

INTRODUCTION TO ANALYSIS

COLLECTION

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Abstract

THIS IS THE COLLECTION OF LECTURE NOTES FOR THE *Introduction to Analysis* COURSE IN SPRING 2019. THE PURPOSE OF THIS COURSE IS TO BRIDGE THE GAP BETWEEN *Calculus* AND *Advanced Calculus*.

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Chapter 1

Completeness of the real numbers

1.1 Real number

Definition 1. Let $S \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, we say that

1. r is an upper (lower) bound of S if $\forall s \in S, r \geq (\leq) s$;
2. r is the greatest (least) element of S if r is an upper (lower) bound of S and $r \in S$, denoted by $r = \max S$ ($\min S$).
3. r is the least upper (greatest lower) bound of S if r is the least (greatest) element of the set of upper (lower) bound of S , denoted by $r = \sup S$ ($\inf S$).

Remark 1. r is a least upper bound of S means any element of S which is smaller than r is not an upper bound of S , that is $\forall \epsilon > 0, \exists s \in S$, s.t. $r - \epsilon < s \leq r$.

We write $\sup S = \infty$ ($\inf S = -\infty$) if and only if S has no upper (lower) bound. If this is the case we say $\sup S$ ($\inf S$) does not exist. We say S is bounded from above (below) iff S has an upper (lower) bound.

Definition 2 (Dedekind Cut). Let $A, B \subseteq \mathbb{R}$, we say that (A, B) is a Dedekind cut if

1. $A, B \neq \emptyset$;
2. $A \cup B = \mathbb{R}$;
3. $\forall a \in A, b \in B, a < b$.

We usually call $A(B)$ the lower (upper) part of (A, B) .

We assume that \mathbb{R} has the **Dedekind's Gapless Property**: If (A, B) is a Dedekind cut of \mathbb{R} , then exactly one of the following happens:

1. $\max A$ exists but $\min B$ does not;
2. $\min B$ exists but $\max A$ does not.

We call $\max A$ in 1. (or $\min B$ in 2.) the **cutting** of (A, B) .

Exercise 1. We may define Dedekind cuts on \mathbb{Q} and \mathbb{Z} similarly, does Dedekind Gapless Property hold for \mathbb{Q} and \mathbb{Z} ?

Proof. 1. Let $A := \{q \in \mathbb{Q} | q^2 < 2\}, B := \{q \in \mathbb{Q} | q^2 > 2\}$. It is direct to see that $A, B \neq \emptyset$.

If $\exists r \in \mathbb{Q}$, s.t. $r^2 = 2$, then $\exists p, q \in \mathbb{N}$, s.t. $r = p/q$ and p, q are not both even. Then $p^2/q^2 = 2 \Rightarrow p^2 = 2q^2 \Rightarrow p^2$ is even $\Rightarrow p$ is even $\Rightarrow p^2$ can be divided by 4 $\Rightarrow q^2$ can be divided by 2 $\Rightarrow q^2$ is even $\Rightarrow q$ is even, which leads to a contradiction. Thus $\forall r \in \mathbb{Q}, r^2 \neq 2$. Thus $A \cup B = \mathbb{Q}$.

Finally $\forall q_a \in A, q_b \in B$ one has $q_a^2 < 2 < q_b^2 \Rightarrow q_a < q_b$. Thus (A, B) is a Dedekind cut of \mathbb{Q} . It is direct to see that (A, B) has no Dedekind's gapless property:

For example, if $p \in A$, then $p \in \mathbb{Q}$ and $p^2 < 2$, put $\epsilon = 2 - p^2$, then we should find a $q \in \mathbb{Q}$ such that $q^2 < 2$ and $q > p$, which means

$$p^2 < q^2 < 2$$

we consider there exists a function r of p, ϵ , such that $r > 0$ and $r \in \mathbb{Q}$, and put $q = p + r$, thus $q > p$ and $q \in \mathbb{Q}$, we now prove that $q^2 < 2$. Since

$$2 - q^2 = 2 - (p^2 + r^2 + 2pr) = \epsilon - r^2 - 2pr$$

We now try to find a suitable structure of r to make $r^2 + 2pr < \epsilon$. Since $p > 0$ and $\epsilon = 2 - p^2, 0 < \epsilon < 2$. Consider $r = \epsilon/2$ then

$$r^2 = \frac{\epsilon^2}{4} = \frac{\epsilon}{2} \cdot \frac{\epsilon}{2} < \frac{\epsilon}{2}.$$

Consider $r = \epsilon / ((2p + 1)2) < \epsilon/2$ and

$$2pr = 2p \cdot \frac{\epsilon}{(2p + 1)2} < \frac{\epsilon}{2},$$

then we have $r^2 + 2pr < \epsilon$ and

$$q^2 < 2,$$

by defining

$$q = p + \frac{\epsilon}{2(2p + 1)} = \frac{3p^2 + 2p + 2}{4p + 2},$$

thus there is no maximal element in A and correspondingly, there is no minimal element in B as well.

2. trivial. □

Theorem 1 (Weierstrass Theorem). *Let $\emptyset \neq S \subseteq \mathbb{R}$, if S has an upper bound, then $\sup S$ exists.*

Proof. Let B be the set of all upper bound of S , and define $A := \mathbb{R} \setminus B$.

CLAIM 1: (A, B) is a Dedekind cut of \mathbb{R} :

1. $S \neq \emptyset \Rightarrow \forall s \in S, s-1 \notin B \Rightarrow s-1 \in A \Rightarrow A \neq \emptyset$; And S has an upper bound $\Rightarrow B \neq \emptyset$;
2. $A = \mathbb{R} \setminus B \Rightarrow A \cup B = \mathbb{R}$;
3. If $\exists a \in A, b \in B$, s.t. $a \geq b$ where b is an upper bound of S while a is not, thus $\exists s' \in S$, s.t. $a < s' \leq b < a$, which leads to a contradiction. Thus $\forall a \in A, b \in B$ one has $a < b$.

CLAIM 2: $\min B$ exists:

If $\min B \nexists$, then by Dedekind's gapless property, $\max A \exists$, denoted by a_0 . $a_0 \in A \Leftrightarrow a_0 \notin B \Leftrightarrow a_0$ is not an upper bound of $S \Leftrightarrow \exists s_0 \in S$, s.t. $a_0 < s_0$. Choose $x \in \mathbb{R}$ such that $a_0 < x < s_0$, thus $\max A < x \Rightarrow x \in B \Rightarrow x$ is an upper bound of S but $x < s_0$ which leads to a contradiction. \square

Exercise 2 (Archimedean Property). Show that $\forall r \in \mathbb{R}, r > 0 \Rightarrow \exists n \in \mathbb{N}$, s.t. $n > r$. (or say $\exists n \in \mathbb{N}$, s.t. $1/n < r$).

Proof. Let $r \in \mathbb{R}$, $S := \{n \in \mathbb{N} | n \leq r\}$, since $r > 0, 0 \in S \Rightarrow S \neq \emptyset$. Then $S \subseteq \mathbb{R}$ and S is bounded above (by r), thus S has a least upper bound in \mathbb{R} , let $s = \sup S$.

