Introduction to Analysis
Collection

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Abstract

This is the collection of lecture notes for the *Introduction to Analysis* course in Spring 2019. The purpose of this course is to bridge the gap between *Calculus* and *Advanced Calculus*.

Reference Materials:
高木貞治, 解析概論 (中译本: 高等微积分 (第 3 版修订版), 人民邮电出版社)
Richard Courant and Fritz John, Introduction to Calculus and Analysis (I) (II)
Protter and Morrey, A first course in real analysis

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Chapter 1

Completeness of the real numbers

1.1 Real number

Definition 1. Let $S \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, we say that
1. r is an upper (lower) bound of S if $\forall s \in S, r \geq (\leq) s$;
2. r is the greatest (least) element of S if r is an upper (lower) bound of S and $r \in S$, denoted by $r = \max S (\min S)$.
3. r is the least upper (greatest lower) bound of S if r is the least (greatest) element of the set of upper (lower) bound of S, denoted by $r = \sup S (\inf S)$.

Remark 1. r is a least upper bound of S means any element of S which is smaller than r is not an upper bound of S, that is $\forall \epsilon > 0, \exists s \in S,$ s.t.

$$r - \epsilon < s \leq r.$$

Thus if for some $l \in \mathbb{R}, S \subseteq \mathbb{R}$ and $\sup S > l$, then $\exists s \in S,$ s.t. $s > l$; In the other word, if $s < (\leq) l$ for $\forall s \in S \Rightarrow \sup S \leq l$.

We write $\sup S = \infty$ ($\inf S = -\infty$) if and only if S has no upper (lower) bound. If this is the case we say $\sup S (\inf S)$ does not exist. We say S is bounded from above (below) iff S has an upper (lower) bound.

Definition 2 (Dedekind Cut)**.** Let $A, B \subseteq \mathbb{R}$, we say that (A, B) is a Dedekind cut if
1. $A, B \neq \emptyset$;
2. $A \cup B = \mathbb{R}$;
3. $\forall a \in A, b \in B, a < b$.

We usually call $A(B)$ the lower (upper) part of (A, B).

We assume that $\mathbb{R}$ has the **Dedekind's Gapless Property**: If (A, B) is a Dedekind cut of $\mathbb{R}$, then exactly one of the following happens:
1. $\max A$ exists but $\min B$ does not;

2. $\min B$ exists but $\max A$ does not.
We call $\max A$ in 1. (or $\min B$ in 2.) the **cutting** of (A, B).

Exercise 1. *We may define Dedekind cuts on $\mathbb{Q}$ and $\mathbb{Z}$ similarly, does Dedekind Gapless Property hold for $\mathbb{Q}$ and $\mathbb{Z}$?*

Proof. 1. Let $A := \{q \in \mathbb{Q} | q^2 < 2\}, B := \{q \in \mathbb{Q} | q^2 > 2\}$. It is direct to see that $A, B \neq \emptyset$.
If $\exists r \in \mathbb{Q}$, s.t. $r^2 = 2$, then $\exists p, q \in \mathbb{N}$, s.t. $r = p/q$ and p, q are not both even. Then $p^2/q^2 = 2 \Rightarrow p^2 = 2q^2 \Rightarrow p^2$ is even $\Rightarrow p$ is even $\Rightarrow p^2$ can be divided by $4 \Rightarrow q^2$ can be divided by $2 \Rightarrow q^2$ is even $\Rightarrow q$ is even, which leads to a contradiction. Thus $\forall r \in \mathbb{Q}, r^2 \neq 2$. Thus $A \cup B = \mathbb{Q}$.
Finally $\forall q_a \in A, q_b \in B$ one has $q_a^2 < 2 < q_b^2 \Rightarrow q_a < q_b$. Thus (A, B) is a Dedekind cut of $\mathbb{Q}$. It is direct to see that (A, B) has no Dedekind's gapless property:
For example, if $p \in A$, then $p \in \mathbb{Q}$ and $p^2 < 2$, put $\epsilon = 2 - p^2$, then we should find a $q \in \mathbb{Q}$ such that $q^2 < 2$ and $q > p$, which means

$$p^2 < q^2 < 2$$

we consider there exists a function r of p, ϵ, such that $r > 0$ and $r \in \mathbb{Q}$, and put $q = p + r$, thus $q > p$ and $q \in \mathbb{Q}$, we now prove that $q^2 < 2$. Since

$$2 - q^2 = 2 - (p^2 + r^2 + 2pr) = \epsilon - r^2 - 2pr$$

We now try to find a suitable structure of r to make $r^2 + 2pr < \epsilon$. Since $p > 0$ and $\epsilon = 2 - p^2$, $0 < \epsilon < 2$. Consider $r = \epsilon/2$ then

$$r^2 = \frac{\epsilon^2}{4} = \frac{\epsilon}{2} \cdot \frac{\epsilon}{2} < \frac{\epsilon}{2}.$$

Consider $r = \epsilon/((2p+1)2) < \epsilon/2$ and

$$2pr = 2p \cdot \frac{\epsilon}{(2p+1)2} < \frac{\epsilon}{2},$$

then we have $r^2 + 2pr < \epsilon$ and

$$q^2 < 2,$$

by defining

$$q = p + \frac{\epsilon}{2(2p+1)} = \frac{3p^2 + 2p + 2}{4p + 2},$$

thus there is no maximal element in A and correspondingly, there is no minimal element in B as well.
2. trivial. $\square$

Theorem 1 (Weierstrass Theorem)**.** *Let $\emptyset \neq S \subseteq \mathbb{R}$, if S has an upper bound, then* $\sup S$ *exists.*

Proof. Let B be the set of all upper bound of S, and define $A := \mathbb{R} \backslash B$.
Claim 1: (A, B) is a Dedekind cut of $\mathbb{R}$:
1. $S \neq \emptyset \Rightarrow \forall s \in S, s - 1 \notin B \Rightarrow s - 1 \in A \Rightarrow A \neq \emptyset$; And S has an upper bound $\Rightarrow B \neq \emptyset$;
2. $A = \mathbb{R} \backslash R \Rightarrow A \cup B = \mathbb{R}$;
3. If $\exists a \in A, b \in B,$ s.t. $a \geq b$ where b is an upper bound of S while a is not, thus $\exists s' \in S,$ s.t. $a < s' \leq b < a$, which leads to a contradiction. Thus $\forall a \in A, b \in B$ one has $a < b$.

Claim 2: $\min B$ exists:
If $\min B \not\exists$, then by Dedekind's gapless property, $\max A \exists$, denoted by a_0. $a_0 \in A \Leftrightarrow a_0 \notin B \Leftrightarrow a_0$ is not an upper bound of $S \Leftrightarrow \exists s_0 \in S,$ s.t. $a_0 < s$. Choose $x \in \mathbb{R}$ such that $a_0 < x < s_0$, thus $\max A < x \Rightarrow x \in B \Rightarrow x$ is an upper bound of S but $x < s_0$ which leads to a contradiction. $\square$

Exercise 2 (Archimedean Property)**.** *Show that $\forall r \in \mathbb{R}, r > 0 \Rightarrow \exists n \in \mathbb{N},$ s.t. $n > r$. (or say $\exists n \in \mathbb{N},$ s.t. $1/n < r$).*

Proof. Let $r \in \mathbb{R}$, $S := \{n \in \mathbb{N} | n \leq r\}$, since $r > 0, 0 \in S \Rightarrow S \neq \emptyset$. Then $S \subseteq \mathbb{R}$ and S is bounded above (by r), thus S has a least upper bound in $\mathbb{R}$, let $s = \sup S$.
Now consider the number $s - 1$. Since s is the supremum of S, $s - 1$ cannot be an upper bound of S by definition. Thus $\exists m \in S$ such that

$$m > s - 1 \Rightarrow m + 1 > s$$

But as $m \in \mathbb{N}$, it follows that $m + 1 \in \mathbb{N}$. Because $m + 1 > s$, it follows that $m + 1 \notin S$ and so $m + 1 > r$. Furthermore, for $\forall r > 0, 1/r > 0$ then $\exists n \in \mathbb{N},$ s.t. $n > 1/r \Rightarrow 1/n < r$. $\square$

1.2 Sequence

Definition 3 (sequence)**.** A sequence $a_n (n \in \mathbb{N})$ is a map $\mathbb{N} \xrightarrow{a.} \mathbb{R}$ where $n \mapsto a(n)$, denoted by a_n.

Definition 4 (Convergence)**.** Let $a_n (n \in \mathbb{N})$ be a sequence in $\mathbb{R}$ and $l \in \mathbb{R}$, we say that a_n converges to l as $n \to \infty$ if $\forall \epsilon > 0, \exists N \in \mathbb{N},$ s.t. $\forall n \geq N, |a_n - l| < \epsilon$, denoted by $a_n \to l$ (as $n \to \infty$).

If such l exists, we call it the limit of $\{a_n\}$ and denote is as $\lim_{n \to \infty} a_n = l$, and call $\{a_n\}$ a convergent sequence; otherwise we call it a **divergent** sequence. Furthermore, we say $\lim_{n \to \infty} a_n = \infty$ if $\forall M > 0, \exists N \in \mathbb{N},$ s.t. $\forall n \geq N, a_n \geq M$.

Exercise 3. *Show that*
1. $\lim_{n\to\infty} a_n = l$ *and* $\lim_{n\to\infty} a_n = m \Rightarrow l = m$;
2. $a_n (n \in \mathbb{N})$ *is convergent* $\Rightarrow \{a_n | n \in \mathbb{N}\}$ *is bounded;*
3. if $a_n < b_n$ *for all* $n \in \mathbb{N}, \lim_{n\to\infty} a_n = l, \lim_{n\to\infty} b_n = m \Rightarrow l \leq m$.

Proof. 1. $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} a_n = m \Rightarrow$ for $\forall \epsilon > 0$, $\exists N, M \in \mathbb{N}$, s.t. $\forall n \geq N$ one has $|a_n - l| < \epsilon/2$ and $\forall n \geq M$ has $|a_n - m| < \epsilon/2$, thus for $\forall n \geq \max\{N, M\}$, has

$$|l - m| = |l - a_n + a_n - m| \leq |a_n - l| + |a_n - m| < \epsilon$$

holds for $\forall \epsilon > 0 \Rightarrow l = m$.
2. Suppose $a_n \to l$ as $n \to \infty$, then given an $\epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ we have $|a_n - l| < \epsilon \Rightarrow l - \epsilon < a_n < l + \epsilon$, thus a_n has upper bound

$$\max\{a_1, \cdots, a_{n-1}, l + \epsilon\},$$

and lower bound

$$\min\{a_1, \cdots, a_{n-1}, l - \epsilon\}.$$

3. if $l > m$, let $\epsilon = l - m$, then $\exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ has $|a_n - l| < \epsilon/2$ and $|b_n - m| < \epsilon/2$ thus

$$a_n < \frac{l + m}{2} < b_n,$$

which leads to a contradiction, thus $l \leq m$. □

Remark 2. Changing or removing finitely many terms in $a_n (n \in \mathbb{N})$ does not effect a_n's being convergent (and its limit)/ divergent.

Proposition 1. *If* $\lim_{n\to\infty} a_n = l$ *and* $\lim_{n\to\infty} b_n = m$ *then*
1. $\lim_{n\to\infty}(a_n \pm b_n) = l \pm m$;
2. $\lim_{n\to\infty} a_n b_n = lm$;
3. if $m \neq 0$ *and* $b_n \neq 0$ *for all but finitely many* n *then* $\lim_{n\to\infty} a_n / b_n = l/m$.

Proof. 1. For $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N, |a_n - l| \leq \epsilon/2$ and $\exists M \in \mathbb{N}$, s.t. $\forall n \geq M, |b_n - m| \leq \epsilon/2$, thus $\forall n \geq \max\{N, M\}$, one has

$$\begin{aligned}|(a_n \pm b_n) - (l \pm m)| &= |(a_n - l) \pm (b_m - m)| \\ &\leq |a_n - l| + |b_n - m| \\ &\leq \epsilon,\end{aligned}$$

thus $(a_n \pm b_n) \to l \pm m$ as $n \to \infty$.
2. Since a_n, b_n are convergent, thus they are bounded. Choose $C > 0$ such that $|b_n| \leq C$ for all $n \in \mathbb{N}$ and $|l| \leq C$, then for $\forall \epsilon > 0, \exists N, M \in \mathbb{N}$, s.t. $\forall n \geq N$ one has $|a_n - l| \leq$

$\epsilon/(2C)$ and $\forall n \geq M$ has $|b_n - m| \leq \epsilon/(2C)$, thus $\forall n \geq \max\{N, M\}$ one has

$$
\begin{aligned}
|a_n b_n - lm| &= |a_n b_n - lb_n + lb_n - lm| \\
&\leq |a_n - l| \cdot |b_n| + |b_n - m| \cdot |l| \\
&\leq (|a_n - l| + |b_n - m|) \cdot |C| \\
&\leq \epsilon
\end{aligned}
$$

thus $a_n b_n \to lm$.
3. all we need to show is $\lim_{n\to\infty} 1/b_n = 1/m$ which is trivial. □

Exercise 4 (Squeeze theorem). *If* $\lim_{n\to\infty} a_n = l$ *and* $\lim_{n\to\infty} b_n = m$ *and* $a_n \leq c_n \leq b_n$, *show that* $l = m \Rightarrow \lim_{n\to\infty} c_n = l$.

