General Topology

Lecture 1

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1 Some Definitions

Definition 1 (Partial Order)**.** Given a set X, a relation $\leq$ on X is a partial order if

1. $\forall x \in X \Rightarrow x \leq x$;
2. $\forall x, x' \in X, x \leq x', x' \leq x \Rightarrow x = x'$;
3. $\forall x, x', x'' \in X, x \leq x', x' \leq x'' \Rightarrow x \leq x''$.

We say that $(X, \leq)$ is a partially ordered set (poset).

Note 1. A relation on X, is a subset of $X \times X$.

Example 1. For example, $\leq$ is a partial order on $\mathbb{T}$; given a set X, $\subseteq$ is a partial on $\mathcal{P}(X)$.

If $(X, \leq)$ is a poset and $A \subseteq X$, then A has a natural partial order induced by $\leq$.

Definition 2 (Total Order, Chain)**.** A poset $(X, \leq)$ is a chain (or totally order set) if $\forall x, x' \in X$, then $x \leq x'$ or $x' \leq x$.

If $(X, \leq)$ is a poset, $A \subseteq X, b \in X$, we say

1. b is an upper (lower) bound of A (in X w.r.t. $\leq$) if $\forall a \in A, a \leq b(b \leq a)$, denoted the set of upper (lower) bound of A by $U_A(L_A)$.
2. b is a greatest (least) element of A (in X w.r.t. $\leq$), if b is an upper (lower) bound of A and $b \in A$.
3. b is the least upper bound (greatest lower bound) of A, if b is the least (greatest) element of the set of upper bound (lower bound) of A, denoted by lub or sup A (glb or inf A).
4. b is a maximal (minimal) element in X if $b \in X, \forall x \in X, b \leq x \Rightarrow b = x(x \leq b \Rightarrow x = b)$.

Note 2 (Maximal vs. Greatest). An element $m \in X$ is **maximal** if there does not exist $x \in X$ such that $x > m$. An element $g \in X$ is **greatest** if for all $x \in X$, $g \geq x$.

1. A set may have no greatest elements and no maximal elements: for example, any open interval of real numbers.
2. If a set has a greatest element, that element is also maximal.
3. A set with two maximal elements and no greatest element: $X = \{a, b, c\}$, where $a \leq b, a \leq c$ and b and c are incomparable, then each of b and b are maximal, and none of the elements of this set are greatest.
4. A set can have exactly one maximal element but no greatest element: $X = \{a + q | 0 \leq q < 1\} \cup \{c\}$, where $a \leq c$ and $a + q$ and c are incomparable for any $0 \leq q < 1$. Then only c is maximal, and the set overall has no greatest element.

Definition 3 (Well Order)**.** If $(X, \leq)$ is a chain, we say that $(X, \leq)$ is a well-ordered set if $\forall A \subseteq X, A \neq \emptyset \Rightarrow A$ has a least element.

For example, $\mathbb{Z}^+$ is a well-ordered set. If $(X, \leq)$ is a well-ordered set, for any $a \in X$, the **successor** of a is $succ_{(X,\leq)}(a) :=$ the least element of $\{x \in X | a < x\}$. So if $\{x \in X | a < x\} \neq \emptyset$, then $succ_{(X,\leq)}(a)$ exists.

Note 3. Given a poset X, $a, b \in X$, we say $a < b$ if $a \leq b$ and $a \neq b$.

Definition 4. Given a poset X, $a \in X$, define initial segment as

$$IS_{(X,\leq)}(a) := \{x \in X | x < a\}$$

and weak initial segment as

$$WIS_{(X,\leq)}(a) := \{x \in X | x \leq a\}.$$

2 Axiom of Choice

Theorem 1 (Bourbaki's fixed point theorem)**.** *Suppose $(X, \leq)$ is a poset, in which every well-ordered subset has lub. Given a map $X \xrightarrow{f} X$,* s.t. *$x \leq f(x)$ for $\forall x \in X$, then $\exists a \in X$,* s.t. *$f(a) = a$.*

Proof. Pick an element $x_0 \in X$. Let S be the collection of subsets $Y \subseteq X$ such that:

- Y is well ordered with the least element x_0 and successor function $f|_{Y \setminus lubY}$,
- $x_0 \neq y \in Y \Rightarrow lub_X(IS_Y(y)) \in Y$.

