

# Point Set Topology

## Lecture 3

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THIS IS THE LECTURE NOTE FOR THE *Point Set Topology*.

### CONTENT:

1. Basis
2. Countable, Separable and Lindelof Compact
3. Examples

## Basis

**Definition 1** (Coarser Topology). Let  $X$  be a set, and  $\mathcal{T}$  and  $\mathcal{T}'$  be two topologies on  $X$ . We say that  $\mathcal{T}$  is coarser/weaker than  $\mathcal{T}'$  if  $\mathcal{T} \subseteq \mathcal{T}'$  (or say  $\mathcal{T}'$  is finer/stronger than  $\mathcal{T}$ ).

Let  $X$  be a set and  $\mathcal{S} \subseteq \mathcal{P}(X)$  be a family of subsets of  $X$ . Are there a smallest topology  $\mathcal{T}'$  on  $X$  s.t. all  $\mathcal{S} \subseteq \mathcal{T}'$ ? It is direct to check that if  $\mathcal{T}_\alpha (\alpha \in A)$  is a family of topologies on  $X$ , then  $\cap_{\alpha \in A} \mathcal{T}_\alpha$  is also a topology on  $X$ . For any  $\alpha \in A$ :

1.  $\emptyset, X \in \mathcal{T}_\alpha \Rightarrow \emptyset, X \in \cap_{\alpha \in A} \mathcal{T}_\alpha$ ;
2.  $U_\beta \in \mathcal{T}_\alpha (\beta \in B) \Rightarrow \cup_{\beta \in B} U_\beta \in \mathcal{T}_\alpha \Rightarrow \cup_{\beta \in B} U_\beta \in \cap_{\alpha \in A} \mathcal{T}_\alpha$ .
3.  $U_1, U_2 \in \mathcal{T}_\alpha \Rightarrow U_1 \cap U_2 \in \mathcal{T}_\alpha \Rightarrow U_1 \cap U_2 \in \cap_{\alpha \in A} \mathcal{T}_\alpha$ .

Define  $\mathcal{T}$  be the family of all topologies on  $X$  containing the elements in  $\mathcal{S}$ , that is for  $\forall \mathcal{T} \in \mathcal{T}, \mathcal{S} \subseteq \mathcal{T}$ . We call

$$\mathcal{T}(\mathcal{S}) := \cap_{\mathcal{T} \in \mathcal{T}} \mathcal{T}$$

the topology induced by  $\mathcal{S}$ , which is clearly the coarsest topology containing  $\mathcal{S}$ .

Let  $\Pi$  be the family of any finite intersection of the element in  $\mathcal{S}$ , then for  $\forall \mathcal{T} \in \mathcal{T}, \Pi \subseteq \mathcal{T}$  by def. Furthermore, for  $\forall \mathcal{T} \in \mathcal{T}$ , the arbitrary union of the elements in  $\Pi$  must in  $\mathcal{T}$ , that is  $\{\cup_{V \in \mathcal{F}} V | \mathcal{F} \subseteq \Pi\} \subseteq \mathcal{T}$ . Thus  $\{\cup_{V \in \mathcal{F}} V | \mathcal{F} \subseteq \Pi\} \subseteq \cap_{\mathcal{T} \in \mathcal{T}} \mathcal{T} = \mathcal{T}(\mathcal{S})$ .

**Proposition 1.** Let  $X$  be a set and  $\mathcal{S} \subseteq \mathcal{P}(X)$  be a family of subsets of  $X$ . Then

$$\mathcal{T}(\mathcal{S}) = \{\cup_{V \in \mathcal{F}} V | \mathcal{F} \subseteq \Pi\},$$

where  $\Pi$  is the family of any finite intersection of elements in  $\mathcal{S}$ , that is

$$\Pi := \{S_1 \cup \dots \cup S_k | S_1, \dots, S_k \in \mathcal{S}, k \in \mathbb{N}\} \cup \{X\}.$$

*Proof.* We have proved that  $\{\cup_{V \in \mathcal{F}} V | \mathcal{F} \subseteq \Pi\} \subseteq \mathcal{T}(\mathcal{S})$ . Note that  $\mathcal{T}(\mathcal{S})$  is the coarsest topology containing  $\mathcal{S}$ , Thus if  $\{\cup_{V \in \mathcal{F}} V | \mathcal{F} \subseteq \Pi\}$  is a topology containing  $\mathcal{S}$ , we are done.

