Introduction to Topology

General Topology, Lecture 12,13

Haoming Wang

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Content:

This is the Lecture note for the *Introduction to Topology*. The course covers the following topics: Naive Set Theory, Elementary Number Theory, Group Theory, Topological Spaces and Continuous Maps, Introduction to Algebraic Topology.

Continuous maps and topology space

Definition 1 (Continuous)**.** Let $(X, d_X), (Y, d_Y)$ be metric spaces. $a \in S \subseteq X, f : S \mapsto Y$, we say map f is continuous at a if for $\forall \epsilon > 0, \exists \delta > 0$, for $\forall x \in B_\delta(a) \cap S$, s.t. $f(x) \in B_\epsilon(f(a))$, that is $f(B_\delta(a)) \subseteq B_\epsilon(f(a))$.

We say f is a continuous map if f is continuous at every $a \in S$.

Exercise 1. Given a map $X \xrightarrow{f} Y, a \in X$, Show that

1. f is continuous at $a \Leftrightarrow$ for $\forall V \subseteq_{open} Y$, where $f(a) \in V$, $\exists U \subseteq_{open} X$, where $a \in U$, such that $f(U) \subseteq V$.
2. f is a continuous map $\Leftrightarrow$ for $\forall V \subseteq_{open} Y, f^{-1}(V) \subseteq_{open} X$.

Proof. 1. $\Rightarrow$: for $\forall V \subseteq_{open} Y$, where $f(a) \in V$, $\exists \epsilon > 0$, s.t. $B_\epsilon(f(a)) \subseteq V$, thus $\exists U = B_\delta(a)$. $\Leftarrow$: trivial.

2. $\Leftarrow$: for $\forall a \in S$, □

Exercise 2. Given maps $X \xrightarrow{f} Y, Y \xrightarrow{g} Z$, show that

1. If f is continuous at x_0, g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.
2. If f, g are continuous maps, then $g \circ f$ is a continuous map.

Proof. □

We replaced open ball with open set in Exercise 1, this is a meaningful operation, which means we could **substitute the metric with set** (here is open set), which drives the concept of topology. Generally, topology is a family of sets which have the basic properties of open set, but not necessarily be open sets. Using these sets, we can no longer rely on metric d.

Definition 2 (Topology)**.** Given a set X, we say $\mathscr{T} \subseteq \mathcal{P}(X)$ is a topology on X if

1. $X, \emptyset \in \mathscr{T}$;
2. $U, V \in \mathscr{T} \Rightarrow U \cap V \in \mathscr{T}$;

3. $U_\alpha \in \mathscr{T} (\alpha \in A) \Rightarrow \cup_{\alpha \in A} U_\alpha \in \mathscr{T}$. ($A$ is an arbitrary index set)

And $(X, \mathscr{T})$ is a topology space.