

Introduction to Analysis

Lecture 1

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1 Real Number

Definition 1. Let $S \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, we say that

1. r is an upper (lower) bound of S if $\forall s \in S, r \geq (\leq) s$;
2. r is the greatest (least) element of S if r is an upper (lower) bound of S and $r \in S$, denoted by $r = \max S$ ($\min S$).
3. r is the least upper (greatest lower) bound of S if r is the least (greatest) element of the set of upper (lower) bound of S , denoted by $r = \sup S$ ($\inf S$).

Remark 1. r is a least upper bound of S means any element of S which is smaller than r is not an upper bound of S , that is $\forall \epsilon > 0, \exists s \in S$, s.t. $r - \epsilon < s \leq r$.

We write $\sup S = \infty$ ($\inf S = -\infty$) if and only if S has no upper (lower) bound. If this is the case we say $\sup S$ ($\inf S$) does not exist. We say S is bounded from above (below) iff S has an upper (lower) bound.

Definition 2 (Dedekind Cut). Let $A, B \subseteq \mathbb{R}$, we say that (A, B) is a Dedekind cut if

1. $A, B \neq \emptyset$;
2. $A \cup B = \mathbb{R}$;
3. $\forall a \in A, b \in B, a < b$.

We usually call $A(B)$ the lower (upper) part of (A, B) .

We assume that \mathbb{R} has the **Dedekind's Gapless Property**: If (A, B) is a Dedekind cut of \mathbb{R} , then exactly one of the following happens:

1. $\max A$ exists but $\min B$ does not;
2. $\min B$ exists but $\max A$ does not.

We call $\max A$ in 1. (or $\min B$ in 2.) the **cutting** of (A, B) .

Exercise 1. We may define Dedekind cuts on \mathbb{Q} and \mathbb{Z} similarly, does Dedekind Gapless Property hold for \mathbb{Q} and \mathbb{Z} ?

Proof. 1. Let $A := \{q \in \mathbb{Q} | q^2 < 2\}, B := \{q \in \mathbb{Q} | q^2 > 2\}$. It is direct to see that $A, B \neq \emptyset$.

If $\exists r \in \mathbb{Q}$, s.t. $r^2 = 2$, then $\exists p, q \in \mathbb{N}$, s.t. $r = p/q$ and p, q are not both even. Then $p^2/q^2 = 2 \Rightarrow p^2 = 2q^2 \Rightarrow p^2$ is even $\Rightarrow p$ is even $\Rightarrow p^2$ can be divided by 4 $\Rightarrow q^2$ can be divided by 2 $\Rightarrow q^2$ is even $\Rightarrow q$ is even, which leads to a contradiction. Thus $\forall r \in \mathbb{Q}, r^2 \neq 2$. Thus $A \cup B = \mathbb{Q}$.

Finally $\forall q_a \in A, q_b \in B$ one has $q_a^2 < 2 < q_b^2 \Rightarrow q_a < q_b$. Thus (A, B) is a Dedekind cut of \mathbb{Q} . It is direct to see that (A, B) has no Dedekind's gapless property:

For example, if $p \in A$, then $p \in \mathbb{Q}$ and $p^2 < 2$, put $\epsilon = 2 - p^2$, then we should find a $q \in \mathbb{Q}$ such that $q^2 < 2$ and $q > p$, which means

$$p^2 < q^2 < 2$$

we consider there exists a function r of p, ϵ , such that $r > 0$ and $r \in \mathbb{Q}$, and put $q = p + r$, thus $q > p$ and $q \in \mathbb{Q}$, we now prove that $q^2 < 2$. Since

$$2 - q^2 = 2 - (p^2 + r^2 + 2pr) = \epsilon - r^2 - 2pr$$

We now try to find a suitable structure of r to make $r^2 + 2pr < \epsilon$. Since $p > 0$ and $\epsilon = 2 - p^2, 0 < \epsilon < 2$. Consider $r = \epsilon/2$ then

$$r^2 = \frac{\epsilon^2}{4} = \frac{\epsilon}{2} \cdot \frac{\epsilon}{2} < \frac{\epsilon}{2}.$$

Consider $r = \frac{\epsilon}{(2p+1)2} < \epsilon/2$ and

$$2pr = 2p \cdot \frac{\epsilon}{(2p+1)2} < \frac{\epsilon}{2},$$

then we have $r^2 + 2pr < \epsilon$ and

$$q^2 < 2,$$

by defining $q = p + \frac{\epsilon}{2(2p+1)} = \frac{3p^2+2p+2}{4p+2}$, thus there is no maximal element in A and correspondingly, there is no minimal element in B as well.

