Introduction to Topology

Naive Set Theory, Lecture 1

Haoming Wang

14 June 2019

This is the Lecture note for the *Introduction to Topology*. The course covers the following topics: Naive Set Theory, Elementary Number Theory, Group Theory, Topological Spaces and Continuous Maps, Introduction to Algebraic Topology.

Content:

Proposition

If p, q are statements, We denote "if p then a" as $p \Rightarrow q$. The only case where this proposition is false is p is true while q is false, thus $p \Rightarrow q \Leftrightarrow (\neg p) \vee q$. When we use proof by contradiction to prove a proposition like $p \Rightarrow q$, what we do is that $\neg(p \Rightarrow q)$ leads to a contradiction, that is $p \wedge \neg q$ leads to a contradiction.

Quantifier

There are two quantifiers : "for all" $\forall$ and "exists" $\exists$. There are two commonly-used Propositions:

1 $\exists x, \forall y,$ s.t. proposition $P(x, y)$ holds;
2 $\forall y, \exists x,$ s.t. proposition $P(x, y)$ holds;

The difference between these proposition is former x in $P(x, y)$ could be constant, but the latter would be not.

Note 1. It is easy to check that the former is the sufficient condition of the latter. For example, suppose $P(x, y) = [\![x < y]\!]$, then the latter holds but the former does not.

Set

Inclusion

Suppose A and B are sets, we say $A \subseteq B$ if $\forall x,$ s.t. $x \in A \Rightarrow x \in B$; and $B \subseteq A$ if $\forall x,$ s.t. $x \in B \Rightarrow x \in A$. Correspondingly, $A = B$ if $\forall x,$ s.t. $x \in A \Leftrightarrow x \in B$.

Note 2. Suppose $\emptyset \nsubseteq A$, which means $\exists x \in \emptyset,$ s.t. $x \notin A$. But there is no element in $\emptyset$, thus $\emptyset \subseteq A$ logically.

Example 1. Suppose $A = \{x \in \mathbb{R} | x = x + 1\}, B = \{x \in \mathbb{Q} | x^2 = 2\}$. There's no element in either A or B, although it is a little wilder, but still fits our definition above, thus $A = B$.

Operations on set

Definition 1 (Difference)**.** Given sets A, B, the difference of sets is $A \backslash B := \{x \in A | x \notin B\}$.

Definition 2 (Union and Intersection)**.** Given $S_j(j \in J)$, a family of sets indexed by a set J. Then, the union of sets is

$$\cup_{j\in J} S_j := \{x | \exists j \in J, x \in S_j\};$$

the intersection of sets is

$$\cap_{j\in J} S_j := \{x | \forall j \in J, x \in S_j\}.$$

Definition 3 (Power set)**.** Given a set S, the power set of S is $\mathcal{P}(S) := \{A | A \subseteq S\}$, that is $\forall A, A \in \mathcal{P}(S) \Leftrightarrow A \subseteq S$.

Example 2. Suppose $S = \{0, 1\}$, then $\mathcal{P}(S) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$. If there are n elements in a finite set S, then there are 2^n elements in its power set $\mathcal{P}(S)$. Thus sometimes we also denote the power set of S by 2^S.

Definition 4 (Cartesian porduct)**.** Given sets X and Y, then the cartesian product of them is $X \times Y := \{(x, y) | x \in X \wedge y \in Y\}$.

Note 3. The pair (x, y) is defined as a set $\{\{x\}, \{x, y\}\}$ which indicates a truth: if $x, x' \in X, y, y' \in Y$, then $(x, y) = (x', y') \Leftrightarrow x = x' \wedge y = y'$.
The reason why we define the pair (x, y) as such form is that there is no order in set, that is $\{\{x\}, \{y\}\} = \{\{y\}, \{x\}\}$.

Relation

Definition 5 (Relation)**.** Given sets X, Y, we say a subset R of $X \times Y$ induces a binary relation among elements of X and Y.