Now consider the number $s - 1$. Since s is the supremum of S , $s - 1$ cannot be an upper bound of S by definition. Thus $\exists m \in S$ such that

$$m > s - 1 \Rightarrow m + 1 > s$$

But as $m \in \mathbb{N}$, it follows that $m + 1 \in \mathbb{N}$. Because $m + 1 > s$, it follows that $m + 1 \notin S$ and so $m + 1 > r$. Furthermore, for $\forall r > 0, 1/r > 0$ then $\exists n \in \mathbb{N}$, s.t. $n > 1/r \Rightarrow 1/n < r$. \square

1.2 Sequence

Definition 3 (Convergence). Let $a_n (n \in \mathbb{N})$ be a sequence in \mathbb{R} and $l \in \mathbb{R}$, we say that a_n converges to l as $n \rightarrow \infty$ if $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N, |a_n - l| < \epsilon$, denoted by $a_n \rightarrow l$ (as $n \rightarrow \infty$).

If such l exists, we call it the limit of $\{a_n\}$ and denote it as $\lim_{n \rightarrow \infty} a_n = l$, and call $\{a_n\}$ a convergent sequence; otherwise we call it a **divergent** sequence. Furthermore, we say $\lim_{n \rightarrow \infty} a_n = \infty$ if $\forall M > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N, a_n \geq M$.

Exercise 3. Show that

1. $\lim_{n \rightarrow \infty} a_n = l$ and $\lim_{n \rightarrow \infty} a_n = m \Rightarrow l = m$;
2. $a_n (n \in \mathbb{N})$ is convergent $\Rightarrow \{a_n | n \in \mathbb{N}\}$ is bounded;
3. if $a_n < b_n$ for all $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} a_n = l, \lim_{n \rightarrow \infty} b_n = m \Rightarrow l \leq m$.

Proof. 1. $\lim_{n \rightarrow \infty} a_n = l$ and $\lim_{n \rightarrow \infty} a_n = m \Rightarrow$ for $\forall \epsilon > 0, \exists N, M \in \mathbb{N}$, s.t. $\forall n \geq N$

one has $|a_n - l| < \epsilon/2$ and $\forall n \geq M$ has $|a_n - m| < \epsilon/2$, thus for $\forall n \geq \max\{N, M\}$, has

$$|l - m| = |l - a_n + a_n - m| \leq |a_n - l| + |a_n - m| < \epsilon$$

holds for $\forall \epsilon > 0 \Rightarrow l = m$.

2. Suppose $a_n \rightarrow l$ as $n \rightarrow \infty$, then given an $\epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ we have $|a_n - l| < \epsilon \Rightarrow l - \epsilon < a_n < l + \epsilon$, thus a_n has upper bound

$$\max\{a_1, \dots, a_{N-1}, l + \epsilon\},$$

and lower bound

$$\min\{a_1, \dots, a_{N-1}, l - \epsilon\}.$$

3. if $l > m$, let $\epsilon = l - m$, then $\exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ has $|a_n - l| < \epsilon/2$ and $|b_n - m| < \epsilon/2$ thus

$$a_n < \frac{l + m}{2} < b_n,$$

which leads to a contradiction, thus $l \leq m$. \square

Remark 2. Changing or removing finitely many terms in $a_n (n \in \mathbb{N})$ does not effect a_n 's being convergent (and its limit)/ divergent.

Proposition 1. If $\lim_{n \rightarrow \infty} a_n = l$ and $\lim_{n \rightarrow \infty} b_n = m$ then

1. $\lim_{n \rightarrow \infty} (a_n \pm b_n) = l \pm m$;
2. $\lim_{n \rightarrow \infty} a_n b_n = lm$;
3. if $m \neq 0$ and $b_n \neq 0$ for all but finitely many n then $\lim_{n \rightarrow \infty} a_n / b_n = l / m$.

Proof. 1. For $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N, |a_n - l| \leq \epsilon/2$ and $\exists M \in \mathbb{N}$, s.t. $\forall n \geq M, |b_n - m| \leq \epsilon/2$, thus $\forall n \geq \max\{N, M\}$, one has

$$\begin{aligned} |(a_n \pm b_n) - (l \pm m)| &= |(a_n - l) \pm (b_n - m)| \\ &\leq |a_n - l| + |b_n - m| \\ &\leq \epsilon, \end{aligned}$$

thus $(a_n \pm b_n) \rightarrow l \pm m$ as $n \rightarrow \infty$.

2. Since a_n, b_n are convergent, thus they are bounded. Choose $C > 0$ such that $|b_n| \leq C$ for all $n \in \mathbb{N}$ and $|l| \leq C$, then for $\forall \epsilon > 0, \exists N, M \in \mathbb{N}$, s.t. $\forall n \geq N$ one has $|a_n - l| \leq \epsilon/(2C)$ and $\forall n \geq M$ has $|b_n - m| \leq \epsilon/(2C)$, thus $\forall n \geq \max\{N, M\}$ one has

$$\begin{aligned} |a_n b_n - lm| &= |a_n b_n - l b_n + l b_n - lm| \\ &\leq |a_n - l| \cdot |b_n| + |b_n - m| \cdot |l| \\ &\leq (|a_n - l| + |b_n - m|) \cdot |C| \\ &\leq \epsilon \end{aligned}$$

thus $a_n b_n \rightarrow lm$.

3. all we need to show is $\lim_{n \rightarrow \infty} 1/b_n = 1/m$ which is trivial. \square

Exercise 4 (Squeeze theorem). If $\lim_{n \rightarrow \infty} a_n = l$ and $\lim_{n \rightarrow \infty} b_n = m$ and $a_n \leq c_n \leq b_n$, show that $l = m \Rightarrow \lim_{n \rightarrow \infty} c_n = l$.

Proof. Since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = l$, for $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ has $|a_n - l| < \epsilon/3$ and $|b_n - l| < \epsilon/3$. And since $a_n \leq c_n \leq b_n$, we have that $0 \leq c_n - a_n \leq b_n - a_n$. Thus for $\forall n \geq N$, we have

$$\begin{aligned} |c_n - l| &= |c_n - a_n + a_n - l| \\ &\leq |c_n - a_n| + |a_n - l| \\ &\leq |b_n - a_n| + |a_n - l| \\ &= |b_n - l + l - a_n| + |a_n - l| \\ &\leq |b_n - l| + 2|a_n - l| \\ &< \epsilon. \end{aligned}$$

thus $\lim_{n \rightarrow \infty} c_n = l$. □

Exercise 5. If $a > 1$ show that $\lim_{n \rightarrow \infty} 1/a^n = 0$.

Proof. Since $a > 1 \Rightarrow b := a - 1 > 0$, thus

$$0 \leq \frac{1}{a^n} = \frac{1}{(1+b)^n} \leq \frac{1}{1+nb} \rightarrow 0$$

as $n \rightarrow \infty$, thus $\lim_{n \rightarrow \infty} 1/a^n = 0$ by Squeeze theorem. □

Definition 4. A seq. $a_n (n \in \mathbb{N})$ in \mathbb{R} is

1. nondecreasing monotone/increasing if $a_n \leq a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n \nearrow$; nonincreasing monotone/decreasing if $a_n \geq a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n \searrow$.
2. strictly increasing if $a_n < a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n \nearrow \nearrow$; strictly decreasing if $a_n > a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n \searrow \searrow$.

