Introduction to Topology

General Topology, Lecture 15

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THIS IS THE LECTURE NOTE FOR THE *Introduction to Topology*. The course covers the following topics: Naive Set Theory, Elementary Number Theory, Group Theory, Topological Spaces and Continuous Maps, Introduction to Algebraic Topology.

CONTENT:

Compactness

Definition 1 (Compact)**.** Let X be a top. sp. we say that X is compact if $\forall U_\alpha \subseteq_{open} X(\alpha \in A), X = \cup_{\alpha\in A} U_\alpha \Rightarrow \exists \alpha_1, \cdots, \alpha_k \in A$, s.t. $X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_k}$.

Definition 2 (Compact Subset)**.** Let X be a top. sp. $K \subseteq X$, we say K is a compact subset in X, if $\forall U_\alpha \subseteq_{open} X(\alpha \in A), K \subseteq \cup_{\alpha\in A} U_\alpha \Rightarrow \exists \alpha_1, \cdots, \alpha_k \in A$, s.t. $K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_k}$.

Exercise 1. Show that K is a compact subset in $X \Leftrightarrow K$ (equipped with the subspace topology) is a compact space.

Proof. $\Rightarrow$: For any $V_\alpha \subseteq_{open} K, \exists U_\alpha \subseteq_{open} X$, s.t. $V_\alpha = U_\alpha \cap K$. For any

$$\begin{aligned} K &= \cup_{\alpha\in A} V_\alpha \\ &= \cup_{\alpha\in A} U_\alpha \cap K \\ &= K \cap \cup_{\alpha\in A} U_\alpha \\ &= K \cap (U_{\alpha_1} \cup \cdots U_{\alpha_k}) \\ &= V_{\alpha_1} \cup \cdots V_{\alpha_k} \end{aligned}$$

Thus K is compact. $\Leftarrow$: for any $K \subseteq \cup_{\alpha\in A} U_\alpha$, we have $\cup_{\alpha\in A}(U_\alpha \cap K) \subseteq K$ and

$$\begin{aligned} K &= K \cap K \\ &\subseteq K \cap \cup_{\alpha\in A} U_\alpha \\ &= \cup_{\alpha\in A}(K \cap U_\alpha) \end{aligned}$$

Thus $K = \cup_{\alpha\in A}(K \cap U_\alpha) = \cup_{\alpha\in A} V_\alpha$, where $V_\alpha \subseteq_{open} K$. And $\exists V_{\alpha_1}, \cdots, V_{\alpha_k} \subseteq_{open} K$, s.t.

$$\begin{aligned} K &= V_{\alpha_1} \cup \cdots V_{\alpha_k} \\ &= K \cap (U_{\alpha_1} \cup \cdots U_{\alpha_k}) \\ &\subseteq U_{\alpha_1} \cup \cdots U_{\alpha_k} \end{aligned}$$

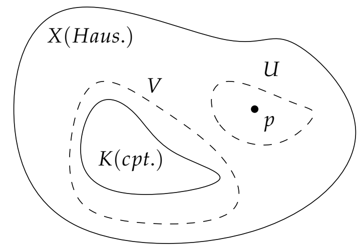
Thus K is a compact subset in X. $\square$

Definition 3 (Hausdorff Topology Space)**.** A top. sp. X is Hausdorff if $\forall p, q \in X, p \neq q \Rightarrow \exists$ open nbds U of p and V of q in X such that $U \cap V = \emptyset$.

Example 1. Let $X = \{1, 2\}$, $\mathscr{T}$ is trivial topology, then $(X, \mathscr{T})$ is not a Hausdorff topology space.

Note 1. Thus the larger top. is, the more likely it is to be Hausdorff.

Proposition 1. *Suppose X is Hausdorff, $K(\subseteq X)$ is compact, $p \in X \backslash K \Rightarrow \exists U, V \subseteq_{open} X$,* s.t. *$K \subseteq V, p \in U$, and $U \cap V = \emptyset$.*



