General Topology
Lecture 6

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This is the Lecture note for the *General Topology* course in Spring 2020.

1 Compactness

Definition 1 (Compact Subset). Let $(X, \mathscr{T})$ be a topology space and $K \subseteq X$, we call K is compact subset of X if $\forall \mathcal{U} \subseteq \mathscr{T}, K \subseteq \cup\mathcal{U} \Rightarrow \exists$ finite $\mathcal{S} \subseteq U,$ s.t. $K \subseteq \cup\mathcal{S}$.

We say $(X, \mathscr{T})$ is a compact space if X is a compact subset of itself.

Exercise 1. *Let $(X, \mathscr{T})$ be a topology space and $K \subseteq X$, show that K is a compact subset of $X \Leftrightarrow (K, \mathscr{T}_K)$ is a compact space, where $\mathscr{T}_K$ is subspace topology.*

Proof. $\Rightarrow$: For any $V_\alpha \subseteq_{open} K, \exists U_\alpha \subseteq_{open} X,$ s.t. $V_\alpha = U_\alpha \cap K$. For any

$$\begin{aligned} K &= \cup_{\alpha \in A} V_\alpha \\ &= \cup_{\alpha \in A}(U_\alpha \cap K) \\ &= K \cap \cup_{\alpha \in A} U_\alpha \\ &= K \cap (U_{\alpha_1} \cup \cdots U_{\alpha_k}) \\ &= V_{\alpha_1} \cup \cdots V_{\alpha_k} \end{aligned}$$

Thus K is compact. $\Leftarrow$: for any $K \subseteq \cup_{\alpha \in A} U_\alpha$, we have $\cup_{\alpha \in A}(U_\alpha \cap K) \subseteq K$ and

$$\begin{aligned} K &= K \cap K \\ &\subseteq K \cap \cup_{\alpha \in A} U_\alpha \\ &= \cup_{\alpha \in A}(K \cap U_\alpha) \end{aligned}$$

Thus $K = \cup_{\alpha \in A}(K \cap U_\alpha) = \cup_{\alpha \in A} V_\alpha$, where $V_\alpha \subseteq_{open} K$. And $\exists V_{\alpha_1}, \cdots, V_{\alpha_k} \subseteq_{open} K,$ s.t.

$$\begin{aligned} K &= V_{\alpha_1} \cup \cdots V_{\alpha_k} \\ &= K \cap (U_{\alpha_1} \cup \cdots U_{\alpha_k}) \\ &\subseteq U_{\alpha_1} \cup \cdots U_{\alpha_k} \end{aligned}$$

Thus K is a compact subset in X. $\square$

Definition 2 (Finite Intersection Property, FIP)**.** Let S be a set and $\mathcal{F} \subseteq \mathcal{P}(S)$ is a family of subsets of S. We say that $\mathcal{F}$ has the finite intersection property (FIP) if $\forall \mathcal{F}_0 \subseteq \mathcal{F}, \mathcal{F}_0$ is finite $\Rightarrow \cap \mathcal{F}_0 \neq \emptyset$.

Exercise 2. *For a set X and a family of subsets $\mathcal{U} \subseteq \mathcal{P}(X)$, let $\mathcal{F} := \{X \backslash U | U \in \mathcal{U}\}$, then $X = \cup \mathcal{U} \Leftrightarrow \cap \mathcal{F} = \emptyset$.*

Proof. $\Rightarrow$: if $\cap \mathcal{F} \neq \emptyset$, then $\exists x \in \cap \mathcal{F}$, that is for $\forall U \in \mathcal{U}, x \in X \backslash U \Rightarrow x \notin U$ but $x \in X$. Thus $\cup \mathcal{U} \neq X$ which leads to a contradiction.
$\Leftarrow$: $\cap \mathcal{F} = \emptyset$, thus for any $x \in X$, $\exists F \in \mathcal{F},$ s.t. $x \notin F$, that is $\exists U \in \mathcal{U},$ s.t. $x \notin X \backslash U \Rightarrow x \in U$. Thus $X \subseteq \cup \mathcal{U} \subseteq X \Rightarrow X = \cup \mathcal{U}$. $\square$