Then we claim:

1. If $Y \in S$ and $Y' \in S$, then Y is an initial segment of Y' or vice versa.
   Let $V = \{x \in Y \cap Y' | WIS_Y(x) = WIS_{Y'}(x)\}$. Suppose first that V has a last element v. If v is not the last element of Y, then $succ_Y(v) = f(v)$; if v is not the

last element of Y' then $succ_{Y'}(v) = f(v)$. Hence if neither of Y, Y' is an initial segment of the other, then $succ_Y(v) = succ_{Y'}(v) = f(v) \in V$, thus $f(v) = v$, and v is the fixed point.

If V has no last element, let $z = lub_X(V)$. If $Y \neq V \neq Y'$, then it follows that $z \in Y \cap Y'$ (because if $y = \inf(Y - V)$ then $V = IS_Y(y)$ and therefore $z = lub_X(IS_Y(y)) \in Y$). Therefore $z \in V$, which is a contradiction.

2. The set $Y_0 = \cup\{Y | Y \in S\} \in S$.

   If $y_0 \in Y \in S$, then it follows from 1. that $\{y \in Y_0 | y < y_0\} = IS_Y(y_0)$ and so this subset is well ordered with successor function f. This implies that Y_0 is well ordered and satisfies first conditions of element in S. Also $lub_X(IS(y_0)) \in Y \subseteq Y_0$ which gives the second condition for Y_0. Thus 2. is proved.

Let $y_0 = lub_X(Y_0)$, if $y_0 \notin Y_0$ then $Y_0 \cup \{y_0\} \in S$ and so $y_0 \in Y_0$ after all. If $f(y_0) > y_0$ then $Y_0 \cup \{f(y_0)\} \in S$ contrary to the definition of Y_0, thus $f(y_0) = y_0$ as desired. □

Note 4. A map $X \xrightarrow{f} Y$ is a subset $\Gamma \subseteq X \times Y$, s.t. $\forall x \in X, \exists! y \in Y, (x, y) \in \Gamma$.

Theorem 2. *The following statement are equivalent:*

1. *For $\forall$ set X, $\exists$ map $\mathcal{P}_o(X) \xrightarrow{f} X$,* s.t. *$\forall S \in \mathcal{P}_o(X), f(S) \in S$. ($\mathcal{P}_o(X) := \{A | A | \subseteq X, A \neq \emptyset\}$)*
2. *If $(X, \leq)$ is a poset, in which every well-ordered subset has a lub in X, then X has a maximal element.*
3. *(Maximal Chain Theorem) $\forall$ poset $(X, \leq)$ has a maximal chain w.r.t $\subseteq$. i.e. a chain such that there is no other chain in $(X, \leq)$ which has it as a proper subset.*
4. *(Zorn's Lemma) If $(X, \leq)$ is a poset in which every chain has an upper bound in X then X has a maximal element.*
5. *(Zermelo's Well-Ordering Theorem) Every set has a well-order.*
6. *$\forall$ surj. $X \xrightarrow{f} Y$, $\exists$ an injection $Y \xrightarrow{g} X$,* s.t. *$f \circ g = id_Y$.*
7. *(Axiom of Choice) Given non-empty sets $S_\alpha (\alpha \in A)$, there exists a map $A \xrightarrow{f} \cup_{\alpha \in A} S_\alpha$,* s.t. *$f(\alpha) \in S_\alpha$.*

Proof. $7 \Rightarrow 1$: We can number each non-empty subset of X by itself, since any element in a set is unique. That is $\mathcal{P}_o(X) = \{S_\alpha := \alpha | \alpha \in \mathcal{P}_o(X)\}$, here $\mathcal{P}_o(X)$ serves as A. Thus Axiom of Choice means $\exists$ a map $\mathcal{P}_o(X) \xrightarrow{f} X$, s.t. $f(\alpha) \in S_\alpha = \alpha (\alpha \in \mathcal{P}_o(X))$. (we emphasize $\mathcal{P}_o(X)$, rather than $\mathcal{P}(X)$, because there is nothing in $\emptyset$)

Note 5. Statement 1 claims that given a set X, any non-empty subset of X can be maps to a point inside this subset.

