General Topology

Lecture 8

Haoming Wang

16 May 2020

This is the Lecture note for the *General Topology* course in Spring 2020.

1 Net

Let X be set, then a sequence $x_n (n \in \mathbb{N})$ in X is such a map $\mathbb{N} \xrightarrow{x.} X$ (denote $x(n)$ by x_n). Now we gonna generalize this concept.

Definition 1 (Directed Set)**.** A directed set $(D, \geq)$ consists of a non-empty set D and a relation $\geq$ on D s.t.

1. $\forall d \in D, d \geq d$;
2. $\forall d, d', d'' \in D, d \geq d', d' \geq d'' \Rightarrow d \geq d''$,

i.e. $(D, \geq)$ is pre-order. And $\forall d, d' \in D, \exists d'' \in D,$ s.t. $d'' \geq d, d'' \geq d'$.

Remark 1. Note that the pre-order is not total order, which means there could exist $d_1, d_2 \in D$ which are not comparable. On the other hand, the pre-order is not partial order yet, which means it does not require $d \geq d' \wedge d' \geq d \Rightarrow d = d'$. Thus the following statement in a directed set may hold: $\exists d_1, d_2, d_3, d_4 \in D$ such that

$$d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_1,$$

but $d_1 \neq d_2 \neq d_3 \neq d_4$.

Example 1. Let X be a topology space, $x \in X$, $D = \{\text{all open nbd.s of } x\}$ and for any $U, V \in D$ define $U \geq V \Leftrightarrow U \subseteq V$, then $(D, \geq)$ is a directed set. (Since for any $U, V \in D$, $\exists W := U \cap V \in D,$ s.t. $W \geq U, W \geq V$)

Definition 2 (Net)**.** Let X be a set, a net $(D, \geq) \xrightarrow{x.} X$, ($x_\alpha (\alpha \in D)$ for short,) in X consists of a directed set $(D, \geq)$ and a map $D \xrightarrow{x.} X$.

Suppose that a net $x.$ ($x_\alpha(\alpha \in D)$) is a net in a set X, and $S \subseteq X$, we say that $x.$ lies in S

- **eventually** if $\exists \delta \in D, \forall \alpha \in D, \alpha \geq \delta \Rightarrow x_\alpha \in S$;
- **frequently** if $\forall \delta \in D, \exists \alpha \in D,$ s.t. $\alpha \geq \delta$ and $x_\alpha \in S$.

Remark 2. $\neg(x.$ lies in S eventually$) \Leftrightarrow x.$ lies in $X \backslash S$ frequently.

Definition 3 (Convergence)**.** Let X be a topology space, $x_\alpha(\alpha \in D)$ is a net in X, $x \in X$. We say that $x.$ converges x (or say x is a limit of $x.$) if $\forall$ open nbd. U of x in X, $x.$ lies in U eventually.

Exercise 1. *Show that X is a Hausdorff space $\Leftrightarrow$ every net has at most one limit.*

Proof. $\Rightarrow$: Suppose a net $D \xrightarrow{x.} X$ converges to x and y in X and $x \neq y$, then $\exists$ open nbd.s U of x and V of y, s.t. $x \in U, y \in V$ and $U \cap V = \emptyset$. Since $x. \to x$ then $\exists \delta_x \in D,$ s.t. $\forall \alpha \in D, \alpha \geq \delta_x \Rightarrow x_\alpha \in U$. And since $x. \to y, \exists \delta_y \in D,$ s.t. $\forall \alpha \in D, \alpha \geq \delta_y \Rightarrow x_\alpha \in V$. Then $\exists \delta \in D,$ s.t. $\delta \geq \delta_x \wedge \delta \geq \delta_y$, thus for $\forall \alpha \in D, \alpha \geq \delta$ has $x_\alpha \in U \subseteq X \backslash V$ and $x_\alpha \in V$ which leads to a contradiction.
$\Leftarrow$: Suppose X is not a Hausdorff space, then $\exists x, y \in X,$ s.t. $\forall$ open nbd.s U of x, V of y has $U \cap V \neq \emptyset$. Thus we can form a net in X.
Define $D = \{U \cap V | x \in U \subseteq_{open} X, y \in V \subseteq_{open} X\}$ and $\forall d_1, d_2 \in D, d_1 \geq d_2 \Leftrightarrow d1 \subseteq d_2$, it is direct to see $(D, \geq)$ is a directed set. And then $D \xrightarrow{x.} X$ where $d \mapsto x_d \in d$ is a net (since $\forall d \in D, d \neq \emptyset$, and hence $x_d \exists$).
Thus given any open nbd. W of x, $W \cap V \in D$ where D is a open nbd. of y, then $\forall \alpha \in D, \alpha \geq W \cap V$ we have

