# *Introduction to Topology*

*General Topology, Lecture 12,13*

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*17 July 2019*

This is the Lecture note for the *Introduction to Topology*. The course covers the following topics: Naive Set Theory, Elementary Number Theory, Group Theory, Topological Spaces and Continuous Maps, Introduction to Algebraic Topology.

Content:



## *Continuous maps and topology space*

**Definition 1** (Continuous)**.** Let $(X, d_X), (Y, d_Y)$ be metric spaces. $a \in S \subseteq X, f : S \mapsto Y$, we say map $f$ is continuous at $a$ if for $\forall \epsilon > 0, \exists \delta > 0$, for $\forall x \in B_\delta(a) \cap S$, s.t. $f(x) \in B_\epsilon(f(a))$, that is $f(B_\delta(a)) \subseteq B_\epsilon(f(a))$.

We say $f$ is a continuous map if $f$ is continuous at every $a \in S$.

**Exercise 1.** Given a map $X \xrightarrow{f} Y, a \in X$, Show that

1. $f$ is continuous at $a \Leftrightarrow$ for $\forall V \subseteq_{open} Y$, where $f(a) \in V$, $\exists U \subseteq_{open} X$, where $a \in U$, such that $f(U) \subseteq V$.
2. $f$ is a continuous map $\Leftrightarrow$ for $\forall V \subseteq_{open} Y, f^{-1}(V) \subseteq_{open} X$.

*Proof.* 1. $\Rightarrow$: for $\forall V \subseteq_{open} Y$, where $f(a) \in V$, $\exists \epsilon > 0$, s.t. $B_\epsilon(f(a)) \subseteq V$, thus $\exists U = B_\delta(a)$. $\Leftarrow$: trivial.

2. $\Rightarrow$: Given an open set $V \subseteq_{open} Y$, for $\forall x \in f^{-1}(V)$, have $f(x) \in V$. Since $V$ is open, $\exists r > 0$ s.t. $B_r(f(x)) \subseteq V$. Since $f(x)$ is continuous map, $\exists \epsilon > 0$, s.t. $f(B_\epsilon(x)) \subseteq B_r(f(x)) \subseteq V \Rightarrow B_\epsilon(x) \in f^{-1}(V)$, thus $f^{-1}(V)$ is open.

$\Leftarrow$: Given $x \in X, f(x) \in Y$, given $r > 0$, s.t. $B_r(f(x)) \subseteq Y$, then $f^{-1}(B_r(f(x))) \subseteq_{open} X$, and $x \in f^{-1}(B_r(f(x)))$. Thus $\exists \epsilon > 0$, s.t. $B_\epsilon(x) \subseteq f^{-1}(B_r(f(x)))$ and $f(B_\epsilon(x)) \subseteq B_r(f(x))$. $\square$

*Note 1.* It can also be proved that $f$ is cont. $\Leftrightarrow$ for $\forall V \subseteq_{close} Y, f^{-1}(V) \subseteq_{close} X$.

Suppose $V \subseteq_{close} Y$, then $X \backslash f^{-1}(V) = f^{-1}(X \backslash V) \subseteq_{open} X$, thus $f^{-1}(V) \subseteq_{close} X$.

**Exercise 2.** Given maps $X \xrightarrow{f} Y, Y \xrightarrow{g} Z$, show that

1. If $f$ is continuous at $x_0$, $g$ is continuous at $f(x_0)$, then $g \circ f$ is continuous at $x_0$.
2. If $f, g$ are continuous maps, then $g \circ f$ is a continuous map.

*Proof.* 1. For any $V$, s.t. $g(f(x_0)) \in V \subseteq_{open} Z$, $\exists U$, s.t. $f(x_0) \in U \subseteq_{open} Y$, $\exists W$, s.t. $x_0 \in W \subseteq X$, thus $g \circ f$ is continuous at $x_0$.

2. For any $V \subseteq_{open} Z, \exists U \subseteq_{open} Y, \exists W \subseteq_{open} X$, thus $g \circ f$ is continuous. $\square$

*Note 2.* Prove this exercise using sets instead of metrics.

We replaced open ball with open set in Exercise 1, this is a meaningful operation, which means we could **substitute the metric with**

**set** (here is open set), which drives the concept of topology. Generally, topology is a family of sets which have the basic properties of open set, but not necessarily be open sets. Using these sets, we can no longer rely on metric $d$.

