Introduction to Topology

Group Theory, Lecture 16, 17, 18

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This is the Lecture note for the *Introduction to Topology*. The course covers the following topics: Naive Set Theory, Elementary Number Theory, Group Theory, Topological Spaces and Continuous Maps, Introduction to Algebraic Topology.

Content:

Abelian Group

Definition 1 (Abelian Group)**.** Given a group $(G, \square)$, we say $(G, \square)$ is a abelian group if $\forall g, g' \in G, g\square g' = g'\square g$.

The set $\mathbb{Z} \times \mathbb{Z}$ is equivalent with $\{\{1,2\} \xrightarrow{f} \mathbb{Z} | f \text{ if a map}\}$. For any $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, it can be represented as $f : 1 \mapsto x, 2 \mapsto y$, $\{1,2\}$ is the ordinate. And for any maps $\{1,2\} \xrightarrow{f} \mathbb{Z}$, it is corresponded by $(f(1), f(2)) \in \mathbb{Z} \times \mathbb{Z}$.

Let S be a set, define

$$\mathbb{Z}^{\oplus S} := \{S \xrightarrow{f} \mathbb{Z} | f \text{ is a map s.t. } f(s) \neq 0 \text{ for finitely many } s \in S\}$$

The element of $\mathbb{Z}^{\oplus S}$ is encoding each element of S with integer, and only finite element of S will be encoded by nonzero integer.

Note 1. Sometimes, we will denote $S \xrightarrow{f} \mathbb{Z}$ by $(x_s)_{s \in S}$.

Example 1. The element of $\mathbb{Z}^{\oplus \mathbb{N}}$ is a series of integer $(x_1, x_2, \cdots)(x_i \in \mathbb{Z}, i \in \mathbb{N})$ which has only finite nonzero integers.

We can define add on $\mathbb{Z}^{\oplus S}$: $(x_s)_{s\in S} + (y_s)_{s\in S} = (x_s + y_s)_{s\in S}$. Then for any $(x_s)_{s\in S}, (y_s)_{s\in S} \in \mathbb{Z}^{\oplus S}$, the binary operation $(\mathbb{Z}^{\oplus S}, +)$ has

1. $(x_s + y_s)_{s\in S} \in \mathbb{Z}^{\oplus S}$ (for $(x_s)_{s\in S}, (y_s)_{s\in S}$ only has finite nonzero integers)
2. $e = (0)_{s\in S} \in \mathbb{Z}^{\oplus S}$
3. $((x_s)_{s\in S})^{-1} = (-x_s)_{s\in S} \in \mathbb{Z}^{\oplus S}$
4. $(x_s)_{s\in S} + (y_s)_{s\in S} = (x_s + y_s)_{s\in S} = (y_s)_{s\in S} + (x_s)_{s\in S}$

Thus $(\mathbb{Z}^{\oplus S}, +)$ is a abelian group, and we call $(\mathbb{Z}^{\oplus S}, +)$ as **Free Abelian Group**.

Definition 2 (Homomorphism)**.** Given two groups $(G, \square), (G', \square')$, a map $G \xrightarrow{T} G'$ is a homomorphism w.r.t. $\square$ and $\square'$ if $\forall g_1, g_2 \in G$, $T(g_1 \square g_2) = T(g_1)\square' T(g_2)$.

Note 2. The purpose of defining a group homomorphism is to create functions that preserve the algebraic structure. An equivalent definition of group homomorphism is: The map $G \xrightarrow{h} G'$ is a group homomorphism if whenever $a\square b = c$ we have $h(a)\square' h(b) = h(c)$.

In other words, the group G' in some sense has a similar algebraic structure as G and the homomorphism h preserves that.

Example 2. Map $\mathbb{Z} \xrightarrow{T} \mathbb{Z}/5\mathbb{Z}$ is a homomorphism, since for any $a, b \in \mathbb{Z}$, $(a + 5\mathbb{Z}) + (b + 5\mathbb{Z}) = (a + b) + 5\mathbb{Z}$.

Definition 3 (Isomorphism)**.** We say a homomorphism T is an isomorphism if T is a bijection.