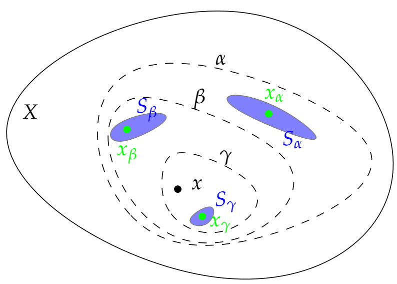
$$x_\alpha \in \alpha \subseteq W \cap V \subseteq W$$

thus $x.$ lies in any open nbd. W of x eventually, and hence $x.$ converges to x. Thus $x.$ converges to y as well in the same way, which leads to a contradiction. $\square$

Remark 3 (A naturally convergent net). If X is a set, $x \in X$, if we define the directed set as $D = \{U | x \in U \subseteq_{open} X\}$ and $\geq \Leftrightarrow \subseteq$, then $(D, \geq)$ is a directed set. And define the net $D \xrightarrow{x.} X$, where $\alpha \mapsto x_\alpha \in S_\alpha \subseteq \alpha$. Then for any open nbd. U of x, $U \in D$ and $\forall \alpha \in D, \alpha \geq U$ has

$$x_\alpha \in S_\alpha \subseteq \alpha \subseteq U.$$

Thus such $x.$ converges to x naturally.



Exercise 2. *Let X be a topology space, $A \subseteq X$, define*

$$\overline{A}'' := \{x \in X | \exists \text{ net } a. \text{ in } A \text{ converging to } x\}$$

and

$$L_A'' := \{x \in X | \exists \text{ net } a. \text{ in } A \backslash \{x\} \text{ converging to } x\}$$

show that $\overline{A} = \overline{A}''$ and $L_A = L_A''$.

Proof. 1. $\subseteq$: if $x \in \overline{A}$, then any open nbd. U of x has $U \cap A \neq \emptyset$, thus we can form a net. Define $D = \{U | x \in U \subseteq_{open} X\}$ and $\geq \Leftrightarrow \subseteq$ then $(D, \geq)$ is a directed set and $D \xrightarrow{x.} A$ where $d \mapsto x_d \in d \cap A$ is a net. And $x.$ converges to $x \Rightarrow x \in \overline{A}''$ by *Remark* 2.
$\supseteq$: if $x \in \overline{A}''$, then $\exists$ a net $D \xrightarrow{x.} A$ s.t. for $\forall$ open nbd. U of x, $\exists \delta \in D$ s.t. $\forall \alpha \in D, \alpha \geq \delta \Rightarrow \alpha \in U$, then $U \cap A \neq \emptyset \Rightarrow x \in \overline{A}$.
2. the same as above. □

Exercise 3. *Let $X \xrightarrow{f} Y$ be a map between topology spaces, $x_0 \in X$, show that f is continuous at $x_0 \Leftrightarrow$ for $\forall$ net $D \xrightarrow{x.} X$ in X that converges to x_0, $f(x.)$ is a net in Y converges to $f(x_0)$.*