Exercise 3. *Let $(X, \mathscr{T})$ be a topology space, show that X is compact space $\Leftrightarrow \forall$ family $\mathcal{F} (\subseteq \mathcal{P}(X))$ of closed subsets of X, $\mathcal{F}$ has FIP $\Rightarrow \cap \mathcal{F} \neq \emptyset$.*

Proof. $\Rightarrow$: For any family $\mathcal{F}$ of closed subset of X, define $\mathcal{U} := \{X \backslash F | F \in \mathcal{F}\}$, thus $\mathcal{U}$ is a family of open subsets of X. If $\cup \mathcal{U} = X$, since X is compact, $\exists$ a finite $\mathcal{U}_0 \subseteq \mathcal{U},$ s.t. $X = \cup \mathcal{U}_0$.
Define $\mathcal{F}_0 := \{X \backslash U | U \in \mathcal{U}_0\}$, thus $\mathcal{F}_0$ is finite and $\cap \mathcal{F}_0 = \emptyset$, which leads to the FIP of X. Thus $\cup \mathcal{U} \neq X \Leftrightarrow \cap \mathcal{F} \neq \emptyset$.
$\Leftarrow$: If X is not a compact set, we will show the statement in the right side is wrong. If X is not a compact set then $\exists$ s family $\mathcal{U}$ of open subsets of X such that any finite $\mathcal{U}_0 \subseteq \mathcal{U}$ has $X \neq \cup \mathcal{U}_0$.
Define $\mathcal{F} := \{X \backslash U | U \in \mathcal{U}\}; \mathcal{F}_0 := \{X \backslash U | U \in \mathcal{U}_0\}$ for any finite $\mathcal{U}_0 \subseteq \mathcal{U}$. Thus $\mathcal{F}$ has FIP, but $\cap \mathcal{F} = \emptyset$. $\square$

Proposition 1. *Let $(X, \mathscr{T})$ be a topology space, $K \subseteq X$, then*

1. *X is Hausdorff space, K is compact $\Rightarrow K \subseteq_{close} X$;*
2. *X is compact space, $K \subseteq_{close} X \Rightarrow K$ is compact.*

Proof.

1. Select a point $x \in X \backslash K$, then for any $k \in K, \exists U_k, V_k \subseteq_{open} X,$ s.t. $k \in U_k, x \in V_k$ and $U_k \cap V_k = \emptyset$. Thus $K \subseteq \cup_{k \in K} U_k$. Since K is compact, $\exists k_1, \cdots, k_n \in K,$ s.t. $K \subseteq \cup_{i=1}^n U_{k_i}$, and $x \in \cap_{i=1}^n V_{k_i} \subseteq_{open} X$. And $(\cup_{i=1}^n U_{k_i}) \cap (\cap_{i=1}^n V_{k_i}) = \emptyset \Rightarrow \cap_{i=1}^n V_{k_i} \subseteq X \backslash K \Rightarrow X \backslash K$ is open $\Rightarrow K$ is close.
2. Suppose $\exists U_\alpha \subseteq_{open} X (\alpha \in A),$ s.t. $K \subseteq \cup_{\alpha \in A} U_\alpha$, thus $X = K \cup X \backslash K = (X \backslash K) \cup \cup_{\alpha \in A} U_\alpha$. Since X is compact thus $\exists$ finite $A_0 \subseteq A,$ s.t. $K \subseteq X = (X \backslash K) \cup \cup_{\alpha \in A_0} U_\alpha \Rightarrow K$ is compact.

