

Introduction to Topology

General Topology, Lecture 12,13

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THIS IS THE LECTURE NOTE FOR THE *Introduction to Topology*. The course covers the following topics: Naive Set Theory, Elementary Number Theory, Group Theory, Topological Spaces and Continuous Maps, Introduction to Algebraic Topology.

CONTENT:

1. Continuous maps and topology space
2. Subspace Topology

Continuous maps and topology space

Definition 1 (Continuous). Let $(X, d_X), (Y, d_Y)$ be metric spaces. $a \in S \subseteq X, f : S \rightarrow Y$, we say map f is continuous at a if for $\forall \epsilon > 0, \exists \delta > 0$, for $\forall x \in B_\delta(a) \cap S$, s.t. $f(x) \in B_\epsilon(f(a))$, that is $f(B_\delta(a)) \subseteq B_\epsilon(f(a))$.

We say f is a continuous map if f is continuous at every $a \in S$.

Exercise 1. Given a map $X \xrightarrow{f} Y, a \in X$, Show that

1. f is continuous at $a \Leftrightarrow$ for $\forall V \subseteq_{\text{open}} Y$, where $f(a) \in V, \exists U \subseteq_{\text{open}} X$, where $a \in U$, such that $f(U) \subseteq V$.
2. f is a continuous map \Leftrightarrow for $\forall V \subseteq_{\text{open}} Y, f^{-1}(V) \subseteq_{\text{open}} X$.

Proof. 1. \Rightarrow : for $\forall V \subseteq_{\text{open}} Y$, where $f(a) \in V, \exists \epsilon > 0$, s.t. $B_\epsilon(f(a)) \subseteq V$, thus $\exists U = B_\delta(a)$. \Leftarrow : trivial.

2. \Rightarrow : Given an open set $V \subseteq_{\text{open}} Y$, for $\forall x \in f^{-1}(V)$, have $f(x) \in V$. Since V is open, $\exists r > 0$ s.t. $B_r(f(x)) \subseteq V$. Since $f(x)$ is continuous map, $\exists \epsilon > 0$, s.t. $f(B_\epsilon(x)) \subseteq B_r(f(x)) \subseteq V \Rightarrow B_\epsilon(x) \in f^{-1}(V)$, thus $f^{-1}(V)$ is open.

\Leftarrow : Given $x \in X, f(x) \in Y$, given $r > 0$, s.t. $B_r(f(x)) \subseteq Y$, then $f^{-1}(B_r(f(x))) \subseteq_{\text{open}} X$, and $x \in f^{-1}(B_r(f(x)))$. Thus $\exists \epsilon > 0$, s.t. $B_\epsilon(x) \subseteq f^{-1}(B_r(f(x)))$ and $f(B_\epsilon(x)) \subseteq B_r(f(x))$. \square

Note 1. It can also be proved that f is cont. \Leftrightarrow for $\forall V \subseteq_{\text{close}} Y, f^{-1}(V) \subseteq_{\text{close}} X$.

Exercise 2. Given maps $X \xrightarrow{f} Y, Y \xrightarrow{g} Z$, show that

1. If f is continuous at x_0, g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0 .
2. If f, g are continuous maps, then $g \circ f$ is a continuous map.

Proof. 1. For any V , s.t. $g(f(x_0)) \in V \subseteq_{\text{open}} Z, \exists U$, s.t. $f(x_0) \in U \subseteq_{\text{open}} Y, \exists W$, s.t. $x_0 \in W \subseteq X$, thus $g \circ f$ is continuous at x_0 .

2. For any $V \subseteq_{\text{open}} Z, \exists U \subseteq_{\text{open}} Y, \exists W \subseteq_{\text{open}} X$, thus $g \circ f$ is continuous. \square

Note 2. Prove this exercise using sets instead of metrics.

We replaced open ball with open set in Exercise 1, this is a meaningful operation, which means we could **substitute the metric with**

set (here is open set), which drives the concept of topology. Generally, topology is a family of sets which have the basic properties of open set, but not necessarily be open sets. Using these sets, we can no longer rely on metric d .

Definition 2 (Topology). Given a set X , we say a family of subsets $\mathcal{T}(\subseteq \mathcal{P}(X))$ is a topology on X if

1. $X, \emptyset \in \mathcal{T}$;
2. $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$;
3. $U_\alpha \in \mathcal{T} (\alpha \in A) \Rightarrow \bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$. (A is an arbitrary index set)

Note 3. From here on, we define the **open sets** as elements in a topology, instead of the previous metric-based definition.