Proof. $\Rightarrow$: if V is an open nbd. of $f(x_0)$, since f is continuous, $f^{-1}(V)$ is an open nbd. of x_0, then $\exists \delta \in D$, s.t. $\forall \alpha \in D, \alpha \geq \delta \Rightarrow x_\alpha \in f^{-1}(V) \Rightarrow f(x_\alpha) \in V \Rightarrow f(x.)$ converges to $f(x_0)$.
$\Leftarrow$: suppose f is not continuous at x_0, then $\exists$ an open nbd. V of $f(x_0)$, $f^{-1}(V)$ is not an open nbd. of x_0, that is $x_0 \notin (f^{-1}(V))^o$, since $x_0 \in f^{-1}(V)$, $x_0 \in f^{-1}(V) \backslash (f^{-1}(V))^o = \partial f^{-1}(V)$. Thus any open nbd. U of x has $U \cap f^{-1}(V) \neq \emptyset$ and $U \cap X \backslash f^{-1}(V) \neq \emptyset$, and hence we can form a net.
Define $D = \{U | x \in U \subseteq_{open} X\}$ and $\geq \Leftrightarrow \subseteq$ then $(D, \geq)$ is a directed set, and define a net $D \xrightarrow{x.} X \backslash f^{-1}(V)$ where $\alpha \mapsto x_\alpha \in \alpha \cap X \backslash f^{-1}(V)$, then $x.$ converges to x by *Remark* 2, and hence $f(x.)$ converges to $f(x_0)$ by assumptions, which means $\exists \delta \in D$, s.t. $\forall \alpha \in D, \alpha \geq \delta \Rightarrow f(x_\alpha) \in V$ which leads to a contradiction with $f(x_\alpha) \in X \backslash f^{-1}(V)$. □

Note 1. $f(x.)$ is a net in Y:

$$D \xrightarrow{x.} X \xrightarrow{f} Y$$

2 Subnet

Recall that given a sequence $x_n (n \in \mathbb{N})$ in a set X, a subsequence $x_{n_k} (k \in \mathbb{N})$ is composite map

$$\mathbb{N} \xrightarrow{n.} \mathbb{N} \xrightarrow{x.} X$$

(denote $x(n(k))$ as x_{n_k}), where $\mathbb{N} \xrightarrow{n.} \mathbb{N}$ is a monotone injection. We now want to generalize this conception.

Definition 4 (Final Map)**.** Let $(D, \geq)$ and $(D', \geq')$ be directed sets, a map $D' \xrightarrow{h} D$ is a final map (w.r.t. $\geq$ and $\geq'$) if $\forall \delta \in D, \exists \delta' \in D',$ s.t. $\forall \alpha \in D', \alpha \geq \delta' \Rightarrow h(\alpha) \geq \delta$.

Note 2. Final map analogizes the monotones of $\mathbb{N} \xrightarrow{n.} \mathbb{N}$. Final map require the `tail` of the map is `monotones`.

Definition 5. Let $D' \xrightarrow{h} D$ is a final map between directed sets, net $x_{h(\cdot)}$:

$$\begin{tikzcd} D' \arrow[r, "h"'] \arrow[rr, bend left, "x_{h(\cdot)}"] & D \arrow[r, "x."'] & X \end{tikzcd}$$

is called a subnet of x.

Exercise 4. *If a net $x.$ converges to x_0 show that the subnet $x_{h(\cdot)}$ converges to x_0 as well.*

Proof. For any open nbd. U of x_0, $\exists \delta \in D,$ s.t. $\forall \alpha \in D, \alpha \geq \delta \Rightarrow x_\alpha \in U \Rightarrow \exists \delta' \in D', \forall \alpha' \geq \delta', h(\alpha') \geq \delta \Rightarrow x_{h(\alpha')} \in U \Rightarrow x_{h(\cdot)}$ converges to x_0. $\square$

Exercise 5. *Let X be a set, $x.$ is a net in X, $S \subseteq X$. Show that $x.$ lies in S frequently $\Leftrightarrow \exists$ subnet of $x.$ lies in S eventually.*

