# Introduction to Analysis

## Lecture 7

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**Abstract**

THIS IS THE LECTURE NOTE FOR THE *Introduction to Analysis* CLASS IN SPRING 2019.

## 1   Signed area and indefinite integral

For $a, b \in \mathbb{R}$, we let

$$\underline{ab} := \begin{cases} [a,b], & a \leq b \\ [b,a], & a \geq b \end{cases}$$

We hope to define and study the properties of 'signed area' $S(f; a, b)$ of region in the $xy$ - plane enclosed by $x = a, x = b, y = 0$ and $y = f(x)$ where $D \xrightarrow{f} \mathbb{R}$ is a (reasonably well behavior) function such that $\underline{ab} \subseteq D$.
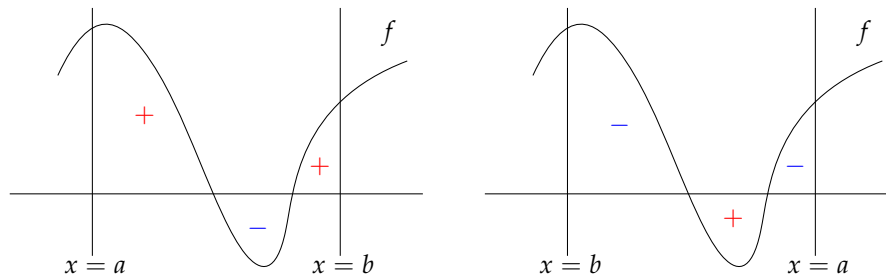


Figure 1: $S(f; a, b)$

If $S(f; a, b)$ is defined, we call $f$ integrable on $\underline{ab}$ w.r.t. the specific definition of $S(\cdot; \cdot, \cdot)$. Assuming we have known the definition of $S(\cdot; \cdot, \cdot)$, we expect it to satisfy several properties (P):

(P1) [Monotonicity]: $f$ is integrable and $\geq 0$ on $[a, b]$, then $S(f; a, b) \geq 0$.

(P2) [Linearity]: $f, g$ are integrable on $[a, b], \alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is integrable on $[a, b]$

and
$$S(\alpha f + \beta g; a, b) = \alpha S(f; a, b) + \beta S(g; a, b).$$

(P3) $f$ is integrable on $\underline{ab}$ and $c \in \underline{ab}$, then $f$ is integrable on $\underline{ac}$ and $\underline{cb}$ and

$$S(f; a, b) = S(f; a, c) + S(f; c, b);$$
$$S(f; a, c) = S(f; a, b) + S(f; b, c).$$

(P4) $f$ is conti. on $\underline{ab}$, then $f$ is integrable on $\underline{ab}$.

(P5) Then constant function 1 is integrable on $\underline{ab}$ for all $a, b \in \mathbb{R}$ and $S(1; a, b) = b - a$.

(P6) All $D \xrightarrow{f} \mathbb{R}$ are integrable on $[a, a]$ if $a \in D$.

(P7) $f$ is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$.

**Exercise 1.** *Using the above properties, show that*

1. *$f, g$ are integrable on $[a, b]$, $f(x) \leq g(x)$ for all $x \in [a, b]$, then $S(f; a, b) \leq S(g; a, b)$.*
2. *$f$ are integrable on $[a, b]$ then $|S(f; a, b)| \leq S(|f|; a, b)$.*
3. *$f$ is integrable on $\underline{ab}$, $\underline{cd} \subseteq \underline{ab}$, then $f$ is integrable on $\underline{cd}$.*
4. *$D \xrightarrow{f} \mathbb{R}, a \in D$, then $S(f; a, a) = 0$.*
5. *$f$ is integrable on $\underline{ab}$, then $S(f; a, b) = -S(f; b, a)$.*

*Proof.* 1. $f(x) \leq g(x) \Rightarrow g(x) - f(x) \geq 0$ for $\forall x \in [a, b]$, thus

$$S(g; a, b) - S(f; a.b) = S(g - f; a, b) \tag{P2}$$
$$\geq 0 \tag{P1}$$

2. $f$ is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and since $-|f| \leq f \leq |f|$, we have that

$$-S(|f|; a, b) = S(-|f|; a, b) \leq S(f; a, b) \leq S(|f|; a, b) \tag{P2, 1}$$

and hence

$$|S(f; a, b)| \leq S(|f|; a, b).$$

3. trivial by P3.