Example 1. Given a set X ,

1. $\mathcal{T} = \{\emptyset, X\}$ is called trivial topology. In this case, we define only X and \emptyset are open sets.
2. Given a metric space (X, d) , the previous definition of open sets is $\mathcal{T}_d = \{U \subseteq \mathcal{P}(X) | \forall x \in U, \exists r > 0, \text{ s.t. } B_r(x) \subseteq U\}$.

Given different metric d , we will obtain different topology. For example, if we use discrete metric, then for $\forall x \in X, \exists r > 0$, such as $r = 0.5$, s.t. $B_r(x) = \{x\} \subseteq \{x\}$, thus $\{x\}$ is an open set. For $\forall U \subseteq X, U = \bigcup \{x | x \in U\}$, thus any subset of X is an open set. In this case, $\mathcal{T} = \mathcal{P}(X)$, and we call it the discrete topology.

Definition 3 (Topological Space). A topological space (X, \mathcal{T}) consists of a set X and a topology \mathcal{T} on X .

Definition 4 (Open set). Let (X, \mathcal{T}) be a topological space, any $A \in \mathcal{T}$ is called an open set in X w.r.t. \mathcal{T} ; and $X \setminus A$ is called a closed set in X w.r.t. \mathcal{T} .

Definition 5. Let (X, \mathcal{T}) be a top. space and $A \subseteq X, x \in X$.

1. x is an interior point of A in X w.r.t. \mathcal{T} , if $\exists U \in \mathcal{T}$, s.t. $x \in U \subseteq A$ (that is $U \cap X \setminus A = U \setminus A = \emptyset$). U is called an open neighborhood of x w.r.t. \mathcal{T} .
2. x is an exterior point of A in X w.r.t. \mathcal{T} , if $\exists U \in \mathcal{T}$, s.t. $x \in U \subseteq X \setminus A$. (i.e. x is an interior of $X \setminus A$).
3. x is a boundary point of A in X w.r.t. \mathcal{T} , if $\forall U \in \mathcal{T}$, if $x \in U$, then $U \cap A \neq \emptyset \wedge U \setminus A \neq \emptyset$.

Note 4. The definition of boundary point is the complementary of interior points union with exterior points.

Definition 6. Let (X, \mathcal{T}) be a top. space and $A \subseteq X$. The set consists of all interior points of A in X w.r.t. \mathcal{T} is called interior (of A in X w.r.t. \mathcal{T}), denote as $\text{int}_X A (= A^\circ)$; the set of all exterior points is called exterior, denoted as $\text{ext}_X A (= A^e)$; and the set of all boundary points is called boundary, denoted as $\text{bdy}_X A (= \partial A)$.

Note 5. Let (X, \mathcal{T}) be a top. space $\forall A \subseteq X, X = A^\circ \cup A^e \cup \partial A$, and $A^\circ, A^e, \partial A$ are disjoint.

A° is the interior of $X \setminus A$, A^e is the exterior of $X \setminus A$, and ∂A is the boundary of $X \setminus A$, which means

$$\begin{aligned} A^\circ &= (X \setminus A)^e \\ A^e &= (X \setminus A)^\circ \\ \partial A &= \partial(X \setminus A). \end{aligned}$$

Example 2. Given a top. space $(\mathbb{R}, \mathcal{T}_d)$, where $d = |x - y|, \forall x, y \in \mathbb{R}$. Let $A = [0, 1]$. Then $A^\circ = (0, 1), A^e = (-\infty, 0) \cup (1, \infty), \partial A = \{0, 1\}$.

Exercise 3. Show that A°, A^e are open sets (on X w.r.t \mathcal{T} , that is $A^\circ, A^e \in \mathcal{T}$); ∂A is close set.

Proof. 1. $\forall x \in A^\circ, \exists U_x \in \mathcal{T}$, s.t. $x \in U_x$, thus $A^\circ = \bigcup_{x \in A^\circ} U_x \in \mathcal{T}$, thus A° is open on X w.r.t. \mathcal{T} .

