Introduction to Topology

General Topology, Lecture 14

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CONTENT:

THIS IS THE LECTURE NOTE FOR THE *Introduction to Topology*. The course covers the following topics: Naive Set Theory, Elementary Number Theory, Group Theory, Topological Spaces and Continuous Maps, Introduction to Algebraic Topology.

Product Topology

Definition 1 (Homeomorphism)**.** A map $X \xrightarrow{f} Y$ between top. sp. is a homeomorphism if f is a bijection, and f and f^{-1} (inverse of f) are continuous.

We say two top. sp. X and Y are homeomorphic if $\exists$ homeomorphism from X onto Y.

Given sets, we can create new sets from them, thus given topologies we also want to create new topologies from them. We just defined the subspace topology, now we want to create new topology on the cartesian product of the sets.

Let X, Y be top. sp., maps $X \times Y \xrightarrow{p_1} X((x,y) \mapsto x)$ and $X \times Y \xrightarrow{p_2} Y((x,y) \mapsto y)$ are called the natural projections. A natural intuition is these two natural projections need be continuous.

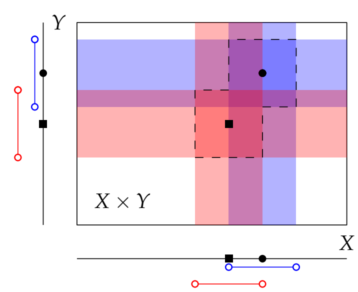
If $\mathscr{T}$ is a topology on $X \times Y$ such that $X \times Y \xrightarrow{p_1} X((x,y) \mapsto x)$ and $X \times Y \xrightarrow{p_2} Y((x,y) \mapsto y)$ both are continuous, then for $\forall U \subseteq_{open} X, V \subseteq_{open} Y$,

$$p_1^{-1}(U) = U \times Y = \subseteq_{open} X \times Y$$

and

$$p_2^{-1}(V) = X \times V = \subseteq_{open} X \times Y$$

thus $(U \times Y) \cap (X \times V) = U \times V \subseteq_{open} X \times Y$. Unfortunately, $\{U \times VU \subseteq_{open} X, V \subseteq_{open} Y\}$ is not a topology, because the union of $U_\alpha \times V_\beta$ are not necessarily covered by $\{U \times V | U \subseteq_{open} X, V \subseteq_{open} Y\}$. (like the margin figure)



Thus for $U_\alpha \subseteq_{open} X (\alpha \in A), V_\beta \subseteq_{open} Y (\beta \in B)$, then it need be true that $U_\alpha \times V_\beta \subseteq_{open} X \times Y ((\alpha \times \beta) \in A \times B)$ and $\cup_{(\alpha,\beta)\in A\times B}(U_\alpha \times V_\beta) \subseteq_{open} X \times Y$.

Proposition 1. *A, B are index sets,*

$$\mathscr{T}_{X\times Y} := \left\{ \bigcup_{(\alpha,\beta)\in A\times B} (U_\alpha \times V_\beta) | U_\alpha \subseteq_{open} X, V_\beta \subseteq_{open} Y \right\}$$

is a topology on $X \times Y$.

Proof. The union of two elements of $\mathscr{T}_{X\times Y}$ is in $\mathscr{T}_{X\times Y}$ is trivial, the only thing we need confirm is the intersection of two elements of $\mathscr{T}_{X\times Y}$ is in $\mathscr{T}_{X\times Y}$. Suppose C, D are index sets, and $U_\alpha, S_\delta \subseteq_{open} X$, $V_\beta, T_\gamma \subseteq_{open} Y$, then

$$\begin{aligned}
&\left(\bigcup_{(\alpha,\beta)\in A\times B}(U_\alpha \times V_\beta)\right)\bigcap\left(\bigcup_{(\delta,\gamma)\in C\times D}(S_\delta \times T_\gamma)\right)\\
&=\bigcup_{(\delta,\gamma)}\left[(S_\delta\times T_\gamma)\bigcap\left(\bigcup_{(\alpha,\beta)}(U_\alpha\times V_\beta)\right)\right]\\
&=\bigcup_{(\delta,\gamma)}\left[\bigcup_{(\alpha,\beta)}(U_\alpha\times V_\beta)\bigcap(S_\delta\times T_\delta)\right]\\
&=\bigcup_{(\delta,\gamma)}\left[\bigcup_{(\alpha,\beta)}(U_\alpha\cap S_\delta)\times(V_\beta\cap T_\gamma)\right]
\end{aligned}$$

