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School of Finance Lecture 1,2
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General Topology: Open set on metric space

Key word: Metric Space, Open Ball, Open Set, Bolzano-Weierstrass theorem, Open cover, Compact set, Heine-Borel theorem, The Lebesgue number of an open cover, Isolated point, limit point and accumulation point, Limit, Limit of composite maps

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1 Metric Space

Definition 1.1: Metric Space

Let $X \times X \xrightarrow{d} \mathbb{R}$ be a function, we cay that d is a metric on X or (X, d) is a metric space if for $\forall x, x', x'' \in X$ have

1. Positivity: $d(x, x') \geq 0$ and $d(x, x'') = 0$ iff $x = x'$;
2. Symmetry: $d(x, x') = d(x', x)$;
3. Triangle inequality: $d(x, x') \leq d(x, x'') + d(x'', x')$.

Notice that the triangle inequality is equivalent with for $\forall x, x', x'' \in X$

$$d(x, x') \geq |d(x, x'') - d(x', x'')|.$$

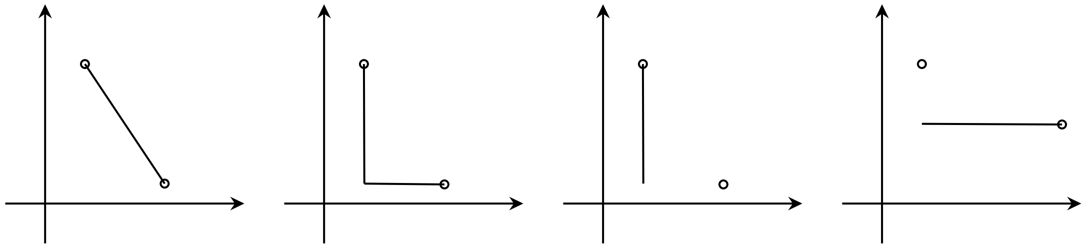
$\geq \Rightarrow \leq$: since $d(x, x') \geq |d(x, x'') - d(x', x'')| \geq d(x, x'') - d(x', x'')$, we have that $d(x, x'') \leq d(x, x') + d(x', x'')$. $\leq \Rightarrow \geq$: if $\exists x, x', x''$ such that $d(x, x') < |d(x, x'') - d(x', x'')|$, then

$$\begin{aligned} d(x, x') &< |d(x, x'') - d(x', x'')| \\ &\leq |d(x, x') + d(x', x'') - d(x', x'')| \\ &\leq d(x, x') \end{aligned}$$

thus $d(x, x') < d(x, x')$, which leads to a contradiction.

Example 1.1. Here are some metric examples:

1. define $d_2(x, y) := \left(\sum_i^m |x_i - y_m|^2\right)^{1/2}$, $x, y \in \mathbb{R}^m$. Then d_2 is a metric on $\mathbb{R}^m$ by cauchy inequality.
2. define $d_1(x, y) := \sum_{i=1}^m |x_i - y_i|$, $x, y \in \mathbb{R}^m$. Then d_1 is a metric on $\mathbb{R}^m$.
3. define $d_\infty(x, y) := \max\{|x_i - y_i|\}$, $i \in \{1, 2, \cdots, m\}$, $x, y \in \mathbb{R}^m$. Then d_∞ is a metric on $\mathbb{R}^m$.



$(\mathbb{R}^2, d_2)$ $(\mathbb{R}^2, d_1)$ $(\mathbb{R}^2, d_\infty)$ $(\mathbb{R}^2, d_\infty)$

Example 1.2 (p-adic). If p is a prime number, $x \in \mathbb{Q}$, define

$$|x|_{p-adic} := \begin{cases} p^{-m}, & x = \frac{a}{b} \cdot p^m, x \neq 0, \\ 0, & x = 0, \end{cases}$$

where $a, b, m \in \mathbb{Z}$, $(a, p) = (b, p) = 1$. For $\forall x, y \in \mathbb{Q}$, define $d_{p-adic}(x, y) = |x - y|_{p-adic}$, then d_{p-adic} is a metric on $\mathbb{Q}$.