Definition 4. Given two groups $(G, \Box), (G', \Box')$, let $G \xrightarrow{T} G'$ be a homomorphism:
1. $ker(T) := T^{-1}(e') = \{g \in G | T(g) = e'\}$;
2. $im(T) := T(G) = \{T(g) | g \in G\}$.

Exercise 1. Show that $ker(T)$ is a subgroup of $(G, \Box)$, $im(T)$ is a subgroup of $(G', \Box')$.

Proof. 1.
(0.) Obviously $ker(T) \subseteq G$.
(1.) for $\forall g_1, g_2 \in ker(T)$:

$$\begin{aligned} T(g_1 \Box g_2) &= T(g_1) \Box' T(g_2) \\ &= e' \Box e' = e' \end{aligned}$$

thus $g_1 \Box g_2 \in ker(T)$.
(2.) for $\forall g \in ker(T)$,

$$\begin{aligned} T(g) &= T(g \Box e) \\ &= T(g) \Box' T(e) \\ &= e' \Box' T(e) = e' \end{aligned}$$

and $T(e) \Box' e' = e'$ in the same way, thus $e \in ker(T)$, and be the unit element of $ker(T)$.
(3.) for $\forall g \in ker(T)$,

$$\begin{aligned} T(e) &= T(g \Box g^{-1}) \\ &= T(g) \Box' T(g^{-1}) \\ &= e' \Box' T(g^{-1}) \\ &= e' \end{aligned}$$

and $T(g^{-1}) \Box' e' = e'$, thus $T(g^{-1}) = e'$, and $g^{-1} \in ker(T)$.
Thus $ker(T)$ is a subgroup of $(G, \Box)$.
2.
0. Obviously $im(T) \subseteq G'$.
1. for $\forall g_1', g_2' \in im(T), \exists g_1, g_2,$ s.t. $T(g_1) = g_1', T(g_2) = g_2'$. Thus

$$\begin{aligned} T(g_1 \Box g_2) &= T(g_1) \Box' T(g_2) \\ &= g_1' \Box' g_2' \end{aligned}$$

thus $g_1' \Box' g_2' \in im(T)$.
(2.) Since $e \in ker(T) \Rightarrow T(e) = e' \Rightarrow e' \in im(T)$.

(3.) for $\forall g' \in im(T), \exists g \in G,$ s.t. $T(g) = g'$, and

$$\begin{aligned} T(e) &= T(g \Box g^{-1}) \\ &= T(g) \Box' T(g^{-1}) \\ &= g' \Box' T(g^{-1}) \\ &= e' \end{aligned}$$

and $T(g^{-1}) \Box' g' = e'$ in the same way, thus $T(g^{-1}) = g'^{-1}, g'^{-1} \in im(T)$.
Thus $im(T)$ is a subgroup of G'. $\Box$

Exercise 2. $G \xrightarrow{T} G'$ is a homomorphism show that $T(e) = e'$ and $T(g^{-1}) = T(g)^{-1}$ for $\forall g \in G$. e' is the unit element of $(G', \Box')$,

Proof. 1. $ker(T)$ is a subgroup of G, thus $e \in ker(T) \Rightarrow T(e) = e'$. 2. $T(g^{-1}) \Box' T(g) = T(g^{-1} \Box g) = T(e) = e'$, thus $T(g^{-1}) = T(g)^{-1}$. $\Box$

Definition 5. Given two groups $(G, \Box), (G', \Box')$, let $G \xrightarrow{T} G'$ be a homomorphism. If $(G', \Box')$ is abelian, $cok(T) := G'/im(T)$.

Normal Subgroup

Consider a group $(G, \Box)$ and natural projection π. Are there is map $\Box'$ such that the following commutative diagram holds? i.e. for $\forall g_1, g_2, \pi(g_1 \Box g_2) = \pi(g_1) \Box' \pi(g_2)$?

$$\begin{tikzcd} {}_{(a,b)}G \times G^{(a,b)} \arrow[r, "\Box"] \arrow[d, "\pi\times\pi"'] & G^{a\Box b} \arrow[d, "\pi"] \\ {}_{(a\Box H, b\Box H)}G/H \times G/H \arrow[r, "\Box'"] & {}_{(a\Box H)\Box'(b\Box H)}G/H^{a\Box b\Box H} \end{tikzcd}$$

Note 3. Given maps f_1, f_2 and a surjection g, we have proved if $g \circ f_1 = g \circ f_2 \Rightarrow f_1 = f_2$, thus if $\Box'$ exists, there would be only one.

