Introduction to Analysis

Lecture 4

Haoming Wang

25 April 2019

Abstract

This is the Lecture note for the *Introduction to Analysis* class in Spring 2019.

1 Rearrangement theorem

Given a seq. $a_n (n \in \mathbb{N})$, we can separate all terms into two subseq.s:

$$a_{n_1}, a_{n_2}, \cdots \text{ and } a_{n'_1}, a_{n'_2}, \cdots$$

where $n_1 < n_2 < \cdots$ and $n'_1 < n'_2 < \cdots$ and $\{n_1, n_2, \cdots\} \cup \{n'_1, n'_2, \cdots\} = \mathbb{N}$, such that $a_{n_j} \geq 0 (j \in \mathbb{N}), a_{n'_k} \leq 0 (k \in \mathbb{N})$. Let $p_j := a_{n_j} (j \in \mathbb{N})$ and $q_k := a_{n'_k} (k \in \mathbb{N})$.

Exercise 1. *Show that $\sum_n |a_n| < \infty \Leftrightarrow \sum_j p_j < \infty$ and $\sum_k q_k < \infty$. Moreover, if any side holds, then*

$$\sum_n |a_n| = \sum_j p_j + \sum_k q_k$$

and

$$\sum_n a_n = \sum_j p_j - \sum_k q_k.$$

Exercise 2. *If $\sum_n a_n$ converges conditionally, show that*

1. $\sum_j p_j = \infty$ *and* $\sum_k q_k = \infty$;
2. $\lim_{j \to \infty} p_j = \lim_{k \to \infty} q_k = 0$.

Exercise 3. *If $\sum_n a_n, \sum_n b_n$ converges, show that $\sum_n (a_n \pm b_n) = \sum_n a_n \pm \sum_n b_n$.*

Exercise 4. *Inserting 0s to a series will not affect its convergence / divergence and its sum (if the sum exists).*

Recall that a sequence a_n is a map $\mathbb{N} \xrightarrow{a.} \mathbb{R}$ where $n \mapsto a(n)$ denoted by a_n. A subsequence a_{n_m} is a composite map

$$\mathbb{N} \xrightarrow{n.} \mathbb{N} \xrightarrow{a.} \mathbb{R}$$

where $n.$ is an injection (or say $n. \nearrow\nearrow$) and $m \mapsto n(m)$ denoted by n_m. A **rearrangement** is also a composite map

$$\mathbb{N} \xrightarrow{n(\cdot)} \mathbb{N} \xrightarrow{a.} \mathbb{R}$$

where $n(\cdot)$ is a bijection. (To distinguish it from the notion of subseq., we don't use subscripts here)
Now we gonna explore two questions: if a series $\sum_n$ converges, $a_{n(m)}(m \in \mathbb{N})$ is a rearrangement of $a_n(n \in \mathbb{N})$, then
1. whether $\sum_m a_{n(m)}$ converges ?
2. whether $\sum_n a_n = \sum_m a_{n(m)}$?

Exercise 5. *Let $\sum_n a_n$ be a positive series, show that*

$$\sum_n a_n = \sup\{a_{n_1} + \cdots + a_{n_k} | n_1 < \cdots < n_k, k \in \mathbb{N}\}$$

including the case $\sum_n a_n = \infty$.

Exercise 6. *If $\sum_n a_n$ is a convergent positive series, show that for every rearrangement $a_{n(m)}(m \in \mathbb{N})$ of $a_n(n \in \mathbb{N})$, we have that $\sum_n a_n = \sum_m a_{n(m)}$.*

Exercise 7 (Dirichlet's Rearrangement Theorem (1829))**.** *If $\sum_n a_n$ converges absolutely, show that for every rearrangement $a_{n(m)}(m \in \mathbb{N})$ of $a_n(n \in \mathbb{N})$, we have that $\sum_n a_n = \sum_m a_{n(m)}$.*

Theorem 1 (Riemann's Rearrangement Theorem(1852))**.** *If $\sum_n a_n$ converges conditionally, then for $\forall r \in \mathbb{R}$, there exists a rearrangement $a_{n(m)}(m \in \mathbb{N})$ of $a_n(n \in \mathbb{N})$ such that $\sum_m a_{n(m)} = r$.*

