Formulation of the Squad Substitution Optimization problem

The Squad Substitution Optimization Problem, where we want to optimize the substitution strategy during a sports match given a set of players, positions and a number of time windows can be defined by the formulation given below. First, define the following variables and parameters:

- x_{pqw} is equal to 1 if player p plays at position q during time window w, and 0 otherwise.
- a_{pq} is the competence level of player p for playing at position q
- P is the set of players
- Q is the set of positions
- W is the set of time windows
- $T_1 = \lfloor \frac{|W|*|Q|}{|P|} \rfloor$, the minimum number of time windows a player should be on the pitch
- $T_2 = \lceil \frac{|W|*|Q|}{|P|} \rceil$, the maximum number of time windows a player should be on the pitch
- m is the minimum number of time windows a player should be on the pitch between subsequent substitution.

$$max \sum_{p \in P} \sum_{q \in Q} \sum_{w \in W} a_{pq} * x_{pqw} \tag{1}$$

$$subject\ to$$

$$\sum_{q \in Q} x_{pqw} \leq 1,\ \forall p \in P, \forall w \in W \tag{2}$$

$$\sum_{p \in P} x_{pqw} = 1,\ \forall q \in Q, \forall w \in W \tag{3}$$

$$x_{pq^*w} + \sum_{q \in Q, q \neq q^*} x_{pqw+1} \leq 1,\ \forall p \in P, \forall q^* \in Q, \forall w \in \{1, 2, \ldots, |W| - 1\} \tag{4}$$

$$\sum_{w \in W} \sum_{q \in Q} x_{pqw} \geq T_1,\ \forall p \in P \tag{5}$$

$$\sum_{w \in W} \sum_{q \in Q} x_{pqw} \leq T_2,\ \forall p \in P \tag{6}$$

$$\sum_{y \in \{0,1,\ldots,m\}} \sum_{q \in Q} x_{pqw+y} \geq m,\ \forall p \in P, \forall w \in \{1, 2, \ldots, |W| - m\} \tag{7}$$

$$x_{pqw} \in \{0, 1\},\ \forall p \in P, \forall q \in Q, \forall w \in W \tag{8}$$

Here, The objective function 1 maximizes the sum of the skills of players that are on the pitch each time window. Constraint 2 ensures that every player occupies at most one position during every time window, i.e. they are either on the pitch or they are a substitute. Constraint 3 states that every position should be filled by exactly one player during every time window. Furthermore, Constraint 4 prohibits players from changing positions on the pitch in subsequent time windows; a player can only play on another position if he is substituted in between. Constraints 5 and 6 aim to balance the amount of times each player is substituted. Constraint 7 ensures that there is a minimum amount of time windows between subsequent substitutions of a single player. Lastly, Constraint 8 states that the x-variables can only take binary values.