Introduction to Analysis

Lecture 9

Haoming Wang

28 May 2019

Abstract

This is the Lecture note for the *Introduction to Analysis* class in Spring 2019.

1 Lebesgue criterion

Definition 1 (Countable set)**.** A set S is countable if $\exists$ a bijection $S_0 \xrightarrow{f} S$ with $S_0 \subseteq \mathbb{N}$.

Example 1. finite set, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable sets. $\mathbb{R}$ is not countable (refer to the note *Introduction to Topology / Cardinality /* $\mathbb{N}$ *and* $\mathbb{R}$).

Definition 2 (Lebesgue d - dimensional measure 0)**.** $S \subseteq \mathbb{R}^d$ is of Lebesgue d - dimensional measure 0, if $\forall \epsilon > 0, \exists$ rectangles $R_n (n \in \mathbb{N})$ s.t. $S \subseteq \cup_{n=1}^{\infty} R_n$ and $\sum_{n=1}^{\infty} vol_d(R_n) < \epsilon$.

Example 2. If $S (\subseteq \mathbb{R}^d)$ is countable, then S is of measure 0.

Proof. Let $S = \{s_1, s_2, \cdots\}$. For any $\epsilon > 0$, choose a rectangle R_n s.t. $s_n \in R_n$ and $vol_d(R_n) < \epsilon/2^n$, then $S \subseteq \cup_{n=1}^{\infty} R_n$ and $\sum_{n=1}^{\infty} vol_d(R_n) < \epsilon$. $\square$

Exercise 1. *If* $[0,1] \subseteq \cup_{j=1}^{\infty} [a_j, b_j]$*, show that* $\sum_{j=1}^{\infty} (b_j - a_j) \geq 1$*, that is* $[0,1]$ *is not of measure* 0.

Proof. Claim 1, if $\exists m \in \mathbb{N}$, s.t. $[0,1] \subseteq \cup_{j=1}^{m} [c_j, d_j] \Rightarrow \sum_{j=1}^{m} (d_j - c_j) \geq 1$. Trivial.
Claim 2, (general cases) **enlarge** $[a_j, b_j]$ to (a_j', b_j') s.t.

$$b_j' - a_j' = b_j - a_j + \frac{\eta}{2^j},$$

$\eta > 0, j \in \mathbb{N}$. Since $[0,1]$ is compact, then $\exists m \in \mathbb{N}$ s.t. $[0,1] \subseteq \cup_{j=1}^{m} (a_j', b_j')$, and by Claim 1, we have

$$1 \leq \sum_{j=1}^{m} (b_j' - a_j')$$

$$
\begin{aligned}
&= \sum_{j=1}^{m} \left(b_j - a_j + \frac{\eta}{2^j} \right) \\
&< \sum_{j=1}^{\infty} \left(b_j - a_j \right) + \eta
\end{aligned}
$$

That is for any $\eta > 0$, $\sum_{j=1}^{\infty} \left(b_j - a_j \right) + \eta \geq 1 \Rightarrow \sum_{j=1}^{\infty} \left(b_j - a_j \right) \geq 1$. $\square$

Lemma 1. *Given $S_j \subseteq \mathbb{R}^n (j \in \mathbb{N})$, if for $\forall j$, S_j is of measure $0 \Rightarrow \cup_{j \in \mathbb{N}} S_j$ is of measure 0.*

Proof. For any j, there exists rectangles $R_{j,k} (k \in \mathbb{N})$, such that

$$\cup_{k \in \mathbb{N}} vol(R_{j,k}) < \frac{\epsilon}{2^j},$$

and then encode them from northeast to southwest as $R_l (l \in \mathbb{N})$, then

$$\cup_{j \in \mathbb{N}} S_j \subseteq \cup_{j \in \mathbb{N}} (\cup_{k \in \mathbb{N}} R_{j,k}) = \cup_{l \in \mathbb{N}} R_l$$

and

$$\sum_{l \in \mathbb{N}} vol(R_l) = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} vol(R_{j,k}) < \sum_{j \in \mathbb{N}} \frac{\epsilon}{2^j} < \epsilon.$$

