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Lecture 3

Game Theory I: Further Solution Concepts

Key word: Removal of Dominated Strategies, Maxmin Strategies, Correlated equilibrium

1 Removal of Dominated Strategies

Just like the way we defined the domination strategy, we define an opposite conception.

Definition 1.1: Dominated strategy

A strategy s_i is strictly (weakly, very weakly) dominated for an agent i, if any other strategy $s_i' \in S_i$ strictly (weakly, very weakly) dominates s_i.

Intuitively, all strictly dominated pure strategies can be ignored, since they can never be BR to any strategies by the other players, thus the elimination process **preserves Nash equilibrium**. There are three point need to be noticed. Firstly, once a pure strategy is eliminated, another strategy that was not dominated can be dominated, thus the process of elimination can be continued. Secondly, a pure strategy might be dominated by a mixture of other pure strategies without being dominated by any of them independently.

Thirdly, the elimination order does not matter when we remove strictly dominated strategies, this is called a *Church-Rosser property*. However, the elimination order can make a difference to the final reduced game if you remove the weakly or very weakly dominated strategies. You might even git rid of Nash equilibrium. There is an example.

Example 1.1. There is a game like

	L	C	R
U	3,1	0,1	0,0
M	1,1	1,1	5,0
D	0,1	4,1	0,0

Column R can be eliminated, since it is dominated by L and C. We are left with the reduced game like

	L	C
U	3,1	0,1
M	1,1	1,1
D	0,1	4,1

Here M is dominated by neither U nor D, but it is dominated by the mixed strategy that selects either U or D with equal probability. Notice that it was not dominated before the elimination of the R column. And so we are left with the maximally reduced game.

	L	C
U	3,1	0,1
D	0,1	4,1

It is easy to check that the Nash equilibrium is (U, L) and (D, C). Notice strategy L very weekly dominates strategy C, if we remove either L or C, one of the NEs would be missed.

Example 1.2 (dominance solvable)**.** We say that a game is dominance solvable, if iterative deletion of strictly dominated strategies yields a unique outcome. For example, the following game is dominance solvable.

	L	M	R
U	3,8	2,0	1,2
D	0,0	1,7	8,2

Firstly, Column R is dominated by the mixture of L and M, so it can be eliminated. Secondly, D is dominated by U, and then M is dominated by L.

Example 1.3 (weakly dominated strategy elimination order)**.** In order to illustrate the problem that arises when iteratively eliminating weakly dominated strategies, consider the following game:

	L	M	R
U	4,3	3,5	3,5
D	3,4	5,3	3,4

so L is weakly dominated by R and we remove it:

	M	R
U	3,5	3,5
D	5,3	3,4

now U is weakly dominated by D, and M is weakly dominated by R. If we remove row U, the unique NE would be (D,R)

	M	R
D	5,3	3,4

If we remove column M, then (U,R) and (D,R) are both NEs.

	R
U	3,5
D	3,4

2 Maxmin Strategies

The *maxmin strategy* s_i of player i is a strategy that maximizes i's worst-case payoff, in the situation where all the other players happen to play the strategies (**w.r.t** s_i) which cause the greatest harm to i.

Definition 2.1: Maxmin

The maxmin strategy for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$, and the maxmin value for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

Notice that the maxmin strategy is a concept that make sense in **simultaneous-move** games. But it can be well-understandable through the following temporal intuition. Consider that player i commits to a strategy s_i, and then the remaining agents $-i$ observe this strategy and choose their own strategies to minimize i's expected payoff. The maxmin strategy is the one that maximizes player i's expected utility during this process.

The maxmin strategy indicate what player i should do in the case that not only the others aim to baffle him, but also the other agents play **arbitrarily** which is closer to **reality**. In this case i will still receive an expected payoff at least his maxmin value.

The *minmax strategy* and *minmax value* play a dual role to their maxmin counterparts. In two player games the minmax strategy for player i against player $-i$ is a strategy that keeps the maximum payoff of $-i$ at a minimum. This is useful when we want to consider the amount that one player can punish another without regard for his own payoff.