4. since $a \in [a, a]$, then

$$S(f; a, a) = S(f; a, a) + S(f; a, a) \tag{P3,P6}$$

and hence $S(f; a, a) = 0$.

5. $S(f; a, b) + S(f; b, a) = S(f; a, a) = 0$. $\qquad \square$

**Theorem 1** (Fundamental theorem of calculus, FTC[1]). *Let $a \leq b$ and $f$ is integrable on $[a, b]$. Let $F(x) := S(f; a, x), x \in [a, b]$. If $c \in [a, b]$ and $f$ is conti. at $c$, then*

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[1] Assume that we have known the definition of $S(\cdot; \cdot, \cdot)$. And if we can actually define $S(\cdot; \cdot, \cdot)$ that satisfies P1 - P7, then all properties we discuss will work for such $S(\cdot; \cdot, \cdot)$.

- $c \in (a, b) \Rightarrow F'(c) = f(c)$;
- $c = a \Rightarrow F'_+(a) = f(a)$;
- $c = b \Rightarrow F'_-(b) = f(b)$.

*Proof.* For $\forall x \in [a, b]$,

$$
\begin{aligned}
F(x) - F(c) - f(c)(x - c) &= S(f; a, x) - S(f; a, c) - f(c) \cdot S(1; c, x) && \text{(P5)} \\
&= S(f; a, x) + S(f; c, a) - S(f(c); c, x) && \text{(P2)} \\
&= S(f; c, x) - S(f(c); c, x) && \text{(P3)} \\
&= S(f - f(c); c, x) && \text{(P2)}
\end{aligned}
$$

And hence

$$
\begin{aligned}
|F(x) - F(c) - f(c)(x - c)| &= |S(f - f(c); c, x)| \\
&\leq S(|f - f(c)|; \min\{c, x\}, \max\{c, x\})
\end{aligned}
$$

Since $f$ is continuous at $c$, then for any $\epsilon > 0, \exists \delta > 0$, s.t. $\forall |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$. So if $|x - c| < \delta$, then

$$
|F(x) - F(c) - f(c)(x - c)| \leq S(\epsilon; \min\{c, x\}, \max\{c, x\}) = \epsilon \cdot \delta
$$

thus

$$
\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| < \epsilon \cdot \delta
$$

if $0 < |x - c| < \delta$. $\qquad\square$

**Corollary 1** (FTC'). *Let $D \xrightarrow{F} \mathbb{R}$ be continuously differentiable on $[a, b]$ where $[a, b] \subseteq D \subseteq \mathbb{R}$, i.e. $f(x) = F'(x)$ exists for all $x \in [a, b]$ and $f(x)$ is continuous on $[a, b]$. Then*

$$
F(b) - F(a) = S(f; a, b).
$$

*Proof.* $f$ is continuous on $[a, b] \Rightarrow f$ is integrable on $[a, b]$. Let $G(x) = S(f; a, x)$. Then FTC $\Rightarrow G'(x) = f(x)$ for $\forall x \in (a, b)$. And since $F'(x) = f(x)$, then $0 = f(x) = f(x) = F'(x) - G'(x) = (F(x) - G(x))'$ and $F(x) - G(x)$ is continuous on $[a, b] \Rightarrow F - G$ is const., thus

$$
F(b) - F(a) = G(b) - G(a) = S(f; a, b).
$$

$\qquad\square$

The FTC' motivates the following definition:

**Definition 1** (Indefinite integral). Given two functions $D \xrightarrow{F} \mathbb{R}$ and $D \xrightarrow{f} \mathbb{R}$, where $D \subseteq \mathbb{R}$, we say that $F$ is a primitive (function) on $D$ of $f$ if $F' = f$ on $D$. We also say

that
$$\int f(x)\,dx = F(x) + C,$$
(where $C \in \mathbb{R}$ is const.) the indefinite integral of $f$ on $D$.

*Remark 1* (integration by part). Recall Leibniz's rule: if $f, g$ is diff. then
$$(fg)' = f'g + fg'$$
and hence
$$\int f'g = fg - \int fg'$$
this is called **integration by part**.