2. trivial. □

Theorem 1 (Weierstrass Theorem). Let $\emptyset \neq S \subseteq \mathbb{R}$, if S has an upper bound, then $\sup S$ exists.

Proof. Let B be the set of all upper bound of S , and define $A := \mathbb{R} \setminus B$.

CLAIM 1: (A, B) is a Dedekind cut of \mathbb{R} :

1. $S \neq \emptyset \Rightarrow \forall s \in S, s-1 \notin B \Rightarrow s-1 \in A \Rightarrow A \neq \emptyset$; And S has an upper bound $\Rightarrow B \neq \emptyset$;
2. $A = \mathbb{R} \setminus B \Rightarrow A \cup B = \mathbb{R}$;
3. If $\exists a \in A, b \in B$, s.t. $a \geq b$ where b is an upper bound of S while a is not, thus $\exists s' \in S$, s.t. $a < s' \leq b < a$, which leads to a contradiction. Thus $\forall a \in A, b \in B$ one has $a < b$.

CLAIM 2: $\min B$ exists:

If $\min B \nexists$, then by Dedekind's gapless property, $\max A \exists$, denoted by a_0 . $a_0 \in A \Leftrightarrow a_0 \notin B \Leftrightarrow a_0$ is not an upper bound of $S \Leftrightarrow \exists s_0 \in S$, s.t. $a_0 < s_0$. Choose $x \in \mathbb{R}$ such that $a_0 < x < s_0$, thus $\max A < x \Rightarrow x \in B \Rightarrow x$ is an upper bound of S but $x < s_0$, which leads to a contradiction. \square

Exercise 2 (Archimedean Property). Show that $\forall r \in \mathbb{R}, r > 0 \Rightarrow \exists n \in \mathbb{N}$, s.t. $n > r$. (or say $\exists n \in \mathbb{N}$, s.t. $1/n < r$).

Proof. Let $r \in \mathbb{R}$, $S := \{n \in \mathbb{N} | n \leq r\}$, since $r > 0, 0 \in S \Rightarrow S \neq \emptyset$. Then $S \subseteq \mathbb{N}$ and S is bounded above (by r), thus S has a least upper bound in \mathbb{R} , let $s = \sup S$.

Now consider the number $s - 1$. Since s is the supremum of S , $s - 1$ cannot be an upper bound of S by definition. Thus $\exists m \in S$ such that

$$m > s - 1 \Rightarrow m + 1 > s$$

But as $m \in \mathbb{N}$, it follows that $m + 1 \in \mathbb{N}$. Because $m + 1 > s$, it follows that $m + 1 \notin S$ and so $m + 1 > r$. Furthermore, for $\forall r > 0, 1/r > 0$ then $\exists n \in \mathbb{N}$, s.t. $n > 1/r \Rightarrow 1/n < r$. \square

2 Sequence

Definition 3 (Convergence). Let $a_n (n \in \mathbb{N})$ be a sequence in \mathbb{R} and $l \in \mathbb{R}$, we say that a_n converges to l as $n \rightarrow \infty$ if $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N, |a_n - l| < \epsilon$, denoted by $a_n \rightarrow l$ (as $n \rightarrow \infty$).

If such l exists, we call it the limit of $\{a_n\}$ and denote it as $\lim_{n \rightarrow \infty} a_n = l$, and call $\{a_n\}$ a convergent sequence; otherwise we call it a **divergent** sequence. Furthermore, we say $\lim_{n \rightarrow \infty} a_n = \infty$ if $\forall M > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N, a_n \geq M$.