Definition 2.2: Minmax, 2 players

The minmax strategy for player i against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$, and player $-i$'s minmax value is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

player i's minmax strategy is the one that minimizes the payoff of $-i$'s BR. In two-player games, a player's minmax value is always equal to his maxmin value, that is player i's maximum payoff in the worst situation (maxmin value) is equal to minimum BR payoff (minmax value). For the games with more than 2 players, player i's maxmin value is always less than or equal to his minmax value. Assume that $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = u_i(s'_i, s'_{-i})$, $\min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i}) = u_i(s^*_i, s^*_{-i})$, then

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = u_i(s'_i, s'_{-i}) \leq u_i(s'_i, s^*_{-i}) \leq u_i(s^*_i, s^*_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i}).$$

Note 1. The maxmin strategy and minmax strategy are both player i's strategy. The maxmin strategy is player i's the best choice in the worst situation (i.e. everyone want to kill him). And the minmax strategy is minimum amount of damage he can do, regardless of cost.

Theorem 2.1: Minmax theorem

In any finite, two-players, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and minmax value.

Minmax theorem demonstrates that maxmin strategies, minmax strategies, and Nash equilibrium coincide in two-player, zero-sum game. In particular, we can conclude that in two-player, zero-sum games:

1. Each player's maxmin value is equal to his minmax value. By convention, the maxmin value for player 1 is called the *value of the game*;
2. For both players, the set of maxmin strategies coincides with the set of minmax strategies;
3. Any maxmin strategy profile (or equivalently, minmax strategy profile) is a Nash equilibrium. Furthermore, these are all the NE, since in NE, every one choose his BR or say maximize his payoff, which means minimize other's payoff in zero-sum, two-player game.

Example 2.1. (Maxmin strategy of matching pennies game) In Matching pennies game, assume that player 1 chose H and T with probability α and $1-\alpha$; player 2 chose H and T with probability β and $1-\beta$, thus the expected utility function of player 1 is that

$$u_1(s) = 4\alpha\beta - 2(\alpha + \beta) + 1.$$

In the one hand, assume player 1 commit to strategy s_1: $\alpha = 0.8$ (blue line), then $\arg\min_{s_2} u_1(s_1, s_2) = 0$, and $u_1(\alpha = 0.8, \beta = 0) = -0.6$ (the intersection of two blue lines). If player 1 want to increase (until reaching the maximum) this value, he need decrease value of α (blue arrow).

In the other hand, if player1 commit to strategy $\alpha = 0.2$ (red line), then $\arg\min_{s_2} u_1(s_1, s_2) = 1$, and $u_1(\alpha = 0.2, \beta = 1) = -0.6$. if player 1 want to increase (until reaching the maximum) this value, he need increase value of α (red arrow). It is obvious, the result of this maxmin strategy adjusting process is that $\alpha = 0.5$ which is also the Nash equilibrium.