$1 \Rightarrow 2$: Assume that X has no maximal element, i.e. $\forall a \in X, X_a := \{x \in X | a < x\} \neq \emptyset$. $\exists$ map $\mathcal{P}_o(X) \xrightarrow{f} X$, s.t. $f(S) \in S$ for all $S \in \mathcal{P}_o(X)$. Define a map $X \xrightarrow{\pi} \mathcal{P}_o(X) (a \mapsto X_a)$

and $X \xrightarrow{g=f\circ\pi} X$. Thus for any $a \in X$, $g(a) = f(X_a) \in X_a$, thus $a < g(a)$, which leads to a contradiction with Bourbaki's fixed point theorem.

$$\begin{tikzcd} \mathcal{P}_o(X) \arrow[r, "f"] & X \\ X \arrow[u, "\pi"] \arrow[ur, "g"'] & \end{tikzcd}$$

$2 \Rightarrow 3$: Given a poset $(X, \leq)$ consider $S = \{C | C \text{ is a chain in } P\ w.r.t.\ \leq\}$. Thus $(S, \subseteq)$ is a poset. We claim that any totally ordered set in S has a lub in S. If $T \subseteq S$ is a totally ordered set, (that is T is a chain w.r.t $\subseteq$ of the chains w.r.t. $\leq$), then $\cup_{C\in T} C = lub_S T$. To show this, we need prove 2 things:

1. $\cup_{C\in T} C \in U_T$;
   For any $C \in T$, $C \subseteq \cup_{C\in T} C$, thus $\cup_{C\in T} C \in U_T$.
2. $\cup_{C\in T} C \in L_{U_T}$.
   For any $v \in \cup_{C\in T} C, O \in U_T$, $\exists C \in T$, s.t. $v \in C \subseteq O$. Thus $\cup_{C\in T} C \subseteq O$, thus $\cup_{C\in T} C \in L_{U_T}$.

Thus every totally ordered subset (including well order subset) of $(S, \subseteq)$ has a lub, and $(S, \subseteq)$ has a maximal element, which implies $(X, \leq)$ has a maximal chain.

Note 6. $(T, \subseteq)$ is a chain, thus any comparison with the element in T need to use relation $\subseteq$.

$3 \Rightarrow 4$: Given a poset $(X, \leq)$, it has a max. chain C, by assumption, C has an upper bound, say a, in X. Then a is a max. element in X, otherwise $\exists x \in X, a < x$, and hence $C \subsetneq C \cup \{x\}$ and $C \cup \{x\}$ is a chain, which leads to a contradiction to the maximality of C.

$4 \Rightarrow 5$: Let Y be a set, consider $X := \{A | A = (S_A, \leq_A)$ where $S_A \subseteq Y$ and $\leq_A$ is a well-ordering on $S_A\}$. We define a relation $\preceq$ on X: $A \preceq A' \Leftrightarrow A = A'$ or A is an initial segment of A' (i.e. $\exists a' \in S_{A'}, S_A = \{x \in S_{A'} | x <_{A'} a'\}$) and $\forall x_1, x_2 \in S_A, x_1 \leq_A x_2 \Leftrightarrow x_1 \leq_{A'} x_2$.

