高等数学基本方法: 不定积分

Collection of Calculus Tips:

Indefinite integral

王浩铭

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这篇笔记的参考资料为全国大学生数学竞赛习题, 历年考研真题, 历年西南财经大学高等数学期末考试真题, 部分内容根据我的理解进行调整. 本笔记系应试技巧集锦, 其中多数定理均在 *Calculus (CN)* 笔记中给出, 因此不再提供证明. 因为本人水平有限, 无法保证本文内容正确性, 这篇笔记仅供参考. 若您发现本文的错误, 请将这些错误发送到我的邮箱 wanghaoming17@163.com, 谢谢! 您可以在我的主页中浏览更多笔记.

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Edition 4

1 基本积分公式

1.

$$\begin{aligned}
&\int \sec^2 x\mathrm{d}x = \tan x + C,\\
&\int \sec x \tan x\mathrm{d}x = \sec x + C,\\
&\int \sec x\mathrm{d}x = \ln|\sec x + \tan x| + C,\\
&\int \mathrm{csc}^2 x\mathrm{d}x = -\cot x + C,\\
&\int \mathrm{csc} x \cot x\mathrm{d}x = -\mathrm{csc} x + C,\\
&\int \mathrm{csc} x\mathrm{d}x = -\ln|\csc x + \cot x| + C.
\end{aligned}$$

2.

$$\begin{aligned}
&\int \frac{1}{a^2+x^2}\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + C,\\
&\int \frac{1}{a^2-x^2}\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C,\\
&\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x = \arcsin\frac{x}{a} + C,\\
&\int \frac{1}{\sqrt{x^2\pm a^2}}\mathrm{d}x = \ln|x+\sqrt{x^2\pm a^2}| + C\\
&\int \frac{Mx+N}{x^2+a^2}\mathrm{d}x = \frac{M}{2}\ln(x^2+a^2) + \frac{N}{a}\arctan\frac{x}{a} + C.
\end{aligned}$$

3.

$$\begin{aligned}
&\int e^{ax}\sin bx\mathrm{d}x = \frac{a\sin bx - b\cos bx}{a^2+b^2}\cdot e^{ax}.\\
&\int e^{ax}\cos bx\mathrm{d}x = \frac{a\cos bx + b\sin bx}{a^2+b^2}\cdot e^{ax}.
\end{aligned}$$

4. Γ 函数

$$\int_0^{+\infty} x^n e^{-x}\mathrm{d}x = n!$$

注 1.1. 对于一些复杂的形式，如 $\int \frac{M(x+A)+N}{(x+A)^2+a^2}\mathrm{d}x$，可先推其一般公式（上列公式），然后再往里代.

2 化简手段

1. 配方法

- 对于二次式：ax^2+bx+c 或 $(ax+b)(cx+d)$ 将其配方为 $(ax+b)^2+c$ 的形式
- 对于 $2n$ 次式 $ax^{2n}+b$ 配方为 $(ax^n+c)^2+\cdots$
- 对于 $2n$ 次式 $x^{2n}+\frac{1}{x^{2n}}=(x^n+\frac{1}{x})^n-2.$

2. 加减凑项

$f(x) = [f(x) + g(x)] - g(x)$

3. 乘项、提取凑项

- $\frac{\mathrm{d}x}{x(ax^n+b)} = \frac{x^{n-1}\mathrm{d}x}{x^n(ax^n+b)} = \frac{\mathrm{d}x^n}{nx^n(ax^n+b)}$
- $\frac{\mathrm{d}x}{x(ax^n+b)} = \frac{\mathrm{d}x}{x^{n+1}(a+bx^{-n})} = -\frac{1}{n}\frac{\mathrm{d}x^{-n}}{a+bx^{-n}}$.

4. 凑微分

- $\frac{f'(x)\mathrm{d}x}{\sqrt{f(x)}} = \frac{\mathrm{d}f(x)}{\sqrt{f(x)}} = 2\mathrm{d}\sqrt{f(x)}$
- $\frac{\mathrm{d}x}{f(x^{-n})x^{n+1}} = -\frac{1}{n}\frac{\mathrm{d}x^{-n}}{f(x^{-n})}$

注 2.1. 有两点注意:

1. 见到分母上的根号 $\sqrt{f(x)}$，一种思路是<u>根式换元</u>；另一种思路是<u>将其凑到分子上去</u>

2. 形如 $\frac{1}{a^2-x^2}, \frac{1}{x^2-a^2}$ 的式子，消去谁则令谁前的系数正负号相反:

$$\frac{1}{a^2-x^2} = \frac{1}{(a-x)(a+x)} = \frac{1}{2a}\left(\frac{1}{a-x} + \frac{1}{a+x}\right) = \frac{1}{2a}\left(\frac{1}{a+x} - \frac{1}{x-a}\right)$$
$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

例 2.1. $I = \int \frac{\mathrm{d}x}{\sqrt{x(4-x)}}$.

法一：配方法

$$I = \int \frac{\mathrm{d}x}{\sqrt{x(4-x)}} = \int \frac{\mathrm{d}x}{\sqrt{4x-x^2}} = \int \frac{\mathrm{d}x}{\sqrt{4-(x-2)^2}} = \arcsin\frac{x-2}{2} + C.$$

法二：凑微分法

$$I = \int \frac{\mathrm{d}x}{\sqrt{x(4-x)}} = \int \frac{2\mathrm{d}\sqrt{x}}{\sqrt{4-x}} = \arcsin\frac{x-2}{2} + C.$$

（看到分母上的 $\sqrt{x}$ 考虑把它凑到分子上.）

例 2.2. $I = \int \frac{x^5}{\sqrt{1+x^2}}\mathrm{d}x$

法一：加减凑项

$$
\begin{aligned}
I &= \int \frac{x^5}{\sqrt{1+x^2}}\mathrm{d}x = \frac{1}{2}\int \frac{x^4}{\sqrt{1+x^2}}\mathrm{d}x^2+1 \\
&= \int x^4 \mathrm{d}\sqrt{x^2+1} = x^4\sqrt{x^2+1} - 4\int x^3\sqrt{x^2+1}\mathrm{d}x \\
&= x^4\sqrt{x^2+1} - 2\int x^2\sqrt{x^2+1}\mathrm{d}x^2 \\
&= x^4\sqrt{x^2+1} - 2\int (x^2+1-1)\sqrt{x^2+1}\mathrm{d}(x^2+1) \\
&= x^4\sqrt{x^2+1} - 2\int (x^2+1)\sqrt{x^2+1}\mathrm{d}(x^2+1) + 2\int \sqrt{x^2+1}\mathrm{d}(x^2+1) \\
&= x^4\sqrt{x^2+1} - \frac{4}{5}(x^2+1)^{\frac{5}{2}} + \frac{4}{3}(1+x^2)^{\frac{3}{2}} + C.
\end{aligned}
$$

法二：令 $x=\tan t$，略.

例 2.3. $I=\int \frac{\mathrm{d}x}{\cos x\sqrt{\sin x}}$.