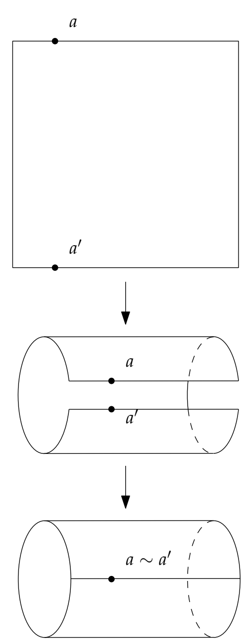
$U_\alpha, S_\delta \subseteq_{open} X \Rightarrow U_\alpha \cap S_\delta \subseteq_{open} X$, and $V_\beta \cap T_\gamma \subseteq_{open} Y$ in the same way $\Rightarrow \cup_{(\alpha,\beta)}(U_\alpha \cap S_\delta) \times (V_\beta \cap T_\gamma) \in \mathscr{T}_{X\times Y} \Rightarrow \cup_{(\delta,\gamma)}\left[\cup_{(\alpha,\beta)}(U_\alpha \cap S_\delta) \times (V_\beta \cap T_\gamma)\right] \in \mathscr{T}_{X\times Y}$. □

Definition 2 (Product Topology)**.** X, Y are top. sp. the topology $\mathscr{T}_{X\times Y}$ is called the Product Topology on $X \times Y$.

Note 1. Thus $\mathscr{T}_{X\times Y}$ is the smallest topology on $X \times Y$ that holds projections p_1, p_2 continuous.

Quotient Topology

We now introduce a new method to create top. which is on the set where some points are viewed as same. For example, when we fold a piece of paper into a column, the corresponding points on the top and bottom edges of the paper are regarded as the same point. If we have already defined a topology on the paper, now we want to define a topology on the column.



If X is a top. sp., and R is a equivalent relation on X. Recall that X is the disjoin union of the element in quotient set X/R (whose elements are distinct equivalent class $R(x)$), we can regard the points in the identical equivalent class as the same. And we want to define topology on the quotient set X/R which is associated with the top. on X.

Thus a fairly intuitive association between X and X/R is the natural projection $X \xrightarrow{\pi} X/R(x \mapsto R(x))$, so we need define a topology $\mathscr{T}$ on X/R such that π is continuous. This is different with the subspace top. and product top. where the undetermined top. is the domain, whereas in this case (quotient top.) the undetermined top. is the codomain. Thus in the previous cases, we need to find the smallest top. which satisfies conditions, in this case, we need to find the largest one.

Clearly, if the top. $\mathscr{T}$ on X/R makes π be continuous, then $\mathscr{T} \subseteq \{S \subseteq X | \pi^{-1}(S) \subseteq_{open} X\}$. Thus if $\{S \subseteq X | \pi^{-1}(S) \subseteq_{open} X\}$ is a top. then it is the largest one what we are looking for.

Exercise 1. Suppose X is a top. sp. Y is a set, $X \xrightarrow{f} Y$ is a map. Show that $\mathscr{S} := \{S \subseteq Y | f^{-1}(S) \subseteq_{open} X\}$ is a topology on Y.

Proof. 1. $f^{-1}(\emptyset) = \emptyset \subseteq_{open} X$ thus $\emptyset \in \mathscr{S}$; $f^{-1}(Y) = X \subseteq_{open} X$, thus $Y \in \mathscr{S}$.