Proof. □

Remark 1 ($2S = S$). Dirichelet made the following observation in 1827: Consider a alternating series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$$

which is convergent (conditionally) by Leibniz's Criterion, and hence

$$\begin{aligned} 2S &= 2 - 1 + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \frac{2}{7} - \frac{2}{8} + \frac{2}{9} - \frac{2}{10} + \cdots \\ &=' (2-1) - \frac{2}{4} + \left(\frac{2}{3} - \frac{2}{6}\right) - \frac{2}{8} + \left(\frac{2}{5} - \frac{2}{10}\right) + \cdots \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots \\ &= S \end{aligned}$$

2 Multiplying absolutely convergent series

Proposition 1. *Let* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *converge absolutely, let*

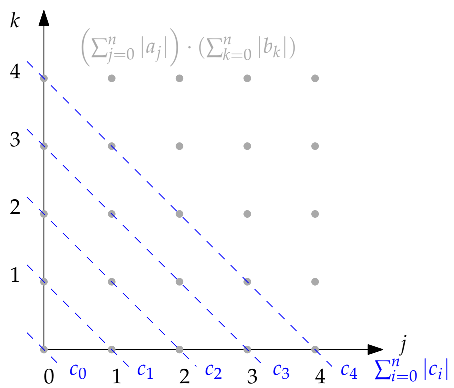
$$c_n = a_n b_0 + \cdots + a_0 b_n = \sum_{j+k=n;\ j,k\geq 0} a_j b_k,$$

then $\sum_n |c_n| < \infty$ *and* $\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right)$.

Proof. 1. $\sum_n |c_n| < \infty$
For all n,

$$\begin{aligned}\sum_{m=0}^{n} |c_m| = \sum_{m=0}^{n} \left| \sum_{\substack{j+k=m\\ j,k\geq 0}} a_j b_k \right| &\leq \sum_{m=0}^{n} \sum_{\substack{j+k=m\\ j,k\geq 0}} |a_j||b_k| \\ &\leq \left(\sum_{j=0}^{n} |a_j|\right) \cdot \left(\sum_{k=0}^{n} |b_k|\right).\end{aligned}$$

Since $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converges absolutely, the partial sums of $|a_n|, |b_n|$ have upper bounds, denoted by M, N respectively, then $\sum_{m=0}^{n} |c_m|$ has a upper bound $M \cdot N$ and hence $\sum_{n=0}^{\infty} c_n$ converges absolutely.

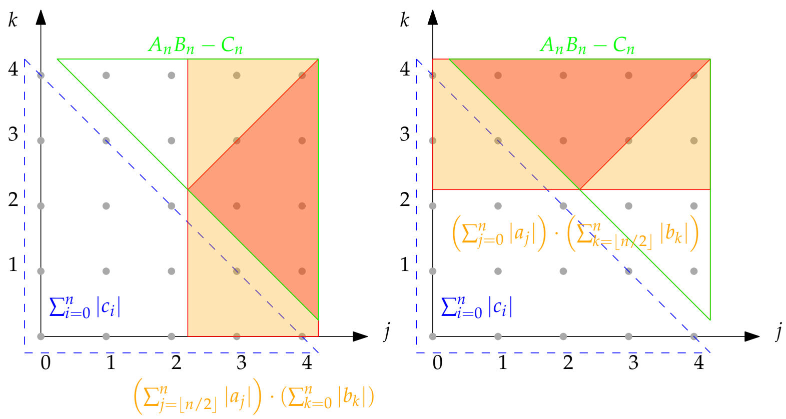


2. $\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right)$.
Let $A_n := a_0 + \cdots + a_n; B_n := b_0 + \cdots + b_n$ and $C_n := c_0 + \cdots + c_n$, we claim that $\lim_{n\to\infty}(A_n B_n - C_n) = 0$. Then

$$\begin{aligned}|A_n B_n - C_n| &= \sum_{\substack{j+k>n\\ 0\leq j,k\leq n}} |a_j b_k| \\ &\leq \left(\sum_{j=\lfloor n/2 \rfloor}^{n} |a_j|\right) \cdot \left(\sum_{k=0}^{n} |b_k|\right) + \left(\sum_{j=0}^{n} |a_j|\right) \cdot \left(\sum_{k=\lfloor n/2 \rfloor}^{n} |b_k|\right)\end{aligned}$$