乘除凑项

$$
\begin{aligned}
I &= \int \frac{\mathrm{d}x}{\cos x\sqrt{\sin x}} = \int \frac{\cos x\mathrm{d}x}{\cos^2 x\sqrt{\sin x}} = \int \frac{\mathrm{d}\sin x}{\cos^2 x\sqrt{\sin x}} \\
&= 2\int \frac{\mathrm{d}\sqrt{\sin x}}{1-\sin^2 x} \xlongequal{\sqrt{\sin x}=t} 2\int \frac{\mathrm{d}t}{1-t^4} = 2\int \frac{\mathrm{d}t}{(1-t^2)(1+t^2)} \\
&= \int \left(\frac{1}{1-t^2}+\frac{1}{1+t^2}\right)\mathrm{d}t = \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + \arctan t + C \\
&= \frac{1}{2}\ln\left|\frac{1+\sqrt{\sin x}}{1-\sqrt{\sin x}}\right| + \arctan\sqrt{\sin x} + C
\end{aligned}
$$

例 2.4. $I=\int \frac{xe^x}{\sqrt{e^x-1}}\mathrm{d}x$

$$
\begin{aligned}
I &= \int \frac{xe^x}{\sqrt{e^x-1}}\mathrm{d}x = \int \frac{x}{\sqrt{e^x-1}}\mathrm{d}(e^x-1) = 2\int x\mathrm{d}\sqrt{e^x-1} \\
&= 2x\sqrt{e^x-1} - 2\int \sqrt{e^x-1}\mathrm{d}x \xlongequal{\sqrt{e^x-1}=t} 2x\sqrt{e^x-1} - 4\int \frac{t^2}{t^2+1}\mathrm{d}t \\
&= 2x\sqrt{e^x-1} - 4(t-\arctan t) + C \\
&= 2x\sqrt{e^x-1} - 4\sqrt{e^x-1} + 4\arctan\sqrt{e^x-1} + C.
\end{aligned}
$$

例 2.5. $I=\int \frac{1}{x+x^9}\mathrm{d}x$

法一：加减凑项

$$
\begin{aligned}
I &= \int \frac{1}{x+x^9}\mathrm{d}x = \int \frac{1+x^8-x^8}{x(1+x^8)}\mathrm{d}x \\
&= \int \frac{1}{x}\mathrm{d}x - \int \frac{x^7}{1+x^8}\mathrm{d}x = \int \frac{1}{x}\mathrm{d}x - \frac{1}{8}\int \frac{\mathrm{d}x^8}{1+x^8} \\
&= \ln x - \frac{1}{8}\ln(1+x^8) + C = \frac{1}{8}\ln\frac{x^8}{1+x^8} + C.
\end{aligned}
$$

法二：乘除凑项

$$
\begin{aligned}
I &= \int \frac{1}{x+x^9}\mathrm{d}x = \int \frac{1}{x(1+x^8)}\mathrm{d}x = \int \frac{x^7}{x^8(1+x^8)}\mathrm{d}x \\
&= \frac{1}{8}\int \frac{1}{x^8(1+x^8)}\mathrm{d}x^8 = \frac{1}{8}\int \left(\frac{1}{x^8} - \frac{1}{1+x^8}\right)\mathrm{d}x^8 \\
&= \frac{1}{8}\ln\frac{x^8}{1+x^8} + C.
\end{aligned}
$$

法三：提项凑项

$$
\begin{aligned}
I &= \int \frac{1}{x+x^9}\mathrm{d}x = \int \frac{1}{x(1+x^8)}\mathrm{d}x = \int \frac{1}{x^9(1+x^{-8})}\mathrm{d}x \\
&= -\frac{1}{8}\int \frac{1}{1+x^{-8}}\mathrm{d}x^{-8} = -\frac{1}{8}\ln|1+x^{-8}| + C.
\end{aligned}
$$

例 2.6. $I = \int \frac{1+x^4}{1+x^6}\mathrm{d}x$

$$
\begin{aligned}
I &= \int \frac{1+x^4}{1+x^6}\mathrm{d}x = \int \frac{1+x^4-x^2+x^2}{1-(-x^2)^3}\mathrm{d}x \\
&= \int \frac{(1+x^4-x^2)+x^2}{(1+x^2)(1-x^2+x^4)}\mathrm{d}x = \int \frac{1}{1+x^2}\mathrm{d}x + \int \frac{x^2}{1+x^6}\mathrm{d}x \\
&= \arctan x + \frac{1}{3}\int \frac{1}{1+(x^3)^2}\mathrm{d}x^3 + C \\
&= \arctan x + \frac{1}{3}\arctan x^3 + C.
\end{aligned}
$$

注 2.2. 关于多项式有以下结论：

1. 任意正整数 n：
$$1-x^n = (1-x)(1+x+x^2+\cdots+x^{n-1})$$

2. 当 n 为奇数时：
$$1+x^n = 1-(-x)^n = (1+x)(1-x+x^2-x^3+\cdots+x^{n-1})$$

3. 当 n 为偶数，且 n 可以表示为一个奇数和偶数的乘积时，如 $n=6$，则
$$1+x^6 = 1+(x^2)^3 = (1+x^2)(1-x^2+x^4)$$

3 组合积分法

应用于可能配对或组合的函数的积分的方法，称为组合积分法. 有三种类型，在构造组合的时候需要灵活应用加减构造、乘除构造、提项构造、凑微分、配方等方法.

3.1 A. 第一组合积分法

基本思想为，欲求 A，则寻找与之对应的 B，使

$$\begin{aligned} aA + bB &= C \\ cA + dB &= D \end{aligned}$$

易求，则可解出 $A = \dfrac{\begin{vmatrix} C & b \\ D & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$.

对于有自导性或互导性的函数（如指数函数，三角函数），还可以通过分别对 A, B 被积函数求导，再两侧积分找到相应的对应关系

例 3.1. 求 $\int \frac{\cos x}{3\cos x + 4\sin x} \mathrm{d}x$

对于三角有理式一般可以使用万能公式，但本题使用该法很复杂. 令

$$I = \int \frac{\cos x}{3\cos x + 4\sin x} \mathrm{d}x, \quad J = \int \frac{\sin x}{3\cos x + 4\sin x} \mathrm{d}x$$

则

$$\begin{aligned} &3I + 4J = \int 1 \ \mathrm{d}x = x + C_1 \\ &4I - 3J = \int \tfrac{4\cos x - 3\sin x}{3\cos x + 4\sin x} \mathrm{d}x = \int \tfrac{\mathrm{d}(4\sin x + 3\cos x)}{3\cos x + 4\sin x} = \ln|3\cos x + 4\sin x| + C_2 \end{aligned}$$

解得

$$\begin{cases} I = \frac{3}{25}x + \frac{4}{25}\ln|3\cos x + 4\sin x| + C \\ J = \frac{4}{25}x - \frac{3}{25}\ln|3\cos x + 4\sin x| + C \end{cases}$$