**Definition 2** (Topology)**.** Given a set $X$, we say a family of subsets $\mathscr{T}(\subseteq \mathcal{P}(X))$ is a topology on $X$ if

1. $X, \emptyset \in \mathscr{T}$;
2. $U, V \in \mathscr{T} \Rightarrow U \cap V \in \mathscr{T}$;
3. $U_\alpha \in \mathscr{T}(\alpha \in A) \Rightarrow \cup_{\alpha \in A} U_\alpha \in \mathscr{T}$. ($A$ is an arbitrary index set)

*Note* 3. From here on, we define the **open sets** as elements in a topology, instead of the previous metric-based definition.

**Example 1.** Given a set $X$,

1. $\mathscr{T} = \{\emptyset, X\}$ is called trivial topology. In this case, we define only $X$ and $\emptyset$ are open sets.
2. Given a metric space $(X, d)$, the previous definition of open sets is $\mathscr{T}_d = \{U \subseteq \mathcal{P}(X) | \forall x \in U, \exists r > 0, \text{ s.t. } B_r(x) \subseteq U\}$.

Given different metric $d$, we will obtain different topology. For example, if we use discrete metric, then for $\forall x \in X$, $\exists r > 0$, such as $r = 0.5$, s.t. $B_r(x) = \{x\} \subseteq \{x\}$, thus $\{x\}$ is an open set. For $\forall U \subseteq X, U = \cup\{x | x \in U\}$, thus any subset of $X$ is an open set. In this case, $\mathscr{T} = \mathcal{P}(X)$, and we call it the discrete topology.

**Definition 3** (Topology Space)**.** A topological space $(X, \mathscr{T})$ consists of a set $X$ and a topology $\mathscr{T}$ on $X$.

**Definition 4** (Open set)**.** Let $(X, \mathscr{T})$ be a topological space, any $A \in \mathscr{T}$ is called an open set in $X$ w.r.t. $\mathscr{T}$; and $X \backslash A$ is called a closed set in $X$ w.r.t. $\mathscr{T}$.

**Definition 5.** Let $(X, \mathscr{T})$ be a top. space and $A \subseteq X, x \in X$.

1. $x$ is an interior point of $A$ in $X$ w.r.t. $\mathscr{T}$, if $\exists U \in \mathscr{T}$, s.t. $x \in U \subseteq A$ (that is $U \cap X \backslash A = U \backslash A = \emptyset$). $U$ is called an open neighborhood of $x$ w.r.t. $\mathscr{T}$.
2. $x$ is an exterior point of $A$ in $X$ w.r.t. $\mathscr{T}$, if $\exists U \in \mathscr{T}$, s.t. $x \in U \subseteq X \backslash A$. (i.e. $x$ is an interior of $X \backslash A$).
3. $x$ is a boundary point of $A$ in $X$ w.r.t. $\mathscr{T}$, if $\forall U \in \mathscr{T}$, if $x \in U$, then $U \cap A \neq \emptyset \wedge U \backslash A \neq \emptyset$.

*Note* 4. The definition of boundary point is the complementary of interior points union with exterior points.

**Definition 6.** Let $(X, \mathscr{T})$ be a top. space and $A \subseteq X$. The set consists of all interior points of $A$ in $X$ w.r.t. $\mathscr{T}$ is called interior (of $A$ in $X$ w.r.t. $\mathscr{T}$), denote as $int_X A (= A^\circ)$; the set of all exterior points is called exterior, denoted as $ext_X A (= A^e)$; and the set of all boundary points is called boundary, denoted as $bdy_X A (= \partial A)$.

*Note* 5. Let $(X, \mathscr{T})$ be a top. space $\forall A \subseteq X$, $X = A^\circ \cup A^e \cup \partial A$, and $A^\circ, A^e, \partial A$ are disjoin.

$A^\circ$ is the exterior of $X \backslash A$, $A^e$ is the interior of $X \backslash A$, and $\partial A$ is the boundary of $X \backslash A$, which means

$$A^\circ = (X \backslash A)^e$$
$$A^e = (X \backslash A)^\circ$$
$$\partial A = \partial(X \backslash A).$$

**Example 2.** Given a top. space $(\mathbb{R}, \mathscr{T}_d)$, where $d = |x - y|, \forall x, y \in \mathbb{R}$. Let $A = [0, 1)$. Then $A^\circ = (0, 1), A^e = (-\infty, 0) \cup (1, \infty), \partial A = \{0, 1\}$.

**Exercise 3.** Show that $A^\circ, A^e$ are open sets (on $X$ w.r.t $\mathscr{T}$, that is $A^\circ, A^e \in \mathscr{T}$); $\partial A$ is close set.

