

# Point Set Topology

## Lecture 4

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THIS IS THE LECTURE NOTE FOR THE *Point Set Topology*.

### CONTENT:

1. Initial Topology
2. Subspace Topology
3. Product Space
4. Cartesian Product
5. Infinite Dimension Product Topology

## Initial Topology

Given maps  $X \xrightarrow{f_\alpha} Y_\alpha (\alpha \in A)$  from a set  $X$  to topology spaces  $Y_\alpha (\alpha \in A)$ . It is direct to see that if  $X$  is topologized by discrete topology, the  $f_\alpha$  are all continuous. Now the question is how coarse the topology  $\mathcal{T}$  on  $X$  could be to ensure  $f_\alpha (\alpha \in A)$  to be continuous.

Let  $\mathcal{S} := \{f_\alpha^{-1}(V) | V \subseteq_{\text{open}} Y_\alpha, \alpha \in A\}$ , then  $\mathcal{T}(\mathcal{S})$  is the expected coarsest topology, called the **initial topology** induced by the family of maps  $\{f_\alpha | \alpha \in A\}$ .

## Subspace Topology

Let  $(Y, \mathcal{T}_Y)$  be a topology space, for a subset  $X \subseteq Y$ . We want to define an natural topology  $\mathcal{T}_X$  on  $X$  from  $Y$ , such that keep **inclusion map**  $X \xrightarrow{id_X} Y (x \mapsto x)$  be continuous.

As we said,  $\mathcal{T}_X$  is the arbitrary union of finite intersection of the pre-image of the open set in  $Y$ . We call this initial topology induced by the inclusion map the **subspace topology** on  $X$  inherited from  $Y$ .

Note that the arbitrary union of finite intersection of the pre-image of the open set in  $Y$  is just the pre-image of arbitrary union of finite intersection of the open set in  $Y$ , which is just the pre-image of the open set in  $Y$ . Thus  $\mathcal{T}_X = \{id_X^{-1}(V) | V \subseteq_{\text{open}} Y\} = \{V \cap X | V \subseteq_{\text{open}} Y\}$ .

**Exercise 1** (The universal property of subspace topologies). Suppose  $Y$  is a topology space,  $X$  is a subspace (i.e. a subset equipped with the subspace topology from  $Y$ ). Given a topology space  $Z$ , for  $\forall$  map  $Z \xrightarrow{g} Y$ , if  $g(Z) \subseteq X$ , show that  $Z \xrightarrow{g} Y$  is conti.  $\Leftrightarrow Z \xrightarrow{g|_X} X$  is conti.

*Proof.*  $\Rightarrow$ : any open set in  $X$  can be represented by  $U \cap X$  where  $U \subseteq_{\text{open}} Y$ , thus  $g^{-1}(U \cap X) = g^{-1}(U) \cap g^{-1}(X) = g^{-1}(U) \cap Z \subseteq_{\text{open}} Z \Rightarrow Z \xrightarrow{g|_X} X$  is conti.  $\Leftarrow$ : Trivial.  $\square$

### Product Space

Let  $(Y_1, \mathcal{T}_1)$  and  $(Y_2, \mathcal{T}_2)$  be topology spaces, we want to create a natural topology  $\mathcal{T}_{Y_1 \times Y_2}$  on  $Y_1 \times Y_2$  which makes the projections

$Y_1 \times Y_2 \xrightarrow{p_i} Y_i (i = 1, 2)$  be continuous.

Suppose  $U_i (i = 1, \dots, k_U) \subseteq_{\text{open}} Y_1$  and  $V_j (j = 1, \dots, k_V) \subseteq_{\text{open}} Y_2$ , then

$$\left( \bigcap_{i=1}^{k_U} f^{-1}(U_i) \right) \cap \left( \bigcap_{j=1}^{k_V} f^{-1}(V_j) \right) = f^{-1} \left( \bigcap_{i=1}^{k_U} U_i \right) \cap f^{-1} \left( \bigcap_{j=1}^{k_V} V_j \right)$$