1.  $\{X\}, \emptyset \subseteq \Pi$ , thus  $X = \cup_{V \in \{X\}} V \in \{\cup_{V \in \mathcal{F}} V | \mathcal{F} \subseteq \Pi\}, \emptyset = \cup_{V \in \{\emptyset\}} V \in \{\cup_{V \in \mathcal{F}} V | \mathcal{F} \subseteq \Pi\}$ .

*Note 1.* In other words,  $\mathcal{T}$  is weaker than  $\mathcal{T}'$  iff  $X \xrightarrow{id_X} X$ , where the former and later  $X$  are equipped with  $\mathcal{T}'$  and  $\mathcal{T}$  respectively, is continuous.

*Note 2.* Orally,  $\mathcal{T}(\mathcal{S})$  consists of arbitrary unions of finite intersection of elements of  $\mathcal{S}$ .

Conventionally, when we talking about the subsets of  $X$ , we define  $\cap \emptyset := X$ .

2. For any  $U_\alpha = \{\cup_{V \in \mathcal{F}_\alpha} V \mid \mathcal{F}_\alpha \subseteq \Pi\} \in \{\cup_{V \in \mathcal{F}} V \mid \mathcal{F} \subseteq \Pi\} (\alpha \in A)$ , we have  $\mathcal{F}_\alpha \subseteq \Pi (\alpha \in A) \Rightarrow \cup_{\alpha \in A} \mathcal{F}_\alpha \subseteq \Pi \Rightarrow \cup_{\alpha \in A} U_\alpha = \{\cup_{V \in \cup_{\alpha \in A} \mathcal{F}_\alpha} V \mid \cup_{\alpha \in A} \mathcal{F}_\alpha \subseteq \Pi\} \in \{\cup_{V \in \mathcal{F}} V \mid \mathcal{F} \subseteq \Pi\}$
3. If  $\cup_{V \in \mathcal{F}_1} V, \cup_{W \in \mathcal{F}_2} W \in \{\cup_{V \in \mathcal{F}} V \mid \mathcal{F} \subseteq \Pi\}$ , then  $(\cup_{V \in \mathcal{F}_1} V) \cap (\cup_{W \in \mathcal{F}_2} W) = \cup_{V \in \mathcal{F}_1, W \in \mathcal{F}_2} (V \cap W)$  where  $V, W \in \Pi$ . Since  $\Pi$  is the family of finite intersection,  $V \cap W$  is the finite intersection of elements of  $\mathcal{S}$  or  $X$ , i.e.  $V \cap W \in \Pi$ . Let  $\mathcal{F}_3 := \{V \cap W \mid V \in \mathcal{F}_1, W \in \mathcal{F}_2\}$ , thus  $\mathcal{F}_3 \subseteq \Pi$ . Then  $\cup_{V \in \mathcal{F}_1, W \in \mathcal{F}_2} (V \cap W) = \cup_{Z \in \mathcal{F}_3} Z \in \{\cup_{V \in \mathcal{F}_3} V \mid \mathcal{F}_3 \subseteq \Pi\}$ .

Thus  $\{\cup_{V \in \mathcal{F}} V \mid \mathcal{F} \subseteq \Pi\}$  is a topology containing  $\mathcal{S}$ , and  $\mathcal{T}(\mathcal{S}) \subseteq \{\cup_{V \in \mathcal{F}} V \mid \mathcal{F} \subseteq \Pi\} \Rightarrow \mathcal{T}(\mathcal{S}) = \{\cup_{V \in \mathcal{F}} V \mid \mathcal{F} \subseteq \Pi\}$ .  $\square$

**Definition 2** (Sub-basis). Given a set  $X$ ,  $\mathcal{S} \subseteq \mathcal{P}(X)$ ,  $\mathcal{S}$  is called a sub-basis of a topology  $\mathcal{T}$  on  $X$  if  $\mathcal{T} = \mathcal{T}(\mathcal{S})$ .

To obtain  $\mathcal{T}(\mathcal{S})$  from  $\mathcal{S}$ , we need two steps: first, perform the finite intersection of elements in  $\mathcal{S}$ ; then perform arbitrary union of the these intersection. But can we construct a topology that contains  $\mathcal{S}$  only by union?