2. A^e is the interior of $X \setminus A$ by definition, thus A^e is open.

3. $A^\circ, A^e \in \mathcal{T} \Rightarrow A^\circ \cup A^e \in \mathcal{T}$, thus $\partial A = X \setminus (A^\circ \cup A^e) \in \mathcal{T}$. \square

Exercise 4. Given a topology space (X, \mathcal{T}) , $A \subseteq X$, show that

$$A^\circ = \bigcup \{U \mid U \subseteq_{\text{open}} A\}.$$

Proof. \subseteq : for $\forall x \in A^\circ, \exists U \in \mathcal{T}$, s.t. $x \in U \subseteq A \Rightarrow x \in \bigcup \{U \mid U \subseteq_{\text{open}} A\}$; \supseteq : for $\forall x \in \bigcup \{U \mid U \subseteq_{\text{open}} A\}, \exists U_x \subseteq_{\text{open}} A$, s.t. $x \in U_x$, thus x is an interior point, and $x \in A^\circ$. \square

Definition 7 (Closure). Given a topology space (X, \mathcal{T}) , $A \subseteq X$, the set

$$\overline{A} = \text{cls}_X A := \bigcap \{C \mid A \subseteq C \subseteq_{\text{close}} X\}$$

is called the closure of A in X w.r.t. \mathcal{T} .

Exercise 5. Given a topology space (X, \mathcal{T}) , $A \subseteq X$, show that $\overline{A} = A^\circ \cup \partial A$.

Proof.

$$\begin{aligned} A^\circ \cup \partial A &= X \setminus A^e \\ &= X \setminus (X \setminus A)^\circ \\ &= X \setminus \bigcup \{U \mid U \subseteq_{\text{open}} X \setminus A\} \\ &= \bigcap \{X \setminus U \mid U \subseteq_{\text{open}} X \setminus A\} \\ &= \bigcap \{C \mid A \subseteq C \subseteq_{\text{close}} X\} \\ &= \overline{A}. \end{aligned}$$

Note 6. A° is the largest open set in X contained in A . Thus,

$$A = A^\circ \Leftrightarrow A \subseteq_{\text{open}} X \Leftrightarrow \partial A \cap A = \emptyset$$

for $\partial A \cap A = \partial A \cap A^\circ = \emptyset$. And furthermore $(A^\circ)^\circ = A^\circ$.

Note 7. \overline{A} is the smallest close set in X containing in A . Thus,

$$A = \overline{A} \Leftrightarrow A \subseteq_{\text{close}} X \Leftrightarrow \partial A \subseteq A$$

for $\partial A \subseteq A^\circ \cup \partial A = \overline{A} = A$. And furthermore $\overline{\overline{A}} = \overline{A}$.

Note 8.

$$\begin{aligned} U &\subseteq X \setminus A \\ &\Rightarrow \forall x \in U \Rightarrow x \in X \setminus A \\ &\Rightarrow \forall x \notin X \setminus A \Rightarrow x \notin U \\ &\Rightarrow \forall x \in A \Rightarrow x \in X \setminus U \\ &\Rightarrow A \subseteq X \setminus U, \end{aligned}$$

U is open $\Rightarrow X \setminus U$ is close, hence $C = X \setminus U \subseteq_{\text{close}} A$.

Exercise 6. Show that $X \setminus \overline{A} = (X \setminus A)^\circ$ and $X \setminus A^\circ = \overline{(X \setminus A)}$.

Proof. 1.

$$\begin{aligned} \overline{A} &= A^\circ \cup \partial A \\ &= X \setminus A^e \\ &= X \setminus (X \setminus A)^\circ \\ X \setminus \overline{A} &= (X \setminus A)^\circ. \end{aligned}$$

2.

$$\begin{aligned} X \setminus A^\circ &= A^e \cup \partial A \\ &= (X \setminus A)^c \cup \partial(X \setminus A) \\ &= \overline{(X \setminus A)}. \end{aligned}$$

Note 9. We denote $X \setminus A$ as A^c if X is clearly given. Thus

$$\begin{aligned} (\overline{A})^c &= (A^c)^\circ \\ (A^\circ)^c &= \overline{A^c} \end{aligned}$$

\square

Exercise 7. If $A \subseteq B$, show that $A^\circ \subseteq B^\circ$, $\overline{A} \subseteq \overline{B}$.