It is direct to see that $(X, \preceq)$ is a poset:

1. For any $A \in X, A \preceq A$;
2. If A is initial segment of A' then $A \neq A'$, since if $\exists a' \in S_{A'}, S_A = \{x \in S_{A'} | x <_{A'} a'\}$ then $a' \in A'$ but $a' \notin A$. Thus $A \preceq A', A' \preceq A \Rightarrow A = A'$
3. Suppose that $A \preceq A' \preceq A''$, and A, A' and A'' are not equal. Thus $\exists a'' \in S_{A''}$, s.t. $S_{A'} = IS_{A''}(a'')$, and $\exists a' \in S_{A'}$, s.t. $S_A = IS_{A'}(a')$. Since $a' <_{A''} a''$, any $a \in S_{A''}, a <_{A''} a' \Rightarrow a \in S_{A'}$. Thus $IS_{A''}(a) = \{x \in S_{A''} | x <_{A''} a'\} = \{x \in S_{A'} | x <_{A''} a'\} = \{x \in S_{A'} | x <_{A'} a'\} = IS_{A'}(a') = A$, thus $A \preceq A''$.

Then, we claim:

1. $(X, \preceq)$ has a maximal element:

Apply Zorn's lemma, let $(C, \preceq)$ be a chain on $(X, \preceq)$. Let $A_0 = (S_{A_0}, \leq_{A_0})$ where $S_{A_0} = \cup_{A \in C} S_A$, and $\leq_{A_0}$: for any $x_1, x_2 \in S_{A_0}$, find $A \in C$, s.t. $x_1, x_2 \in S_A$, we say that $x_1 \leq_{A_0} x_2$ if $x_1 \leq_A x_2$. Then we claim:

- Such A exists:
  For any $x_1, x_2 \in S_{A_0}, \exists A_1, A_2 \in C$, s.t. $x_1 \in S_{A_1}, x_2 \in S_{A_2}$ and S_{A_1} and S_{A_2} are comparable on X w.r.t. $\preceq$, since C is a chain. Assume that S_{A_1} is an initial segment of S_{A_2}, then $x_1, x_2 \in S_{A_2}$.
- $x_1 \leq_{A_0} x_2$ is independent of the choice of A, s.t. $x_1, x_2 \in S_A$:
  If $\exists A, A' \in C$, s.t. $x_1, x_2 \in S_A, S_{A'}$, then A, A' are comparable. Assume that $A \preceq A'$, that is A is an initial segment of A', then in S_A, we have $x_1 \leq_A x_2 \Leftrightarrow x_1 \leq_{A'} x_2$.
- $(S_{A_0}, \leq_{A_0})$ is a total order set :
  Any $x_1, x_2 \in S_{A_0}$ will be covered by a S_A where A is an element of a chain C on X. Thus x_1 and x_2 are comparable by $x_1 \leq_{A_0} x_2 \Leftrightarrow x_1 \leq_A x_2$.
- $(S_{A_0}, \leq_{A_0})$ is a well order set :
  Let $T \subseteq S_{A_0}$ and $T \neq \emptyset$. Then $T = T \cap S_{A_0} = T \cap \cup_{A \in C} S_A = \cup_{A \in C}(T \cap S_A) \neq \emptyset$. Thus $\exists A \in C$, s.t. $T \cap S_A \neq \emptyset$. Since A is well ordering, $T \cap S_A$ has least element, denoted by t.
  Any $A' \in C$, it is either $A' = A$ or $A' \preceq A$ or $A \preceq A'$. If $A' \preceq A$, then $S_{A'}$ is an initial segment of S_A, that is $\exists a \in S_A$, s.t. $S_{A'} = \{x \in S_A | x <_A a\}$. Thus $S'_A \subseteq S_A$, and $T \cap S_{A'} \subseteq T \cap S_A$, thus t is the least element of $T \cap S_A \Rightarrow t$ is the least element of $T \cap S_{A'}$;
  If $A \preceq A'$, then S_A is an initial segment of $S_{A'}$, thus $\exists a' \in S_{A'}$, s.t. $S_A = \{x \in S_{A'} | x <_{A'} a'\}$ and $T \cap S_A = T \cap \{x \in S_{A'} | x <_{A'} a'\} = \{x \in T \cap S_{A'} | x <_{A'} a'\}$. For any $s \in T \cap S_{A'}$, if $a' \leq_{A'} s$, then $t <_{A'} a' \leq_{A'} s$; if $s <_{A'} a'$, then $s \in T \cap S_A$, and $t \leq_A s \Rightarrow t \leq_{A'} s$. Thus t is the least element of $T \cap S_{A'}$.
  Thus t is the least element of $T \cap S_{A_0} = T$, thus $\leq_{A_0}$ is a well order on S_{A_0}. Furthermore, $(S_{A_0}, \leq_{A_0}) \in X$.
- S_{A_0} is an upper bound of C on X, w.r.t. $\preceq$:
  Given $A \in C$, since C is a chain, any $A' \in C$ admits 3 cases: $A' = A, A' \preceq A, A \preceq A'$. Define $\Pi := \{A' \in C | A \preceq A'\} \backslash \{A\}$ and $\Gamma := \{A' \in C | A' \preceq A\} \backslash \{A\}$.