Assume $x = (a/b)p^m, y = (s/t)p^n \in \mathbb{Q}$ where $a, b, s, t, m, n \in \mathbb{Z}, (a, p) = (b, p) = (s, p) = (t, p) = 1, m > n$, then $|x|_{p-adic} = p^{-m} < |y|_{p-adic} = p^{-n}$, and

$$\begin{aligned} |x - y|_{p-adic} &= |(a/b)p^m - (s/t)p^n|_{p-adic} \\ &= \left| \frac{adp^{m-n} - bc}{bd} p^n \right|_{p-adic}. \end{aligned}$$

it is easy to check $adp^{m-n} - bc, bd \in \mathbb{Z}$ and $(adp^{m-n} - bc, p) = (bd, p) = 1$, thus

$$|x - y|_{p-adic} = p^{-n} = |y|_{p-adic} = \max\{|x|_{p-adic}, |y|_{p-adic}\}.$$

Thus for any $x, y, z \in \mathbb{Q}$, we have that

$$\begin{aligned} d_{p-adic}(x, y) &= \max\{|x|_{p-adic}, |y|_{p-adic}\} \\ &\leq \max\{|x|_{p-adic}, |z|_{p-adic}\} + \max\{|z|_{p-adic}, |y|_{p-adic}\} \\ &= d_{p-adic}(x, z) + d_{p-adic}(y, z), \end{aligned}$$

which follows the triangle inequality, the other two conditions is trivial.

2 Open set on metric space

Definition 2.1: Open Ball

Let (X, d) be a metric space, for $\forall r > 0$ and $x_0 \in X$, we let

$$B_r(x_0) := \{x \in X | d(x, x_0) < r\},$$

and call it the open ball with center x_0 and radius r; let

$$\overline{B_r}(x_0) := \{x \in X | d(x, x_0) \leq r\},$$

and call it the close ball with center x_0 and radius r.

Example 2.1 (discrete metric). For $\forall x, x' \in \mathbb{R}^2$, define metric $d(x, x') = 0$ if $x = x'$, and $d(x, x') = 1$ if $x \neq x'$, then $B_1(x) = \{x\}$, $\overline{B_1}(x) = \mathbb{R}^2$, $B_{1.1}(x) = \mathbb{R}^2$.

Definition 2.2: Open Set

$S(\subseteq X)$ is called an Open Set of X with respect to d, if $\forall x_0 \in S$, $\exists r > 0$ such that $B_r(x_0) \subseteq S$; $F(\subseteq X)$ is Close Set of X w.r.t. d if $X \backslash F$ is open set of X w.r.t. d.

Example 2.2. Prove that $B_r(x)$ is open set and $\overline{B_r}(x)$ is close.

For $\forall x' \in B_r(x)$, we have $d(x,x') < r$, donate $r - d(x,x')$ by s, then for $\forall x'' \in B_{s/2}(x')$ satisfy

$$\begin{aligned} d(x,x'') &\leq d(x,x') + d(x',x'') \\ &\leq d(x,x') + \frac{s}{2} \\ &< r, \end{aligned}$$

thus $x'' \in B_r(x)$ and $B_{s/2}(x') \subseteq B_r(x)$ and $B_r(x)$ is a open set. For $\forall x' \in X \backslash \overline{B_r}(x)$ has $d(x,x') > r$. Denote $d(x,x') - r$ by t, then for $\forall x'' \in B_{t/2}(x')$ satisfy

$$\begin{aligned} d(x,x'') &\geq |d(x,x') - d(x',x'')| \\ &\geq d(x,x') - d(x',x'') \\ &\geq d(x,x;) - \frac{t}{2} \\ &> r. \end{aligned}$$

Thus $B_{t/2}(x') \subseteq X \backslash \overline{B}_r$ and $X \backslash \overline{B_r}$ is an open set, thus $\overline{B_r}$ is a close set.

Example 2.3. Let (X,d) be a metric space. show that

1. $X, \emptyset \subseteq_{open} X$;
2. $O_1, O_2 \subseteq_{open} X \Rightarrow O_1 \cap O_2 \subseteq_{open} X$;
3. $O_\alpha \subseteq_{open} X, (\alpha \in A) \Rightarrow \cup_{\alpha \in A} O_\alpha \subseteq_{open} X$ (α not necessarily be integral or countable);
4. All above corresponding statements for close set are true.

Proof.

