General Topology
Lecture 3

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This is the Lecture note for the *General Topology* course in Spring 2020.

1 Basis

Definition 1 (Coarser Topology)**.** Let X be a set, and $\mathscr{T}$ and $\mathscr{T}'$ be two topologies on X. We say that $\mathscr{T}$ is coarser/weaker than $\mathscr{T}'$ if $\mathscr{T} \subseteq \mathscr{T}'$ (or say $\mathscr{T}'$ is finer/stronger than $\mathscr{T}$).

Note 1. In other words, $\mathscr{T}$ is weaker than $\mathscr{T}'$ iff $X \xrightarrow{id_X} X$, where the former and later X are equipped with $\mathscr{T}'$ and $\mathscr{T}$ respectively, is continuous.

Let X be a set and $\mathcal{S} \subseteq \mathcal{P}(X)$ be a family of subsets of X. Are there a smallest topology $\mathscr{T}'$ on X s.t. all $\mathcal{S} \subseteq \mathscr{T}'$? It is direct to check that if $\mathscr{T}_\alpha(\alpha \in A)$ is a family of topologies on X, then $\cap_{\alpha\in A}\mathscr{T}_\alpha$ is also a topology on X. For any $\alpha \in A$:

1. $\emptyset, X \in \mathscr{T}_\alpha \Rightarrow \emptyset, X \in \cap_{\alpha\in A}\mathscr{T}_\alpha$;
2. $U_\beta \in \mathscr{T}_\alpha(\beta \in B) \Rightarrow \cup_{\beta\in B}U_\beta \in \mathscr{T}_\alpha \Rightarrow \cup_{\beta\in B}U_\beta \in \cap_{\alpha\in A}\mathscr{T}_\alpha$.
3. $U_1, U_2 \in \mathscr{T}_\alpha \Rightarrow U_1 \cap U_2 \in \mathscr{T}_\alpha \Rightarrow U_1 \cap U_2 \in \cap_{\alpha\in A}\mathscr{T}_\alpha$.

Define $\mathcal{T}$ be the family of all topologies on X containing the elements in $\mathcal{S}$, that is for $\forall \mathscr{T} \in \mathcal{T}, \mathcal{S} \subseteq \mathscr{T}$. We call

$$\mathscr{T}(\mathcal{S}) := \cap_{\mathscr{T}\in\mathcal{T}}\mathscr{T}$$

the topology induced by $\mathcal{S}$, which is clearly the coarsest topology containing $\mathcal{S}$.

Let Π be the family of any finite intersection of the element in $\mathcal{S}$, then for $\forall \mathscr{T} \in \mathcal{T}, \Pi \subseteq \mathscr{T}$ by def. Furthermore, for $\forall \mathscr{T} \in \mathcal{T}$, the arbitrary union of the elements in Π must in $\mathscr{T}$, that is $\{\cup_{V\in\mathcal{F}}V | \mathcal{F} \subseteq \Pi\} \subseteq \mathscr{T}$. Thus $\{\cup_{V\in\mathcal{F}}V | \mathcal{F} \subseteq \Pi\} \subseteq \cap_{\mathscr{T}\in\mathcal{T}}\mathscr{T} = \mathscr{T}(\mathcal{S})$.

Proposition 1. *Let X be a set and $\mathcal{S} \subseteq \mathcal{P}(X)$ be a family of subsets of X. Then*

$$\mathscr{T}(\mathcal{S}) = \{\cup_{V\in\mathcal{F}}V | \mathcal{F} \subseteq \Pi\},$$

where Π is the family of any finite intersection of elements in $\mathcal{S}$, that is

$$\Pi := \{S_1 \cup \cdots \cup S_k | S_1, \cdots, S_k \in \mathcal{S}, k \in \mathbb{N}\} \cup \{X\}.$$

Proof. We have proved that $\{\cup_{V\in\mathcal{F}} V | \mathcal{F} \subseteq \Pi\} \subseteq \mathscr{T}(\mathcal{S})$. Note that $\mathscr{T}(\mathcal{S})$ is the coarsest topology containing $\mathcal{S}$, Thus if $\{\cup_{V\in\mathcal{F}} V | \mathcal{F} \subseteq \Pi\}$ is a topology containing $\mathcal{S}$, we are done.

