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spring, 2020

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Lecture 2

Game Theory I: Mixed Strategy Nash Equilibrium

Key word: Mixed Strategies Nash Equilibrium, Brouwer's Fixed Point Theorem, Nash Theorem, Examples

1 Mixed Strategies and Nash Equilibrium

We have so for defined the actions available to each player in a game, but not yet his set of strategy (i.e. how to choice his action). Certainly one kind of strategy is to select a single action and play it, we will call this strategy a **pure strategy**, and a choice of pure strategy for each agent a pure strategy profile.

It would be a bad idea to play any deterministic strategy in some game, such as Matching pennies game. Since if player 1 declare action H, then the BR for player 2 is T, then the BR for player 1 is T, then the BR for player 2 is H,... This iteration is an infinite loop.

But player can confuse the opponent by playing **randomly**. For instance, Player 1 could choose action by taking the result of flipping a coin.

Definition 1.1: Mixed strategy

Let (N, A, O, μ, u) be a normal form game, and for any set X let $\Pi(X)$ be the set of all probability distributions over X. Then the set of mix strategies for player i is $S_i = \Pi(A_i)$. The set of mixed strategy profiles is simply the Cartesian product of the individual mixed strategy sets, $S = S_1 \times \cdots \times S_n$.

Collectively, Strategy s_i for agent i is a **probability distribution** over the actions A_i. If there exists only one action $a_i' \in A_i$ with probability 1, then call this distribution **Pure Strategy**. If there are more than one actions is played with positive probability, then the distribution is called **Mixed Strategy**. Note that a pure strategy is a special case of a mixed strategy, in which the support is a single action.

By $s_i(a_i)$ we denote the probability that an action a_i will be played under mixed strategy s_i. The set of actions that are assigned positive probability by the mixed strategy s_i is called the **support** of s_i, $\{a_i | s_i(a_i) > 0\}$.

If players follow mixed strategy profile $s \in S$, then we can not read the payoff of each one from the game matrix anymore. Instead, we use the idea of expected utility to calculate payoff. Review that we define the utility in previous chapter as $u_i(a_i, a_{-i}) = u_i(a)$.

Definition 1.2: Expected utility

The expected utility of player i given the strategy profile $s \in \mathcal{S}$ is

$$u_i(s) = \sum_{a \in \mathcal{A}} u_i(a) \cdot \mathbb{P}\left(a | s\right),$$

where $a = \langle a_1, a_2, \cdots, a_n \rangle$ and $\mathbb{P}\left(a|s\right) = \prod_{j \in N} s_j(a_j)$.

Intuitively, we first calculate the probability of reaching each outcome given the strategy profile, and then we calculate the average of the payoffs of outcomes, weighted by the probability of each outcome.

Example 1.1. Consider each player in Matching pennies game chooses H with probability .5 and T with .5. The strategy profile is

$$s = \langle s_1, s_2 \rangle = \langle (\mathbb{P}_1(H) = 0.5, \mathbb{P}_1(T) = 0.5), (\mathbb{P}_2(H) = 0.5, \mathbb{P}_2(T) = 0.5) \rangle,$$

then the expected utility of player 1 given strategy profile s is

$$\begin{aligned} u_1(s) = &\sum_{a \in \mathcal{A}} u_i(a) \cdot \mathbb{P}\left(a|s\right) \\ =& u_1(H,H) \cdot \mathbb{P}\left(H,H|s\right) + u_1(H,T) \cdot \mathbb{P}\left(H,T|s\right) \\ &+ u_1(T,H) \cdot \mathbb{P}\left(T,H|s\right) + u_1(T,T) \cdot \mathbb{P}\left(T,T|s\right) \end{aligned}$$

where $\mathbb{P}\left(H,H|s\right) = s_1(a_1) \cdot s_2(a_2) = s_1(H) \cdot s_2(H) = 0.25$, others are the same. In summary $u_1(s) = 0$.

If an agent knew how the others were going to play, his strategy problem would be simple. Define $s_{-i} = \langle s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n \rangle \in S_{-i}$, a strategy profile s without agent i's strategy. Thus we can write $s = \langle s_i, s_{-i} \rangle$. If the agents other than i were to commit play s_{-i}, a rational agent i would face the problem of determining his best response.

Definition 1.3: Best response

We say s_i^* is the Best responce for player i given s_{-i}, denote by $s_i^* \in r_i(s_{-i})$, iff $\forall s_i \in S_i$, $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$.

Notice that Best response is a value-set function $r_i(s_{-i}) : S_{-i} \mapsto 2^{S_i}$, which means that Best response (BR) is not necessarily unique. When support of a BR s^* includes not only one actions, the agent must be **indifferent** between them, otherwise the agent would prefer to reduce the probability of playing at least one of actions to zero. Thus any mixture of these actions must be a BR, not only the particular mixture in s^* and similarly, if there are two pure strategies that are individually BR, any mixture of the two is necessarily also a BR.

Note 1. In general, an agent won't know what strategies the other players will adopt, thus the notion of BR does not identify an interesting set of outcomes in general case. However, we can leverage the idea of BR to define what is arguably the most central notion in non-cooperative game theory, the Nash equilibrium.