例 3.2. 求 $\int \frac{\cos^2 x}{a\cos x + b\sin x} \mathrm{d}x$

令 $A=\int \frac{\cos^2 x}{a\cos x+b\sin x}\mathrm{d}x, B=\int \frac{\sin^2 x}{a\cos x+b\sin x}\mathrm{d}x$，令 $\frac{a}{\sqrt{a^2+b^2}}=\cos\alpha, \frac{b}{\sqrt{a^2+b^2}}=\sin\alpha$ 则

$$
\begin{aligned}
A+B&=\int \frac{\cos^2 x+\sin^2 x}{a\cos x+b\sin x}\mathrm{d}x=\int \frac{1}{a\cos x+b\sin x}\mathrm{d}x\\
&=\frac{1}{\sqrt{a^2+b^2}}\int \frac{1}{\cos\alpha\sin x+\sin\alpha\cos x}dx\\
&=\frac{1}{\sqrt{a^2+b^2}}\int \frac{1}{\sin(x+\alpha)}dx=\frac{1}{2\sqrt{a^2+b^2}}\int \frac{1}{\sin\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x+\alpha}{2}\right)}dx\\
&=\frac{1}{\sqrt{a^2+b^2}}\int \frac{1}{\tan\left(\frac{x+\alpha}{2}\right)}d\left(\tan\left(\frac{x+\alpha}{2}\right)\right)\\
&=\frac{1}{\sqrt{a^2+b^2}}\ln\left(\tan\left(\frac{x+\alpha}{2}\right)\right)+c=M\\
a^2A-b^2B&=\int \frac{a^2\cos^2 x-b^2\sin^2 x}{a\cos x+b\sin x}\mathrm{d}x=\int \frac{(a\cos x-b\sin x)(a\cos x+b\sin x)}{a\cos x+b\sin x}\mathrm{d}x\\
&=\int (a\cos x-b\sin x)\mathrm{d}x=-a\cos x-b\sin x+c=N
\end{aligned}
$$

因此 $A=\dfrac{\begin{vmatrix} M & 1\\ N & -b^2\end{vmatrix}}{\begin{vmatrix} 1 & 1\\ a^2 & -b^2\end{vmatrix}}$

例 3.3. 求 $\int_0^{\frac{\pi}{6}} \frac{\cos^2 x}{\cos x+\sqrt{3}\sin x}\mathrm{d}x$

设 $I=\int_0^{\frac{\pi}{6}} \frac{\cos^2 x}{\cos x+\sqrt{3}\sin x}\mathrm{d}x, J=\int_0^{\frac{\pi}{6}} \frac{\sin^2 x}{\cos x+\sqrt{3}\sin x}\mathrm{d}x$，则

$$
\begin{aligned}
I+J&=\int_0^{\frac{\pi}{6}} \frac{\cos^2 x}{\cos x+\sqrt{3}\sin x}dx+\int_0^{\frac{\pi}{6}} \frac{\sin^2 x}{\cos x+\sqrt{3}\sin x}dx\\
&=\int_0^{\frac{\pi}{6}} \frac{\cos^2 x+\sin^2 x}{\cos x+\sqrt{3}\sin x}dx=\int_0^{\frac{\pi}{6}} \frac{1}{\cos x+\sqrt{3}\sin x}dx\\
&=\int_0^{\frac{\pi}{6}} \frac{1}{2\left(\frac{1}{2}\cos x+\frac{\sqrt{3}}{2}\sin x\right)}dx=\frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{1}{\sin\frac{\pi}{6}\cos x+\cos\frac{\pi}{6}\sin x}dx\\
&=\frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{1}{\sin\left(x+\frac{\pi}{6}\right)}dx=\frac{1}{2}\int_0^{\frac{\pi}{6}} \csc\left(x+\frac{\pi}{6}\right)dx\\
&=-\frac{1}{2}\ln\left|\csc\left(x+\frac{\pi}{6}\right)+\cot\left(x+\frac{\pi}{6}\right)\right|\Big|_0^{\frac{\pi}{6}}\\
&=\frac{1}{2}\ln(2+\sqrt{3})-\frac{\ln 3}{4}
\end{aligned}
$$

以及

$$
\begin{aligned}
I-3J &= \int_0^{\frac{\pi}{6}} \frac{\cos^2 x}{\cos x+\sqrt{3}\sin x}dx - \int_0^{\frac{\pi}{6}} \frac{3\sin^2 x}{\cos x+\sqrt{3}\sin x}dx \\
&= \int_0^{\frac{\pi}{6}} \frac{\cos^2 x-3\sin^2 x}{\cos x+\sqrt{3}\sin x}dx = \int_0^{\frac{\pi}{6}} \frac{(\cos x-\sqrt{3}\sin x)(\cos x+\sqrt{3}\sin x)}{\cos x+\sqrt{3}\sin x}dx \\
&= \int_0^{\frac{\pi}{6}} (\cos x-\sqrt{3}\sin x)dx = (\sin x+\sqrt{3}\cos x)\Big|_0^{\frac{\pi}{6}} \\
&= 2-\sqrt{3}
\end{aligned}
$$

解得

$$
\begin{aligned}
I &= \frac{1}{4}[3(I+J)+(I-3J)] = \frac{1}{4}\left[\left(\frac{1}{2}\ln(2+\sqrt{3})-\frac{\ln 3}{4}\right)+(2-\sqrt{3})\right] \\
&= \frac{3}{8}\ln(2+\sqrt{3})-\frac{3}{16}\ln 3+\frac{1}{2}-\frac{\sqrt{3}}{4}
\end{aligned}
$$

例 3.4. 求 $\int e^{ax}\cos bx\mathrm{d}x$

令

$$
I=\int e^{ax}\cos bx\mathrm{d}x,\quad J=\int e^{ax}\sin bx\mathrm{d}x
$$

则

$$
\begin{aligned}
(e^{ax}\cos bx)' &= ae^{ax}\cos bx-be^{ax}\sin bx \\
(e^{ax}\sin bx)' &= ae^{ax}\sin bx+be^{ax}\cos bx
\end{aligned}
$$

两侧各自积分有

$$
\left.\begin{aligned}
e^{ax}\cos bx+C_1 &= aI-bJ \\
e^{ax}\sin bx+C_2 &= aJ+bI
\end{aligned}\right\}
$$

从而

$$
\left\{\begin{aligned}
I &= \tfrac{e^x}{a^2+b^2}(a\cos bx+b\sin bx)+C \\
J &= \tfrac{e^x}{a^2+b^2}(a\sin bx-b\cos bx)+C
\end{aligned}\right.
$$

例 3.5. 求 $\int xe^{ax}\cos bx\mathrm{d}x$

令

$$
I=\int xe^{ax}\cos bx\mathrm{d}x,\quad J=\int xe^{ax}\sin bx\mathrm{d}x
$$

则则

$$
\begin{aligned}
(xe^{ax}\cos bx)' &= e^{ax}\cos bx+x(ae^{ax}\cos bx-be^{ax}\sin bx) \\
(xe^{ax}\sin bx)' &= e^{ax}\sin bx+x(ae^{ax}\sin bx+be^{ax}\cos bx)
\end{aligned}
$$

两侧各自积分有

$$
\left.\begin{aligned}
xe^{ax}\cos bx+C_1 &= \int e^{ax}\cos bx\ \mathrm{d}x+aI-bJ \\
xe^{ax}\sin bx+C_2 &= \int e^{ax}\sin bx\ \mathrm{d}x+aJ+bI
\end{aligned}\right\}
$$

结合已有结果解二元一次方程组即可.