Theorem 2 (Monotone Seq. Property). If $a_n \nearrow$ and $\{a_n | n \in \mathbb{N}\}$ has an upper bound, then $\lim_{n \rightarrow \infty} a_n = \sup\{a_n | n \in \mathbb{N}\}$; $a_n \searrow$ and $\{a_n | n \in \mathbb{N}\}$ has a lower bound, then $\lim_{n \rightarrow \infty} a_n = \inf\{a_n | n \in \mathbb{N}\}$.

Proof. $\{a_n | n \in \mathbb{N}\}$ has an upper bound $\Rightarrow l := \sup\{a_n | n \in \mathbb{N}\}$ exists by Weierstrass theorem. Thus for $\forall \epsilon > 0, l - \epsilon$ is not an upper bound of $\{a_n\}$, then $\exists N \in \mathbb{N}$, s.t. $a_N > l - \epsilon$ and since $a_n \nearrow$, we have that $\forall n \geq N, l - \epsilon < a_n \leq l \Rightarrow \lim_{n \rightarrow \infty} a_n = l$. □

Example 1 (Decimal expression gives real number). Suppose $d_i \in \mathbb{N}$ and $0 \leq d_i \leq 9$ for $i \in \mathbb{N}$, and define

$$a_n = \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

for $n \in \mathbb{N}$, then it is direct to see that $a_n \nearrow$ and

$$\begin{aligned} a_n &\leq \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} \\ &= \frac{9}{10} \left(\frac{1 - \frac{1}{10^{n+1}}}{1 - \frac{1}{10}} \right) \\ &< \frac{9}{10} \left(\frac{1}{1 - \frac{1}{10}} \right) \\ &= 1 \end{aligned}$$

and hence $\lim_{n \rightarrow \infty} a_n$ exists, and we can define a real number by $\lim_{n \rightarrow \infty} a_n =: 0.d_1d_2 \cdots$

Example 2 (The natural base e). Define a seq. $a_n = (1 + 1/n)^n (n \in \mathbb{N})$, then we have

$$\begin{aligned} a_n &= \left(1 + \frac{1}{n} \right)^n \\ &= \binom{n}{0} + \binom{n}{1} \frac{1}{n} + \binom{n}{2} \frac{1}{n^2} + \cdots + \binom{n}{n} \frac{1}{n^n} \\ &= \sum_{j=0}^n \binom{n}{j} \frac{1}{n^j} = \sum_{j=0}^n \frac{n!}{j!(n-j)!} \cdot \frac{1}{n^j} \\ &= \sum_{j=0}^n \frac{1}{j!} \left(1 - \frac{1}{n} \right) \left(1 - \frac{2}{n} \right) \cdots \left(1 - \frac{j-1}{n} \right) \\ &< \sum_{j=0}^{n+1} \frac{1}{j!} \left(1 - \frac{1}{n+1} \right) \left(1 - \frac{2}{n+1} \right) \cdots \left(1 - \frac{j-1}{n+1} \right) \end{aligned}$$

Thus $a_n \nearrow$. On the other hand, for $\forall n \in \mathbb{N}$, we have

$$\begin{aligned} a_n &< \sum_{j=0}^{n+1} \frac{1}{j!} = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} \\ &< 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} \\ &= 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} \\ &< 3 \end{aligned}$$

Thus a_n has an upper bound and hence a_n converges, and we define $\lim_{n \rightarrow \infty} a_n =: e$.

1.3 Nested Intervals

Definition 5 (Nested). A seq. of intervals $I_n (n \in \mathbb{N})$ is nested if $I_n \neq \emptyset$ and $I_{n+1} \subseteq I_n$ for $\forall n \in \mathbb{N}$.

Example 3. If we have a seq. of nested intervals $I_n (n \in \mathbb{N})$, do we have $\cap_{n \in \mathbb{N}} I_n \neq \emptyset$? The answer is not sure. For example,

1. $I_n = (0, 1/n), n \in \mathbb{N}$, then for any $r \in \mathbb{R}, \exists N \in \mathbb{N}$, s.t. $1/N < r$ by Archimedean Property, thus $r \notin I_N$, and hence $\cap_{n \in \mathbb{N}} I_n = \emptyset$;
2. $I_n = [n, \infty), n \in \mathbb{N}$, then for any $r \in \mathbb{R}, \exists N \in \mathbb{N}$, s.t. $r < N$ by Archimedean Property, thus $r \notin I_N$, and hence $\cap_{n \in \mathbb{N}} I_n = \emptyset$;

Theorem 3 (Theorem of Nested Interval). *If $I_n (n \in \mathbb{N})$ is a seq. of bounded closed nested intervals, then $\cap_{n \in \mathbb{N}} I_n \neq \emptyset$. (In the other word, there exists a real number $c \in \mathbb{R}$ such that $c \in \cap_{n \in \mathbb{N}} I_n$)*

Proof. Write $I_n = [a_n, b_n] (n \in \mathbb{N})$, then $I_n (n \in \mathbb{N})$ is nested $\Leftrightarrow a_n \leq b_n$ and $a_n \nearrow$ and $b_n \searrow$. And furthermore, for $\forall n, m \in \mathbb{N}$,

$$a_n \leq a_{\max\{m, n\}} \leq b_{\max\{m, n\}} \leq b_m,$$

in the other word, for $\forall m \in \mathbb{N}$, b_m is an upper bound of $\{a_n | n \in \mathbb{N}\}$, thus seq. a_n converges. Let $c = \lim_{n \rightarrow \infty} a_n$, then given $m \in \mathbb{N}$, for $\forall n \in \mathbb{N}, a_n \leq b_m$ thus

$$c = \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_m = b_m.$$

On the other hand, $c = \sup\{a_n | n \in \mathbb{N}\}$, thus for all $m \in \mathbb{N}$, we have

$$a_m \leq c \leq b_m$$

thus $c \in I_m$ for $\forall m \in \mathbb{N} \Rightarrow c \in \cap_{n \in \mathbb{N}} I_n$. □

Exercise 6. Show that $\cap_{n \in \mathbb{N}} I_n \neq \emptyset$, if

1. $I_n = (a_n, b_n)$, nested and $a_n \nearrow$ and $b_n \searrow$?
2. $I_n = (a_n, \infty)$, nested and $\{a_n | n \in \mathbb{N}\}$ is bounded from above.

Proof. 1. Just as analyzed before, there exist $c \in \mathbb{R}$ such that $c = \lim_{n \rightarrow \infty} a_n$, and $c = \sup\{a_n | n \in \mathbb{N}\}$ and hence $a_n \leq c \leq b_m$ for $\forall n, m \in \mathbb{N}$. Note that $a_n \leq c$ implies that $a_n < c$ for $\forall n \in \mathbb{N}$, otherwise if $\exists n' \in \mathbb{N}$, s.t. $a_{n'} = c$ then

$$a_{n'+1} \geq a_{n'} = c,$$

which leads to the contradiction. In the same way $c \leq b_m$ implies that $c < b_m$ for $\forall m \in \mathbb{N}$. Thus there $\exists c \in \mathbb{R}$ such that

$$a_n < c < b_m$$

for $\forall n, m \in \mathbb{N} \Rightarrow c \in \cap_{n \in \mathbb{N}} I_n$.

