Point Set Topology

Lecture 2

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Content:

This is the Lecture note for the *Point Set Topology*.

Topology Space

Definition 1 (Topology Space)**.** A topology space $X = (\underline{X}, \mathscr{T}_X)$ consists of a set $\underline{X}$, called the underlying space of X and a family $\mathscr{T}_X$ of subset of $\underline{X}$ (i.e. $\mathscr{T}_X \subseteq \mathcal{P}(X)$) s.t.

1. $\underline{X}, \emptyset \in \mathscr{T}_X$;
2. $U_\alpha \in \mathscr{T}_X (\alpha \in A) \Rightarrow \cup_{\alpha \in A} U_\alpha \in \mathscr{T}_X$;
3. $U, U' \in \mathscr{T}_X \Rightarrow U \cap U' \in \mathscr{T}_X$.

$\mathscr{T}_X$ is called a topology on $\underline{X}$, the element in $\mathscr{T}_X$ is called the open set on $\underline{X}$ w.r.t. $\mathscr{T}_X$.

Note 1. Conventionally, we usually use X to indicate the set $\underline{X}$ and omit the subscript X in $\mathscr{T}_X$ by saying "a topology space $(X, \mathscr{T})$".

Exercise 1. Let X be a topology space, $U \subseteq X$, show that U is open $\Leftrightarrow$ for any $u \in U$, $\exists O_u \subseteq U$, s.t. $u \in O_u \subseteq_{open} X$.

Proof. $\Rightarrow$: define $O_u := U$ for $\forall u \in U$; $\Leftarrow$: since $O_u \subseteq U$, $\cup_{u \in U} O_u \subseteq U$; on the other hand, for any $v \in U$, $v \in O_v \subseteq \cup_{u \in U} O_u \Rightarrow U \subseteq \cup_{u \in U} O_u$. Thus $U = \cup_{u \in U} O_u \subseteq_{open} X$. □

Definition 2 (Continuous)**.** Let X and Y are top. spaces and $\underline{X} \xrightarrow{f} \underline{Y}$ is a map. We say f is conti. at a point $x_0 \in X$ (from X to Y), if for $\forall f(x_0) \in V \in \mathscr{T}_Y, \exists x \in U \in \mathscr{T}_X$, s.t. $f(U) \subseteq V$.

We say f is continuous (from X to Y) if it is continuous at every point of $\underline{X}$.

Note 2. We will denote $U \in \mathscr{T}_X$ as $U \subseteq_{open} X$, and denote $X \backslash A \subseteq_{open} X$ as $A \subseteq_{close} X$.

Definition 3 (Compact)**.** X is a top. sp. $K \subseteq \underline{X}$. We say K is compact in X if $\forall U_\alpha \subseteq_{open} X (\alpha \in A), K \subseteq \cup_{\alpha \in A} U_\alpha \Rightarrow \exists$ finite set $S \subseteq A$, s.t. $K \subseteq \cup_{\alpha \in S} U_\alpha$, and denote by $K \subseteq_{cpt.} X$. We say X is a compact space if $\underline{X}$ is compact in X.

Definition 4 (Neighborhood)**.** Let X be a top. sp. and $x \in X$. A subset N of X is called a neighborhood of x if $\exists U \subseteq N$, s.t. $x \in U \subseteq_{open} X$.

Exercise 2. $X \xrightarrow{f} Y$ is a map between top. sp., $x_0 \in X$, show that f is conti. at $x_0 \Leftrightarrow \forall$ nbd. V of $f(x_0), \exists$ nbd. U of x_0, s.t. $f(U) \subseteq V \Leftrightarrow \forall$ nbd. V of $f(x_0)$, $f^{-1}(V)$ is a nbd. of x_0.

Proof. 1. $\Rightarrow$: Suppose $V \subseteq Y$ is a nbd. of $f(x_0)$, then $\exists V_0 \subseteq V$, s.t. $f(x_0) \in V_0 \subseteq_{open} Y \Rightarrow \exists U_0 \subseteq_{open} X$, s.t. $x \in U_0$ and $f(U_0) \subseteq V_0$, since f is conti. at x_0. Thus U_0 is the nbd. that we desire.