1. Obviously X is an Open set thus $\emptyset$ is a close set. If $\emptyset$ is not an open set, then $\exists x \in \emptyset$, $\forall r > 0$ such that $B_r(x) \nsubseteq \emptyset$, which is impossible. Thus $\emptyset$ is an open set (logically) and X is a close set;
2. $\forall x \in O_1 \cap O_2$, $\exists r_1, r_2 > 0$, s.t. $B_{r_1}(x) \subseteq O_1$ and $B_{r_2}(x) \subseteq O_2$. Thus $\forall x' \in B_{\min\{r_1,r_2\}}(x) = B_{r_1}(x) \cap B_{r_2}(x) \Rightarrow x' \in O_1 \cap O_2 \Rightarrow B_{\min\{r_1,r_2\}}(x) \subseteq O_1 \cap O_2$, thus $O_1 \cap O_2$ is open. Collectively, the intersection of any finite open sets is an open set;
3. For $\forall x \in \cup_{\alpha \in A} O_\alpha, \exists$ at least one $\alpha' \in A$, s.t. $x \in O_{\alpha'}$, then $\exists r > 0$, s.t. $B_r(x) \subseteq O_{\alpha'} \subseteq \cup_{\alpha \in A} O_\alpha$, thus $\cup_{\alpha \in A} O_\alpha$ is an open set;
4. Suppose $F_1, F_2 \subseteq_{close} X$, then $X \backslash F_1, X \backslash F_2 \subseteq_{open} X$, thus $(X \backslash F_1) \cup (X \backslash F_2) = X \backslash (F_1 \cap F_2) \subseteq_{open} X$ and $F_1 \cap F_2 \subseteq_{close} X$.
5. Suppose $F_\alpha (\alpha \in A)$ is (an arbitrary family of) close set, for any $x \in X \backslash \cup_{\alpha \in A} F_\alpha \Rightarrow x \notin \cup_{\alpha \in A} F_\alpha \Rightarrow x \notin F_\alpha (\forall \alpha \in A) \Rightarrow x \in X \backslash F_\alpha (\alpha \in A)$. Since F_α is close, there exists $r_\alpha > 0$, s.t. $B_{r_\alpha}(x) \subseteq X \backslash F_\alpha (\forall \alpha \in A)$, and $B_{\min r_\alpha}(x) = \cap_{\alpha \in A} B_{r_\alpha}(x) \subseteq \cap_{\alpha \in A} X \backslash F_\alpha = X \backslash \cup_{\alpha \in A} F_\alpha$, thus $X \backslash \cup_{\alpha \in A} F_\alpha$ is open, and $\cup_{\alpha \in A} F_\alpha$ is close.

$\square$

Definition 2.1: Convergence

Let (X, d) be a metric space, $a_n \in X, (n \in \mathbb{N})$, $L \in X$, define $\lim_{n\to\infty} a_n = L$ w.r.t. d, if $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N$ s.t. $d(a_n, L) < \epsilon$, that is $a_n \in B_\epsilon(L)$.

Example 2.4. Show that

1. $\lim_{n\to\infty} a_n = L \Leftrightarrow \lim_{n\to\infty} d(a_n, L) = 0$;
2. $\lim_{n\to\infty} a_n = L \Leftrightarrow \forall L \in U \subseteq_{open} X, \exists N \in \mathbb{N}, \forall n \geq N$ s.t. $a_n \in U$.

Proof. (1) Trivial; (2) $\Rightarrow$: Suppose that $\lim_{n\to\infty} a_n = L$, for $\forall U$ that $L \in U$, $\exists r > 0$, s.t. $B_r(L) \subseteq U$, and $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, s.t. $a_n \in B_r(L) \subseteq U$; $\Leftarrow$: Suppose $L \in U \subseteq_{open} X$, then $\exists r > 0$ such that $B_r(L) \subseteq U$. Since $B_r(L)$ is also an open set, then $\exists N \in \mathbb{N}$, for $\forall n \geq N$, s.t. $a_n \in B_r(L) \subseteq U$. $\square$

We say $S \subseteq X$ is bounded w.r.t. d, if $\exists r > 0$ and $x_0 \in X$, s.t. $S \subseteq B_r(x_0)$.

Theorem 2.1: Bolzano-Weierstrass theorem

If $a_n \in \mathbb{R}^m (n \in \mathbb{N})$ is bounded w.r.t. d_2, then $\exists$ a subsequence $a_{n_m} (m \in \mathbb{N})$ which converges.

Proof. We only prove $\mathbb{R}^2$ case. If we want to prove that the vector $a = (a_1, \cdots, a_m) \in \mathbb{R}^m \to L = (l_1, \cdots, l_m) \in \mathbb{R}^m$, all we need to prove is $\lim_{n\to\infty} a_i = l_i, (i = 1, \cdots, m)$.

Choose $M > 0$, s.t. $a_n \in Q = [-M, M] \times [-M, M]$ for all $n \in \mathbb{N}$. Divide Q into 4 squares with equal size and choose one, say Q_1, such that $|\{n | a_n \in Q\}| = \infty$. Select $n_1 \in \mathbb{N}$, such that $a_{n_1} \in Q_1$. Repeat this and we have $\cap_{k=1}^{\infty} Q_k = \{a\}$. By theorem of nested interval we have that $\lim_{k\to\infty} a_{n_k} = a$. $\square$

Example 2.5. Let (X, d) be a metric space, $F \subseteq X$ show that $F \subseteq_{close} X \Leftrightarrow \forall a_n \in F (n \in \mathbb{N})$ and $\lim_{n\to\infty} a_n = a \in X$ then $a \in F$.