Proof. Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = l$, for $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ has $|a_n - l| < \epsilon/3$ and $|b_n - l| < \epsilon/3$. And since $a_n \leq c_n \leq b_n$, we have that $0 \leq c_n - a_n \leq b_n - a_n$. Thus for $\forall n \geq N$, we have

$$
\begin{aligned}
|c_n - l| &= |c_n - a_n + a_n - l| \\
&\leq |c_n - a_n| + |a_n - l| \\
&\leq |b_n - a_n| + |a_n - l| \\
&= |b_n - l + l - a_n| + |a_n - l| \\
&\leq |b_n - l| + 2|a_n - l| \\
&< \epsilon.
\end{aligned}
$$

thus $\lim_{n\to\infty} c_n = l$. □

Exercise 5. *If* $a > 1$ *show that* $\lim_{n\to\infty} 1/a^n = 0$.

Proof. Since $a > 1 \Rightarrow b := a - 1 > 0$, thus

$$
0 \leq \frac{1}{a^n} = \frac{1}{(1+b)^n} \leq \frac{1}{1+nb} \to 0
$$

as $n \to \infty$, thus $\lim_{n\to\infty} 1/a^n = 0$ by Squeeze theorem. □

Definition 5. A seq. $a_n (n \in \mathbb{N})$ in $\mathbb{R}$ is

1. nondecreasing monotone/increasing if $a_n \leq a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n \nearrow$; nonincreasing monotone/decreasing if $a_n \geq a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n \searrow$.
2. strictly increasing if $a_n < a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n \nearrow\nearrow$; strictly decreasing if $a_n > a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n \searrow\searrow$.

Theorem 2 (Monotone Seq. Property). *If* $a_n \nearrow$ *and* $\{a_n | n \in \mathbb{N}\}$ *has an upper bound, then* $\lim_{n\to\infty} a_n = \sup\{a_n | n \in \mathbb{N}\}$; $a_n \searrow$ *and* $\{a_n | n \in \mathbb{N}\}$ *has an lower bound, then* $\lim_{n\to\infty} a_n = \inf\{a_n | n \in \mathbb{N}\}$.

Proof. $\{a_n | n \in \mathbb{N}\}$ has an upper bound $\Rightarrow l := \sup\{a_n | n \in \mathbb{N}\}$ exists by Weierstrass theorem. Thus for $\forall \epsilon > 0, l - \epsilon$ is not an upper bound of $\{a_n\}$, then $\exists N \in \mathbb{N}$, s.t. $a_N > l$ and since $a_n \nearrow$, we have that $\forall n \geq N, l - \epsilon < a_n \leq l \Rightarrow \lim_{n\to\infty} a_n = l$. $\square$

Example 1 (Decimal expression gives real number)**.** Suppose $d_i \in \mathbb{N}$ and $0 \leq d_i \leq 9$ for $i \in \mathbb{N}$, and define

$$a_n = \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

for $n \in \mathbb{N}$, then it is direct to see that $a_n \nearrow$ and

$$\begin{aligned} a_n &\leq \frac{9}{10} + \frac{9}{10^2} + \cdots \frac{9}{10^n} \\ &= \frac{9}{10}\left(\frac{1 - \frac{1}{10^{n+1}}}{1 - \frac{1}{10}}\right) \\ &< \frac{9}{10}\left(\frac{1}{1 - \frac{1}{10}}\right) \\ &= 1 \end{aligned}$$

and hence $\lim_{n\to\infty} a_n$ exists, and we can define a real number by $\lim_{n\to\infty} a_n =: 0.d_1 d_2 \cdots$

Example 2 (The natural base e)**.** Define a seq. $a_n = (1 + 1/n)^n (n \in \mathbb{N})$, then we have

$$\begin{aligned} a_n &= \left(1 + \frac{1}{n}\right)^n \\ &= \binom{n}{0} + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{n}\frac{1}{n^n} \\ &= \sum_{j=0}^{n} \binom{n}{j}\frac{1}{n^j} = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \cdot \frac{1}{n^j} \\ &= \sum_{j=0}^{n} \frac{1}{j!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{j-1}{n}\right) \\ &< \sum_{j=0}^{n+1} \frac{1}{j!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{j-1}{n+1}\right) \end{aligned}$$

Thus $a_n \nearrow\nearrow$. On the other hand, for $\forall n \in \mathbb{N}$, we have

$$\begin{aligned} a_n &< \sum_{j=0}^{n+1} \frac{1}{j!} = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} \\ &< 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} \\ &= 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} \\ &< 3 \end{aligned}$$

Thus a_n has an upper bound and hence a_n converges, and we define $\lim_{n\to\infty} a_n =: e$.

Definition 6 (subsequence)**.** Let $\mathbb{N} \xrightarrow{a.} \mathbb{R}$ be a sequence, a subsequence $a_{n_m}(m \in \mathbb{N})$ is a composite map

$$\mathbb{N} \xrightarrow{n.} \mathbb{N} \xrightarrow{a.} \mathbb{R}$$

where $n.$ is a strictly monotone injection and $m \mapsto n(m)$ denoted by n_m.

That is for any $m_1, m_2 \in \mathbb{N}, m_1 > m_2 \Rightarrow n(m_1) = n_{m_1} > n_{m_2} = n(m_2)$.

Exercise 6. *Let $\mathbb{N} \xrightarrow{a.} X$ be a sequence in metric space*[1] (X, d)*,* $a_{n_m}(m \in \mathbb{N})$ *is a subsequence of* $a_n(n \in \mathbb{N})$*, show that if* $\exists l \in X$ *s.t.* $\lim_{n\to\infty} a_n = l \Rightarrow \lim_{m\to\infty} a_{n_m} = l$*.*

Proof. For any $\epsilon > 0, \exists N \in \mathbb{N},$ s.t. $\forall n \geq N, d(x_n, l) < \epsilon$. On the other hand, $n. \nearrow\nearrow \Rightarrow \lim_{m\to\infty} n(m) = \infty \Rightarrow$ and hence $\exists M \in \mathbb{N},$ s.t. $\forall m \geq M \Rightarrow n_m \geq N \Rightarrow d(a_{n_m}, l) < \epsilon \Rightarrow \lim_{m\to\infty} a_{n_m} = l$. $\square$

1.3 Nested Intervals

Definition 7 (Nested)**.** A seq. of intervals $I_n(n \in \mathbb{N})$ is nested if $I_n \neq \emptyset$ and $I_{n+1} \subseteq I_n$ for $\forall n \in \mathbb{N}$.

Example 3. If we have a seq. of nested intervals $I_n(n \in \mathbb{N})$, do we have $\cap_{n\in\mathbb{N}} I_n \neq \emptyset$? The answer is not sure. For example,

1. $I_n = (0, 1/n), n \in \mathbb{N}$, then for any $r \in \mathbb{R}, \exists N \in \mathbb{N},$ s.t. $1/N < r$ by Archimedean Property, thus $r \notin I_N$, and hence $\cap_{n\in\mathbb{N}} I_n = \emptyset$;
2. $I_n = [n, \infty), n \in \mathbb{N}$, then for any $r \in \mathbb{R}, \exists N \in \mathbb{N},$ s.t. $r < N$ by Archimedean Property, thus $r \notin I_N$, and hence $\cap_{n\in\mathbb{N}} I_n = \emptyset$;

Theorem 3 (Theorem of Nested Interval)**.** *If $I_n(n \in \mathbb{N})$ is a seq. of bounded closed nested intervals, then $\cap_{n\in\mathbb{N}} I_n \neq \emptyset$. (In the other word, there exists a real number $c \in \mathbb{R}$ such that $c \in \cap_{n\in\mathbb{N}} I_n$)*

Proof. Write $I_n = [a_n, b_n](n \in \mathbb{N})$, then $I_n(n \in \mathbb{N})$ is nested $\Leftrightarrow a_n \leq b_n$ and $a_n\nearrow$ and $b_n\searrow$. And furthermore, for $\forall n, m \in \mathbb{N}$,

$$a_n \leq a_{\max\{m,n\}} \leq b_{\max\{m,n\}} \leq b_m,$$

in the other word, for $\forall m \in \mathbb{N}$, b_m is an upper bound of $\{a_n | n \in \mathbb{N}\}$, thus seq. a_n converges. Let $c = \lim_{n\to\infty} a_n$, then given $m \in \mathbb{N}$, for $\forall n \in \mathbb{N}, a_n \leq b_m$ thus

$$c = \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_m = b_m.$$

[1]The concept of metric space will be given in Chapter 3.

On the other hand, $c = \sup\{a_n | n \in \mathbb{N}\}$, thus for all $m \in \mathbb{N}$, we have

$$a_m \leq c \leq b_m$$

thus $c \in I_m$ for $\forall m \in \mathbb{N} \Rightarrow c \in \cap_{n \in \mathbb{N}} I_n$. $\square$

Exercise 7. *Show that* $\cap_{n \in \mathbb{N}} I_n \neq \emptyset$, *if*
1. $I_n = (a_n, b_n)$, *nested and* $a_n \nearrow\nearrow$ *and* $b_n \searrow\searrow$?
2. $I_n = (a_n, \infty)$, *nested and* $\{a_n | n \in \mathbb{N}\}$ *is bounded from above.*

Proof. 1. Just as analyzed before, there exist $c \in \mathbb{R}$ such that $c = \lim_{n \to \infty} a_n$, and $c = \sup\{a_n | n \in \mathbb{N}\}$ and hence $a_n \leq c \leq b_m$ for $\forall n, m \in \mathbb{N}$. Note that $a_n \leq c$ implies that $a_n < c$ for $\forall n \in N$, otherwise if $\exists n' \in \mathbb{N}$, s.t. $a_{n'} = c$ then

$$a_{n'+1} \geq a_{n'} = c,$$

which leads to the contradiction. In the same way $c \leq b_m$ implies that $c < b_m$ for $\forall m \in \mathbb{N}$. Thus there $\exists c \in \mathbb{R}$ such that

$$a_n < c < b_m$$

for $\forall n, m \in \mathbb{N} \Rightarrow c \in \cap_{n \in \mathbb{N}} I_n$.
2. Since $I_n = (a_n, \infty)$ is a nested interval, $a_n \nearrow \Rightarrow a_n$ converges since a_n is upper bounded. That is $\exists c \in \mathbb{R}$, s.t. $c = \lim_{n \to \infty} a_n = \sup\{a_n\}$, thus for $\forall n \in \mathbb{N}, c \geq a_n$, that is

$$c + 1 > c \geq a_n$$

for $\forall n \in \mathbb{N} \Rightarrow c + 1 \in \cap_{n \in \mathbb{N}} I_n$. $\square$

Exercise 8. *Show that Theorem of Nested Interval implies Dedekind's Gapless Property.*

Proof. Let (A, B) be a Dedekind cut of $\mathbb{R}$, pick a from A and b from B, and form an interval $I_0 = [a, b]$. Then $(a + b)/2$ lies in the middle of I_0 and must belong to A or B. If $(a + b)/2$ belongs to A, we let

$$a_1 = \frac{a+b}{2}, \quad b_1 = b$$

and if $(a + b)/2$ belongs to B, let

$$a_1 = a, \quad b_1 = \frac{a+b}{2}$$

and hence we can form a new interval $I_1 = [a_1, b_1]$ whose length is half of the former I_0. Repeat this process, we obtain a seq. of nested intervals

$$I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots,$$

where $I_n = [a_n, b_n], b_n - a_n = (b_{n-1} - a_{n-1})/2$. Thus there exists $s \in \mathbb{R}$ lies in the $\cap_{n \in \mathbb{N}} I_n$ by the theorem of nested intervals, and either $s \in A$ or $s \in B$.
Assume that $s \in A$, for any $s' \in \mathbb{R}, s < s'$, exists b_n such that $s < b_n < s'$ since $b_n \to s$, thus $s' \in B$. That is $s \in A$ and for any $s' > s, s' \in B$. In the other word, s is the maximal element of A and B has no minimal element in this case, since assume s' is the minimal element of B then $\exists b_n$, s.t. $b_n < s'$ and $b_n \in B$, which is a contradiction. $\square$

Remark 3. Summary, we have discussed
1) Dedekind's Gapless Property;
2) Weierstrass Theorem;
3) Monotone Seq. Property;
4) Theorem of Nested Interval.
which have the relationship:

$$\begin{array}{ccc} 1) & \Longrightarrow & 2) \\ \Uparrow & & \Downarrow \\ 4) & \Longleftarrow & 3) \end{array}$$

These 5 properties are equivalent and we call the these the **Completeness of the real numbers**.

1.4 Limit superior / inferior

Let $a_n (n \in \mathbb{N})$ be a bounded (upper bdd. and lower bdd.) seq. in $\mathbb{R}$, we define **upper seq. of** a_n as

$$u_n := \sup\{a_m | m \geq n\},$$

and **lower seq. of** a_n as

$$l_n := \inf\{a_m | m \geq n\},$$

for $n \in \mathbb{N}$. Thus give $n \in \mathbb{N}$, we have that for $\forall m \geq n$

$$l_n \leq a_m \leq u_n,$$

We now show that l_n and u_n is monotone. Assume that $\exists n \in \mathbb{N}$, s.t. $u_n < u_{n+1}$, let $\epsilon = (u_{n+1} - u_n)/2$, then

$$u_{n+1} - \epsilon > u_n = \sup\{a_m | m \geq n\},$$

thus for $\forall m \geq n$, $u_{n+1} - \epsilon > a_m$ and hence $u_{n+1} - \epsilon$ is an upper bound of $\{a_m | m \geq n+1\}$, which leads to a contradiction. Thus for $\forall n \in \mathbb{N}, u_n \geq u_{n+1} \Rightarrow u_n \searrow$, and $l_n \nearrow$ in the same way.

