General Topology
Lecture 9

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24 May 2020

This is the Lecture note for the *General Topology* course in Spring 2020.

1 Generalization of Ascoli's Theorem

Recall that for metric spaces X and Y, a family $\mathcal{F}$ of maps from X to Y (i.e. Y - valued functions on X) is **equicontinuous** at a point $x_0 \in X$ if $\forall \epsilon > 0, \exists \delta > 0, \forall f \in \mathcal{F}$ and $x \in X$, $d(x_0, x) < \delta \Rightarrow d(f(x_0), f(x)) < \epsilon$. We now generalize this concept.

Definition 1 (Equicontinuous)**.** Let X be a topology space and Y be a metric space, a family $\mathcal{F}$ of maps from X to Y is equicontinuous at a point $x_0 \in X$ if $\forall \epsilon > 0, \exists$ open nbd. U of x_0 s.t. $\forall f \in \mathcal{F}$ and $x \in X$, $x \in U \Rightarrow d(f(x_0), f(x)) < \epsilon$.

Definition 2 (Point-wise convergence)**.** Let X, Y be metric spaces, and $X \xrightarrow{f_n} Y (n \in \mathbb{N})$ is a sequence of functions, then f_n converges point-wise to $X \xrightarrow{f} Y$ if for $\forall x \in X$ one has $f_n(x) \to f(x)$ as $n \to \infty$.

Definition 3 (Uniform convergence)**.** Let X, Y be metric spaces, a sequence of functions $X \xrightarrow{f_n} Y (n \in \mathbb{N})$ converges uniformly to $X \xrightarrow{f} Y$ if for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that for $\forall n \geq N$ and $\forall x \in X$ one has $d(f_n(x), f(x)) < \epsilon$.

Definition 4 (Compact convergence)**.** Let $(X, \mathscr{T})$ be a topological space and (Y, d_Y) be a metric space. A sequence of functions $X \xrightarrow{f_n} Y (n \in \mathbb{N})$ is said to converge compactly to some function $X \xrightarrow{f} Y$ if, for every compact set $K \subseteq X$, $f_n|_K \to f|_K$ uniformly.

Theorem 1 (A generalization of Ascoli's theorem)**.** *Let X be a topology space and $\mathcal{F}$ be a family of $\mathbb{R}$ - valued functions on X, if*

1. *X is separable;*
2. *$\mathcal{F}$ is equicontinuous for $\forall x \in X$;*
3. *for $\forall x \in X$, $\{f(x) | f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{R}$,*

then every seq. in $\mathcal{F}$ has a subseq. which converges compactly, i.e. uniformly on every compact subset of X.

Proof. Let $A = \{a_1, a_2, \cdots\}$ be a countable dense subset, suppose $f_n (n \in \mathbb{N})$ is a seq. in $\mathcal{F}$.

Claim 1: $\exists$ subseq. $f_{n_m}(m \in \mathbb{N})$ which converges point-wise on A:

For $a_1 \in A$, we have that $\{f_n(a_1) | n \in \mathbb{N}\} \subseteq \{f(a_1) | f \in \mathcal{F}\} \subseteq_{bdd.} \mathbb{R}$. Then by Bolzano-Weierstrass theorem, there exists a $n_m^{(1)}(m \in \mathbb{N})$, which is strictly monotone, such that $f_{n_\cdot^{(1)}}(a_1)$ converges. Inductively, we can construct $n_m^{(j)}(m \in \mathbb{N})(j \in \mathbb{N}_0)$, and let $n_m^{(0)} = m$, such that

1. $n_\cdot^{(j)}$ monotone strictly;
2. $\{n_m^{(j)} | m \in \mathbb{N}\} \subseteq \{n_m^{(j-1)} | m \in \mathbb{N}\}$;
3. $f_{n_m^{(j)}}(a_j)$ converges as $m \to \infty$.

Let $n_m := n_m^{(m)}(m \in \mathbb{N})$, then $f_{n_m}(m = k, k+1, \cdots)$ is a subseq. of $f_{n_m^{(k)}}(m \in \mathbb{N})$ and hence $f_{n_m}(a_k)$ converges as $m \to \infty$ for every $k \in \mathbb{N}$.