If $x \in X, y \in Y$ fit $(x, y) \in R \subset X \times Y$, we say x, y has relation R, denote as xRy. Different subsets of $X \times Y$ induce different relation, the $\emptyset$, also a subset of $X \times Y$, means elements in X have no relationship with elements in Y.

Definition 6 (Equivalence relation)**.** $R \in X \times X$ is an equivalence relation on X if:

1. $\forall x \in X, (x, x) \in R$;
2. $\forall x, x' \in X, (x, x') \in R \Rightarrow (x', x) \in R$;
3. $\forall x, x', x'' \in X, (x, x') \in R \wedge (x', x'') \in R \Rightarrow (x, x'') \in R$.

Example 3.
1. If $R = \{(x, x) | x \in X\}$, that is R is the diagonal of $X \times X$, then R induces Equal relation.
2. If $X = \mathbb{Z}, R = \{(x, x') | x \equiv x'(\text{mod}3)\}$, then R is an equivalence relation.

Definition 7 (Partial order)**.** $R \subseteq X \times X$ is a partial order on X if:

1. $\forall x \in X, (x, x) \in R$;
2. $\forall x, x' \in X, (x, x') \in R \wedge (x', x) \in R \Rightarrow x = x'$;
3. $\forall x, x', x'' \in X, (x, x') \in R \wedge (x', x'') \in R \Rightarrow (x, x'') \in R$.

Note 4. If we eliminate the second condition, then R is a Pre-order.

Example 4. Less than or equal ($\leq$), as well as greater than or equal ($\geq$), are partial order on $\mathbb{Z}$. Given a set S, inclusion ($\subseteq$) is a partial order on $\mathcal{P}(S)$.

Definition 8 (Total order)**.** A total order on X is a partial order R on X such that $\forall x, x', (x, x') \in R \vee (x', x) \in R$.

Example 5. Given a set S, inclusion ($\subseteq$) is not a total order on $\mathcal{P}(S)$, e.g. neither $(\{0\}, \{1\})$ nor $(\{0\}, \{1\})$ in relation $\subseteq$ on $\mathcal{P}(\{0, 1\})$.

Definition 9 (Well order)**.** A well order on X is a total order R on X such that: $\forall S, S \subseteq X \wedge S \neq \emptyset \Rightarrow \exists s := \min_R S \in S, \forall s' \in S,$ s.t. $(s, s') \in R$.

Example 6. $\leq$ is a well order on $\mathbb{N}_0$, but not on $\mathbb{Z}$. But we can define a new relation R, such that R is a well order on $\mathbb{Z}$. For example, define

$$n(p) = \begin{cases} 2p - 1 & p > 0, \\ -2p & p < 0, \\ 0 & p = 0 \end{cases}$$

where $p \in \mathbb{Z}$, thus $n(p) \in \mathbb{N}$, define

$$R = \{(x, x') \in X \times X | n(x) \leq n(x')\},$$

then R is a well order on $\mathbb{Z}$. For example $(3, -10) \in R$, since $n(3) = 5$ and $n(-10) = 20$. And $\min_R\{x \in \mathbb{Z} | x \leq 4\} = 0$.

Note 5. Actually, For any non-empty set, there exists a well order on it by *Axiom of Choice*.

Maps

Definition 10 (Map)**.** Given sets X, Y, A relation $f \subseteq X \times Y$ is called a map from X to Y, if $\forall x \in X, \exists! y \in Y, (x, y) \in f$.

Note 6. $\exists! y \in Y$ represents there is one and only one $y \in Y$.

Definition 11. Given a map: $X \xrightarrow{f} Y$, for $A \subseteq X$, $B \subseteq Y$, we say:

1. The domain of f, $D_f := X$;
2. The codomain of f, $C_f := Y$;
3. The image of A under f, $f(A) := \{f(a) | a \in A\}$;
4. The pre-image of B under f, $f^{-1}(B) := \{x \in X | f(x) \in B\}$;
5. The range of f, $R_f := f(X)$.

Note 7. Notice that f^{-1} is not a map. $f^{-1}(Y) = X$.