例 3.6. $I=\int \frac{1}{1+x^6}\mathrm{d}x.$

记 $A=\int \frac{x^4}{1+x^6}\mathrm{d}x, B=\int \frac{1}{1+x^6}\mathrm{d}x$，则

$$\begin{aligned}
A+B&=\int \frac{x^4+1}{x^6+1}\ \mathrm{d}x\\
&=\int \frac{\left(x^4-x^2+1\right)+x^2}{x^6+1}\ \mathrm{d}x\\
&=\int \frac{x^4-x^2+1}{\left(x^2+1\right)\left(x^4-x^2+1\right)}\mathrm{d}x+\int \frac{x^2}{x^6+1}\ \mathrm{d}x\\
&=\int \frac{\mathrm{d}x}{x^2+1}+\frac{1}{3}\int \frac{\mathrm{d}x^3}{x^6+1}\\
&=\arctan x+\frac{1}{3}\arctan x^3+C\\
A-B&=\int \frac{x^4-1}{x^6+1}\ \mathrm{d}x=\int \frac{\left(x^2-1\right)\left(x^2+1\right)}{\left(x^2+1\right)\left(x^4-x^2+1\right)}\mathrm{d}x\\
&=\int \frac{x^2-1}{x^4-x^2+1}\ \mathrm{d}x=\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}-1}\ \mathrm{d}x\\
&=\int \frac{\mathrm{d}\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-3}=\frac{1}{2\sqrt{3}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{3}}{x+\frac{1}{x}+\sqrt{3}}\right|+C\\
&=\frac{1}{2\sqrt{3}}\ln\left|\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1}\right|+C
\end{aligned}$$

解出

$$B=\frac{1}{2}\left[\arctan x+\frac{1}{3}\arctan x^3-\frac{1}{2\sqrt{3}}\ln\left|\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1}\right|\right]+C$$

例 3.7. $I=\int \ln\left(\sqrt{1+x}-\sqrt{1-x}\right)\mathrm{d}x$

令 $I_0=\int \ln\left(\sqrt{1+x}+\sqrt{1-x}\right)\mathrm{d}x$，则

$$\begin{aligned}
I+I_0&=\int \ln(\sqrt{1+x}-\sqrt{1-x})\mathrm{d}x+\int \ln(\sqrt{1+x}+\sqrt{1-x})\mathrm{d}x\\
&=\int \ln(2x)\mathrm{d}x=\frac{1}{2}\int \ln(2x)\mathrm{d}(2x)=\frac{1}{2}[(2x)\ln(2x)-2x]+C\\
&=x\ln 2x-x+C\\
I_0-I&=\int \ln(\sqrt{1+x}+\sqrt{1-x})\mathrm{d}x-\int \ln(\sqrt{1+x}-\sqrt{1-x})\mathrm{d}x\\
&=\int \ln\frac{1+\sqrt{1-x^2}}{x}\ \mathrm{d}x=\int \ln(1+\sqrt{1-x^2})\ \mathrm{d}x-\int \ln x\mathrm{d}x
\end{aligned}$$

令 $x=\sin t$ 则

$$\begin{aligned}
\int \ln\left(1+\sqrt{1-x^2}\right)\mathrm{d}x &= \int \ln(1+\cos t)\mathrm{d}(\sin t) = \sin t\ln(1+\cos t) + \int \frac{\sin^2 t}{1+\cos t}\ \mathrm{d}t \\
&= \sin t\ln(1+\cos t) + \int 1-\cos t\ \mathrm{d}t \\
&= \sin t\ln(1+\cos t) + t - \sin t + C \\
&= x\ln\left(1+\sqrt{1-x^2}\right) + \arcsin x - x + C
\end{aligned}$$

因此

$$\begin{aligned}
I_0 - I &= \int \ln\left(1+\sqrt{1-x^2}\right) - \ln x\ \mathrm{d}x \\
&= x\ln\left(1+\sqrt{1-x^2}\right) + \arcsin x - x\ln x \\
&= x\ln\frac{1+\sqrt{1-x^2}}{x} + \arcsin x + C \\
2I &= x\ln 2x - x - x\ln\frac{1+\sqrt{1-x^2}}{x} - \arcsin x + C \\
I &= \frac{1}{2}\left(x\ln 2x - x - x\ln\frac{1+\sqrt{1-x^2}}{x} - \arcsin x\right) + C
\end{aligned}$$

3.2 B. 第二组合积分法

基本思想为

$$A = \frac{2A+B-B}{2} = \frac{A+B}{2} + \frac{A-B}{2}$$

常与凑微分、配方法、乘除构造等技巧相结合

例 3.8. $I = \int \frac{1}{1+x^4}\mathrm{d}x$

方法一：组合积分法（加减构造、除项构造、配方法、凑微分）

因为

$$\begin{aligned}
\frac{1}{1+x^4} &= \frac{1}{2}\frac{1+x^2}{1+x^4} + \frac{1}{2}\frac{1-x^2}{1+x^4} \\
&= \frac{1}{2}\frac{\frac{1}{x^2}+1}{x^2+\frac{1}{x^2}} + \frac{1}{2}\frac{\frac{1}{x^2}-1}{x^2+\frac{1}{x^2}} \\
&= \frac{1}{2}\frac{\frac{1}{x^2}+1}{\left(x-\frac{1}{x}\right)^2+2} + \frac{1}{2}\frac{\frac{1}{x^2}-1}{\left(x+\frac{1}{x}\right)^2-2}
\end{aligned}$$

所以

$$\begin{aligned}
\int \frac{1}{1+x^4}\mathrm{d}x =& \frac{1}{2}\int \frac{\frac{1}{x^2}+1}{\left(x-\frac{1}{x}\right)^2+2}\ \mathrm{d}x + \frac{1}{2}\int \frac{\frac{1}{x^2}-1}{\left(x+\frac{1}{x}\right)^2-2}\ \mathrm{d}x \\
=& \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{x}\right)^2+2}\ \mathrm{d}\left(x-\frac{1}{x}\right) - \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{x}\right)^2-2}\ \mathrm{d}\left(x+\frac{1}{x}\right) \\
=& \frac{\sqrt{2}}{4}\arctan\left[\frac{\sqrt{2}}{2}\left(x-\frac{1}{x}\right)\right] - \frac{\sqrt{2}}{8}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C
\end{aligned}$$

方法二：配方法、部分分式法

$$
\begin{aligned}
I &= \int \frac{1}{1+x^4}\mathrm{d}x = \int \frac{1}{(1+x^2)^2-2x^2}\mathrm{d}x \\
&= \int \frac{1}{(1+x^2-\sqrt{2}x)(1+x^2+\sqrt{2}x)}\mathrm{d}x \\
&= \int \frac{M_1x+N_1}{1+x^2+\sqrt{2}x} + \frac{M_2x+N_2}{1+x^2-\sqrt{2}x}\mathrm{d}x
\end{aligned}
$$