$\Leftarrow$: For any open set $V_0 \subseteq_{open} Y$ and $f(x_0) \in V_0$, V_0 is a nbd. of $f(x_0)$. Thus $\exists$ a nbd. U of x_0 such that $\exists U_0 \subseteq U, x_0 \in U_0 \subseteq_{open} X$. And $f(U_0) \subseteq f(U) \subseteq V_0$. Thus f is conti.

2. $\Rightarrow$: For any nbd. V of $f(x_0)$, $\exists$ nbd. U of x_0 and $\exists U_0 \subseteq U$, s.t. $x_0 \in U_0 \subseteq_{open} X$ and $f(U) \subseteq V$. Thus $x_0 \in U_0 \subseteq U \subseteq f^{-1}(V)$, that is $U \in f^{-1}(V)$ and $x_0 \in U_0 \subseteq_{open} X$, thus $f^{-1}(V)$ is a nbd. of x_0.

$\Leftarrow$: Trivial. □

Definition 5 (Separation Axioms)**.** Let X be a top. space:

(T_0 or Kolmogorov Space) For any distinct $x, y \in X, \exists U \subseteq_{open} X$, s.t. $x \in U \not\ni y$ or $y \in U \not\ni x$.

(T_1 or Fréchet Space) For any distinct $x, y \in X, \exists U, V \subseteq_{open} X, x \in U \not\ni y$ and $y \in V \not\ni x$.

(T_2 or Hausdorff Space) For any distinct $x, y \in X, \exists U, V \subseteq_{open} X$, s.t. $x \in U, y \in V$ and $U \cap V = \emptyset$.

(T_3 or Regular Space) If X is a T_1 space, and $\forall x \in X, C \subseteq_{close} X, x \notin C \Rightarrow \exists U, V \subseteq_{open} X$, s.t. $x \in U, C \in V$ and $U \cap V = \emptyset$.

(T_4 or Normal Space) If X is a T_1 space, and $\forall C_1, C_2 \subseteq_{close} X, C_1 \cap C_2 = \emptyset \Rightarrow \exists U, V \subseteq_{open} X$, s.t. $C_1 \subseteq U, C_2 \subseteq V$ and $U \cap V = \emptyset$.

Exercise 3. Show that X is a T_1 space $\Leftrightarrow \forall x \in X, \{x\} \subseteq_{close} X$.

Proof. $\Rightarrow$: Given $x \in X$, for any $y \in X \backslash \{x\}$, there exists $U_y \subseteq_{open} X$, s.t. $y \in U_y \not\ni x$. Thus $\cup_{y \in X \backslash \{x\}} U_y \subseteq_{open} X$. If $z \in \cup_{y \in X \backslash \{x\}} U_y$, $\exists y' \in X$, s.t. $z \in U_{y'} \subseteq_{open} X$ and $x \notin U_{y'} \Rightarrow z \neq x \Rightarrow z \in X \backslash \{x\}$. For any $z \in X \backslash \{x\} \Rightarrow \exists U_z \subseteq_{open} X$, s.t. $z \in U_z \not\ni x \Rightarrow z \in \cup_{y \in X \backslash \{x\}} U_y$. Thus $X \backslash \{x\} = \cup_{y \in X \backslash \{x\}} U_y \subseteq_{open} X \Rightarrow \{x\} \subseteq_{close} X$.