Proof. $\Rightarrow$: $D \xrightarrow{x.} X$ lies in S frequently, then $\forall \delta \in D, \exists \alpha_\delta \in D,$ s.t. $\alpha_\delta \geq \delta$ and $x_{\alpha_\delta} \in S$. Then we can for a final map $D \xrightarrow{h} D$ where $\delta \mapsto \alpha_\delta$. Thus for any $\alpha_\delta \in D, \exists \alpha_\delta \in D,$ s.t. $\forall \alpha \in D, \alpha \geq \alpha_\delta \Rightarrow \alpha \geq \alpha_\delta$, thus h is a final map, and $x_{h(\cdot)}$ is a subnet of $x.$ and for any $\alpha \in D, x_{h(\alpha)} = x_{\alpha_\delta} \in S \Rightarrow x_{h(\cdot)}$ lies in S eventually.
$\Leftarrow$: if $D \xrightarrow{x.} X$ has an subnet $D' \xrightarrow{x_{h(\cdot)}} X$ which lies in S eventually. Then $\exists \beta \in D',$ s.t. $\forall \alpha' \in D', \alpha' \geq \beta \Rightarrow x_{h(\alpha')} \in S$. On the other hand, $\forall \delta \in D, \exists \delta' \in D',$ s.t. $\forall \alpha' \in D', \alpha' \geq \delta' \Rightarrow h(\alpha') \geq \delta$. Since D' is directed set, $\exists \gamma \in D',$ s.t. $\gamma \geq \beta$ and $\gamma \geq \delta'$, then $h(\gamma) \geq \delta$ and $x_{h(\gamma)} \in S$.
Collectively, $\forall \delta \in D, \exists h(\gamma) \in D,$ s.t. $h(\gamma) \geq \delta$ and $x_{h(\gamma)} \in S \Rightarrow x.$ lies in S frequently. $\square$

Definition 6 (Universal Net)**.** A net $x.$ in a set X is universal if $\forall A \subseteq X$ either $x.$ lies in A eventually or $x.$ lies in $X \backslash A$ eventually.

Exercise 6. *$X \xrightarrow{f} Y$ is a map, show that $x.$ is a universal net in $X \Rightarrow f(x.)$ is universal net in Y.*

Proof. For any $B \subseteq Y, f^{-1}(B) \subseteq X$, since $D \xrightarrow{x.} X$ is a universal net, $x.$ lies in $f^{-1}(B)$ eventually or $X \backslash f^{-1}(B)$.
If $x.$ lies in $f^{-1}(B)$ eventually, $\exists \delta \in D,$ s.t. $\forall \alpha \in D, \alpha \geq \delta, \Rightarrow x_\alpha \in f^{-1}(B) \Rightarrow f(x_\alpha) \in B$;
If $x.$ lies in $X \backslash f^{-1}(B)$ eventually, $\exists \delta \in D,$ s.t. $\forall \alpha \in D, \alpha \geq \delta, \Rightarrow x_\alpha \in X \backslash f^{-1}(B) = f^{-1}(Y) \backslash f^{-1}(B) = f^{-1}(Y \backslash B) \Rightarrow f(x_\alpha) \in Y \backslash B$. Thus $f(x.)$ is a universal net in Y. $\square$

Exercise 7. *Show that every subnet of a universal net is universal.*

Proof. Suppose $D \xrightarrow{x.} X$ is a universal net in X which has a subnet $D' \xrightarrow{x_{h(\cdot)}} X$. And for any $A \subseteq X$, $x.$ lies in A or $X \backslash A$ eventually. Suppose $x.$ lies in A, then $\exists \delta \in D$, s.t. $\forall \alpha \in D, \alpha \geq \delta \Rightarrow x_\alpha \in A$. On the other hand, $\exists \delta' \in D'$, s.t. $\forall \alpha' \in D', \alpha' \geq \delta' \Rightarrow h(\alpha') \geq \delta \Rightarrow x_{h(\alpha')} \in A \Rightarrow x_{h(\cdot)}$ lies in A eventually as well $\Rightarrow x_{h(\cdot)}$ is universal. □

Theorem 1. *Every net has a universal subnet.*

Proof. Let $(D, \geq_D) \xrightarrow{x.} X$ be a net in a set X, where $(D, \geq_D)$ is a directed set.

1. Define Y as the family of some families $\mathcal{A}(\subseteq \mathcal{P}(X))$ of subsets of X such that
   (a) $\forall A \in \mathcal{A}$, $x.$ lies in A frequently;
   (b) $\forall A_1, A_2 \in \mathcal{A}, A_1 \cap A_2 \in \mathcal{A}$.

   That is the element of Y is the family of subsets of X that satisfies the above conditions. Thus $Y \neq \emptyset$ (since $\{X\} \in Y$) and $(Y, \subseteq)$ is a poset. We now apply Zorn's lemma to get a maximal element of Y.

