General Topology
Lecture 7

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This is the Lecture note for the *General Topology* course in Spring 2020.

1 Closure, Limit Point, Continuity

Definition 1 (Convergence). Let $(X, \mathscr{T})$ be a topology space, $x \in X$ and $x_n \in X(n \in \mathbb{N})$, we say $x_n \to x$ as $n \to \infty$ if for any open nbd. U_x of x, $\exists N$, s.t. $\forall n \in \mathbb{N}, n \geq N \Rightarrow x_n \in U$.

We define

$$\overline{A}' := \{x \in X | \exists \textit{ seq. } a_n \in A(n \in \mathbb{N}), \text{ s.t. } a_n \to x \textit{ as } n \to \infty\}$$

and

$$L'_A := \{x \in X | \exists \textit{ seq. } a_n \in A \backslash \{x\} (n \in \mathbb{N}), \text{ s.t. } a_n \to x \textit{ as } n \to \infty\}.$$

Exercise 1. *Let (X, d) be a metric space, $A \subseteq X$, show that*

1. $\overline{A} = \overline{A}'$;
2. $L_A = L'_A$

Proof. 1. $\subseteq$: if $x \in \overline{A}$, then $B_{\frac{1}{n}}(x) \cap A \neq \emptyset$ for $\forall n \in \mathbb{N}$. Then we can form a seq. $x_n(n \in \mathbb{N})$ such that $x_n \in B_{\frac{1}{n}}(x) \cap A$ for $\forall n \in \mathbb{N}$. Thus for any open nbd. U_x of x, since X is metric space, $\exists r > 0$, s.t. $B_r(x) \subseteq U_x$. Let $N = \lceil \frac{1}{r} \rceil$, then for any $n \in \mathbb{N}, n \geq N \Rightarrow x_n \in B_{\frac{1}{n}}(x) \subseteq B_r(x) \subseteq U_x \Rightarrow x_n \to x$ as $n \to \infty \Rightarrow x \in \overline{A}'$.
$\supseteq$: If $x \in \overline{A}' \Rightarrow \exists$ a seq. $x_n(n \in \mathbb{R})$, s.t. $x_n \to x$ as $n \to \infty$. Thus $\forall$ open nbd. U_x of x, $\exists N$, s.t. $\forall n \in \mathbb{N}, n \geq N \Rightarrow x_n \in U_x \Rightarrow$ such $x_n \in U_x \cap A \neq \emptyset \Rightarrow x \in \overline{A}$.
2. The same as above. $\square$

Exercise 2. *Let $X \xrightarrow{f} Y$ is a map between metric spaces and $x_0 \in X$, show that f is continuous at $x_0 \Leftrightarrow \forall$ seq. $x_n \in X(n \in \mathbb{N})$, $x_n \to x$ as $n \to \infty \Rightarrow f(x_n) \to f(x)$ as $n \to \infty$.*

Proof. $\Rightarrow$: For any open nbd. V of $f(x_0), f^{-1}(V) \subseteq_{open} X$ is an open nbd. of x_0, since $x_n \to x$ as $n \to \infty$, $\exists N$ s.t. $\forall n \in \mathbb{N}, n \geq N \Rightarrow x_n \in f^{-1}(V) \Rightarrow f(x_n) \in V \Rightarrow f(x_n) \to f(x_0)$ as $n \to \infty$.
$\Leftarrow$: Form a seq. $x_n(n \in \mathbb{N})$ such that $x_n \in B_{\frac{1}{n}}(x_0)$ for any $n \in \mathbb{N}$, then $x_n \to x_0$ and $f(x_n) \to f(x_0)$ as $n \to \infty$. Thus for any open nbd. V of $f(x_0), \exists N$, s.t. $\forall n \in \mathbb{N}, n \geq N \Rightarrow f(x_n) \in V$, which means for any $x \in B_{\frac{1}{n}}(x_0), f(x) \in V \Rightarrow B_{\frac{1}{n}}(x_0) \subseteq V \Rightarrow f$ is continuous at x_0. $\square$

As we shown, given metric spaces, then we can re-define the concept of *closure, limit points* and *continuity of the map* with sequential description. But if given topology spaces, instead of metric spaces, we only have

1. $\overline{A}' \subseteq \overline{A}$;
2. $L'_A \subseteq L_A$;
3. f is continuous at $x_0 \Rightarrow \forall$ seq. $x_n \in X(n \in \mathbb{N})$, $x_n \to x$ as $n \to \infty$ then $f(x_n) \to f(x)$ as $n \to \infty$

since the proofs of these properties are independent with the features of metric space. And in general, the converses do not hold.

