Point Set Topology

Lecture 1

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Content:

This is the Lecture note for the *Point Set Topology*.

Some Definitions

Definition 1 (Partial Order)**.** Given a set X, a relation $\leq$ on X is a partial order if

1. $\forall x \in X \Rightarrow x \leq x$;
2. $\forall x, x' \in X, x \leq x', x' \leq x \Rightarrow x = x'$;
3. $\forall x, x', x'' \in X, x \leq x', x' \leq x'' \Rightarrow x \leq x''$.

We say that $(X, \leq)$ is a partially ordered set (poset).

Note 1. A relation on X, is a subset of $X \times X$.

Example 1. For example, $\leq$ is a partial order on $\mathbb{T}$; given a set X, $\subseteq$ is a partial on $\mathcal{P}(X)$.

If $(X, \leq)$ is a poset and $A \subseteq X$, then A has a natural partial order induced by $\leq$.

Definition 2 (Total Order, Chain)**.** A poset $(X, \leq)$ is a chain (or totally order set) if $\forall x, x' \in X$, then $x \leq x'$ or $x' \leq x$.

If $(X, \leq)$ is a poset, $A \subseteq X, b \in X$, we say

1. b is an upper (lower) bound of A (in X w.r.t. $\leq$) if $\forall a \in A, a \leq b (b \leq a)$, denoted the set of upper (lower) bound of A by $U_A(L_A)$.
2. b is a greatest (least) element of A (in X w.r.t. $\leq$), if b is an upper (lower) bound of A and $b \in A$.
3. b is the least upper bound (greatest lower bound) of A, if b is the least (greatest) element of the set of upper bound (lower bound) of A, denoted by lub or sup A (glb or inf A).
4. b is a maximal (minimal) element in X if $b \in X, \forall x \in X, b \leq x \Rightarrow b = x (x \leq b \Rightarrow x = b)$.

Note 2 (Maximal vs. Greatest). An element $m \in X$ is **maximal** if there does not exist $x \in X$ such that $x > m$. An element $g \in X$ is **greatest** if for all $x \in X$, $g \geq x$.

1. A set may have no greatest elements and no maximal elements: for example, any open interval of real numbers.
2. If a set has a greatest element, that element is also maximal.
3. A set with two maximal elements and no greatest element: $X = \{a, b, c\}$, where $a \leq b, a \leq c$ and b and c are incomparable, then each of b and b are maximal, and none of the elements of this set are greatest.

4. A set can have exactly one maximal element but no greatest element: $X = \{a+q|0 \leq q < 1\} \cup \{c\}$, where $a \leq c$ and $a+q$ and c are incomparable for any $0 \leq q < 1$. Then only c is maximal, and the set overall has no greatest element.

Definition 3 (Well Order)**.** If $(X, \leq)$ is a chain, we say that $(X, \leq)$ is a well-ordered set if $\forall A \subseteq X, A \neq \emptyset \Rightarrow A$ has a least element.

Note 3. Given a poset X, $a, b \in X$, we say $a < b$ if $a \leq b$ and $a \neq b$.

For example, $\mathbb{Z}^+$ is a well-ordered set. If $(X, \leq)$ is a well-ordered set, for any $a \in X$, the **successor** of a is $succ_{(X,\leq)}(a) :=$ the least element of $\{x \in X|a < x\}$. So if $\{x \in X|a < x\} \neq \emptyset$, then $succ_{(X,\leq)}(a)$ exists.

Definition 4. Given a poset X, $a \in X$, define initial segment as

$$IS_{(X,\leq)}(a) := \{x \in X|x < a\}$$

and weak initial segment as

$$WIS_{(X,\leq)}(a) := \{x \in X|x \leq a\}.$$

Axiom of Choice

Theorem 1 (Bourbaki's fixed point theorem)**.** *Suppose $(X, \leq)$ is a poset, in which every well-ordered subset has lub. Given a map $X \xrightarrow{f} X$, s.t. $x \leq f(x)$ for $\forall x \in X$, then $\exists a \in X$, s.t. $f(a) = a$.*

Proof. Pick an element $x_0 \in X$. Let S be the collection of subsets $Y \subseteq X$ such that:

- Y is well ordered with the least element x_0 and successor function $f|_{Y \setminus lubY}$,
- $x_0 \neq y \in Y \Rightarrow lub_X(IS_Y(y)) \in Y$.