$\square$

Proposition 2 (Continuous Maps Preserve Compactness)**.** *Suppose X, Y are top. sp. $X \xrightarrow{f} Y$ is continuous. $K \subseteq_{cpt.} X \Rightarrow f(K) \subseteq_{cpt.} Y$.*

Proof. Suppose $\exists U_\alpha \subseteq_{open} Y(\alpha \in A)$, s.t. $f(K) \subseteq \cup_{\alpha\in A} U_\alpha \Rightarrow K \subseteq f^{-1}(\cup_{\alpha\in A} U_\alpha) = \cup_{\alpha\in A} f^{-1}(U_\alpha)$. Since K is compact, $\exists$ finite $A_0 \subseteq A$, s.t. $K \subseteq \cup_{\alpha\in A_0} U_\alpha \Rightarrow f(K) \subseteq \cup_{\alpha\in A_0} U_\alpha \Rightarrow f(K)$ is compact. $\square$

Proposition 3. *Let $X \xrightarrow{f} Y$ be a continuous map with X is compact and Y is Hausdorff, then*

1. *f is a close map (i.e. $\forall C \subseteq_{close} X, f(C) \subseteq_{close} Y$);*
2. *f is a surjection $\Rightarrow$ f is a quotient map;*
3. *f is a bijection $\Rightarrow$ f is a homeomorphism (i.e. a bijection which and whose inverse are both continuous).*

Proof.

1. Any close set C in X is compact, thus $f(C)$ is compact, since Y is Hausdorff, $f(C)$ is close;
2. For any $V \subseteq Y$, if $f^{-1}(V)$ is closed $\Rightarrow f(f^{-1}(V))$ is closed, and $V = f(f^{-1}(V))$ is closed since f is surjection.
   On the other hand, if V is closed, since f is continuous, $f^{-1}(V)$ is closed. Thus f is quotient map.
3. All we need to prove is the inverse of f, denoted by $Y \xrightarrow{\overline{f}} X$ is continuous.
   Note that for any $y \in f(U)$, $\exists x \in U$, s.t. $y = f(x)$ and $x = \overline{f}(y)$, thus $y \in \overline{f}^{-1}(x) \subseteq \overline{f}^{-1}(U)$, thus $f(U) \subseteq \overline{f}^{-1}(U)$. On the other hand, for any $y \in \overline{f}^{-1}(U)$, $\overline{f}(y) \in U \Rightarrow \exists x \in U$, s.t. $x = \overline{f}(y)$ and $y = f(x) \in f(U)$. Thus $\overline{f}^{-1}(U) \subseteq f(U)$. Thus we have for any $U \in X$,
   $$f(U) = \overline{f}^{-1}(U),$$
   For any $V \subseteq_{close} X$, $\overline{f}^{-1}(V) = f(V) \subseteq_{close} Y$, since f is a close map, thus $\overline{f}$ is continuous and f is a homeomorphism.

$\square$

Remark 1. Given a map $X \xrightarrow{f} Y$, for any $A \subseteq X, B \subseteq Y$:

1. f is injection $\Rightarrow f^{-1}(f(A)) = A$;
2. f is surjection $\Rightarrow f(f^{-1}(B)) = B$;

Exercise 4. *Let R be an equiv. rel. on $[0,1] \times [0,1]$ whose equiv. classes are exactly*

$$\{(x,y)\}, \quad \textit{if } (x,y) \in (0,1) \times [0,1]$$
$$\{(0,y),(1,1-y)\}, \quad \textit{if } y \in [0,1]$$

Define

$$\begin{aligned} Y := \{&(2 + t\cos(\theta/2))\cos(\theta), \\ &(2 + t\cos(\theta/2))\sin(\theta), \\ &t\sin(\theta/2) \\ &|(\theta,t) \in [0,2\pi] \times [-0.5,0.5]\} \end{aligned}$$

as a subspace of $\mathbb{R}^3$. Show that there exists a homeomorphism from $X := [0,1] \times [0,1]/R$ to Y.