1. $\{X\}, \emptyset \subseteq \Pi$, thus $X = \cup_{V\in\{X\}} V \in \{\cup_{V\in\mathcal{F}} V | \mathcal{F} \subseteq \Pi\}$, $\emptyset = \cup_{V\in\{\emptyset\}} V \in \{\cup_{V\in\mathcal{F}} V | \mathcal{F} \subseteq \Pi\}$.
2. For any $U_\alpha = \{\cup_{V\in\mathcal{F}_\alpha} V | \mathcal{F}_\alpha \subseteq \Pi\} \in \{\cup_{V\in\mathcal{F}} V | \mathcal{F} \subseteq \Pi\} (\alpha \in A)$, we have $\mathcal{F}_\alpha \subseteq \Pi (\alpha \in A) \Rightarrow \cup_{\alpha\in A} \mathcal{F}_\alpha \subseteq \Pi \Rightarrow \cup_{\alpha\in A} U_\alpha = \{\cup_{V\in\cup_{\alpha\in A}\mathcal{F}_\alpha} V | \cup_{\alpha\in A} \mathcal{F}_\alpha \subseteq \Pi\} \in \{\cup_{V\in\mathcal{F}} V | \mathcal{F} \subseteq \Pi\}$
3. If $\cup_{V\in\mathcal{F}_1} V, \cup_{W\in\mathcal{F}_2} W \in \{\cup_{V\in\mathcal{F}} V | \mathcal{F} \subseteq \Pi\}$, then $(\cup_{V\in\mathcal{F}_1} V) \cap (\cup_{W\in\mathcal{F}_2} W) = \cup_{V\in\mathcal{F}_1, W\in\mathcal{F}_2} (V \cap W)$ where $V, W \in \Pi$. Since Π is the family of finite intersection, $V \cap W$ is the finite intersection of elements of $\mathcal{S}$ or X, i.e. $V \cap W \in \Pi$. Let $\mathcal{F}_3 := \{V \cap W | V \in \mathcal{F}_1, W \in \mathcal{F}_2\}$, thus $\mathcal{F}_3 \subseteq \Pi$. Then $\cup_{V\in\mathcal{F}_1, W\in\mathcal{F}_2} (V \cap W) = \cup_{Z\in\mathcal{F}_3} Z \in \{\cup_{V\in\mathcal{F}_3} V | \mathcal{F}_3 \subseteq \Pi\}$.

Thus $\{\cup_{V\in\mathcal{F}} V | \mathcal{F} \subseteq \Pi\}$ is a topology containing $\mathcal{S}$, and $\mathscr{T}(S) \subseteq \{\cup_{V\in\mathcal{F}} V | \mathcal{F} \subseteq \Pi\} \Rightarrow \mathscr{T}(S) = \{\cup_{V\in\mathcal{F}} V | \mathcal{F} \subseteq \Pi\}$. □

Note 2. Orally, $\mathscr{T}(\mathcal{S})$ consists of arbitrary unions of finite intersection of elements of $\mathcal{S}$.

Conventionally, when we talking about the subsets of X, we define $\cap\emptyset := X$.

Definition 2 (Sub-basis)**.** Given a set X, $\mathcal{S} \subseteq \mathcal{P}(X)$, $\mathcal{S}$ is called a sub-basis of a topology $\mathscr{T}$ on X if $\mathscr{T} = \mathscr{T}(S)$.

To obtain $\mathscr{T}(\mathcal{S})$ from $\mathcal{S}$, we need two steps: first, perform the finite intersection of elements in $\mathcal{S}$; then perform arbitrary union of the these intersection. But can we construct a topology that contains $\mathcal{S}$ only by union?