Proof. X is Hausdorff $\Rightarrow$ $\forall q \in K, \exists U_q, V_q \subseteq_{open} X$ s.t. $q \in V_q, p \in U_q, U_q \cap V_q = \emptyset$. Thus $K \subseteq \cup_{q \in K} V_q = \cup_{i=1}^{k} V_{q_i}$, Let $V = \cup_{i=1}^{k} V_{q_i}, U = \cap_{i=1}^{k} U_{q_i}$, then

$$\begin{aligned} U \cap V &= \left(\cap_{j=1}^{k} U_{q_j}\right) \cap \left(\cup_{i=1}^{k} V_{q_i}\right) \\ &= \cup_{i=1}^{k} \left[\cap_{j=1}^{k} \left(U_{q_j} \cap V_{q_i}\right)\right] \\ &= \cup_{i=1}^{k} \emptyset = \emptyset. \end{aligned}$$

$\square$

Corollary 1. *Suppose X is Hausdorff, $K \subseteq_{cpt.} X$ is compact $\Rightarrow$ K is closed.*

Proof. For $\forall p \in X \backslash K, \exists W_p \subseteq_{open} X$, s.t. $x \in W_p$ and $W_p \cap K = \emptyset$, that is $W_p \subseteq X \backslash K$. And because

$$X \backslash K = \cup_{p \in X \backslash K} \{p\} \subseteq \cup_{p \in X \backslash K} W_p \subseteq X \backslash K$$

we have that $X \backslash K = \cup_{p \in X \backslash K} W_p \subseteq_{open} X$, and then $K \subseteq_{close} X$. $\square$

Exercise 2. Suppose X is locally compact Hausdorff, $K \subseteq_{cpt.} X$, $C \subseteq_{close} X$, show that if $C \cap K = \emptyset \Rightarrow \exists U, V \subseteq_{open} X$, s.t. $K \subseteq V, C \subseteq U$ and $U \cap V = \emptyset$.

Proposition 2. *Suppose X is a compact space, $K \subseteq_{close} X \Rightarrow K \subseteq_{cpt.} X$.*

Note 2. If X is Haus. $K \subseteq_{cpt.} X$ is close; If X is cpt. $K \subseteq_{close} X$ is cpt.

Proof. Suppose that $U_\alpha \subseteq_{open} X (\alpha \in A)$ cover K, i.e. $K \subseteq \cup_{\alpha \in A} U_\alpha$. Then

Note 3. So the standard routines for proving a set K is cpt. is suppose $U_\alpha (\alpha \in A)$ cover it at first, and then try to argue $K \subseteq \cup_{i=1}^{k} U_{\alpha_i}$.

$$\begin{aligned} X = K \cup (X \backslash K) &\subseteq \\ &(\cup_{\alpha \in A} U_\alpha) \cup (X \backslash K) \\ &= \left[\cup_{i=1}^{k} U_{\alpha_i}\right] \cup (X \backslash K) \\ &= \cup_{i=1}^{k} \left[U_{\alpha_i} \cup (X \backslash K)\right] \end{aligned}$$

and

$$\begin{aligned} K = K \cap X &= K \cap \left[\cup_{i=1}^{k} U_{\alpha_i} \cup (X \backslash K)\right] \\ &= \cup_{i=1}^{k} \left[K \cap ((X \backslash K) \cup U_{\alpha_i})\right] \\ &= \cup_{i=1}^{k} \left[(K \cap (X \backslash K)) \cup (K \cap U_{\alpha_i})\right] \\ &= \cup_{i=1}^{k} (K \cap U_{\alpha_i}) \\ &\subseteq \cup_{i=1}^{k} U_{\alpha_i}. \end{aligned}$$

Thus K is compact. □

Exercise 3. X is locally compact Hausdorff (LCH) space, $C \subseteq_{close} X$, show that $\forall c \in C, \exists T_c \subseteq_{cpt.} C,$ s.t. $c \in T_c$.

Proof. For $\forall c \in C, \exists S_c \subseteq_{cpt.} X,$ s.t. $c \in S_c$ and $c \in S_c \cap C$. Since $S_c \subseteq_{cpt.} X \Rightarrow S_c \subseteq_{close} X \Rightarrow S_c \cap C \subseteq_{close} X$

$$\begin{aligned}
&X \backslash (S_c \cap C) \subseteq_{open} X \\
&\Rightarrow S_c \cap (X \backslash (S_c \cap C)) \subseteq_{open} S_c \\
&\Rightarrow S_c \backslash [S_c \cap (X \backslash (S_c \cap C))] \subseteq_{close} S_c \\
&\Rightarrow (S_c \backslash S_c) \cup [S_c \backslash X \backslash (S_c \cap C)] \subseteq_{close} S_c \\
&\Rightarrow S_c \backslash X \backslash (S_c \cap C) \\
&= S_c \cap C \subseteq_{close} S_c.
\end{aligned}$$

Note 4. $A \subseteq_{close} X$, $A \subseteq B \subseteq X$, then $A \subseteq_{close} B$.