Thus we have that for any $n, m \in \mathbb{N}$,

$$l_m \leq l_{\max\{m,n\}} \leq u_{\max\{m,n\}} \leq u_n,$$

thus l_1 is a lower bound for $\{u_n | n \in \mathbb{N}\}$ and u_1 is an upper bound of $\{l_n | n \in \mathbb{N}\}$ and hence $u_n, l_n (n \in \mathbb{N})$ are convergent by Monotone seq. property. We define the **limit superior** of a_n as the limit of u_n:

$$\overline{\lim_{n\to\infty}} a_n := \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sup_{m\geq n} a_m = \inf_{n\in\mathbb{N}} \sup_{m\geq n} a_m$$

The last equals sign is because $u_n\searrow$ and hence converges to its greatest lower bound by Monotone seq. property. And the **limit inferior** of a_n as the limit of l_n:

$$\underline{\lim_{n\to\infty}} a_n := \lim_{n\to\infty} l_n = \lim_{n\to\infty} \inf_{m\geq n} a_m = \sup_{n\in\mathbb{N}} \inf_{m\geq n} a_m$$

Exercise 9. *Let $a_n (n \in \mathbb{N})$, show that*

$$a_n \textit{ converges} \Leftrightarrow \overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n$$

and if any of both sides holds, then

$$\lim_{n\to\infty} a_n = \overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n$$

Proof. $\Rightarrow$: Suppose that $\lim_{n\to\infty} a_n = s$. Then for any $\epsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $\forall n \geq N, |a_n - s| < \epsilon/2$, thus $s - \epsilon/2 < a_n < s + \epsilon/2$ for $\forall n \geq N$. Thus the upper seq. u_n of a_n has

$$s - \frac{\epsilon}{2} < a_n \leq u_n \leq s + \frac{\epsilon}{2},$$

for $\forall n \geq N$. The third inequality symbol is because if $\exists n' \geq N$ such that $u_{n'} > s + \epsilon/2$, then there exist a real number q such that $s + \epsilon/2 < q < u_{n'}$ and $q > s + \epsilon/2 > a_n$ for $\forall n \geq N$ and hence $q > a_n$ for $\forall n \geq n'$, and then $u_{n'}$ is not the least upper bound of $\{a_n | n \geq n'\}$ which is contrary. Thus $|u_n - s| \leq \epsilon/2 < \epsilon$, thus

$$\lim_{n\to\infty} u_n = \overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} a_n = s,$$

and $\lim_{n\to\infty} l_n = \underline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty} a_n = s$ in the same way.
$\Leftarrow$: Suppose $\lim_{n\to\infty} u_n = \lim_{n\to\infty} l_n = s$, then for $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ one has $|u_n - s| < \epsilon/3$ and $|l_n - s| < \epsilon/3$ and $|u_n - l_n| \leq |u_n - s| + |l_n - s| < 2\epsilon/3$, since $l_n \leq a_n \leq u_n$ then $0 \leq a_n - l_n \leq u_n - l_n$. Then we have that

$$\begin{aligned}
|a_n - s| &= |a_n - l_n + l_n - s| \\
&\leq |a_n - l_n| + |l_n - s| \\
&\leq |u_n - l_n| + |l_n - s| \\
&< \epsilon
\end{aligned}$$

for $\forall n \geq N \Rightarrow \lim_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n = \underline{\lim}_{n\to\infty} a_n = s$. $\square$

Exercise 10. *Let $a_n, b_n (n \in \mathbb{N})$ be two bdd. seq. show that*
1. $\overline{\lim}_{n\to\infty}(a_n + b_n) \le \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n$*;*
2. $\underline{\lim}_{n\to\infty} a_n + \underline{\lim}_{n\to\infty} b_n \le \underline{\lim}_{n\to\infty}(a_n + b_n)$*.*

Proof. 1. Let $u_n = \sup_{m\ge n} a_m, v_n = \sup_{m\ge n} b_m, w_n = \sup_{m\ge n}(a_m + b_m)$. If $\exists n' \in \mathbb{N}$ such that $w_{n'} > u_{n'} + v_{n'}$, then $\exists r \in \mathbb{R}$ s.t. $u_{n'} + v_{n'} < r < w_{n'}$ and hence for any $m \ge n'$, $a_m \le u_{n'}, b_m \le v_{n'}$ and

$$a_m + b_m \le u_{n'} + v_{n'} < r$$

which means r is an upper bound of $\{a_m | m \ge n'\}$ which leads to a contradiction with $w_{n'}$ is the least upper bound of $\{a_m | m \ge n'\}$. Thus for $\forall n \in \mathbb{N}, u_n + v_n \le w_n$, and since $\lim_{n\to\infty} u_n, \lim_{n\to\infty} v_n$ exists, we have that

$$\lim_{n\to\infty}(u_n + v_n) = \lim_{n\to\infty} u_n + \lim_{n\to\infty} v_n \le \lim_{n\to\infty} w_n$$

that is

$$\overline{\lim_{n\to\infty}}\, a_n + \overline{\lim_{n\to\infty}}\, b_n \le \overline{\lim_{n\to\infty}}\,(a_n + b_n).$$

2. The same as 1. $\square$

And in the same way, we can prove that
1. $\overline{\lim}_{n\to\infty}(a_n \cdot b_n) \le \overline{\lim}_{n\to\infty} a_n \cdot \overline{\lim}_{n\to\infty} b_n$;
2. $\underline{\lim}_{n\to\infty} a_n \cdot \underline{\lim}_{n\to\infty} b_n \le \underline{\lim}_{n\to\infty}(a_n \cdot b_n)$.
In general, the properties does not hold for subtraction.

1.5 Cauchy seq.

Given a seq. $a_n (n \in \mathbb{N})$ in $\mathbb{R}$, can we determine whether a_n converges or not without referring a limit candidate l, but concluding according to the mutual behavior of the terms of $a_n (n \in \mathbb{N})$?

Definition 8 (Cauchy Sequence)**.** A seq. $a_n (n \in \mathbb{N})$ in $\mathbb{R}$ is a Cauchy seq. if $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n, m \ge N \Rightarrow |a_n - a_m| < \epsilon$.

Exercise 11. *Show that*
1. a_n is convergent $\Rightarrow$ a_n is Cauchy seq.
2. a_n is Cauchy seq. $\Rightarrow$ a_n is bounded.

Proof. 1. assume that a_n converges to l, then for any $\epsilon > 0, \exists N \in \mathbb{N}, \forall n \ge N$ one has $|a_n - l| < \epsilon/2$, then for any $m, n \ge N$ we have

$$|a_m - a_n| \le |a_m - l| - |a_n - l| < \epsilon$$

thus $a_n (n \in \mathbb{N})$ is Cauchy seq.

2. For $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n, m \geq N$ one has $|a_m - a_n| \leq \epsilon$, thus for $\forall m \geq N \Rightarrow |a_N - a_m| \leq \epsilon \Rightarrow a_n - \epsilon \leq a_m \leq a_N + \epsilon$, thus $a_n (n \in N)$ has upper and lower bound

$$\max\{a_1, \cdots, a_N, a_N + \epsilon\}, \quad \min\{a_1, \cdots, a_N, a_N - \epsilon\},$$

thus a_n is bounded. □

Theorem 4. *Let $a_n (n \in \mathbb{N})$ be a seq. in $\mathbb{R}$, then a_n is convergent $\Leftrightarrow a_n$ is Cauchy seq.*

Proof. $\Leftarrow$: a_n is Cauchy seq. $\Rightarrow a_n$ is bdd. $\Rightarrow$ the upper/lower seq. u_n, l_n of a_n converges. Thus $\lim_{n\to\infty} u_n - \lim_{n\to\infty} l_n = \lim_{n\to\infty}(u_n - l_n)$. For $\forall \epsilon > 0$, there is some $N \in \mathbb{N}$ s.t. $\forall m, n \geq N, |a_n - a_m| < \epsilon/3$. In particular, $\forall n \geq N \Rightarrow |a_n - a_N| < \epsilon/3$ and hence

$$a_N - \frac{\epsilon}{3} < a_n < a_N + \frac{\epsilon}{3}$$

which means $a_N - \epsilon/3$ is a lower bound of $a_n (n \geq N)$ and is not greater that $\{a_n | n \geq N\}$'s greatest lower bound l_N, and the same to $a_N + \epsilon/3$, thus

$$a_N - \frac{\epsilon}{3} \leq l_N \leq u_N \leq a_N + \frac{\epsilon}{3}$$

and since $l_n \nearrow$ and $u_n \searrow$, we have that for $\forall n \geq N$

$$0 \leq u_n - l_n \leq u_N - l_N \leq \frac{2\epsilon}{3} < \epsilon$$

thus $\lim_{n\to\infty}(u_n - l_n) = 0 \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} l_n \Rightarrow a_n$ converges. □

Exercise 12. *Let $S \subseteq \mathbb{R}$, if $|s - s'| \leq 3$ for $\forall s, s' \in S$, show that*
1. S is bdd.;
2. $\sup S - \inf S \leq 3$;

Proof. 1. If S has no upper bound, then for any $s \in S$, define $M = s + 4$, then $\exists s' \in S$ s.t. $s' > M = s + 4 \Rightarrow s' - s = |s' - s| > 4$, which is contrary.
2. Let $u = \sup S, l = \inf S$, suppose $u - l > 3$, then let $\epsilon = u - l - 3$, we have that $\exists s \in S$, s.t.

$$u - \frac{\epsilon}{3} < s \leq u,$$

and $\exists s' \in S$ s.t.

$$l \leq s' < l + \frac{\epsilon}{3},$$

then

$$u - l - \frac{2\epsilon}{3} < s - s' \leq u - l$$

thus $3 + \epsilon/3 < s - s' = |s - s'| \leq 3 + \epsilon \Rightarrow |s - s'| > 3$, which is contrary. □

Chapter 2

Series

2.1 Positive series

Definition 9. Let $a_n(n \in \mathbb{N})$ be a seq. in $\mathbb{R}$, we say that the series $\sum_n^\infty a_n$ (or $\sum_n a_n$) converges to a real number s if

$$\lim_{n\to\infty} s_n = s,$$

where $s_n := \sum_{j=1}^n a_j$ is called the n - th partial sum of $\sum_n a_n$.

If such s exists (resp. does not exist), we say that the series $\sum_n a_n$ convergent (resp. divergent). For a series $\sum_n a_n$ and $l, m \in \mathbb{N}, l < m$, we let $s_{l,m} := \sum_{j=l}^m a_j$ the (l, m) - tail of $\sum_n a_n$. If a series $\sum_n a_n$ converges, we denote it as $\sum_n a_n < \infty$.

Exercise 13. *If a series $\sum_n a_n < \infty$, show that $\lim_{n\to\infty} a_n = 0$.*

Proof. Trivial. □

$\sum_n a_n$ converges $\Leftrightarrow$ s_n converges by definition and $\Leftrightarrow$ s_n is Cauchy seq., i.e. $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geq N$, (assume that $n > m$)

$$\begin{aligned} |s_n - s_m| &= |a_{m+1} + \cdots + a_n| \\ &= |a_{m+1} + a_{m+2} + \cdots + a_{m+1+(n-1)}| \\ &\leq \epsilon. \end{aligned}$$

In particular, for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ and for $\forall k > N, \forall l \geq 0$, then $|a_k| + \cdots + |a_{k+l}| < \epsilon \Leftrightarrow \sum_n |a_n|$ convergent $\Leftrightarrow \exists M > 0, \forall n \in \mathbb{N}, s_n = \sum_{j=1}^n a_j \leq M$, since $s_n \nearrow$. Collectively, we have some conclusions:

1. series $\sum_n a_n$ converges $\Leftrightarrow$ for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ and for $\forall k > N, \forall l \geq 0$, $|a_k + \cdots + a_{k+l}| < \epsilon$;
2. series $\sum_n b_n$, where $b_n \geq 0$, converges $\Leftrightarrow \exists M > 0, \forall n \in \mathbb{N}, \sum_{j=1}^n b_j \leq M$.
3. series $\sum_n |a_n|$ converges $\Rightarrow \sum_n a_n$ converges.

Example 4. Given series $\sum_n 1/n$. we have that

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + \frac{1}{2} \\ s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2} \\ s_8 &\geq 1 + \frac{2}{2} + \frac{1}{5} + \cdots + \frac{1}{8} \geq 1 + \frac{2}{2} + \frac{4}{8} = 1 + \frac{3}{2} \end{aligned}$$

In general, for $\forall m \in \mathbb{N}, s_{2^m} \geq 1 + m/2$ which has no upper bound $\Leftrightarrow \sum_n 1/n$ diverges.

Example 5. Given series $\sum_n 1/n^2$. we have that $1/n^2 < 1/((n-1)n) = 1/(n-1) - 1/n$. Then

$$\begin{aligned} s_n &= 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \\ &< 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} \\ &= 2 - \frac{1}{n} \\ &< 2. \end{aligned}$$

thus s_n has upper bound 2 $\Leftrightarrow \sum_n 1/n^2$ converges.

Definition 10. Given a seq. $a_n (n \in \mathbb{N})$, we say that
1. $\sum_n a_n$ converges absolutely if $\sum_n |a_n|$ converges;
2. $\sum_n a_n$ converges conditionally if $\sum_n |a_n|$ diverges but $\sum_n a_n$ converges.