where  $\bigcap_{i=1}^{k_U} U_i \subseteq_{\text{open}} Y_1$  and  $\bigcap_{j=1}^{k_V} V_j \subseteq_{\text{open}} Y_2$ . Thus the desired initial topology can be represented as the arbitrary union of the intersection of the pre-image of an open set in  $Y_1$  and the pre-image of an open set in  $Y_2$ . (instead of the finite intersection of pre-image of open sets in  $Y_1$  and  $Y_2$ , it is subtle) Thus the basis of the expected initial topology is

$$\begin{aligned} \Pi &= \{p_1^{-1}(W_1) \cap p_2^{-1}(W_2) | W_1 \subseteq_{\text{open}} Y_1, W_2 \subseteq_{\text{open}} Y_2\} \\ &= \{(W_1 \times Y_2) \cap (Y_1 \times W_2) | W_1 \subseteq_{\text{open}} Y_1, W_2 \subseteq_{\text{open}} Y_2\} \\ &= \{W_1 \times W_2 | W_1 \subseteq_{\text{open}} Y_1, W_2 \subseteq_{\text{open}} Y_2\} \end{aligned}$$

*Note 1.* We call such a  $W_1 \times W_2$  a rectangle.

Thus the topology desired is all unions of rectangle:

$$\mathcal{T}_{Y_1 \times Y_2} = \{\cup \mathcal{F} | \mathcal{F} \subseteq \Pi\}.$$

We call such initial topology **product topology** of  $Y_1$  and  $Y_2$ , denote as  $\mathcal{T}_1 \times \mathcal{T}_2$ .

In particular, the open set  $O$  in  $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$  can be written by  $O = \cup U \times V$  where  $U, V \subseteq_{\text{open}} \mathbb{R}$ .

### Cartesian Product

Let's recall the definition of Cartesian product. Given two sets  $Y_1, Y_2$ , there exists a **bijection** between  $Y_1 \times Y_2$  and the family of maps  $\{\{1, 2\} \xrightarrow{s} Y_1 \cup Y_2 | s(1) \in Y_1, s(2) \in Y_2\} =: \mathcal{M}_{Y_1 \times Y_2}$ . First, there is an injection from left to right: for any  $(s_1, s_2) \in Y_1 \times Y_2$ , define  $s$  as  $s(1) = s_1, s(2) = s_2$ . Thus different points in  $Y_1 \times Y_2$  reflect to different maps in  $\mathcal{M}_{Y_1 \times Y_2}$ .

On the other hand, there exists an injection from right to left as well: for any  $s', s \in \mathcal{M}_{Y_1 \times Y_2}$ , correspond to  $(s(1), s(2)), (s'(1), s'(2)) \in Y_1 \times Y_2$ , and  $(s(1), s(2)) \neq (s'(1), s'(2))$  if  $s \neq s'$ .

Furthermore, when we project a point  $(y_1, y_2) \in Y_1 \times Y_2$  to  $y_1 \in Y_1$  (using projection  $Y_1 \times Y_1 \xrightarrow{p_1} Y_1$ ), it is equivalent with mapping the

corresponding map  $s$  to  $s(1)$ .

$$\begin{array}{ccccc}
 & & p_1 & & \\
 & \swarrow & & \searrow & \\
 Y_1 \times Y_2 & \ni & (y_1, y_2) & \longmapsto & y_1 \in Y_1 \\
 & \updownarrow & & \parallel & \\
 \mathcal{M}_{Y_1 \times Y_2} & \ni & s & \longmapsto & s(1) \in Y_1 \\
 & \searrow & & \swarrow &
 \end{array}$$

Similarly, we can define infinite dimension Cartesian product as

$$\prod_{\alpha \in A} Y_\alpha := \{A \xrightarrow{s} \cup_{\alpha \in A} Y_\alpha \mid \forall \alpha \in A, s(\alpha) \in Y_\alpha\} =: \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha},$$

according to the axiom of choice, if  $Y_\alpha \neq \emptyset$  for any  $\alpha \in A$ , then such map  $s$  exists, then  $\prod_{\alpha \in A} Y_\alpha \neq \emptyset$ . For  $\alpha \in A$ , we often denote the value of  $s$  at  $\alpha$  by  $s_\alpha$  rather than  $s(\alpha)$ ; we call it the  $\alpha$ -th **coordinate** of  $s$ . And we often denote the function  $s$  itself by the symbol

$$(s_\alpha)_{\alpha \in A},$$

which is as close as we can come to a "tuple notation" for an arbitrary index set  $A$ .

