Introduction to Topology

General Topology, Lecture 16

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This is the Lecture note for the *Introduction to Topology*. The course covers the following topics: Naive Set Theory, Elementary Number Theory, Group Theory, Topological Spaces and Continuous Maps, Introduction to Algebraic Topology.

Content:

Abelian Group

Definition 1 (Abelian Group)**.** Given a group $(G, \square)$, we say $(G, \square)$ is a abelian group if $\forall g, g' \in G, g \square g' = g' \square g$.

The set $\mathbb{Z} \times \mathbb{Z}$ is equivalent with $\{\{1,2\} \xrightarrow{f} \mathbb{Z} | f \text{ if a map}\}$. For any $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, it can be represented as $f : 1 \mapsto x, 2 \mapsto y$, $\{1,2\}$ is the ordinate. And for any maps $\{1,2\} \xrightarrow{f} \mathbb{Z}$, it is corresponded by $(f(1), f(2)) \in \mathbb{Z} \times \mathbb{Z}$.

Let S be a set, define

$$\mathbb{Z}^{\oplus S} := \{S \xrightarrow{f} \mathbb{Z} | f \text{ is a map s.t. } f(s) \neq 0 \text{ for finitely many } s \in S\}$$

The element of $\mathbb{Z}^{\oplus S}$ is encoding each element of S with integer, and only finite element of S will be encoded by nonzero integer.

Note 1. Sometimes, we will denote $S \xrightarrow{f} \mathbb{Z}$ by $(x_s)_{s \in S}$.

Example 1. The element of $\mathbb{Z}^{\oplus \mathbb{N}}$ is a series of integer $(x_1, x_2, \cdots)(x_i \in \mathbb{Z}, i \in \mathbb{N})$ which has only finite nonzero integers.

We can define add on $\mathbb{Z}^{\oplus S}$: $(x_s)_{s \in S} + (y_s)_{s \in S} = (x_s + y_s)_{s \in S}$. Then for any $(x_s)_{s \in S}, (y_s)_{s \in S} \in \mathbb{Z}^{\oplus S}$, the binary operation $(\mathbb{Z}^{\oplus S}, +)$ has

1. $(x_s + y_s)_{s \in S} \in \mathbb{Z}^{\oplus S}$ (for $(x_s)_{s \in S}, (y_s)_{s \in S}$ only has finite nonzero integers)
2. $e = (0)_{s \in S} \in \mathbb{Z}^{\oplus S}$
3. $((x_s)_{s \in S})^{-1} = (-x_s)_{s \in S} \in \mathbb{Z}^{\oplus S}$
4. $(x_s)_{s \in S} + (y_s)_{s \in S} = (x_s + y_s)_{s \in S} = (y_s)_{s \in S} + (x_s)_{s \in S}$

Thus $(\mathbb{Z}^{\oplus S}, +)$ is a abelian group, and we call $(\mathbb{Z}^{\oplus S}, +)$ as **Free Abelian Group**.

Definition 2 (Homomorphism)**.** Given two groups $(G, \square), (G', \square')$, a map $G \xrightarrow{T} G'$ is a homomorphism w.r.t. $\square$ and $\square'$ if $\forall g_1, g_2 \in G$, $T(g_1 \square g_2) = T(g_1) \square' T(g_2)$.

Example 2. Map $\mathbb{Z} \xrightarrow{T} \mathbb{Z}/5\mathbb{Z}$ is a homomorphism, since for any $a, b \in \mathbb{Z}$, $(a + 5\mathbb{Z}) + (b + 5\mathbb{Z}) = (a + b) + 5\mathbb{Z}$.

Definition 3 (Isomorphism)**.** We say a homomorphism T is an isomorphism if T is a bijection.

Definition 4. Given two groups $(G, \square), (G', \square')$, let $G \xrightarrow{T} G'$ be a homomorphism:
1. $ker(T) := T^{-1}(e') = \{g \in G | T(g) = e'\}$;
2. $im(T) := T(G) = \{T(g) | g \in G\}$.

Exercise 1. Show that $ker(T)$ is a subgroup of $(G, \square)$, $im(T)$ is a subgroup of $(G', \square')$.