Theorem 5 (Comparison Test). *If $a_n, b_n \geq 0 (n \in \mathbb{N})$, then $\exists C > 0$ and $N \in \mathbb{N}, n \geq N \Rightarrow a_n \leq C b_n \Rightarrow [\sum_n b_n < \infty \Rightarrow \sum_n a_n < \infty]$.*

Proof. If $\sum_n b_n$ converges, then for $\forall n \geq N$,

$$\begin{aligned} a_1 + \cdots + a_n &= a_1 + \cdots + a_N + a_{N+1} + \cdots + a_n \\ &\leq a_1 + \cdots + a_N + C \cdot (b_{N+1} + \cdots + b_n) \\ &\leq a_1 + \cdots + a_N + C \cdot M =: H, \end{aligned}$$

where M is an upper bound of $\sum_{j=1}^n b_j$, thus $\sum_j^n a_j$ as upper bound $H \Leftrightarrow \sum_n a_n$ converges. $\square$

Theorem 6 (Limit Form of Comparison Test). *If $a_n, b_n \geq 0 (n \in \mathbb{N})$, and if $\lim_{n\to\infty} a_n/b_n$ exists $\Rightarrow [\sum_n b_n < \infty \Rightarrow \sum_n a_n < \infty]$.*

Proof. Let $l = \lim_{n\to\infty} a_n/b_n$, then for $\epsilon = 1, \exists N \in \mathbb{N},$ s.t. $\forall n \geq N, a_n/b_m < l + 1 \Rightarrow a_n < (l+1)b_n$, which follows the proof by Comparison test. Furthermore if $l \neq 0$, then for $\epsilon = l/2, \exists N_l \in \mathbb{N}$, s.t. $\forall n \geq N_l$, s.t.

$$\frac{l}{2} \leq \frac{a_n}{b_n} \leq \frac{3l}{2},$$

and hence $b_n \leq a_n \cdot 2/l$ and $a_n \leq b_n \cdot 3l/2$, therefore $\sum_n b_n$ converges $\Leftrightarrow \sum_n a_n$ converges. □

Exercise 14. *If $a_n, b_n \geq 0 (n \in \mathbb{N})$, show that if $\overline{\lim}_{n\to\infty} a_n/b_n$ exists $\Rightarrow [\sum_n b_n < \infty \Rightarrow \sum_n a_n < \infty]$.*

Proof. Let $u_n = a_n/b_n$ and $\lim_{n\to\infty} u_n = l$, then $l = \inf_{n\in\mathbb{N}} u_n$ and for $\epsilon = 1, \exists n' \in \mathbb{N}$ s.t.

$$l \leq u_{n'} < l + 1$$

and hence for $\forall n \geq n'$ we have that

$$\frac{a_n}{b_n} \leq u_{n'} < l + 1$$

thus $a_n < (l+1) \cdot b_n$ for $\forall n \geq n'$ and finish the proof by comparison test. □

Exercise 15 (Ratio and Root test). *If $a_n, b_n \geq 0 (n \in \mathbb{N})$, show that*
1. $\lim_{n\to\infty} a_{n+1}/a_n < 1 \Rightarrow \sum_n a_n < \infty$; $\lim_{n\to\infty} a_{n+1}/a_n > 1 \Rightarrow \sum_n a_n$ diverges.
2. $\lim_{n\to\infty} (a_n)^{1/n} < 1 \Rightarrow \sum_n a_n < \infty$; $\lim_{n\to\infty} (a_n)^{1/n} > 1 \Rightarrow \sum_n a_n$ diverges.

Proof. Trivial. □

2.2 Alternating series

Definition 11. A series $\sum_n a_n$ is called alternating series, if $\exists b_n > 0 (n \in \mathbb{N})$ s.t. $a_n = (-1)^{n-1} b_n (n \in \mathbb{N})$.

Theorem 7 (Leibniz's Criterion). *Let $\sum_n a_n$ be an alternating series, and $b_n = |a_n|_{\searrow 0}$ as $n \to \infty$, then $\sum_n a_n < \infty$.*

Proof. Since $b_n = (-1)^{n-1} a_n$, for any $k, l \in \mathbb{N}$ the tail of $\sum_n a_n$ is

$$\begin{aligned} |a_k + \cdots + a_{k+l}| &= (-1)^{k-1} |b_k - b_{k+1} + \cdots + (-1)^l b_{k+l}| \\ &= |b_k - b_{k+1} + \cdots + (-1)^l b_{k+l}| \end{aligned}$$

define $\lambda_{k,l} = b_k - b_{k+1} + \cdots + (-1)^l b_{k+l}$. Then if l is even, then

$$\lambda_{k,l} = (b_k - b_{k+1}) + \cdots + (b_{k+l-2} - b_{k+l-1}) + b_{k+l} \geq 0,$$

and if l is odd, then

$$\lambda_{k,l} = (b_k - b_{k+1}) + \cdots + (b_{k+l-1} - b_{k+l}) \geq 0,$$

thus $\lambda_{k,l} \geq 0$ for $\forall k, l \in \mathbb{N}$. And hence

$$\begin{aligned}
|a_k + \cdots + a_{k+l}| &= |\lambda_{k,l}| = \lambda_{k,l} \\
&= \begin{cases} b_k - (b_{k+1} - b_{k+2}) - \cdots - (b_{k+l-1} - b_{k+l}), & l \text{ is even} \\ b_k - (b_{k+1} - b_{k+2}) - \cdots - (b_{k+l-2} - b_{k+k-1}) - b_{k+l}, & l \text{ is odd} \end{cases} \\
&\leq b_k
\end{aligned}$$

Since $\lim_{n\to\infty} b_n = 0 \Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N},$ s.t. $n \geq N \Rightarrow |b_n| < \epsilon \Rightarrow |a_n + \cdots + a_{n+l}| \leq b_n < \epsilon$ for $\forall l \in \mathbb{N}$, thus $\sum_n a_n$ converges. □

2.3 Rearrangement theorem

Given a seq. $a_n (n \in \mathbb{N})$, we can separate all terms into two subseq.s:

$$a_{n_1}, a_{n_2}, \cdots \text{ and } a_{n'_1}, a_{n'_2}, \cdots$$

where $n_1 < n_2 < \cdots$ and $n'_1 < n'_2 < \cdots$ and $\{n_1, n_2, \cdots\} \cup \{n'_1, n'_2, \cdots\} = \mathbb{N}$, such that $a_{n_j} \geq 0 (j \in \mathbb{N}), a_{n'_k} \leq 0 (k \in \mathbb{N})$. Let $p_j := a_{n_j} (j \in \mathbb{N})$ and $q_k := a_{n'_k} (k \in \mathbb{N})$.

Exercise 16. *Show that $\sum_n |a_n| < \infty \Leftrightarrow \sum_j p_j < \infty$ and $\sum_k q_k < \infty$. Moreover, if any side holds, then*

$$\sum_n |a_n| = \sum_j p_j + \sum_k q_k$$

and

$$\sum_n a_n = \sum_j p_j - \sum_k q_k.$$

Proof. 1. $\Rightarrow$: since $\sum_n |a_n| < \infty$, any partial sum of a_n has upper bound such as M, then for any $j \in \mathbb{N}$:

$$\begin{aligned}
p_1 + \cdots + p_j &= |a_{n_1}| + \cdots + |a_{n_j}| \\
&\leq \sum_{n=1}^{n_j} |a_n| \\
&\leq M,
\end{aligned}$$

Thus any partial sum of p_j has upper bound M and hence $\sum_j p_j < \infty$. And $\sum_k q_k < \infty$ in the same way.

2. $\Leftarrow$: The partial sum of $\sum_n |a_n|$ can be decompose by the partial sums of $\sum_n p_n$ and $\sum_n q_n$ which have upper bounds, thus partial sum of $\sum_n |a_n|$ has upper bound, and $\sum_n |a_n| < \infty$.
3. Define the partial sum of $\sum_n |a_n|, \sum_n a_n, \sum_n p_n, \sum_n q_n$ as

$$A_m := \sum_{i=1}^{m} |a_i|, \quad S_m := \sum_{i=1}^{m} a_i$$
$$P_m := \sum_{i=1}^{m} p_i = \sum_{i=1}^{m} |a_{n_i}|, \ Q_m := \sum_{i=1}^{m} q_i = \sum_{i=1}^{m} |a_{n'_i}|$$

Then for any $m \in \mathbb{N}$, we have that

$$A_m \le P_m + Q_m \le A_{\max\{n_m, n'_m\}},$$

thus $\lim_{n\to\infty} A_n = \lim_{n\to\infty}(P_n + Q_n) = \lim_{n\to\infty} P_n + \lim_{n\to\infty} Q_n$ since $\sum_n p_n, \sum_n q_n$ exists, and the squeeze theorem. And hence $\sum_n |a_n| = \sum_j p_j + \sum_k q_k$.
On the contrary, for any $m \in \mathbb{N}$, we can represent the partial sum of $\sum_n a_n$ as

$$s_m = P_l - Q_v$$

where $l, v \to \infty$ as $m \to \infty$, thus $\sum_n a_n = \sum_n p_n - \sum_n q_n$. □

Exercise 17. *If $\sum_n a_n$ converges conditionally, show that*
1. $\sum_j p_j = \infty$ and $\sum_k q_k = \infty$;
2. $\lim_{j\to\infty} p_j = \lim_{k\to\infty} q_k = 0$.

Proof. 1. Denote the partial sum of $\sum_n a_n, \sum_j p_j, \sum_k q_k$ as s_n, P_j, Q_k respectively, then we have that $\lim_{n\to\infty} s_n = \lim_{n\to\infty}(P_j - Q_k)$ exists, then either both $\lim_{n\to\infty} P_j, \lim_{n\to\infty} Q_k$ exist or neither exists, since $\sum_n a_n$ converges conditionally $\Rightarrow \lim_{n\to\infty} P_j = \infty$ and $\lim_{n\to\infty} Q_k = \infty$.
2. Since

$$\lim_{j\to\infty} p_j = \lim_{j\to\infty} a_{n_j} = \lim_{n\to\infty} a_n = 0,$$

and $\lim k \to \infty q_k = 0$ as well in the same way. □

Exercise 18. *If $\sum_n a_n, \sum_n b_n$ converges, show that $\sum_n (a_n \pm b_n) = \sum_n a_n \pm \sum_n b_n$.*

Proof. Denote the partial sum of $\sum_n (a_n + b_n), \sum_n a_n, \sum_n b_n$ as S_n, A_n, B_n respectively, then for any $n \in \mathbb{N}$

$$S_n = A_n + B_n$$

and hence

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (A_n + B_n) = \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n$$

since $\lim_{n\to\infty} A_n, \lim_{n\to\infty} B_n$ exists, thus $\sum_n (a_n + b_n) = \sum_n a_n + \sum_n b_n$, and $\sum_n (a_n - b_n) = \sum_n a_n - \sum_n b_n$ in the same way. □

Exercise 19. *Inserting* 0*s to a series will not affect its convergence / divergence and its sum (if the sum exists).*

Proof. Consider the tail of series. Trivial. $\square$

Recall that a sequence a_n is a map $\mathbb{N} \xrightarrow{a.} \mathbb{R}$ where $n \mapsto a(n)$ denoted by a_n. A subsequence a_{n_m} is a composite map

$$\mathbb{N} \xrightarrow{n.} \mathbb{N} \xrightarrow{a.} \mathbb{R}$$

where $n.$ is a strictly monotone injection and $m \mapsto n(m)$ denoted by n_m. A **rearrangement** is also a composite map

$$\mathbb{N} \xrightarrow{n(\cdot)} \mathbb{N} \xrightarrow{a.} \mathbb{R}$$

where $n(\cdot)$ is a bijection. (To distinguish it from the notion of subseq., we don't use subscripts here)
Now we gonna explore two questions: if a series $\sum_n$ converges, $a_{n(m)}(m \in \mathbb{N})$ is a rearrangement of $a_n(n \in \mathbb{N})$, then
1. whether $\sum_m a_{n(m)}$ converges ?
2. whether $\sum_n a_n = \sum_m a_{n(m)}$?

Exercise 20. *Let $\sum_n a_n$ be a positive series, show that*

$$\sum_n a_n = \sup \Lambda$$

including the case $\sum_n a_n = \infty$. Here $\Lambda = \{a_{n_1} + \cdots + a_{n_k} | n_1 < \cdots < n_k, k \in \mathbb{N}\}$ represents the set of every sum of finite terms of $a_n(n \in \mathbb{N})$.

