Introduction to Topology

Group Theory, Lecture 5

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This is the Lecture note for the *Introduction to Topology*. The course covers the following topics: Naive Set Theory, Elementary Number Theory, Group Theory, Topological Spaces and Continuous Maps, Introduction to Algebraic Topology.

Content:

Subgroup

Definition 1 (Cyclic Subgroup)**.** Given a group $(G, \square)$, for any $g \in G$, and $k \in \mathbb{Z}$, we define

$$g^k := \begin{cases} \underbrace{g\square\cdots\square g}_{k}, & k > 0 \\ e, & k = 0 \\ \underbrace{g^{-1}\square\cdots\square g^{-1}}_{k}, & k < 0. \end{cases}$$

$\{g^k | k \in \mathbb{Z}\}$ constructs a subgroup of $(G, \square)$. We call $\{g^k | k \in \mathbb{Z}\}$ is a cyclic subgroup generated by g, denote as $\langle g \rangle$.

Given $g \in G$, for any $g^{k_1}, g^{k_2} \in \langle g \rangle$, $g^{k_1} \square g^{k_2} = g^{k_1+k_2} \in \langle g \rangle$, $e \in \langle g \rangle$ and for any $g^k \in \langle g \rangle$, $(g^k)^{-1} = g^{-k} \in \langle g \rangle$, thus $\langle g \rangle$ constructs a subgroup of $(G, \square)$.

Note 1. The concept of cyclic subgroup provide a method to construct a subgroup of $(G, \square)$.

Example 1. Given $g \in G$, suppose $g^k \neq e(k = \{1, 2, 3, 4, 5, 6\})$, and $g^7 = e$. Then $g^7 = g^6 \square g = g \square g^6 = e$, thus $g^6 = g^{-1}$.

It is easy to check that $\langle g \rangle = \{g, g^2, g^3, g^4, g^5, g^6, e\}$. For any other element such as $g^9 = g^7 \square g^2 = g^2$, thus $\langle g \rangle$ has at most 7 elements. If $g^3 = g^5$, then $g^3 = g^3 \square g^2 = g^2 \square g^3 \Rightarrow g^2 = e$, it is a contradiction, thus $\langle g \rangle$ has at least 7 elements.

So if $\exists k \in \mathbb{Z}$, such that $g^k = e$, then $\langle g \rangle$ is finite and $\langle g \rangle = \{g, g^2, \cdots, g^k\}$, otherwise $\langle g \rangle$ is infinite.

There would be two occasions of $\langle g \rangle$:

1. $\exists a, b \in \mathbb{Z}, a < b,$ s.t. $g^a = g^b$.
   In this case, $g^a = g^b = g^a \square g^{b-a} = g^{b-a} \square g^a$, thus $g^{b-a} = e$, we say $\min\{b - a\}$ is the degree of g, denote as $|\langle g \rangle|$.
2. $\forall a, b \in \mathbb{Z}, a < b, g^a \neq g^b$
   In this case, we say $|\langle g \rangle| = \infty$.

Given a finite group $(G, \square)$, $g \in G$, $\langle g \rangle$ is a subgroup of G. Thus $\langle g \rangle$ is a finite group, which means $\exists a < b,$ s.t. $g^a = g^b \Rightarrow g^{b-a} = e$.

Thus $\langle g \rangle$ has total $|\langle g \rangle| = \min\{b - a\}$ elements, and $g^{|\langle g \rangle|} = e$. Notice that $\langle g \rangle$ is the subgroup of $(G, \Box)$, thus $|\langle g \rangle| \,|\, |G|$ ($\exists q \in \mathbb{Z}$, s.t. $|G| = q|\langle g \rangle|$). Thus for a finite group $(G, \Box)$, $\forall g \in G, g^{|G|} = e$.

Definition 2 (Equivalence Class)**.** Given a set X, an equivalence relation R on X and $\forall x \in X$, we say $\{x' \in X | x' R x\}$ the equivalence class of x under R, denote as $R(x)$.

Definition 3 (Quotient Set)**.** We say the set whose elements are all equivalence class of the elements in X, that is $\{R(x) | \forall x \in X\}$, the quotient set of X under R, denote as X/R.