*Remark 2* (substitution). Recall chain rule: if $F, h$ are diff. on the relevant domain, then
$$(F \circ h)'(t) = F'(h(t)) \cdot h'(t).$$

if $F'(x) = f(x)$, then
$$\int f(x)\,dx \bigg|_{x=h(t)} = \int f(h(t))h'(t)\,dt.$$

this is called **substitution**.

## 2 Darboux integral

### 2.1 Definitions

**Definition 2** (Partition). A finite subset $\Delta \subseteq [a, b]$ is a partition of $[a, b]$ if $a, b \in \Delta$. We usually list the elements of $\Delta$ in order as
$$\Delta: \quad a = x_0^\Delta < x_1^\Delta < \cdots < x_{k(\Delta)}^\Delta = b,$$
and let $I_j^\Delta := [x_{j-1}^\Delta, x_j^\Delta], j = 1, \cdots, k(\Delta)$. We may write $x_j^\Delta$, $I_j^\Delta$ and $k(\Delta)$ as $x_j$, $I_j$ and $k$ if no confusion will be caused.

If $D \xrightarrow{f} \mathbb{R}$ is bounded on $[a, b]$, where $[a, b] \subseteq D$, and the signed area $S(f; a, b)$ is defined (with properties P1 - P7). Since $f$ is bdd. $\Rightarrow \sup f$ exists, and thus for any partition of $[a, b]$:
$$S(f; a, b) = \sum_{j=1}^{k} S(f; x_{j-1}, x_j)$$
$$\leq \sum_{j=1}^{k} S(\sup_{I_j} f; x_{j-1}, x_j)$$

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$$= \sum_{j=1}^{k} (\sup_{I_j} f) \cdot (x_j - x_{j-1})$$
$$:= \overline{S}(f, \Delta)$$

We call $\overline{S}(f, \Delta)$ the **upper sum** (上和) of $f$ w.r.t. $\Delta$. Similarly, we define

$$\underline{S}(f, \Delta) := \sum_{j=1}^{k} (\inf_{I_j} f) \cdot (x_j - x_{j-1})$$

and call it the **lower sum** (下和) of $f$ w.r.t. $\Delta$. Then for any partition $\Delta$ of $[a, b]$, we have

$$\underline{S}(f, \Delta) \leq S(f; a, b) \leq \overline{S}(f, \Delta)$$

**Definition 3** (Refine). Let $\Delta', \Delta$ are partitions of $[a, b]$, we say that
1. $\Delta'$ refines $\Delta$, if $\Delta \subseteq \Delta'$;
2. $\Delta \cup \Delta'$ the common refinement of $\Delta$ and $\Delta'$.

**Proposition 1.** *Let $\Delta_1, \Delta_2$ are partitions of $[a, b]$ and $\Delta_1 \subseteq \Delta_2$, then*

$$\underline{S}(f, \Delta_1) \leq \underline{S}(f, \Delta_2) \leq \overline{S}(f, \Delta_2) \leq \overline{S}(f, \Delta_1).$$

*Proof.* w.l.o.g. let $\Delta_1 = \{a, b\}$ and $\Delta_2 = \{a, c, b\}$ where $c \in (a, b)$. Then

$$\begin{aligned}
\overline{S}(f, \Delta_2) &= \sup_{[a,c]} f \cdot (c - a) + \sup_{[c,b]} f \cdot (b - c) \\
&\leq \sup_{[a,b]} f \cdot (c - a) + \sup_{[a,b]} f \cdot (b - c) \qquad \text{(Exercise ??)} \\
&= \sup_{[a,b]} f \cdot (b - a) \\
&= \overline{S}(f, \Delta_1).
\end{aligned}$$

and $\underline{S}(f, \Delta_2) \geq \underline{S}(f, \Delta_1)$ in the same way. $\qquad \square$

*Remark* 3. In particular, for $\forall$ partitions $\Delta, \Delta'$ of $[a, b]$, we have that

$$\begin{aligned}
\underline{S}(f, \Delta) &\leq \underline{S}(f, \Delta \cup \Delta') \\
&\leq \overline{S}(f, \Delta \cup \Delta') \\
&\leq \overline{S}(f, \Delta'),
\end{aligned}$$

that is any lower sum is smaller than any upper sum, thus the set of all lower/upper sum has upper/lower bound, and hence has l.u.b/g.l.b. which is called **upper/lower integral**.