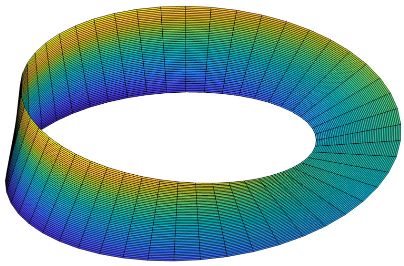


Figure 1: Y: a subspace of $\mathbb{R}^3$

Proof.

1. Y, equipped with subspace topology, is a Hausdorff space:
For any $y_1, y_2 \in Y, \exists U_1, U_2 \subseteq_{open} \mathbb{R}^3$, s.t. $y_1 \in U_1, y_2 \in U_2$ and $U_1 \cap U_2$. Thus $y_1 \in Y \cap U_1 \subseteq_{open} Y$ and $y_2 \in Y \cap U_2 \subseteq_{open} Y$ and $(Y \cap U_1) \cap (Y \cap U_2) = Y \cap (U_1 \cap U_2) = \emptyset \Rightarrow Y$ is Hausdorff space.
2. X, equipped with quotient topology, is a compact space:
Since X is equipped with the quotient topology, thus the natural projection $[0,1] \times [0,1] \xrightarrow{\pi} [0,1] \times [0,1]/R$ is continuous. Since $[0,1] \times [0,1]$ is a compact subset of $\mathbb{R}^2 \Leftrightarrow [0,1] \times [0,1]$ is a compact set, thus $X = \pi([0,1] \times [0,1])$ is a compact set.
3. $\exists$ a bijection $X \xrightarrow{m} Y$:
For any $(x,y) \in (0,1) \times [0,1]$, define a map $h : \{(x,y)\} \mapsto (\theta, t)$ where $\theta = 2\pi x, t = y - 0.5$;
For any $x = 0$ (or 1), define $h : \{(0,y), (1, 1-y)\} \mapsto (2\pi, t)$ (or $(0, -t)$) where $t = y - 0.5$; It is direct to see $X \xrightarrow{h} \{(\theta, t) | \theta \in [0, 2\pi], t \in [-0.5, 0.5]\}$ is a bijection.
Finally, define $\{(\theta, t) | \theta \in [0, 2\pi], t \in [-0.5, 0.5]\} \xrightarrow{g} Y$ which is a bijection as well,
Thus $m = g \circ h$ is a bijection.

Collectively, $X \xrightarrow{m} Y$ is a bijection from compact space to Hausdorff space, thus m is a homeomorphism. □

Definition 3 (Proper Map)**.** A map $X \xrightarrow{f} Y$ between topology spaces is called a proper map if $f^{-1}(K) \subseteq_{cpt.} X$ for $\forall K \subseteq_{cpt.} Y$.

Proposition 4. *X, Y are compact spaces $\Rightarrow$ $X \times Y$ equipped with the product topology is compact.*

Thus if Y is compact, X is topology space, then the projection $X \times Y \xrightarrow{\pi_X} X$ is a proper map.

Exercise 5. *Let $X \xrightarrow{f} Y$ is a map between topology spaces, $\mathcal{B}$ is a basis of the topology of X, show that f is an open map $\Leftrightarrow \forall B \in \mathcal{B}, f(B) \subseteq_{open} Y$.*

Proof. $\Rightarrow$: $\forall B \in \mathcal{B}, B \subseteq_{open} X \Rightarrow f(B) \subseteq_{open} Y$. $\Leftarrow$: $\forall U \subseteq_{open} X$ can be represented as $U = \cup_{F \in \mathcal{F}} F$ where $\mathcal{F} \subseteq \mathcal{B}$. Thus $f(U) = f(\cup_{F \in \mathcal{F}} F) = \cup_{F \in \mathcal{F}} f(F) \subseteq_{open} Y$. □

Thus if X, Y are topology, then map $X \times Y \xrightarrow{\pi} X$ is an open map.