  Note 7. Recall the proof of $2 \Rightarrow 3$.

  For any $B \in \Pi$, $\exists b \in S_B$, s.t. $S_A = IS_B(b)$. Define $\Phi := \{A' \in \Pi | A' \preceq B\} \backslash \{B\}$. If $\Phi \neq \emptyset$, then $\exists C \in \Phi, \exists c \in S_C$, s.t. $S_A = IS_C(c)$. Collect all these kind of c and form a set Δ, then Δ is a non-empty subset of S_B. Since S_B is a well ordering set, Δ has a least element μ, and exists the corresponding

$D \in \Phi$, s.t. $S_A = IS_D(\mu)$. Thus

$$S_A = IS_D(\mu) = \{x \in S_D | x <_D \mu\}$$
$$\xlongequal{x,\mu \in S_{A_0}} \{x \in S_D | x <_{A_0} \mu\}$$

Since any $A' \in \Pi$, the corresponding $\mu \leq_{A'} a'$, thus

$$\begin{aligned} \{x \in S_{A'} | x <_{A_0} \mu\} &= \{x \in S_{A'} | x <_{A'} \mu\} \\ &\subseteq \{x \in S_{A'} | x <_{A'} a'\} \\ &= IS_{A'}(a') \\ &= S_A = IS_D(\mu) \end{aligned}$$

On the other hand, For any $A'' \in \Gamma$, $A'' \preceq A \Rightarrow S_{A''} \subseteq S_A$, thus $\{x \in S_{A''} | x <_{A_0} \mu\} \subseteq S_A$. Thus

$$\begin{aligned} S_A &= IS_D(\mu) \\ &= \cup_{A' \in \Pi} \{x \in S_{A'} | x <_{A_0} \mu\} \cup (\cup_{A'' \in \Gamma} \{x \in S_{A''} | x <_{A_0} \mu\}) \\ &= \{x \in \cup_{A' \in \Pi \cup \Gamma} S_{A'} | x <_{A_0} \mu\} \\ &= \{x \in \cup_{A' \in C} S_{A'} | x <_{A_0} \mu\} \\ &= IS_{A_0}(\mu) \end{aligned}$$

Thus $A \preceq A_0$ for any $A \in C$, and A_0 is an upper bound of C. $(X, \preceq)$, as a poset, whose any chain C has an upper bound A_0, thus X has a maximal element by Zorn's lemma.

2. A maximal element in $(X, \preceq)$ is $(Y, \leq_Y)$.
If $(Y_0, \leq_{Y_0})$ is a max. element in X w.r.t. $\preceq$ and $Y_0 \neq Y$, then $\exists y \in Y \backslash Y_0$. Define $Y_1 := Y_0 \cup \{y\}$ and a partial order: $v \leq_{Y_1} y, v_1 \leq_{Y_1} v_2 \Leftrightarrow v_1 \leq_{Y_0} v_2$ for $\forall v, v_1, v_2 \in Y_0$.
Then $(Y_1, \leq_{Y_1})$ admits a well-ordering which makes $(Y_0, \leq_{Y_0})$ an initial segment, because any non-empty subset ϕ of Y_1 is either $\{y\}$ or $(\phi \cap Y_0) \cup (\phi \cap \{y\})$, clearly ϕ has least element.
Thus $(Y_1, \leq_{Y_1}) \in X$ and $(Y_0, \leq_{Y_0}) \preceq (Y_1, \leq_{Y_1})$, which leads to a contradiction to the maximality of $(Y_0, \leq_{Y_0})$.