Remark 1. For instance, $f_{n_m^{(2)}}$ is a subseq. of $f_{n_m^{(1)}}$ and $f_{n_m^{(1)}}(a_1)$ converges hence $f_{n_m^{(2)}}(a_1)$ converges as well. Thus $f_{n_m^{(2)}}(a_1)$ and $f_{n_m^{(2)}}(a_2)$ both converge.

Since given $j \in \mathbb{N}$, the `tail` of seq. $f_{n_m} = f_{n_m^{(m)}}$ is subseq. of $f_{n_m^{(j)}}$, for example, $f_{n_m}(m = 3, 4, \cdots)$ is subseq. of $f_{n_m^{(3)}}(m \in \mathbb{N})$, thus $f_{n_m}(a_j)$ converges for all $j \in \mathbb{N}$.

$$
\begin{array}{cccccc}
 & a_1 & a_2 & a_3 & a_4 & \cdots \\
f_1 & f_{n_1^{(1)}} & f_{n_1^{(2)}} & f_{n_1^{(3)}} & f_{n_1^{(4)}} & \cdots \\
f_2 & f_{n_2^{(1)}} & f_{n_2^{(2)}} & f_{n_2^{(3)}} & f_{n_2^{(4)}} & \cdots \\
f_3 & f_{n_3^{(1)}} & f_{n_3^{(2)}} & f_{n_3^{(3)}} & f_{n_3^{(4)}} & \cdots \\
f_4 & f_{n_4^{(1)}} & f_{n_4^{(2)}} & f_{n_4^{(3)}} & f_{n_4^{(4)}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Figure 1: $f_m(m \in \mathbb{N}), a_j(j \in \mathbb{N})$

Claim 2: $\forall \epsilon > 0$ and $x \in X, \exists$ (open) nbd. U_x of x in X and a number $N_x > 0$ s.t. if $x' \in U_x$ and $k, l \geq N_x \Rightarrow |f_{n_k}(x') - f_{n_l}(x')| < \epsilon$.

Since $\mathcal{F}$ is equicontinuous at x, thus for $\forall \epsilon > 0, \exists$ (open) nbd. U_x of x, s.t. $|f(z) - f(x)| < \epsilon/6$ for $f \in \mathcal{F}, z \in U_x$. Since $A \subseteq_{dense} X, \exists a \in U_x \cap A$. For any $x' \in U_x$ we have that

$$\begin{aligned}|f_{n_k}(x') - f_{n_l}(x')| &\leq |f_{n_k}(x') - f_{n_k}(x)| \\ &+ |f_{n_k}(x) - f_{n_k}(a)| \\ &+ |f_{n_k}(a) - f_{n_l}(a)| \\ &+ |f_{n_l}(a) - f_{n_l}(x)| \\ &+ |f_{n_l}(x) - f_{n_l}(x')| \\ &< |f_{n_k}(a) - f_{n_l}(a)| + \frac{2}{3}\epsilon.\end{aligned}$$

since $f_{n.}(a)$ converges $\Rightarrow \exists N_x > 0$, s.t. $\forall k, l \geq N \Rightarrow |f_{n_k}(a) - f_{n_l}(a)| < \epsilon/3 \Rightarrow |f_{n_k}(x') - f_{n_l}(x')| < \epsilon$.

Claim 3: $\forall K \subseteq_{cpt.} X, f_{n_m}|_K (m \in \mathbb{N})$ converges uniformly.