Exercise 3. *If X are 1-st countable topology space, $A \subseteq X$, show that $\overline{A}' = \overline{A}$ and $L'_A = L_A$.*

Proof. All we need to prove is $\overline{A} \subseteq \overline{A}'$ and $L_A \subseteq L'_A$:
1. For any $x \in X, \exists$ a countable local basis $\mathcal{B}_x$ of x such as $\mathcal{B}_x = \{V_1, V_2, \cdots\}$, thus we can form a seq. $x_n(n \in \mathbb{N})$ such that $x_n \in A \cap (\cap_{i=1}^n V_i)$ for any $n \in \mathbb{N}$. Note that $x \in \overline{A} \Rightarrow A \cap (\cap_{i=1}^n V_i) \neq \emptyset$, thus x_n exists and $x_n \in A$.
Thus for any open nbd. U of x, $\exists V_m \in \mathcal{B}_x$ such that $x \in V_m \subseteq U$, and for any $n \geq m, x_n \subseteq V_m \subseteq U \Rightarrow x_n \to x$ as $n \to \infty$. Thus $x \in \overline{A}'$.
2. The same as 1. $\square$

2 Sequentially Compact, Totally Bounded

Definition 2. Let (X, d) be a metric space, we say

1. (X, d) is a sequentially compact if every sequence in X has a convergent subsequence.
2. (X, d) is a totally bounded if $\forall \epsilon > 0, \exists$ finite set $S \subseteq X$, s.t. $X = \cup_{s \in S} B_\epsilon(s)$.

Exercise 4. *Let (X, d) be a totally bounded metric space, that is for any $n \in \mathbb{N}$, there exist a finite set $S_n \subseteq X$,* s.t. *$X = \cup_{s \in S} B_{\frac{1}{n}}(s)$, show that $S := \cup_{n \in \mathbb{N}} S_n$ is a countable dense subset in X w.r.t. d.*

Proof. S is countable is trivial, we will show that S is dense. If U is an un-empty open set in X, then $\exists x \in U$ and $\exists r > 0$, s.t. $B_r(x) \subseteq U$, define $N = \lceil \frac{1}{r} \rceil$ then for any given

$n \geq N$, $x \in U \subseteq \cup_{s \in S_n} B_{\frac{1}{n}}(s)$. And $\exists s' \in S_n$, s.t.

$$x \in B_{\frac{1}{n}}(s') \Rightarrow s' \in B_{\frac{1}{n}}(x) \subseteq B_r(x) \subseteq U$$

since $s' \in S_n \subseteq S$, $s' \in U \cap S \Rightarrow U \cap S \neq \emptyset \Rightarrow S$ is dense. $\square$

Thus Total boundedness $\Rightarrow$ separability (and hence 2-nd countability and Lindelof since X is a metric space).

Proposition 1. *Let (X, d) be metric space, the following are equivalent:*

1. *X is compact (w.r.t $\mathscr{T}_d$);*
2. *X is sequentially compact (w.r.t. d);*
3. *X is complete and totally bounded (w.r.t. d).*

Proof. $1 \Rightarrow 2$: Assume that $\exists$ seq. $x_n \in X (n \in \mathbb{N})$ such that any subseq. of it is not convergent, that is $\forall x \in X, x$ is not the limit of any subseq. of $x_n (n \in \mathbb{N})$. Thus for any $x \in X, \exists$ open nbd. U_x, s.t. $\{n \in \mathbb{N} | x_n \in U_x\}$ is finite.

Remark 1. We highlight that the index number $\{n \in \mathbb{N} | x_n \in U_x\}$ is finite instead of the point. Since the same point can be encoded by multi index number, and we want to expile that case.

Since X is compact, $X = \cup_{x \in X} U_x \Rightarrow \exists$ finite $X_0 \subseteq X$, s.t. $X = \cup_{x \in X_0} U_x$. Thus $\mathbb{N} = \{n \in \mathbb{N} | x_n \in X\} = \cup_{x \in X_0} \{n \in \mathbb{N} | x_n \in U_x\}$ which leads to a contradiction since $\cup_{x \in X_0} \{n \in \mathbb{N} | x_n \in U_x\}$ is finite.

$2 \Rightarrow 3$: Let $x_n (n \in \mathbb{N})$ be a Cauchy seq. in X, it is suffices to show that $x_n (n \in \mathbb{N})$ has a convergent subseq. and this is implied by 2.