**Definition 3** (Basis). Given a set  $X$ , let  $\mathcal{B} \subseteq \mathcal{P}(X)$  and  $\mathcal{T}$  is a topology on  $X$ . We say that  $\mathcal{B}$  is a basis of  $\mathcal{T}$  if  $\mathcal{B} \subseteq \mathcal{T}$  and for any  $U \in \mathcal{T}$ ,  $\exists \mathcal{F} \subseteq \mathcal{B}$ , s.t.  $U = \cup \mathcal{F} (= \cup_{B \in \mathcal{F}} B)$ .

Note that if  $\mathcal{B}$  is a basis of  $\mathcal{T}$ , then  $B \in \mathcal{T}$  for any  $B \in \mathcal{B}$ , thus any union of elements of  $\mathcal{B}$  is in  $\mathcal{T}$ . Thus we can define the  $\mathcal{B}$  is a basis of  $\mathcal{T}$  directly:

$$\mathcal{T} = \{\cup \mathcal{F} \mid \mathcal{F} \subseteq \mathcal{B}\}.$$

In general, a topological space  $(X, \mathcal{T})$  can have many bases. The whole topology  $\mathcal{T}$  is always a base for itself (that is,  $\mathcal{T}$  is a base for  $\mathcal{T}$ ).

**Definition 4** (Local Basis). For a given  $x \in X$ , we say that  $\mathcal{B}_x$  is a local basis of  $\mathcal{T}$  at  $x$ , if

1. for  $\forall V \in \mathcal{B}_x, x \in V \in \mathcal{T}$  and
2. for  $\forall U \in \mathcal{T}$  where  $x \in U$ ,  $\exists V \in \mathcal{B}_x$ , s.t.  $x \in V \subseteq U$ .

**Example 1.** Let  $X$  be a metric space and  $\mathcal{T}$  is the topology defined by metric. Then  $\mathcal{B} = \{B_r(x) \mid r > 0\}$  is a local basis of  $\mathcal{T}$  at  $x$ .

**Exercise 1.** Let  $(X, \mathcal{T})$  be a topology space and  $\mathcal{B} \subseteq \mathcal{P}(X)$ . For  $x \in X$ , define  $\mathcal{B}_x := \{U \in \mathcal{B} \mid x \in U\}$ . Show that  $\mathcal{B}$  is a basis of  $\mathcal{T}$  on  $X \Leftrightarrow \forall x \in X, \mathcal{B}_x$  is a local basis of  $\mathcal{T}$  on  $X$  at  $x$ .

*Proof.*  $\Rightarrow$ : pick a  $x \in X$  and  $U \in \mathcal{T}$  where  $x \in U$ , then  $\exists \mathcal{F} \subseteq \mathcal{B}$ , s.t.  $x \in U = \cup \mathcal{F}$ , since  $\mathcal{B}$  is a basis of  $\mathcal{T}$ . Then  $\exists B \in \mathcal{F}$  such that

*Note 3.* Thus given a sub-basis  $\mathcal{S}$ , we can induce the basis  $\Pi$ , and then perform the union on basis to obtain the topology  $\mathcal{T}(\mathcal{S})$ .

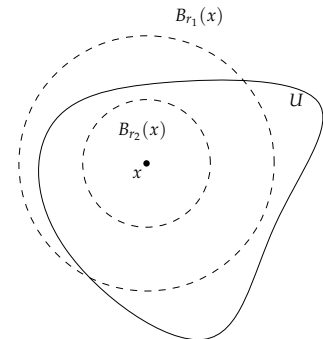


Figure 1: Local Basis

$x \in B \subseteq \cup \mathcal{F} = U$ , it is clear to see  $B \in \mathcal{B}_x$ . Since  $\mathcal{B}$  is a basis of  $\mathcal{T}$ ,  $B \in \mathcal{T}$  for  $\forall B \in \mathcal{B}$ . Thus  $\mathcal{B}_x$  is a local basis of  $\mathcal{T}$  at  $x$  for any  $x \in X$ .

$\Leftarrow$ : On the one hand, given a  $x \in X, \mathcal{B}_x \subseteq \mathcal{B} \Rightarrow \cup_{x \in X} \mathcal{B}_x \subseteq \mathcal{B}$ . For any  $B \in \mathcal{B}$ , if  $B \neq \emptyset$ , there exists  $x' \in B$ , thus  $B \in \mathcal{B}_{x'} \subseteq \cup_{x \in X} \mathcal{B}_x$ . Thus  $\mathcal{B} = \cup_{x \in X} \mathcal{B}_x$ .  $\mathcal{B}_x$  is a local basis of  $\mathcal{T}$  at any  $x \in X \Rightarrow \mathcal{B}_x \subseteq \mathcal{T}$  for any  $x \in X$ . Thus  $\mathcal{B} \subseteq \mathcal{T}$ .