Thus $\cup_{j \in \mathbb{N}} S_j$ is of measure 0. $\square$

Lemma 2. *Let $X \xrightarrow{f} \mathbb{R}$ be a bdd. function, X is a metric space, for any $a \in X$, define*

$$o_f(a) := \lim_{\delta \to 0} \left(\sup_{B_\delta(a)} f - \inf_{B_\delta(a)} f \right),$$

then

1. *f is conti. at $a \in X \Leftrightarrow o_f(a) = 0$.*
2. *for $c \in \mathbb{R}$, $\Lambda_c = \{x \in X | o_f(x) < c\} \subseteq_{open} X$. Correspondingly, $\Omega_c := \{x \in X | o_f(x) \geq c\} \subseteq_{close} X$.*
3. *if $a \in B_r(a) \subseteq S \subseteq X$, for some r, i.e. $a \in S^o$, then $\sup_S f - \inf_S f \geq o_f(a)$.*

Proof. See Proposition **??**. $\square$

Theorem 1 (Lebesgue's criterion of Darboux integrability)**.** *Let $S \xrightarrow{f} \mathbb{R}$ be a bdd. function, with $R := \prod_{i=1}^{d} [a_i, b_i] \subseteq S \subseteq \mathbb{R}^d$. Then f is Darboux integrable on $R \Leftrightarrow D := \{x \in R | f \text{ is disconti. at } x\}$ is of d - dim measure 0.*

Proof. $\Rightarrow$: For any $\eta > 0$, there exists a partition Δ of R s.t.

$$\overline{S}(f, \Delta) - \underline{S}(f, \Delta) = \sum_{R_\cdot^\Delta} \left(\sup_{R_\cdot^\Delta} f - \inf_{R_\cdot^\Delta} f \right) \cdot vol(R_\cdot^\Delta) < \eta$$

Where $R^{\Delta}_{\cdot} = \prod_{i=1}^{d}[c_i, d_i]$ is a subinterval of R w.r.t. partition Δ. Define $\tilde{R}^{\Delta}_{\cdot} = \prod_{i=1}^{d}(c_i, d_i)$. It is direct to see that

$$D = \cup_{c>0}\Omega_c = \cup_{n\in\mathbb{N}} D_{1/n}$$

by Archimedean Property. Then

$$\begin{aligned}
\eta &> \sum_{R^{\Delta}_{\cdot}} \left(\sup_{R^{\Delta}_{\cdot}} f - \inf_{R^{\Delta}_{\cdot}} f \right) \cdot vol(R^{\Delta}_{\cdot}) \\
&\geq \sum_{\substack{R^{\Delta}_{\cdot} \text{ s.t.} \\ \tilde{R}^{\Delta}_{\cdot}\cap\Omega_c\neq\emptyset}} \left(\sup_{R^{\Delta}_{\cdot}} f - \inf_{R^{\Delta}_{\cdot}} f \right) \cdot vol(R^{\Delta}_{\cdot}) \\
&\geq \sum_{\substack{R^{\Delta}_{\cdot} \text{ s.t.} \\ \tilde{R}^{\Delta}_{\cdot}\cap\Omega_c\neq\emptyset}} c \cdot vol(R^{\Delta}_{\cdot}) \qquad (\star) \\
&= c \cdot \sum_{\substack{R^{\Delta}_{\cdot} \text{ s.t.} \\ \tilde{R}^{\Delta}_{\cdot}\cap\Omega_c\neq\emptyset}} vol(R^{\Delta}_{\cdot})
\end{aligned}$$