Definition 3 (Basis)**.** Given a set X, let $\mathcal{B} \subseteq \mathcal{P}(X)$ and $\mathscr{T}$ is a topology on X. We say that $\mathcal{B}$ is a basis of $\mathscr{T}$ if $\mathcal{B} \subseteq \mathscr{T}$ and for any $U \in \mathscr{T}, \exists \mathcal{F} \subseteq \mathcal{B}$, s.t. $U = \cup\mathcal{F} (:= \cup_{B\in\mathcal{F}} B)$.

Note 3. Thus given a sub-basis $\mathcal{S}$, we can induce the basis Π, and then perform the union on basis to obtain the topology $\mathscr{T}(\mathcal{S})$.

Note that if $\mathcal{B}$ is a basis of $\mathscr{T}$, then $B \in \mathscr{T}$ for any $B \in \mathcal{B}$, thus any union of elements of $\mathcal{B}$ is in $\mathscr{T}$. Thus we can define the $\mathcal{B}$ is a basis of $\mathscr{T}$ directly:

$$\mathscr{T} = \{\cup\mathcal{F} | \mathcal{F} \subseteq \mathcal{B}\}.$$

In general, a topological space $(X, \mathscr{T})$ can have many bases. The whole topology $\mathscr{T}$ is always a base for itself (that is, $\mathscr{T}$ is a base for $\mathscr{T}$).

Definition 4 (Local Basis)**.** For a given $x \in X$, we say that $\mathcal{B}_x$ is a local basis of $\mathscr{T}$ at x, if

1. for $\forall V \in \mathcal{B}_x, x \in V \in \mathscr{T}$ and

2. for $\forall U \in \mathscr{T}$ where $x \in U$, $\exists V \in \mathcal{B}_x$, s.t. $x \in V \subseteq U$.

Example 1. Let X be a metric space and $\mathscr{T}$ is the topology defined by metric. Then $\mathcal{B} = \{B_r(x) | r > 0\}$ is a local basis of $\mathscr{T}$ at x.

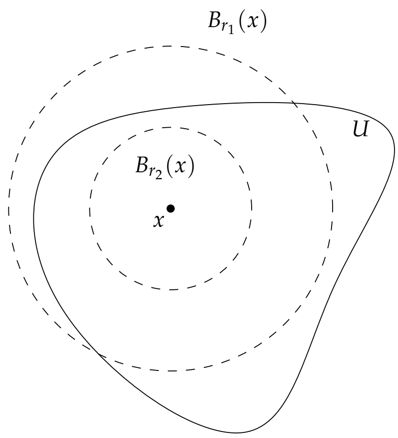

Figure 1: Local Basis

Exercise 1. *Let $(X, \mathscr{T})$ be a topology space and $\mathcal{B} \subseteq \mathcal{P}(X)$. For $x \in X$, define $\mathcal{B}_x := \{U \in \mathcal{B} | x \in U\}$. Show that $\mathcal{B}$ is a basis of $\mathscr{T}$ on $X \Leftrightarrow \forall x \in X, \mathcal{B}_x$ is a local basis of $\mathscr{T}$ on X at x.*

Proof. $\Rightarrow$: pick a $x \in X$ and $U \in \mathscr{T}$ where $x \in U$, then $\exists \mathcal{F} \subseteq \mathcal{B}$, s.t. $x \in U = \cup \mathcal{F}$, since $\mathcal{B}$ is a basis of $\mathscr{T}$. Then $\exists B \in \mathcal{F}$ such that $x \in B \subseteq \cup \mathcal{F} = U$, it is clear to see $B \in \mathcal{B}_x$. Since $\mathcal{B}$ is a basis of $\mathscr{T}$, $B \in \mathscr{T}$ for $\forall B \in \mathcal{B}$, Thus $\mathcal{B}_x$ is a local basis of $\mathscr{T}$ at x for any $x \in X$.
$\Leftarrow$: On the one hand, given a $x \in X, \mathcal{B}_x \subseteq \mathcal{B} \Rightarrow \cup_{x \in X} \mathcal{B}_x \subseteq \mathcal{B}$. For any $B \in \mathcal{B}$, if $B \neq \emptyset$, there exists $x' \in B$, thus $B \in \mathcal{B}_{x'} \subseteq \cup_{x \in X} \mathcal{B}_c$. Thus $\mathcal{B} = \cup_{x \in X} \mathcal{B}_x$. $\mathcal{B}_x$ is a local basis of $\mathscr{T}$ at any $x \in X \Rightarrow \mathcal{B}_x \subseteq \mathscr{T}$ for any $x \in X$. Thus $\mathcal{B} \subseteq \mathscr{T}$.
On the other hand, given a non-empty $U \in \mathscr{T}$, for any $x \in U$, $\exists B_x \in \mathcal{B}_x$, such that $x \in B_x \subseteq U$. Thus $\cup_{x \in U} B_x \subseteq U$. For any $x' \in U$, $\exists B_{x'} \in \mathcal{B}_{x'}$, s.t. $x' \in B_{x'} \subseteq U \Rightarrow x' \in \cup_{x \in U} B_x \Rightarrow \cup_{x \in U} B_x = U$, where $B_x \in \mathcal{B}$. Thus $\mathcal{B}$ is a basis of $\mathscr{T}$. $\square$