2. Since $I_n = (a_n, \infty)$ is a nested interval, $a_n \nearrow \Rightarrow a_n$ converges since a_n is upper bounded. That is $\exists c \in \mathbb{R}$, s.t. $c = \lim_{n \rightarrow \infty} a_n = \sup\{a_n\}$, thus for $\forall n \in \mathbb{N}, c \geq a_n$, that is

$$c + 1 > c \geq a_n$$

for $\forall n \in \mathbb{N} \Rightarrow c + 1 \in \cap_{n \in \mathbb{N}} I_n$. □

Exercise 7. Show that Theorem of Nested Interval implies Dedekind's Gapless Property.

Proof. Let (A, B) be a Dedekind cut of \mathbb{R} , pick a from A and b from B , and form an interval $I_0 = [a, b]$. Then $(a + b)/2$ lies in the middle of I_0 and must belong to A or B . If $(a + b)/2$ belongs to A , we let

$$a_1 = \frac{a + b}{2}, \quad b_1 = b$$

and if $(a + b)/2$ belongs to B , let

$$a_1 = a, \quad b_1 = \frac{a + b}{2}$$

and hence we can form a new interval $I_1 = [a_1, b_1]$ whose length is half of the former I_0 . Repeat this process, we obtain a seq. of nested intervals

$$I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots,$$

where $I_n = [a_n, b_n], b_n - a_n = (b_{n-1} - a_{n-1})/2$. Thus there exists $s \in \mathbb{R}$ lies in the $\cap_{n \in \mathbb{N}} I_n$ by the theorem of nested intervals, and either $s \in A$ or $s \in B$.

Assume that $s \in A$, for any $s' \in \mathbb{R}, s < s'$, exists b_n such that $s < b_n < s'$ since $b_n \rightarrow s$, thus $s' \in B$. That is $s \in A$ and for any $s' > s, s' \in B$. In the other word, s is the maximal element of A and B has no minimal element in this case, since assume s' is the minimal element of B then $\exists b_n$, s.t. $b_n < s'$ and $b_n \in B$, which is a contradiction. □

Remark 3. Summary, we have discussed

- 1) Dedekind's Gapless Property;
- 2) Weierstrass Theorem;
- 3) Monotone Seq. Property;
- 4) Theorem of Nested Interval.

which have the relationship:

$$\begin{array}{ccc} 1) & \implies & 2) \\ \uparrow & & \downarrow \\ 4) & \impliedby & 3) \end{array}$$

These 5 properties are equivalent and we call these the **Completeness of the real numbers**.

1.4 Limit superior / inferior

Let $a_n (n \in \mathbb{N})$ be a bounded (upper bdd. and lower bdd.) seq. in \mathbb{R} , we define **upper seq. of a_n** as

$$u_n := \sup\{a_m | m \geq n\},$$

and **lower seq. of a_n** as

$$l_n := \inf\{a_m | m \geq n\},$$

for $n \in \mathbb{N}$. Thus give $n \in \mathbb{N}$, we have that for $\forall m \geq n$

$$l_n \leq a_m \leq u_n,$$

We now show that l_n and u_n is monotone. Assume that $\exists n \in \mathbb{N}$, s.t. $u_n < u_{n+1}$, let $\epsilon = (u_{n+1} - u_n)/2$, then

$$u_{n+1} - \epsilon > u_n = \sup\{a_m | m \geq n\},$$

thus for $\forall m \geq n$, $u_{n+1} - \epsilon > a_m$ and hence $u_{n+1} - \epsilon$ is an upper bound of $\{a_m | m \geq n+1\}$, which leads to a contradiction. Thus for $\forall n \in \mathbb{N}$, $u_n \geq u_{n+1} \Rightarrow u_n \searrow$, and $l_n \nearrow$ in the same way.

Thus we have that for any $n, m \in \mathbb{N}$,

$$l_m \leq l_{\max\{m, n\}} \leq u_{\max\{m, n\}} \leq u_n,$$

thus l_1 is a lower bound for $\{u_n | n \in \mathbb{N}\}$ and u_1 is an upper bound of $\{l_n | n \in \mathbb{N}\}$ and hence $u_n, l_n (n \in \mathbb{N})$ are convergent by Monotone seq. property. We define the **limit superior** of a_n as the limit of u_n :

$$\overline{\lim}_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \sup_{m \geq n} a_m = \inf_{n \in \mathbb{N}} \sup_{m \geq n} a_m$$

The last equals sign is because $u_n \searrow$ and hence converges to its greatest lower bound by Monotone seq. property. And the **limit inferior** of a_n as the limit of l_n :

$$\underline{\lim}_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} l_n = \lim_{n \rightarrow \infty} \inf_{m \geq n} a_m = \sup_{n \in \mathbb{N}} \inf_{m \geq n} a_m$$

Exercise 8. Let $a_n (n \in \mathbb{N})$, show that

$$a_n \text{ converges} \Leftrightarrow \overline{\lim}_{n \rightarrow \infty} a_n = \underline{\lim}_{n \rightarrow \infty} a_n$$

and if any of both sides holds, then

$$\lim_{n \rightarrow \infty} a_n = \overline{\lim}_{n \rightarrow \infty} a_n = \underline{\lim}_{n \rightarrow \infty} a_n$$

Proof. \Rightarrow : Suppose that $\lim_{n \rightarrow \infty} a_n = s$. Then for any $\epsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $\forall n \geq N$, $|a_n - s| < \epsilon/2$, thus $s - \epsilon/2 < a_n < s + \epsilon/2$ for $\forall n \geq N$. Thus the upper seq. u_n of a_n has

$$s - \frac{\epsilon}{2} < a_n \leq u_n \leq s + \frac{\epsilon}{2},$$

for $\forall n \geq N$. The third inequality symbol is because if $\exists n' \geq N$ such that $u_{n'} > s + \epsilon/2$, then there exist a real number q such that $s + \epsilon/2 < q < u_{n'}$ and $q > s + \epsilon/2 > a_n$ for $\forall n \geq N$ and hence $q > a_n$ for $\forall n \geq n'$, and then $u_{n'}$ is not the least upper bound of $\{a_n | n \geq n'\}$ which is contrary. Thus $|u_n - s| \leq \epsilon/2 < \epsilon$, thus

$$\lim_{n \rightarrow \infty} u_n = \overline{\lim}_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_n = s,$$

and $\lim_{n \rightarrow \infty} l_n = \underline{\lim}_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_n = s$ in the same way.

\Leftarrow : Suppose $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} l_n = s$, then for $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ one has $|u_n - s| < \epsilon/3$ and $|l_n - s| < \epsilon/3$ and $|u_n - l_n| \leq |u_n - s| + |l_n - s| < 2\epsilon/3$, since $l_n \leq a_n \leq u_n$ then $0 \leq a_n - l_n \leq u_n - l_n$. Then we have that

$$\begin{aligned} |a_n - s| &= |a_n - l_n + l_n - s| \\ &\leq |a_n - l_n| + |l_n - s| \\ &\leq |u_n - l_n| + |l_n - s| \\ &< \epsilon \end{aligned}$$

for $\forall n \geq N \Rightarrow \lim_{n \rightarrow \infty} a_n = \overline{\lim}_{n \rightarrow \infty} a_n = \underline{\lim}_{n \rightarrow \infty} a_n = s$. □

Exercise 9. Let $a_n, b_n (n \in \mathbb{N})$ be two bdd. seq. show that

1. $\overline{\lim}_{n \rightarrow \infty} (a_n + b_n) \leq \overline{\lim}_{n \rightarrow \infty} a_n + \overline{\lim}_{n \rightarrow \infty} b_n$;
2. $\underline{\lim}_{n \rightarrow \infty} a_n + \underline{\lim}_{n \rightarrow \infty} b_n \leq \underline{\lim}_{n \rightarrow \infty} (a_n + b_n)$.