Suppose (X, d) is not totally bounded, then $\exists \epsilon > 0$, such that pick any $x_1 \in X$ we have that

$$B_\epsilon(x_1) \subsetneq X \Rightarrow X \backslash B_\epsilon(x_1) \neq \emptyset,$$

and pick $x_2 \in X \backslash B_\epsilon(x_1)$ have

$$B_\epsilon(x_1) \cup B_\epsilon(x_2) \subsetneq X \Rightarrow X \backslash (B_\epsilon(x_1) \cup B_\epsilon(x_2)) \neq \emptyset,$$

and pick $x_3 \in X \backslash (B_\epsilon(x_1) \cup B_\epsilon(x_2))$, and so on.

Thus we can find a seq. $x_1, x_2, \cdots$ such that $d(x_i, x_j) \geq \epsilon$ for $i \neq j$ (since $x_i \in X \backslash B_\epsilon(x_j)$). Thus any subseq. of $x_n (n \in \mathbb{N})$ is not Cauchy seq. and hence is not convergent, which leads to a contradiction with 2.

$3 \Rightarrow 2$: Let $x_n (n \in \mathbb{N})$ be a seq. in X, since (X, d) is totally bounded $\Rightarrow$ For any given $n \in \mathbb{N}$, X can be covered by finitely many $\frac{1}{n}$ balls.

Thus X can be covered by finite many 1-balls, $x_n \in X (n \in \mathbb{N}) \Rightarrow \exists$ a 1-ball B_1, s.t.

$$\{n \in \mathbb{N} | x_n \in B_1\} \text{ is infinite;}$$

X can be covered by finite many 1/2-balls, and so do B_1, thus $\exists$ a 1/2-ball B_2, s.t.

$$\{n \in \mathbb{N} | x_n \in B_1 \cap B_2\} \text{ is infinite.}$$

And if $\exists$ 1/m-ball B_m, s.t. $\{n \in \mathbb{N} | x_n \in \cap_{i=1}^m B_i\}$ is infinite, then since $\cap_{i=1}^m B_i$, which covers infinite points of the seq., can be covered by finitely many 1/(m+1) balls, there $\exists$ a 1/(m+1) ball B_{m+1} s.t.

$$\{n \in \mathbb{N} | x_n \in \cap_{i=1}^{m+1} B_i\} \text{ is infinite.}$$

Thus $\exists$ subseq. $x_{n_k}(k \in \mathbb{N})$, s.t. $x_{n_k} \in B_1 \cap \cdots \cap B_k$ for every $k \in \mathbb{N}$. And for every $l, l' \geq k, x_{n_l}, x_{n'_l} \in B_k$ and hence $d(x_{n_l}, x_{n'_l}) \leq \frac{1}{k}$. Thus $x_{n_k}(k \in \mathbb{N})$ is a Cauchy seq., and since X is complete, $x_{n_k}(k \in \mathbb{N})$ is convergent.

Remark 2. Refer to the proof of *Bolzano-Weierstrass theorem* in *Introduction to Topology, Lecture 8,9*.

$2 \Rightarrow 1$: Let $\mathcal{F}$ be a family of closed subsets of X which satisfies the FIP, we need to show that $\cap\mathcal{F} \neq \emptyset$. Suppose that $\cap\mathcal{F} = \emptyset$. Then $\{X \backslash C | C \in \mathcal{F}\}$ is an open cover of X, since X is sequentially compact, then X is totally bounded, and hence X is Lindelof countable.

Thus $\exists$ a countable $\mathcal{F}_0 = \{C_1, C_2, \cdots\} \subseteq \mathcal{F}$ s.t. $\{X \backslash C | C \in \mathcal{F}_0\}$ still cover X, and hence $\cap_{C \in \mathcal{F}_0} C = \emptyset$. Note that $\mathcal{F}$ satisfies FIP, thus $\mathcal{F}_0$ satisfies FIP as well. Thus any finite intersection of the elements in $\mathcal{F}_0$ is not empty, thus exists

$$\begin{aligned}
&x_1 \in C_1, \\
&x_2 \in C_1 \cap C_2, \\
&\quad \cdots \\
&x_n \in \cap_{i=1}^n C_i, \\
&\quad \cdots
\end{aligned}$$

which forms a seq. $x_n(n \in \mathbb{N})$ in X, and since X is seq. cpt., there exists a convergent subseq. $x_{n_k}(k \in \mathbb{N})$. And $x_{n_k} \to x \in X$ as $k \to \infty$.