Then we claim:

1. If $Y \in S$ and $Y' \in S$, then Y is an initial segment of Y' or vice versa. Let $V = \{x \in Y \cap Y'|WIS_Y(x) = WIS_{Y'}(x)\}$. Suppose first that V has a last element v. If v is not the last element of Y, then $succ_Y(v) = f(v)$; if v is not the last element of Y' then $succ_{Y'}(v) = f(v)$. Hence if neither of Y, Y' is an initial segment of the other, then $succ_Y(v) = succ_{Y'}(v) = f(v) \in V$, thus $f(v) = v$, and v is the fixed point.
   If V has no last element, let $z = lub_X(V)$. If $Y \neq V \neq Y'$, then it follows that $z \in Y \cap Y'$ (because if $y = \inf(Y - V)$ then $V = IS_Y(y)$ and therefore $z = lub_X(IS_Y(y)) \in Y$). Therefore $z \in V$, which is a contradiction.
2. The set $Y_0 = \cup\{Y|Y \in S\} \in S$.
   If $y_0 \in Y \in S$, then it follows from 1. that $\{y \in Y_0|y < y_0\} = IS_Y(y_0)$ and so this subset is well ordered with successor function

f. This implies that Y_0 is well ordered and satisfies first conditions of element in S. Also $lub_X(IS(y_0)) \in Y \subseteq Y_0$ which gives the second condition for Y_0. Thus 2. is proved.

Let $y_0 = lub_X(Y_0)$, if $y_0 \notin Y_0$ then $Y_0 \cup \{y_0\} \in S$ and so $y_0 \in Y_0$ after all. If $f(y_0) > y_0$ then $Y_0 \cup \{f(y_0)\} \in S$ contrary to the definition of Y_0, thus $f(y_0) = y_0$ as desired. $\square$

Theorem 2. *The following statement are equivalent:*

1. *For $\forall$ set X, $\exists$ map $\mathcal{P}_o(X) \xrightarrow{f} X$,* s.t. *$\forall S \in \mathcal{P}_o(X), f(S) \in S$. ($\mathcal{P}_o(X) := \{A | A| \subseteq X, A \neq \emptyset\}$)*
2. *If $(X, \leq)$ is a poset, in which every well-ordered subset has a lub in X, then X has a maximal element.*
3. *(Maximal Chain Theorem) $\forall$ poset $(X, \leq)$ has a maximal chain w.r.t $\subseteq$. i.e. a chain such that there is no other chain in $(X, \leq)$ which has it as a proper subset.*
4. *(Zorn's Lemma) If $(X, \leq)$ is a poset in which every chain has an upper bound in X then X has a maximal element.*
5. *(Zermelo's Well-Ordering Theorem) Every set has a well-order.*
6. *$\forall$ surj. $X \xrightarrow{f} Y$, $\exists$ an injection $Y \xrightarrow{g} X$,* s.t. *$f \circ g = id_Y$.*
7. *(Axiom of Choice) Given non-empty sets $S_\alpha (\alpha \in A)$, there exists a map $A \xrightarrow{f} \cup_{\alpha \in A} S_\alpha$,* s.t. *$f(\alpha) \in S_\alpha$.*

Note 4. A map $X \xrightarrow{f} Y$ is a subset $\Gamma \subseteq X \times Y$, s.t. $\forall x \in X, \exists! y \in Y, (x, y) \in \Gamma$.

Proof. $7 \Rightarrow 1$: We can number each non-empty subset of X by itself, since any element in a set is unique. That is $\mathcal{P}_o(X) = \{S_\alpha := \alpha | \alpha \in \mathcal{P}_o(X)\}$, here $\mathcal{P}_o(X)$ serves as A. Thus Axiom of Choice means $\exists$ a map $\mathcal{P}_o(X) \xrightarrow{f} X$, s.t. $f(\alpha) \in S_\alpha = \alpha(\alpha \in \mathcal{P}_o(X))$. (we emphasize $\mathcal{P}_o(X)$, rather than $\mathcal{P}(X)$, because there is nothing in $\emptyset$)

Note 5. Statement 1 claims that given a set X, any non-empty subset of X can be maps to a point inside this subset.