Definition 1.4: Nash equilibrium

$s = \langle s_1, s_2, \cdots, s_n \rangle$ is a Nash equilibrium iff $\forall i, s_i \in r_i(s_{-i})$.

In a mixed strategy Nash equilibrium, Player 1 should select strategy so that Player 2 feels **indifferent** for the actions in his support, which means whatever strategy player 2 chooses, the expected utility is always constant. And player 2 chooses a strategy from his indifferent support so that Player 1 feels indifferent for the actions in his support. As a result, No agent want to change his strategy is he knew what strategies the other agents were following. Since in a Nash equilibrium all of the agents **simultaneously** play BRs to each other's strategies.

2 Nash Theorem

To prove Nash Theorem, we need to add some mathematical concepts.

Definition 2.1: Convexity

set $C \subset \mathbb{R}^m$ is convex if for $\forall x, y \in C$ and $\lambda \in [0,1]$, $\lambda x + (1-\lambda) y \in C$. For vectors $\mathbf{x}^0, \cdots, \mathbf{x}^n$ and nonnegative scalars $\lambda_0, \cdots, \lambda_n$ satisfying $\sum_{i=0}^n \lambda_i = 1$, the vector $\sum_{i=0}^n \lambda_i \mathbf{x}^i$ is called a convex combination of $\mathbf{x}^0, \cdots, \mathbf{x}^n$.

Definition 2.2: Affine independence

A finite set of vectors $\{\mathbf{x}^0, \cdots, \mathbf{x}^n\}$ in an Euclidean space is affine independence if $\sum_{i=0}^n \lambda_i \mathbf{x}^i = \mathbf{0}$ and $\sum_{i=0}^n \lambda_i = 0$ imply that $\lambda_i \equiv 0, i = 1, \cdots, n$.

Definition 2.3: n-simplex

A n-simplex, denoted $\mathbf{x}^0 \cdots \mathbf{x}^n$, is the set of all convex combinations of the affine independence set of vectors $\{\mathbf{x}^0, \cdots, \mathbf{x}^n\}$

$$\mathbf{x}^0 \cdots \mathbf{x}^n = \left\{ \sum_{i=0}^n \lambda_i \mathbf{x}^i : \lambda_i \geq 0, \sum_{i=0}^n \lambda_i = 1. \right\}$$

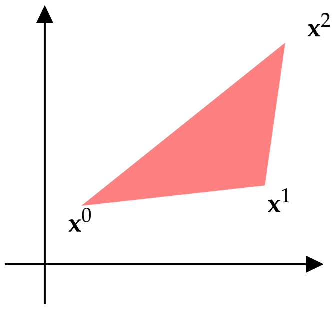
Each $\mathbf{x}^i$ is called a vertex of the simplex $\mathbf{x}^0 \cdots \mathbf{x}^n$ and each k simplex $\mathbf{x}^{i_0} \cdots \mathbf{x}^{i_k}$ is called a $k - face$ of $\mathbf{x}^0 \cdots \mathbf{x}^n$. Notice that we do note specify the dimension of $\mathbf{x}^i$, which imply that a 2-simplex could be any (narrow) triangle in $\mathbb{R}^m$ $(m \geq 2)$.

Definition 2.4: Standard n-simplex

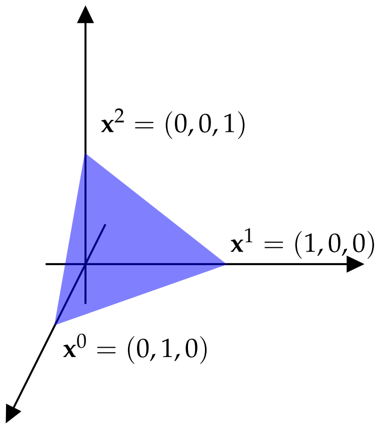
The standard n-simplex $\triangle_n$ is $\{\mathbf{y} \in \mathbb{R}^{n+1} : \sum_{i=0}^n y_i = 1, y_i \geq 0\}$.

In other words, standard n-simplex is the set of convex combinations of the $n + 1$ unit vectors $e^0, \cdots, e^n$.

2-simplex



standard 2-simplex



Definition 2.5: Simplicial subdivision

A simplicial subdivision of an n-simplex T is a finite set of simplexes $\{T\}_i$ for which $\cup_{T_i \subset T} T_i = T$, and for any $T_i, T_j \subset T$, $T_i \cap T_j$ is either $\emptyset$ or equal to a common face.

Theorem 2.1: Nash Theorem

Every finite game with mixed strategy has a Nash equilibrium.

3 Examples

We now calculate the Nash equilibrium of the Matching pennies game with mixed strategy.