2 HLC Space

Definition 4 (Locally Compact)**.** X is a locally compact space if $\forall x \in X$ has a compact nbd. (i.e. $\forall x \in X, \exists K \subseteq_{cpt.} X$, s.t. $x \in K^o$, or equivalently, $\forall x \in X, \exists U \subseteq_{open} X, x \in U \subseteq \overline{U} \subseteq_{cpt.} X$)

Exercise 6. *If X is a locally compact Hausdorff (LCH) space and $x \in X$ has an open nbd. U, show that, there is a compact nbd. of x which is a subset of U. (That is $x \in U \subseteq_{open} X$, then $\exists W \subseteq_{open} X$,* s.t. *$x \in W \subseteq \overline{W} \subseteq U$ where $\overline{W} \subseteq_{cpt.} X$).*

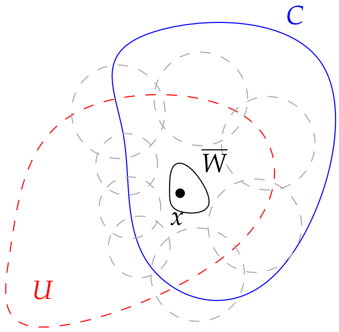
Proof. Given a $x \in X$ and an open nbd. U of x. Since X is locally compact, $\exists C \subseteq_{cpt.} X$, s.t. $x \in C$. Since X is Hausdorff $\Rightarrow C$ is closed $\Rightarrow x \in U \cap C^o \subseteq_{open} X$.

Denote $\partial[U \cap C^o]$ as ∂, since $\partial = X \backslash ((U \cap C^o)^o \cup (U \cap C^o)^e)$, ∂ is closed. Since $\partial \subseteq \partial[U \cap C^o] \cup (U \cap C^o)^o = \overline{U \cap C^o} \subseteq \overline{C} = C$, ∂ is a closed subset of compact set C, thus ∂ is compact.

Since $x \in U \cap C^o$, thus $x \notin \partial$. Since X is Hausdorff, for any $s \in \partial, \exists V_s, W_s \subseteq_{open} X$, s.t. $s \in V_s$ and $x \in W_s$ and $V_s \cap W_s = \emptyset$. Thus $\partial \subseteq \cup_{s \in \partial} V_s \Rightarrow \exists$ finite $\partial_0 \subseteq \partial$, s.t. $\partial \subseteq \cup_{s \in \partial_0} V_s \subseteq_{open} X$ and $x \in \cap_{s \in \partial_0} W_s \subseteq_{open} X$.

Denote $\cap_{s \in \partial_0} W_s =: W$ and $\cup_{s \in \partial_0} V_s =: V$, thus $W \cap V = \emptyset \Rightarrow W \subseteq X \backslash V \Rightarrow \overline{W} \subseteq \overline{X \backslash V} = X \backslash V \Rightarrow \overline{W} \cap V = \emptyset \Rightarrow \overline{W} \cap \partial = \emptyset$. Since $\overline{W} \subseteq \overline{U \cap C^o} = (U \cap C^o) \cup \partial \Rightarrow \overline{W} \subseteq U \cap C^o \subseteq U$ and $\overline{W} \subseteq C$.

Finally, since C is compact, $\overline{W}$ is closed $\Rightarrow \overline{W}$ is compact. Thus $x \in W \subseteq \overline{W} \subseteq U$ and $\overline{W} \subseteq_{cpt.} X$. □



Exercise 7. *More generally, we can replace the point x with a compact set, i.e. X is HLC space, $\forall K \subseteq_{cpt.} X$ if $\exists U \subseteq_{open} X$, s.t. $K \subseteq U$ show that $\exists W \subseteq_{open} X$, s.t. $K \subseteq W \subseteq \overline{W} \subseteq U$ where $\overline{W} \subseteq_{cpt.} X$.*