Note 4. Very useful routine. We use it to prove the open set, in metric space, is the union of open balls as well.

Exercise 2. *Let X be a set and $\mathcal{B} \subseteq \mathcal{P}(X)$. Show that there exists a topology $\mathscr{T}$ such that $\mathcal{B}$ is a basis of $\mathscr{T} \Leftrightarrow$*

1. *$\cup \mathcal{B} = X$ and*
2. *$\forall U, V \in \mathcal{B}$ and $x \in U \cap V, \exists W \in \mathcal{B}$,* s.t. *$x \in W \subseteq U \cap V$.*

(Hint: if such $\mathscr{T}$ exists, it must be $\{\cup \mathcal{F} | \mathcal{F} \subseteq \mathcal{B}\}$.)

Proof. $\Rightarrow$: 1) $X \in \mathscr{T} \Rightarrow \exists \mathcal{F} \subseteq \mathcal{B}$, s.t. $X = \cup\mathcal{F} \subseteq \cup\mathcal{B} \subseteq X \Rightarrow X = \cup\mathcal{B}$; 2) $\mathcal{B}$ is a basis of $\mathscr{T} \Rightarrow \forall U, V \in \mathcal{B}, U, V \in \mathscr{T}$, thus $U \cap V \in \mathscr{T}$. Pick $x \in U \cap V$, $\mathcal{B}_x$ is a local basis of $\mathscr{T}$ at x. Thus $\exists B \in \mathcal{B}_x \subseteq \mathcal{B}$, s.t. $x \in B \subseteq U \cap V$.
$\Leftarrow$: Define $\mathscr{T} = \{\cup\mathcal{F} | \mathcal{F} \subseteq \mathcal{B}\}$, all we need to de is show $\mathscr{T}$ is a topology:

1. $\varnothing \subseteq \mathcal{B} \Rightarrow \varnothing = \cup\varnothing \in \mathscr{T}$; $\mathcal{B} \subseteq \mathcal{B} \Rightarrow X = \cup\mathcal{B} \in \mathscr{T}$.
2. for any $\mathcal{F}_\alpha \subseteq \mathcal{B}(\alpha \in A)$,
$$\begin{aligned}\cup_{\alpha\in A}(\cup\mathcal{F}_\alpha) &= \cup_{\alpha\in A}(\cup_{B\in\mathcal{F}_\alpha} B)\\ &= \cup_{B\in\cup_{\alpha\in A}\mathcal{F}_\alpha} B\\ &= \cup(\cup_{\alpha\in A}\mathcal{F}_\alpha)\\ &\in \mathscr{T},\end{aligned}$$
since $\cup_{\alpha\in A}\mathcal{F}_\alpha \subseteq \mathcal{B}$.
3. for any $U = \cup\mathcal{F}_1, V = \cup\mathcal{F}_2 \in \mathscr{T}$,
$$\begin{aligned}U \cap V &= (\cup\mathcal{F}_1) \cap (\cup\mathcal{F}_2)\\ &= \cup_{B\in\mathcal{F}_1, C\in\mathcal{F}_2}(B \cap C)\end{aligned}$$
where $B, C \in \mathcal{B}$, thus for any $x \in B \cap C, \exists D_x \in \mathcal{B}$ such that $x \in D_x \subseteq B \cap C$. Thus it is direct to see that $B \cap C = \cup_{x\in B\cap C} D_x$. Thus
$$\begin{aligned}D_x \in \mathcal{B} &\Rightarrow D_x \in \mathscr{T}\\ &\Rightarrow \cup_{x\in B\cap C} D_x \in \mathscr{T}\\ &\Rightarrow U \cap V = \cup_{B\in\mathcal{F}_1, C\in\mathcal{F}_2}(B \cap C)\\ &= \cup_{B\in\mathcal{F}_1, C\in\mathcal{F}_2}(\cup_{x\in B\cap C} D_x) \in \mathscr{T}.\end{aligned}$$

