Introduction to Analysis

Lecture 1

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1 Real number

Definition 1. Let $S \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, we say that
1. r is an upper (lower) bound of S if $\forall s \in S, r \geq (\leq) s$;
2. r is the greatest (least) element of S if r is an upper (lower) bound of S and $r \in S$, denoted by $r = \max S(\min S)$.
3. r is the least upper (greatest lower) bound of S if r is the least (greatest) element of the set of upper (lower) bound of S, denoted by $r = \sup S(\inf S)$.

Remark 1. r is a least upper bound of S means any element of S which is smaller than r is not an upper bound of S, that is $\forall \epsilon > 0, \exists s \in S,$ s.t. $r - \epsilon < s \leq r$.

We write $\sup S = \infty$ ($\inf S = -\infty$) if and only if S has no upper (lower) bound. If this is the case we say $\sup S(\inf S)$ does not exist. We say S is bounded from above (below) iff S has an upper (lower) bound.

Definition 2 (Dedekind Cut)**.** Let $A, B \subseteq \mathbb{R}$, we say that (A, B) is a Dedekind cut if
1. $A, B \neq \emptyset$;
2. $A \cup B = \mathbb{R}$;
3. $\forall a \in A, b \in B, a < b$.

We usually call $A(B)$ the lower (upper) part of (A, B).

We assume that $\mathbb{R}$ has the **Dedekind's Gapless Property**: If (A, B) is a Dedekind cut of $\mathbb{R}$, then exactly one of the following happens:
1. $\max A$ exists but $\min B$ does not;
2. $\min B$ exists but $\max A$ does not.

We call $\max A$ in 1. (or $\min B$ in 2.) the **cutting** of (A, B).

Exercise 1. *We may define Dedekind cuts on $\mathbb{Q}$ and $\mathbb{Z}$ similarly, does Dedekind Gapless Property hold for $\mathbb{Q}$ and $\mathbb{Z}$?*

Proof. 1. Let $A := \{q \in \mathbb{Q} | q^2 < 2\}, B := \{q \in \mathbb{Q} | q^2 > 2\}$. It is direct to see that $A, B \neq \emptyset$.
If $\exists r \in \mathbb{Q}$, s.t. $r^2 = 2$, then $\exists p, q \in \mathbb{N}$, s.t. $r = p/q$ and p, q are not both even. Then $p^2/q^2 = 2 \Rightarrow p^2 = 2q^2 \Rightarrow p^2$ is even $\Rightarrow p$ is even $\Rightarrow p^2$ can be divided by $4 \Rightarrow q^2$ can be divided by $2 \Rightarrow q^2$ is even $\Rightarrow q$ is even, which leads to a contradiction. Thus $\forall r \in \mathbb{Q}, r^2 \neq 2$. Thus $A \cup B = \mathbb{Q}$.
Finally $\forall q_a \in A, q_b \in B$ one has $q_a^2 < 2 < q_b^2 \Rightarrow q_a < q_b$. Thus (A, B) is a Dedekind cut of $\mathbb{Q}$. It is direct to see that (A, B) has no Dedekind's gapless property:
For example, if $p \in A$, then $p \in \mathbb{Q}$ and $p^2 < 2$, put $\epsilon = 2 - p^2$, then we should find a $q \in \mathbb{Q}$ such that $q^2 < 2$ and $q > p$, which means

$$p^2 < q^2 < 2$$

we consider there exists a function r of p, ϵ, such that $r > 0$ and $r \in \mathbb{Q}$, and put $q = p + r$, thus $q > p$ and $q \in \mathbb{Q}$, we now prove that $q^2 < 2$. Since

$$2 - q^2 = 2 - (p^2 + r^2 + 2pr) = \epsilon - r^2 - 2pr$$

We now try to find a suitable structure of r to make $r^2 + 2pr < \epsilon$. Since $p > 0$ and $\epsilon = 2 - p^2$, $0 < \epsilon < 2$. Consider $r = \epsilon/2$ then

$$r^2 = \frac{\epsilon^2}{4} = \frac{\epsilon}{2} \cdot \frac{\epsilon}{2} < \frac{\epsilon}{2}.$$