Proof. For any $k \in K, k \in U$, thus $\exists W^{(k)} \subseteq_{open} X$, s.t. $k \in W^{(k)} \subseteq \overline{W^{(k)}} \subseteq U$ where $\overline{W^{(k)}} \subseteq_{cpt.} X$. Thus $K \subseteq \cup_{k \in K} W^{(k)}$ and since K is compact, there exists a finite $K_0 \subseteq K$, s.t.

$$K \subseteq \cup_{k \in K_0} W^{(k)} \subseteq \cup_{k \in K_0} \overline{W^{(k)}} \subseteq U.$$

Since $\overline{W^{(k)}}$ is compact for $\forall k \in K \Rightarrow \cup_{k \in K_0} \overline{W^{(k)}}$ is compact. And since K_0 is finite, $\cup_{k \in K_0} \overline{W^{(k)}} = \overline{\cup_{k \in K_0} W^{(k)}}$. Thus $W := \cup_{k \in K_0} W^{(k)}$ and

$$K \subseteq W \subseteq \overline{W} \subseteq U$$

where $W \subseteq_{open} X$ and $\overline{W} \subseteq_{cpt.} X$. □

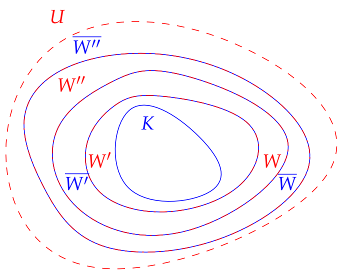
Note that there is an iteration process, that is if $K \subseteq_{cpt.} X, and K \subseteq U \subseteq_{open} X$, and then $\exists W \subseteq_{open} X$ and $\overline{W} \subseteq_{cpt.} X$, s.t. $K \subseteq W \subseteq \overline{W} \subseteq U$. Then $\exists W', W'' \subseteq_{open} X$ and $\overline{W'}, \overline{W''} \subseteq_{cpt.} X$, s.t.

$$K \subseteq W' \subseteq \overline{W'} \subseteq W,$$

and

$$\overline{W} \subseteq W'' \subseteq \overline{W''} \subseteq U$$

and so on.



3 Continuous $\mathbb{R}$ - value maps

Let X be a topology space, consider a $\mathbb{R}$ - value map $X \xrightarrow{f} \mathbb{R}$ on it. Now we want to explore the relationship between the continuity of f and the topology structure of X.

Exercise 8. *Given a trivial topology space X, show that $X \xrightarrow{f} \mathbb{R}$ is constant $\Leftrightarrow X \xrightarrow{f} \mathbb{R}$ is continuous.*

Proof. $\Rightarrow$: Suppose for $\forall x \in X, f(x) \equiv r \in \mathbb{R}$. For any $U \subseteq_{open} \mathbb{R}$ containing r, $f^{-1}(U) = X \subseteq_{open} X$; and for any $V \subseteq_{open} \mathbb{R}$ that do not contain r, $f^{-1}(V) = \emptyset \subseteq_{open} X$, thus f is continuous.
$\Leftarrow$: If f is not a constant map, then $\exists x_1, x_2 \in X,$ s.t. $f(x_1) = r_1, f(x_2) = r_2$ and $r_1 \neq r_2$. Since $r_1, r_2 \in \mathbb{R}$, and $\mathbb{R}$ is Hausdorff space, then $\exists U, V \subseteq_{open} \mathbb{R}$, s.t. $r_1 \in U, r_2 \in V$ and $U \cap V = \emptyset$. Thus $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset$. And $x_1 \in f^{-1}(U) \neq \emptyset$ and $x_2 \in f^{-1}(V) \neq \emptyset$.
If f is continuous, then $f^{-1}(U) \subseteq_{open} X \Rightarrow f^{-1}(U) = X$ which leads to a contradiction with $f^{-1}(V) = f^{-1}(U \cap V) = \emptyset$, thus f is not continuous. $\square$

As we can see that if X is a trivial topology space, then the $\mathbb{R}$ - value map f on it is continuous iff f is constant map. Now if we add some sets into the trivial topology, can we obtain more continuous $\mathbb{R}$ - value maps that are not constant?