解得 $M_1=\frac{\sqrt{2}}{4}, M_2=-\frac{\sqrt{2}}{4}, N_1=N_2=\frac{1}{2}$，因此

$$
\begin{aligned}
I &= \int \frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{1+x^2+\sqrt{2}x} + \frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{1+x^2-\sqrt{2}x}\mathrm{d}x \\
&= \int \frac{\frac{\sqrt{2}}{4}\left(x+\frac{1}{\sqrt{2}}\right)+\frac{1}{4}}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2} + \int \frac{-\frac{\sqrt{2}}{4}\left(x-\frac{1}{\sqrt{2}}\right)+\frac{1}{4}}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}\mathrm{d}x \\
&= \frac{1}{4\sqrt{2}}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right| + \frac{1}{2\sqrt{2}}\arctan(1+\sqrt{2}x) - \frac{1}{2\sqrt{2}}\arctan(1-\sqrt{2}x)+C
\end{aligned}
$$

例 3.9. $\int \frac{1}{e^{-3x}+e^{x}}\mathrm{d}x$

分子分母同乘以 e^x，则

$$
\begin{aligned}
I &= \int \frac{e^x}{e^{-2x}+e^{2x}}\ \mathrm{d}x \\
&= \frac{1}{2}\int \frac{e^x+e^{-x}}{e^{-2x}+e^{2x}}\ \mathrm{d}x + \frac{1}{2}\int \frac{e^x-e^{-x}}{e^{-2x}+e^{2x}}\ \mathrm{d}x \\
&= \frac{1}{2}\int \frac{\mathrm{d}\left(e^x+e^{-x}\right)}{\left(e^x+e^{-x}\right)^2-2} + \frac{1}{2}\int \frac{\mathrm{d}\left(e^x-e^{-x}\right)}{\left(e^x-e^{-x}\right)^2+2} \\
&= \frac{1}{4a}\ln\left|\frac{(e^x+e^{-x})-\sqrt{2}}{(e^x+e^{-x})+\sqrt{2}}\right| + \frac{1}{2\sqrt{2}}\arctan\frac{(e^x+e^{-x})}{\sqrt{2}}+C
\end{aligned}
$$

3.3 C. 定积分组合积分

1. 通过变换处理一般定积分. 基本思想为

$$
A \xlongequal{\text{变换}} B = \frac{A+B}{2}
$$

解出 $\frac{A+B}{2}$，从而得到 A. 主要的变换技巧为：<u>轮换法、正反代换、凑微分、分部积分</u>等.

对于反代换，强调如下结论：$\int_0^a f(x)\mathrm{d}x = -\int_a^0 f(a-u)\mathrm{d}u = \int_0^a f(a-u)\mathrm{d}x$，所以

$$
\boxed{\int_0^a f(x)\mathrm{d}x = \frac{1}{2}\int_0^a [f(x)+f(a-x)]\mathrm{d}x}
$$

对于特殊的形式有：

$$
A \xlongequal{\text{变换}} C = B + k\cdot A \Rightarrow A = \frac{B}{1-k}
$$

对于某些含有三角函数的积分、三角函数与指数函数的乘积的积分、带有趋势的周期函数的无穷积分常用到该性质.

例 3.10. $I=\int^{\frac{\pi}{2}}\frac{\sin x}{\sin x+\cos x}\mathrm{d}x$.

法一:: 第三组合积分法（反代换）

易知

$$I=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x+\cos x}\mathrm{d}x=\int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x+\sin x}\mathrm{d}x=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{\sin x+\cos x}\mathrm{d}x=\frac{\pi}{4}.$$

法二: 第一组合积分法

令

$$\begin{aligned}I=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x+\cos x}\mathrm{d}x&=\int_0^{\frac{\pi}{2}}\frac{A(\sin x+\cos x)'+B(\sin x+\cos x)}{\sin x+\cos x}\mathrm{d}x\\&=\int_0^{\frac{\pi}{2}}\frac{A(\cos x-\sin x)+B(\sin x+\cos x)}{\sin x+\cos x}\mathrm{d}x\end{aligned}$$

即

$$A(\cos x-\sin x)+B(\sin x+\cos x)=\sin x,$$

则 $A=-\frac{1}{2},B=\frac{1}{2}$，因此

$$I=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{-(\sin x+\cos x)'+(\sin x+\cos x)}{\sin x+\cos x}\mathrm{d}x=\frac{1}{2}[-\ln(\sin x+\cos x)+x]_0^{\frac{\pi}{2}}=\frac{\pi}{4}.$$

例 3.11. $\int_0^{\frac{\pi}{2}}\frac{\sin^p x}{\sin^p x+\cos^p x}\mathrm{d}x,(p>0)$.

$$I=\int_0^{\frac{\pi}{2}}\frac{\sin^p x}{\sin^p x+\cos^p(x)}\mathrm{d}x=\int_0^{\frac{\pi}{2}}\frac{\cos^p x}{\cos^p x+\sin^p(x)}\mathrm{d}x=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sin^p x+\cos^p x}{\sin^p x+\cos^p x}\mathrm{d}x=\frac{\pi}{4}.$$

例 3.12. $\int_0^{\frac{\pi}{2}}\frac{1}{1+\tan^p x}\mathrm{d}x,(p>0)$.

$$I=\int_0^{\frac{\pi}{2}}\frac{1}{1+\tan^p x}\mathrm{d}x=\int_0^{\frac{\pi}{2}}\frac{\cos^p x}{\sin^p x+\cos^p x}\mathrm{d}x=\int_0^{\frac{\pi}{2}}\frac{\sin^p x}{\sin^p x+\cos^p x}\mathrm{d}x=\frac{\pi}{4}.$$

例 3.13. $\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{1+\sqrt[5]{\tan x}}\mathrm{d}x$.

$$
\begin{aligned}
I&=\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{1+\sqrt[5]{\tan x}}\mathrm{d}x \xlongequal{t=-x+\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{0}\frac{\sin\left(-x+\frac{\pi}{2}\right)+\cos\left(-x+\frac{\pi}{2}\right)}{1+\sqrt[5]{\tan\left(-x+\frac{\pi}{2}\right)}}\mathrm{d}\left(-x+\frac{\pi}{2}\right)\\
&=\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{1+\sqrt[5]{\cot x}}\mathrm{d}x=\frac{1}{2}\cdot\int\left[(\sin x+\cos x)\cdot\left(\frac{1}{1+\sqrt[5]{\tan x}}+\frac{1}{1+\sqrt[5]{\cot x}}\right)\right]\mathrm{d}x\\
&=\frac{1}{2}\cdot\int\left[(\sin x+\cos x)\cdot\frac{\sqrt[5]{\sin x}+\sqrt[5]{\cos x}}{\sqrt[5]{\sin x}+\sqrt[5]{\cos x}}\right]\mathrm{d}x\\
&=\frac{1}{2}\cdot\int\sin x+\cos x\mathrm{d}x=1.
\end{aligned}
$$

2. 处理对称区间定积分. 在对称区间的定积分中，有以下技巧：

$$
\boxed{\int_{-a}^{a}f(x)\mathrm{d}x=\int_{-a}^{a}\left[\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}\right]\mathrm{d}x=\int_{-a}^{a}\frac{f(x)+f(-x)}{2}\mathrm{d}x}
$$

这是一种十分重要的技巧.

对于非对称区间的一般定积分，可以通过正代换将其平移至对称区间，再进行奇偶性的探讨.