Example 3.1 (Matching pennies game). Assume the general strategy profile s has following structure

	H	T
Player 1	α	$1-\alpha$
Player 2	β	$1-\beta$

Then the expected utility of Player 1 is

$$\begin{aligned} u_1(s) &= \sum_{a\in\mathcal{A}} u_1(a)\cdot \mathbb{P}\left(a|s\right) \\ &= 4\alpha\beta - 2\alpha - 2\beta + 1 \end{aligned}$$

And correspondingly,

$$u_2(s) = -4\alpha\beta + 2\alpha + 2\beta - 1,$$

For any β player 2 decided, player 1 will choose α that maximize $u_1(s)$ and minimize $u_2(s)$ at the same time since $u_1(s) = -u_2(s)$. Thus player 2 will change β if player 1 chooses α, unless $\exists\beta'$ such that u_1 is constant whatever α is. and let $\alpha = \beta'$, then u_2 is constant. thus (β', β') is Nash equilibrium. Solve the equation, we have $\beta' = 0.5$, thus the strategy profile

	H	T
Player 1	0.5	0.5
Player 2	0.5	0.5

is Nash equilibrium.

Example 3.2 (BoS). Review the Battle of sexes game

		β B	$1-\beta$ F	
α	B	2,1	0,0	$\mathbb{E}_{1B} = \beta\cdot 2 + (1-\beta)\cdot 0$
$1-\alpha$	F	0,0	1,2	$\mathbb{E}_{1F} = \beta\cdot 0 + (1-\beta)\cdot 1$
		$\mathbb{E}_{2F} = \alpha\cdot 1 + (1-\alpha)\cdot 0$	$\mathbb{E}_{2F} = \alpha\cdot 0 + (1-\alpha)\cdot 2$	

If player 1 best-responds with a mixed strategy, player 2 must choose β to make player 1 be indifferent between F and B a.k.a. $\mathbb{E}_{1B} = \mathbb{E}_{1F}$, otherwise, assume $\mathbb{E}_{1B} > \mathbb{E}_{1F}$, player 1 will prefer B than F and will add more probability from F to B until the mixed strategy degenerate to pure strategy. Correspondingly, player 1 should make $\mathbb{E}_{2B} = \mathbb{E}_{2F}$. Thus $\alpha = 2/3$ and $\beta = 1/3$. according to the definition,

$$\begin{aligned} u_1(s) &= \sum_{a\in\mathcal{A}} u_1(a)\cdot s_1(a_1)\cdot s_2(a_2) \\ &= \sum_{a_1\in A_1} \mathbb{E}_{1a_1}\cdot s_1(a_1) \\ &= s_1(B)\cdot\mathbb{E}_{1B} + s_1(F)\cdot\mathbb{E}_{1F} \\ &\overset{\mathbb{E}_{1B}+\mathbb{E}_{1F}}{=\!=\!=\!=} \mathbb{E}_{1B}\cdot(s_1(B)+s_1(F)) \\ &= \mathbb{E}_{1B}. \end{aligned}$$

Easy to check, if $\alpha = 2/3$, then $u_2(s)$ is constant, then $\beta = 1/3$ is one of the BR, or vice versa. thus $s = \langle\alpha = 2/3, \beta = 1/3\rangle$ is Nash equilibrium.

Example 3.3 (Soccer Penalty Kicks). Let's consider an example of Mixed strategies in sports and competitive games, Soccer Penalty Kicks. Assume player 1 is kicker and player 2 is goalie, who form the following game

	Left	Right
Left	0, 1	1, 0
Right	1, 0	0,1

This means if kicker and goalie choose the same direction, then the goalie will keep the door, otherwise the kicker will shoot successfully. But what if the kicker kicks worse to the right than left, for example

	Left	Right
Left	0, 1	1, 0
Right	.75, .25	0,1

The Nash equilibrium is easy to calculate:

	Left	Right
Kicker	4/7	3/7
Goalie	3/7	4/7

We can see by adjusting the strategy to keep the opponent indifferent, the Goalie takes advantage of the kicker's weak right kick and wins more often.

Example 3.4. There are 2 firms, each advertising an available job opening. Firms offer different wages: Firm 1 offers $w_1 = 4$ and 2 offers $w_2 = 6$. There are two unemployed workers looking for jobs. They simultaneously apply to either of the firms. If only one worker applies to a firm, then he/she gets the job. If both workers apply to the same firm, the firm 1 hires a worker 1 with probability α (worker 2 with $1 - \alpha$)and the firm 2 hires a worker 1 with probability β (worker 2 with $1 - \beta$), the other worker remains unemployed (and receives a payoff of 0).

Find a mixed strategy Nash Equilibrium where p is the probability that worker 1 applies to firm 1 and q is the probability that worker 2 applies to firm 1.

If worker 1 and 2 both apply firm 1, then the expected wage for worker 1 is $4\alpha + 0(1 - \alpha) = 4\alpha$; the expected wage for worker 2 is $0\alpha + 4(1 - \alpha) = 4(1 - \alpha)$. The situation that both workers choose firm 2 is similar. Thus we have game matrix

	Firm 1	Firm 2
Firm 1	$4\alpha, 4(1-\alpha)$	4,6
Firm 2	6,4	$6\beta, 6(1-\beta)$

thus we can get the Nash Equilibrium is $p = \frac{3\beta-1}{2\alpha+3\beta}$ and $q = \frac{3\beta-2}{2\alpha+3\beta-5}$. If $\alpha = \beta = .5$, then the Nash Equilibrium is $p = .5$ and $q = .5$.