Exercise 1. Given maps $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$. Show that for $A \subseteq X$, $(g \circ f)(A) = g(f(A))$; for $C \subseteq Z$, $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$.

Proof. 1. Trivial; 2. By definition, we have $(g \circ f)^{-1}(C) = \{x \in X | g(f(x)) \in C\} =: U$, $f^{-1}(g^{-1}(C)) = \{x \in X | f(x) \in \{y \in Y | g(y) \in C\}\} =: K$. if $x \in U, x \notin K$, then $f(x) \notin \{y \in Y | g(y) \in C\}$ and $g(f(x)) \notin C$, which leads to a contradiction, thus $U \subseteq K$. Correspondingly, we can prove $K \subseteq U$ by contradiction, thus $U = K$. $\square$

Exercise 2. Given a map $X \xrightarrow{f} Y$, show that:

1. For a family of subset $T_j \subseteq Y (j \in J)$, have

$$f^{-1}(\cup_{j\in J} T_j) = \cup_{j\in J} f^{-1}(T_j) \text{ and } f^{-1}(\cap_{j\in J} T_j) = \cap_{j\in J} f^{-1}(T_j);$$

2. For $B, B' \in Y$, $f^{-1}(B \backslash B') = f^{-1}(B) \backslash f^{-1}(B')$;
3. For a family of subset $S_j \subseteq X (j \in J)$, have

$$f(\cup_{j\in J} S_j) = \cup_{j\in J} f(S_j) \text{ and } f(\cap_{j\in J} S_j) \subseteq \cap_{j\in J} f(S_j);$$

4. For $A, A' \in X$, $f(A) \backslash f(A') \subseteq f(A \backslash A')$.

Proof. 1. $\cup$: If

$$\begin{aligned} x \in f^{-1}(\cup_{j\in J} T_j) &\Rightarrow f(x) \in \cup_{j\in J} T_j \\ &\Rightarrow f(x) \in T_j (\exists j \in J) \\ &\Rightarrow x \in f^{-1}(T_j) \subseteq \cup_{j\in J} f^{-1}(T_j) \end{aligned}$$

thus $f^{-1}(\cup_{j\in J} T_j) \subseteq \cup_{j\in J} f^{-1}(T_j)$. If

$$\begin{aligned} x \in \cup_{j\in J} f^{-1}(T_j) &\Rightarrow x \in f^{-1}(T_j)(\exists j \in J) \\ &\Rightarrow f(x) \in T_j \subseteq \cup_{j\in J} T_j \\ &\Rightarrow x \in f^{-1}(\cup_{j\in J} T_j) \end{aligned}$$

thus $\cup_{j\in J} f^{-1}(T_j) \subseteq f^{-1}(\cup_{j\in J} T_j)$. Thus $f^{-1}(\cup_{j\in J} T_j) = \cup_{j\in J} f^{-1}(T_j)$.
$\cap$: If

$$\begin{aligned} x \in f^{-1}(\cap_{j\in J} T_j) &\Rightarrow f(x) \in \cap_{j\in J} T_j \\ &\Rightarrow f(x) \in T_j (\forall j \in J) \\ &\Rightarrow x \in f^{-1}(T_j)(\forall j \in J) \\ &\Rightarrow x \in \cap_{j\in J} f^{-1}(T_j), \end{aligned}$$

thus $f^{-1}(\cap_{j\in J} T_j) \subseteq \cap_{j\in J} f^{-1}(T_j)$; If

$$\begin{aligned} x \in \cap_{j\in J} f^{-1}(T_j) &\Rightarrow x \in f^{-1}(T_j)(\forall j \in J) \\ &\Rightarrow f(x) \in T_j (\forall j \in J) \\ &\Rightarrow f(x) \in \cap_{j\in J} T_j \\ &\Rightarrow x \in f^{-1}(\cap_{j\in J} T_j), \end{aligned}$$

thus $\cap_{j\in J} f^{-1}(T_j) \subseteq f^{-1}(\cap_{j\in J} T_j)$, and $f^{-1}(\cap_{j\in J} T_j) = \cap_{j\in J} f^{-1}(T_j)$.