Proof. $\Rightarrow$: Assume that F is close and $a_n \in F$. If $a_n \to a \in X \backslash F$, then $\exists r > 0$, s.t. $B_r(a) \in X \backslash F$. Since $\lim_{n\to\infty} a_n = a$, for r, there exists $N \in \mathbb{N}, \forall n \geq N$, s.t. $d(a_n, a) < r$, i.e. $a_n \in B_r(a) \subseteq X \backslash F$, which leads to a contradiction. $\Leftarrow$: Suppose that $\forall a_n \in F (n \in \mathbb{N})$ and $\lim_{n\to\infty} a_n = a \in X$ then $a \in F$, and F is not close, which means $X \backslash F$ is not open, and $\exists x \in X \backslash F, \forall r > 0, B_r(x) \cap F \neq \emptyset$. Select $n \in \mathbb{N}$ such that $a_n = B_{\frac{1}{n}}(x) \cap F$. Thus $\lim_{n\to\infty} a_n = x \notin F$, which leads to a contradiction. $\square$

Note 1. Set family of sets as $\{B_{1/n}(x)\}_{n\in\mathbb{N}}$ is a very useful skill.

Definition 2.2: Open cover, Compact set

Let (X, d) be a metric space, $S \subseteq X$, $O_\alpha \in X (\alpha \in A)$, we say that $O_\alpha (\alpha \in A)$ form an open cover of S, if $S \subseteq \cup_{\alpha \in A} O_\alpha$. S is called a compact set if $\forall$ open cover $O_\alpha (\alpha \in A)$ of S, $\exists \alpha_1, \cdots, \alpha_m \in A$, s.t. $S \subseteq \cup_{i=1}^m O_{\alpha_i}$, where $\cup_{i=1}^m O_{\alpha_i}$ is called a finite subcover.

If there exists an open cover of F whose any finite subcover can not cover it, then F is not a compact set. for instance, let $F = (0, 1), O_n = (1/n, 2), n \in \mathbb{N}$, then O_n is an open cover of F, however any finite subcover of O_n can not cover F.

Theorem 2.2: Heine-Borel theorem

Let $S \subseteq \mathbb{R}^n$, then S is compact $\Leftrightarrow$ S is bounded and closed.

Proof. $\Rightarrow$: Suppose that S is compact, select a point $s \in S$ arbitrarily, define $O_i = B_i(s)$, it is easy to check that $S \subseteq \cup_{i\in\mathbb{N}} O_i$, since for any $s' \in S$, we have that $s' \in B_{2d(s,s')}(s) \subseteq O_{\lceil 2d(s,s')\rceil}$. Since S is compact, there exists a finite subcover, thus S is bounded.

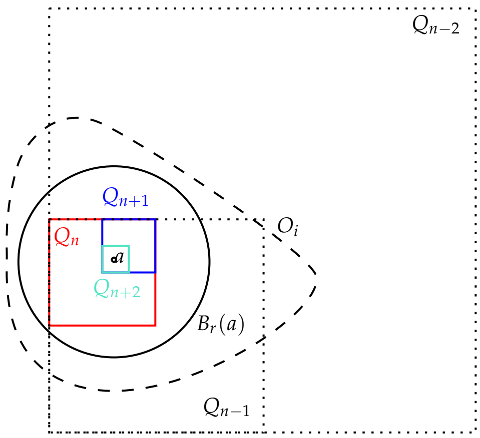
Suppose S is compact, but S is not closed, which means $X\backslash S$ is not open and $\exists x \in X\backslash S$, s.t. $\forall r > 0, B_r(x) \cap S \neq 0$. Since S is bounded, define $\iota = \sup_{s\in S} d(s,x)$, define open cover

$$O_n = B_{\frac{\iota}{n}}(x) - B_{\frac{\iota}{n+1}}(x),$$

thus $O_i \cap O_j = \varnothing (i \neq j)$ and $O_i \cap S \neq \varnothing (\forall i)$. Thus O_n has no finite subcover, which leads to a contradiction and S is closed.

$\Leftarrow$: Suppose that S is bounded and closed, and $\exists$ an open cover $O_\alpha (\alpha \in A)$ of S which admits no finite subcover. Choose a cube Q containing S (S is bounded), divide Q into 4 equal-sized cubes and select one of them denoted by Q_1, s.t. $Q_1 \cap S$ can not be covered by finitely many Q_α, select a point such that $s_1 \in Q_1 \cap S$. Repeat.

Obviously, $\lim_{n\to\infty} s_n = a$, notice that $s_n \in Q_n \cap S \subseteq S$, thus $\lim_{n\to\infty} s_n = a \in S$ for S is closed. Thus there exist O_i such that $a \in O_i$. Since O_i is open, $\exists r > 0$, s.t. $B_r(a) \subseteq O_i$. Then $\exists N \in \mathbb{N}, \forall n \geq N$, s.t. $Q_n \subseteq B_r(a) \subseteq O_i$. Since $Q_n \cap S$ can not be covered by finitely many O_α, but could be covered by O_i, which leads to a contradiction.