Proof. 1. $\leq$: since $\sum_n a_n$ is the limit of the partial sum s_n (which is the sum of finite terms, i.e. $s_n \in \Lambda$ for any $n \in \mathbb{N}$), and since $a_n \geq 0$, s_n monotone, then

$$\sum_n a_n = \lim_{n\to\infty} s_n = \sup_{n\in\mathbb{N}} s_n \leq \sup \Lambda$$

2. $\geq$: If $\sup \Lambda > \sup s_n$, let $\epsilon := \sup \Lambda - \sup s_n$, then $\exists \lambda = a_{n_1} + \cdots + a_{n_{k_\lambda}} \in \Lambda$ such that

$$s_m \leq \sup_{n\in\mathbb{N}} s_n < \sup \Lambda - \frac{\epsilon}{2} < \lambda \leq \sup \Lambda$$

for $\forall m \in \mathbb{N}$, but

$$\lambda = a_{n_1} + \cdots + a_{n_{k_\lambda}} \leq s_{n_{k_\lambda}}$$

which leads to a contradiction.
3. If $\sum_n a_n = \infty$, it is direct to see that $\sup \Lambda = \infty$ as well by 1. $\square$

Exercise 21. *If $\sum_n a_n$ is a convergent positive series, show that for every rearrangement $a_{n(m)}(m \in \mathbb{N})$ of $a_n(n \in \mathbb{N})$, we have that $\sum_n a_n = \sum_m a_{n(m)}$.*

Proof. If $\sum_n a_n$ is positive series, then $\sum_m a_{n(m)}$ is positive series as well.

$$\sum_n a_n = \sup \Lambda_{a_n} = \sup \Lambda_{a_{n(m)}} = \sum_m a_{n(m)}$$

where Λ_{a_n} and $\Lambda_{a_{n(m)}}$ are the set of every sum of finite terms of a_n and $a_{n(m)}$ respectively. That is the proof follows by the $\Lambda_{a_n} = \Lambda_{a_{n(m)}}$. □

Exercise 22 (Dirichlet's Rearrangement Theorem (1829))**.** *If $\sum_n a_n$ converges absolutely, show that for every rearrangement $a_{n(m)}(m \in \mathbb{N})$ of $a_n(n \in \mathbb{N})$, we have that $\sum_n a_n = \sum_m a_{n(m)}$.*

Proof. $\sum_n a_n$ converges absolutely $\Rightarrow \sum_m a_{n(m)}$ converges absolutely. Furthermore

$$\begin{aligned}\sum_n a_n &= \sum_j p_j - \sum_k q_k \\ &= \sum_\mu p_{j_\mu} - \sum_\nu q_{k_\nu} \\ &= \sum_m a_{n_m}.\end{aligned}$$

□

Theorem 8 (Riemann's Rearrangement Theorem(1852))**.** *If $\sum_n a_n$ converges conditionally, then for $\forall r \in \mathbb{R}$, there exists a rearrangement $a_{n(m)}(m \in \mathbb{N})$ of $a_n(n \in \mathbb{N})$ such that $\sum_m a_{n(m)} = r$.*

Proof. We will only use two known fact:
1. $\sum_j p_j = \infty$ and $\sum_k q_k = \infty$;
2. $\lim_{j\to\infty} p_j = \lim_{k\to\infty} q_k = 0$.

Given a $L \in \mathbb{R}$, start with p_1, plus by p_2 and so on till p_{m_1-1} where

$$\sum_i^{m_1-1} p_i \leq L \quad \text{but} \quad \sum_i^{m_1} p_i > L.$$

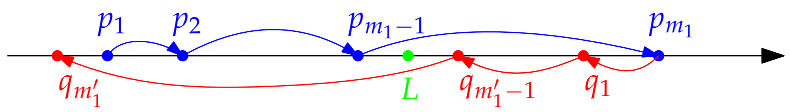
Then minus by q_1, q_2 and so on till $q_{m_1'-1}$ where

$$\sum_i^{m_1} p_1 - \sum_{j=1}^{m_1'-1} q_j > L \quad \text{but} \quad \sum_i^{m_1} p_1 - \sum_{j=1}^{m_1'} q_j \leq L.$$

This process can be repeat since $\sum_j p_j = \infty$ and $\sum_k q_k = \infty$ and hence any tail of $\sum_j p_j, \sum_k q_k$ has no upper bound, therefore the *cross* action can always happen, in the other word, $m_i, m_i'(i \in \mathbb{N})$ exists.

Thus we can form a rearrangement χ_n of $\sum_n a_n$ as

$$p_1, \cdots, p_{m_1}, -q_1, \cdots, -q_{m'_1}, \cdots$$



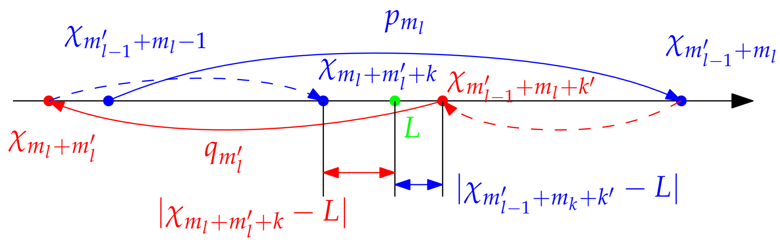
Now we will show that this rearrangement converges to L, i.e. $\lim_{n\to\infty} \chi_n = L$. Consider $\chi_{\cdots+m'_{l-1}+m_l-1}$ which implies the point lies in the left of L and will cross the l in next jump, and we denote it by $\chi_{m'_{l-1}+m_l-1}$ for simply. And hence we have that

$$|\chi_{m'_{l-1}+m_k+k'} - L| < p_{m_l}$$

if $0 \leq k' < m'_l - m'_{l-1}$. And similarly

$$|\chi_{m_l+m'_l+k} - L| < q_{m'_l}$$

if $0 \leq k < m_{m+1-m_l}$.



And since $\lim_{l\to\infty} p_{m_l} = \lim_{l\to\infty} q_{m'_l} = 0$, for $\forall\epsilon > 0, \exists N_0 \in \mathbb{N}$ s.t. $l \geq N_0 \Rightarrow p_{m_l}$ and $q_{m'_l} < \epsilon$. Let $N = m'_{N_0-1} + m_{N_0}$, then $n \geq N \Rightarrow |\chi_n - L| < \epsilon$. $\square$

Remark 4 ($2S = S$). Dirichelet made the following observation in 1827: Consider a alternating series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$$

which is convergent (conditionally) by Leibniz's Criterion, and hence

$$\begin{aligned}
2S &= 2 - 1 + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \frac{2}{7} - \frac{2}{8} + \frac{2}{9} - \frac{2}{10} + \cdots \\
&=' (2-1) - \frac{2}{4} + \left(\frac{2}{3} - \frac{2}{6}\right) - \frac{2}{8} + \left(\frac{2}{5} - \frac{2}{10}\right) + \cdots \\
&= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots \\
&= S
\end{aligned}$$

Remark 5. In summary, given a series $\sum_n a_n$, and its any rearrangement $\sum_m a_{n(m)}$, then
1. If $a_n \geq 0$ for $\forall n \in \mathbb{N} \Rightarrow \sum_n a_n = \sum_m a_{n(m)}$;
2. If $\sum_n |a_n| < \infty \Rightarrow \sum_n a_n = \sum_m a_{n(m)}$;
3. If $\sum_n |a_n| = \infty$ but $\sum_n a_n < \infty \Rightarrow \sum_m a_{n(m)}$ could be anything.

2.4 Multiplying absolutely convergent series

Proposition 2. *Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely, let*

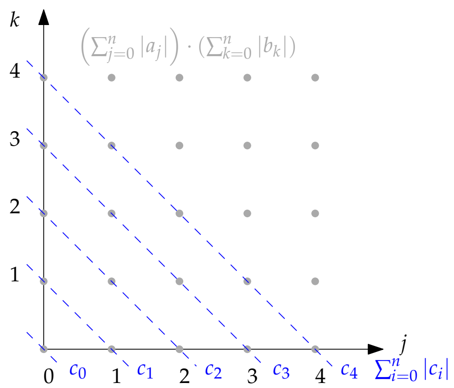
$$c_n = a_n b_0 + \cdots + a_0 b_n = \sum_{j+k=n;\ j,k\geq 0} a_j b_k,$$

then $\sum_n |c_n| < \infty$ and $\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right)$.

Proof. 1. $\sum_n |c_n| < \infty$
For all n,

$$\sum_{m=0}^{n} |c_m| = \sum_{m=0}^{n} \left| \sum_{\substack{j+k=m \\ j,k\geq 0}} a_j b_k \right| \leq \sum_{m=0}^{n} \sum_{\substack{j+k=m \\ j,k\geq 0}} |a_j||b_k|$$
$$\leq \left(\sum_{j=0}^{n} |a_j| \right) \cdot \left(\sum_{k=0}^{n} |b_k| \right).$$

Since $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converges absolutely, the partial sums of $|a_n|, |b_n|$ have upper bounds, denoted by M, N respectively, then $\sum_{m=0}^{n} |c_m|$ has a upper bound $M \cdot N$ and hence $\sum_{n=0}^{\infty} c_n$ converges absolutely.

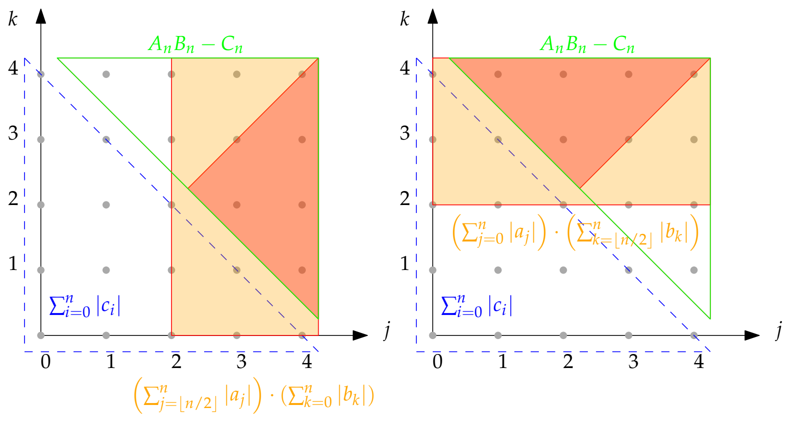


2. $\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right)$.
Let $A_n := a_0 + \cdots + a_n; B_n := b_0 + \cdots + b_n$ and $C_n := c_0 + \cdots + c_n$, we claim that $\lim_{n\to\infty}(A_n B_n - C_n) = 0$. Then

$$
\begin{aligned}
|A_n B_n - C_n| &= \sum_{\substack{j+k>n \\ 0\le j,k\le n}} |a_j b_k| \\
&\le \left(\sum_{j=\lfloor n/2\rfloor}^{n} |a_j|\right)\cdot\left(\sum_{k=0}^{n}|b_k|\right) + \left(\sum_{j=0}^{n}|a_j|\right)\cdot\left(\sum_{k=\lfloor n/2\rfloor}^{n}|b_k|\right)
\end{aligned}
$$

where $\sum_{k=0}^{n}|b_k|, \sum_{j=0}^{n}|a_j|$ are bounded, and tails $\sum_{j=\lfloor n/2\rfloor}^{n}|a_j|, \sum_{k=\lfloor n/2\rfloor}^{n}|b_k| \to 0$ as $n\to\infty$ since $\sum_n a_n, \sum_n b_n$ are converges abs. Thus $\lim_{n\to\infty}|A_nB_n - C_n| = 0$ and since $\lim_{n\to\infty} A_n, \lim_{n\to\infty} B_n, \lim_{n\to\infty} C_n$ exists, we have that

$$
\begin{aligned}
\sum_{n=0}^{\infty} c_n &= \lim_{n\to\infty} C_n \\
&= \lim_{n\to\infty} A_n B_b = \lim_{n\to\infty} A_n \cdot \lim_{n\to\infty} B_n \\
&= \left(\sum_{n=0}^{\infty} a_n\right)\cdot\left(\sum_{n=0}^{\infty} b_n\right)
\end{aligned}
$$



$\square$

Theorem 9. *If $\sum_n a_n, \sum_n b_n$ cvg. abs., $\mathbb{N} \xrightarrow{(j(\cdot),k(\cdot))} \mathbb{N}\times\mathbb{N}$ is bijection where $n \mapsto (j(n),k(n))$, let $c_n := a_{j(n)}b_{k(n)} (n\in\mathbb{N})$, then $\sum_n |c_n| < \infty$ (cvg. abs.) and $\sum_n c_n = (\sum_n a_n)(\sum_n b_n)$.*

Proof. 1. $\sum_n c_n$ cvg. abs.
For $\forall n\in\mathbb{N}$, let $l = \max\{j(1),\cdots,j(n),k(1),\cdots,k(n)\}$. Then

$$
\begin{aligned}
|c_1| + \cdots + |c_n| &= |a_{j(1)}b_{k(1)}| + \cdots + |a_{j(n)}b_{k(n)}| \\
&\le \left(\sum_{j=1}^{l}|a_j|\right)\cdot\left(\sum_{k=1}^{l}|b_k|\right) \\
&\le M\cdot N
\end{aligned}
$$

Thus $\sum_n c_n$ cvg. abs.
2. $\sum_n c_n = (\sum_n a_n)(\sum_n b_n)$.
Let $A_n = a_1 + \cdots + a_n$, $B_n = b_1 + \cdots + b_n$ and $C_n = c_1 + \cdots + c_n (n \in \mathbb{N})$. And define the bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by the second one in Figure 2.1. Then

$$\begin{aligned} A_n B_n &= (a_1 + \cdots + a_n)(b_1 + \cdots + b_n) \\ &= \sum_{1 \le j,k \le n} a_j b_k \\ &= C_{n^2} \end{aligned}$$

Thus $\lim_{n\to\infty} A_n B_n = \lim_{n\to\infty} C_{n^2} = \lim_{n\to\infty} C_n \Rightarrow \sum_n c_n = (\sum_n a_n)(\sum_n b_n)$. $\square$

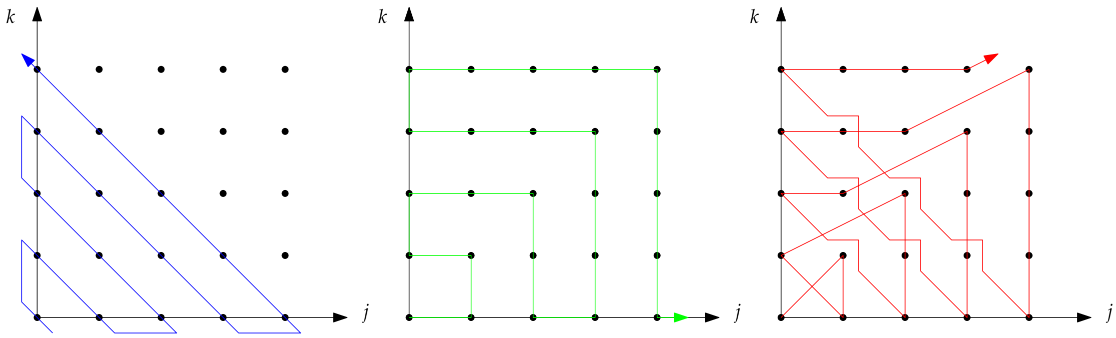


Figure 2.1: 3 kinds of bijections $(j(\cdot), k(\cdot))$

Chapter 3

Metric space

This chapter refers to *Chapter 2 of General Topology* Notes for details.