In the other word, for $(a, b) \in G \times G$, we can define map $\Box'$ as

$$(a \Box H) \Box' (b \Box H) := a \Box b \Box H$$

But there is not well-defined, because there would exists $a', b' \in G$ such that $a' \Box H = a \Box H, b' \Box H = b \Box H$, thus $(a \Box H) \Box' (b \Box H) = (a' \Box H) \Box' (b' \Box H)$, but $a' \Box b' \Box H \neq a \Box b \Box H$.

Definition 6 (Normal Subgroup)**.** Given a group $(G, \Box)$, $(H, \Box)$ is a subgroup of $(G, \Box)$ (denote by $H \leq G$). We call H is a normal subgroup, denote by $H \trianglelefteq G$, if $\forall g \in G, \forall h \in H, g^{-1} \Box h \Box g \in H$.

Note 4. The definition of normal subgroup is equivalent with
1. $\forall g \in G, \forall h \in H, g \Box h \Box g^{-1} \in H$.
2. $g^{-1} \Box H \Box g \subseteq H$
3. $g \Box H \Box g^{-1} \subseteq H$
4. $g^{-1} \Box H \Box g = H$
5. $g \Box H \Box g^{-1} = H$

Exercise 3. Show that the definition of normal subgroup is equivalent with $g^{-1} \Box H \Box g = H$.

Proof. $\forall g \in G, \forall h \in H$, $g^{-1}\Box h\Box g \in H \Leftrightarrow g^{-1}\Box H\Box g \subseteq H$ by the definition of coset. And then for $\forall g \in G, H\Box g \subseteq g\Box H$ and $g^{-1}\Box H \subseteq H\Box g^{-1} \Rightarrow g\Box H \subseteq H\Box g$ Because $g = (g^{-1})^{-1}$. So $g\Box H = H\Box g$ and $g^{-1}\Box H\Box g = H$. ☐

Exercise 4. If $H \trianglelefteq G$, show that $a^{-1}\Box a' \in H, b^{-1}\Box b' \in H \Rightarrow (a\Box b)^{-1}\Box(a'\Box b') \in H$, that is $H \trianglelefteq G$ is the **sufficient condition**.

Proof. Denote $a^{-1}\Box a' = h \in H$, $\exists h' \in H$, s.t. $b^{-1}\Box b' = h' \Rightarrow b' = b\Box h'$, thus

$$\begin{aligned}&(a\Box b)^{-1}\Box(a'\Box b')\\&= b^{-1}\Box a^{-1}\Box a'\Box b'\\&= b^{-1}\Box h\Box b\Box h'\\&= (b^{-1}\Box h\Box b)\Box h'\end{aligned}$$

$H \trianglelefteq G \Rightarrow b^{-1}\Box h\Box b \in H \Rightarrow (a\Box b)^{-1}\Box(a'\Box b') \in H$. ☐

Note 5. a, a' belong to the same coset of $H \Leftrightarrow a\Box H = a'\Box H \Leftrightarrow a^{-1}a' \in H \Leftrightarrow a' = a\Box h$.

We have seen that if $H \trianglelefteq G$ then there is a binary operation $G/H \times G/H \xrightarrow{\Box'} G/H((a\Box H, b\Box H) \mapsto a\Box b\Box H)$, such that the commutative diagram

$$\begin{array}{ccc} G\times G & \xrightarrow{\quad\Box\quad} & G \\ {\scriptstyle\pi\times\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ G/H\times G/H & \xrightarrow{\quad\Box'\quad} & G/H \end{array}$$

holds.

Exercise 5 (Quotient Group)**.** $H \trianglelefteq G$, show that $(G/H, \Box')$ is a group.