Note that since $C_n(n \in \mathbb{N})$ are closed, then for any given $N \in \mathbb{N}, \cap_{i=1}^N C_i$ is still closed. Since $x_{n_k} \in \cap_{i=1}^{n_k} C_i$ and for any $k \geq$ given $K \in \mathbb{N}$ have that $x_{n_k} \in \cap_{i=1}^{n_K} C_i$ and $\cap_{i=1}^{n_K} C_i$ is closed $\Rightarrow x \in \cap_{i=1}^{n_K} C_i$ for any $K \in \mathbb{N}$. Since $n_k \to \infty$ as $k \to \infty$, thus $x \in \cap_{i=1}^N C_i$ for any $N \in \mathbb{N} \Rightarrow x \in \lim_{n\to\infty} \cap_{i=1}^n C_i = \cap_{C \in \mathcal{F}_0} C \Rightarrow \cap_{C \in \mathcal{F}_0} C \neq \emptyset$ which leads to the contradiction with the assumption. $\square$

Exercise 5. *Let (X, d) be a complete metric space, $K \subseteq X$, show that*

1. *(K, d) is complete $\Leftrightarrow K \subseteq_{close} X$;*
2. *(K, d) is compact $\Leftrightarrow K \subseteq_{close} X$ and (K, d) is totally bounded;*
3. *(K, d) is totally bounded $\Leftrightarrow \forall \epsilon > 0, \exists$ finite set $S \subseteq X$,* s.t. $K \subseteq \cup_{s \in S} B_\epsilon(s)$.

Proof. 1. This will be proved by demonstrating the contrapositive: K is not complete if and only if K is not closed.
$\Rightarrow$: Suppose that K is not complete. Then there exists a Cauchy sequence x_n in K such that the limit $x = \lim_{n\to\infty} x_n$, which exists in the complete metric space X, is not a member of K.
For all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ has $d(x, x_n) < \epsilon$, and hence $X \backslash K$ is not open (if $X \backslash K$ is open then $\exists r > 0$, s.t. $B_r(x) \subseteq X \backslash K \Rightarrow d(x, x_n) > r$ for all $n \in \mathbb{N}$). Therefore, K is not closed.
$\Leftarrow$: Suppose that K is not closed. Then $X \backslash K$ is not open. Therefore, there exists a $x \in X \backslash K$ such that for all $\epsilon > 0$, there exists a $y \in K$ such that $d(x, y) < \epsilon$. Thus we can form a seq. $y_n (n \in \mathbb{N})$ in K such that $y_n \in K \cap B_{\frac{1}{n}}(x)$ for all $n \in \mathbb{N}$ and hence $d(x, y_n) < \frac{1}{n}$.
Now, we show that y_n is a Cauchy sequence. Given an $\epsilon > 0$, let $N \in \mathbb{N}$ be such that for all $n \geq N$ has $d(x, y_n) < \frac{\epsilon}{2}$. Let $m, n \geq N$, then by the triangle inequality:

$$d(y_n, y_m) \leq d(x, y_m) + d(x, y_n) \leq \epsilon,$$

Hence y_n is a Cauchy sequence. Because (X, d) is a complete metric space by assumption, the limit $\lim_{n\to\infty} y_n$ exists and is in X. Denote this limit by y. By the definition of y_n we have that $\lim_{n\to\infty} d(x, y_n) = 0$. From Distance Function of Metric Space is Continuous and Composite of Continuous Mappings is Continuous we have $d(x, y) = 0 \Rightarrow x = y$, since $x \notin K \Rightarrow y \notin K \Rightarrow K$ is not complete.
2. trivial
3. $\Rightarrow$ is trivial; $\Leftarrow$: Since given any $\epsilon > 0, \exists$ finite $S \subseteq X$ s.t. $K \subseteq \cup_{s \in S} B_\epsilon(s)$. Define $S_0 = \{s_1, \cdots, s_n\} \subseteq S$ where $B_\epsilon(s) \cap K \neq \varnothing$ for any $s \in S_0$. Then pick $k_i \in K \cap B_\epsilon(s_i)$ for $i = 1, \cdots, 2$, then we have that

$$k_i \in B_\epsilon(s_i) \Rightarrow d(s_i, k_i) < \epsilon,$$

thus for any $k \in K$, $\exists s_i \in S_0$, s.t. $k \in B_\epsilon(s_i) \Rightarrow d(k, s_i) < \epsilon$, thus

$$d(k, k_i) \leq d(k, s_i) + d(s_i + k_i) \leq 2\epsilon$$

thus $k \in B_{2\epsilon}(k_i) \Rightarrow K \subseteq \cup_{i=1}^n B_{2\epsilon}(k_i) \Rightarrow K$ is totally bounded. □

Remark 3. Let (X, d) be a metric space, define $d'(x_1, x_2) := \min\{1, d(x_1, x_2)\}$, then d' is still a metric. And

- $\{$the Cauchy seq.s in $(X, d)\} = \{$the Cauchy seq.s in $(X, d')\}$
- $\mathscr{T}_d = \mathscr{T}_{d'}$
- (X, d') is always a **bounded** metric space.