On the other hand, given a non-empty  $U \in \mathcal{T}$ , for any  $x \in U$ ,  $\exists B_x \in \mathcal{B}_x$ , such that  $x \in B_x \subseteq U$ . Thus  $\cup_{x \in U} B_x \subseteq U$ . For any  $x' \in U$ ,  $\exists B_{x'} \in \mathcal{B}_{x'}$ , s.t.  $x' \in B_{x'} \subseteq U \Rightarrow x' \in \cup_{x \in U} B_x \Rightarrow \cup_{x \in U} B_x = U$ , where  $B_x \in \mathcal{B}$ . Thus  $\mathcal{B}$  is a basis of  $\mathcal{T}$ .  $\square$

*Note 4.* Very useful routine. We use it to prove the open set, in metric space, is the union of open balls as well.

**Exercise 2.** Let  $X$  be a set and  $\mathcal{B} \subseteq \mathcal{P}(X)$ . Show that there exists a topology  $\mathcal{T}$  such that  $\mathcal{B}$  is a basis of  $\mathcal{T} \Leftrightarrow$

1.  $\cup \mathcal{B} = X$  and
2.  $\forall U, V \in \mathcal{B}$  and  $x \in U \cap V, \exists W \in \mathcal{B}$ , s.t.  $x \in W \subseteq U \cap V$ .

(Hint: if such  $\mathcal{T}$  exists, it must be  $\{\cup \mathcal{F} \mid \mathcal{F} \subseteq \mathcal{B}\}$ .)

*Proof.*  $\Rightarrow$ : 1)  $X \in \mathcal{T} \Rightarrow \exists \mathcal{F} \subseteq \mathcal{B}$ , s.t.  $X = \cup \mathcal{F} \subseteq \cup \mathcal{B} \subseteq X \Rightarrow X = \cup \mathcal{B}$ ;  
2)  $\mathcal{B}$  is a basis of  $\mathcal{T} \Rightarrow \forall U, V \in \mathcal{B}, U, V \in \mathcal{T}$ , thus  $U \cap V \in \mathcal{T}$ . Pick  $x \in U \cap V$ ,  $\mathcal{B}_x$  is a local basis of  $\mathcal{T}$  at  $x$ . Thus  $\exists B \in \mathcal{B}_x \subseteq \mathcal{B}$ , s.t.  $x \in B \subseteq U \cap V$ .

$\Leftarrow$ : Define  $\mathcal{T} = \{\cup \mathcal{F} \mid \mathcal{F} \subseteq \mathcal{B}\}$ , all we need to do is show  $\mathcal{T}$  is a topology:

1.  $\emptyset \subseteq \mathcal{B} \Rightarrow \emptyset = \cup \emptyset \in \mathcal{T}; \mathcal{B} \subseteq \mathcal{B} \Rightarrow X = \cup \mathcal{B} \in \mathcal{T}$ .
2. for any  $\mathcal{F}_\alpha \subseteq \mathcal{B} (\alpha \in A)$ ,

$$\begin{aligned} \cup_{\alpha \in A} (\cup \mathcal{F}_\alpha) &= \cup_{\alpha \in A} (\cup_{B \in \mathcal{F}_\alpha} B) \\ &= \cup_{B \in \cup_{\alpha \in A} \mathcal{F}_\alpha} B \\ &= \cup (\cup_{\alpha \in A} \mathcal{F}_\alpha) \\ &\in \mathcal{T}, \end{aligned}$$

since  $\cup_{\alpha \in A} \mathcal{F}_\alpha \subseteq \mathcal{B}$ .

3. for any  $U = \cup \mathcal{F}_1, V = \cup \mathcal{F}_2 \in \mathcal{T}$ ,

$$\begin{aligned} U \cap V &= (\cup \mathcal{F}_1) \cap (\cup \mathcal{F}_2) \\ &= \cup_{B \in \mathcal{F}_1, C \in \mathcal{F}_2} (B \cap C) \end{aligned}$$

where  $B, C \in \mathcal{B}$ , thus for any  $x \in B \cap C, \exists D_x \in \mathcal{B}$  such that  $x \in D_x \subseteq B \cap C$ . Thus it is direct to see that  $B \cap C = \cup_{x \in B \cap C} D_x$ .