$\Leftarrow$: For any distinct $x, y \in X$, $x \in X \backslash \{y\} \subseteq_{open} X$ and $y \in X \backslash \{x\} \subseteq_{open} X$ where $x \notin X \backslash \{x\}$ and $y \notin X \backslash \{y\}$. □

Closure

Definition 6. X is a top. sp., $p \in X$, $A \subseteq X$:

1. p is an interior point of A in X, if $\exists$ nbd. U of p, s.t. $U \subseteq A$;
2. p is an exterior point of A in X, if $\exists$ nbd. U of p, s.t. $U \subseteq X \backslash A$, i.e. $U \cap A = \emptyset$;
3. p is a boundary point of A in X, if $\forall$ nbd. U of p, s.t. $U \cap A \neq \emptyset \neq U \cap (X \backslash A)$;
4. p is an isolated point of A in X, if $\exists$ nbd. U of p, s.t. $U \cap A = \{p\}$;
5. p is a limit point of A in X, if $\forall$ nbd. U of $p, U \cap (A \backslash \{p\}) \neq \emptyset$.

Correspondingly, define

1. $int_X A = A^o := \{$all interior point of A in $X\}$,

2. $ext_X A = A^e :=$ {all exterior point of A in X},
3. $bd_X A = \partial A :=$ {all boundary point of A in X}

It is direct to see

1. $A^o = (X \backslash A)^e$, $A^e = (X \backslash A)^o$ and $\partial A = \partial(X \backslash A)$;
2. $A^o = \cup\{U | U \subseteq A, U \subseteq_{open} X\}$ is the largest open set of X contained in A.
3. $cls_X A = \overline{A} := \cap\{C | A \subseteq C \subseteq_{close} X\}$ is the smallest closed set of X containing A;
4. $\overline{A} = A^o \cup \partial A = X \backslash A^e$;
5. $A \subseteq_{open} X \Leftrightarrow A^o = A$;
6. $A \subseteq_{close} X \Leftrightarrow \overline{A} = A$.

The proies of these statements has been given in *Introduction of Topology, Lecture 12,13*.

Exercise 4. Show that $\partial A \backslash A \subseteq L_A$, where $L_A :=$ {limit points of A in X}.

Proof. $x \in \partial A \backslash A \Rightarrow x \in \partial A$ and $x \notin A \Rightarrow$ for any nbd. U of x, $U \cap A = U \cap (A \backslash \{x\}) = U \cap A \backslash \{x\} \neq \emptyset \Rightarrow x \in L_A$. □

Note 3. In general, $\partial A \not\subseteq L_A$. For example, if x is an isolate point of A, then it is a boundary point of A, but not be the limit point of A.

Exercise 5. Show that $\overline{A} = A \cup L_A$.

Proof 1. 1. $\overline{A} \subseteq A \cup L_A$: If $x \in A \Rightarrow x \in A \cup L_A$; If $x \in \overline{A} \backslash A$: since $x \in \overline{A} = A^o \cup \partial A = X \backslash A^e$, any nbd. U of x has $U \not\subseteq X \backslash A \Rightarrow U \cap A \neq \emptyset$. Since $x \notin A$, $U \cap A = U \cap (A \backslash \{x\}) \neq \emptyset \Rightarrow x \in L_A$.

2. $A \cup L_A \subseteq \overline{A}$: If $x \in A \Rightarrow x \in \overline{A}$; If $x \in L_A \Rightarrow$ any nbd. U of x has $U \cap A \neq \emptyset \Rightarrow x \notin A^e \Rightarrow x \in X \backslash A^e = \overline{A}$. □

Proof 2.
1. $\overline{A} = A^o \cup \partial A = A^o \cup (\partial A \cap A) \cup (\partial A \backslash A)$. If $x \in A^o \cup (\partial A \cap A) \Rightarrow x \in A$; if $x \in \partial A \backslash A \Rightarrow x \in L_A$. Thus $\overline{A} \subseteq A \cup L_A$.
2. If $x \in X \backslash \overline{A} = (X \backslash A)^o$, then $\exists$ a nbd. U of x, s.t. $U \subseteq X \backslash A \Rightarrow U \cap A = \emptyset \Rightarrow x$ is not a limit point of A in $X \Rightarrow x \in X \backslash L_A \Rightarrow X \backslash \overline{A} \subseteq X \backslash L_A \Rightarrow L_A \subseteq \overline{A} \Rightarrow A \cup L_A \subseteq A \cup \overline{A} = \overline{A}$.