Proof. 0. $H \trianglelefteq G \Rightarrow \Box'$ is well-defined by $(g_1\Box H)\Box'(g_2\Box H) := (g_1\Box g_2)\Box H$ for any $g_1, g_2 \in G$.

1. $\forall g_1, g_2 \in G, g_1\Box H, g_2\Box H \in G/H$, then $(g_1\Box H)\Box'(g_2\Box H) = (g_1\Box g_2)\Box H$. $g_1\Box g_2 \in G$ thus $(g_1\Box g_2)\Box H \in G/H$.

2. $\forall g \in G, g\Box H \in G/H$, then $(g\Box H)\Box H = (g\Box e)\Box H = g\Box H$, thus $e_{G/H} = H \in G/H$.

3. $(g\Box H)^{-1} = g^{-1}\Box H \in G/H$. ☐

Exercise 6. $G \xrightarrow{T} G'$ is a homomorphism, show that $ker(T) \trianglelefteq G$ and $im(T) \leq G'$.

Proof. 1. For $\forall g \in G, k \in ker(T)$,

$$\begin{aligned}T(g^{-1}\Box k\Box g) &= T(g^{-1})\Box' e'\Box' T(g)\\&= T(g)^{-1}\Box' T(g)\\&= e'\end{aligned}$$

Thus $g^{-1}\Box k\Box g \in ker(T) \Rightarrow ker(T) \trianglelefteq G$.

2. (1.) $T(g_1)\Box' T(g_2) = T(g_1\Box g_2) \in im(T)$; (2.) $e' = T(e) \in im(T)$; (3) $T(g)^{-1} = T(g^{-1}) \in im(T)$. ☐

Thus if subgroup $(H, \Box)$ is normal then $(G/H, \Box')$ is a group. Conversely, if $(G, \Box)$ is abelian, then any subgroup $(H, \Box)$ is normal, for $ghg^{-1} = gg^{-1}h = h \in H$; and $(G/H, \Box')$ is abelian, for

$$\begin{aligned} &(a \Box H) \Box' (b \Box H) \\ &= a \Box b \Box H = b \Box a \Box H \\ &= (b \Box H) \Box' (a \Box H). \end{aligned}$$

Exercise 7. $G \xrightarrow{T} G'$ is a homomorphism, show that T is injection $\Leftrightarrow ker(T) = \{e\}$.

Proof. $\Rightarrow$: $\forall g \in G, k \in ker(T)$, $T(g \Box k) = T(g) \Box' T(k) = T(g) \Box' e' = T(g) \Rightarrow g = g \Box k$. Similarly, $g = k \Box g$, thus $k = e (\forall k \in ker(T))$ and $ker(T) = \{e\}$.

$\Leftarrow$: For any $g_1, g_2 \in G$, if $T(g_1) = T(g_2)$, then

$$\begin{aligned} T(g_2) \Box T(g_2)^{-1} &= T(g_1) \Box' T(g_2)^{-1} \\ &= T(g_1) \Box' T(g_2^{-1}) \\ &= T(g_1 \Box g_2^{-1}) \\ &= e' \end{aligned}$$

Thus $g_1 \Box g_2^{-1} \in ker(T) = \{e\} \Rightarrow g_1 \Box g_2^{-1} = e \Rightarrow g_1 = g_2$. $\Box$

Theorem of Isomorphism

Theorem 1 (Theorem of homomorphism)**.** *Given groups $(G, \Box)$ and $(G', \Box')$, suppose $G \xrightarrow{T} G'$ is a homomorphism, $H \leq G$. Then*

1. *$T(H) = \{e'\}$, i.e. $H \subseteq ker(T) \Leftrightarrow \exists!$ map $G/H \xrightarrow{\tilde{T}} G'$ s.t.*

$$\begin{tikzcd} G \arrow[rr, "T"] \arrow[dr, "\pi"'] & & G' \\ & G/H \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

2. *If $H \subseteq ker(T)$ and $H \trianglelefteq G$ then $G/H \xrightarrow{\tilde{T}} G'$ is a homomorphism.*
3. *$H = ker(T) \Leftrightarrow \tilde{T}$ is injection.*
4. *T is surjection $\Leftrightarrow \tilde{T}$ is surjection.*

Proof. 1. $\Leftarrow$: for $\forall h \in H$, $\pi(h) = \pi(e) = H$, thus

$$T(h) = \tilde{T} \circ \pi(h) = \tilde{T} \circ \pi(e) = T(e) = e',$$

thus $T(h) = e' (\forall h \in H)$, that is $H \subseteq ker(T)$.

