hat{\beta^n}(\xi+2\pi a)e^{i\pi a} = e^{i\xi\frac{1}{2}}\sum_{a\in\mathbb{Z}}\hat{\beta^n}(\xi+2\pi a)(-1)^a \\
&= e^{i\xi\frac{1}{2}}(\sum_{a\in\mathbb{Z}}\hat{\beta^n}(\xi+2\pi a) - 2\sum_{b\in\mathbb{Z}}\hat{\beta^n}(\xi+4\pi b+2\pi)) \\
&= e^{i\xi\frac{1}{2}}(\hat{\beta^n_d}(\xi) - 2\hat{\beta^n_{\frac{d}{2}}}(\xi+2\pi))
\end{aligned}$$

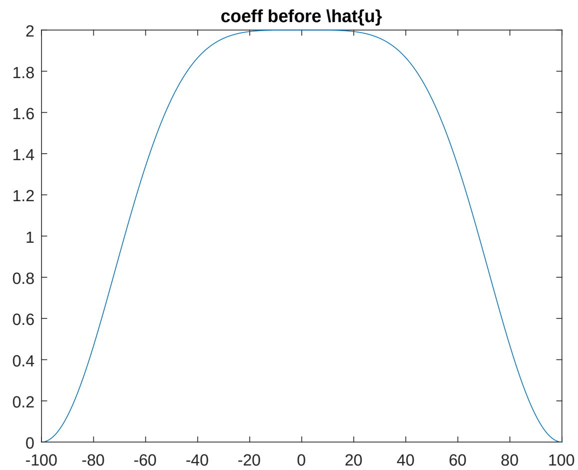


Figure 1: Suppression of high frequency

where $\hat{\beta^n_{\frac{d}{2}}}$ is $4\pi$-periodic of $\hat{\beta^n}$.

So

$$\widehat{\beta^n_{-\frac{1}{2}d}}(\frac{2\pi p}{N}) = e^{i\frac{\pi p}{N}}(\hat{\beta^n_d}(\frac{2\pi p}{N}) - 2\hat{\beta^n_{\frac{d}{2}}}(\frac{2\pi p}{N} + 2\pi))$$

Then

$$\begin{aligned}\forall 0 \le p \le 2N-1,\ \hat{u_2}[p] &= \hat{u}[p] + \hat{c}[p](\hat{\beta^n_d}(\frac{2\pi p}{N}) - 2\hat{\beta^n_{\frac{d}{2}}}(\frac{2\pi p}{N} + 2\pi)) \\ &= \hat{u}[p] + \frac{\hat{u}[p]}{\hat{\beta^n_d}(\frac{2\pi p}{N})}(\hat{\beta^n_d}(\frac{2\pi p}{N}) - 2\hat{\beta^n_{\frac{d}{2}}}(\frac{2\pi p}{N} + 2\pi)) \\ &= \hat{u}[p](2 - 2\hat{\beta^n_{\frac{d}{2}}}(\frac{2\pi p}{N} + 2\pi)/\hat{\beta^n_d}(\frac{2\pi p}{N}))\end{aligned}$$

We know that $\hat{\beta^n}(\xi) = (\frac{\sin(\xi/2)}{\xi/2})^{n+1}$ Then

$$\hat{\beta^n_{\frac{d}{2}}}(\xi) = \sum_{b\in\mathbb{Z}} \hat{\beta^n}(\xi + 4\pi b) = \sum_{b\in\mathbb{Z}} (\frac{\sin(\xi/2 + 2\pi b)}{\xi/2 + 2\pi b})^{n+1} = \sum_{b\in\mathbb{Z}} (\frac{\sin(\xi/2)}{\xi/2 + 2\pi b})^{n+1}$$

Figure 1 shows the coeff on $\hat{u}$. Different from zero padding of Shannon interpolation, cubic spline copies $\hat{u}$ then suppress it by this function.

# Ex 18

## 1

It's direct, because $\phi$ is zero for all non-zero integer.