Proof. 1.
(0.) Obviously $ker(T) \subseteq G$.
(1.) for $\forall g_1, g_2 \in ker(T)$:

$$\begin{aligned} T(g_1 \square g_2) &= T(g_1) \square' T(g_2) \\ &= e' \square e' = e' \end{aligned}$$

thus $g_1 \square g_2 \in ker(T)$.
(2.) for $\forall g \in ker(T)$,

$$\begin{aligned} T(g) &= T(g \square e) \\ &= T(g) \square' T(e) \\ &= e' \square' T(e) = e' \end{aligned}$$

and $T(e) \square' e' = e'$ in the same way, thus $e \in ker(T)$, and be the unit element of $ker(T)$.
(3.) for $\forall g \in ker(T)$,

$$\begin{aligned} T(e) &= T(g \square g^{-1}) \\ &= T(g) \square' T(g^{-1}) \\ &= e' \square' T(g^{-1}) \\ &= e' \end{aligned}$$

and $T(g^{-1}) \square' e' = e'$, thus $T(g^{-1}) = e'$, and $g^{-1} \in ker(T)$.
Thus $ker(T)$ is a subgroup of $(G, \square)$.
2.
0. Obviously $im(T) \subseteq G'$.
1. for $\forall g'_1, g'_2 \in im(T), \exists g_1, g_2,$ s.t. $T(g_1) = g'_1, T(g_2) = g'_2$. Thus

$$\begin{aligned} T(g_1 \square g_2) &= T(g_1) \square' T(g_2) \\ &= g'_1 \square' g'_2 \end{aligned}$$

thus $g'_1 \square' g'_2 \in im(T)$.
(2.) Since $e \in ker(T) \Rightarrow T(e) = e' \Rightarrow e' \in im(T)$.

(3.) for $\forall g' \in im(T), \exists g \in G,$ s.t. $T(g) = g'$, and

$$\begin{aligned} T(e) &= T(g\square g^{-1}) \\ &= T(g)\square' T(g^{-1}) \\ &= g'\square' T(g^{-1}) \\ &= e' \end{aligned}$$

and $T(g^{-1})\square' g' = e'$ in the same way, thus $T(g^{-1}) = g'^{-1}, g'^{-1} \in im(T)$.

Thus $im(T)$ is a subgroup of G'. $\square$

Exercise 2. $G \xrightarrow{T} G'$ is a homomorphism show that $T(e) = e'$ and $T(g^{-1}) = T(g)^{-1}$ for $\forall g \in G$. e' is the unit element of $(G', \square')$,

Proof. 1. $ker(T)$ is a subgroup of G, thus $e \in ker(T) \Rightarrow T(e) = e'$. 2. $T(g^{-1})\square' T(g) = T(g^{-1}\square g) = T(e) = e'$, thus $T(g^{-1}) = T(g)^{-1}$. $\square$

Definition 5. Given two groups $(G, \square), (G', \square')$, let $G \xrightarrow{T} G'$ be a homomorphism. If $(G', \square')$ is abelian, $cok(T) := G'/im(T)$.

Normal Subgroup

Consider a group $(G, \square)$ and natural projection π. Are there is map $\square'$ such that the following commutative diagram holds? i.e. for $\forall g_1, g_2, \pi(g_1 \square g_2) = \pi(g_1)\square' \pi(g_2)$?

$$\begin{tikzcd} {}_{(a,b)}G \times G^{(a,b)} \arrow[r, "\square"] \arrow[d, "\pi\times\pi"'] & G^{a\square b} \arrow[d, "\pi"] \\ {}_{(a\square H, b\square H)}G/H \times G/H \arrow[r, "\square'"] & {}_{(a\square H)\square'(b\square H)}G/H^{a\square b\square H} \end{tikzcd}$$

Note 2. Given maps f_1, f_2 and a surjection g, we have proved if $g \circ f_1 = g \circ f_2 \Rightarrow f_1 = f_2$, thus if $\square'$ exists, there would be only one.

In the other word, for $(a, b) \in G \times G$, we can define map $\square'$ as

$$(a\square H)\square'(b\square H) := a\square b\square H$$

But there is not well-defined, because there would exists $a', b' \in G$ such that $a'\square H = a\square H, b'\square H = b\square H$, thus $(a\square H)\square'(b\square H) = (a'\square H)\square'(b'\square H)$, but $a'\square b'\square H \neq a\square b\square H$.

Definition 6 (Normal Subgroup)**.** Given a group $(G, \square)$, $(H, \square)$ is a subgroup of $(G, \square)$ (denote by $H \leq G$). We call H is a normal subgroup, denote by $H \trianglelefteq G$, if $\forall g \in G, \forall h \in H, g^{-1}hg \in H$.

Exercise 3. If $H \trianglelefteq G$, show that $a^{-1}\square a' \in H, b^{-1}\square b' \in H \Rightarrow (a\square b)^{-1}\square(a'\square b') \in H$, that is $H \trianglelefteq G$ is the **sufficient condition**.

Proof. Denote $a^{-1}\Box a' = h \in H$, $\exists h' \in H$, s.t. $b^{-1}\Box b' = h' \Rightarrow b' = b\Box h'$, thus

$$\begin{aligned}&(a\Box b)^{-1}\Box(a'\Box b')\\&= b^{-1}\Box a^{-1}\Box a'\Box b'\\&= b^{-1}\Box h\Box b\Box h'\\&= (b^{-1}\Box h\Box b)\Box h'\end{aligned}$$

$H \trianglelefteq G \Rightarrow b^{-1}\Box h\Box b \in H \Rightarrow (a\Box b)^{-1}\Box(a'\Box b') \in H.$ □

Note 3. a, a' belong to the same coset of $H \Leftrightarrow a\Box H = a'\Box H \Leftrightarrow a^{-1}a' \in H \Leftrightarrow a' = a\Box h$.