$1 \Rightarrow 2$: Assume that X has no maximal element, i.e. $\forall a \in X, X_a := \{x \in X | a < x\} \neq \emptyset$. $\exists$ map $\mathcal{P}_o(X) \xrightarrow{f} X$, s.t. $f(S) \in S$ for all $S \in \mathcal{P}_o(X)$. Define a map $X \xrightarrow{\pi} \mathcal{P}_o(X)(a \mapsto X_a)$ and $X \xrightarrow{g = f \circ \pi} X$. Thus for any $a \in X$, $g(a) = f(X_a) \in X_a$, thus $a < g(a)$, which leads to a contradiction with Bourbaki's fixed point theorem.

$$\begin{tikzcd} \mathcal{P}_o(X) \arrow[r, "f"] & X \\ X \arrow[u, "\pi"] \arrow[ur, "g"'] & \end{tikzcd}$$

$2 \Rightarrow 3$: Given a poset $(X, \leq)$ consider $S = \{C | C$ is a chain in P $w.r.t. \leq\}$. Thus $(S, \subseteq)$ is a poset. We claim that any totally ordered set in S has a lub in S. If $T \subseteq S$ is a totally ordered set, (that is T is a chain w.r.t $\subseteq$ of the chains w.r.t. $\leq$), then $\cup_{C \in T} C = lub_S T$. To show this, we need prove 2 things:

1. $\cup_{C \in T} C \in U_T$;

   For any $C \in T$, $C \subseteq \cup_{C \in T} C$, thus $\cup_{C \in T} C \in U_T$.

Note 6. $(T, \subseteq)$ is a chain, thus any comparison with the element in T need to use relation $\subseteq$.

2. $\cup_{C \in T} C \in L_{U_T}$.
   For any $v \in \cup_{C \in T} C, O \in U_T$, $\exists C \in T$, s.t. $v \in C \subseteq O$. Thus $\cup_{C \in T} C \subseteq O$, thus $\cup_{C \in T} C \in L_{U_T}$.

Thus every totally ordered subset (including well order subset) of $(S, \subseteq)$ has a lub, and $(S, \subseteq)$ has a maximal element, which implies $(X, \leq)$ has a maximal chain.

$3 \Rightarrow 4$: Given a poset $(X, \leq)$, it has a max. chain C, by assumption, C has an upper bound, say a, in X. Then a is a max. element in X, otherwise $\exists x \in X, a < x$, and hence $C \subsetneq C \cup \{x\}$ and $C \cup \{x\}$ is a chain, which leads to a contradiction to the maximality of C.

$4 \Rightarrow 5$: Let Y be a set, consider $X := \{A | A = (S_A, \leq_A)$ where $S_A \subseteq Y$ and $\leq_A$ is a well-ordering on $S_A\}$. We define a relation $\preceq$ on X: $A \preceq A' \Leftrightarrow A = A'$ or A is an initial segment of A' (i.e. $\exists a' \in S_{A'}, S_A = \{x \in S_{A'} | x <_{A'} a'\}$) and $\forall x_1, x_2 \in S_A, x_1 \leq_A x_2 \Leftrightarrow x_1 \leq_{A'} x_2$.

It is direct to see that $(X, \preceq)$ is a poset:

1. For any $A \in X, A \preceq A$;
2. If A is initial segment of A' then $A \neq A'$, since if $\exists a' \in S_{A'}, S_A = \{x \in S_{A'} | x <_{A'} a'\}$ then $a' \in A'$ but $a' \notin A$. Thus $A \preceq A', A' \preceq A \Rightarrow A = A'$
3. Suppose that $A \preceq A' \preceq A''$, and A, A' and A'' are not equal. Thus $\exists a'' \in S_{A''}$, s.t. $S_{A'} = IS_{A''}(a'')$, and $\exists a' \in S_{A'}$, s.t. $S_A = IS_{A'}(a')$. Since $a' <_{A''} a''$, any $a \in S_{A''}, a <_{A''} a' \Rightarrow a \in S_{A'}$. Thus $IS_{A''}(a) = \{x \in S_{A''} | x <_{A''} a'\} = \{x \in S_{A'} | x <_{A''} a'\} = \{x \in S_{A'} | x <_{A'} a'\} = IS_{A'}(a') = A$, thus $A \preceq A''$.
   Then, we claim:

1. $(X, \preceq)$ has a maximal element:
   Apply Zorn's lemma, let $(C, \preceq)$ be a chain on $(X, \preceq)$. Let $A_0 = (S_{A_0}, \leq_{A_0})$ where $S_{A_0} = \cup_{A \in C} S_A$, and $\leq_{A_0}$: for any $x_1, x_2 \in S_{A_0}$, find $A \in C$, s.t. $x_1, x_2 \in S_A$, we say that $x_1 \leq_{A_0} x_2$ if $x_1 \leq_A x_2$.
   Then we claim:
   - Such A exists:
     For any $x_1, x_2 \in S_{A_0}, \exists A_1, A_2 \in C$, s.t. $x_1 \in S_{A_1}, x_2 \in S_{A_2}$ and S_{A_1} and S_{A_2} are comparable on X w.r.t. $\preceq$, since C is a chain. Assume that S_{A_1} is an initial segment of S_{A_2}, then $x_1, x_2 \in S_{A_2}$.
   - $x_1 \leq_{A_0} x_2$ is independent of the choice of A, s.t. $x_1, x_2 \in S_A$:
     If $\exists A, A' \in C$, s.t. $x_1, x_2 \in S_A, S_{A'}$, then A, A' are comparable. Assume that $A \preceq A'$, that is A is an initial segment of A', then in S_A, we have $x_1 \leq_A x_2 \Leftrightarrow x_1 \leq_{A'} x_2$.
   - $(S_{A_0}, \leq_{A_0})$ is a total order set :
     Any $x_1, x_2 \in S_{A_0}$ will be covered by a S_A where A is an element of a chain C on X. Thus x_1 and x_2 are comparable by $x_1 \leq_{A_0} x_2 \Leftrightarrow x_1 \leq_A x_2$.
   - $(S_{A_0}, \leq_{A_0})$ is a well order set :

Let $T \subseteq S_{A_0}$ and $T \neq \emptyset$. Then $T = T \cap S_{A_0} = T \cap \cup_{A \in C} S_A = \cup_{A \in C}(T \cap S_A) \neq \emptyset$. Thus $\exists A \in C$, s.t. $T \cap S_A \neq \emptyset$. Since A is well ordering, $T \cap S_A$ has least element, denoted by t.
Any $A' \in C$, it is either $A' = A$ or $A' \preceq A$ or $A \preceq A'$. If $A' \preceq A$, then $S_{A'}$ is an initial segment of S_A, that is $\exists a \in S_A$, s.t. $S_{A'} = \{x \in S_A | x <_A a\}$. Thus $S'_A \subseteq S_A$, and $T \cap S_{A'} \subseteq T \cap S_A$, thus t is the least element of $T \cap S_A \Rightarrow t$ is the least element of $T \cap S_{A'}$;
If $A \preceq A'$, then S_A is an initial segment of $S_{A'}$, thus $\exists a' \in S_{A'}$, s.t. $S_A = \{x \in S_{A'} | x <_{A'} a'\}$ and $T \cap S_A = T \cap \{x \in S_{A'} | x <_{A'} a'\} = \{x \in T \cap S_{A'} | x <_{A'} a'\}$. For any $s \in T \cap S_{A'}$, if $a' \leq_{A'} s$, then $t <_{A'} a' \leq_{A'} s$; if $s <_{A'} a'$, then $s \in T \cap S_A$, and $t \leq_A s \Rightarrow t \leq_{A'} s$. Thus t is the least element of $T \cap S_{A'}$.
Thus t is the least element of $T \cap S_{A_0} = T$, thus $\leq_{A_0}$ is a well order on S_{A_0}. Furthermore, $(S_{A_0}, \leq_{A_0}) \in X$.

- S_{A_0} is an upper bound of C on X, w.r.t. $\preceq$:
Given $A \in C$, since C is a chain, any $A' \in C$ admits 3 cases: $A' = A, A' \preceq A, A \preceq A'$. Define $\Pi := \{A' \in C | A \preceq A'\} \backslash \{A\}$ and $\Gamma := \{A' \in C | A' \preceq A\} \backslash \{A\}$.
For any $B \in \Pi$, $\exists b \in S_B$, s.t. $S_A = IS_B(b)$. Define $\Phi := \{A' \in \Pi | A' \preceq B\} \backslash \{B\}$. If $\Phi \neq \emptyset$, then $\exists C \in \Phi, \exists c \in S_C$, s.t. $S_A = IS_C(c)$. Collect all these kind of c and form a set Δ, then Δ is a non-empty subset of S_B. Since S_B is a well ordering set, Δ has a least element μ, and exists the corresponding $D \in \Phi$, s.t. $S_A = IS_D(\mu)$. Thus

$$\begin{aligned} S_A = IS_D(\mu) &= \{x \in S_D | x <_D \mu\} \\ &\overset{x,\mu \in S_{A_0}}{=\!=\!=} \{x \in S_D | x <_{A_0} \mu\} \end{aligned}$$