Proof. 1. Given $x \in A^\circ = \cup\{U \mid U \subseteq_{\text{open}} A\}$, $\exists U_x \subseteq_{\text{open}} A$, s.t. $x \in U_x \subseteq_{\text{open}} A \subseteq B$, thus $x \in \cup\{V \mid V \subseteq_{\text{open}} B\}$, and $x \in B^\circ$. 2. the same way with 1. \square

Exercise 8. Given a set U , (denote \overline{U} as U^-), show that $U \subseteq_{\text{open}} X \Rightarrow U^- = U^{-c-c-}$.

Proof.

$$\begin{aligned} U^{-c-c-} &= (U^-)^{c-c-} \\ &= (U^-)^{\circ c c -} \\ &= U^{-\circ -} \end{aligned}$$

$U \subseteq U^- \Rightarrow U = U^\circ \subseteq U^{-\circ} \Rightarrow U^- \subseteq U^{-\circ-}$. Let $C = U^- \subseteq_{\text{close}} X$, thus $C^\circ \subseteq C \Rightarrow C^{\circ-} \subseteq C^- = C \Rightarrow U^{-\circ-} \subseteq U^-$, thus $U^- = U^{-\circ-} = U^{-c-c-}$. \square

Exercise 9 (Kuratowski's 14 sets). Given a top. sp. X , $A \subseteq X$, Show that among

$$\begin{aligned} A, A^-, A^{-c}, A^{-c-}, A^{-c-c} \dots \\ A^c, A^{c-}, A^{c-c}, A^{c-c-} \dots \end{aligned}$$

there are at most 14 different subsets of A .

Proof. On the one hand,

$$A, A^-, \underbrace{A^{-c}}_{\text{open}}, A^{-c-}, A^{-c-c}, A^{-c-c-}, A^{-c-c-c}, \underbrace{A^{-c-c-c-}}_{=(A^{-c})^{-c-c-}}, \dots$$

On the other hand,

$$A^c, A^{c-}, \underbrace{A^{c-c}}_{\text{open}}, A^{c-c-}, A^{c-c-c}, A^{c-c-c-}, \underbrace{A^{c-c-c-c-}}_{=(A^{c-c})^{-c-c-}}, \dots$$

thus there are at most 14 different subsets of A . \square

Definition 8 (Continuous Map). Let X, Y be top. spaces. A map

$X \xrightarrow{f} Y$ is continuous at a point $x_0 \in X$ if \forall open neighborhood (nbd.) V of $f(x_0)$, \exists open nbd. U of x_0 , s.t. $f(U) \subseteq V$. f is a continuous map, if f is continuous at every $x_0 \in X$.

Note 10. We have discussed that f is conti. \Leftrightarrow for $\forall V \subseteq_{\text{open}} Y, f^{-1}(V) \subseteq_{\text{open}} X \Leftrightarrow$ for $\forall V \subseteq_{\text{close}} Y, f^{-1}(V) \subseteq_{\text{close}} X$.

Exercise 10. Let X, Y be top. spaces, $X \xrightarrow{f} Y$ is a conti. map, show that $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$, and $f(\overline{A}) \subseteq \overline{f(A)}$.

Proof. 1. $B \subseteq \overline{B} \Rightarrow f^{-1}(B) \subseteq f^{-1}(\overline{B})$ where $f^{-1}(\overline{B})$ is close, thus $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) = f^{-1}(\overline{B})$.

2. $f(A) \subseteq \overline{f(A)} \Rightarrow A \subseteq f^{-1}(\overline{f(A)})$ where $f^{-1}(\overline{f(A)})$ is close, thus $\overline{A} \subseteq f^{-1}(\overline{f(A)}) \subseteq f^{-1}(\overline{f(A)}) \Rightarrow f(\overline{A}) \subseteq \overline{f(A)}$. \square

Note 11. $f(A) \subseteq B \Rightarrow A \subseteq f^{-1}(B)$ by the definition of pre-image.

Subspace Topology

Let X be a top. space and $A \subseteq X$. A top. space is a set which has been specified some subsets are open but the others are not. Not consider how to transform a subset A into a top. space in a reasonable way. And the issue is that what kind of subsets of A should be defined as open.

Consider the inclusion map $A \xrightarrow{i} X$ where $a \mapsto a$. Thus an intuitive motivation is we need select open sets in the top. space of A such that keep i is continuous. Because, for any point a in the codomain of i , if \exists an open set $U \in X$, such that covers a , then it covers the pre-image of a (in the top. space of X), since $i^{-1}(a) = a \in U$. So if any point $a \in X$ has an open nbd. U then it's pre-image should have an open nbd. U_A , otherwise the subspace top. would be too simple or wried to show the inheritance of the "sub".