Since X is the set of well ordering subset on Y, $(Y, \leq_Y) \in X$, thus $(Y, \leq_Y)$ is well ordering.

$5 \Rightarrow 6$: Choose a well ordering $\leq$ on X, For any $y \in Y$, define $g(y) :=$ the least element of $f^{-1}(y)$, then $f \circ g(y) = y$. For any $y_1, y_2 \in Y, y_1 \neq y_2 \Rightarrow f(g(y_1)) \neq f(g(y_2)) \Rightarrow g(y_1) \neq g(y_2) \Rightarrow g$ is injective.

$6 \Rightarrow 7$: Let $S := \cup_{\alpha \in A} S_\alpha$, define $X := \{(s, \alpha) \in S \times A | s \in S_\alpha\}$. Consider two projection $X \xrightarrow{f} A((s,\alpha) \mapsto \alpha)$ and $X \xrightarrow{p} S((s,\alpha) \mapsto s)$, thus f is a surjection, then $\exists A \xrightarrow{g} X$ such that $f \circ g(\alpha) = \alpha$ for any $\alpha \in A$.

Define s_α is the least element of S_α, then $g(\alpha) = (s_\alpha, \alpha)$ and $p \circ g(\alpha) = p(s_\alpha, \alpha) = s_\alpha \in S_\alpha$. Thus $A \xrightarrow{p \circ g} S = \cup_{\alpha \in A} S_\alpha$ is desired.

$$\begin{tikzcd}
 & & A_{\alpha} \arrow[dll, shift left, "g"] \\
X \arrow[urr, shift left, "f"] \arrow[drr, "p"'] & \subseteq & S \times A_{(s,\alpha)} \arrow[u, "\pi"'] \arrow[d, "\pi"] \\
 & & S_{s}
\end{tikzcd}$$

$\square$

3 Applications of Zorn's Lemma

3.1 Cardinality

Definition 5 (Cardinality)**.** Let X and Y be two sets, we say $|X| = |Y|$ if there exists a bijection $X \to Y$; $|X| \le |Y|$ if exist an injection $X \to Y$.

Exercise 1. *Let X and Y be two sets, show that $\exists$ an injection $X \to Y \Leftrightarrow \exists$ a surjection $Y \to X$.*

Proof. $\Leftarrow$: If $Y \xrightarrow{f} X$ is a surjection, then $\exists$ an injection $X \xrightarrow{g} Y$ by equivalent statements 6 of AC. $\Rightarrow$: If $X \xrightarrow{f} Y$ is an injection, then $X \xrightarrow{f} f(X)$ is a bijection, and there exists an inverse $f(X) \xrightarrow{f^{-1}} X$. Select $x \in X$, define $g(y) \equiv x, y \in Y \backslash f(X)$, Then $Y \xrightarrow{g} X$ where $y \mapsto f^{-1}(y)$ if $y \in f(X)$ and $y \mapsto x$ if $y \in Y \backslash f(X)$ is as desired. $\square$

Exercise 2. *Let X and Y be two sets, show that there exist an injection from X to Y or from Y to X.*

Proof. Consider $\Pi := \{S_f \xrightarrow{f} Y | f$ is an injection on a subset S_f of $X\}$ and $f \preceq f' \Leftrightarrow S_f \subseteq S_{f'}$ and $f'|_{S_f} = f$. Thus $(\Pi, \preceq)$ is a poset.
If $\Pi = \emptyset$, which implies there is only one element in Y, thus there exists a surjection from X to $Y \Rightarrow$ there exists an injection from Y to X.
If $\Pi \neq \emptyset$:
suppose $(C, \preceq)$ is a chain on $(\Pi, \preceq)$, define $Z = \cup_{S \in C} S$, and for any $z \in Z$, $f_o(z) = f(z)$ if $z \in S_f$. As always: (1) S_f exists by the def. of Z; (2) the def. of f_o is well-defined, that is the value of $f_o(z)$ is independent with the choice of S_f, because any S_f, S'_f that cover z are in the chain C, thus they are comparable, and one is the extension of the other.