$\Rightarrow$: Define $\tilde{T}(g \Box H) := T(g)$. For any $g, g_1 \in G$, s.t. $\pi(g) = \pi(g_1)$, that is $g \Box H = g_1 \Box H \Leftrightarrow \exists h \in H$ s.t. $g = g_1 \Box h$. Thus $T(g) = T(g_1 \Box h) = T(g_1) \Box' T(h) = T(g_1)$. Thus the definition of $\tilde{T}$ is **well defined**. π is surjection $\Rightarrow \tilde{T}$ has uniqueness.

2. $H \trianglelefteq G$, thus $(G/H, \Box^*)$ is a group, where $(g_1 \Box G)\Box^*(g_2 \Box H) = g_1 \Box g_2 \Box H$ for any $g_1, g_2 \in G$. Thus

$$\begin{aligned}\tilde{T}((g_1\Box H)\Box^*(g_2\Box H)) &= \tilde{T}(g_1\Box g_2\Box H)\\ &= T(g_1\Box g_2) = T(g_1)\Box' T(g_2)\\ &= \tilde{T}(g_1\Box H)\Box'\tilde{T}(g_2\Box H).\end{aligned}$$

So $\tilde{T}$ is a homomorphism.

3. We now explore the structure of $ker(\tilde{T})$. Given $a \in G$, then

$$\begin{aligned}a\Box H \in ker(\tilde{T}) &\Leftrightarrow \tilde{T}(a\Box H) = T(a) = e'\\ &\Leftrightarrow a \in ker(T)\\ &\Rightarrow a\Box H \in ker(T)/H\end{aligned}$$

Note 6. Easy to check: $H \leq G, ker(T) \leq G, H \subseteq ker(T) \Rightarrow H \leq ker(T)$.
If $H \leq ker(T)$, then $ker(T)/H := \{k\Box H | k \in ker(T)\}$.

If $a\Box H \in ker(T)/H$, then $\exists k \in ker(T)$, s.t. $a\Box H = k\Box H$, then $\exists h \in H \subseteq ker(T)$, s.t. $a = k\Box h \in ker(T)$ (for $k, h \in ker(T)$, $ker(T) \leq G$ and enclosed with $\Box$) Thus $a\Box H \in ker(\tilde{T}) \Leftrightarrow a\Box H \in ker(T)/H$, thus $ker(\tilde{T}) = ker(T)/H$.

Thus $\tilde{T}$ is injection $\Leftrightarrow ker(\tilde{T}) = \{H\}$ (for H is unit element of G/H) $\Leftrightarrow ker(T) = H$.

4.$\Rightarrow$: $\tilde{T} \circ \pi$ is surj. $\Rightarrow \tilde{T}$ is surj. $\Leftarrow$: Composite of surj. is surj. $\Box$

Collectively, $\tilde{T}$ is inj. $\Leftrightarrow H = ker(T)$; $\tilde{T}$ is surj. $\Leftrightarrow T$ is surj. Thus $\tilde{T}$ is isomorphism (bij. + homomorphism) $\Leftrightarrow T$ is surj and $H = ker(T)$.

So $G \xrightarrow{T} G'$ is a homomorphism then exists an isomorphism $G/ker(T) \xrightarrow{\tilde{T}} im(T)$, denote by $G/ker(T) \simeq im(T)$. This conclusion is called **1st theorem of isomorphism**.

Example 3. Define $S_3 := \{\{1,2,3\} \xrightarrow{\sigma} \{1,2,3\} | \sigma \text{ is bij.}\}$, then $(S_3, \circ)$ is a group. And the element of $(S_3, \circ)$ is $e' = (1)(2)(3)$.