例 3.14. $I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{e^x}{1+e^x}\sin^4 x\mathrm{d}x.$

法一：反代换

令 $t=-x$，则

$$
\begin{aligned}
I&=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{e^x}{1+e^x}\sin^4 x\mathrm{d}x=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{e^{-t}}{1+e^{-t}}\sin^4 -t\mathrm{d}t\\
&=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{1+e^t}\sin^4 t\mathrm{d}t=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{e^t+1}{1+e^t}\sin^4 t\mathrm{d}t\\
&=2\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^4 t\mathrm{d}t=\frac{3\pi}{16}
\end{aligned}
$$

法二：对称区间非奇非偶函数定积分组合积分技巧

$$
\begin{aligned}
I&=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}f(x)\mathrm{d}x=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}\right]\mathrm{d}x\\
&=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{f(x)+f(-x)}{2}\mathrm{d}x=\int_0^{\frac{\pi}{2}}f(x)+f(-x)\mathrm{d}x\\
&=\int_0^{\frac{\pi}{2}}\sin^4 x\cdot\left[\frac{e^x}{1+e^x}+\frac{e^{-x}}{1+e^{-x}}\right]\mathrm{d}x\\
&=\int_0^{\frac{\pi}{2}}\sin^4 x\cdot\frac{e^x+1}{1+e^x}\mathrm{d}x=\frac{3\pi}{16}
\end{aligned}
$$

例 3.15. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos x}{e^{\sin^3 x}+1}\mathrm{d}x$

$$
\begin{aligned}
I &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos x}{e^{\sin^3 x}+1}\mathrm{d}x = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{\cos x}{e^{\sin^3 x}+1} + \frac{\cos x}{e^{-\sin^3 x}+1}\right]\mathrm{d}x \\
&= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x \frac{e^{\sin^3 x}+1}{e^{\sin^3 x}+1}\mathrm{d}x = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x\mathrm{d}x \\
&= \frac{\sqrt{2}}{2}.
\end{aligned}
$$

例 3.16. $\int_{-\pi}^{\pi} \frac{x^2}{1+\sin x+\sqrt{1+\sin^2 x}}\mathrm{d}x$

$$
\begin{aligned}
I &= \int_{-\pi}^{\pi} \frac{x^2}{1+\sin x+\sqrt{1+\sin^2 x}}\mathrm{d}x \\
&= \frac{1}{2}\int_{-\pi}^{\pi} x^2 \cdot \left[\frac{1}{1-+\sin x+\sqrt{1+\sin^2 x}} + \frac{1}{1+\sin x+\sqrt{1+\sin^2 x}}\right]\mathrm{d}x \\
&= \frac{1}{2}\int_{-\pi}^{\pi} x^2 \cdot \frac{2+2\sqrt{1+\sin^2 x}}{(1+\sqrt{1+\sin^2 x})^2-\sin^2 x}\mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} x^2 \cdot \frac{2+2\sqrt{1+\sin^2 x}}{2+2\sqrt{1+\sin^2 x}}\mathrm{d}x \\
&= \frac{1}{2}\int_{-\pi}^{\pi} x^2\mathrm{d}x = \frac{\pi^3}{3}.
\end{aligned}
$$

例 3.17. $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\mathrm{d}x$

$$
\begin{aligned}
I &= \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} \left[\sqrt{\frac{1+x}{1-x}} + \sqrt{\frac{1-x}{1+x}}\right]\mathrm{d}x \\
&= \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \pi.
\end{aligned}
$$

例 3.18. 设 $f(x)$ 二阶可导，证明：

1. 当 $f'(x) < 0$ 时，$\int_{-\pi}^{\pi} f(x)\sin x\mathrm{d}x < 0$；

2. 当 $f''(x) < 0$ 时，$\int_{-\pi}^{\pi} f(x)\cos x\mathrm{d}x < 0$.

1. 因为

$$
\begin{aligned}
\int_{-\pi}^{\pi} f(x)\sin x\mathrm{d}x &= \int_{-\pi}^{\pi} \left[\frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}\right]\sin x\mathrm{d}x \\
&= \int_{-\pi}^{\pi} \frac{f(x)+f(-x)}{2}\sin x\mathrm{d}x + \int_{-\pi}^{\pi} \frac{f(x)-f(-x)}{2}\sin x\mathrm{d}x \\
&= \int_{-\pi}^{\pi} \frac{f(x)-f(-x)}{2}\sin x\mathrm{d}x = \int_{0}^{\pi} [f(x)-f(-x)]\sin x\mathrm{d}x
\end{aligned}
$$

因为 $f'(x) < 0$，所以 $f(x)-f(-x) < 0$，所以 $\int_{-\pi}^{\pi} f(x)\sin x\mathrm{d}x < 0$；

2. 因为

$$
\begin{aligned}
\int_{-\pi}^{\pi} f(x)\cos x\mathrm{d}x &= \int_{-\pi}^{\pi}\left[\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}\right]\cos x\mathrm{d}x \\
&= \int_{-\pi}^{\pi}\frac{f(x)+f(-x)}{2}\cos x\mathrm{d}x+\int_{-\pi}^{\pi}\frac{f(x)-f(-x)}{2}\cos x\mathrm{d}x \\
&= \int_{-\pi}^{\pi}\frac{f(x)+f(-x)}{2}\cos x\mathrm{d}x=\int_{0}^{\pi}[f(x)+f(-x)]\mathrm{d}\sin x \\
&= [\sin x\cdot[f(x)-f(-x)]]_0^{\pi}-\int_0^{\pi}\sin x\mathrm{d}[f(x)+f(-x)] \\
&= -\int_0^{\pi}\sin x[f'(x)-f'(-x)]\mathrm{d}x
\end{aligned}
$$

因为 $f''(x)<0$，所以 $f'(x)-f'(-x)<0$，所以 $\int_{-\pi}^{\pi} f(x)\cos x\mathrm{d}x>0$；

例 3.19. 设 $\delta>0$，在 $(-\delta,\delta)$ 内有 $|f(x)|\le x^2, f''(x)>0, I=\int_{-\delta}^{\delta} f(x)\mathrm{d}x$，证明 $I>0$.

法一：（泰勒公式）

因为 $|f(x)|\le x^2$，所以 $f(0)=f'(0)=0$，所以 $f(x)=f(0)+f'(0)x+\frac{1}{2}f''(\xi)x^2=\frac{1}{2}f''(\xi)x^2>0$. 所以 $I>0$.

法二：（导函数单调性）

因为

$$
\begin{aligned}
I=\int_{-\delta}^{\delta} f(x)\mathrm{d}x &= \int_{-\delta}^{\delta}\left[\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}\right]\mathrm{d}x \\
&= \int_{-\delta}^{\delta}\frac{f(x)+f(-x)}{2}\mathrm{d}x+\int_{-\delta}^{\delta}\frac{f(x)-f(-x)}{2}\mathrm{d}x \\
&= \int_{-\delta}^{\delta}\frac{f(x)+f(-x)}{2}\mathrm{d}x=\int_0^{\delta}[f(x)+f(-x)]\mathrm{d}(x-\delta) \\
&= [f(x)+f(-x)](x-\delta)|_0^{\delta}-\int_0^{\delta}(x-\delta)\mathrm{d}[f(x)+f(-x)] \\
&= -\int_0^{\delta}(x-\delta)[f'(x)-f'(-x)]\mathrm{d}x=\int_0^{\delta}(\delta-x)[f'(x)-f'(-x)]\mathrm{d}x
\end{aligned}
$$

因为 $f''(x), f'(x)-f'(-x)>0$，所以 $I>0$.