Since any $A' \in \Pi$, the corresponding $\mu \leq_{A'} a'$, thus

$$\begin{aligned} \{x \in S_{A'} | x <_{A_0} \mu\} &= \{x \in S_{A'} | x <_{A'} \mu\} \\ &\subseteq \{x \in S_{A'} | x <_{A'} a'\} \\ &= IS_{A'}(a') \\ &= S_A = IS_D(\mu) \end{aligned}$$

On the other hand, For any $A'' \in \Gamma$, $A'' \preceq A \Rightarrow S_{A''} \subseteq S_A$, thus $\{x \in S_{A''} | x <_{A_0} \mu\} \subseteq S_A$. Thus

$$\begin{aligned} S_A &= IS_D(\mu) \\ &= \cup_{A' \in \Pi}\{x \in S_{A'} | x <_{A_0} \mu\} \cup \left(\cup_{A'' \in \Gamma}\{x \in S_{A''} | x <_{A_0} \mu\}\right) \\ &= \{x \in \cup_{A' \in \Pi \cup \Gamma} S_{A'} | x <_{A_0} \mu\} \\ &= \{x \in \cup_{A' \in C} S_{A'} | x <_{A_0} \mu\} \\ &= IS_{A_0}(\mu) \end{aligned}$$

Note 7. Recall the proof of $2 \Rightarrow 3$.

Thus $A \preceq A_0$ for any $A \in C$, and A_0 is an upper bound of C. $(X, \preceq)$, as a poset, whose any chain C has an upper bound A_0, thus X has a maximal element by Zorn's lemma.

2. A maximal element in $(X, \preceq)$ is $(Y, \leq_Y)$.
   If $(Y_0, \leq_{Y_0})$ is a max. element in X w.r.t. $\preceq$ and $Y_0 \neq Y$, then $\exists y \in Y \backslash Y_0$. Define $Y_1 := Y_0 \cup \{y\}$ and a partial order: $v \leq_{Y_1} y, v_1 \leq_{Y_1} v_2 \Leftrightarrow v_1 \leq_{Y_0} v_2$ for $\forall v, v_1, v_2 \in Y_0$.
   Then $(Y_1, \leq_{Y_1})$ admits a well-ordering which makes $(Y_0, \leq_{Y_0})$ an initial segment, because any non-empty subset ϕ of Y_1 is either $\{y\}$ or $(\phi \cap Y_0) \cup (\phi \cap \{y\})$, clearly ϕ has least element.
   Thus $(Y_1, \leq_{Y_1}) \in X$ and $(Y_0, \leq_{Y_0}) \preceq (Y_1, \leq_{Y_1})$, which leads to a contradiction.

Since X is the set of well ordering subset on Y, $(Y, \leq_Y) \in X$, thus $(Y, \leq_Y)$ is well ordering.

5 $\Rightarrow$ 6: Choose a well ordering $\leq$ on X, For any $y \in Y$, define $g(y) :=$ the least element of $f^{-1}(y)$, then $f \circ g(y) = y$.

6 $\Rightarrow$ 7: Let $S := \cup_{\alpha \in A} S_\alpha$, define $X := \{(s, \alpha) \in S \times A | s \in S_\alpha\}$. Consider two projection $X \xrightarrow{f} A((s, \alpha) \mapsto \alpha)$ and $X \xrightarrow{p} S((s, \alpha) \mapsto s)$, thus f is a surjection, then $\exists A \xrightarrow{g} X$ such that $f \circ g(\alpha) = \alpha$ for any $\alpha \in A$.

Define s_α is the least element of S_α, then $g(\alpha) = (s_\alpha, \alpha)$ and $p \circ g(\alpha) = p(s_\alpha, \alpha) = s_\alpha \in S_\alpha$. Thus $A \xrightarrow{p \circ g} S = \cup_{\alpha \in A} S_\alpha$ is desired.

$$\begin{tikzcd}
 & A_{\alpha} \arrow[dl, shift left, "g"] \\
X \arrow[ur, shift left, "f"] \arrow[r, phantom, "\subseteq"] \arrow[dr, "p"'] & S \times A_{(s,\alpha)} \arrow[u, "\pi"'] \arrow[d, "\pi"] \\
 & S_{s}
\end{tikzcd}$$

$\square$