Given a group $(\mathbb{Z}, +)$, define a homomorphism $\mathbb{Z} \xrightarrow{T} S_3$. So if $1 \mapsto (12)$, then $T(2) = T(1+1) = T(1) \circ T(1) = e'$, $T(-1) = T(1)^{-1} = T(1) = (12)$. Furthermore $T(2\mathbb{Z}) = e'$, $T(2\mathbb{Z}+1) = (12)$. And $ker(T) = 2\mathbb{Z}, im(T) = \{(12), e'\}$. So $\mathbb{Z}/2\mathbb{Z} \simeq \{(12), e'\}$. Similarly, $\mathbb{Z}/3\mathbb{Z} \simeq \{e', (123), (132)\}$ (Define $T(1) = (123)$).

$$\begin{tikzcd} \mathbb{Z} \arrow[rr, "T"] \arrow[dr, "\pi"'] & & S_3 \\ & \mathbb{Z}/2\mathbb{Z} \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

Homotopy

Definition 7 (Path)**.** Assume X is a top. sp. $p, q \in X$.

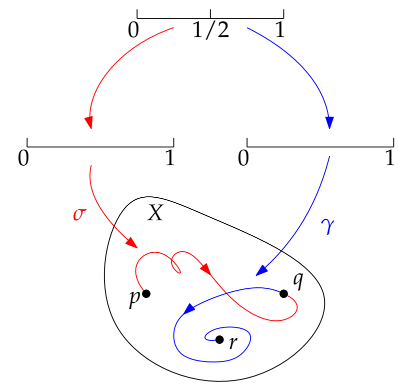
1. A path from p to q in X is a continuous map $[0,1] \xrightarrow{\gamma} X$, s.t. $\gamma(0) = p, \gamma(1) = q$.

2. Define $\Omega(X,p,q) := \{[0,1] \xrightarrow{\gamma} |\gamma \text{ is conti.}, \gamma(0) = p, \gamma(1) = q\}$.
3. $\forall \gamma \in \Omega(X,p,q)$, define inverse path $[0,1] \xrightarrow{\gamma^-} X(t \mapsto \gamma(1-t))$.

Thus we attain a map $\Omega(X,p,q) \to \Omega(X,q,p)(\gamma \mapsto \gamma^-)$, which is a bijection.

Definition 8. Assume X is a top. sp. $p,q,r,s \in X$. For $\sigma \in \Omega(X,p,q), \gamma \in \Omega(X,q,r)$, define $[0,1] \xrightarrow{\sigma\cdot\gamma} X$ by

$$(\sigma\cdot\gamma)(t) := \begin{cases} \sigma(2t), & t \in [0,1/2], \\ \gamma(2t-1), & t \in [1/2,1]. \end{cases}$$



Note 7. Notice that the open set in subspace topology is not necessarily open set in (parent) topology.

Exercise 8. Given a top. sp. X and subspace A, B of X, s.t. $X = A \cup B$ and either $A, B \subseteq_{open} X$ or $A, B \subseteq_{close} X$. Show that a map $X \xrightarrow{f} Y$ to a top. sp. Y is conti. $\Leftrightarrow A \xrightarrow{f|_A} Y$ and $B \xrightarrow{f|_B} Y$ are conti.

Proof. $\Rightarrow$: f is conti, thus $\forall U \subseteq_{open} Y, f^{-1}(U) \subseteq_{open} X$. And $f|_A^{-1}(U) = f^{-1}(U) \cap A \subseteq_{open} A$, since A is equipped by subspace top. So $f|_A$ is conti. and the same thing to $f|_B$.