## 2

$$\widehat{\mathbb{1}_{[-p,p]}}(\xi) = \int_{-p}^{p} e^{i\xi x}dx = \frac{1}{-i\xi}(e^{-ip\xi} - e^{ip\xi}) = 2psinc(\frac{p\xi}{\pi})$$

$$\begin{aligned}\hat{\phi}(\xi) &= \mathscr{F}(sinc \cdot \mathbb{1}_{[-p,p]}) = \widehat{sinc} \star \mathscr{F}(\mathbb{1}_{[-p,p]}) \\ &= \mathbb{1}_{[-\pi,\pi]} \star (2p\ sinc(\frac{p\xi}{\pi}))\end{aligned}$$

## 3

In interral $[n, n+1],\ n \in \mathbb{N}$, $\sin(\pi x)$ has sign $(-1)^n$, and $\pi x$ is always positive, so $sinc(x)$ has sign $(-1)^n$. So its integral on this interval has the same sign.

## 4

$\hat{\phi}(0) = \int \phi = 2\sum_{k=0}^{p-1}\int_k^{k+1} sinc = 2\sum_{k=0}^{p-1}(-1)^k|\int_k^{k+1} sinc|$

Note $a_k = |\int_k^{k+1} sinc|$. Then $\hat{\phi}(0) = \sum_{k=0}^{p-1}(-1)^k a_k$

From Fourier transform of $sinc$, we know that $\int sinc = 1 = \sum_{k\geq 0}(-1)^k a_k$. We note $S_p = \hat{\phi}(0) = \sum_{k=0}^{p-1}(-1)^k a_k,\ S_0 = 0$.

Notice that $a_k$ is strictly decreasing w.r.t $k$. So $S_{2p},\ p \geq 0$ is strictly increasing. With same argument, we can show $S_{2p+1}$ is strictly decreasing.

And we know $S_p \to 1$ when $p \to \infty$, so $\forall p, S_p \neq 1$.

## 5

$$\begin{aligned} f_\delta(x) &= \sum_{k\in\mathbb{Z}} f(k\delta)\phi(x-k) \\ \forall \xi \in \mathbb{R},\ \hat{f}_\delta(\xi) &= \int \sum_{k\in\mathbb{Z}} f(k\delta)\phi(x-k)e^{-i\xi x}dx \\ &= \sum_{k\in\mathbb{Z}} f(k\delta)e^{-i\xi k}\int \phi(x-k)e^{-i\xi(x-k)}dx \\ &= \hat{\phi}(\xi)\sum_{k\in\mathbb{Z}} f(k\delta)e^{-i\xi k}\end{aligned}$$

Note $\forall k \in \mathbb{Z},\ u[k] = f(k\delta)$. $u$ is $\delta$ discretization of $f$, so $\hat{u}$ is $2\pi\delta^{-1}$ periodic of $\hat{f}$

$$\hat{f}_\delta(\xi) = \hat{\phi}(\xi)\hat{u}(\xi) = \hat{\phi}(\xi)\sum_{k\in\mathbb{Z}}\hat{f}(\xi + 2k\pi\delta^{-1})$$

Set $\xi = 0$, we proved the formula.

## 6

Let's calculate $\widehat{f_\delta - f}(0)$.

$\widehat{f_\delta - f}(0) = \hat{f}_\delta(0) - \hat{f}(0) = \hat{\phi}(0)\sum_{k\in\mathbb{Z}} \hat{f}(2k\pi\delta^{-1}) - \hat{f}(0)$

$= (\hat{\phi}(0) - 1)\hat{f}(0) + \hat{\phi}(0)\sum_{k\in\mathbb{Z},k\neq 0} \hat{f}(2k\pi\delta^{-1})$

Because $f$ is in Schwartz space, so $\hat{f}$ is also in it. With the property of rapidly decreasing, when $\delta \to 0$, the term $\sum_{k\in\mathbb{Z},k\neq 0} \hat{f}(2k\pi\delta^{-1})$ vanishes.

So $\lim_{\delta\to 0} \widehat{f_\delta - f}(0) \neq 0$. $\lim_{\delta\to 0} \int |f_\delta - f| > \lim_{\delta\to 0} |\int f_\delta - f| > 0$