□

Note 4. Useful routines:
1. $A \subseteq B \Leftrightarrow X \backslash A \supseteq X \backslash B$
2. $x \notin \overline{A} \Leftrightarrow \exists$ nbd. U of x, s.t. $U \cap A = \emptyset$.

Exercise 6. X is a top. sp., $A_i \subseteq X (i \in I)$, show that

$$\cup_{i \in I} \overline{A_i} \subseteq \overline{\cup_{i \in I} A_i}$$

and

$$\overline{\cap_{i \in I} A_i} \subseteq \cap_{i \in I} \overline{A_i}.$$

Note 5. If I is finite, then $\cup_{i \in I} \overline{A_i} = \overline{\cup_{i \in I} A_i}$.
Since $A_i \subseteq \overline{A_i} \Rightarrow \cup_{i \in I} A_i \subseteq \cup_{i \in I} \overline{A_i} \Rightarrow \overline{\cup_{i \in I} A_i} \subseteq \overline{\cup_{i \in I} \overline{A_i}}$, and since I is finite, $\cup_{i \in I} \overline{A_i}$ is closed, thus $\overline{\cup_{i \in I} A_i} \subseteq \cup_{i \in I} \overline{A_i}$.

Proof. 1. For any $i \in I, A_i \subseteq \cup_{i \in I} A_i \Rightarrow \overline{A_i} \subseteq \overline{\cup_{i \in I} A_i} \Rightarrow \cup_{i \in I} \overline{A_i} \subseteq \overline{\cup_{i \in I} A_i}$.

2. For any $i \in I$, $A_i \subseteq \overline{A_i} \subseteq_{close} X \Rightarrow \cap_{i \in I} A_i \subseteq \cap_{i \in I} \overline{A_i} \subseteq_{close} X \Rightarrow \overline{\cap_{i \in I} A_i} \subseteq \overline{\cap_{i \in I} \overline{A_i}} = \cap_{i \in I} \overline{A_i}$. □

Note that the '=' doer not necessary hold. For example, let $A_r = (1/r, 1-1/r), r > 2$, then $\cup_{r>2} A_r = \cup_{r>2}\overline{A_r} = (0,1) \subsetneq \overline{\cup_{r>2} A_r} = [0,1]$.

Let $B_1 = (0,1/2), B_2 = (1/2,1)$, then $\overline{B_1 \cap B_2} = B_1 \cap B_2 = \emptyset$, but $\overline{B_1} \cap \overline{B_2} = [0,1/2] \cap [1/2,1] = 1/2$.

Definition 7 (Locally Finite)**.** A family $\mathcal{S}$ of some subsets of a top. space X is locally finite if $\forall p \in X, \exists$ nbd. U of p s.t. $\{S \in \mathcal{S} | U \cap S \neq \emptyset\}$ is a finite set.

Exercise 7. If $\mathcal{S}$ is locally finite family, show that

$$\overline{\cup_{S\in\mathcal{S}} S} = \cup_{S\in\mathcal{S}} \overline{S}.$$

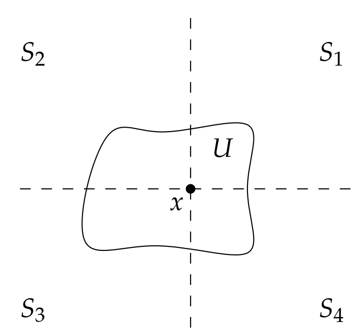
Proof 1. We claim $\overline{\cup_{S\in\mathcal{S}} S} \subseteq \cup_{S\in\mathcal{S}} \overline{S}$, i.e. $\cap_{S\in\mathcal{S}}(X\backslash\overline{S}) = X\backslash \cup_{S\in\mathcal{S}} \overline{S} \subseteq X\backslash\overline{\cup_{S\in\mathcal{S}} S}$. Note that $x \in X\backslash\overline{\cup_{S\in\mathcal{S}} S} \Leftrightarrow \exists$ a nbd. W of x, s.t. $W \cap S = \emptyset$ for $\forall S \in \mathcal{S}$. That is, we want to find a nbd of x such that has no intersection with any S in $\mathcal{S}$, the locally finiteness of $\mathcal{S}$ tells us there exists a nbd. U of x that intersects with only finite sets $S_1, \cdots, S_k \in \mathcal{S}$. Thus all we need to do is eliminate these intersected part from U.