Thus $\mathscr{T}$ is such topology as desired. $\square$

Recall that when we check whether a map $X \xrightarrow{f} Y$ is conti., we need show that for $\forall V \subseteq_{open} Y$, $f^{-1}(V) \subseteq_{open} X$. But if Y is equipped with a topology induced by some sub-basis, we can only check some subset of Y, instead of any subset of Y.

Exercise 3. *Let Z be a topology space and $Z \xrightarrow{f} X$ is a map. Show that f is continuous when X is topologized by $\mathscr{T}(\mathcal{S}) \Leftrightarrow \forall S \in \mathcal{S}, f^{-1}(S) \subseteq_{open} Z$.*

Proof. $\Rightarrow$: $\forall S \in \mathcal{S} \Rightarrow S \in \mathscr{T}(\mathcal{S})$, that is $S \subseteq_{open} X \Rightarrow f^{-1}(S) \subseteq_{open} Z$.
$\Leftarrow$: for any $U \in \mathscr{T}(\mathcal{S})$, it can be represented by the union of some finite intersections of elements of $\mathcal{S}$, that is $U = \cup_{F\in\mathcal{F}} F$, where $\mathcal{F} \subseteq \Pi$, and $F = \cap_{i=1}^{k_F} S_i, S_i \in \mathcal{S}$. Thus
$$\begin{aligned}f^{-1}(U) &= f^{-1}(\cup_{F\in\mathcal{F}} F)\\ &= \cup_{F\in\mathcal{F}} f^{-1}(\cap_{i=1}^{k_F} S_i)\\ &= \cup_{F\in\mathcal{F}} \left(\cap_{i=1}^{k_F} f^{-1}(S_i)\right)\\ &\subseteq_{open} Z.\end{aligned}$$

Thus $Z \xrightarrow{f} X$ is continuous. $\square$

2 Countable, Separable and Lindelof Compact

Definition 5. A topology space $(X, \mathscr{T})$ is

1. 1st-countable if $\forall x \in X, \exists$ countable local basis of $\mathscr{T}$ at x;
2. 2nd-countable if $\exists$ countable basis of $\mathscr{T}$. (That is $\exists$ countable open set in X such that any element in $\mathscr{T}$ is the union of these open set.)

Note 5. $\mathcal{B}$ is a basis of $\mathscr{T} \Rightarrow \mathcal{B}_x$ is a local basis of $\mathscr{T}$ at x. Thus $(X, \mathscr{T})$ is 2nd-countable $\Rightarrow (X, \mathscr{T})$ is 1st-countable.

Example 2.

1. Let X be a metric space and $\mathscr{T}$ is the topology defined by metric. Then $\mathcal{B} = \{B_r(x) | r > 0, r \in \mathbb{Q}\}$ is a countable local basis of $\mathscr{T}$ at x, Thus metric space is 1st-countable.
2. Note that the open set in $\mathbb{R}$ is the union of disjoined open intervals in $\mathbb{R}$. Any open interval can be represented by the union of countable open intervals that start and end at rational number. Thus any open set in $\mathbb{R}$ is the union of countable open intervals. Thus $\mathbb{R}$ is 2nd-countable.