2. For any $S_1, S_2 \in \mathscr{S}$, $f^{-1}(S_1 \cap S_2) = f^{-1}(S_1) \cap f^{-1}(S_2) \subseteq_{open} X$, thus $S_1 \cap S_2 \in \mathscr{S}$.

3. For any $S_\alpha (\alpha \in A) \in \mathscr{S}$, $f^{-1}(\cup_{\alpha \in A} S_\alpha) = \cup_{\alpha \in A} f^{-1}(S_\alpha) \subseteq_{open} X$, thus $\cup_{\alpha \in A} S_\alpha \in \mathscr{S}$. □

Note 2. Actually, $\mathscr{S}$ is the largest top. on Y such that f is continuous.

Thus $\{S \subseteq X | \pi^{-1}(S) \subseteq_{open} X\}$ is the largest topology on X/R.

Definition 3 (Quotient Topology)**.** Suppose X is a top. sp., and R is a equivalent relation on X. $\{S \subseteq X | \pi^{-1}(S) \subseteq_{open} X\}$ is called the quotient topology on X/R.

Example 1. Let $X = [0,1] \times [0,1]$ is a subspace topology inherited from $\mathbb{R}^2$. We now fold X up and down to form a column, this operation can be view as defining an equivalent class R on X:

$$R(x,y) = \begin{cases} \{(x,y)\}, & 0 \leq x \leq 1, 0 < y < 1, \\ \{(x,0),(x,1)\}, & 0 \leq x \leq 1. \end{cases}$$

And the topology on the column is $\{S \subseteq X | \pi^{-1}(X) \subseteq_{open} X\}$.

Abelian Group

Definition 4 (Abelian Group)**.** Given a group $(G, \square)$, we say $(G, \square)$ is a abelian group if $\forall g, g' \in G, g \square g' = g' \square g$.

The set $\mathbb{Z} \times \mathbb{Z}$ is equivalent with $\{\{1,2\} \xrightarrow{f} \mathbb{Z} | f \text{ if a map}\}$. For any $(x,y) \in \mathbb{Z} \times \mathbb{Z}$, it can be represented as $f : 1 \mapsto x, 2 \mapsto y$, $\{1,2\}$ is the ordinate. And for any maps $\{1,2\} \xrightarrow{f} \mathbb{Z}$, it is corresponded by $(f(1), f(2)) \in \mathbb{Z} \times \mathbb{Z}$.

Let S be a set, define

$$\mathbb{Z}^{\oplus S} := \{S \xrightarrow{f} \mathbb{Z} | f \text{ is a map s.t. } f(s) \neq 0 \text{ for finitely many } s \in S\}$$

The element of $\mathbb{Z}^{\oplus S}$ is encoding each element of S with integer, and only finite element of S will be encoded by nonzero integer.

Note 3. Sometimes, we will denote $S \xrightarrow{f} \mathbb{Z}$ by $(x_s)_{s \in S}$.

Example 2. The element of $\mathbb{Z}^{\oplus \mathbb{N}}$ is a series of integer $(x_1, x_2, \cdots)(x_i \in \mathbb{Z}, i \in \mathbb{N})$ which has only finite nonzero integers.

We can define add on $\mathbb{Z}^{\oplus S}$: $(x_s)_{s\in S} + (y_s)_{s\in S} = (x_s + y_s)_{s\in S}$. Then for any $(x_s)_{s\in S}, (y_s)_{s\in S} \in \mathbb{Z}^{\oplus S}$, the binary operation $(\mathbb{Z}^{\oplus S}, +)$ has

1. $(x_s + y_s)_{s\in S} \in \mathbb{Z}^{\oplus S}$ (for $(x_s)_{s\in S}, (y_s)_{s\in S}$ only has finite nonzero integers)
2. $e = (0)_{s\in S} \in \mathbb{Z}^{\oplus S}$
3. $((x_s)_{s\in S})^{-1} = (-x_s)_{s\in S} \in \mathbb{Z}^{\oplus S}$
4. $(x_s)_{s\in S} + (y_s)_{s\in S} = (x_s + y_s)_{s\in S} = (y_s)_{s\in S} + (x_s)_{s\in S}$

Thus $(\mathbb{Z}^{\oplus S}, +)$ is a abelian group, and we call $(\mathbb{Z}^{\oplus S}, +)$ as **Free Abelian Group**.