$x \in \cap_{S\in\mathcal{S}}(X\backslash\overline{S}) \Rightarrow x \in X\backslash\overline{S}$ for any $S \in \mathcal{S}$. Thus for any $S \in \mathcal{S}$, $\exists$ a nbd V of x, s.t. $V \cap S = \emptyset$. And $\exists$ a nbd U of x, s.t. U only intersects with finite set $S_1, \cdots, S_k \in \mathcal{S}$. Note that $W := U \cap V_1 \cap \cdots \cap V_k$ is still a nbd. of x, since the finite union of open set is open. And $W \cap S = \emptyset$ for any $S \in \mathcal{S}$, thus for $\exists$ a nbd. W of x, s.t. $W \cap \cup_{S\in\mathcal{S}} S = \emptyset \Rightarrow x \in X\backslash\overline{\cup_{S\in\mathcal{S}} S} \Rightarrow \overline{\cup_{S\in\mathcal{S}} S} \subseteq \cup_{S\in\mathcal{S}}\overline{S}$. □

Note 6. There is no similar feature for the intersection, for example, $S_1 = (0,1)$ and $S_2 = (1,2)$.

Proof 2. Pick $x \notin \bigcup_{S\in\mathcal{S}} \overline{S}$. Due to local finiteness, there is an (open) neighborhood U of x, such that U intersects only finitely many of S: let's say $S_1, S_2, \ldots, S_n$. Now create a new neighborhood $U' = U \setminus (\overline{S_1} \cup \overline{S_2} \cup \cdots \cup \overline{S_n})$, which is an open set containing x, and U' does not intersect any of $S \in \mathcal{S}$. Thus for any $S \in \mathcal{S}, S \subseteq X\backslash U' \Rightarrow \overline{S} \subseteq \overline{X\backslash U'} \xlongequal{X\backslash U' \subseteq_{close} X} X\backslash U'$. Thus U' also does not intersect any of $\overline{S}$.

Thus, for any $x \in X\backslash \bigcup_{S\in\mathcal{S}} \overline{S}, \exists$ an open nbd. U' of x, such that $U' \cap \bigcup_{S\in\mathcal{S}} \overline{S} = \emptyset$. Thus $X\backslash \bigcup_{S\in\mathcal{S}} \overline{S}$ is open, i.e. $\bigcup_{S\in\mathcal{S}} \overline{S}$ is closed. Thus $\bigcup_{S\in\mathcal{S}} S \subseteq \bigcup_{S\in\mathcal{S}} \overline{S} \Rightarrow \overline{\bigcup_{S\in\mathcal{S}} S} \subseteq \overline{\bigcup_{S\in\mathcal{S}} \overline{S}} = \bigcup_{S\in\mathcal{S}} \overline{S}$. □



If $\mathcal{S}$ is locally finite, given a $x \in X$, then $\exists$ a nbd. U of x, s.t. U intersects only finite, such as k, Ss in $\mathcal{S}$. Clearly k has a minimal number, such as 3. Note that it does not imply x is covered by 3 Ss in $\mathcal{S}$.

Exercise 8. Let $X \xrightarrow[conti.]{f} Y, A \subseteq X, B \subseteq Y$, show that:

1. $f^{-1}(\overline{B}) \supseteq \overline{f^{-1}(B)}$; $f(\overline{A}) \subseteq \overline{f(A)}$
2. $f^{-1}(B^o) \subseteq f^{-1}(B)^o$; $f(A^o) \supseteq f(A)^o$.
3. $f^{-1}(B^e) \subseteq f^{-1}(B)^e$; if f is a surjection, $f(A^e) \supseteq f(A)^e$.