*Proof.* 1. $\forall x \in A^\circ, \exists U_x \in \mathscr{T}$, s.t. $x \in U_x$, thus $A^\circ = \cup_{x\in A^\circ} U_x \in \mathscr{T}$, thus $A^\circ$ is open on $X$ w.r.t. $\mathscr{T}$.

2. $A^e$ is the interior of $X\backslash A$ by definition, thus $A^e$ is open.

3. $A^\circ, A^e \in \mathscr{T} \Rightarrow A^\circ \cup A^e \in \mathscr{T}$, thus $\partial A = X\backslash(A^\circ \cup A^e) \in \mathscr{T}$. □

**Exercise 4.** Given a topology space $(X, \mathscr{T}), A \subseteq X$, show that

$$A^\circ = \cup\{U | U \subseteq_{open} A\}.$$

*Proof.* $\subseteq$: for $\forall x \in A^\circ, \exists U \in \mathscr{T}$, s.t. $x \in U \subseteq A \Rightarrow x \in \cup\{U|U \subseteq_{open} A\}$; $\supseteq$: for $\forall x \in \cup\{U|U \subseteq_{open} A\}, \exists U_x \subseteq_{open} A$, s.t. $x \in U_x$, thus $x$ is an interior point, and $x \in A^\circ$. □

*Note 6.* $A^\circ$ is the largest open set in $X$ contained in $A$. Thus,

$$A = A^\circ \Leftrightarrow A \subseteq_{open} X \Leftrightarrow \partial A \cap A = \emptyset$$

for $\partial A \cap A = \partial A \cap A^\circ = \emptyset$. And furthermore $(A^\circ)^\circ = A^\circ$.

**Definition 7** (Closure)**.** Given a topology space $(X, \mathscr{T}), A \subseteq X$, the set

$$\overline{A} = cls_x A := \cap\{C | A \subseteq C \subseteq_{close} X\}$$

is called the closure of $A$ in $X$ w.r.t. $\mathscr{T}$.

*Note 7.* $\overline{A}$ is the smallest close set in $X$ containing in $A$. Thus,

$$A = \overline{A} \Leftrightarrow A \subseteq_{close} X \Leftrightarrow \partial A \subseteq A$$

for $\partial A \subseteq A^\circ \cup \partial A = \overline{A} = A$. And furthermore $\overline{\overline{A}} = \overline{A}$.

**Exercise 5.** Given a topology space $(X, \mathscr{T}), A \subseteq X$, show that $\overline{A} = A^\circ \cup \partial A$.

*Proof.*

$$\begin{aligned} A^\circ \cup \partial A &= X\backslash A^e \\ &= X\backslash(X\backslash A)^\circ \\ &= X\backslash \cup \{U | U \subseteq_{open} X\backslash A\} \\ &= \cap\{X\backslash U | U \subseteq_{open} X\backslash A\} \\ &= \cap\{C | A \subseteq C \subseteq_{close} X\} \\ &= \overline{A}. \end{aligned}$$

□

*Note 8.*

$$\begin{aligned} &U \subseteq X\backslash A \\ &\Rightarrow \forall x \in U \Rightarrow x \in X\backslash A \\ &\Rightarrow \forall x \notin X\backslash A \Rightarrow x \notin U \\ &\Rightarrow \forall x \in A \Rightarrow x \in X\backslash U \\ &\Rightarrow A \subseteq X\backslash U, \end{aligned}$$

$U$ is open $\Rightarrow$ $X\backslash U$ is close, hence $C = X\backslash U \subseteq_{close} A$.

**Exercise 6.** Show that $X\backslash\overline{A} = (X\backslash A)^\circ$ and $X\backslash A^\circ = \overline{(X\backslash A)}$.

*Proof.* 1.

$$\begin{aligned} \overline{A} &= A^\circ \cup \partial A \\ &= X\backslash A^e \\ &= X\backslash(X\backslash A)^\circ \\ X\backslash\overline{A} &= (X\backslash A)^\circ. \end{aligned}$$

2.

$$\begin{aligned} X\backslash A^\circ &= A^e \cup \partial A \\ &= (X\backslash A)^c \cup \partial(X\backslash A) \\ &= \overline{(X\backslash A)}. \end{aligned}$$

□

*Note 9.* We denote $X\backslash A$ as $A^c$ if $X$ is clearly given. Thus

$$(\overline{A})^c = (A^c)^\circ$$
$$(A^\circ)^c = \overline{A^c}$$

**Exercise 7.** If $A \subseteq B$, show that $A^\circ \subseteq B^\circ$, $\overline{A} \subseteq \overline{B}$.

*Proof.* 1. Given $x \in A^\circ = \cup\{U | U \subseteq_{open} A\}$, $\exists U_x \subseteq_{open} A$, s.t. $x \in U_x \subseteq_{open} A \subseteq B$, thus $x \in \cup\{V | V \subseteq_{open} B\}$, and $x \in B^\circ$. 2. the same way with 1. $\square$

**Exercise 8.** Given a set $U$, (denote $\overline{U}$ as $U^-$,) show that $U \subseteq_{open} X \Rightarrow U^- = U^{-c-c-}$.