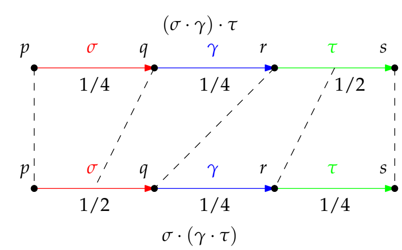
$\Leftarrow$: Suppose $A, B \subseteq_{open} X$, for any $U \subseteq_{open} Y$, since $f|_A$ conti., $f|_A^{-1}(U) \subseteq_{open} A$, thus $\exists V \subseteq_{open} X$, s.t. $f|_A^{-1}(U) = V \cap A \subseteq_{open} X$, and similarly $f|_B^{-1}(U) \subseteq_{open} X$. Thus

$$\begin{aligned} f^{-1}(U) &= \{x \in X | f(x) \in U\} \\ &= \{x \in A | f(x) \in U\} \cup \{x \in B | f(x) \in U\} \\ &= f|_A^{-1}(U) \cup f|_B^{-1}(U) \\ &\subseteq_{open} X. \end{aligned}$$

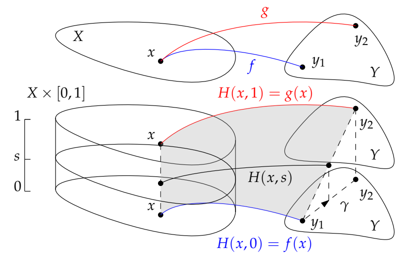
Thus f is conti. If $A, B \subseteq_{close} X$, the argument is similar, because $X \xrightarrow{f} Y$ is conti. $\Leftrightarrow \forall U \subseteq_{open} Y, f^{-1} \subseteq_{open} X \Leftrightarrow \forall U \subseteq_{close} Y, f^{-1} \subseteq_{close} X$. □

Thus map $[0,1] \xrightarrow{\sigma\cdot\gamma} X$ is also conti. and $\sigma\cdot\gamma \in \Omega(X,p,r)$. And we can define another map: $\Omega(X,p,q) \times \Omega(X,q,r) \to \Omega(X,p,r)((\sigma,\gamma) \mapsto \sigma\cdot\gamma)$.

Notice that the "composite" path does not have "associative" property, that is for any 3 paths $\sigma \in \Omega(X,p,q), \gamma \in \Omega(X,q,r), \tau \in \Omega(X,r,s)$, $(\sigma\cdot\gamma)\cdot\tau$ is not necessarily equal to $\sigma\cdot(\gamma\cdot\tau)$. Because the time (or say speed) distribution on these two composite paths is different, the former is $\sigma(0-1/4) \to \gamma(1/4-1/2) \to \tau(1/2-1)$, whereas the later is $\sigma(0-1/2) \to \gamma(1/2-3/4) \to \tau(3/4-1)$.



Definition 9 (Homotopy)**.** Given two conti. maps $X \xrightarrow{f} Y, X \xrightarrow{g} Y$ between top. sp. X and Y. A map $X \times [0,1] \xrightarrow{H} Y$ is a homotopy from f to g if H is conti. and $\forall x \in X, H(x,0) = f(x), H(x,1) = g(x)$.



The intuition of homotopy is creating a map H that starts from f and generally approximates to g as time goes on. On the other hand, we can also view H as creating a path from y_1 to y_2. If H exists, we say f and g are homotopy, denote by $f \underset{H}{\sim} g$.

Definition 10. Suppose X is a top. sp. $p, q \in X, \sigma, \gamma \in \Omega(X, p, q)$, we say σ and γ are homotopic with fixed initial and end point if $\exists$ homotopy $[0,1] \times [0,1] \xrightarrow{H} X$ from σ to γ, s.t. $\forall s \in [0,1], H(0,s) = p, H(1,s) = q$, and denote by $p \underset{\cdot\cdot}{\sim} q$.

Notice that homotopy is only required to be two maps at beginning and end, that is $H(x,0) = \sigma(x)$ and $H(x,1) = \gamma(x)$. But when we say two paths are homotopic, we need for any $s \in [0,1]$, path $H(x,s)$ is from p to q. $\underset{\cdot\cdot}{\sim}$ is an equivalence relation on $\Omega(X, p, q)$, and define $\pi_1(X, p, q) := \Omega(X, p, q) / \underset{\cdot\cdot}{\sim}$, and denote the equivalence class of $\gamma \in \Omega(X, p, q)$ as $[\gamma]$. So if $\gamma \underset{\cdot\cdot}{\sim} \sigma$, then $[\gamma] = [\sigma] \in \pi_1(X, p, q)$.