Since $S_c \cap C \subseteq_{close} S_c, S_c$ is cpt. $\Rightarrow S_c \cap C \subseteq_{cpt.} S_c \Rightarrow S_c \cap C$ is cpt. $\Rightarrow S_c \cap C \subseteq_{cpt.} C$. □

Note 5. Remember that $A \subseteq_{cpt.} B$ means A is a cpt. subset of B, which is equivalent with A is a cpt. set.

Proposition 3. *X, Y are cpt. space $\Rightarrow X \times Y$ (equipped with the product topology) is compact.*

Proof. Trivial. Fix x, consider $\{x\} \times Y$. then utilize the definition of product topology, and then use the compactness of Y.Thus $\{x\} \times Y$ could be covered by finite open set. For detailed argument, see here(0:52:00). □

Proposition 4. *Suppose X, Y are top. sp. $X \xrightarrow{f} Y$ is continuous. $K \subseteq_{cpt.} X \Rightarrow f(K) \subseteq_{cpt.} Y$.*

Proof. Suppose $V_\alpha (\alpha \in A)$ cover $f(K)$, that is $f(K) \subseteq \cup_{\alpha \in A} U_\alpha$, thus

$$\begin{aligned}
K &\subseteq f^{-1}(\cup_{\alpha \in A} U_\alpha) \\
&= \cup_{\alpha \in A} f^{-1}(U_\alpha) \\
&= \cup_{i=1}^k f^{-1}(U_{\alpha_i}) \\
&= f^{-1}(\cup_{i=1}^k U_{\alpha_i}).
\end{aligned}$$

Note 6. Since f is continuous, $f^{-1}(U_\alpha)(\alpha \in A)$ are open.

Thus $f(K) \subseteq \cup_{i=1}^k U_{\alpha_i}$ and it is compact. □

Corollary 2. *Suppose map $X \xrightarrow{f} Y$ is conti. X is compact, Y is Hausdorff, then f is a closed map, i.e. $C \subseteq_{close} X \Rightarrow f(C) \subseteq_{close} Y$.*

Proof. $C \subseteq_{close} X$, X is compact $\Rightarrow C \subseteq_{cpt.} X \Rightarrow f(C) \subseteq_{cpt.} Y$, Y is Hausdorff $\Rightarrow f(C)$ is close. □

Corollary 3. *Suppose map $X \xrightarrow{f} Y$ is conti. bijection, X is compact, Y is Hausdorff $\Rightarrow f$ is a homeomorphism.*

Proof. f is closed map, and f is a bijection, thus f^{-1} is continuous. f is bijection, f and f^{-1} are continuous, thus f is a homeomorphism. □

Bound

Definition 4 (Upper Bound)**.** Given $A \subseteq \mathbb{R}$, we call $u \in \mathbb{R}$ is a upper bound of A if $a \leq u$ for $\forall a \in A$; $l \in \mathbb{R}$ is a lower bound of A if $l \leq a$ for $\forall a \in A$.

Definition 5 (Greatest Element)**.** $x \in \mathbb{R}$ is the greatest (smallest) element of A if $x \in A$ and x is a upper (lower) bound of A.

Definition 6 (Least Upper Bound)**.** $u \in \mathbb{R}$ is the least upper bound (or supremum) of A, if u is the smallest element of the set of all upper bounds of A, denote as $u = \sup A$.

$l \in \mathbb{R}$ is the greatest lower bound (or infimum) of A, if l is the greatest element of the set of all lower bounds of A, denote as $l = \inf A$.

Example 2. Let $A = [0, 1)$, the set of upper bound of A is $[1, \infty)$, the set of lower bound of A is $(-\infty, 0]$. Thus $\sup A = 1, \inf A = 0$.

Suppose we admit that the gapless property of real number: if $\emptyset \neq S \subseteq \mathbb{R}$ has upper bound (lower bound), then $\sup S (\inf S) \exists$.