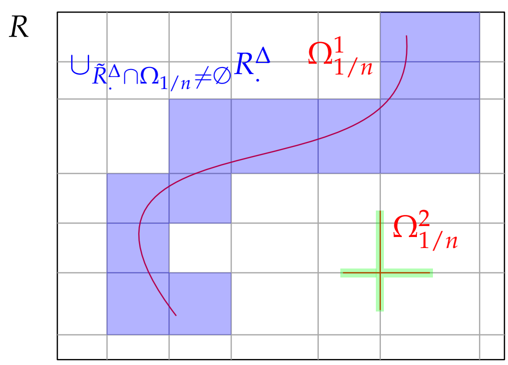
$(\star)$ is since $\tilde{R}^{\Delta}_{\cdot} \cap \Omega_c \neq \emptyset \Rightarrow \exists x \in \Omega_c$ s.t. $x \in \tilde{R}^{\Delta}_{\cdot} \Rightarrow x \in (R^{\Delta}_{\cdot})^o \Rightarrow \exists r > 0$, s.t. $B_r(x) \subseteq R^{\Delta}_{\cdot} \Rightarrow \sup_{R^{\Delta}_{\cdot}} f - \inf_{R^{\Delta}_{\cdot}} f \geq o_f(x) \geq c$, by the third conclusion of second Lemma. Since for any given $n \in \mathbb{N}$

$$\begin{aligned}
\Omega_{1/n} &= \{x \in \Omega_{1/n} | x \in \tilde{R}^{\Delta}_{\cdot}\} \cup \{x \in \Omega_{1/n} | x \in R^{\Delta}_{\cdot} \backslash \tilde{R}^{\Delta}_{\cdot}\} \\
&:= \Omega^1_{1/n} \cup \Omega^2_{1/n}
\end{aligned}$$

And $\Omega^1_{1/n} \subseteq \cup_{\tilde{R}^{\Delta}_{\cdot}\cap\Omega_{1/n}\neq\emptyset} R^{\Delta}_{\cdot}$ and

$$\sum_{\substack{R^{\Delta}_{\cdot} \text{ s.t.} \\ \tilde{R}^{\Delta}_{\cdot}\cap\Omega_{1/n}\neq\emptyset}} vol(R^{\Delta}_{\cdot}) \leq n\eta$$

for any $\eta > 0$, thus $\Omega^1_{1/n}$ is of measure 0. $\Omega^2_{1/n}$ is of measure if trivial. Thus $\Omega_{1/n}$ is of measure 0, then $D = \cup_{n\in\mathbb{N}}\Omega_{1/n}$ is of measure 0.



$\Leftarrow$:
(A) For any $n \in \mathbb{N}$, since $D_{1/n}$ is closed (by Lemma) and bdd. in Euclidean space $\Rightarrow D_{1/n}$ is cpt. And $D_{1/n}$ is of measure $0 \Rightarrow \forall \epsilon > 0, \exists$ (closed) rectangles $R_j (j \in \mathbb{N})$ s.t.

$$D_{1/n} \subseteq \cup_{j=1}^{\infty} R_j, \quad \sum_{j=1}^{\infty} vol(R_j) < \epsilon.$$

Enlarge all R_j to be open rectangle R'_j (like Exercise 1) with $vol(R'_j) = vol(R_j) + 1/2^j$ and hence

$$\sum_{j \in \mathbb{N}} vol(R'_j) < 2\epsilon,$$

then $D_{1/n} \subseteq \cup_{j\in\mathbb{N}} R_j = \cup_{j\in\mathbb{N}} R'_j$.
(B) $D_{1/n} \subseteq_{close} R \Rightarrow R\backslash D_{1/n} \subseteq_{open} R$, then for $\forall a \in R\backslash D_{1/n}$, i.e. $a \notin D_{1/n}$ and hence $o_f(a) = \inf_\delta \left(\sup_{B_\delta(a)} f - \inf_{B_\delta(a)} f \right) < 1/n \Rightarrow \exists \delta(a) > 0$ s.t.

$$\sup_{B_{\delta(a)}(a)} f - \inf_{B_{\delta(a)}(a)} f < \frac{1}{n}$$

And hence $\{R'_j | j \in \mathbb{N}\} \cup \{B_{\delta(a)}(a) | a \in R\backslash D_{1/n}\}$ is an open cover of the cpt. set R. Let $\delta > 0$ be a lebesgue number of this open cover.
(C) Choose a partition Δ of R s.t. for every $R^\Delta_\cdot$ we have $\forall x, x' \in R^\Delta_\cdot \Rightarrow d(x,x') < \delta$. Then $R^\Delta_\cdot \subseteq R'_j$ for some j or $R^\Delta_\cdot \subseteq B_{\delta(a)}(a)$ for some $a \in R\backslash D_{1/n}$ by Theorem **??**. And hence