4. $f^{-1}(\partial B) \supseteq \partial f^{-1}(B)$; $f(\partial A) \subseteq \partial f(A)$.

Proof. 1. $B \subseteq \overline{B} \Rightarrow f^{-1}(B) \subseteq f^{-1}(\overline{B}) \subseteq_{close} X \Rightarrow \overline{f^{-1}(B)} \subseteq \overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B})$;
$f(A) \subseteq \overline{f(A)} \Rightarrow A \subseteq f^{-1}(\overline{f(A)}) \subseteq_{close} X \Rightarrow \overline{A} \subseteq \overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)}) \Rightarrow f(\overline{A}) \subseteq \overline{f(A)}$.

2. $B^o \subseteq B \Rightarrow X_{\ open} \supseteq f^{-1}(B^o) \subseteq f^{-1}(B) \Rightarrow f^{-1}(B^o) = f^{-1}(B^o)^o \subseteq f^{-1}(B)^o$;
$f(A)^o \subseteq f(A) \Rightarrow f^{-1}(f(A)^o) \subseteq A \Rightarrow f^{-1}(f(A)^o) = f^{-1}(f(A)^o)^o \subseteq A^o \Rightarrow f(A)^o \subseteq f(A^o)$.

3. Since $B^e = (Y \backslash B)^e$,

$$\begin{aligned} f^{-1}(B^e) &= f^{-1}((Y\backslash B)^o) \\ &\subseteq f^{-1}(Y\backslash B)^o \\ &= [f^{-1}(Y)\backslash f^{-1}(B)]^o \\ &= [X\backslash f^{-1}(B)]^o \\ &= f^{-1}(B)^e. \end{aligned}$$

and

$$\begin{aligned} f(A^e) &= f((X\backslash A)^o) \\ &\supseteq f(X\backslash A)^o \\ &\supseteq [f(X)\backslash f(A)]^o \\ &\overset{f \text{ is surj.}}{=\!=\!=} [Y\backslash f(A)]^o \\ &= f(A)^e. \end{aligned}$$

4. Since $\overline{B} = B^o \cup \partial B$,

$$\begin{aligned} &\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \\ &\Rightarrow f^{-1}(B)^o \cup \partial f^{-1}(B) \subseteq f^{-1}(B^o \cup \partial B) \\ &\Rightarrow f^{-1}(B)^o \cup \partial f^{-1}(B) \subseteq f^{-1}(B^o) \cup f^{-1}(\partial B). \end{aligned}$$

since $f^{-1}(B)^o \supseteq f^{-1}(B^o)$, $\partial f^{-1}(B) \subseteq f^{-1}(\partial B)$.
and

$$\begin{aligned} &f(\overline{A}) \subseteq \overline{f(A)} \\ &\Rightarrow f(\partial A) \cup f(A^o) = f(\partial A \cup A^o) \\ &\subseteq \partial f(A) \cup f(A)^o \end{aligned}$$

since $f(A^o) \supseteq f(A)^o$, $f(\partial A) \subseteq \partial f(A)$. □

Note 7. Recall that:

1. $X \xrightarrow{f} Y$ is conti. $\Leftrightarrow$ for any $B \subseteq_{open} Y (\subseteq_{close} Y)$, $f^{-1}(B) \subseteq_{open} X (\subseteq_{close} X)$.
2. $A^o \subseteq A \subseteq \overline{A}$.
3. $A \subseteq_{close} X \Rightarrow \overline{A} = A$; $A \subseteq_{open} X \Rightarrow A^o = A$.

Note 8. $A \subseteq B, A \cup C \supseteq B \cup D \Rightarrow C \supseteq D$.

Proof. $A \cup C \supseteq B \cup D \supseteq A \cup D \Rightarrow A^c \cup A \cup C \supseteq A^c \cup A \cup D \Rightarrow X \cup C \supseteq X \cup D \Rightarrow C \supseteq D$. □