Theorem 1. $[0, 1]$ *(as a subspace of* $\mathbb{R}$*) is compact.*

Proof. Suppose that $V_\alpha \subseteq_{open} \mathbb{R} (\alpha \in A)$ cover $[0, 1]$. Consider

$$S := \{s \in [0,1] | [0, s] \text{ can be covered by finitely many } V_\alpha\}$$

Thus $0 \in S$, $S \neq \emptyset$. $S \subseteq [0, 1]$, thus S has an upper bound $\Rightarrow \sup S \exists$. Let $s_0 := \sup S$. Since 1 is an upper bound of S, $s_0 \leq 1$. For $\forall t \leq s_0$, t is not an upper bound of S, $\exists s' \in S$, s.t. $t < s'$, thus $[0, t]$ could be covered by finitely many V_α.

Since $s_0 \leq 1$, $\exists \alpha_0$, s.t. $s_0 \subseteq V_{\alpha_0}$, $\exists r > 0$, s.t. $B_r(s_0) \subseteq V_{\alpha_0}$. Thus $[0, s_0 - r]$ can be covered by finitely many of V_α, and $(s_0 - r, s_0 + r)$ can be covered by V_{α_0}. Thus $[0, s_0 + r)$ can be covered by finitely many V_α. Thus $s_0 = 1$ and $s_0 \in S \Rightarrow S = [0, 1]$. □

Thus $[0, 1] \times [0, 1]$, as a subspace of $\mathbb{R}^2$, which coincides with the product space of $[0, 1]$ and $[0, 1]$, is compact.

More generally, we can reprove the **Heine–Borel theorem**: for $K \subseteq \mathbb{R}^n$, then $K \subseteq_{cpt.} \mathbb{R}^n \Leftrightarrow K \subseteq_{close} \mathbb{R}^n$ and K is bdd.

Proof. $\Rightarrow$: $\mathbb{R}^n$ is metric space $\Rightarrow$ $\mathbb{R}^n$ is Hausdorff $\Rightarrow$ $K \subseteq_{close} \mathbb{R}^n$. Since $K \subseteq \cup_{n \in \mathbb{N}} B_n(0) \Rightarrow \exists r_1, \cdots, r_k$, s.t. $K \subseteq \cup_{i=1}^k B_{n_{r_i}}(0) \Rightarrow K$ is bdd.

Note 7. Actually, In any metric space X, $K \subseteq_{cpt.} X \Rightarrow K \subseteq_{close} X$ and be bdd.

$\Leftarrow$: K is bdd. $\Rightarrow, \exists r > 0$, s.t. $K \subseteq B_r(0), \Rightarrow \exists [a_1, b_1], \cdots, [a_n, b_n] \in \mathbb{R}$, s.t. $K \subseteq B_r(0) \subseteq \times_{i=1}^n [a_i, b_i]$. Since $K \subseteq_{close} \mathbb{R}^n \Rightarrow K \subseteq_{close} \times_{i=1}^n [a_i, b_i] \subseteq_{cpt.} \mathbb{R}^n \Rightarrow K \subseteq_{cpt.} \times_{i=1}^n [a_i, b_i] \Rightarrow K$ is cpt. $\square$

Exercise 4. Suppose $S \subseteq_{close} \mathbb{R}$ and $S \neq \emptyset$, S has an upper bound, show that $\sup S \in S$.

Proof. Let $s_0 := \sup S$. If $s_0 \notin S \Rightarrow s_0 \in \mathbb{R} \backslash S \subseteq_{open} \mathbb{R}$. Thus $\exists r > 0$, s.t. $B_r(s_0) \in \mathbb{R} \backslash S$, that is $s_0 - r \in \mathbb{R} \backslash S \Rightarrow s_0 - r \geq s (\forall s \in S)$. But s_0 is the smallest upper bound, then $\forall s' < s_0, \exists s \in S$, s.t. $s > s'$, which leads to a contradiction. $\square$

Corollary 4. *Given a conti. map $K \xrightarrow{f} \mathbb{R}$, K is cpt. $\Rightarrow$ f has a max. and min.*

Proof. K is cpt., f is conti. $\Rightarrow f(K) \subseteq_{cpt.} \mathbb{R} \Rightarrow f(K) \subseteq_{close} \mathbb{R}$ and be bdd. Thus $f(K)$ has a upper bound and lower bound, thus $\max f(K) = \sup f(K) \in f(K)$ and $\min f(K) = \inf f(K) \in f(K)$. $\square$