$$\begin{aligned}
\overline{S}(f,\Delta) - \underline{S}(f,\Delta) &= \sum_{R^\Delta_\cdot} \left(\sup_{R^\Delta_\cdot} f - \inf_{R^\Delta_\cdot} f \right) \cdot vol(R^\Delta_\cdot) \\
&= \sum_{R^\Delta_\cdot \subseteq R'_j} \left(\sup_{R^\Delta_\cdot} f - \inf_{R^\Delta_\cdot} f \right) \cdot vol(R^\Delta_\cdot) \\
&\quad + \sum_{R^\Delta_\cdot \subseteq B_{\delta(a)}(a)} \left(\sup_{R^\Delta_\cdot} f - \inf_{R^\Delta_\cdot} f \right) \cdot vol(R^\Delta_\cdot) \\
&:= (I) + (II).
\end{aligned}$$

Since f is bdd., then $\exists M > 0$, s.t. $\forall x \in R, |f| \leq M \Rightarrow$

$$\begin{aligned}
(I) &= \sum_{R^\Delta_\cdot \subseteq R'_j} \left(\sup_{R^\Delta_\cdot} f - \inf_{R^\Delta_\cdot} f \right) \cdot vol(R^\Delta_\cdot) \\
&\leq \sum_{R^\Delta_\cdot \subseteq R'_j} 2M \cdot vol(R^\Delta_\cdot)
\end{aligned}$$

$$
\begin{aligned}
&\leq 2M \sum_{j \in \mathbb{N}} vol(R'_j) \\
&< 4M\epsilon.
\end{aligned}
$$

And

$$
\begin{aligned}
(II) &= \sum_{R_\cdot^\Delta \subseteq B_{\delta(a)}(a)} \left(\sup_{R_\cdot^\Delta} f - \inf_{R_\cdot^\Delta} f \right) \cdot vol(R_\cdot^\Delta) \\
&\leq \sum_{R_\cdot^\Delta \subseteq B_{\delta(a)}(a)} \left(\sup_{B_{\delta(a)}(a)} f - \inf_{B_{\delta(a)}(a)} f \right) \cdot vol(R_\cdot^\Delta) && (\star) \\
&< \sum_{R_\cdot^\Delta \subseteq B_{\delta(a)}(a)} \frac{1}{n} \cdot vol(R_\cdot^\Delta) \\
&\leq \frac{1}{n} \cdot vol(R).
\end{aligned}
$$

$(\star)$ is since $R_\cdot^\Delta \subseteq B_{\delta(a)}(a)$. In summary, we have

$$
\overline{S}(f, \Delta) - \underline{S}(f, \Delta) \leq 4M\epsilon + \frac{1}{n} \cdot vol(R).
$$

Thus for any $\mu > 0$, select n so large and ϵ so small that $4M\epsilon + \frac{1}{n} \cdot vol(R) < \mu$, then we can form a Δ of R, s.t. $\overline{S}(f, \Delta) - \underline{S}(f, \Delta) < \mu \Rightarrow f$ is integrable on R. $\square$

Remark 1. Recall that Thomae function, the set of disconti. point is $\mathbb{Q}$ which is of measure 0, thus Thomae function is integrable.

2 Convergence and integration

Proposition 1. *Let $[a, b] \xrightarrow{f} \mathbb{R} (n \in \mathbb{N})$ be integrable on $[a, b]$, and $f_n \to f$ uni., show that f is integrable on $[a, b]$, and*

$$
\lim_{n \to \infty} \int_a^b f_n(x) \, \mathrm{d}x = \int_a^b f(x) \, \mathrm{d}x.
$$