Thus

$$\begin{aligned} D_x \in \mathcal{B} &\Rightarrow D_x \in \mathcal{T} \\ &\Rightarrow \cup_{x \in B \cap C} D_x \in \mathcal{T} \\ &\Rightarrow U \cap V = \cup_{B \in \mathcal{F}_1, C \in \mathcal{F}_2} (B \cap C) \\ &= \cup_{B \in \mathcal{F}_1, C \in \mathcal{F}_2} (\cup_{x \in B \cap C} D_x) \in \mathcal{T}. \end{aligned}$$

Thus  $\mathcal{T}$  is such topology as desired.  $\square$

Recall that when we check whether a map  $X \xrightarrow{f} Y$  is conti., we need show that for  $\forall V \subseteq_{\text{open}} Y, f^{-1}(V) \subseteq_{\text{open}} X$ . But if  $Y$  is equipped with a topology induced by some sub-basis, we can only check some subset of  $Y$ , instead of any subset of  $Y$ .

**Exercise 3.** Let  $Z$  be a topology space and  $Z \xrightarrow{f} X$  is a map. Show that  $f$  is continuous when  $X$  is topologized by  $\mathcal{T}(\mathcal{S}) \Leftrightarrow \forall S \in \mathcal{S}, f^{-1}(S) \subseteq_{\text{open}} Z$ .

*Proof.*  $\Rightarrow$ :  $\forall S \in \mathcal{S} \Rightarrow S \in \mathcal{T}(\mathcal{S})$ , that is  $S \subseteq_{\text{open}} X \Rightarrow f^{-1}(S) \subseteq_{\text{open}} Z$ .

$\Leftarrow$ : for any  $U \in \mathcal{T}(\mathcal{S})$ , it can be represented by the union of some finite intersections of elements of  $\mathcal{S}$ , that is  $U = \cup_{F \in \mathcal{F}} F$ , where  $\mathcal{F} \subseteq \Pi$ , and  $F = \cap_{i=1}^{k_F} S_i, S_i \in \mathcal{S}$ . Thus

$$\begin{aligned} f^{-1}(U) &= f^{-1}(\cup_{F \in \mathcal{F}} F) \\ &= \cup_{F \in \mathcal{F}} f^{-1}(\cap_{i=1}^{k_F} S_i) \\ &= \cup_{F \in \mathcal{F}} \left( \cap_{i=1}^{k_F} f^{-1}(S_i) \right) \\ &\subseteq_{\text{open}} Z. \end{aligned}$$

Thus  $Z \xrightarrow{f} X$  is continuous. □

### Countable, Separable and Lindelof Compact

**Definition 5.** A topology space  $(X, \mathcal{T})$  is

1. 1st-countable if  $\forall x \in X, \exists$  countable local basis of  $\mathcal{T}$  at  $x$ ;
2. 2nd-countable if  $\exists$  countable basis of  $\mathcal{T}$ . (That is  $\exists$  countable open set in  $X$  such that any element in  $\mathcal{T}$  is the union of these open set.)

*Note 5.*  $\mathcal{B}$  is a basis of  $\mathcal{T} \Rightarrow \mathcal{B}_x$  is a local basis of  $\mathcal{T}$  at  $x$ . Thus  $(X, \mathcal{T})$  is 2nd-countable  $\Rightarrow (X, \mathcal{T})$  is 1st-countable.

**Example 2.** 1. Let  $X$  be a metric space and  $\mathcal{T}$  is the topology defined by metric. Then  $\mathcal{B} = \{B_r(x) | r > 0, r \in \mathbb{Q}\}$  is a countable local basis of  $\mathcal{T}$  at  $x$ . Thus metric space is 1st-countable.

2. Note that the open set in  $\mathbb{R}$  is the union of disjointed open intervals in  $\mathbb{R}$ . Any open interval can be represented by the union of countable open intervals that start and end at rational number. Thus any open set in  $\mathbb{R}$  is the union of countable open intervals. Thus  $\mathbb{R}$  is 2nd-countable.