   Let C be a chain in Y w.r.t. $\subseteq$. Then we claim that $\cup_{\mathcal{A} \in C} \mathcal{A} \in Y$ and is an upper bound of C.
   (a) For any $A \in \cup_{\mathcal{A} \in C} \mathcal{A}$ there $\exists \mathcal{A}' \in C$, s.t. $A \in \mathcal{A}'$, thus $x.$ lies in A eventually;
   (b) For any $A_1, A_2 \in \mathcal{A}$ there $\exists \mathcal{A}_1, \mathcal{A}_2 \in C$, s.t. $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$ and $\mathcal{A}_1$ is comparable with $\mathcal{A}_2$ w.r.t. $\subseteq$, for example $\mathcal{A}_1 \subseteq \mathcal{A}_2$, then $A_1, A_2 \in \mathcal{A}_2 \Rightarrow A_1 \cap A_2 \in \mathcal{A}_2 \subseteq \cup_{\mathcal{A} \in C} \mathcal{A}$.

   Thus $\exists$ maximal element $\mathcal{A}_0$ of Y.
2. Let $D_0 := \{(A, \alpha) \in \mathcal{A}_0 \times D | x_\alpha \in A\}$ with the pre-order $\geq_0$ on D_0: $(A', \alpha') \geq_0 (A, \alpha) \Leftrightarrow A' \subseteq A$ and $\alpha' \geq_D \alpha$. Since
   (a) For any $A \in \mathcal{A}_{\prime}, \alpha \in D, A \subseteq A$ and $\alpha \geq_D \alpha \Rightarrow (A, \alpha) \geq_0 (A, \alpha)$;
   (b) For any $(A_1, \alpha_1), (A_2, \alpha_2), (A_3, \alpha_3) \in D_0$, $(A_1, \alpha_1) \geq_0 (A_2, \alpha_2)$ and $(A_2, \alpha_2) \geq_0 (A_3, \alpha_3)$ means
   $$\alpha_1 \geq_D \alpha_2 \geq_D \alpha_3$$
   and
   $$A_1 \subseteq A_2 \subseteq A_3$$
   thus $(A_1, \alpha_1) \geq_0 (A_3, \alpha_3)$
   (c) For any $(A_1, \alpha_1), (A_2, \alpha_2) \in D_0, \mathcal{A}_0 \ni A_1 \cap A_2 \subseteq A_1$ and A_2; and $\exists \alpha' \geq_D \alpha_1$ and $\alpha_2 \Rightarrow D_0 \ni (A_1 \cap A_2, \alpha') \geq_0 (A_1, \alpha_1)$ and (A_2, α_2).

   Thus $(D_0, \geq_0)$ is a directed set.
3. And then we can define a final map $D_0 \xrightarrow{h} D$ where $(A, \alpha) \mapsto \alpha$. Given $\delta \in D$, for any $A \in \mathcal{A}_0$, since $x.$ lies in A frequently, $\exists \alpha \in D$, s.t. $\alpha \geq \delta$ and $x_\alpha \in A$, and hence $(A, \alpha) \in D_0$. For any $(A', \alpha') \geq_0 (A, \alpha)$, we have that $h((A', \alpha')) = \alpha' \geq \alpha \geq \delta$, thus h is a final map.

In particular, we donate the subnet of $x.$, i.e. the composite of $D_0 \xrightarrow{h} D \xrightarrow{x.} X$ as $D_0 \xrightarrow{y.:=x.\circ h} X$ where $(A,\alpha) \mapsto x_\alpha = y_{(A,\alpha)}$.

4. Let $S \subseteq X$, we will show that the subnet $y.$ is universal: if $\neg$ ($y.$ lies in $X\backslash S$ eventually) $\Leftrightarrow$ ($y.$ lies in S frequently) then we will show that it implies $y.$ lies in S eventually.