Consider $r = \epsilon/((2p+1)2) < \epsilon/2$ and

$$2pr = 2p \cdot \frac{\epsilon}{(2p+1)2} < \frac{\epsilon}{2},$$

then we have $r^2 + 2pr < \epsilon$ and

$$q^2 < 2,$$

by defining

$$q = p + \frac{\epsilon}{2(2p+1)} = \frac{3p^2 + 2p + 2}{4p + 2},$$

thus there is no maximal element in A and correspondingly, there is no minimal element in B as well.
2. trivial. □

Theorem 1 (Weierstrass Theorem). *Let $\emptyset \neq S \subseteq \mathbb{R}$, if S has an upper bound, then* $\sup S$ *exists.*

Proof. Let B be the set of all upper bound of S, and define $A := \mathbb{R}\backslash B$.
CLAIM 1: (A, B) is a Dedekind cut of $\mathbb{R}$:
1. $S \neq \emptyset \Rightarrow \forall s \in S, s-1 \notin B \Rightarrow s-1 \in A \Rightarrow A \neq \emptyset$; And S has an upper bound $\Rightarrow B \neq \emptyset$;
2. $A = \mathbb{R}\backslash R \Rightarrow A \cup B = \mathbb{R}$;
3. If $\exists a \in A, b \in B,$ s.t. $a \geq b$ where b is an upper bound of S while a is not, thus $\exists s' \in S,$ s.t. $a < s' \leq b < a$, which leads to a contradiction. Thus $\forall a \in A, b \in B$ one has $a < b$.

CLAIM 2: $\min B$ exists:
If $\min B \nexists$, then by Dedekind's gapless property, $\max A \exists$, denoted by a_0. $a_0 \in A \Leftrightarrow a_0 \notin B \Leftrightarrow a_0$ is not an upper bound of $S \Leftrightarrow \exists s_0 \in S,$ s.t. $a_0 < s$. Choose $x \in \mathbb{R}$ such that $a_0 < x < s_0$, thus $\max A < x \Rightarrow x \in B \Rightarrow x$ is an upper bound of S but $x < s_0$ which leads to a contradiction. □

Exercise 2 (Archimedean Property)**.** *Show that $\forall r \in \mathbb{R}, r > 0 \Rightarrow \exists n \in \mathbb{N},$ s.t. $n > r$. (or say $\exists n \in \mathbb{N},$ s.t. $1/n < r$).*

Proof. Let $r \in \mathbb{R}$, $S := \{n \in \mathbb{N} | n \leq r\}$, since $r > 0, 0 \in S \Rightarrow S \neq \emptyset$. Then $S \subseteq \mathbb{R}$ and S is bounded above (by r), thus S has a least upper bound in $\mathbb{R}$, let $s = \sup S$.
Now consider the number $s-1$. Since s is the supremum of S, $s-1$ cannot be an upper bound of S by definition. Thus $\exists m \in S$ such that

$$m > s - 1 \Rightarrow m + 1 > s$$

But as $m \in \mathbb{N}$, it follows that $m+1 \in \mathbb{N}$. Because $m+1 > s$, it follows that $m+1 \notin S$ and so $m+1 > r$. Furthermore, for $\forall r > 0, 1/r > 0$ then $\exists n \in \mathbb{N},$ s.t. $n > 1/r \Rightarrow 1/n < r$. □

2 Sequence

Definition 3 (Convergence)**.** Let $a_n (n \in \mathbb{N})$ be a sequence in $\mathbb{R}$ and $l \in \mathbb{R}$, we say that a_n converges to l as $n \to \infty$ if $\forall \epsilon > 0, \exists N \in \mathbb{N},$ s.t. $\forall n \geq N, |a_n - l| < \epsilon$, denoted by $a_n \to l$ (as $n \to \infty$).

If such l exists, we call it the limit of $\{a_n\}$ and denote is as $\lim_{n\to\infty} a_n = l$, and call $\{a_n\}$ a convergent sequence; otherwise we call it a **divergent** sequence. Furthermore, we say $\lim_{n\to\infty} a_n = \infty$ if $\forall M > 0, \exists N \in \mathbb{N},$ s.t. $\forall n \geq N, a_n \geq M$.