3. 处理无穷积分. 处理形如 $\int_0^{\infty} f(x)\mathrm{d}x$ 的无穷积分，可以将其拆为两个区间积分的和

$$\int_0^{\infty} f(x)\mathrm{d}x=\int_0^1 f(x)\mathrm{d}x+\int_1^{\infty} f(x)\mathrm{d}x$$

再进行倒代换：

$$\int_1^{\infty} f(x)\mathrm{d}x \xlongequal{x=\frac{1}{t}} \int_1^0 f\left(\frac{1}{t}\right)\mathrm{d}\frac{1}{t}=\int_0^1 f\left(\frac{1}{t}\right)\cdot\frac{1}{t^2}\mathrm{d}t$$

从而得到

$$\boxed{\int_0^{\infty} f(x)\mathrm{d}x=\int_0^1\left[f(x)+f\left(\frac{1}{t}\right)\cdot\frac{1}{t^2}\right]\mathrm{d}x}$$

- 对于 $f\left(\frac{1}{t}\right)$ 形式比较简单的无穷积分常考虑这种方法，如对数函数、幂函数等；
- 对于抽象函数而言，若有 $f\left(\frac{1}{t}\right)$ 结构，也考虑这种方法.
- 对于一些一般的情况（例??），也可以采取类似的思想如：

$$\begin{aligned}\int_0^{2a} f(x)\mathrm{d}x &= \int_0^a f(x)\mathrm{d}x + \int_a^{2a} f(x)\\ &\xlongequal{\text{正代换}} \int_0^a f(x)\mathrm{d}x + \int_0^a f(x+a)\mathrm{d}x\\ &= \int_0^a [f(x)+f(x+a)]\mathrm{d}x\end{aligned}$$

或

$$\begin{aligned}\int_0^{2a} f(x)\mathrm{d}x &= \int_0^a f(x)\mathrm{d}x + \int_a^{2a} f(x)\\ &\xlongequal{\text{反代换}} \int_0^a f(x)\mathrm{d}x + \int_a^0 f(-x+2a)\mathrm{d}(-x+2a)\\ &= \int_0^a [f(x)+f(-x+2a)]\mathrm{d}x\end{aligned}$$

例 3.20. $\int_0^\infty \frac{\mathrm{d}x}{(x^2+1)(x^\pi+1)}$

$$\begin{aligned}I &= \int_0^\infty \frac{\mathrm{d}x}{(x^2+1)(x^\pi+1)} = \int_0^1 \frac{\mathrm{d}x}{(x^2+1)(x^\pi+1)} + \int_1^\infty \frac{\mathrm{d}x}{(x^2+1)(x^\pi+1)}\\ &= \int_0^1 \left[\frac{1}{(x^2+1)(x^\pi+1)} + \frac{1}{x^2}\frac{x^2\cdot x^\pi}{(x^2+1)(x^\pi+1)}\right]\mathrm{d}x\\ &= \int_0^1 \frac{1}{x^2+1}\mathrm{d}x = \arctan x|_0^1 = \frac{\pi}{4}.\end{aligned}$$

例 3.21. $\int_0^\infty \frac{\ln\frac{1+x^{11}}{1+x^3}}{(1+x^2)\ln x}\mathrm{d}x$

$$\begin{aligned}I &= \int_0^\infty \frac{\ln\frac{1+x^{11}}{1+x^3}}{(1+x^2)\ln x}\mathrm{d}x = \int_0^1 \frac{\ln\frac{1+x^{11}}{1+x^3}}{(1+x^2)\ln x}\mathrm{d}x + \int_1^\infty \frac{\ln\frac{1+x^{11}}{1+x^3}}{(1+x^2)\ln x}\mathrm{d}x\\ &= \int_0^1 \left[\frac{\ln\frac{1+x^{11}}{1+x^3}}{(1+x^2)\ln x} + \frac{1}{x^2}\cdot\frac{\ln\frac{1+x^{11}}{1+x^3} - 8\ln x}{-(1+x^2)\ln x}\cdot x^2\right]\mathrm{d}x\\ &= 8\int_0^1 \frac{1}{1+x^2}\mathrm{d}x = 2\pi\end{aligned}$$

例 3.22. 假定所涉及的广义积分收敛，证明

$$\int_{-\infty}^{+\infty} f\left(x-\frac{1}{x}\right)\mathrm{d}x = \int_{-\infty}^{+\infty} f(x)\mathrm{d}x.$$

$$
\begin{aligned}
I &= \int_{-\infty}^{+\infty} f\left(x-\frac{1}{x}\right)\mathrm{d}x = \int_0^{+\infty} f\left(x-\frac{1}{x}\right)\mathrm{d}x + \int_{-\infty}^{0} f\left(x-\frac{1}{x}\right)\mathrm{d}x \\
&= \int_0^{1} f\left(x-\frac{1}{x}\right)\mathrm{d}x + \int_1^{+\infty} f\left(x-\frac{1}{x}\right)\mathrm{d}x + \int_{-1}^{0} f\left(x-\frac{1}{x}\right)\mathrm{d}x + \int_{-\infty}^{-1} f\left(x-\frac{1}{x}\right)\mathrm{d}x \\
&= I_1 + I_2 + I_3 + I_4
\end{aligned}
$$

因为

$$
\begin{aligned}
I_2 &= \int_1^{+\infty} f\left(x-\frac{1}{x}\right)\mathrm{d}x \xlongequal{x=\frac{1}{x}} \int_0^1 f\left(\frac{1}{x}-x\right)\frac{1}{x^2}\mathrm{d}x \\
&\xlongequal{x=-x} \int_0^{-1} f\left(x-\frac{1}{x}\right)\frac{1}{x^2}\mathrm{d}(-x) = \int_{-1}^{0} f\left(x-\frac{1}{x}\right)\frac{1}{x^2}\mathrm{d}x = I_5 \\
I_4 &= \int_{-\infty}^{-1} f\left(x-\frac{1}{x}\right)\mathrm{d}x \xlongequal{x=\frac{1}{x}} \int_{-1}^{0} f\left(\frac{1}{x}-x\right)\frac{1}{x^2}\mathrm{d}x \\
&\xlongequal{x=-x} \int_1^{0} f\left(x-\frac{1}{x}\right)\frac{1}{x^2}\mathrm{d}(-x) = \int_0^{1} f\left(x-\frac{1}{x}\right)\frac{1}{x^2}\mathrm{d}x = I_6
\end{aligned}
$$