3.1 Metric space

Definition 12 (Metric Space)**.** Let $X \times X \xrightarrow{d} \mathbb{R}$ be a function, we cay that d is a metric on X or (X, d) is a metric space if for $\forall x, x', x'' \in X$ have
1. Positivity: $d(x, x') \geq 0$ and $d(x, x'') = 0$ iff $x = x'$;
2. Symmetry: $d(x, x') = d(x', x)$;
3. Triangle inequality: $d(x, x') \leq d(x, x'') + d(x'', x')$.

Exercise 23. *Show that the triangle inequality is equivalent with for* $\forall x, x', x'' \in X$

$$d(x, x') \geq |d(x, x'') - d(x', x'')|.$$

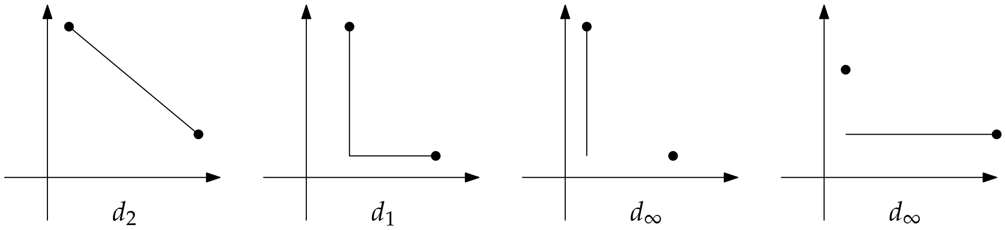
Proof. $\geq \Rightarrow \leq$: since $d(x, x') \geq |d(x, x'') - d(x', x'')| \geq d(x, x'') - d(x', x'')$, we have that $d(x, x'') \leq d(x, x') + d(x', x'')$.
$\leq \Rightarrow \geq$: if $\exists x, x', x''$ such that $d(x, x') < |d(x, x'') - d(x', x'')|$, then

$$\begin{aligned} d(x, x') &< |d(x, x'') - d(x', x'')| \\ &\leq |d(x, x') + d(x', x'') - d(x', x'')| \\ &\leq d(x, x') \end{aligned}$$

thus $d(x, x') < d(x, x')$, which leads to a contradiction. □

Example 6. Here are some metric examples:
1. define $d_2(x, y) := \left(\sum_i^m |x_i - y_m|^2\right)^{1/2}$, $x, y \in \mathbb{R}^m$. Then d_2 is a metric on $\mathbb{R}^m$ by cauchy inequality.
2. define $d_1(x, y) := \sum_{i=1}^m |x_i - y_i|$, $x, y \in \mathbb{R}^m$. Then d_1 is a metric on $\mathbb{R}^m$.

3. define $d_\infty(x,y) := \max\{|x_i - y_i|\}$, $i \in \{1,2,\cdots,m\}$, $x,y \in \mathbb{R}^m$. Then d_∞ is a metric on $\mathbb{R}^m$.



d_2 can be proved to be a metric by Cauchy inequality:

Exercise 24 (Cauchy inequality). *For any $(x_1,\cdots,x_n),(y_1,\cdots,y_n) \in \mathbb{R}^n$, show that*

$$(x_1y_1 + \cdots, x_ny_n)^2 \le (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2)$$

and \$=\$ holds iff $\exists a,b \in \mathbb{R}$ which are not all 0.

Proof. Consider the polynomial $p(t) = \sum_{i=1}^n (x_i t + y)^2 = t^2 \sum_{i=1}^n x_i^2 + 2\sum_{i=1}^n x_i y_i + \sum_{i=1}^n y_i^2 \ge 0$, thus $\Delta = 4\left(\sum_{i=1}^n x_i y_i\right)^2 - 4\sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2 \le 0 \Rightarrow \left(\sum_{i=1}^n x_i y_i\right)^2 \le \sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2$. $\square$

Example 7 (p-adic). If p is a prime number, $x \in \mathbb{Q}$, define

$$|x|_{p-adic} := \begin{cases} p^{-m}, & x = \frac{a}{b}\cdot p^m, x \ne 0, \\ 0, & x = 0, \end{cases}$$

where $a,b,m \in \mathbb{Z}$, $(a,p) = (b,p) = 1$. For $\forall x,y \in \mathbb{Q}$, define $d_{p-adic}(x,y) = |x-y|_{p-adic}$, then d_{p-adic} is a metric on $\mathbb{Q}$.

Assume $x = (a/b)p^m, y = (s/t)p^n \in \mathbb{Q}$ where $a,b,s,t,m,n \in \mathbb{Z}, (a,p) = (b,p) = (s,p) = (t,p) = 1, m > n$, then $|x|_{p-adic} = p^{-m} < |y|_{p-adic} = p^{-n}$, and

$$\begin{aligned} |x-y|_{p-adic} &= |(a/b)p^m - (s/t)p^n|_{p-adic} \\ &= \left|\frac{adp^{m-n} - bc}{bd}p^n\right|_{p-adic}. \end{aligned}$$

it is easy to check $adp^{m-n} - bc, bd \in \mathbb{Z}$ and $(adp^{m-n} - bc, p) = (bd,p) = 1$, thus

$$|x-y|_{p-adic} = p^{-n} = |y|_{p-adic} = \max\{|x|_{p-adic}, |y|_{p-adic}\}.$$

Thus for any $x,y,z \in \mathbb{Q}$, we have that

$$\begin{aligned} d_{p-adic}(x,y) &= \max\{|x|_{p-adic}, |y|_{p-adic}\} \\ &\le \max\{|x|_{p-adic}, |z|_{p-adic}\} + \max\{|z|_{p-adic}, |y|_{p-adic}\} \\ &= d_{p-adic}(x,z) + d_{p-adic}(y,z), \end{aligned}$$

which follows the triangle inequality, the other two conditions is trivial.

3.2 Open and compact on metric space

Definition 13 (Open Ball)**.** Let (X, d) be a metric space, for $\forall r > 0$ and $x_0 \in X$, we let

$$B_r(x_0) := \{x \in X | d(x, x_0) < r\},$$

and call it the open ball with center x_0 and radius r; let

$$\overline{B_r(x_0)} := \{x \in X | d(x, x_0) \le r\},$$

and call it the close ball with center x_0 and radius r.

Example 8 (discrete metric)**.** For $\forall x, x' \in \mathbb{R}^2$, define metric $d(x, x') = 0$ if $x = x'$, and $d(x, x') = 1$ if $x \neq x'$, then $B_1(x) = \{x\}$, $\overline{B_1(x)} = \mathbb{R}^2$, $B_{1.1}(x) = \mathbb{R}^2$.

Definition 14 (Open Set)**.** $S(\subseteq X)$ is called an Open Set of X with respect to d, if $\forall x_0 \in S$, $\exists r > 0$ such that $B_r(x_0) \subseteq S$; $F(\subseteq X)$ is Close Set of X w.r.t. d if $X \backslash F$ is open set of X w.r.t. d.

Exercise 25. *Prove that $B_r(x)$ is open set and $\overline{B_r(x)}$ is close.*

Proof. For $\forall x' \in B_r(x)$, we have $d(x, x') < r$, donate $r - d(x, x')$ by s, then for $\forall x'' \in B_{s/2}(x')$ satisfy

$$\begin{aligned} d(x, x'') &\le d(x, x') + d(x', x'') \\ &\le d(x, x') + \frac{s}{2} \\ &< r, \end{aligned}$$

thus $x'' \in B_r(x)$ and $B_{s/2}(x') \subseteq B_r(x)$ and $B_r(x)$ is a open set. For $\forall x' \in X \backslash \overline{B_r(x)}$ has $d(x, x') > r$. Denote $d(x, x') - r$ by t, then for $\forall x'' \in B_{t/2}(x')$ satisfy

$$\begin{aligned} d(x, x'') &\ge |d(x, x') - d(x', x'')| \\ &\ge d(x, x') - d(x', x'') \\ &\ge d(x, x;) - \frac{t}{2} \\ &> r. \end{aligned}$$

Thus $B_{t/2}(x') \subseteq X \backslash \overline{B_r}$ and $X \backslash \overline{B_r}$ is an open set, thus $\overline{B_r}$ is a close set. □

Exercise 26. *Let (X, d) be a metric space. show that*

1. $X, \emptyset \subseteq_{open} X$;
2. $O_1, O_2 \subseteq_{open} X \Rightarrow O_1 \cap O_2 \subseteq_{open} X$;
3. $O_\alpha \subseteq_{open} X, (\alpha \in A) \Rightarrow \cup_{\alpha \in A} O_\alpha \subseteq_{open} X$ *(α not necessarily be integral or countable);*
4. *All above corresponding statements for close set are true.*

Proof. 1. Obviously X is an Open set thus $\emptyset$ is a close set. If $\emptyset$ is not an open set, then $\exists x \in \emptyset$, $\forall r > 0$ such that $B_r(x) \nsubseteq \emptyset$, which is impossible. Thus $\emptyset$ is an open set (logically) and X is a close set;
2. $\forall x \in O_1 \cap O_2, \exists r_1, r_2 > 0$, s.t. $B_{r_1}(x) \subseteq O_1$ and $B_{r_2}(x) \subseteq O_2$. Thus $\forall x' \in B_{\min\{r_1,r_2\}}(x) = B_{r_1}(x) \cap B_{r_2}(x) \Rightarrow x' \in O_1 \cap O_2 \Rightarrow B_{\min\{r_1,r_2\}}(x) \subseteq O_1 \cap O_2$, thus $O_1 \cap O_2$ is open. Collectively, the intersection of any finite open sets is an open set;
3. For $\forall x \in \cup_{\alpha \in A} O_\alpha, \exists$ at least one $\alpha' \in A$, s.t. $x \in O_{\alpha'}$, then $\exists r > 0$, s.t. $B_r(x) \subseteq O_{\alpha'} \subseteq \cup_{\alpha \in A} O_\alpha$, thus $\cup_{\alpha \in A} O_\alpha$ is an open set;
4. Take complementary set: The union of finite close sets is close; the intersection of any close sets is close.

□

Remark 6. First 3 statements are the essential intuition for the definition of *Topology*.

Exercise 27. *Show that an open set is the union of open balls.*

Proof. Given an open set O, for any $o \in O, \exists r_o > 0$, s.t. $B_{r_o}(o) \subseteq O$, define $O' = \cup_{o \in O} B_{r_o}(o)$. Thus for $\forall x \in O', \exists o'$, s.t. $x \in B_{r'_o}(o') \subseteq O \Rightarrow O' \subseteq O$;
On the other hand, for any $y \in O, \exists r_y > 0$, s.t. $B_{r_y}(y) \subseteq O \Rightarrow y \in B_{r_y}(y) \subseteq O' \Rightarrow O \subseteq O'$. Thus $O = O' = \cup_{o \in O} B_{r_o}(o)$. □

Definition 15 (Convergence)**.** Let (X, d) be a metric space, $a_n \in X, (n \in \mathbb{N})$, $L \in X$, define $\lim_{n\to\infty} a_n = L$ w.r.t. d, if $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N$ s.t. $d(a_n, L) < \epsilon$, that is $a_n \in B_\epsilon(L)$.

Exercise 28. *Show that*
1. $\lim_{n\to\infty} a_n = L \Leftrightarrow \lim_{n\to\infty} d(a_n, L) = 0$;
2. $\lim_{n\to\infty} a_n = L \Leftrightarrow \forall L \in U \subseteq_{open} X, \exists N \in \mathbb{N}, \forall n \geq N$ s.t. $a_n \in U$.

Proof. (1) Trivial; (2) $\Rightarrow$: Suppose that $\lim_{n\to\infty} a_n = L$, for $\forall U$ that $L \in U$, $\exists r > 0$, s.t. $B_r(L) \subseteq U$, and $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, s.t. $a_n \in B_r(L) \subseteq U$; $\Leftarrow$: Suppose $L \in U \subseteq_{open} X$, then $\exists r > 0$ such that $B_r(L) \subseteq U$. Since $B_r(L)$ is also an open set, then $\exists N \in \mathbb{N}$, for $\forall n \geq N$, s.t. $a_n \in B_r(L) \subseteq U$. □

We say $S \subseteq X$ is bounded w.r.t. d, if $\exists r > 0$ and $x_0 \in X$, s.t. $S \subseteq B_r(x_0)$.

Theorem 10 (Bolzano-Weierstrass theorem)**.** *If $a_n \in \mathbb{R}^m (n \in \mathbb{N})$ is bounded w.r.t. d_2, then $\exists$ a subsequence $a_{n_m} (m \in \mathbb{N})$ which converges.*

Proof. We only prove $\mathbb{R}^2$ case. If we want to prove that the vector $a = (a_1, \cdots, a_m) \in \mathbb{R}^m \to L = (l_1, \cdots, l_m) \in \mathbb{R}^m$, all we need to prove is $\lim_{n\to\infty} a_i = l_i, (i = 1, \cdots, m)$.
Choose $M > 0$, s.t. $a_n \in Q = [-M, M] \times [-M, M]$ for all $n \in \mathbb{N}$. Divide Q into 4 squares with equal size and choose one, say Q_1, such that $|\{n | a_n \in Q\}| = \infty$. Select $n_1 \in \mathbb{N}$, such that $a_{n_1} \in Q_1$. Repeat this and we have $\cap_{k=1}^{\infty} Q_k = \{a\}$. By theorem of nested interval we have that $\lim_{k\to\infty} a_{n_k} = a$. □

Remark 7. The conclusion of Bolzano-Weierstrass theorem can be generalize to metric space (Remark 13).