**Definition 6** (Dense). Given a topology space  $X$ , we say a subset  $A \subseteq X$  is dense if  $\overline{A} = X$ .

**Exercise 4.**  $X$  is a topology space,  $A \subseteq X$ , show that  $A$  is dense  $\Leftrightarrow \forall U \subseteq_{\text{open}} X, U \neq \emptyset$ , then  $U \cap A \neq \emptyset$ .

*Proof.*  $\Rightarrow$ :  $\bar{A} = A^o \cup \partial A = X$ , thus  $X \setminus A^o = \partial A$  as  $A^o$  and  $\partial A$  are disjoint. For any  $U \subseteq_{\text{open}} X$ , if  $U \neq \emptyset$ , pick  $x \in U$ , then either  $x \in A^o$  or  $x \in X \setminus A^o = \partial A$ .

If  $x \in A^o \Rightarrow x \in U \cap A \neq \emptyset$ ; If  $x \in \partial A$ ,  $U$  is a nbd. of  $x \Rightarrow U \cap A \neq \emptyset$ .

$\Leftarrow$ : If  $\bar{A} \neq X \Rightarrow W := X \setminus \bar{A} \neq \emptyset$ , and  $W \subseteq_{\text{open}} X$ ,  $W \cap \bar{A} = (X \setminus \bar{A}) \cap \bar{A} = \emptyset$ , which leads to a contradiction.  $\square$

**Definition 7** (Separable). A topology space  $(X, \mathcal{T})$  is separable if  $X$  has a countable dense subset.

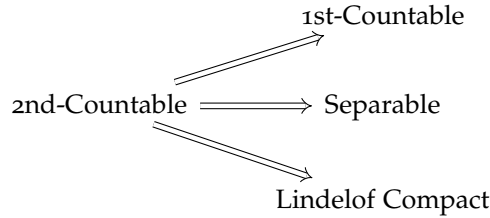
**Exercise 5.** If  $\mathcal{B}$  is a basis of a topology space  $X$  and pick a point  $x_B$  in  $B$  for any non-empty set  $B \in \mathcal{B}$ . Show that  $\{x_B \in B \mid B \in \mathcal{B}, B \neq \emptyset\} \subseteq_{\text{dense}} X$ .

*Proof.* If  $U \subseteq_{\text{open}} X$  and  $U \neq \emptyset$ , then  $\exists \mathcal{F} \subseteq \mathcal{S}$ , s.t.  $U = \cup \mathcal{F}$ . Then  $x_F \in F \in \mathcal{F} \subseteq \cup \mathcal{F} = U \Rightarrow x_F \in U \cap \{x_B \in B \mid B \in \mathcal{B}\} \neq \emptyset \Rightarrow \{x_B \in B \mid B \in \mathcal{B}\} \subseteq_{\text{dense}} X$ .  $\square$

*Note 6.* Thus if  $\mathcal{B}$  is a countable basis of  $\mathcal{T}$  on  $X$ , then  $\{x_B \in B \mid B \in \mathcal{B}\}$  is a countable dense subset of  $X$ , and  $(X, \mathcal{T})$  is a separable topology space.

**Definition 8** (Lindelof Compact). A topology space  $(X, \mathcal{T})$  is Lindelof compact if  $\forall U_\alpha \subseteq_{\text{open}} X (\alpha \in A), \cup_{\alpha \in A} U_\alpha = X \Rightarrow \exists$  countable set  $A_0 \subseteq A$ , s.t.  $\cup_{\alpha \in A_0} U_\alpha = X$ .

It is direct to see that 2nd-countable  $\Rightarrow$  Lindelof Compact, since if  $\mathcal{B}$  is a basis of  $\mathcal{T}$  on  $X$ , then  $X = \cup \mathcal{B}$ . Collectively, we have



**Exercise 6.** If  $X$  is topologized by a metric (a.k.a.  $X$  is a metrizable topology space) then 2nd-Countable  $\Leftrightarrow$  Separable  $\Leftrightarrow$  Lindelof Compact.

*Proof.* 1. Separable  $\Rightarrow$  2nd-Countable: To prove this statement, we need to track back to the  $\Leftarrow$  case: If  $D$  is the countable dense subset of  $X$ , we claim that  $\mathcal{B} := \{B_{\frac{1}{n}}(s) \mid s \in D, n \in \mathbb{N}\}$  is the basis of metric topology on  $X$ .