Definition 5 (Homomorphism)**.** Given two groups $(G, \Box), (G', \Box')$, a map $G \xrightarrow{T} G'$ is a homomorphism w.r.t. $\Box$ and $\Box'$ if $\forall g_1, g_2 \in G$, $T(g_1 \Box g_2) = T(g_1) \Box' T(g_2)$.

Example 3. Map $\mathbb{Z} \xrightarrow{T} \mathbb{Z}/5\mathbb{Z}$ is a homomorphism, since for any $a, b \in \mathbb{Z}$, $(a + 5\mathbb{Z}) + (b + 5\mathbb{Z}) = (a + b) + 5\mathbb{Z}$.

Definition 6 (Isomorphism)**.** We say a homomorphism T is an isomorphism if T is a bijection.

Definition 7. Given two groups $(G, \Box), (G', \Box')$, let $G \xrightarrow{T} G'$ be a homomorphism:

1. $ker(T) := \{g \in G | T(g) = e'\}$, e' is the unit element of $(G', \Box')$;
2. $im(T) := \{T(g) | g \in G\}$.

Exercise 2. Show that $ker(T)$ is a subgroup of $(G, \Box)$, $im(T)$ is a subgroup of $(G', \Box')$.

Proof. 1.

(0.) Obviously $ker(T) \subseteq G$.

(1.) for $\forall g_1, g_2 \in ker(T)$:

$$\begin{aligned} T(g_1 \Box g_2) &= T(g_1) \Box' T(g_2) \\ &= e' \Box e' = e' \end{aligned}$$

thus $g_1 \Box g_2 \in ker(T)$.

(2.) for $\forall g \in ker(T)$,

$$\begin{aligned} T(g) &= T(g \Box e) \\ &= T(g) \Box' T(e) \\ &= e' \Box' T(e) = e' \end{aligned}$$

and $T(e) \Box' e' = e'$ in the same way, thus $e \in ker(T)$, and be the unit element of $ker(T)$.

(3.) for $\forall g \in ker(T)$,

$$\begin{aligned} T(e) &= T(g \Box g^{-1}) \\ &= T(g) \Box' T(g^{-1}) \\ &= e' \Box' T(g^{-1}) \\ &= e' \end{aligned}$$

and $T(g^{-1}) \Box' e' = e'$, thus $T(g^{-1}) = e'$, and $g^{-1} \in ker(T)$.
Thus $ker(T)$ is a subgroup of $(G, \Box)$.

2.

0. Obviously $im(T) \subseteq G'$.
1. for $\forall g_1', g_2' \in im(T), \exists g_1, g_2,$ s.t. $T(g_1) = g_1', T(g_2) = g_2'$. Thus

$$\begin{aligned} T(g_1 \Box g_2) &= T(g_1) \Box' T(g_2) \\ &= g_1' \Box' g_2' \end{aligned}$$

thus $g_1' \Box' g_2' \in im(T)$.

(2.) Since $e \in ker(T) \Rightarrow T(e) = e' \Rightarrow e' \in im(T)$.

(3.) for $\forall g' \in im(T), \exists g \in G,$ s.t. $T(g) = g'$, and

$$\begin{aligned} T(e) &= T(g \Box g^{-1}) \\ &= T(g) \Box' T(g^{-1}) \\ &= g' \Box' T(g^{-1}) \\ &= e' \end{aligned}$$

and $T(g^{-1}) \Box' g' = e'$ in the same way, thus $T(g^{-1}) = g'^{-1}$, $g'^{-1} \in im(T)$.
Thus $im(T)$ is a subgroup of G'. $\Box$

Definition 8. Given two groups $(G, \Box), (G', \Box')$, let $G \xrightarrow{T} G'$ be a homomorphism. If $(G', \Box')$ is abelian, $cok(T) := G'/im(T)$.