Exercise 9. *Let X be an infinite set, define $\mathscr{T} := \{U \subseteq X | U = \emptyset \vee X \backslash U \text{ is finite}\}$ which is called **Cofinite topology**. Show that The only $\mathbb{R}$ - valued continuous maps on $(X, \mathscr{T})$ are constant maps.*

Proof. We have proved that any $\mathbb{R}$ - valued constants map on X is continuous, we will show that any $\mathbb{R}$ - valued un-constants maps on X is not continuous.
Just as we shown before, If f is not a constant map, then $\exists x_1, x_2 \in X,$ s.t. $f(x_1) = r_1, f(x_2) = r_2$ and $r_1 \neq r_2$. Since $r_1, r_2 \in \mathbb{R}$, and $\mathbb{R}$ is Hausdorff space, then $\exists U, V \subseteq_{open} \mathbb{R}$, s.t. $r_1 \in U, r_2 \in V$ and $U \cap V = \emptyset$. Thus $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset$. And $x_1 \in f^{-1}(U) \neq \emptyset$ and $x_2 \in f^{-1}(V) \neq \emptyset$.
Then if f is continuous, then $f^{-1}(U) \in \mathscr{T}$, since $f^{-1}(U) \neq \emptyset \Rightarrow X \backslash f^{-1}(U)$ is finite. Since $f^{-1}(U) \cap f^{-1}(V) = \emptyset \Rightarrow f^{-1}(V) \subseteq X \backslash f^{-1}(U)$ and thus $f^{-1}(V)$ is finite. Since X is infinite, $X \backslash f^{-1}(V)$ is infinite $\Rightarrow f^{-1}(V) \notin \mathscr{T} \Rightarrow f$ is not continuous. $\square$

As we can see that, even though we add some sets into the topology of X, we can not construct some 'nontrivial' $\mathbb{R}$ - valued maps. Actually, if X is uncountable, even if we add sets into $\mathscr{T}$ again, such as define $\mathscr{T}' := \{U \subseteq X | U = \emptyset \vee X \backslash U \text{ is countable}\}$ which is called **Cocountable topology**, the only $\mathbb{R}$ - valued continuous maps on $(X, \mathscr{T}')$ are still constant maps.

These examples tell us, there is a 'threshold' of the topology, if the topology is too coarse to achieve the threshold, then the only $\mathbb{R}$ - valued continuous maps on X are constant maps.

Let X be a topology space and $A, B \subseteq X$ be disjoint. We say a **Chain** $\mathcal{C}$ from A to B consists of a sequence of subsets C_k of $X (k = 0, 1, \cdots, r)$, s.t.

$$A = C_0 \subseteq \overline{C_0} \subseteq C_1^o \subseteq \overline{C_1} \subseteq \cdots \subseteq \overline{C_{r-1}} \subseteq C_r^o \subseteq \overline{C_r} \subseteq X \backslash B.$$

For a chain $\mathcal{C} : C_k (k = 0, \cdots, r)$, we let $C_0 := \emptyset$ and $C_{r+1} := X$ and define

$$f_{\mathcal{C}}(x) := \begin{cases} 1 - \dfrac{k}{r}, & \text{if } x \in C_k \backslash C_{k-1}, k = 0, \cdots, r \\ 0, & \text{if } x \notin C_r, \end{cases}$$

It is direct to see that $|f_{\mathcal{C}}(x) - f_{\mathcal{C}}(x')| \leq \frac{1}{r}$ if $x, x' \in C_{k+1}^o \backslash \overline{C_{k-1}} =: \Omega_k$ for any $k = 0, \cdots, r$. And $\Omega_k \subseteq_{open} X$ and $\cup_{i=0}^r \Omega_k = X$.