Proof. 1. Let $u_n = \sup_{m \geq n} a_m, v_n = \sup_{m \geq n} b_m, w_n = \sup_{m \geq n} (a_m + b_m)$. If $\exists n' \in \mathbb{N}$ such that $w_{n'} > u_{n'} + v_{n'}$, then $\exists r \in \mathbb{R}$ s.t. $u_{n'} + v_{n'} < r < w_{n'}$ and hence for any $m \geq n'$, $a_m \leq u_{n'}, b_m \leq v_{n'}$ and

$$a_m + b_m \leq u_{n'} + v_{n'} < r$$

which means r is an upper bound of $\{a_m | m \geq n'\}$ which leads to a contradiction with $w_{n'}$ is the least upper bound of $\{a_m | m \geq n'\}$. Thus for $\forall n \in \mathbb{N}$, $u_n + v_n \leq w_n$, and since $\lim_{n \rightarrow \infty} u_n, \lim_{n \rightarrow \infty} v_n$ exists, we have that

$$\lim_{n \rightarrow \infty} (u_n + v_n) = \lim_{n \rightarrow \infty} u_n + \lim_{n \rightarrow \infty} v_n \leq \lim_{n \rightarrow \infty} w_n$$

that is

$$\overline{\lim}_{n \rightarrow \infty} a_n + \overline{\lim}_{n \rightarrow \infty} b_n \leq \overline{\lim}_{n \rightarrow \infty} (a_n + b_n).$$

2. The same as 1. □

And in the same way, we can prove that

1. $\overline{\lim}_{n \rightarrow \infty} (a_n \cdot b_n) \leq \overline{\lim}_{n \rightarrow \infty} a_n \cdot \overline{\lim}_{n \rightarrow \infty} b_n$;
2. $\underline{\lim}_{n \rightarrow \infty} a_n \cdot \underline{\lim}_{n \rightarrow \infty} b_n \leq \underline{\lim}_{n \rightarrow \infty} (a_n \cdot b_n)$.

In general, the properties does not hold for subtraction.

1.5 Cauchy seq.

Given a seq. $a_n (n \in \mathbb{N})$ in \mathbb{R} , can we determine whether a_n converges or not without referring a limit candidate l , but concluding according to the mutual behavior of the terms of $a_n (n \in \mathbb{N})$?

Definition 6 (Cauchy Sequence). A seq. $a_n (n \in \mathbb{N})$ in \mathbb{R} is a Cauchy seq. if $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n, m \geq N \Rightarrow |a_n - a_m| < \epsilon$.

Exercise 10. Show that

1. a_n is convergent $\Rightarrow a_n$ is Cauchy seq.
2. a_n is Cauchy seq. $\Rightarrow a_n$ is bounded.

Proof. 1. assume that a_n converges to l , then for any $\epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N$ one has $|a_n - l| < \epsilon/2$, then for any $m, n \geq N$ we have

$$|a_m - a_n| \leq |a_m - l| + |a_n - l| < \epsilon$$

thus $a_n (n \in \mathbb{N})$ is Cauchy seq.

2. For $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n, m \geq N$ one has $|a_m - a_n| \leq \epsilon$, thus for $\forall m \geq N \Rightarrow |a_N - a_m| \leq \epsilon \Rightarrow a_n - \epsilon \leq a_m \leq a_N + \epsilon$, thus $a_n (n \in \mathbb{N})$ has upper and lower bound

$$\max\{a_1, \dots, a_N, a_N + \epsilon\}, \quad \min\{a_1, \dots, a_N, a_N - \epsilon\},$$

thus a_n is bounded. □

Theorem 4. Let $a_n (n \in \mathbb{N})$ be a seq. in \mathbb{R} , then a_n is convergent $\Leftrightarrow a_n$ is Cauchy seq.

Proof. \Leftarrow : a_n is Cauchy seq. $\Rightarrow a_n$ is bdd. \Rightarrow the upper/lower seq. u_n, l_n of a_n converges. Thus $\lim_{n \rightarrow \infty} u_n - \lim_{n \rightarrow \infty} l_n = \lim_{n \rightarrow \infty} (u_n - l_n)$. For $\forall \epsilon > 0$, there is some $N \in \mathbb{N}$ s.t. $\forall m, n \geq N, |a_n - a_m| < \epsilon/3$. In particular, $\forall n \geq N \Rightarrow |a_n - a_N| < \epsilon/3$ and hence

$$a_N - \frac{\epsilon}{3} < a_n < a_N + \frac{\epsilon}{3}$$

which means $a_N - \epsilon/3$ is a lower bound of $a_n (n \geq N)$ and is not greater than $\{a_n | n \geq N\}$'s greatest lower bound l_N , and the same to $a_N + \epsilon/3$, thus

$$a_N - \frac{\epsilon}{3} \leq l_N \leq u_N \leq a_N + \frac{\epsilon}{3}$$

and since $l_n \nearrow$ and $u_n \searrow$, we have that for $\forall n \geq N$

$$0 \leq u_n - l_n \leq u_N - l_N \leq \frac{2\epsilon}{3} < \epsilon$$

thus $\lim_{n \rightarrow \infty} (u_n - l_n) = 0 \Rightarrow \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} l_n \Rightarrow a_n$ converges. \square

Exercise 11. Let $S \subseteq \mathbb{R}$, if $|s - s'| \leq 3$ for $\forall s, s' \in S$, show that

1. S is bdd.;
2. $\sup S - \inf S \leq 3$;

Proof. 1. If S has no upper bound, then for any $s \in S$, define $M = s + 4$, then $\exists s' \in S$ s.t. $s' > M = s + 4 \Rightarrow s' - s = |s' - s| > 4$, which is contrary.

2. Let $u = \sup S, l = \inf S$, suppose $u - l > 3$, then let $\epsilon = u - l - 3$, we have that $\exists s \in S$, s.t.

$$u - \frac{\epsilon}{3} < s \leq u,$$

and $\exists s' \in S$ s.t.

$$l \leq s' < l + \frac{\epsilon}{3},$$

then

$$u - l - \frac{2\epsilon}{3} < s - s' \leq u - l$$

thus $3 + \epsilon/3 < s - s' = |s - s'| \leq 3 + \epsilon \Rightarrow |s - s'| > 3$, which is contrary. \square

Chapter 2

Series

2.1 Positive series

Definition 7. Let $a_n (n \in \mathbb{N})$ be a seq. in \mathbb{R} , we say that the series $\sum_n^\infty a_n$ (or $\sum_n a_n$) converges to a real number s if

$$\lim_{n \rightarrow \infty} s_n = s,$$

where $s_n := \sum_{j=1}^n a_j$ is called the n -th partial sum of $\sum_n a_n$.