Thus $Z \xrightarrow{f_o} Y$ is an upper bound of $(C, \preceq)$, because for any $S_f \xrightarrow{f} Y \in C$, $S_f \subseteq Z$ by def. and $f_o|_{S_f} = f$ by the independence. Thus any chain on $(\Pi, \preceq)$ has an upper bound, and $(\Pi, \preceq)$ has a maximal element $X_0 \xrightarrow{f_0} Y$. Suppose $X_0 \neq X$:
If f_0 is not surj: Then select $y_0 \in Y \backslash f(X_0)$ and $x \in X \backslash X_0$. Define $X_1 = X_0 \cup \{x\}$, and define $f_1|_{X_0} = f_0$, $f_1(x_0) = y_0$. Then $f_0 \preceq f_0$, which against the maximality of $X_0 \xrightarrow{f_0} Y$.
If f_0 is surj: Then select any $y_0 \in Y$ and define $f_1(x) \equiv y_0$ for any $x \in X \backslash X_0$, thus $X \xrightarrow{f_1} Y$ is a surj. Then there exists an injection $Y \xrightarrow{g} X$, and we are done.

□

Note 8. A very useful routine:

1. transform the existence of the target to the existence of the maximal element on some poset
2. use Zorn's Lemma (show any chain on the poset has an upper bound, which is usually the union on all elements in the chain)
3. check that the maximal element = target (use contradiction).

Proposition 1 (Bernstein-Schroeder)**.** $|X| \leq |Y|$ *and* $|Y| \leq |X| \Rightarrow |X| = |Y|$.

Proof. The proof of the proposition has been given in *Introduction to Topology, Lecture 2, Proposition 4*. □

3.2 Vector Space

3.3 Hahn-Banach Theorem

Lemma 1. *Let X be a vector space over $K(= \mathbb{R})$, and $X \xrightarrow{p} \mathbb{R}$ is a func. s.t. $\forall x, x' \in X, t > 0, p(x + x') \leq p(x) + p(x')$ and $p(tx) = tp(x)$.*
For any linear func. $Z \xrightarrow{\Xi_o} \mathbb{R}$ on a vector subspace Z of X s.t. $\Xi_o(z) \leq p(z)$ for any $z \in Z$. If $x_0 \in X \backslash Z$, then there exists a linear func. $Z + \mathbb{R}x_0 \xrightarrow{\Xi} \mathbb{R}$ s.t. $\Xi|_Z = \Xi_o$ and $\Xi(u) \leq p(u)$ for any $u \in Z + \mathbb{R}x_0$.

Proof. All linear func.s $Z + \mathbb{R}x_0 \xrightarrow{\Xi} \mathbb{R}$ such that $\Xi|_Z = \Xi_o$ is of the form $\Xi(z + tx_0) = \Xi_o(z) + t\Xi(x_0)$. It suffices to determine the value of $\Xi(x_0)$ (denoted as a) s.t. $\Xi(u) \leq p(u)$ for any $u \in Z + \mathbb{R}x_0$ holds.
Any $u \in Z + \mathbb{R}x_0$ can be uniquely written as $z + tx_0, z \in Z, t \in \mathbb{R}$. We hope to find $a \in \mathbb{R}$ such that

$$\Xi(u) = \Xi_o(z) + ta \leq p(u) = p(z + tx_0)$$

for all $z \in Z, t \in \mathbb{R}$, or equivalently (if $t < 0$, denote $t = -t'$, $t' > 0$)

$$a \leq \frac{p(z + tx_0) - \Xi_o(z)}{t}, \quad z \in Z, t > 0$$
$$a \geq \frac{p(z' - t'x_0) - \Xi_o(z')}{-t'}, \quad z' \in Z, t' > 0$$