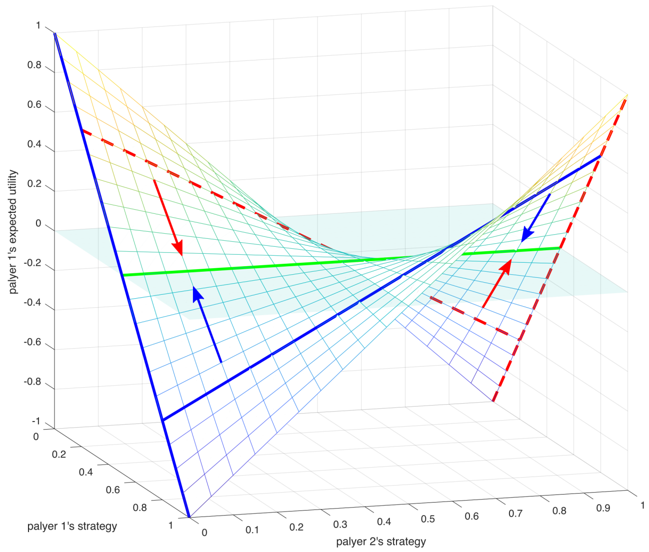

Figure 1: Maxmin strategy of MPG

Here is the MATLAB code:

```
t=0:.05:1;
[x,y]=meshgrid(t);
z=4.*x.*y-2.*(x+y)+1;
z1 = 0.*x+0.*y;
figure(1)
mesh(x,y,z);

hold on;
s = surf(x,y,z1,'FaceAlpha',0.1);
s.EdgeColor = 'none';

hold on;
y2 = 0:.05:1;
x2 = ones(1,21).*0.2;
x3 = ones(1,21).*0.8;
x4 = ones(1,21).*0.5;

z2 = 4.*x2.*y2-2.*(x2+y2)+1;
z3 = 4.*x3.*y2-2.*(x3+y2)+1;
z4 = 4.*x4.*y2-2.*(x4+y2)+1;

p1 = plot3(x2,y2,z2, '--', 'Color','r');
p2 = plot3(x3,y2,z3, '-', 'Color','b');
p3 = plot3(x4,y2,z4, 'Color','g');

xx = 0:.05:1;
yy1 = ones(1,21).*0;
yy2 = ones(1,21).*1;
zz1 = -2.*xx +1;
zz2 = 2.*xx -1;
p4 = plot3(xx,yy1,zz1, '-', 'Color','b') ;
p5 = plot3(xx,yy2,zz2, '--', 'Color','r') ;

p1.LineWidth = 3;
```

```
p2.LineWidth = 3;
p3.LineWidth = 3;
p4.LineWidth = 3;
p5.LineWidth = 3;

axis([0 1 0 1 -1 1]);
xlabel("palyer 1's strategy")
ylabel("palyer 2's strategy")
zlabel("palyer 1's expected utility")
```

Example 2.2 (Penalty Kick Game)**.** Recall Penalty Kick Game we have discussed previously:

	L	R
L	.6,.4	.8,.2
R	.9,.1	.7,.3

Now we compute the maxmin strategy of player 1. Since kicker (player 1)'s expect utility is

$$u_1(s) = [s_1(L)s_2(L) \cdot 0.6 + s_1(L)s_2(R) \cdot 0.8 + s_1(R)s_2(L) \cdot 0.9 + s_1(R)s_2(R) \cdot 0.7]$$

thus kicker's minimum is

$$\begin{aligned}\min_{s_2} u_1(s) &= \min_{s_2}[s_1(L)s_2(L) \cdot 0.6 + s_1(L)s_2(R) \cdot 0.8 + s_1(R)s_2(L) \cdot 0.9 + s_1(R)s_2(R) \cdot 0.7] \\ &= \min_{s_2}[(0.2 - s_1(L) \cdot 0.4) \cdot s_2(L) + (0.7 + s_1(L) \cdot 0.1)]\end{aligned}$$

thus $0.2 - s_1(L) \cdot 0.4 = 0$ and $s_1(L) = 0.5$; Now calculate goalie's minmax strategy. the kicker's maximum is

$$\begin{aligned}\max_{s_1} u_1(s) &= \max_{s_1}[s_1(L)s_2(L) \cdot 0.6 + s_1(L)s_2(R) \cdot 0.8 + s_1(R)s_2(L) \cdot 0.9 + s_1(R)s_2(R) \cdot 0.7] \\ &= \max_{s_1}[(0.1 - s_2(L) \cdot 0.4) \cdot s_1(L) + (0.7 + s_2(L) \cdot 0.2)]\end{aligned}$$

thus $0.1 - s_2(L) \cdot 0.4 = 0$ and $s_2(L) = 0.25$.

3 Correlated equilibrium

Definition 3.1: Correlated Equilibrium (informal)

a randomized assignment of (potentially correlated) action recommendations to agents, such that nobody wants to deviate.