$\square$

Theorem 2.3: The Lebesgue number of an open cover

Let (X,d) be a metric space and $K(\subseteq X)$ a compact set. For any given open cover $O_\alpha(\alpha \in A)$ of K, there exists $\delta > 0$, s.t. *for every $x \in K$ we have $B_\delta(x) \subseteq O'_\alpha$ for some $\alpha' \in A$ (α' depending on x).*

Proof. Since K is compact, for any open cover of K, there exists an finite subcover of K, that is $\exists O_{\alpha_i}, i = 1, \cdots, N$ such that

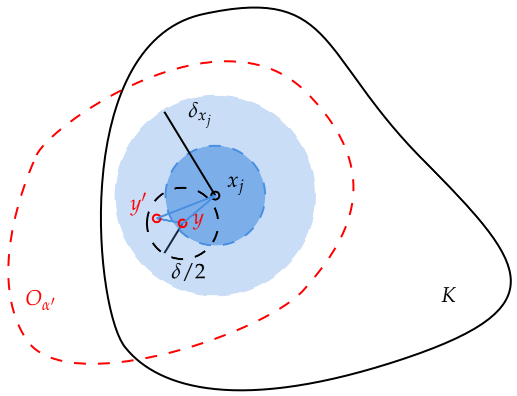
$$K \subseteq \bigcup_{i=1}^{N} O_{\alpha_i}.$$

For any point $x \in K$ there exist $O_{\alpha_j} \in \{O_{\alpha_i}\}_{i=1}^N$, s.t. $x \in O_{\alpha_j}$ and exists $\delta_x > 0$, s.t. $B_{\delta_x/2}(x) \subseteq B_{\delta_x}(x) \subseteq O_{\alpha_j}$ for O_{α_j} is open. Obviously we have that $B_{\delta_x/2}(x)$ is an open cover of K, i.e.

$$K \subseteq \bigcup_{x\in K} B_{\delta_x/2}(x),$$

and $B_{\delta_x/2}(x)$ has an finite subcover of K, donate as $\{B_{\delta_{x_i}/2}(x_i)\}_{i=1}^M$. Let $\delta = \min\{\delta_{x_i}\}_{i=1}^M$. Then for any $y \in K$, there exists $B_{\delta_{x_j}/2}(x_j) \in \{B_{\delta_{x_i}/2}(x_i)\}_{i=1}^M$ such that $y \in B_{\delta_{x_j}/2}(x_j)$ and $d(y, x_j) < \delta_{x_j}/2$. and for any

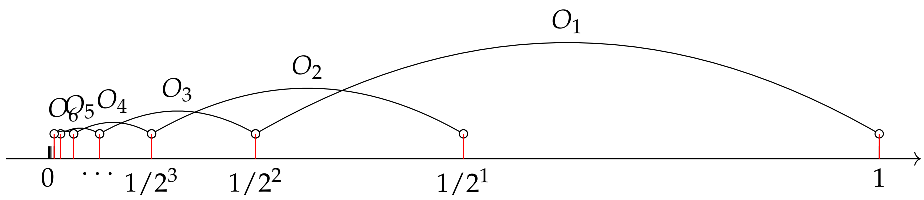
y' where $d(y', y) < \delta/2 < \delta_{x_j}/2$, we have $d(x_j, y') \le d(x_j, y) + d(y, y') < \delta_{x_j}$, thus $y, y' \in B_{\delta_{x_j}}(x_j) \subseteq O_{\alpha'}$, or say $y, y' \in B_{\delta/2}(y) \subseteq B_{\delta_{x_j}}(x_j) \subseteq O_{\alpha'}$.



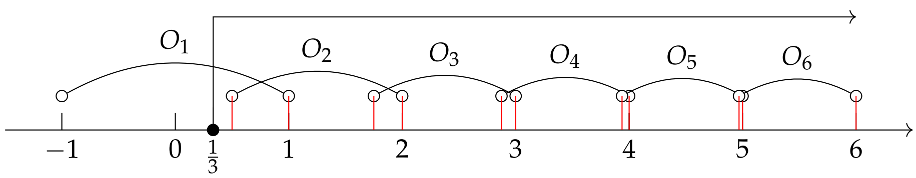
□

The theorem indicates for any open cover O_α of K, $\exists \delta > 0$, s.t. for $\forall x \in K, x' \in X$, is $d(x, x') < \delta$, then $\exists \alpha \in A$ we have $x, x' \in O_\alpha$. Such a $\delta > 0$ is called a **Lebesgue number** of the given open cover $O_\alpha(\alpha \in A)$. Notice that the statement could be false if the compactness assumption is dropped.