*Proof.*

$$\begin{aligned} U^{-c-c-} &= (U^-)^{c-c-} \\ &= (U^-)^{\circ cc-} \\ &= U^{-\circ-} \end{aligned}$$

$U \subseteq U^- \Rightarrow U = U^\circ \subseteq U^{-\circ} \Rightarrow U^- \subseteq U^{-\circ-}$. Let $C = U^- \subseteq_{close} X$, thus $C^\circ \subseteq C \Rightarrow C^{\circ-} \subseteq C^- = C \Rightarrow U^{-\circ-} \subseteq U^-$, thus $U^- = U^{-\circ-} = U^{-c-c-}$. $\square$

**Exercise 9** (Kuratowski's 14 sets)**.** Given a top. sp. $X$, $A \subseteq X$, Show that among

$$\begin{gathered} A\ , A^-, A^{-c}, A^{-c-}, A^{-c-c} \cdots \\ A^c, A^{c-}, A^{c-c}, A^{c-c-} \cdots \end{gathered}$$

there are at most 14 different subsets of $A$.

*Proof.* On the one hand,

$$A\ , A^-, \underbrace{A^{-c}}_{open}, A^{-c-}, A^{-c-c}, A^{-c-c-}, A^{-c-c-c}, \underbrace{A^{-c-c-c-}}_{=(A^{-c})^{-c-c-}}, \cdots$$

On the other hand,

$$A^c, A^{c-}, \underbrace{A^{c-c}}_{open}, A^{c-c-}, A^{c-c-c}, A^{c-c-c-}, \underbrace{A^{c-c-c-c-}}_{=(A^{c-c})^{-c-c-}}, \cdots$$

thus there are at most 14 different subsets of $A$. $\square$

**Definition 8** (Continuous Map)**.** Let $X, Y$ be top. spaces. A map $X \xrightarrow{f} Y$ is continuous at a point $x_0 \in X$ if $\forall$ open neighborhood (nbd.) $V$ of $f(x_0)$, $\exists$ open nbd. $U$ of $x_0$, s.t. $f(U) \subseteq V$. $f$ is a continuous map, if $f$ is continuous at every $x_0 \in X$.

*Note* 10. We have discussed that $f$ is conti. $\Leftrightarrow$ for $\forall V \subseteq_{open} Y, f^{-1}(V) \subseteq_{open} X \Leftrightarrow$ for $\forall V \subseteq_{close} Y, f^{-1}(V) \subseteq_{close} X$.

**Exercise 10.** Let $X, Y$ be top. spaces, $X \xrightarrow{f} Y$ is a conti. map, show that $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$, and $f(\overline{A}) \subseteq \overline{f(A)}$.

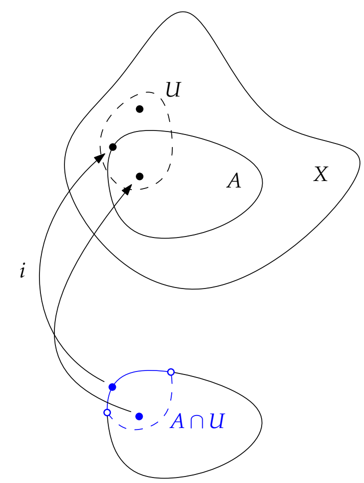
*Proof.* 1. $B \subseteq \overline{B} \Rightarrow f^{-1}(B) \subseteq f^{-1}(\overline{B})$ where $f^{-1}(\overline{B})$ is close, thus $\overline{f^{-1}(B)} \subseteq \overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B})$.

2. $f(A) \subseteq \overline{f(A)} \Rightarrow A \subseteq f^{-1}(\overline{f(A)})$ where $f^{-1}(\overline{f(A)})$ is close, thus $\overline{A} \subseteq \overline{f^{-1}(\overline{f(A)})} \subseteq f^{-1}(\overline{f(A)}) \Rightarrow f(\overline{A}) \subseteq \overline{f(A)}$. $\square$

*Note* 11. $f(A) \subseteq B \Rightarrow A \subseteq f^{-1}(B)$ by the definition of pre-image.