Proof. 1. f is integrable: has been proved before.
2. $f_n \xrightarrow{uni.} f \Rightarrow$ for any $\epsilon > 0, \exists N \in \mathbb{N}, \forall x \in [a, b], n \geq N \Rightarrow |f_n(x) - f(x)| < \epsilon$, thus

$$
\begin{aligned}
\left| \int_a^b f_n(x) \, \mathrm{d}x - \int_a^b f(x) \, \mathrm{d}x \right| &= \left| \int_a^b f_n(x) - f(x) \, \mathrm{d}x \right| \\
&\leq \left| \int_a^b |f_n(x) - f(x)| \, \mathrm{d}x \right|
\end{aligned}
$$

$$\leq \epsilon \cdot (b-a)$$

Thus $\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$. $\square$

Proposition 2. *Let $[a,b] \xrightarrow{f} \mathbb{R}(n \in \mathbb{N})$ be C^1 on $[a,b]$, and $f_n' \to g$ uni., and $\exists c \in [a,b]$ s.t. $f_n(c)$ converges as $n \to \infty$. Then*
1. for $\forall x \in [a,b]$, $f_n(x)$ converges to a number $h(x)$ as $n \to \infty$
2. $g(x) = h'(x)$ for $\forall x \in (a,b)$, that is

$$\lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}x} f_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\lim_{n\to\infty} f_n(x)\right).$$

and $h'_+(a) = f(a), h'_-(b) = g(b)$.

Proof. Since f_n' is continuous, then by FTC' we have that

$$f_n(x) = f_n(c) + \int_c^x f_n'(t)\,\mathrm{d}t$$

for all $x \in [a,b]$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} f_n(c) + \lim_{n\to\infty} \int_c^x f_n'(t)\,\mathrm{d}t \\
&= h(c) + \int_c^x g(t)\,\mathrm{d}t \\
&:= h(x).
\end{aligned}$$

And since f_n is conti. $\Rightarrow g$ is conti. then by FTC, we have $h'(x) = g(x)$ and $h'_+(a) = f(a), h'_-(b) = g(b)$. $\square$

Remark 2. These two props are both sufficient conditions.

Corollary 1. *Let $a_n(x)(n \in \mathbb{N})$ be integrable on $[a,b]$, if $\sum_{n=1}^\infty a_n(x)$ converges uniformly, then*

$$\sum_{n=1}^\infty \int_a^b a_n(x)\,\mathrm{d}x = \int_a^b \left(\sum_{n=1}^\infty a_n(x)\right)\mathrm{d}x.$$

Proof. Let $f_n = \sum_{m=1}^n a_m(x)$, then $f_n \xrightarrow{uni.} f = \sum_{m=1}^\infty a_n(x)$. By Proposition 1, we have that

$$\begin{aligned}
\int_a^b \left(\sum_{n=1}^\infty a_n(x)\right)\mathrm{d}x &= \int_a^b f(x)\,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \int_a^b \sum_{m=1}^{n} a_m(x)\,\mathrm{d}x \\
&= \lim_{n\to\infty} \sum_{m=1}^{n} \int_a^b a_m(x)\,\mathrm{d}x \\
&= \sum_{m=1}^{\infty} \int_a^b a_m(x)\,\mathrm{d}x.
\end{aligned}
$$

□

Corollary 2. *If $a_n(x)(n \in \mathbb{N})$ are C^1, $\sum_{n=1}^{\infty} a_n'(x)$ cvg. uni. and $\exists c \in [a,b]$,* s.t. *$\sum_{n=1}^{\infty} a_n(c)$ cvg. then*
1. $\sum_{n=1}^{\infty} a_n(x)$ cvg. for all $x \in [a,b]$;
2.

$$
\sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} a_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=1}^{\infty} a_n(x)\right).
$$

Proof. Let $g_n = \sum_{m=1}^{n} a_m(x)$, then $\exists c \in [a,b]$ s.t. $g_n(c)$ cvg. And $g_n'(x) = \sum_{m=1}^{n} a_m'(x)$ and hence $g_n'(x)$ cvg. uni. and be continuous. Thus

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} a_n(x) &= \sum_{n=1}^{\infty} a_n'(x) \\
&= \lim_{n\to\infty} \sum_{m=1}^{n} a_m'(x) \\
&= \lim_{n\to\infty} g_n'(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\lim_{n\to\infty} g_n(x)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=1}^{\infty} a_n(x)\right).
\end{aligned}
$$

□