Example 2. Define an equivalence relation $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} | x \equiv y (\text{mod } 5)\}$. The equivalence class of 1 under R is $R(1) = \{1 + q \cdot 5 | q \in \mathbb{Z}\}$, and the Quotient set of $\mathbb{Z}$ under R is $\mathbb{Z}/R = \{R(1), R(2), R(3), R(4), R(5)\}$.

Given a set X and an equivalence relation R, for $\forall x \in X, x \in R(x)$; for $\forall x_1, x_2 \in X$, either $R(x_1) = R(x_2)$ or $R(x_1) \cap R(x_2) = \emptyset$ (this can be proved by transitivity). Collectively,

1. X is the union of all equivalence classes;
2. different equivalence classes are disjoined.

Conversely, if we can divide a set X into many blocks, then the disjoined blocks(subsets) define an equivalence relation.

Group actions

Given a set X, a group $(G, \Box)$ and a map $G \times X \xrightarrow{\alpha} X$, for any $g \in G, x \in X$, we denote $\alpha(g, x)$ as $g * x$ for simplifying the notations. You can view g and x as a driver and an item respectively. So the map $G \times X \xrightarrow{\alpha} X$ means the process where a driver drives an old item into a new item.

Definition 4 (Left group actions)**.** We call the map α is a (left) group action on $(G, \Box)$ if

1. for $\forall g, g' \in G, x \in X, g * (g' * x) = (g \Box g') * x$;
2. for $\forall x \in X, e * x = x$.

Example 3. Given a set X, for $g \in Perm(X)$ and $x \in X$, $g * x = g(x)$ is a group action on X.

Definition 5 (Orbit)**.** Given a group $(G, \Box)$ which actions on a set X, for $x \in X$, we call the set $G(x) = \{g * x | g \in G\}$ is the orbit of x.

Example 4. Group $(\mathbb{Z}, +)$ actions on $\mathbb{R}$ as for $\forall t \in \mathbb{Z}, (x, y) \in \mathbb{R}^2$, let $t * (x, y) = (x + 2t, y - t)$, then the orbits of $x_1, x_2, x_3, x_4 \in \mathbb{R}^2$ are like the margin figure.

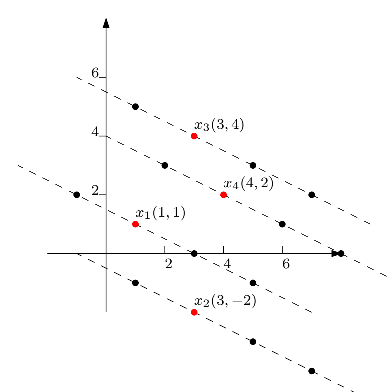


Figure 1: Orbits of x_1, x_2, x_3, x_4

Exercise 1. Suppose a group $(G, \square)$ actions on a set X, define a relation $R := \{(x, x') \in X \times X | \exists g \in G, \text{ s.t. } x' = g * x\}$, that is $R := \{(x, x') \in X \times X | x' \in G(x)\}$. Show that R is an equivalence relation.

Proof. All we need to prove is:
1. $x \in G(x)$;
2. $x' \in G(x) \Rightarrow x \in G(x')$;
3. $x' \in G(x), x'' \in G(x') \Rightarrow x'' \in G(x)$.

1. Since $e \in G$, and $x = e * x$, thus $x \in G(x)$; 2. $x' \in G(x)$, thus $\exists g \in G, \text{ s.t.}$

$$\begin{aligned} x' &= g * x \\ &\Rightarrow g^{-1} * x' = g^{-1} * (g * x) \\ &\Rightarrow g^{-1} * x' = (g^{-1} \square g) * x \\ &\Rightarrow g^{-1} * x' = e * x = x \end{aligned}$$

since $g^{-1} \in G$, $x \in G(x')$; 3. $x' \in G(x), x'' \in G(x')$, thus $\exists g_1, g_2 \in G, \text{ s.t.}$

$$\begin{aligned} x'' &= g_2 * x' \\ &= g_2 * (g_1 * x) \\ &= (g_2 \square g_1) * x \end{aligned}$$

$g_1, g_2 \in G$, thus $g_2 \square g_1 \in G$, and $x'' \in G(x)$. $\square$

This exercise shows the orbit of x is an equivalence class of x, thus the difference orbits are disjoin, and the union of all orbits is X.