Definition 6 (Dense)**.** Given a topology space X, we say a subset $A \subseteq X$ is dense if $\overline{A} = X$.

Exercise 4. *X is a topology space, $A \subseteq X$, show that A is dense $\Leftrightarrow \forall U \subseteq_{open} X, U \neq \emptyset$, then $U \cap A \neq \emptyset$.*

Proof. $\Rightarrow$: $\overline{A} = A^o \cup \partial A = X$, thus $X \backslash A^o = \partial A$ as A^o and ∂A are disjoined. For any $U \subseteq_{open} X$, if $U \neq \emptyset$, pick $x \in U$, then either $x \in A^o$ or $x \in X \backslash A^o = \partial A$.
If $x \in A^o \Rightarrow x \in U \cap A \neq \emptyset$; If $x \in \partial A$, U is a nbd. of $x \Rightarrow U \cap A \neq \emptyset$.
$\Leftarrow$: If $\overline{A} \neq X \Rightarrow W := X \backslash \overline{A} \neq \emptyset$, and $W \subseteq_{open} X, W \cap \overline{A} = (X \backslash \overline{A}) \cap \overline{A} = \emptyset$, which leads to a contradiction. $\square$

Definition 7 (Separable)**.** A topology space $(X, \mathscr{T})$ is separable if X has a countable dense subset.

Exercise 5. *If $\mathcal{B}$ is a basis of a topology space X and pick a point x_B in B for any non-empty set $B \in \mathcal{B}$. Show that $\{x_B \in B | B \in \mathcal{B}, B \neq \emptyset\} \subseteq_{dense} X$.*

Proof. If $U \subseteq_{open} X$ and $U \neq \emptyset$, then $\exists \mathcal{F} \subseteq \mathcal{S}$, s.t. $U = \cup \mathcal{F}$. Then $x_F \in F \in \mathcal{F} \subseteq \cup \mathcal{F} = U \Rightarrow x_F \in U \cap \{x_B \in B | B \in \mathcal{B}\} \neq \emptyset \Rightarrow \{x_B \in B | B \in \mathcal{B}\} \subseteq_{dense} X$. $\square$

Note 6. Thus if $\mathcal{B}$ is a countable basis of $\mathscr{T}$ on X, then $\{x_B \in B | B \in \mathcal{B}\}$ is a countable dense subset of X, and $(X, \mathscr{T})$ is a separable topology space.

Definition 8 (Lindelof Compact)**.** A topology space $(X, \mathscr{T})$ is Lindelof compact if $\forall U_\alpha \subseteq_{open} X(\alpha \in A), \cup_{\alpha \in A} U_\alpha = X \Rightarrow \exists$ countable set $A_0 \subseteq A$, s.t. $\cup_{\alpha \in A_0} U_\alpha = X$.

It is direct to see that 2nd-countable $\Rightarrow$ Lindelof Compact, since if $\mathcal{B}$ is a basis of $\mathscr{T}$ on X, then $X = \cup \mathcal{B}$. Collectively, we have

$$
\begin{tikzcd}
 & \text{1st-Countable} \\
\text{2nd-Countable} \arrow[ur, Rightarrow] \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{Separable} \\
 & \text{Lindelof Compact}
\end{tikzcd}
$$

Exercise 6. *If X is topologized by a metric (a.k.a. X is a metrizable topology space) then 2nd-Countable $\Leftrightarrow$ Separable $\Leftrightarrow$ Lindelof Compact.*