Since

$$\begin{aligned}
&\frac{p(z + tx_0) - \Xi_o(z)}{t} - \frac{p(z' - t'x_0) - \Xi_o(z')}{-t'} \\
&= \frac{p(z + tx_0) - \Xi_o(z)}{t} + \frac{p(z' - t'x_0) - \Xi_o(z')}{t'} \\
&= \frac{t'p(z + tx_0) - t'\Xi_o(z) + tp(z' - t'x_0) - t\Xi_o(z')}{tt'} \\
&= \frac{p(t'z + tt'x_0) - \Xi_o(t'x) + p(tz' - tt'x_0) - \Xi_o(tz')}{tt'} \\
&\geq \frac{p(t'z + tt'x_0 + tz' - tt'x_0) - \Xi_o(t'z + tz')}{tt'} \\
&= \frac{p(t'z + tz') - \Xi_o(t'z + tz')}{tt'} \geq 0.
\end{aligned}$$

$\Rightarrow$ such $a\exists$. ☐

Theorem 3 (Hahn-Banach Theorem)**.** *Let X be a vector space over $K(= \mathbb{R})$, and $X \xrightarrow{p} \mathbb{R}$ is a func. s.t. $\forall x, x' \in X, t > 0, p(x + x') \leq p(x) + p(x')$ and $p(tx) = tp(x)$.*
For any linear func. $Y \xrightarrow{\Lambda_o} \mathbb{R}$ on a vector subspace Y of X s.t. $\Lambda_o(y) \leq p(y)$ for any $y \in Y$. Then there exists a linear func. $X \xrightarrow{\Lambda} \mathbb{R}$ s.t. $\Lambda|_Y = \Lambda_o$ and $\Lambda(x) \leq p(x)$ for any $x \in X$.

Proof. Consider P is the collection of $W_\Theta \xrightarrow{\Theta} \mathbb{R}$ such that Θ is a linear func. on a vec. subspace W_Θ of X containing Y s.t. $\Theta|_Y = \Lambda_o$ and $\Theta(w) \leq p(w)$ for all $w \in W_\Theta$. And define $\preceq$: $\Theta \preceq \Theta' \Leftrightarrow W_\Theta \subseteq W_{\Theta'}$ and $\Theta'|_{W_\Theta} = \Theta$. It is direct to see $(P, \preceq)$ is a poset.
If $(P, \preceq)$ has a maximal element $Z \xrightarrow{\Theta} \mathbb{R}$, then $Z = X$ by Lemma 1. otherwise we can extent Z to $Z + \mathbb{R}x_0$ where $x_0 \in X \backslash Z$ which against the maximality of $Z \xrightarrow{\Theta} \mathbb{R}$.
Note 9. Recall the proof of Well-Ordering Theorem by Zorn's Lemma.
Thus it suffices to show $(P, \preceq)$ has a max. element. Let $(C, \preceq)$ is a chain in $(P, \preceq)$. We take $W = \cup_{\Theta \in C} W_\Theta$ which is a vector subspace of X containing Y. And define $W \xrightarrow{\Pi} \mathbb{R}$ where then $w \mapsto \Theta(w)$ if $w \in W_\theta$. This is well-defined, $\Pi(w)$ is independence of the choice of Θ s.t. $w \in W_\Theta$, since C is a chain, and one of any $W_\Theta, W_{\Theta'}$ that covers w is the extension of the other. Thus for any $\Theta \in C$, $W_\Theta \subseteq W$ and $\Pi|_{W_\Theta} = \Theta$, thus $\Theta \preceq \Pi$. Thus Π is the upper bound of C, and $W = X$ and $X \xrightarrow{\Pi} \mathbb{R}$ is as desired. ☐