Exercise 29. *Let (X,d) be a metric space, $F \subseteq X$ show that $F \subseteq_{close} X \Leftrightarrow \forall a_n \in F(n \in \mathbb{N})$ and $\lim_{n\to\infty} a_n = a \in X$ then $a \in F$.*

Proof. $\Rightarrow$: Assume that F is close and $a_n \in F$. If $a_n \to a \in X\backslash F$, then $\exists r > 0$, s.t. $B_r(a) \in X\backslash F$. Since $\lim_{n\to\infty} a_n = a$, for r, there exists $N \in \mathbb{N}, \forall n \geq N$, s.t. $d(a_n, a) < r$, i.e. $a_n \in B_r(a) \subseteq X\backslash F$, which leads to a contradiction. $\Leftarrow$: Suppose that $\forall a_n \in F(n \in \mathbb{N})$ and $\lim_{n\to\infty} a_n = a \in X$ then $a \in F$, and F is not close, which means $X\backslash F$ is not open, and $\exists x \in X\backslash F, \forall r > 0, B_r(x) \cap F \neq \emptyset$. Select $n \in \mathbb{N}$ such that $a_n = B_{\frac{1}{n}}(x) \cap F$. Thus $\lim_{n\to\infty} a_n = x \notin F$, which leads to a contradiction. □

Remark 8. Set family of sets as $\{B_{1/n}(x)\}_{n\in\mathbb{N}}$ is a very useful skill.

Definition 16 (Open cover, Compact set)**.** Let (X,d) be a metric space, $S \subseteq X$, $O_\alpha \in X(\alpha \in A)$, we say that $O_\alpha(\alpha \in A)$ form an open cover of S, if $S \subseteq \cup_{\alpha\in A} O_\alpha$. S is called a compact set if $\forall$ open cover $O_\alpha(\alpha \in A)$ of S, $\exists \alpha_1, \cdots, \alpha_m \in A$, s.t. $S \subseteq \cup_{i=1}^m O_{\alpha_i}$, where $\cup_{i=1}^m O_{\alpha_i}$ is called a finite subcover.

If there exists an open cover of F whose any finite subcover can not cover it, then F is not a compact set. for instance, let $F = (0,1), O_n = (1/n, 2), n \in \mathbb{N}$, then O_n is an open cover of F, however any finite subcover of O_n can not cover F.

Theorem 11 (Heine-Borel theorem)**.** *Let $S \subseteq \mathbb{R}^n$, then S is compact $\Leftrightarrow$ S is bounded and closed.*

Proof. $\Rightarrow$: Suppose that S is compact, select a point $s \in S$ arbitrarily, define $O_i = B_i(s)$, it is easy to check that $S \subseteq \cup_{i\in\mathbb{N}} O_i$, since for any $s' \in S$, we have that $s' \in B_{2d(s,s')}(s) \subseteq O_{\lceil 2d(s,s')\rceil}$. Since S is compact, there exists a finite subcover, thus S is bounded.
Suppose S is compact, but S is not closed, which means $X\backslash S$ is not open and $\exists x \in X\backslash S$, s.t. $\forall r > 0, B_r(x) \cap S \neq 0$. Since S is bounded, define $\iota = \sup_{s\in S} d(s,x)$, define open cover

$$O_n = B_{\frac{\iota}{n}}(x) - B_{\frac{\iota}{n+1}}(x),$$

thus $O_i \cap O_j = \emptyset(i \neq j)$ and $O_i \cap S \neq \emptyset(\forall i)$. Thus O_n has no finite subcover, which leads to a contradiction and S is closed.
$\Leftarrow$: Suppose that S is bounded and closed, and $\exists$ an open cover $O_\alpha(\alpha \in A)$ of S which admits no finite subcover. Choose a cube Q containing S (S is bounded), divide Q into 4 equal-sized cubes and select one of them denoted by Q_1, s.t. $Q_1 \cap S$ can not be covered by finitely many Q_α, select a point such that $s_1 \in Q_1 \cap S$. Repeat.
Obviously, $\lim_{n\to\infty} s_n = a$, notice that $s_n \in Q_n \cap S \subseteq S$, thus $\lim_{n\to\infty} s_n = a \in S$ for S is closed. Thus there exist O_i such that $a \in O_i$. Since O_i is open, $\exists r > 0$, s.t. $B_r(a) \subseteq O_i$.

Then $\exists N \in \mathbb{N}, \forall n \geq N$, s.t. $Q_n \subseteq B_r(a) \subseteq O_i$. Since $Q_n \cap S$ can not be covered by finitely many O_α, but could be covered by O_i, which leads to a contradiction. $\square$

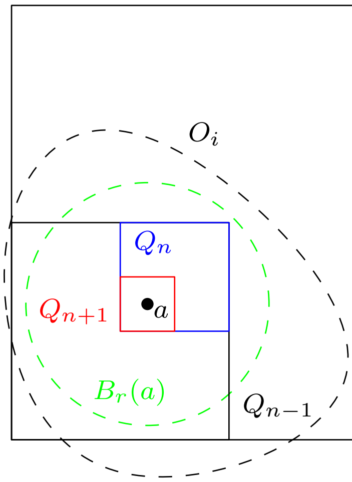


Figure 3.1: Heine-Borel theorem

Theorem 12 (The Lebesgue number of an open cover). *Let (X, d) be a metric space and $K(\subseteq X)$ a compact set. For any given open cover $O_\alpha(\alpha \in A)$ of K, there exists $\delta > 0$,* s.t. *for every $x \in K$ we have $B_\delta(x) \subseteq O'_\alpha$ for some $\alpha' \in A$ (α' depending on x).*

Proof. Since K is compact, for any open cover of K, there exists an finite subcover of K, that is $\exists O_{\alpha_i}, i = 1, \cdots, N$ such that

$$K \subseteq \bigcup_{i=1}^{N} O_{\alpha_i}.$$

For any point $x \in K$ there exist $O_{\alpha_j} \in \{O_{\alpha_i}\}_{i=1}^N$, s.t. $x \in O_{\alpha_j}$ and exists $\delta_x > 0$, s.t. $B_{\delta_x/2}(x) \subseteq B_{\delta_x}(x) \subseteq O_{\alpha_j}$ for O_{α_j} is open. Obviously we have that $B_{\delta_x/2}(x)$ is an open cover of K, i.e.

$$K \subseteq \bigcup_{x \in K} B_{\delta_x/2}(x),$$

and $B_{\delta_x/2}(x)$ has an finite subcover of K, donate as $\{B_{\delta_{x_i}/2}(x_i)\}_{i=1}^M$. Let $\delta = \min\{\delta_{x_i}\}_{i=1}^M$. Then for any $y \in K$, there exists $B_{\delta_{x_j}/2}(x_j) \in \{B_{\delta_{x_i}/2}(x_i)\}_{i=1}^M$ such that $y \in B_{\delta_{x_j}/2}(x_j)$ and $d(y, x_j) < \delta_{x_j}/2$. and for any y' where $d(y', y) < \delta/2 < \delta_{x_j}/2$, we have $d(x_j, y') \leq d(x_j, y) + d(y, y') < \delta_{x_j}$, thus $y, y' \in B_{\delta_{x_j}}(x_j) \subseteq O_{\alpha'}$, or say $y, y' \in B_{\delta/2}(y) \subseteq B_{\delta_{x_j}}(x_j) \subseteq O_{\alpha'}$. $\square$

The theorem indicates for any open cover O_α of K, $\exists \delta > 0$, s.t. for $\forall x \in K, x' \in X$, is $d(x, x') < \delta$, then $\exists \alpha \in A$ we have $x, x' \in O_\alpha$. Such a $\delta > 0$ is called a **Lebesgue**

number of the given open cover $O_\alpha(\alpha \in A)$. Notice that the statement could be false if the compactness assumption is dropped.

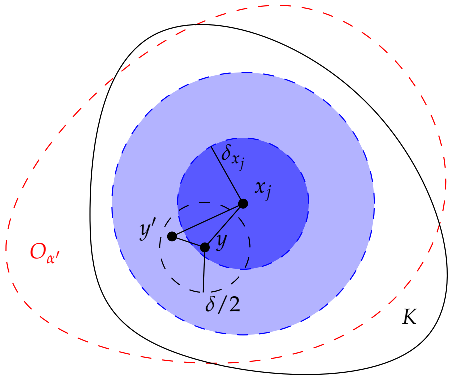
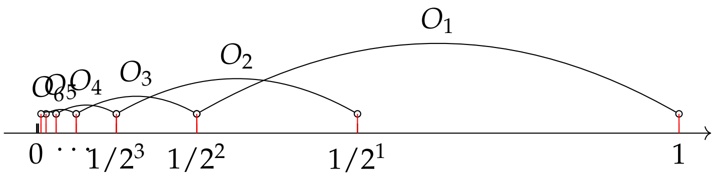
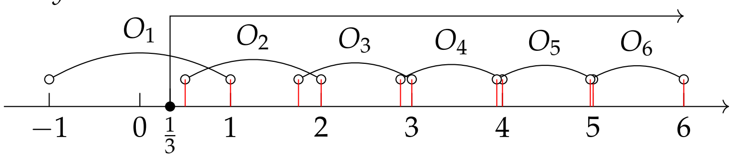


Figure 3.2: The Lebesgue number of an open cover

Exercise 30 (Open set). *Let* $(X,d) = (\mathbb{R}, d_2), K = (0,1), O_\alpha = (1/2^{\alpha+1}, 1/2^{\alpha-1})(\alpha \in \mathbb{N})$. *Thus* $1/2^\alpha \in O_\alpha$ *and* $\notin O_{\alpha'}$ *if* $\alpha' \neq \alpha(\alpha, \alpha' \in \mathbb{N})$. *It is easy to check* O_α *is an open cover of* K, *but* $|1/2^\alpha - 1/2^{\alpha+1}| = 1/2^{\alpha+1}$ *can be arbitrarily small if* $\alpha \uparrow$. *Thus there exists* $x \in K, x' \in X$ *can not be covered one* O_α, *no matter how close they are.*



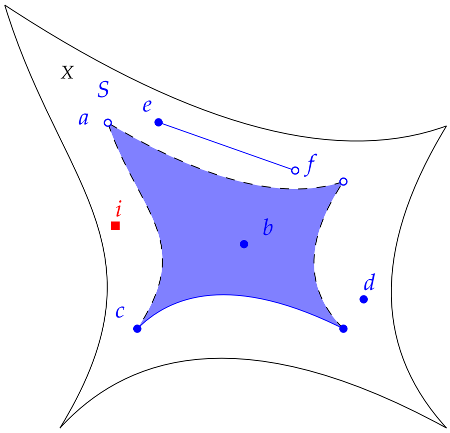
Exercise 31 (Unbounded set). *Let* $(X,d) = (\mathbb{R}, d_2), K = [1/3, \infty), O_\alpha = (\alpha - 1 - 1/2^{\alpha-1}, \alpha)(\alpha \in \mathbb{N})$. *Thus* $x = \alpha - 1/2^\alpha \in O_\alpha$ *and* $x' = \alpha \in O_{\alpha+1}$ *and* $d(x,x')$ *could be arbitrarily small as* $\alpha \uparrow$. *Thus there exists* $x \in K, x' \in X$ *can not be covered one* O_α, *no matter how close they are.*



Definition 17 (Isolated point, limit point and accumulation point)**.** Let (X,d) be a metric space and $S \subseteq X$. A point $x \in X$ is called:

- an **isolated point** of S, if $\exists \epsilon > 0$, s.t. $B_\epsilon(x) \cap S = \{x\}$ ($\Rightarrow x \in S$);
- a **limit point** of S, if $\forall \epsilon > 0$, s.t. $B_\epsilon(x) \cap S \backslash \{x\} \neq \emptyset$;
- an **accumulation point** of S, if $\exists$ seq. $a_n \in S(n \in \mathbb{N})$, s.t. $x = \lim_{n\to\infty} a_n$.

Example 9. $S \subseteq X$ is as the figure, point $i \notin S$:



Then

point	iso. pts. of S	limit pts. of S	acc. pts. of S	$\in S$
i	$\times$	$\times$	$\times$	$\times$
a	$\times$	$\surd$	$\surd$	$\times$
b	$\times$	$\surd$	$\surd$	$\surd$
c	$\times$	$\surd$	$\surd$	$\surd$
d	$\surd$	$\times$	$\surd$	$\surd$
e	$\times$	$\surd$	$\surd$	$\surd$
h	$\times$	$\surd$	$\surd$	$\times$

Notice that x is a isolated point of $S \Rightarrow x \in S$; but x is a limit/accumulate point of $S \not\Rightarrow x \in S$.

Exercise 32. *Let (X, d) be a metric space, $S \subseteq X$,*

1. *Show that x is an isolated/limit point of $S \Rightarrow x$ is a accumulate point of S;*
2. *Donate $\{$iso. pts. of$S\}$, $\{$limit pts. of$S\}$ and $\{$acc. pts. of$S\}$ by I_S, L_S, A_S respectively. Show that $I_S \cup L_S = A_S$;*
3. *Suppose $E \subseteq K \subseteq X$, where E is infinite and K is compact, show that $L_E \neq \emptyset$; (Prove by contradiction)*

Proof. 1. If x is an isolated point of S, thus $x \in S$. Let $a_n \equiv x$, then $\lim_{n\to\infty} a_n = x$, thus x is an accumulate point of S; If x is a limit point of S, then for any $\epsilon > 0, B_\epsilon(x) \cap S \backslash \{x\} \neq \emptyset$. Let $a_n \in B_{1/n}(x)(n \in \mathbb{N})$, then $d(a_n, x) < 1/n$ for $\forall n \in N$, thus $\lim_{n\to\infty} a_n = x$, and x is an accumulate point of S.