For any given $\epsilon > 0$, we have found U_x and N_x as in Claim 2, $K = \cup_{x \in K}\{x\} \subseteq \cup_{x \in K} U_x, K \subseteq_{cpt.} X \Rightarrow \exists x_1, \cdots, x_p$, s.t. $K \subseteq U_{x_1} \cup \cdots \cup U_{x_p}$. Let $N = \max\{N_{x_1}, \cdots, N_{x_p}\}$, then for any $q \in K$ and $k, l \geq N$ we have $|f_{n_k}(q) - f_{n_l}(q)| < \epsilon$. Thus $f_{n_m}(q)(m \in \mathbb{N})$ is a Cauchy seq. in compact set $K \Rightarrow f_{n_m}(q) \to f(q)$ as $m \to \infty$, thus $f_{n.}$ converges uniformly in K. □

Remark 2. The condition (3) is equivalent to that $\mathcal{F}$ is uniformly bounded when X is assumed to be compact and $\mathcal{F}$ is equicontinuous everywhere.

2 Relatively Compact

Definition 5. Let X be a topology space and $A \subseteq X$, A is relatively compact if $\overline{A}$ is compact.

Example 1. Every subset of a compact subset of a Hausdorff space is relatively compact: Suppose X is Hausdorff, $Y \subseteq_{cpt.} X \Rightarrow Y \subseteq_{close} X$. For $\forall Z \subseteq Y, \overline{Z} \subseteq \overline{Y} = Y$. And since $\overline{Z} \subseteq_{close} Y$, Z is compact $\Rightarrow \overline{Z}$ is compact.

Exercise 1. *Let (X, d) is a metric space, $A \subseteq X$, show that A is rel. cpt. $\Leftrightarrow$ any seq. in A has a subseq. which converges in X.*

Proof. $\Rightarrow$: A is rel. cpt. $\Rightarrow \overline{A}$ is cpt. $\Leftrightarrow \overline{A}$ is sequential compact $\Rightarrow$ every seq. in $\overline{A}$ converges $\Rightarrow$ every seq. in A converges in $\overline{A}$ (or in X).

$\Leftarrow$: Suppose that $\overline{A}$ is not compact then there is a seq. $\{a_n\}$ in $\overline{A}$ which is not convergent. So then for each $n \in \mathbb{N}$ define $A_n := A \cap B_{\frac{1}{n}}(a_n) \neq \emptyset$. Then pick a b_n from each A_n so that $\{b_n\}$ is a sequence in A, where for any $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that for any $n > N$,

$$d(a_n, b_n) < 1/n < \epsilon.$$

Then $\{b_n\}$ has a convergent subseq. $\{b_{n_k}\}$ with limit b by assumption. Thus for any $\epsilon > 0$, there exists a $K \in \mathbb{N}$ such that for all $k > K$,

$$d(a_{n_k}, b) \leq d(a_{n_k}, b_{n_k}) + d(b_{n_k, b}) < \epsilon/2 + \epsilon/2,$$

where $\{a_{n_k}\}$ is the corresponding subseq. of $\{a_n\}$, thus $a_{n_k} \to b$ as $k \to \infty$, implying $\overline{A}$ is seq. cpt. and hence cpt. and contradicting the supposition. $\square$

Remark 3. At first glance, the definition of A being **rel. cpt.** appears to be the same as **seq. cpt.** (which is equivalent to cpt. in metric space), but there is a difference:
the subseq. are required to converge in X (or $\overline{A}$ since it is closed), not necessarily in A, while actual seq. cpt. does require it to be in A. (more)

Example 2. Let X be a compact topology space, $C(X, \mathbb{R}) := \{X \xrightarrow{f} \mathbb{R} | f \text{ if continuous}\}$ and $d_{\text{sup}} := \sup_{x \in X} |f(x) - g(x)|$ ($= \max_{x \in X} |f(x) - g(x)|$ since X is compact)
Then $(C(X, \mathbb{R}), d_{\text{sup}})$ is a complete metric space, and $f_n (n \in \mathbb{N})$ converges w.r.t. $d_{\text{sup}} \Leftrightarrow$ f_n converges uniformly on $X \Leftrightarrow f_n$ is uniformly Cauchy seq.
By the generalization of Ascoli's theorem, when X is compact and separable, $\mathcal{F} \subseteq C(X, \mathbb{R})$ is equiconti. and uniformly bdd. (or satisfies condition 3.) $\Rightarrow \mathcal{F}$ is rel. cpt. by Ex1.