   For any $A \in \mathcal{A}_0$, $x.$ lies in A frequently $\Rightarrow$ for any $\delta \in D$, there exists $\alpha \in D$, s.t. $\alpha \geq_D \delta$ and $x_\alpha \in A$ and hence $(A,\alpha) \in D_0$. And since $y.$ lies in S frequently, $\exists (A_1,\alpha_1) \in D_0$, s.t. $(A_1,\alpha_1) \geq_0 (A,\alpha)$, (i.e. $A_1 \subseteq A$ and $\alpha_1 \geq_D \alpha_0$) and $y_{(A_1,\alpha_1)} \in S$. And $y_{(A_1,\alpha_1)} = x_{\alpha_1} \in A_1$ since $(A_1,\alpha_1) \in D_0$. Thus

   $$x_{\alpha_1} = y_{(A_1,\alpha_1)} \in S \cap A_1 \subseteq S \cap A$$

   thus $x.$ lies in $S \cap A$ frequently for any $A \in \mathcal{A}_0$ and thus $x.$ lies in S frequently, thus we have that

   $$\mathcal{A}_0 \cup \{S \cap A | A \in \mathcal{A}_0\} \cup \{S\} \in Y$$

   by the definition of Y, and since $\mathcal{A}_0$ is the maximal element of $Y \Rightarrow S \in \mathcal{A}_0$.

   If $\neg$ ($y.$ lies in S eventually) holds, then $y.$ lies in $X\backslash S$ frequently holds $\Rightarrow X\backslash S \in \mathcal{A}_0$, thus $S, X\backslash S \in \mathcal{A}_0 \Rightarrow \emptyset = S \cap (X\backslash S) \in \mathcal{A}_0$, which leads to a contradiction with $x.$ lies in it frequently.

$\square$

Note 3. Thus we have a corollary: if $x.$ is a universal net in X, $S \subseteq X$, then $\neg$ ($x.$ lies in S eventually) $\Rightarrow$ $x.$ lies in $X\backslash S$ eventually.

3 Net and Compactness

Proposition 1. *Let X be a topology space, the following are equivalent:*

1. *X is a compact space;*
2. *$\forall$ family $\mathcal{F}$ of closed subsets of X, $\mathcal{F}$ has FIP $\Leftrightarrow \cap\mathcal{F} \neq \emptyset$;*
3. *$\forall$ universal net in X converges;*
4. *$\forall$ net in X has a convergent subnet.*

Proof. We will prove this in order $1 \Rightarrow 3 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$.

$1 \Rightarrow 3$: Suppose that $x_\alpha (\alpha \in D)$ is a universal net in X which does not converge to any $x \in X$, thus $\exists$ open nbd. U_x of x in X s.t. $\neg$ ($x.$ lies in U_x eventually) $\Rightarrow$ $x.$ lies in $X\backslash U_x$ frequently. Since $X = \cup_{x\in X} U_x$ and X is compact, there $\exists$ finite $X_0 \subseteq X$, s.t. $X = \cup_{x\in X_0} U_x \Rightarrow \emptyset = \cup_{x\in X_0}(X\backslash U_x)$ which leads to a contradiction with $x.$ lines in $X\backslash U_x$ frequently.

$3 \Rightarrow 4$: $\forall$ net in X has a universal subnet and it is convergent by 3.

$4 \Rightarrow 2$: Let $\mathcal{F}$ be a family of closed subsets of X which has FIP, we can **expand** it as $\mathcal{F}' := \{\cap_{i=1}^m F_i | m \in \mathbb{N}, F_i \in \mathcal{F}, i = 1, \cdots, m\}$. Note that there are 3 facts for $\mathcal{F}'$:

1. $\mathcal{F}'$ also has FIP;
   since finite intersection of $\mathcal{F}'$ is a finite intersection of $\mathcal{F}$;
2. $\cap\mathcal{F}' = \cap\mathcal{F}$;
   since for any $c \in \cap\mathcal{F}' \Rightarrow c \in$ every finite intersection of $\mathcal{F} \Rightarrow c \in \cap_{F\in\{F\}} F = F$ for $\forall F \in \mathcal{F} \Rightarrow c \in \cap\mathcal{F}$. On the contrary, for any $c \in \cap\mathcal{F} \Rightarrow c \in F$ for any $F \in \mathcal{F} \Rightarrow c \in \mathcal{F}'$.
3. $\mathcal{F}'$ is closed under $\cap$.