Exercise 3. *Show that*
1. $\lim_{n\to\infty} a_n = l$ *and* $\lim_{n\to\infty} a_n = m \Rightarrow l = m$;
2. $a_n(n \in \mathbb{N})$ *is convergent* $\Rightarrow \{a_n | n \in \mathbb{N}\}$ *is bounded;*
3. *if* $a_n < b_n$ *for all* $n \in \mathbb{N}, \lim_{n\to\infty} a_n = l, \lim_{n\to\infty} b_n = m \Rightarrow l \leq m$.

Proof. 1. $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} a_n = m \Rightarrow$ for $\forall\epsilon > 0$, $\exists N, M \in \mathbb{N}$, s.t. $\forall n \geq N$ one has $|a_n - l| < \epsilon/2$ and $\forall n \geq M$ has $|a_n - m| < \epsilon/2$, thus for $\forall n \geq \max\{N, M\}$, has

$$|l - m| = |l - a_n + a_n - m| \leq |a_n - l| + |a_n - m| < \epsilon$$

holds for $\forall\epsilon > 0 \Rightarrow l = m$.
2. Suppose $a_n \to l$ as $n \to \infty$, then given an $\epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ we have $|a_n - l| < \epsilon \Rightarrow l - \epsilon < a_n < l + \epsilon$, thus a_n has upper bound

$$\max\{a_1, \cdots, a_{n-1}, l + \epsilon\},$$

and lower bound

$$\min\{a_1, \cdots, a_{n-1}, l - \epsilon\}.$$

3. if $l > m$, let $\epsilon = l - m$, then $\exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ has $|a_n - l| < \epsilon/2$ and $|b_n - m| < \epsilon/2$ thus

$$a_n < \frac{l + m}{2} < b_n,$$

which leads to a contradiction, thus $l \leq m$. $\square$

Remark 2. Changing or removing finitely many terms in $a_n (n \in \mathbb{N})$ does not effect a_n's being convergent (and its limit)/ divergent.

Proposition 1. *If* $\lim_{n\to\infty} a_n = l$ *and* $\lim_{n\to\infty} b_n = m$ *then*
1. $\lim_{n\to\infty}(a_n \pm b_n) = l \pm m$*;*
2. $\lim_{n\to\infty} a_n b_n = lm$*;*
3. if $m \neq 0$ *and* $b_n \neq 0$ *for all but finitely many* n *then* $\lim_{n\to\infty} a_n/b_n = l/m$*.*

Proof. 1. For $\forall\epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N, |a_n - l| \leq \epsilon/2$ and $\exists M \in \mathbb{N}$, s.t. $\forall n \geq M, |b_n - m| \leq \epsilon/2$, thus $\forall n \geq \max\{N, M\}$, one has

$$\begin{aligned} |(a_n \pm b_n) - (l \pm m)| &= |(a_n - l) \pm (b_m - m)| \\ &\leq |a_n - l| + |b_n - m| \\ &\leq \epsilon, \end{aligned}$$

thus $(a_n \pm b_n) \to l \pm m$ as $n \to \infty$.
2. Since a_n, b_n are convergent, thus they are bounded. Choose $C > 0$ such that $|b_n| \leq C$ for all $n \in \mathbb{N}$ and $|l| \leq C$, then for $\forall\epsilon > 0, \exists N, M \in \mathbb{N}$, s.t. $\forall n \geq N$ one has $|a_n - l| \leq \epsilon/(2C)$ and $\forall n \geq M$ has $|b_n - m| \leq \epsilon/(2C)$, thus $\forall n \geq \max\{N, M\}$ one has

$$\begin{aligned} |a_n b_n - lm| &= |a_n b_n - l b_n + l b_n - lm| \\ &\leq |a_n - l| \cdot |b_n| + |b_n - m| \cdot |l| \\ &\leq (|a_n - l| + |b_n - m|) \cdot |C| \\ &\leq \epsilon \end{aligned}$$

thus $a_n b_n \to lm$.
3. all we need to show is $\lim_{n\to\infty} 1/b_n = 1/m$ which is trivial. $\square$