Proof. 1. Separable $\Rightarrow$ 2nd-Countable: To prove this statement, we need to track back to the $\Leftarrow$ case: If D is the countable dense subset of X, we claim that $\mathcal{B} := \{B_{\frac{1}{n}}(s) | s \in D, n \in \mathbb{N}\}$ is the basis of metric topology on X.
Given a $U \subseteq_{open} X$ and $U \neq \emptyset$, we have $U \cap D \neq \emptyset$. For any $u \in U \cap D$, exists $n_u \in \mathbb{N}$, s.t. $B_{\frac{1}{n_u}}(u) \subseteq U$. Obviously,

$$W := \bigcup_{u \in U \cap D} B_{\frac{1}{n_u}}(u) \subseteq U.$$

For any $v \in U$, if $v \in U \cap D \Rightarrow v \in W$; if $v \notin D \Rightarrow v \in L_D$, since $X = D \cup L_D$. Thus $\exists n_v \in \mathbb{N}$, s.t. $\exists u \in B_{\frac{1}{n_v}}(v) \cap D \backslash \{v\}$, where $B_{\frac{1}{n_v}}(v) \subseteq U$ and $u \in U \cap D$ whose $1/n_u > 1/n_v$. Thus $v \in B_{\frac{1}{n_u}}(u) \subseteq W \Rightarrow U = W = \bigcup_{u \in U \cap D} B_{\frac{1}{n_u}}(u)$, where $\{B_{\frac{1}{n_u}}(u) | u \in U \cap D, B_{\frac{1}{n_u}}(u) \subseteq U\} \subseteq \mathcal{B}$. Thus $\mathcal{B}$ is a basis of metric topology on X.
2. Lindelof Compact $\Rightarrow$ Separable: For any $x \in X, \exists r_x > 0$, s.t. $B_{r_x}(x) \subseteq X$, it is direct to see that $X = \cup_{x \in X} B_{r_x}(x)$. X is Lindelof Compact, thus exist countable subset D of X such that $X = \cup_{x \in D} B_{r_x}(x)$. For any non-empty $U \subseteq_{open} X$, any $u \in U \subseteq X = \cup_{x \in D} B_{r_x}(x)$, thus $U \cap D \neq \emptyset \Rightarrow D$ is dense $\Rightarrow X$ is separable. $\square$

3 Examples

There are some examples of topologies:

Example 3. X is a set, $\mathcal{P}(X)$ is called discrete topology; $(\emptyset, X)$ is called trivial topology. Note that discrete topology is defined by discrete metric:

$$d(x, x') = \begin{cases} 1, & x = x' \\ 0, & x \neq x' \end{cases}$$

Thus $\{x\} \subseteq B_{1/2}(x)$ for any $x \in X$, and any $S \in \mathcal{P}(X)$ is the union of these balls, i.e. $S = \cup_{x \in S} B_{1/2}(x)$, and holds an open set in discrete topology.
But trivial topology can not be defined by metric. If it can, then $\forall x \in X, \exists r_x > 0$, s.t. $B_{r_x}(x) \subseteq X$, which implies $B_{r_x}(x) \in (\emptyset, X)$ and leads to a contradiction.

Example 4. X is an uncountable set. $\mathscr{T}_c := \{U \subseteq X | U = \emptyset \vee X \backslash U \text{ is countable}\}$ is called **co-countable** topology. Thus any countable set in X is the close set on topology space $(X, \mathscr{T}_c)$.
Similarly, $\mathscr{T}_f := \{U \subseteq X | U = \emptyset \vee X \backslash U \text{ is finite}\}$ is called **co-finite** topology. Thus any finite set in X is the close set on topology space $(X, \mathscr{T}_f)$.
It is direct to see $\mathscr{T}_c$ and $\mathscr{T}_f$ are topology:

1. $\emptyset \in \mathscr{T}_c, X \in \mathscr{T}_c$ for $X \backslash X = \emptyset$ is countable;
2. Any $U_\alpha \in \mathscr{T}_c (\alpha \in A) \Rightarrow X \backslash U_\alpha$ is countable $\Rightarrow X \backslash \cup_{\alpha \in A} U_\alpha = \cap_{\alpha \in A} (X \backslash U_\alpha)$ is the intersection of countable sets, thus be countable $\Rightarrow \cup_{\alpha \in A} U_\alpha \in \mathscr{T}_c$.
3. $U, V \in \mathscr{T}_c \Rightarrow X \backslash (U \cap V) = (X \backslash U) \cup (X \backslash V)$ is countable, thus $U \cap V \in \mathscr{T}_c$.