Example 2.6 (Open set)**.** Let $(X, d) = (\mathbb{R}, d_2), K = (0, 1), O_\alpha = (1/2^{\alpha+1}, 1/2^{\alpha-1})(\alpha \in \mathbb{N})$. Thus $1/2^\alpha \in O_\alpha$ and $\notin O_{\alpha'}$ if $\alpha' \neq \alpha(\alpha, \alpha' \in \mathbb{N})$. It is easy to check O_α is an open cover of K, but $|1/2^\alpha - 1/2^{\alpha+1}| = 1/2^{\alpha+1}$ can be arbitrarily small if $\alpha \uparrow$. Thus there exists $x \in K, x' \in X$ can not be covered one O_α, no matter how close they are.



Example 2.7 (Unbounded set)**.** Let $(X, d) = (\mathbb{R}, d_2), K = [1/3, \infty), O_\alpha = (\alpha - 1 - 1/2^{\alpha-1}, \alpha)(\alpha \in \mathbb{N})$. Thus $x = \alpha - 1/2^\alpha \in O_\alpha$ and $x' = \alpha \in O_{\alpha+1}$ and $d(x, x')$ could be arbitrarily small as $\alpha \uparrow$. Thus there exists $x \in K, x' \in X$ can not be covered one O_α, no matter how close they are.

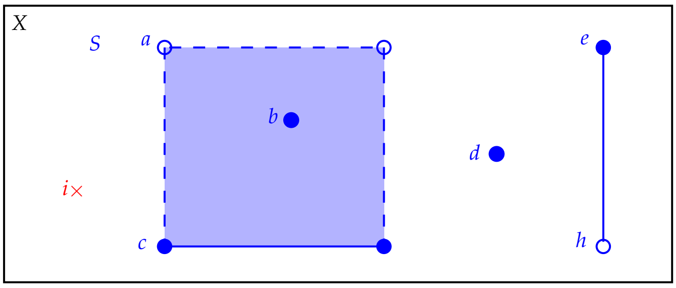


Definition 2.3: Isolated point, limit point and accumulation point

Let (X, d) be a metric space and $S \subseteq X$. A point $x \in X$ is called:

- an **isolated point** of S, if $\exists \epsilon > 0$, s.t. $B_\epsilon(x) \cap S = \{x\}$ ($\Rightarrow x \in S$);
- a **limit point** of S, if $\forall \epsilon > 0$, s.t. $B_\epsilon(x) \cap S \backslash \{x\} \neq \emptyset$;
- an **accumulation point** of S, if $\exists$ seq. $a_n \in S(n \in \mathbb{N})$, s.t. $x = \lim_{n\to\infty} a_n$.

Example 2.8. $S \subseteq X$ is as followed, point $i \notin S$:



we have that

point	iso. pts. of S	limit pts. of S	acc. pts. of S	$\in S$
i	$\times$	$\times$	$\times$	$\times$
a	$\times$	$\surd$	$\surd$	$\times$
b	$\times$	$\surd$	$\surd$	$\surd$
c	$\times$	$\surd$	$\surd$	$\surd$
d	$\surd$	$\times$	$\surd$	$\surd$
e	$\times$	$\surd$	$\surd$	$\surd$
h	$\times$	$\surd$	$\surd$	$\times$

Notice that x is a isolated point of $S \Rightarrow x \in S$; but x is a limit/accumulate point of $S \nRightarrow x \in S$.

Example 2.9. Let (X, d) be a metric space, $S \subseteq X$,

1. Show that x is an isolated/limit point of $S \Rightarrow x$ is a accumulate point of S;
2. Donate $\{\text{iso. pts. of} S\}, \{\text{limit pts. of} S\}$ and $\{\text{acc. pts. of} S\}$ by I, L, A respectively. Show that $I \cup L = A$;
3. Suppose $S \subseteq K \subseteq X$, where S is infinite and K is compact, show that $\{\text{limit pts. of} S\} \neq \emptyset$; (Prove by contradiction)

Proof. 1. If x is an isolated point of S, thus $x \in S$. Let $a_n \equiv x$, then $\lim_{n\to\infty} a_n = x$, thus x is an accumulate point of S; If x is a limit point of S, then for any $\epsilon > 0, B_\epsilon(x) \cap S \backslash \{x\} \neq \emptyset$. Let $a_n \in B_{1/n}(x) (n \in \mathbb{N})$, then $d(a_n, x) < 1/n$ for $\forall n \in N$, thus $\lim_{n\to\infty} a_n = x$, and x is an accumulate point of S.