It is direct to see that $(\mathcal{F}', \geq')$ with $\geq' := \subseteq$ is a directed set. For any $C \in \mathcal{F}'$, (it is finite intersection of $\mathcal{F}$ and hence $C \neq \emptyset$,) choose $x_C \in C$ and form a net $\mathcal{F}' \xrightarrow{x.} X$ where $C \mapsto x_C$.
By 4, net $x.$ has a convergent subnet, that is $\exists$ a final map $D \xrightarrow{h} \mathcal{F}'$ for some directed set $(D, \geq_D)$, s.t. subnet $D \xrightarrow{y.} X$ (where $\alpha \mapsto x_{h(\alpha)} = y_\alpha$) converges to some point $x \in X$.
Since h is finial, $\forall C \in \mathcal{F}', \exists \alpha \in D, \forall \beta \in D, \beta \geq_D \alpha \Rightarrow h(\beta) \geq C \Leftrightarrow h(\beta) \subseteq C$ and thus

$$y_\beta = x_{h(\beta)} \in h(\beta) \subseteq C$$

thus $y.$ lies in C eventually. For any $C \in \mathcal{F}', y.$ converges to $x \Rightarrow x \in C$ since C is closed, thus $x \in \cap_{C\in\mathcal{F}'} C = \cap\mathcal{F} \Rightarrow \mathcal{F} \neq \emptyset$.
$2 \Rightarrow 1$: has been given in *Point Set Topology Lecture 6*. ☐

Remark 4. $1 \Rightarrow 3$: A common routine to utilize the compactness of X: find an open nbd. U_x for any $x \in X$, and then $X = \cup_{x\in X} U_x$.
$4 \Rightarrow 2$: The key to form a net is to find some sets $\neq \emptyset$.

Lemma 1. *Let $X_j (j \in J)$ be a family of topology spaces and $D \xrightarrow{x.} \prod_{j\in J} X_j$ where $\alpha \mapsto x_\alpha = (x_{\alpha_j})_{j\in J}$ be a net. There are groups of corresponding projective nets $D \xrightarrow{x_{\cdot j}} X_j$ where $\alpha \mapsto x_{\alpha_j}$ for $j \in J$.*
Then $x.$ converges to x in $\prod_{j\in J} X_j$ (equipped with the product topology) $\Leftrightarrow \forall j \in J, x_{\cdot j}$ converges x_j in X_j where $x_j = \pi_j(x)$.

Proof. $\Rightarrow$: Since $\prod_{j\in J} X_j \xrightarrow{\pi_k} X_k$ where $(x_j)_{j\in J} \mapsto x_k$ is continuous and $x_{\cdot k} = \pi_k(x.)$, then $x. \to x \in \prod_{j\in J} X_j \Rightarrow x_{\cdot k} = \pi_k(x.) \to \pi_k(x) = x_k$.
$\Leftarrow$: Recall that $\mathcal{B} := \{\prod_{j\in J} Y_j | Y_j \subseteq_{open} X_j (j \in J) \wedge \{j \in J | Y_j \neq X_j\}$ is finite$\}$ is a basis of the product space $\prod_{j\in J} X_j$. For any open nbd. U of x, there exists $\prod_{j\in J} Y_j \in \mathcal{B}$ s.t.

$$x \in \prod_{j\in J} Y_j \subseteq U$$

Let $J_0 = \{j \in J | Y_j \subsetneq X_j\}$, which is a finite set. $x_{\cdot j}$ converges to $x_j \in X_j \Rightarrow x_{\cdot j}$ lies in Y_j eventually i.e. $\exists \alpha_j \in D$, s.t. $\forall \alpha \in D, \alpha \geq \alpha_j \Rightarrow x_{\alpha_j} \in Y_j$ for all $j \in J_0$.
Choose $\tilde{\alpha} \in D$, s.t. $\tilde{\alpha} \geq \alpha_j$ for all $j \in J_0$, then for $D \ni \alpha \geq \tilde{\alpha}, x_{\alpha_j} \in Y_j$ for all $j \in J_0$ and hence for all $j \in J$. ☐

Theorem 2 (Tychonoff Theorem). *For compact space $X_j (j \in J)$ the product space $\prod_{j \in J} X_j =: X$ is also compact.*

Proof. Let $x.$ be a universal net in X, then $x_{\cdot j} = \pi_j(x.)$ is a universal net in X_j, for every $j \in J \Rightarrow x_{\cdot j}$ converges in X_j since X_j is compact $\Rightarrow x.$ converges by Lemma $\Rightarrow X$ is compact. $\square$