Given a  $U \subseteq_{\text{open}} X$  and  $U \neq \emptyset$ , we have  $U \cap D \neq \emptyset$ . For any  $u \in U \cap D$ , exists  $n_u \in \mathbb{N}$ , s.t.  $B_{\frac{1}{n_u}}(u) \subseteq U$ . Obviously,

$$W := \bigcup_{u \in U \cap D} B_{\frac{1}{n_u}}(u) \subseteq U.$$

For any  $v \in U$ , if  $v \in U \cap D \Rightarrow v \in W$ ; if  $v \notin D \Rightarrow v \in L_D$ , since  $X = D \cup L_D$ . Thus  $\exists n_v \in \mathbb{N}$ , s.t.  $\exists u \in B_{\frac{1}{n_v}}(v) \cap D \setminus \{v\}$ ,

where  $B_{\frac{1}{n_v}}(v) \subseteq U$  and  $u \in U \cap D$  whose  $1/n_u > 1/n_v$ . Thus  $v \in B_{\frac{1}{n_u}}(u) \subseteq W \Rightarrow U = W = \bigcup_{u \in U \cap D} B_{\frac{1}{n_u}}(u)$ , where  $\{B_{\frac{1}{n_u}}(u) | u \in U \cap D, B_{\frac{1}{n_u}}(u) \subseteq U\} \subseteq \mathcal{B}$ . Thus  $\mathcal{B}$  is a basis of metric topology on  $X$ .

2. Lindelof Compact  $\Rightarrow$  Separable: For any  $x \in X, \exists r_x > 0$ , s.t.  $B_{r_x}(x) \subseteq X$ , it is direct to see that  $X = \bigcup_{x \in X} B_{r_x}(x)$ .  $X$  is Lindelof Compact, thus exist countable subset  $D$  of  $X$  such that  $X = \bigcup_{x \in D} B_{r_x}(x)$ . For any non-empty  $U \subseteq_{\text{open}} X$ , any  $u \in U \subseteq X = \bigcup_{x \in D} B_{r_x}(x)$ , thus  $U \cap D \neq \emptyset \Rightarrow D$  is dense  $\Rightarrow X$  is separable.  $\square$

### Examples

There are some examples of topologies:

**Example 3.**  $X$  is a set,  $\mathcal{P}(X)$  is called discrete topology;  $(\emptyset, X)$  is called trivial topology. Note that discrete topology is defined by discrete metric:

$$d(x, x') = \begin{cases} 1, & x = x' \\ 0, & x \neq x' \end{cases}$$

Thus  $\{x\} \subseteq B_{1/2}(x)$  for any  $x \in X$ , and any  $S \in \mathcal{P}(X)$  is the union of these balls, i.e.  $S = \bigcup_{x \in S} B_{1/2}(x)$ , and holds an open set in discrete topology.

But trivial topology can not be defined by metric. If it can, then  $\forall x \in X, \exists r_x > 0$ , s.t.  $B_{r_x}(x) \subseteq X$ , which implies  $B_{r_x}(x) \in (\emptyset, X)$  and leads to a contradiction.

**Example 4.**  $X$  is an uncountable set.  $\mathcal{T}_c := \{U \subseteq X | U = \emptyset \vee X \setminus U \text{ is countable}\}$  is called **co-countable** topology. Thus any countable set in  $X$  is the close set on topology space  $(X, \mathcal{T}_c)$ .

Similarly,  $\mathcal{T}_f := \{U \subseteq X | U = \emptyset \vee X \setminus U \text{ is finite}\}$  is called **co-finite** topology. Thus any finite set in  $X$  is the close set on topology space  $(X, \mathcal{T}_f)$ .

It is direct to see  $\mathcal{T}_c$  and  $\mathcal{T}_f$  are topology:

1.  $\emptyset \in \mathcal{T}_c, X \in \mathcal{T}_c$  for  $X \setminus X = \emptyset$  is countable;
2. Any  $U_\alpha \in \mathcal{T}_c (\alpha \in A) \Rightarrow X \setminus U_\alpha$  is countable  $\Rightarrow X \setminus \bigcup_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A} (X \setminus U_\alpha)$  is the intersection of countable sets, thus be countable  $\Rightarrow \bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_c$ .
3.  $U, V \in \mathcal{T}_c \Rightarrow X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$  is countable, thus  $U \cap V \in \mathcal{T}_c$ .