If such s exists (resp. does not exist), we say that the series $\sum_n a_n$ convergent (resp. divergent). For a series $\sum_n a_n$ and $l, m \in \mathbb{N}, l < m$, we let $s_{l,m} := \sum_{j=l}^m a_j$ the (l, m) -tail of $\sum_n a_n$. If a series $\sum_n a_n$ converges, we denote it as $\sum_n a_n < \infty$.

Exercise 12. If a series $\sum_n a_n < \infty$, show that $\lim_{n \rightarrow \infty} a_n = 0$.

Proof. Trivial. □

$\sum_n a_n$ converges $\Leftrightarrow s_n$ converges by definition and $\Leftrightarrow s_n$ is Cauchy seq., i.e. $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geq N$, (assume that $n > m$)

$$\begin{aligned} |s_n - s_m| &= |a_{m+1} + \cdots + a_n| \\ &= |a_{m+1} + a_{m+2} + \cdots + a_{m+1+(n-1)}| \\ &\leq \epsilon. \end{aligned}$$

In particular, for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ and for $\forall k > N, \forall l \geq 0$, then $|a_k| + \cdots + |a_{k+l}| < \epsilon \Leftrightarrow \sum_n |a_n|$ convergent $\Leftrightarrow \exists M > 0, \forall n \in \mathbb{N}, s_n = \sum_{j=1}^n a_j \leq M$, since $s_n \nearrow$. Collectively, we have some conclusions:

1. series $\sum_n a_n$ converges \Leftrightarrow for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ and for $\forall k > N, \forall l \geq 0, |a_k + \cdots + a_{k+l}| < \epsilon$;
2. series $\sum_n b_n$, where $b_n \geq 0$, converges $\Leftrightarrow \exists M > 0, \forall n \in \mathbb{N}, \sum_{j=1}^n b_j \leq M$.
3. series $\sum_n |a_n|$ converges $\Rightarrow \sum_n a_n$ converges.

Example 4. Given series $\sum_n 1/n$. we have that

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + \frac{1}{2} \\ s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2} \\ s_8 &\geq 1 + \frac{2}{2} + \frac{1}{5} + \cdots + \frac{1}{8} \geq 1 + \frac{2}{2} + \frac{4}{8} = 1 + \frac{3}{2} \end{aligned}$$

In general, for $\forall m \in \mathbb{N}$, $s_{2^m} \geq 1 + m/2$ which has no upper bound $\Leftrightarrow \sum_n 1/n$ diverges.

Example 5. Given series $\sum_n 1/n^2$. we have that $1/n^2 < 1/((n-1)n) = 1/(n-1) - 1/n$. Then

$$\begin{aligned} s_n &= 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \\ &< 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} \\ &= 2 - \frac{1}{n} \\ &< 2. \end{aligned}$$

thus s_n has upper bound 2 $\Leftrightarrow \sum_n 1/n^2$ converges.

Definition 8. Given a seq. $a_n (n \in \mathbb{N})$, we say that

1. $\sum_n a_n$ converges absolutely if $\sum_n |a_n|$ converges;
2. $\sum_n a_n$ converges conditionally if $\sum_n |a_n|$ diverges but $\sum_n a_n$ converges.

Theorem 5 (Comparison Test). If $a_n, b_n \geq 0 (n \in \mathbb{N})$, then $\exists C > 0$ and $N \in \mathbb{N}$, $n \geq N \Rightarrow a_n \leq Cb_n \Rightarrow [\sum_n b_n < \infty \Rightarrow \sum_n a_n < \infty]$.

Proof. If $\sum_n b_n$ converges, then for $\forall n \geq N$,

$$\begin{aligned} a_1 + \cdots + a_n &= a_1 + \cdots + a_N + a_{N+1} + \cdots + a_n \\ &\leq a_1 + \cdots + a_N + C \cdot (b_{N+1} + \cdots + b_n) \\ &\leq a_1 + \cdots + a_N + C \cdot M =: H, \end{aligned}$$

where M is an upper bound of $\sum_{j=1}^n b_j$, thus $\sum_j^n a_j$ as upper bound $H \Leftrightarrow \sum_n a_n$ converges. \square

Theorem 6 (Limit Form of Comparison Test). If $a_n, b_n \geq 0 (n \in \mathbb{N})$, and if $\lim_{n \rightarrow \infty} a_n/b_n$ exists $\Rightarrow [\sum_n b_n < \infty \Rightarrow \sum_n a_n < \infty]$.

Proof. Let $l = \lim_{n \rightarrow \infty} a_n/b_n$, then for $\epsilon = 1, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N, a_n/b_n < l + 1 \Rightarrow a_n < (l + 1)b_n$, which follows the proof by Comparison test. Furthermore if $l \neq 0$, then

for $\epsilon = l/2, \exists N_l \in \mathbb{N}$, s.t. $\forall n \geq N_l$, s.t.

$$\frac{l}{2} \leq \frac{a_n}{b_n} \leq \frac{3l}{2},$$

and hence $b_n \leq a_n \cdot 2/l$ and $a_n \leq b_n \cdot 3l/2$, therefore $\sum_n b_n$ converges $\Leftrightarrow \sum_n a_n$ converges. \square

Exercise 13. If $a_n, b_n \geq 0 (n \in \mathbb{N})$, show that if $\overline{\lim}_{n \rightarrow \infty} a_n/b_n$ exists $\Rightarrow [\sum_n b_n < \infty \Rightarrow \sum_n a_n < \infty]$.

Proof. Let $u_n = a_n/b_n$ and $\lim_{n \rightarrow \infty} u_n = l$, then $l = \inf_{n \in \mathbb{N}} u_n$ and for $\epsilon = 1, \exists n' \in \mathbb{N}$ s.t.

$$l \leq u_{n'} < l + 1$$

and hence for $\forall n \geq n'$ we have that

$$\frac{a_n}{b_n} \leq u_{n'} < l + 1$$

thus $a_n < (l + 1) \cdot b_n$ for $\forall n \geq n'$ and finish the proof by comparison test. \square

Exercise 14 (Ratio and Root test). If $a_n, b_n \geq 0 (n \in \mathbb{N})$, show that

1. $\lim_{n \rightarrow \infty} a_{n+1}/a_n < 1 \Rightarrow \sum_n a_n < \infty$; $\lim_{n \rightarrow \infty} a_{n+1}/a_n > 1 \Rightarrow \sum_n a_n$ diverges.
2. $\lim_{n \rightarrow \infty} (a_n)^{1/n} < 1 \Rightarrow \sum_n a_n < \infty$; $\lim_{n \rightarrow \infty} (a_n)^{1/n} > 1 \Rightarrow \sum_n a_n$ diverges.

Proof. Trivial. \square

2.2 Alternating series

Definition 9. A series $\sum_n a_n$ is called alternating series, if $\exists b_n > 0 (n \in \mathbb{N})$ s.t. $a_n = (-1)^{n-1} b_n (n \in \mathbb{N})$.

Theorem 7 (Leibniz's Criterion). Let $\sum_n a_n$ be an alternating series, and $b_n = |a_n| \searrow 0$ as $n \rightarrow \infty$, then $\sum_n a_n < \infty$.