Definition 6 (Stablizer)**.** Suppose a group $(G, \square)$ actions on a finite set X, for any $x \in X$, we call $G_x = \{g \in G | g * x = x\}$ the stablizer of x.

Exercise 2. Show that G_x constructs a subgroup of $(G, \square)$.

Proof. All we need to prove is 1) G_x is enclosed; 2) $(G_x, \square)$ is a group.

1. for $\forall x \in X, \forall g_1, g_2 \in G_x$:

$$\begin{aligned} (g_1 \square g_2) * x &= g_1 * (g_2 * x) \\ &= g_1 * x \\ &= x, \end{aligned}$$

thus $g_1 \square g_2 \in G_x$, similarly, $g_2 \square g_1 \in G_x$.

2. since $\forall g_1, g_2, g_3 \in G_x \subseteq G$, the associative follows; 3. since $e * x = x$, $e \in G_x$; 4. for $\forall x \in X, \forall g \in G_x$:

$$\begin{aligned} g^{-1} * x &= g^{-1} * (g * x) \\ &= (g^{-1} \square g) * x \\ &= e * x \\ &= x, \end{aligned}$$

thus $g^{-1} \in G_x$. The first proof shows G_x is enclosed, the last 3 proofs show $(G_x, \Box)$ is a group, thus $(G_x, \Box)$ is a subgroup of $(G, \Box)$. $\Box$

Definition 7 (Orbit Map)**.** Suppose a group $(G, \Box)$ actions on a finite set X, given $x \in X$, we say the map $G \xrightarrow{o_x} G(x)$ with $g \mapsto g * x$ is the orbit map of x.

Note 2. o_x is a surjection.

Exercise 3. For $\forall g, g' \in G$, show that $o_x(g) = o_x(g') \Leftrightarrow g \Box G_x = g' \Box G_x$.

Proof. $\Rightarrow$:

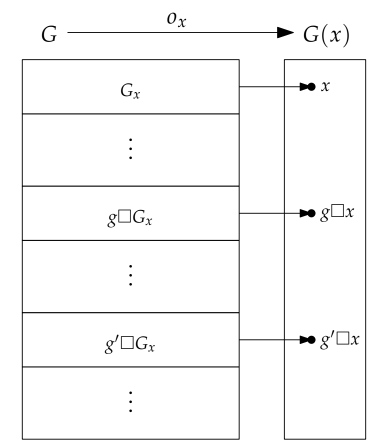
$$\begin{aligned}(g^{-1} \Box g') * x &= g^{-1} * (g' * x) \\ &= g^{-1} * (g * x) \\ &= (g^{-1} \Box g) * x \\ &= e * x \\ &= x,\end{aligned}$$

thus $g^{-1} \Box g' \in G_x \Rightarrow g \Box G_x = g' \Box G_x$.

$\Leftarrow$: $g \Box G_x = g' \Box G_x \Rightarrow \forall h \in G_x$ s.t. $g \Box h = g' \Box h$. Thus

$$\begin{aligned}g * x &= g * (h * x) \\ &= (g \Box h) * x \\ &= (g' \Box h) * x \\ &= g' * (h * x) \\ &= g' * x.\end{aligned}$$

$\Box$



So if $\exists g, g' \in G$, s.t. $g' \Box x = g \Box x$ then g, g' come from the same coset of G_x; conversely, if g, g' are from the same coset of G_x, then $g' \Box x = g \Box x$. That means o_x is a **bijection** from the quotient set G/G_x to the orbit $G(x)$. Thus for finite group $(G, \Box)$ actioning on X, and $x \in X$, have

$$|G/G_x| = |G|/|G_x| = |G(x)|.$$

Note 3. Remember that the cosets of G_x have the same cardinality.