2. If

$$\begin{aligned} f^{-1}(B \backslash B') &\Rightarrow f(x) \in B \backslash B' \\ &\Rightarrow f(x) \in B \wedge f(x) \notin B' \\ &\Rightarrow x \in f^{-1}(B) \wedge x \notin f^{-1}(B') \\ &\Rightarrow x \in f^{-1}(B) \backslash f^{-1}(B'). \end{aligned}$$

If

$$
\begin{aligned}
x \in f^{-1}(B) \backslash f^{-1}(B') &\Rightarrow f \in f^{-1}(B) \wedge x \notin f^{-1}(B') \\
&\Rightarrow f(x) \in B \wedge f(x) \notin B' \\
&\Rightarrow f(x) \in B \backslash B' \\
&\Rightarrow x \in f^{-1}(B \backslash B');
\end{aligned}
$$

Thus $f^{-1}(B \backslash B') = f^{-1}(B) \backslash f^{-1}(B')$.

3. $\cup$: If

$$
\begin{aligned}
y \in f(\cup_{j \in J} S_j) &\Rightarrow y \in f(S_j)(\exists j \in J) \\
&\Rightarrow y \in \cup_{j \in J} f(S_j)
\end{aligned}
$$

and if

$$
\begin{aligned}
y \in \cup_{j \in J} f(S_j) &\Rightarrow y \in f(S_j)(\exists j \in J) \\
&\Rightarrow y \in f(\cup_{j \in J} S_j).
\end{aligned}
$$

Thus $f(\cup_{j \in J} S_j) = \cup_{j \in J} f(S_j)$.
$\cap$: for $\forall j \in J$, we have $f(\cap_{j \in J} S_j) \subseteq f(S_j)$, thus $f(\cap_{j \in J} S_j) \subseteq \cap_{j \in J} f(S_j)$. If $y \in \cap_{j \in J} f(S_j)$ then for $\forall j \in J$, there exists $s_j \in S_j$ such that $s_j \in f^{-1}(y)$. BUT, we can not confirm that s_j are the same number in different S_j, thus $\cap_{j \in J} S_j$ could be $\emptyset$. For example, assume that $f(x) = |x|$ with domain $X = [-2, 2]$. Set $S_1 = (-2, 0), S_2 = (0, 2), y = 1$, then $y \in f(S_1) \cap f(S_2) = (0, 2)$ but $f(S_1 \cap S_2) = f(\emptyset) = \emptyset \subseteq f(S_1) \cap f(S_2) = (0, 2)$.

4. If $y \in f(A) \backslash f(A')$ then $y \in f(A) \wedge y \notin f(A')$. Thus $\exists a \in A$, s.t. $a \in f^{-1}(y)$ and $\forall a' \in A'$, s.t. $a' \notin f^{-1}(y)$, which means $a \notin A'$, and $a \in A \backslash A'$, thus $y \in f(A \backslash A')$. Thus $f(A) \backslash f(A') \subseteq f(A \backslash A')$.
Set $A = (-2, 0), A' = (1, 2)$, then $f(A \backslash A') = f(A) = (0, 2)$. But $f(A) \backslash f(A') = (0, 2) \backslash (1, 2) = (0, 1] \subseteq f(A \backslash A')$.

$\square$

Note 8. It is easy to prove that if $S_1 \subseteq S_2$ then $f(S_1) \subseteq f(S_2)$ and $f^{-1}(S_1) \subseteq f^{-1}(S_2)$.

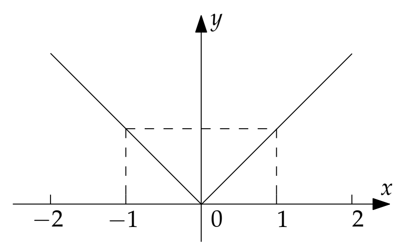

Figure 1: $f(x) = |x|$