所以

$$
\begin{aligned}
I_1 + I_4 = I_1 + I_6 &= \int_0^1 f\left(x-\frac{1}{x}\right)\mathrm{d}x + \int_0^1 f\left(x-\frac{1}{x}\right)\frac{1}{x^2}\mathrm{d}x \\
&= \int_0^1 f\left(x-\frac{1}{x}\right) + f\left(x-\frac{1}{x}\right)\frac{1}{x^2}\mathrm{d}x = \int_0^1 f\left(x-\frac{1}{x}\right)\cdot\left(1+\frac{1}{x^2}\right)\mathrm{d}x \\
&= \int_0^1 f\left(x-\frac{1}{x}\right)\mathrm{d}x\left(x-\frac{1}{x}\right) \\
I_2 + I_3 = I_3 + I_5 &= \int_{-1}^0 f\left(x-\frac{1}{x}\right)\mathrm{d}x + \int_{-1}^0 f\left(x-\frac{1}{x}\right)\frac{1}{x^2}\mathrm{d}x \\
&= \int_{-1}^0 f\left(x-\frac{1}{x}\right) + f\left(x-\frac{1}{x}\right)\frac{1}{x^2}\mathrm{d}x = \int_{-1}^0 f\left(x-\frac{1}{x}\right)\cdot\left(1+\frac{1}{x^2}\right)\mathrm{d}x \\
&= \int_{-1}^0 f\left(x-\frac{1}{x}\right)\mathrm{d}x\left(x-\frac{1}{x}\right)
\end{aligned}
$$

令 $u = x - \frac{1}{x}$，则

$$
\begin{aligned}
I_1 + I_4 &= \int_0^1 f\left(x-\frac{1}{x}\right)\mathrm{d}x\left(x-\frac{1}{x}\right) = \int_{-\infty}^{0} f(u)\mathrm{d}u \\
I_2 + I_3 &= \int_{-1}^0 f\left(x-\frac{1}{x}\right)\mathrm{d}x\left(x-\frac{1}{x}\right) = \int_0^{+\infty} f(u)\mathrm{d}u
\end{aligned}
$$

所以

$$
I = I_1 + I_2 + I_3 + I_4 = \int_{-\infty}^{+\infty} f(u)\mathrm{d}u = \int_{-\infty}^{+\infty} f(x)\mathrm{d}x
$$

4 有理函数

有理函数积分一般有两种方法:

1. 部分分式方法

   不常用；

2. 构造法

   加项减项拆分、乘除提项凑微分降幂. 构造法中有一类凑微分需要注意：$\frac{1}{\sqrt{x}}\mathrm{d}x = 2\mathrm{d}\sqrt{x}$，以及 $\frac{1}{x^m}\mathrm{d}x = \frac{1}{-m+1}\mathrm{d}x^{-m+1}$，如 $\frac{1}{x^9}\mathrm{d}x = \frac{-1}{8}\mathrm{d}x^{-8}$.

5 三角有理式

一般有两种方法：

1. 万能公式

   一般用于一阶的三角有理式，令 $\tan\frac{x}{2} = t$，则

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2},$$

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2},$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{2t}{1-t^2}.$$

$$\mathrm{d}x = \mathrm{d}2\arctan t = \frac{2}{1+t^2}\mathrm{d}t.$$

2. 特殊方法

   恒等变形、换元、分部、提取等方法，注意以下恒等式：$1 = \sin^2 x + \cos^2 x$，$1 + \tan^2 x = \sec^2 x$，$\mathrm{d}\tan x = \sec^2 x\mathrm{d}x$，$\mathrm{d}\sec x = \sec x\tan x\mathrm{d}x$.

   几种常用的换元法

   - 若 $R(-\sin x, \cos x) = -R(\sin x, \cos x)$，令 $u = \cos x$，即凑 $\mathrm{d}\cos x$；
   - 若 $R(\sin x, -\cos x) = -R(\sin x, \cos x)$，令 $u = \sin x$，即凑 $\mathrm{d}\sin x$；
   - 若 $R(-\sin x, -\cos x) = R(\sin x, \cos x)$，令 $u = \tan x$，即凑 $\mathrm{d}\tan x$，一般提取 $\cos^2 x$ 构造.

6 简单无理函数积分

令 $\sqrt[n]{\frac{ax+b}{cx+d}} = t$.

例 6.1. 计算积分 $I = \int \frac{x^2e^x}{(x+2)^2}\mathrm{d}x$

$$
\begin{aligned}
I &= -\int x^2e^x\mathrm{d}\frac{1}{x+2} = -\frac{x^2e^x}{x+2} + \int\frac{1}{x+2}\mathrm{d}x^2e^x \\
&= -\frac{x^2e^x}{x+2} + \int\frac{2xe^x+x^2e^x}{x+2}\mathrm{d}x = -\frac{x^2e^x}{x+2} + \int xe^x\mathrm{d}x \\
&= -\frac{x^2e^x}{x+2} + xe^x - e^x + C.
\end{aligned}
$$

例 6.2. 计算积分 $I = \int\frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\mathrm{d}x$

令 $\sqrt{\frac{1-x}{1+x}} = t$，则 $x = \frac{1-t^2}{1+t^2}$，则

$$
\begin{aligned}
I &= \int\frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\mathrm{d}x = -\int\frac{4t^2}{(1-t^2)^2}\mathrm{d}t \\
&= -2\int\frac{t\mathrm{d}t^2}{(1-t^2)^2} = -2\int t\mathrm{d}\frac{1}{1-t^2} \\
&= -2\frac{2t}{1-t^2} + 2\int\frac{1}{1-t^2}\mathrm{d}t = -\frac{2t}{1-t^2} + \ln\left|\frac{1+t}{1-t}\right| + C \\
&= -\frac{\sqrt{1-x^2}}{x} + \ln\left|\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right| + C.
\end{aligned}
$$

例 6.3. 计算积分 $I = \int_0^1 x^4\sqrt{\frac{1+x}{1-x}}\mathrm{d}x$

因为

$$
\begin{aligned}
I &= \int_0^1 x^4\sqrt{\frac{1+x}{1-x}}\mathrm{d}x = \int_0^1 x^4\sqrt{\frac{1-x^2}{(1-x)^2}}\mathrm{d}x \\
&= \int_0^1 x^4\frac{\sqrt{1-x^2}}{(1-x)^2}\mathrm{d}x \xlongequal{x=\sin t} \int_0^{\frac{\pi}{2}}\sin^4 t\frac{\cos t}{1-\sin t}\mathrm{d}\sin t \\
&= \int_0^{\frac{\pi}{2}}\sin^4 t\frac{\cos^2 t}{1-\sin t}\mathrm{d}t = \int_0^{\frac{\pi}{2}}\sin^4 t\frac{1-\sin^2 t}{1-\sin t}\mathrm{d}t \\
&= \int_0^{\frac{\pi}{2}}\sin^4 t(1+\sin t)\mathrm{d}t \\
&= \int_0^{\frac{\pi}{2}}\sin^4 t\mathrm{d}t + \int_0^{\frac{\pi}{2}}\sin^5 t\mathrm{d}t \\
&= \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{4}{5}\cdot\frac{2}{3} = \frac{3\pi}{16} + \frac{8}{15}.
\end{aligned}
$$

注 6.1. 本题不能令 $\sqrt{\frac{1+x}{1-x}} = t$，因为 x^4 的存在，如此换元将十分麻烦；本题使用的技巧是 $A-B = \frac{A^2-B^2}{A+B}, A+B = \frac{A^2-B^2}{A-B}$.