2. We have obtained that $I, L \subseteq A$. Suppose $x \in A \backslash (I \cup L) \neq \emptyset$. Which means : (1) there exists seq. $a_n \in S$ such that $\lim_{n\to\infty} a_n = x$; (2) $\forall \epsilon > 0$, s.t. $B_\epsilon(x) \cap S \neq \{x\}$ $(\neg I)$;(3) $\exists \epsilon' > 0$, s.t. $B_{\epsilon'}(x) \cap S \backslash \{x\} = \emptyset$ $(\neg L)$. Let $B_\epsilon(x) \cap S = Q_\epsilon \neq \{x\}$, if $x \in Q_\epsilon$, then it leads to a contradiction with (3); If $x \notin Q_\epsilon$, then $Q_{\epsilon'} = \emptyset$, that is $B_{\epsilon'}(x) \cap S = \emptyset$ and for $\forall s \in S \Rightarrow d(s, x) \geq \epsilon'$, which leads to a contradiction with (1). Thus $A \backslash (I \cup L) = \emptyset$. Because $I, L \subseteq A$, we have $I \cup L = A$.

3. Since S is infinite, there exists an infinite seq. $a_n \in S$. By Bolzano-Weierstrass theorem, there exists a subseq. $a_{n_i} \in S$ such that $\lim_{i\to\infty} a_{n_i} = a$. Suppose $L_S = \emptyset$, which means for $\forall x, \exists \epsilon' > 0$, s.t. $B_{\epsilon'}(x) \cap S \backslash \{x\} = \emptyset$, thus there exists ϵ_a, s.t. $B_{\epsilon_a}(a) \cap S \backslash \{a\} = \emptyset$, which means $\forall s \in S, d(s, a) \geq \epsilon_a$ and leads to a contradiction.

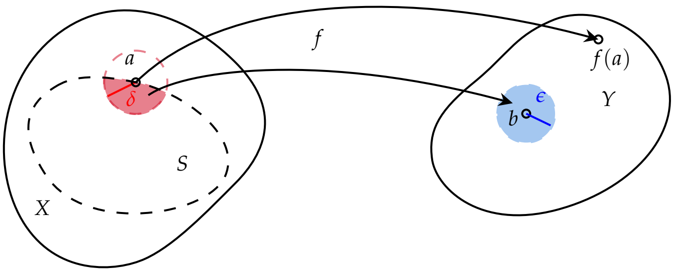
$\square$

3 Limits of functions/maps

Let (X, d_X) and (Y, d_Y) be metric spaces and $S \subseteq X$. We consider a map $f : S \mapsto Y$. e.g. $X = \mathbb{R}^2, Y = \mathbb{R}, S = \mathbb{R}^2 \backslash \{(0,0)\}, f : (x, y) \mapsto 1/x^2 + y^2$. (the reason why shrink X)

Definition 3.1: Limit

Let $a \in X$ (not necessarily $\in S$) and $b \in Y$. We say that $\lim_{x \to a} f(x) = b$ if $\forall \epsilon > 0, \exists \delta > 0, \forall x \in S$, s.t. $0 < d_X(x, a) < \delta \Rightarrow d_Y(f(x), b) < \epsilon$.



Note 2. There are 3 points deserved mention.

1. 3 conditions of x: 1. $x \in B_\delta(a)$; 2. $x \neq a$; 3. $x \in S$. Collectively, $x \in B_\delta(a) \cap S \backslash \{a\}$.
2. We require $d_X(x, a) > 0$, since $f(a)$ could be totally unconnective with $f(B_\delta(a) \cap S \backslash \{a\})$.
3. If $\exists r > 0$, s.t. $B_r(a) \cap S = \emptyset$, then $\lim_{x \to a} f(x) = b$ (logically) holds for every $b \in Y$. Otherwise $\exists \epsilon > 0, \forall \delta > 0, \exists x \in S, 0 < d_X(x, a) < \delta, \cdots$, but if let $\delta < r$, then any $x \in S$ commits $d(x, a) > r > \delta$, which leads to a contradiction.

Example 3.1. Show that

1. If a is a limit point of S and $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} f(x) = b'$ then $b = b'$;
2. Let $(Y, d_Y) = (\mathbb{R}^m, d_2)$ and $f : S \mapsto Y, g : S \mapsto Y$, where $S \subseteq X, a \in X$. If $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$, then $\lim_{x \to a}(f(x) \pm g(x)) = b \pm c$. If furthermore $(Y, d_2) = (\mathbb{R}, d_2)$, then $\lim_{x \to a} f(x)g(x) = bc$; if $c \neq 0$, then $\exists \delta > 0$, s.t. $g(x) \neq 0$ for all $x \in B_\delta(a) \backslash \{a\}$ and $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{b}{c}$.