Corresponding, we can define the projection on infinite dimension cartesian product: for any  $\beta \in A$ ,

$$\prod_{\alpha \in A} Y_\alpha \xrightarrow{p_\beta} Y_\beta$$

as a map

$$\mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} \longrightarrow Y_\beta$$

with  $s \mapsto s_\beta$ .

### *Infinite Dimension Product Topology*

Now we can define the product topology on infinite dimension. As we discussed, the topology is arbitrary union of finite intersection of pre-image of the open set in  $Y_\alpha (\alpha \in A)$ . Since the intersection is finite, we can still exchange the order of pre-image and intersection, and then represent the open sets from the same  $Y_\alpha (\alpha \in A)$  as one open set. Note that the pre-image of  $U_\beta \subseteq_{open} Y_\beta$  can be represented by

$$\{s \in \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} \mid s_\beta \in U_\beta\}.$$

Thus finite intersection of the pre-image of open sets, i.e. the basis of the infinite dimension product topology is

$$\prod_{\alpha \in A} Y_\alpha = \{s \in \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} \mid s_{\beta_1} \in U_{\beta_1}, \dots, s_{\beta_k} \in U_{\beta_k}, k \in \mathbb{N}\}.$$

That the basis of infinite product topology is set of maps that only map **finite** points in domain to the open sets of codomain. Alternatively, we can represent it as

$$\Pi_{\alpha \in A} Y_\alpha = \left\{ \prod_{\alpha \in A} V_\alpha \mid \forall \alpha \in A, V_\alpha \subseteq_{\text{open}} Y_\alpha \wedge \{\alpha \in A \mid V_\alpha \neq Y_\alpha\} \text{ is finite} \right\}.$$

And the topology is

$$\mathcal{T}_{\Pi_{\alpha \in A} Y_\alpha} = \{\cup \mathcal{F} \mid \mathcal{F} \subseteq \Pi_{\alpha \in A} Y_\alpha\}$$

**Exercise 2** (The universal property of product topology). Let  $Z, Y_\alpha (\alpha \in A)$  are topology spaces,  $\Pi_{\alpha \in A} Y_\alpha$  is equipped with product topology, show that for any group of maps

$$Z \xrightarrow[\text{conti.}]{g_\alpha} Y_\alpha (\alpha \in A)$$

$\exists ! Z \xrightarrow[\text{conti.}]{g} \Pi_{\alpha \in A} Y_\alpha$ , s.t.  $p_\alpha \circ g = g_\alpha$  for  $\forall \alpha \in A$ . That is, such commutative diagram holds

$$\begin{array}{ccc} Z & \xrightarrow[\text{conti.}]{g} & \Pi_{\alpha \in A} Y_\alpha \\ & \searrow \text{conti.} \quad \downarrow p_\alpha & \\ & g_\alpha & Y_\alpha \end{array}$$

*Proof.* Existence: Select a group of  $g_\alpha (\alpha \in A)$  such that for a given  $z \in Z$  has

$$g_\alpha(z) = y_\alpha \in Y_\alpha.$$

Define a map  $Z \xrightarrow{g} \mathcal{M}_{\Pi_{\alpha \in A} Y_\alpha}$  with  $z \mapsto s$  where  $s_\alpha = y_\alpha (\alpha \in A)$ . Thus for any  $\beta \in A$ , we have

$$p_\beta \circ g(z) = p_\beta(s) = s_\beta = y_\beta = g_\beta(z)$$

Thus  $p_\alpha \circ g = g_\alpha$  for any  $\alpha \in A$ . We now show  $g$  is continuous.