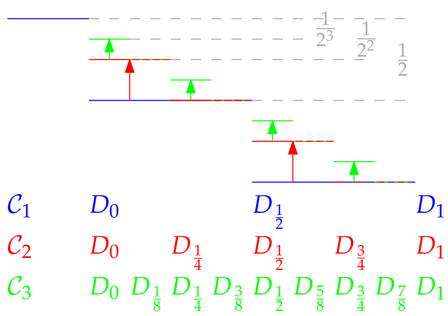
Lemma 1. *Suppose X is a topology space, $A, B \subseteq X$ are disjoint. $D_q \subseteq X$ where*

$$q \in \left\{ \frac{l}{2^m} \middle| l, m \in \mathbb{N}_0, l \leq 2^m \right\} =: Q,$$

s.t. $q \leq q' \Rightarrow \overline{D_q} \subseteq D_{q'}^o$ and $A = D_0, D_1 \subseteq X \backslash B$. Then $\exists$ a continuous map $X \xrightarrow{f} [0, 1]$ s.t. $f(A) = \{1\}$ and $f(B) = \{0\}$.

Proof. Let $\mathcal{C}_m$ be the chain $D_0, D_{\frac{1}{2^m}}, \cdots, D_{\frac{2^m - 1}{2^m}}, D_1$ from A to B. Thus

$$\begin{aligned} \mathcal{C}_0 &= D_0 (= A), D_1 \\ \mathcal{C}_1 &= D_0, D_{\frac{1}{2}}, D_1 \\ \mathcal{C}_2 &= D_0, D_{\frac{1}{4}}, D_{\frac{1}{2}}, D_{\frac{3}{4}}, D_1 \\ &\cdots \end{aligned}$$



Define $f_m := f_{\mathcal{C}_m} : X \to \mathbb{R}(m \in \mathbb{N}_0)$. Since for any $x \in X, m, m' \in \mathbb{N}_0$, $f_m(x) \leq 1$, and if $m \leq m' \Rightarrow f_m(x) \leq f_{m'}(x)$. Thus $f_m \to f$ as $m \to \infty$. And

$$f(x) - f_m(x) = \lim_{k \to \infty} \sum_{n=m}^{k} (f_{n+1}(x) - f_n(x))$$

where $f_{n+1}(x) - f_n(x) \leq \frac{1}{2^{n+1}}$ for $\forall x \in X$. Thus

$$f(x) - f_m(x) \leq \sum_{n=m}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2^m}$$

for any $x \in X$ and $m \in \mathbb{N}_0$. Thus for a given $x_0 \in X$ and any $x \in X$, we have

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_m(x)| + |f_m(x) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\ &\leq \frac{1}{2^m} + \frac{1}{2^m} + |f_m(x) - f_m(x_0)| \end{aligned}$$

For any $\epsilon > 0$, we can choose and fix a large enough m such that $\frac{1}{2^m} < \frac{\epsilon}{3}$. Assume that $x_0 \in \Omega_s$ of $\mathcal{C}_m$ (that is $x_0 \in C^o_{\frac{s+1}{2^m}} \backslash \overline{C_{\frac{s-1}{2^m}}}$), then for any $x \in \Omega_s \subseteq_{open} X, we have that |f_m(x) - f_m(x_0)| \leq \frac{1}{2^m}$ and

$$|f(x) - f(x_0)| \leq \frac{1}{2^m} + \frac{1}{2^m} + \frac{1}{2^m} \leq \epsilon,$$

thus f is continuous, and $f(A) = \{1\}, f(B) = \{0\}$. □

Thus if X is a HLC space, $A, B \subseteq_{cpt.} X$ are disjoint, then there exists a continuous $\mathbb{R}$ - valued map $X \xrightarrow{f} \mathbb{R}$ such that $f(A) = \{1\}$ and $f(B) = \{0\}$.