Proof. 1. Since $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} f(x) = b'$, for $\forall \epsilon > 0$, $\exists \delta_1, \delta_2 > 0$, s.t. $\forall x \in B_{\delta_1}(a) \cap S \backslash \{a\} \Rightarrow d(f(x), b) < \epsilon$, and the same thing for δ_2. Let $\delta = \min\{\delta_1, \delta_2\}$, then for $\forall x \in B_\delta(a) \cap S \backslash \{a\}$, we have $d(f(x), b) < \epsilon$ and $d(f(x), b') < \epsilon$ simultaneously , thus $d(b, b') < \epsilon$ for $\forall \epsilon > 0$, thus $b = b'$.

2. for $\forall \epsilon > 0, \exists \delta > 0, \forall x \in B_\delta(a) \cap S \backslash \{a\} \Rightarrow d_2(f(x), b) < \epsilon$ and $d_2(g(x), c) < \epsilon$. Thus

$$\begin{aligned} d_2(f(x) + g(x), b + c) &= [(f(x) + g(x) - b - c)^T (f(x) + g(x) - b - c)]^{1/2} \\ &= [(f(x) - b)^T (f(x) - b) + (g(x) - c)^T (g(x) - c) + 2(f(x) - b)^T (g(x) - c)]^{1/2} \\ &< [2\epsilon^2 + 2(f(x) - b)^T (g(x) - c)]^{1/2}. \end{aligned}$$

Notice that $(f(x) - b)^T (g(x) - c) = (g(x) - c)^T (f(x) - b)$, thus $(f(x) - b)^T (g(x) - c) = [(g(x) - c)^T (f(x) - b)(f(x) - b)^T (g(x) - c)]^{1/2} = \epsilon^2$. and $d_2(f(x) + g(x), b + c) < (4\epsilon^2)^{1/2} = 2\epsilon$, thus $\lim_{x \to a}(f(x) + g(x)) = b + c$. Others are trivial.

□

Example 3.2 (Composite maps)**.** Let X, Y, Z be metric space and $f : S \mapsto T, g : T \mapsto Z$, where $S \subseteq X, T \subseteq Y$. Show that if $\lim_{x\to a} f(x) = b, \lim_{y\to b} g(y) = c$ and $b \notin f(S)$, then $\lim_{x\to a}(g \circ f)(x) = c$. If condition $b \notin f(S)$ is dropped, find an example s.t. $\lim_{x\to a}(g \circ f)(x) \neq c$.

Proof.

1. Since $\lim_{y\to b} g(y) = c$, then for $\forall \epsilon > 0, \exists \delta_y > 0, \forall y \in B_{\delta_y}(b) \cap T \backslash \{b\} \Rightarrow d(c, g(y)) < \epsilon$. And because $\lim_{x\to a} f(x) = b$, then $\exists \delta_x > 0$, for $\forall x \in B_{\delta_x}(a) \cap S \backslash \{a\}$, s.t. $d(b, f(x)) < \delta_y \Rightarrow f(x) \in B_{\delta_y}(b) \cap T$ (Since $f : S \mapsto T$). If $\forall x \in S$ has $f(x) \neq b$, that is $b \notin f(S)$, then $f(x) \in B_{\delta_y}(b) \cap T \backslash \{b\}$, and then $d(g(f(x)), c) < \epsilon$, i.e. $\lim_{x\to a}(g \circ f)(x) \neq c$.
2. Intuitively, If $g(y)$ is un-continuous as $y = b$, and $f(x)$ touches b with an extremely frequence as $x \to a$ then $g \circ f$ would be oscillating as $x \to a$. For example, let $f(x) = \sin(1/x), g(y) = y$ for $y \neq 0$ and 1 for $y = 0$, then $g \circ f(x)$ has no limit as $x \to 0$.

□

Example 3.3 (Example of nonexistence of limit)**.** Let $f : \mathbb{R}^2 \backslash \{(0, 0)\} \mapsto \mathbb{R}$ where $f(x, y) = \frac{xy}{x^2+y^2}$. Show that $\lim_{(x,y)\to(0,0)} f(x, y) \nexists$, using property of composite maps.

Proof. Consider map $g : \mathbb{R}\backslash\{0\} \mapsto \mathbb{R}^2\backslash\{(0, 0)\}$ thus $(0, 0) \notin g(\mathbb{R}\backslash\{0\})$. Let $g(t) = (at, bt)$ then $\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{t\to 0} f(g(t)) = \frac{ab}{a^2+b^2}$ depending on g. Thus if you set different g, that is different parameters a, b then you get different limit of composite maps $f \circ g$ which is equal to the limit of f, thus $\lim f \nexists$. □