We have seen that if $H \trianglelefteq G$ then there is a binary operation $G/H \times G/H \xrightarrow{\Box'} G/H((a\Box H, b\Box H) \mapsto a\Box b\Box H)$, such that the commutative diagram

$$\begin{array}{ccc} G\times G & \xrightarrow{\Box} & G \\ {\scriptstyle \pi\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G/H\times G/H & \xrightarrow{\Box'} & G/H \end{array}$$

holds.

Exercise 4 (Quotient Group)**.** $H \trianglelefteq G$, show that $(G/H, \Box')$ is a group.

Proof. 0. $H \trianglelefteq G \Rightarrow \Box'$ is well-defined by $(g_1\Box H)\Box'(g_2\Box H) := (g_1\Box g_2)\Box H$ for any $g_1, g_2 \in G$.

1. $\forall g_1, g_2 \in G, g_1\Box H, g_2\Box H \in G/H$, then $(g_1\Box H)\Box'(g_2\Box H) = (g_1\Box g_2)\Box H$. $g_1\Box g_2 \in G$ thus $(g_1\Box g_2)\Box H \in G/H$.

2. $\forall g \in G, g\Box H \in G/H$, then $(g\Box H)\Box H = (g\Box e)\Box H = g\Box H$, thus $e_{G/H} = H \in G/H$.

3. $(g\Box H)^{-1} = g^{-1}\Box H \in G/H$. □

Exercise 5. $G \xrightarrow{T} G'$ is a homomorphism, show that $ker(T) \trianglelefteq G$ and $im(T) \leq G'$.

Proof. 1. For $\forall g \in G, k \in ker(T)$,

$$\begin{aligned}T(g^{-1}\Box k\Box g) &= T(g^{-1})\Box' e'\Box' T(g)\\&= T(g)^{-1}\Box' T(g)\\&= e'\end{aligned}$$

Thus $g^{-1}\Box k\Box g \in ker(T) \Rightarrow ker(T) \trianglelefteq G$.

2. (1.) $T(g_1)\Box' T(g_2) = T(g_1\Box g_2) \in im(T)$; (2.) $e' = T(e) \in im(T)$; (3) $T(g)^{-1} = T(g^{-1}) \in im(T)$. □

Thus if subgroup $(H, \Box)$ is normal then $(G/H, \Box')$ is a group. Conversely, if $(G, \Box)$ is abelian, then any subgroup $(H, \Box)$ is normal, for $ghg^{-1} = gg^{-1}h = h \in H$; and $(G/H, \Box')$ is abelian, for

$$\begin{aligned}&(a\Box H)\Box'(b\Box H)\\&= a\Box b\Box H = b\Box a\Box H\\&= (b\Box H)\Box'(a\Box H).\end{aligned}$$

Theorem 1 (1st theorem of homomorphism)**.** *Suppose $G \xrightarrow{T} G'$ is a homomorphism, $H \leq G$. Then*

1. *$T(H) = \{e'\}$, i.e. $H \subseteq ker(T) \Leftrightarrow \exists!$ map $G/H \xrightarrow{\tilde{T}} G'$ s.t.*

$$\begin{tikzcd} G \arrow[rr, "T"] \arrow[dr, "\pi"'] & & G' \\ & G/H \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

2. *If $H \subseteq ker(T)$ and $H \trianglelefteq G$ then $G/H \xrightarrow{\tilde{T}} G'$ is a homomorphism.*
3. *$H = ker(T) \Leftrightarrow \tilde{T}$ is injection.*
4. *T is surjection $\Leftrightarrow \tilde{T}$ is surjection.*

Proof. 1. $\Leftarrow$: Suppose $\exists \tilde{T}$, then $e \in H \Rightarrow \{e'\} \subseteq T(H) \subseteq G'$; $\pi(H) = H \in G/H$, $\tilde{T}(H) \in G'$. Thus

$$\{e'\} \subseteq T(H) = \{\tilde{T}(H)\} \subseteq G'$$

and $\tilde{T}(H) \in G \Rightarrow T(H) = \{e'\}, \tilde{T}(H) = e'$. $\Rightarrow$: $\tilde{T}(g\square H) := T(g)$. π is surjection $\Rightarrow \tilde{T}$ has uniqueness.

2,3,4 skip. $\square$

Note 4. $H \xrightarrow{\pi} H$ is an element of G/H and then $\tilde{T}(H)$ is an element of G'.