Thus we wish create a corresponding open set U_A of U in the top. space of A , thus for any point in the codomain, if it has open nbd. in the top. space of X , then it's pre-image has open nbd. in the top. space of A , and i is continuous. Specially, if we define $\mathcal{T}_A = \mathcal{P}(A)$, that is discrete topology, then any point forms an open set, thus i is continuous. But we want to find the concisest situation that fits the demand. The concisest way to construct topology of A is selecting the pre-image of the open sets in X , that is for any $U \in \mathcal{T}_X$, $i^{-1}(U) = U \cap A \in \mathcal{T}_A$.

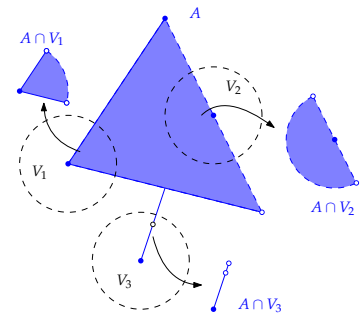
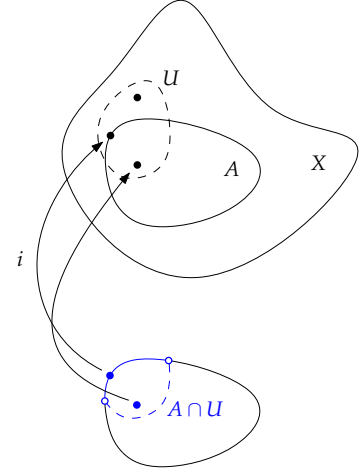
1. $\emptyset \in \mathcal{T}_X \Rightarrow \emptyset \cap A = \emptyset \in \mathcal{T}_A$, $X \in \mathcal{T}_X \Rightarrow X \cap A = A \in \mathcal{T}_A$.
2. $\forall U_1, U_2 \in X, U_1 \cap U_2 \in X$, thus $(U_1 \cap A) \cap (U_2 \cap A) = (U_1 \cap U_2) \cap A \in \mathcal{T}_A$.
3. $\forall U_\alpha \in X (\alpha \in I), \cup_{\alpha \in I} U_\alpha \in X$, thus $\cup_{\alpha \in I} (U_\alpha \cap A) = A \cap (\cup_{\alpha \in I} U_\alpha) \in A$.

thus $\{U \cap A \mid \forall U \subseteq_{\text{open}} X\}$ is a topology which is the smallest topology that satisfies our demand.

Definition 9. The subspace topology on A inherited from X is $\mathcal{T}_A = \{U \cap A \mid U \subseteq_{\text{open}} X\}$.

Example 3. Given a top. space $(\mathbb{R}^2, \mathcal{T}_d)$ where $d = d_2$, a subset A of X like the margin figure. we can see that the elements of \mathcal{T}_A : $A \cap V_1$, $A \cap V_2$ and $A \cap V_3$ are all open sets on (A, \mathcal{T}_A) , even though they are not open sets on $(\mathbb{R}^2, \mathcal{T}_d)$.

Exercise 11. Given a map $X \xrightarrow{f} Y$, X, Y are top. spaces. Suppose $\exists B \subseteq Y$ is a subspace top. inherited from Y . If $f(X) \subseteq B$, we denote the map $X \xrightarrow{f} B$ by $f|_B$. Show that f is continuous $\Leftrightarrow f|_B$ is continuous.



Proof. \Rightarrow : f is conti. then $\forall V \subseteq_{\text{open}} Y$ has $f^{-1}(V) \subseteq_{\text{open}} X$, and $V \cap B \subseteq_{\text{open}} B$. Since:

$$\begin{aligned} f^{-1}(V \cap B) &= f^{-1}(V) \cap f^{-1}(B) \\ &= f^{-1}(V) \cap X \\ &= f^{-1}(V) \subseteq_{\text{open}} X \end{aligned}$$

thus $f|_B$ is conti.

\Leftarrow : $\forall V \subseteq_{\text{open}} Y, f^{-1}(V \cap B) = f^{-1}(V) \cap f^{-1}(B) = f^{-1}(V) \cap X = f^{-1}(V) \subseteq_{\text{open}} X$. Thus f is conti. \square