Proof. Since $b_n = (-1)^{n-1} a_n$, for any $k, l \in \mathbb{N}$ the tail of $\sum_n a_n$ is

$$\begin{aligned} |a_k + \dots + a_{k+l}| &= (-1)^{k-1} |b_k - b_{k+1} + \dots + (-1)^l b_{k+l}| \\ &= |b_k - b_{k+1} + \dots + (-1)^l b_{k+l}| \end{aligned}$$

define $\lambda_{k,l} = b_k - b_{k+1} + \dots + (-1)^l b_{k+l}$. Then if l is even, then

$$\lambda_{k,l} = (b_k - b_{k+1}) + \dots + (b_{k+l-2} - b_{k+l-1}) + b_{k+l} \geq 0,$$

and if l is odd, then

$$\lambda_{k,l} = (b_k - b_{k+1}) + \cdots + (b_{k+l-1} - b_{k+l}) \geq 0,$$

thus $\lambda_{k,l} \geq 0$ for $\forall k, l \in \mathbb{N}$. And hence

$$\begin{aligned} |a_k + \cdots + a_{k+l}| &= |\lambda_{k,l}| = \lambda_{k,l} \\ &= \begin{cases} b_k - (b_{k+1} - b_{k+2}) - \cdots - (b_{k+l-1} - b_{k+l}), & l \text{ is even} \\ b_k - (b_{k+1} - b_{k+2}) - \cdots - (b_{k+l-2} - b_{k+l-1}) - b_{k+l}, & l \text{ is odd} \end{cases} \\ &\leq b_k \end{aligned}$$

Since $\lim_{n \rightarrow \infty} b_n = 0 \Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $n \geq N \Rightarrow |b_n| < \epsilon \Rightarrow |a_n + \cdots + a_{n+l}| \leq b_n < \epsilon$ for $\forall l \in \mathbb{N}$, thus $\sum_n a_n$ converges. \square

2.3 Rearrangement theorem

Given a seq. $a_n (n \in \mathbb{N})$, we can separate all terms into two subseq.s:

$$a_{n_1}, a_{n_2}, \cdots \text{ and } a_{n'_1}, a_{n'_2}, \cdots$$

where $n_1 < n_2 < \cdots$ and $n'_1 < n'_2 < \cdots$ and $\{n_1, n_2, \cdots\} \cup \{n'_1, n'_2, \cdots\} = \mathbb{N}$, such that $a_{n_j} \geq 0 (j \in \mathbb{N}), a_{n'_k} \leq 0 (k \in \mathbb{N})$. Let $p_j := a_{n_j} (j \in \mathbb{N})$ and $q_k := a_{n'_k} (k \in \mathbb{N})$.

Exercise 15. Show that $\sum_n |a_n| < \infty \Leftrightarrow \sum_j p_j < \infty$ and $\sum_k q_k < \infty$. Moreover, if any side holds, then

$$\sum_n |a_n| = \sum_j p_j + \sum_k q_k$$

and

$$\sum_n a_n = \sum_j p_j - \sum_k q_k.$$

Exercise 16. If $\sum_n a_n$ converges conditionally, show that

1. $\sum_j p_j = \infty$ and $\sum_k q_k = \infty$;
2. $\lim_{j \rightarrow \infty} p_j = \lim_{k \rightarrow \infty} q_k = 0$.

Exercise 17. If $\sum_n a_n, \sum_n b_n$ converges, show that $\sum_n (a_n \pm b_n) = \sum_n a_n \pm \sum_n b_n$.

Exercise 18. Inserting 0s to a series will not affect its convergence / divergence and its sum (if the sum exists).

Recall that a sequence a_n is a map $\mathbb{N} \xrightarrow{a} \mathbb{R}$ where $n \mapsto a(n)$ denoted by a_n . A

subsequence a_{n_m} is a composite map

$$\mathbb{N} \xrightarrow{n.} \mathbb{N} \xrightarrow{a.} \mathbb{R}$$

where $n.$ is an injection (or say $n. \nearrow \nearrow$) and $m \mapsto n(m)$ denoted by n_m . A **rearrangement** is also a composite map

$$\mathbb{N} \xrightarrow{n(\cdot)} \mathbb{N} \xrightarrow{a.} \mathbb{R}$$

where $n(\cdot)$ is a bijection. (To distinguish it from the notion of subseq., we don't use subscripts here)

Now we gonna explore two questions: if a series \sum_n converges, $a_{n(m)} (m \in \mathbb{N})$ is a rearrangement of $a_n (n \in \mathbb{N})$, then

1. whether $\sum_m a_{n(m)}$ converges ?
2. whether $\sum_n a_n = \sum_m a_{n(m)}$?

Exercise 19. Let $\sum_n a_n$ be a positive series, show that

$$\sum_n a_n = \sup \{a_{n_1} + \cdots + a_{n_k} | n_1 < \cdots < n_k, k \in \mathbb{N}\}$$

including the case $\sum_n a_n = \infty$.

Exercise 20. If $\sum_n a_n$ is a convergent positive series, show that for every rearrangement $a_{n(m)} (m \in \mathbb{N})$ of $a_n (n \in \mathbb{N})$, we have that $\sum_n a_n = \sum_m a_{n(m)}$.

Exercise 21 (Dirichlet's Rearrangement Theorem (1829)). If $\sum_n a_n$ converges absolutely, show that for every rearrangement $a_{n(m)} (m \in \mathbb{N})$ of $a_n (n \in \mathbb{N})$, we have that $\sum_n a_n = \sum_m a_{n(m)}$.

Theorem 8 (Riemann's Rearrangement Theorem (1852)). If $\sum_n a_n$ converges conditionally, then for $\forall r \in \mathbb{R}$, there exists a rearrangement $a_{n(m)} (m \in \mathbb{N})$ of $a_n (n \in \mathbb{N})$ such that $\sum_m a_{n(m)} = r$.

Proof. □

Remark 4 ($2S = S$). Dirichelet made the following observation in 1827: Consider a alternating series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

which is convergent (conditionally) by Leibniz's Criterion, and hence

$$\begin{aligned} 2S &= 2 - 1 + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \frac{2}{7} - \frac{2}{8} + \frac{2}{9} - \frac{2}{10} + \cdots \\ &= (2 - 1) - \frac{2}{4} + \left(\frac{2}{3} - \frac{2}{6}\right) - \frac{2}{8} + \left(\frac{2}{5} - \frac{2}{10}\right) + \cdots \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \\ &= S \end{aligned}$$

2.4 Multiplying absolutely convergent series

Proposition 2. Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely, let

$$c_n = a_n b_0 + \cdots + a_0 b_n = \sum_{j+k=n; j,k \geq 0} a_j b_k,$$

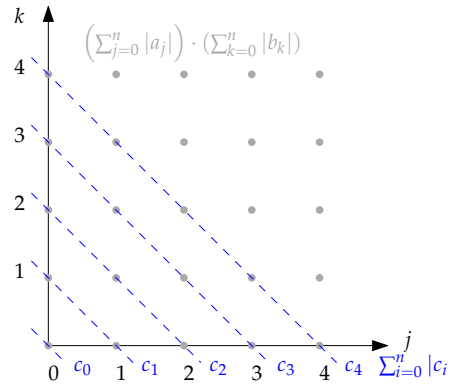
then $\sum_n |c_n| < \infty$ and $\sum_{n=0}^{\infty} c_n = (\sum_{n=0}^{\infty} a_n) \cdot (\sum_{n=0}^{\infty} b_n)$.