Exercise 3. Show that

1. $\lim_{n \rightarrow \infty} a_n = l$ and $\lim_{n \rightarrow \infty} a_n = m \Rightarrow l = m$;
2. $a_n (n \in \mathbb{N})$ is convergent $\Rightarrow \{a_n | n \in \mathbb{N}\}$ is bounded;
3. if $a_n < b_n$ for all $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} a_n = l, \lim_{n \rightarrow \infty} b_n = m \Rightarrow l \leq m$.

Proof. 1. $\lim_{n \rightarrow \infty} a_n = l$ and $\lim_{n \rightarrow \infty} a_n = m \Rightarrow$ for $\forall \epsilon > 0, \exists N, M \in \mathbb{N}$, s.t. $\forall n \geq N$

one has $|a_n - l| < \epsilon/2$ and $\forall n \geq M$ has $|a_n - m| < \epsilon/2$, thus for $\forall n \geq \max\{N, M\}$, has

$$|l - m| = |l - a_n + a_n - m| \leq |a_n - l| + |a_n - m| < \epsilon$$

holds for $\forall \epsilon > 0 \Rightarrow l = m$.

2. Suppose $a_n \rightarrow l$ as $n \rightarrow \infty$, then given an $\epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ we have $|a_n - l| < \epsilon \Rightarrow l - \epsilon < a_n < l + \epsilon$, thus a_n has upper bound

$$\max\{a_1, \dots, a_{N-1}, l + \epsilon\},$$

and lower bound

$$\min\{a_1, \dots, a_{N-1}, l - \epsilon\}.$$

3. if $l > m$, let $\epsilon = l - m$, then $\exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ has $|a_n - l| < \epsilon/2$ and $|b_n - m| < \epsilon/2$ thus

$$a_n < \frac{l + m}{2} < b_n,$$

which leads to a contradiction, thus $l \leq m$. □

Remark 2. Changing or removing finitely many terms in $a_n (n \in \mathbb{N})$ does not effect a_n 's being convergent (and its limit)/ divergent.

Proposition 1. If $\lim_{n \rightarrow \infty} a_n = l$ and $\lim_{n \rightarrow \infty} b_n = m$ then

1. $\lim_{n \rightarrow \infty} (a_n \pm b_n) = l \pm m$;
2. $\lim_{n \rightarrow \infty} a_n b_n = lm$;
3. if $m \neq 0$ and $b_n \neq 0$ for all but finitely many n then $\lim_{n \rightarrow \infty} a_n / b_n = l / m$.

Proof. 1. For $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N, |a_n - l| \leq \epsilon/2$ and $\exists M \in \mathbb{N}$, s.t. $\forall n \geq M, |b_n - m| \leq \epsilon/2$, thus $\forall n \geq \max\{N, M\}$, one has

$$\begin{aligned} |(a_n \pm b_n) - (l \pm m)| &= |(a_n - l) \pm (b_n - m)| \\ &\leq |a_n - l| + |b_n - m| \\ &\leq \epsilon, \end{aligned}$$

thus $(a_n \pm b_n) \rightarrow l \pm m$ as $n \rightarrow \infty$.

2. Since a_n, b_n are convergent, thus they are bounded. Choose $C > 0$ such that $|b_n| \leq C$ for all $n \in \mathbb{N}$ and $|l| \leq C$, then for $\forall \epsilon > 0, \exists N, M \in \mathbb{N}$, s.t. $\forall n \geq N$ one has $|a_n - l| \leq \epsilon/(2C)$ and $\forall n \geq M$ has $|b_n - m| \leq \epsilon/(2C)$, thus $\forall n \geq \max\{N, M\}$ one has

$$\begin{aligned} |a_n b_n - lm| &= |a_n b_n - lb_n + lb_n - lm| \\ &\leq |a_n - l| \cdot |b_n| + |b_n - m| \cdot |l| \\ &\leq (|a_n - l| + |b_n - m|) \cdot |C| \\ &\leq \epsilon \end{aligned}$$

thus $a_n b_n \rightarrow lm$.

3. all we need to show is $\lim_{n \rightarrow \infty} 1/b_n = 1/m$. Trivial. □