where $\sum_{k=0}^{n}|b_k|, \sum_{j=0}^{n}|a_j|$ are bounded, and tails $\sum_{j=\lfloor n/2 \rfloor}^{n}|a_j|, \sum_{k=\lfloor n/2 \rfloor}^{n}|b_k| \to 0$ as $n \to \infty$ since $\sum_n a_n, \sum_n b_n$ are converges abs. Thus $\lim_{n\to\infty}|A_nB_n - C_n| = 0$ and since $\lim_{n\to\infty} A_n, \lim_{n\to\infty} B_n, \lim_{n\to\infty} C_n$ exists, we have that

$$\begin{aligned}\sum_{n=0}^{\infty} c_n &= \lim_{n\to\infty} C_n \\ &= \lim_{n\to\infty} A_nB_b = \lim_{n\to\infty} A_n \cdot \lim_{n\to\infty} B_n \\ &= \left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right)\end{aligned}$$



□

Theorem 2. *If $\sum_n a_n, \sum_n b_n$ cvg. abs., $\mathbb{N} \xrightarrow{(j(\cdot),k(\cdot))} \mathbb{N}\times\mathbb{N}$ is bijection where $n \mapsto (j(n), k(n))$, let $c_n := a_{j(n)}b_{k(n)} (n \in \mathbb{N})$, then $\sum_n |c_n| < \infty$ (cvg. abs.) and $\sum_n c_n = (\sum_n a_n)(\sum_n b_n)$.*

Proof. 1. $\sum_n c_n$ cvg. abs.
For $\forall n \in \mathbb{N}$, let $l = \max\{j(1), \cdots, j(n), k(1), \cdots, k(n)\}$. Then

$$\begin{aligned}|c_1| + \cdots + |c_n| &= |a_{j(1)}b_{k(1)}| + \cdots + |a_{j(n)}b_{k(n)}| \\ &\le \left(\sum_{j=1}^{l}|a_j|\right) \cdot \left(\sum_{k=1}^{l}|b_k|\right) \\ &\le M \cdot N\end{aligned}$$

Thus $\sum_n c_n$ cvg. abs.
2. $\sum_n c_n = (\sum_n a_n)(\sum_n b_n)$.
Let $A_n = a_1 + \cdots + a_n, B_n = b_1 + \cdots + b_n$ and $C_n = c_1 + \cdots + c_n (n \in \mathbb{N})$. And define the bijection $\mathbb{N} \to \mathbb{N}\times\mathbb{N}$ by the second one in Figure 1. Then

$$\begin{aligned} A_n B_n &= (a_1 + \cdots + a_n)(b_1 + \cdots + b_n) \\ &= \sum_{1 \le j,k \le n} a_j b_k \\ &= C_{n^2} \end{aligned}$$

Thus $\lim_{n\to\infty} A_n B_n = \lim_{n\to\infty} C_{n^2} = \lim_{n\to\infty} C_n \Rightarrow \sum_n c_n = (\sum_n a_n)(\sum_n b_n)$. $\square$

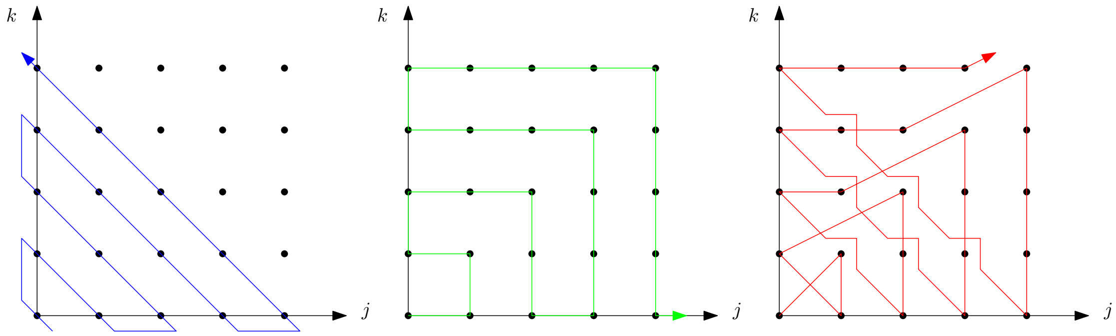


Figure 1: 3 kinds of bijections $(j(\cdot), k(\cdot))$