**Definition 4** (Upper/lower integral, 上/下积分). For a function $D \xrightarrow{f} \mathbb{R}$ which is bounded on $[a, b]$, where $[a, b] \subseteq D \subseteq \mathbb{R}$, we define upper/lower integral of $f$ on $[a, b]$ as

$$\overline{\int_a^b} f(x) \, dx := \inf_\Delta \overline{S}(f, \Delta)$$

$$\underline{\int_a^b} f(x) \, dx := \sup_\Delta \underline{S}(f, \Delta)$$

It is direct to see that

$$\underline{\int_a^b} f(x) \, dx \leq \overline{\int_a^b} f(x) \, dx.$$

**Definition 5** (Darboux integrable, 达布可积). For a function $D \xrightarrow{f} \mathbb{R}$ which is bounded on $[a, b]$, where $[a, b] \subseteq D \subseteq \mathbb{R}$, we say $f$ is Darboux integrable on $[a, b]$ is

$$\underline{\int_a^b} f(x) \, dx = \overline{\int_a^b} f(x) \, dx = S,$$

if this is the case, we call the $S$ the (definite) integral of $f$ on $[a, b]$, denoted as

$$\int_a^b f(x) \, dx.$$

And if $f$ is Darboux integrable on $[a, b]$, we define

$$\int_b^a f(x) \, dx := - \int_a^b f(x) \, dx.$$

**Exercise 2.** *If $f$ is Darboux integrable on $[a, b] \Leftrightarrow \forall \epsilon > 0, \exists \Delta$,* s.t.

$$0 \leq \overline{S}(f, \Delta) - \underline{S}(f, \Delta) < \epsilon.$$

*Proof.* $\Rightarrow$: assume that $\int_b^a f(x) \, dx = L = \overline{\int_b^a} f(x) \, dx = \underline{\int_b^a} f(x) \, dx$ , and $L$ is l.u.b. of the lower sums and g.l.b. of the upper sums, then for $\forall \epsilon, \exists \Delta$ s.t.

$$L - \frac{\epsilon}{2} < \underline{S}(f, \Delta) \leq L \leq \overline{S}(f, \Delta) < L + \frac{\epsilon}{2}$$

Then $0 \leq \overline{S}(f, \Delta) - \underline{S}(f, \Delta) < \epsilon$.
$\Leftarrow$: since for any $\Delta$, $\overline{\int_b^a} f(x) \, dx \leq \overline{S}(f, \Delta)$ and $\underline{\int_b^a} f(x) \, dx \geq \underline{S}(f, \Delta)$, then

$$0 \leq \overline{\int_b^a} f(x) \, dx - \underline{\int_b^a} f(x) \, dx \leq \overline{S}(f, \Delta) - \underline{S}(f, \Delta) < \epsilon$$

thus $\overline{\int_b^a} f(x) \, dx = \underline{\int_b^a} f(x) \, dx$. $\qquad\qquad \square$

**Example 1** (Dirichlet function, 狄利克雷函数). Consider the Dirichlet function $\mathbb{R} \xrightarrow{f} \mathbb{R}$ where

$$x \mapsto \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Then for any $[a, b] \subseteq \mathbb{R}$ and any partition $\Delta$ of $[a, b]$, we have that

$$\overline{S}(f, \Delta) = \sum_{j=1}^{k} 1 \cdot (x_j - x_{j-1}) = b - a$$

$$\underline{S}(f, \Delta) = \sum_{j=1}^{k} 0 \cdot (x_j - x_{j-1}) = 0$$

and hence

$$\overline{\int_a^b} f(x) \, \mathrm{d}x = b - a > 0 = \underline{\int_a^b} f(x) \, \mathrm{d}x,$$

thus Dirichlet function $f$ is non - Darboux integral on any interval $[a, b] \subseteq \mathbb{R}$.