Theorem 1 (Burnside)**.** *Suppose a finite group $(G, \Box)$ actions on a finite set X, then X has*

$$\frac{1}{|G|} \cdot \sum_{g \in G} |\{x \in X | g * x = x\}|$$

orbits.

Proof. Denote the number of orbits as τ. We now compute $\delta = |\{(g, x) \in G \times X | g * x = x\}|$ in two orders. The meaning of these operations is like the right margin figure.

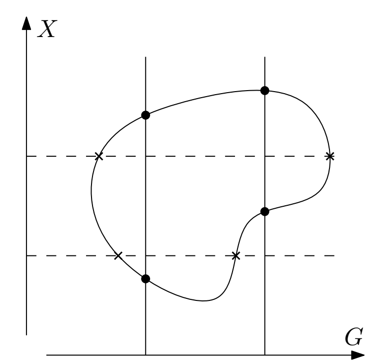


Figure 2: $|\{(g, x) \in G \times X | g * x = x\}|$

1. Fix x: so

$$\begin{aligned}
\delta &= \sum_{x\in X} |\{g \in G | g * x = x\}| \\
&= \sum_{x\in X} |G_x| = \sum_{x\in X} \frac{|G|}{|G(x)|} \\
&= |G| \cdot \sum_{x\in X} \frac{1}{|G(x)|} \\
&= |G| \cdot \tau.
\end{aligned}$$

Notice the last equation, remember that X is the disjoin union of the all orbits. So the sum of $\frac{1}{|G(x)|}$ where xs are in the same orbit is 1. And the sum of $\frac{1}{|G(x)|}$ of all $x \in X$ is the number of orbits τ.

2. Fix g: so

$$\delta = \sum_{g\in G} |\{x \in X | g * x = x\}|.$$

Simultaneous equations, we have

$$|G| \cdot \tau = \sum_{g\in G} |\{x \in X | g * x = x\}|,$$

thus $\tau = \frac{1}{|G|} \cdot \sum_{g\in G} |\{x \in X | g * x = x\}|$. □

Example 5. Color four vertices of a square in black or white, allowing rotation, How many different coloring methods are there?

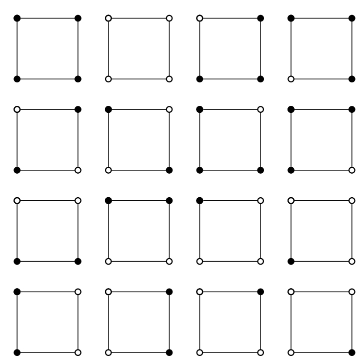
Solution. Let X be the all coloring method without rotation, so $|X| = 2^4 = 16$. Let G be the set that are all rotations of X, let r represents rotate 90 degrees in clockwise, then $G = \{e, r, r^2, r^3\} = \langle r \rangle$. Now the question is how many orbits of X under G are there?

Fix $g = e$, then $|\{x \in X | e * x = x\}| = 16$, these are the all element if X; Fix $g = r$, then $|\{x \in X | r * x = x\}| = 2$; Fix $g = r^2$, then $|\{x \in X | r * x = x\}| = 4$; Fix $g = r^3$, then $|\{x \in X | r * x = x\}| = 2$.

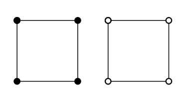
Thus the number of the orbits is

$$\begin{aligned}
\tau &= \frac{1}{|G|} \cdot \sum_{g\in G} |\{x \in X | g * x = x\}| \\
&= \frac{1}{4} \cdot (16 + 2 + 4 + 2) \\
&= \frac{24}{4} = 6.
\end{aligned}$$

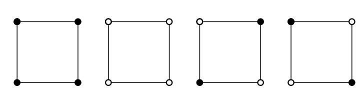
So there are totally 6 coloring methods. □



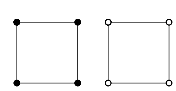
$\{x \in X | e * x = x\}$



$\{x \in X | r * x = x\}$



$\{x \in X | r^2 * x = x\}$



$\{x \in X | r^3 * x = x\}$