Proof. 1. $\sum_n |c_n| < \infty$

For all n ,

$$\begin{aligned} \sum_{m=0}^n |c_m| &= \sum_{m=0}^n \left| \sum_{\substack{j+k=m \\ j,k \geq 0}} a_j b_k \right| \leq \sum_{m=0}^n \sum_{\substack{j+k=m \\ j,k \geq 0}} |a_j| |b_k| \\ &\leq \left(\sum_{j=0}^n |a_j| \right) \cdot \left(\sum_{k=0}^n |b_k| \right). \end{aligned}$$

Since $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converges absolutely, the partial sums of $|a_n|, |b_n|$ have upper bounds, denoted by M, N respectively, then $\sum_{m=0}^n |c_m|$ has a upper bound $M \cdot N$ and hence $\sum_{n=0}^{\infty} c_n$ converges absolutely.



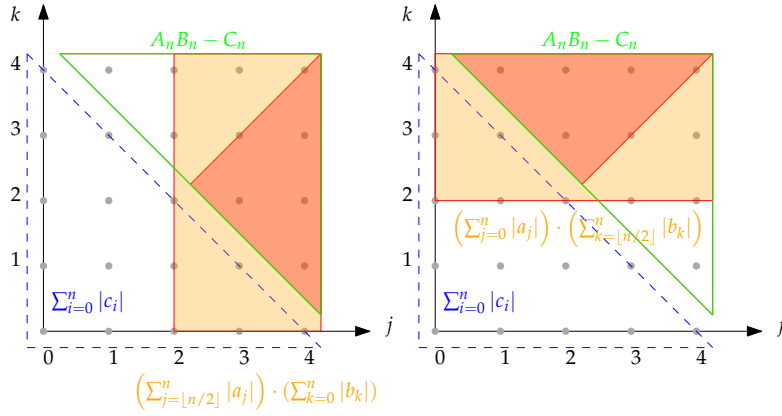
$$2. \sum_{n=0}^{\infty} c_n = (\sum_{n=0}^{\infty} a_n) \cdot (\sum_{n=0}^{\infty} b_n).$$

Let $A_n := a_0 + \cdots + a_n$; $B_n := b_0 + \cdots + b_n$ and $C_n := c_0 + \cdots + c_n$, we claim that $\lim_{n \rightarrow \infty} (A_n B_n - C_n) = 0$. Then

$$\begin{aligned} |A_n B_n - C_n| &= \sum_{\substack{j+k > n \\ 0 \leq j, k \leq n}} |a_j b_k| \\ &\leq \left(\sum_{j=\lfloor n/2 \rfloor}^n |a_j| \right) \cdot \left(\sum_{k=0}^n |b_k| \right) + \left(\sum_{j=0}^n |a_j| \right) \cdot \left(\sum_{k=\lfloor n/2 \rfloor}^n |b_k| \right) \end{aligned}$$

where $\sum_{k=0}^n |b_k|, \sum_{j=0}^n |a_j|$ are bounded, and tails $\sum_{j=\lfloor n/2 \rfloor}^n |a_j|, \sum_{k=\lfloor n/2 \rfloor}^n |b_k| \rightarrow 0$ as $n \rightarrow \infty$ since $\sum_n a_n, \sum_n b_n$ are converges abs. Thus $\lim_{n \rightarrow \infty} |A_n B_n - C_n| = 0$ and since $\lim_{n \rightarrow \infty} A_n, \lim_{n \rightarrow \infty} B_n, \lim_{n \rightarrow \infty} C_n$ exists, we have that

$$\begin{aligned} \sum_{n=0}^{\infty} c_n &= \lim_{n \rightarrow \infty} C_n \\ &= \lim_{n \rightarrow \infty} A_n B_n = \lim_{n \rightarrow \infty} A_n \cdot \lim_{n \rightarrow \infty} B_n \\ &= \left(\sum_{n=0}^{\infty} a_n \right) \cdot \left(\sum_{n=0}^{\infty} b_n \right) \end{aligned}$$



□

Theorem 9. If $\sum_n a_n, \sum_n b_n$ cvg. abs., $\mathbb{N} \xrightarrow{(j(\cdot), k(\cdot))} \mathbb{N} \times \mathbb{N}$ is bijection where $n \mapsto (j(n), k(n))$, let $c_n := a_{j(n)} b_{k(n)} (n \in \mathbb{N})$, then $\sum_n |c_n| < \infty$ (cvg. abs.) and $\sum_n c_n = (\sum_n a_n)(\sum_n b_n)$.

Proof. 1. $\sum_n c_n$ cvg. abs.

For $\forall n \in \mathbb{N}$, let $l = \max\{j(1), \dots, j(n), k(1), \dots, k(n)\}$. Then

$$\begin{aligned} |c_1| + \dots + |c_n| &= |a_{j(1)} b_{k(1)}| + \dots + |a_{j(n)} b_{k(n)}| \\ &\leq \left(\sum_{j=1}^l |a_j| \right) \cdot \left(\sum_{k=1}^l |b_k| \right) \\ &\leq M \cdot N \end{aligned}$$

Thus $\sum_n c_n$ cvg. abs.

2. $\sum_n c_n = (\sum_n a_n)(\sum_n b_n)$.

Let $A_n = a_1 + \dots + a_n, B_n = b_1 + \dots + b_n$ and $C_n = c_1 + \dots + c_n (n \in \mathbb{N})$. And define the bijection $\mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ by the second one in Figure 2.1. Then

$$\begin{aligned}
A_n B_n &= (a_1 + \cdots + a_n)(b_1 + \cdots + b_n) \\
&= \sum_{1 \leq j, k \leq n} a_j b_k \\
&= C_{n^2}
\end{aligned}$$

Thus $\lim_{n \rightarrow \infty} A_n B_n = \lim_{n \rightarrow \infty} C_{n^2} = \lim_{n \rightarrow \infty} C_n \Rightarrow \sum_n c_n = (\sum_n a_n)(\sum_n b_n)$. □

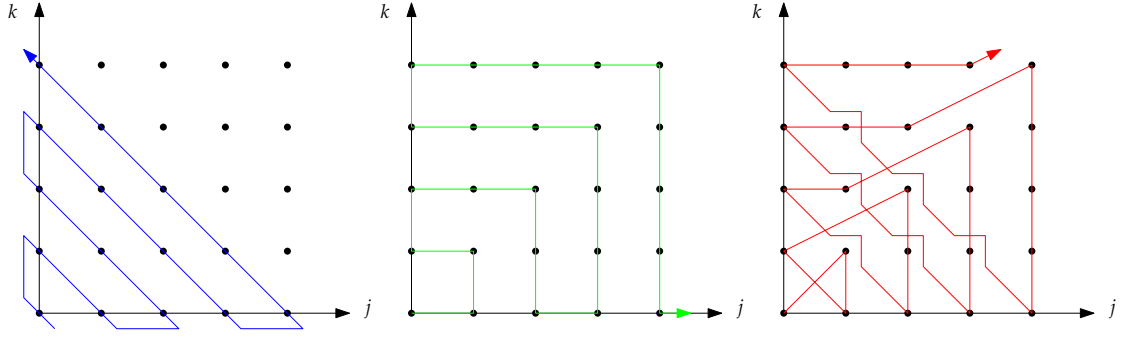


Figure 2.1: 3 kinds of bijections $(j(\cdot), k(\cdot))$

Chapter 3

\mathbb{R} - valued functions