## *Subspace Topology*

Let $X$ be a top. space and $A \subseteq X$. A top. space is a set which has been specified some subsets are open but the others are not. Not consider how to transform a subset $A$ into a top. space in a reasonable way. And the issue is that what kind of subsets of $A$ should be defined as open.



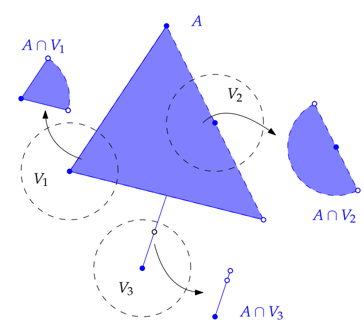

Consider the inclusion map $A \xrightarrow{i} X$ where $a \mapsto a$. Thus an intuitive motivation is we need select open sets in the top. space of $A$ such that keep $i$ is continuous. Because, for any point $a$ in the codomain of $i$, if $\exists$ an open set $U \in X$, such that covers $a$, then it covers the pre-image of $a$ (in the top. space of $X$), since $i^{-1}(a) = a \in U$. So if any point $a \in X$ has an open nbd. $U$ then it's pre-image should have an open nbd. $U_A$, otherwise the subspace top. would be too simple or wried to show the inheritance of the "sub".

Thus we wish create a corresponding open set $U_A$ of $U$ in the top. space of $A$, thus for any point in the codomain, if it has open nbd. in the top. space of $X$, then it's pre-image has open nbd. in the top. space of $A$, and $i$ is continuous. Specially, if we define $\mathscr{T}_A = \mathcal{P}(A)$, that is discrete topology, then any point forms an open set, thus $i$ is continuous. But we want to find the concisest situation that fits the demand. The concisest way to construct topology of $A$ is selecting the pre-image of the open sets in $X$, that is for any $U \in \mathscr{T}_X$, $i^{-1}(U) = U \cap A \in \mathscr{T}_A$.

1. $\emptyset \in \mathscr{T}_X \Rightarrow \emptyset \cap A = \emptyset \in \mathscr{T}_A$, $X \in \mathscr{T}_X, \Rightarrow X \cap A = A \in \mathscr{T}_A$.
2. $\forall U_1, U_2 \in X, U_1 \cap U_2 \in X$, thus $(U_1 \cap A) \cap (U_2 \cap A) = (U_1 \cap U_2) \cap A \in \mathscr{T}_A$.
3. $\forall U_\alpha \in X(\alpha \in I), \cup_{\alpha \in I} \in X$, thus $\cup_{\alpha \in I}(U\alpha \cap A) = A \cap (\cup_{\alpha \in I} U_\alpha) \in A$.

thus $\{U \cap A | \forall U \subseteq_{open} X\}$ is a topology which is the smallest topology that satisfies our demand.

**Definition 9.** The subspace topology on $A$ inherited from $X$ is $\mathscr{T}_A = \{U \cap A | U \subseteq_{open} X\}$.




**Example 3.** Given a top. space $(\mathbb{R}^2, \mathscr{T}_d)$ where $d = d_2$, a subset $A$ of $X$ like the margin figure. we can see that the elements of $\mathscr{T}_A$: $A \cap V_1$, $A \cap V_2$ and $A \cap V_3$ are all open sets on $(A, \mathscr{T}_A)$, even thought they are not open sets on $(\mathbb{R}^2, \mathscr{T}_d)$.

**Exercise 11.** Given a map $X \xrightarrow{f} Y$, $X, Y$ are top. spaces. Suppose $\exists B \subseteq Y$ is a subspace top. inherited from $Y$. If $f(X) \subseteq B$, we denote the map $X \xrightarrow{f} B$ by $f|^B$. Show that $f$ is continuous $\Leftrightarrow f|^B$ is continuous.

*Proof.* $\Rightarrow$: $f$ is conti. then $\forall V \subseteq_{open} Y$ has $f^{-1}(V) \subseteq_{open} X$, and $V \cap B \subseteq_{open} B$. Since:

$$\begin{aligned} f^{-1}(V \cap B) &= f^{-1}(V) \cap f^{-1}(B) \\ &= f^{-1}(V) \cap X \\ &= f^{-1}(V) \subseteq_{open} X \end{aligned}$$

thus $f|^B$ is conti.

$\Leftarrow$: $\forall V \subseteq_{open} Y, f^{-1}(V \cap B) = f^{-1}(V) \cap f^{-1}(B) = f^{-1}(V) \cap X = f^{-1}(V) \subseteq_{open} X$. Thus $f$ is conti. $\square$