**Exercise 3** (Thomae function, 托梅函数). *Consider the Thomae function $\mathbb{R} \xrightarrow{f} \mathbb{R}$ where*

$$x \mapsto \begin{cases} \dfrac{1}{m}, & x = \dfrac{n}{m} \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

*where $(n, m) = 1$ and $m \in \mathbb{N}, n \in \mathbb{Z}$.*
*1. find the points at which $f$ is disconti.;*
*2. find $\overline{\int_0^1} f(x) \, \mathrm{d}x$ and $\underline{\int_0^1} f(x) \, \mathrm{d}x$.*

*Proof.* (1.1) $f$ is discontinuous at $x \in \mathbb{Q}$:
Assume that $c = n/m \in \mathbb{Q}, m \in \mathbb{N}, (n, m) = 1$, then $f(c) = 1/m$. Let $\epsilon = 1/2m$, then for any $\delta$, $\exists x \in \mathbb{R} \backslash \mathbb{Q}$ s.t. $|x - c| < \delta$, thus $f(x) = 0$ and $|f(x) - f(c)| = 1/m > \epsilon = 1/2m$.
(1.2) $f$ is continuous at $x \in \mathbb{R} \backslash \mathbb{Q}$:
Assume that $c \in \mathbb{R} \backslash \mathbb{Q}$, then for any $\epsilon > 0$, choose $m_0 \in \mathbb{N}$ so large that $1/m_0 < \epsilon$. Let $\delta = \min\{|c - n/m| n \in \mathbb{Z}, m \in \mathbb{N}, (m, n) = 1, m < m_0\}$. $\delta$ exists, because for any $m \in \mathbb{N}$, $c$ is sandwiched between at most two rational numbers that divide by $m$. Then for any $|x - c| < \delta$, $x$ is either irrational numbers or can only be represented by $n/m$ where $m \geq m_0$, thus $|f(x) - f(c)| \leq 1/m_0 < \epsilon$.
(2) Let $\Delta$ be a partition of $[0, 1]$, for any $m_0 \in \mathbb{N}$, consider $J_1 = \{j | I_j^\Delta$ contains some $n/m$ for some $m = 1, \cdots, m_0, (n, m) = 1\}$ and $J_2 = \{1, \cdots, k(\Delta)\} \backslash J_1$. Then

$$\overline{S}(f, \Delta) = \sum_{j \in J_1} \sup_{I_j^\Delta} f \cdot (x_j^\Delta - x_{j-1}^\Delta) + \sum_{j \in J_2} \sup_{I_j^\Delta} f \cdot (x_j^\Delta - x_{j-1}^\Delta)$$

If $j \in J_2$, then $\sup_{I_j^\Delta} f \leq 1/(m_0 + 1)$, and hence

$$\sum_{j \in J_2} \sup_{I_j^\Delta} f \cdot (x_j^\Delta - x_{j-1}^\Delta) \leq \frac{1}{m_0 + 1} \cdot \sum_{j \in J_2} (x_j^\Delta - x_{j-1}^\Delta)$$

$$\leq \frac{1}{m_0 + 1} \cdot (1 - 0)$$

$$= \frac{1}{m_0 + 1}.$$

On the other hand

$$\sum_{j \in J_1} \sup_{I_j^\Delta} f \cdot (x_j^\Delta - x_{j-1}^\Delta) \leq \left( 1 + 2 \cdot \frac{1}{2} + \cdots + m_0 \cdot \frac{1}{m_0} \right) \cdot \max_{1 \leq j \leq k(\Delta)} (x_j^\Delta - x_{j-1}^\Delta)$$

$$= m_0 \cdot mesh(\Delta).$$

thus

$$\overline{S}(f, \Delta) \leq m_0 \cdot mesh(\Delta) + \frac{1}{m_0 + 1}$$

Then for any $\epsilon > 0$, choose $m_0$ so large s.t. $1/(m_0 + 1) < \epsilon/2$ and $\Delta$ so fine that $mesh(\Delta) < \epsilon/2m_0$, then $\overline{S}(f, \Delta) < \epsilon$ and hence $\overline{\int_0^1} f = 0$. ($\underline{\int_0^1} f = 0$ is trivial) $\square$
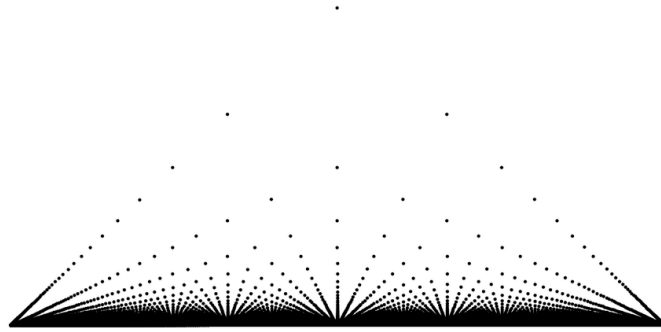


Figure 2: Thomae function