Any open set  $U$  in  $\mathcal{M}_{\Pi_{\alpha \in A} Y_\alpha}$  can be written as  $U = \cup \mathcal{F} = \cup_{V \in \mathcal{F}} V$ , where  $\mathcal{F} \subseteq \Pi_{\alpha \in A} Y_\alpha$ . Thus

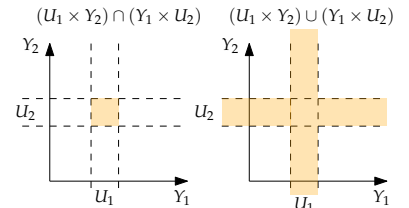
$$g^{-1}(U) = g^{-1}(\cup_{V \in \mathcal{F}} V) = \cup_{V \in \mathcal{F}} g^{-1}(V).$$

Here  $V$  is the element in the basis, and can be represented as

$$V = \{s \in \mathcal{M}_{\Pi_{\alpha \in A} Y_\alpha} \mid s_{\alpha_1} \in U_{\alpha_1}, \dots, s_{\alpha_k} \in U_{\alpha_k}\},$$

where  $U_{\alpha_i} \subseteq_{\text{open}} Y_{\alpha_i} (i = 1, \dots, k)$ , thus

$$\begin{aligned} g^{-1}(V) &= \{z \in Z \mid g_{\alpha_1}(z) \in U_{\alpha_1}, \dots, g_{\alpha_k}(z) \in U_{\alpha_k}\} \\ &= \cap_{i=1}^k g_{\alpha_i}^{-1}(U_{\alpha_i}) \\ &\subseteq_{\text{open}} Z \end{aligned}$$



There is a trap:

- $(U_1 \times Y_2) \cap (Y_1 \times U_2) = U_1 \times U_2$ ;
- $(U_1 \times Y_2) \cup (Y_1 \times U_2) \neq Y_1 \times Y_2$ ;

Thus  $g^{-1}(U) = \cup_{V \in \mathcal{F}} g^{-1}(V) \subseteq_{\text{open}} Z \Rightarrow g$  is continuous.

Uniqueness: for any  $h$  such that  $p_\alpha \circ h = g_\alpha$ , given a  $z \in Z$ , we have  $p_\alpha(h(z)) = g_\alpha(z)$  for  $\forall \alpha \in A$ . Thus

$$\begin{aligned} h(z) &\in \cap_{\alpha \in A} p_\alpha^{-1}(g_\alpha(z)) \\ &= \cap_{\alpha \in A} \{s \in \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} \mid s_\alpha = g_\alpha(z)\} \\ &= \{s \in \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} \mid s_\alpha = g_\alpha(z), \alpha \in A\} \end{aligned}$$

Thus  $h(z) = s$  where  $s_\alpha = g_\alpha(z), \alpha \in A \Rightarrow h = g$ .  $\square$

The conclusion of the universal property of product topology is : for any group of maps  $Z \xrightarrow{g_\alpha} Y_\alpha (\alpha \in A)$ , if they can be substitute by another group of map  $f_\alpha \circ g'$  where  $Z \xrightarrow{g'} Z'$  and  $Z' \xrightarrow{f_\alpha} Y_\alpha$ , we say  $Z'$  is **closer** to  $Y_\alpha (\alpha \in A)$  than  $Z$ .

Then  $\prod_{\alpha \in A} Y_\alpha$  is the **closest** set to  $Y_\alpha (\alpha \in A)$ .

**Exercise 3.** Let  $Z, Y_\alpha (\alpha \in A)$  are top. spaces. Show that  $Z \xrightarrow{g} \prod_{\alpha \in A} Y_\alpha$  is continuous  $\Leftrightarrow p_\alpha \circ g (\alpha \in A)$  are continuous.

*Proof.*  $\Rightarrow$ : Since  $p_\alpha \circ g = g_\alpha$ , we need to prove  $g$  is continuous  $\Rightarrow g_\alpha$  is continuous.

For any open set  $U_\alpha \subseteq_{\text{open}} Y_\alpha$ .  $p_\alpha^{-1}(U_\alpha) = \{s \in \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} \mid s_\alpha \in U_\alpha\} \subseteq_{\text{open}} \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} = \prod_{\alpha \in A} Y_\alpha$ . And  $g^{-1}(\{s \in \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} \mid s_\alpha \in U_\alpha\}) = \{z \in Z \mid g_\alpha(z) \in U_\alpha\} = g_\alpha^{-1}(U_\alpha) \subseteq_{\text{open}} Z$ , since  $g$  is continuous, thus  $g_\alpha$  is continuous.  $\Leftarrow$ : has been given in Ex2.  $\square$