2. We have obtained that $I_S, L_S \subseteq A_S$. Suppose $x \in A_S \backslash (I_S \cup L_S) \neq \emptyset$. Which means : (1) there exists seq. $a_n \in S$ such that $\lim_{n\to\infty} a_n = x$; (2) $\forall \epsilon > 0$, s.t. $B_\epsilon(x) \cap S \neq \{x\}$ $(\neg I_S)$;(3) $\exists \epsilon' > 0$, s.t. $B_{\epsilon'}(x) \cap S \backslash \{x\} = \emptyset$ $(\neg L_S)$. Let $B_\epsilon(x) \cap S = Q_\epsilon \neq \{x\}$, if $x \in Q_\epsilon$, then it leads to a contradiction with (3); If $x \notin Q_\epsilon$, then $Q_{\epsilon'} = \emptyset$, that is $B_{\epsilon'}(x) \cap S = \emptyset$ and for $\forall s \in S \Rightarrow d(s, x) \geq \epsilon'$, which leads to a contradiction with (1). Thus $A_S \backslash (I_S \cup L_S) = \emptyset$. Because $I_S, L_S \subseteq A_S$, we have $I_S \cup L_S = A_S$.

3. We claim there exists a limit point s of E in K, i.e. $\exists s \in K$ s.t. $\forall r > 0, B_r(s) \cap E\backslash\{s\} \neq \emptyset$.
   Assume the contrary, that is $\forall s \in K, \exists r_s > 0$ s.t. $B_{r_s}(s) \cap E\backslash\{s\} = \emptyset$, and $B_{r_s}(s)(s \in K)$ form an open cover of K: $K = \cup_{s\in K} B_{r_s}(s)$. Since K is compact, there exists $s_1, \cdots, s_n \in K$ s.t. $K = \cup_{i=1}^n B_{r_{s_i}}(s_i)$.
   Define $S = \{s_1, \cdots, s_n\}$, then

$$\begin{aligned} K \cap E\backslash S &= \left(\cup_{i=1}^n B_{r_{s_i}}(s_i)\right) \cap E\backslash S \\ &= \cup_{i=1}^n B_{r_{s_i}}(s_i E\backslash S) \\ &= \emptyset \end{aligned}$$

   but since E is infinite set, S is finite set and $E \subseteq K \Rightarrow K \cap E\backslash S = E\backslash S \neq \emptyset$, which is contrary.

□

Remark 9. Refer to the proof method.

Exercise 33. *Let* $(X, d) = (\mathbb{R}, d_2), S \subseteq \mathbb{R}$, *show that if* $\sup S$ *(*$\inf S$*) exists, then it is an accumulate point.*

Proof. If $\sup S \exists$, then for $\forall x \in S$, s.t. $x \leq \sup S$ and for $\forall \epsilon > 0$, $\exists x' \in S$, s.t. $\sup S - \epsilon < x'$. For any $n \in \mathbb{N}$, there exists $x_n \in S$ s.t. $\sup S - 1/n < x' \leq \sup S$, and $d(x_n, \sup S) < 1/n$, thus $x_n \to \sup S$ as $n \to \infty$. □

Exercise 34. *Show that, if* (X, d) *be a metric space, then*

$$S \subseteq_{close} X \Leftrightarrow A_S = S \Leftrightarrow L_S \subseteq S.$$

Proof. For any $x \in S$, let $a_n = x$, then $\lim_{n\to\infty} a_n = x$, thus $S \subseteq A_S$. Since example (**??**), we have $S \subseteq_{close} X \Leftrightarrow A_S = S$. $\Rightarrow$ Since $I_S \cup L_S = A_S$, we have $L_S \subseteq A_S = S$; $\Leftarrow$, for $L_S \subseteq A_S \subseteq S$, we have $S \subseteq A_S \Rightarrow S = A_S$. □

3.3 Functions on metric space

Definition 18 (Limit of function)**.** Let $(X, d_X), (Y, d_Y)$ be metric spaces. $a \in S \subseteq X$, $f : S \mapsto Y$, we say map f has limit at a if $\exists b \in Y$ s.t. for $\forall \epsilon > 0, \exists \delta > 0, \forall x \in S \cap B_\delta(a)\backslash\{a\} \Rightarrow f(x) \in B_\epsilon(b)$. Denoted as $\lim_{x\to a} f(x) = b$ and $B_\delta(a)\backslash\{a\} =: B_\delta^*(a)$, then

$$\lim_{x\to a} f(x) = b \Leftrightarrow f(S \cap B_\delta^*(a)) \subseteq B_\epsilon(b).$$

Definition 19 (Continuous)**.** Let $(X, d_X), (Y, d_Y)$ be metric spaces. $a \in S \subseteq X$, $f : S \mapsto Y$, we say

1. map f is continuous at a if for $\forall \epsilon > 0, \exists \delta > 0$, for $\forall x \in B_\delta(a) \cap S$, s.t. $f(x) \in B_\epsilon(f(a))$, that is $f(B_\delta(a)) \subseteq B_\epsilon(f(a))$.
2. map f is a continuous map if f is continuous at every $a \in S$.

Exercise 35. *Let $X \xrightarrow{f} Y$ be a continuous map between metric spaces, a sequence $x_n (n \in \mathbb{N})$ in X converges to $x \in X$, show that $f(x_n)(n \in \mathbb{N})$ in Y converges to $f(x) \in Y$. In the other word:*

$$\lim_{n\to\infty} f(x_n) = f(x) = f(\lim_{n\to\infty} x_n).$$

Proof. Since f is continuous, then for $\forall \epsilon > 0, \exists \delta > 0$ s.t. $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$. And since $x_n \to x$, then $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N, x_n \in B_\delta(x) \Rightarrow f(x_n) \in B_\epsilon(f(x))$.
Thus for $\epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N \Rightarrow d(f(x_n), f(x)) < \epsilon \Rightarrow \lim_{n\to\infty} f(x_n) = f(x) = f(\lim_{n\to\infty} x_n)$. □

Exercise 36. *(Y, d) is a metric space, $y_0 \in Y$, show that $Y \xrightarrow{d} \mathbb{R}$ where $y \mapsto d(y, y_0)$ is a continuous map.*

Proof. Assume that the map d is not continuous, then $\exists y \in Y, \exists \epsilon > 0, \forall \delta > 0, \exists y' \in B_\delta(y)$ s.t.

$$|d(y) - d(y')| = |d(y, y_0) - d(y', y_0)| \geq \epsilon.$$

select $\delta < \epsilon$, then $d(y', y) < \delta < \epsilon$ and hence

$$|d(y, y_0) - d(y', y_0)| \geq \epsilon > d(y', y)$$

which leads to the contradiction with triangle inequality. □

Remark 10. Thus if there exists a seq $y_n \to y$, then

$$d(y, y_0) = d(\lim_{n\to\infty} y_n, y_0) = \lim_{n\to\infty} d(y_n, y_0),$$

for any $y_0 \in Y$.

Exercise 37. *Given a map $X \xrightarrow{f} Y, a \in X$, Show that*
1. *f is continuous at $a \Leftrightarrow$ for $\forall V \subseteq_{open} Y$, where $f(a) \in V$, $\exists U \subseteq_{open} X$, where $a \in U$, such that $f(U) \subseteq V$.*
2. *f is a continuous map $\Leftrightarrow$ for $\forall V \subseteq_{open} Y, f^{-1}(V) \subseteq_{open} X$.*

Proof. 1. $\Rightarrow$: for $\forall V \subseteq_{open} Y$, where $f(a) \in V$, $\exists \epsilon > 0$, s.t. $B_\epsilon(f(a)) \subseteq V$, thus $\exists U = B_\delta(a)$. $\Leftarrow$: trivial.
2. $\Rightarrow$: Given an open set $V \subseteq_{open} Y$, for $\forall x \in f^{-1}(V)$, have $f(x) \in V$. Since V is open, $\exists r > 0$ s.t. $B_r(f(x)) \subseteq V$. Since $f(x)$ is continuous map, $\exists \epsilon > 0$, s.t. $f(B_\epsilon(x)) \subseteq B_r(f(x)) \subseteq V \Rightarrow B_\epsilon(x) \in f^{-1}(V)$, thus $f^{-1}(V)$ is open.
$\Leftarrow$: Given $x \in X, f(x) \in Y$, given $r > 0$, s.t. $B_r(f(x)) \subseteq Y$, then $f^{-1}(B_r(f(x))) \subseteq_{open} X$, and $x \in f^{-1}(B_r(f(x)))$. Thus $\exists \epsilon > 0$, s.t. $B_\epsilon(x) \subseteq f^{-1}(B_r(f(x)))$ and $f(B_\epsilon(x)) \subseteq B_r(f(x))$. □

Remark 11. It can also be proved that f is cont. $\Leftrightarrow$ for $\forall V \subseteq_{close} Y, f^{-1}(V) \subseteq_{close} X$. Suppose $V \subseteq_{close} Y$, then $X \backslash f^{-1}(V) = f^{-1}(X \backslash V) \subseteq_{open} X$, thus $f^{-1}(V) \subseteq_{close} X$.

Exercise 38. *Given maps $X \xrightarrow{f} Y, Y \xrightarrow{g} Z$, show that*
1. If f is continuous at x_0, g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.
2. If f, g are continuous maps, then $g \circ f$ is a continuous map.

Proof. 1. For any V, s.t. $g(f(x_0)) \in V \subseteq_{open} Z$, $\exists U$, s.t. $f(x_0) \in U \subseteq_{open} Y$, $\exists W$, s.t. $x_0 \in W \subseteq X$, thus $g \circ f$ is continuous at x_0.
2. For any $V \subseteq_{open} Z, \exists U \subseteq_{open} Y, \exists W \subseteq_{open} X$, thus $g \circ f$ is continuous. □

Remark 12. Note that we prove this exercise using general sets instead of metrics. We replaced open ball with open set in last Exercise, this is a meaningful operation, which means we could **substitute the metric with set** (here is open set), which drives the concept of topology. Generally, topology is a family of sets which have the basic properties, using these sets, we can define open set no longer rely on metric d.

Theorem 13. *Let $X \xrightarrow{f} \mathbb{R}$ be a continuous map between metric space, X is compact, then* $\max_{x \in X} f(x), \min_{x \in X} f(x)$ *exists.*

Proof. 1. f is bdd. and hence $\sup_{x \in X} f(x)$ exists (l.u.b. property):
Assume the contrary. Then $\forall n \in \mathbb{N}, \exists x_n \in X$ s.t. $f(x_n) > n$ and we can form a seq. $x_n (n \in \mathbb{N})$ which is a infinite subset of a compact set, thus there exists $a \in X$ and a convergent subseq. $x_{n_k} (k \in \mathbb{N}) \to a$ as $k \to \infty$ (see Remark 13). And hence $\lim_{k \to \infty} f(x_{n_k}) = f(a)$ since f is continuous, which leads to a contradiction with $f(x_{n_k}) \geq n_k$. Thus f is bdd. (Thus continuous map on compact set is bounded)
2. Let $M = \sup_{x \in X} f(x)$, then $\exists x \in X$, s.t. $f(x) = M$:
Assume the contrary, i.e. $\forall x \in X, f(x) < M$. Then the map $X \xrightarrow{\phi} \mathbb{R}$ where $x \mapsto 1/(M - f(x))$ is well-defined continuous map, and hence ϕ is bounded by 1. Then for any $R \in \mathbb{R}_+, 1/R > 0$ and $\exists x \in X$ s.t.

$$M - \frac{1}{R} < f(x) \leq M$$

thus $\phi(x) = 1/(M - f(x)) > R$ which leads to a contradiction with ϕ is bdd. □

Remark 13 (Generalize B-W theorem to metric space). Two facts:
1. Any infinite subset of a compact set K has a limit point in K (Exercise 32);
2. x is a limit point of $A \subseteq X$, where X is a metric space $\Leftrightarrow \exists$ *seq.* $a_n \in A \backslash \{x\} (n \in \mathbb{N})$, s.t. $a_n \to x$ as $n \to \infty$.

These two facts generalize the Bolzano-Weierstrass theorem (Theorem 10) from $\mathbb{R}^n$ space to general metric space as: *A sequence $a_n (n \in N)$ in a compact metric space has a convergent subsequence.*

3.4 Uniformly continuous function

Recall that the concept of continuous map: let $X \xrightarrow{f} Y$ be a map between metric space,
- f is continuous
- $\Leftrightarrow f$ is continuous at every $x \in X$
- $\Leftrightarrow \forall \epsilon > 0, \forall x \in X, \exists \delta > 0$ s.t. $\forall x' \in X, d(x, x') < \delta \Rightarrow d(f(x), f'(x)) < \epsilon$

(or say $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$). **Note that here the order of x and ϵ does not matter, and δ relies on the choice of x and ϵ.**

Definition 20 (Uniformly continuous, 均匀连续)**.** Let $X \xrightarrow{f} Y$ be a map between metric space, we say f is uniformly continuous if
- $\forall \epsilon > 0, \exists \delta > 0$ s.t. for $\forall x, x' \in X, d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \epsilon$.

Remark 14. Now, δ only relies on the choice of ϵ. If f is uniformly continuous $\Rightarrow f$ is continuous.

For a given $\epsilon > 0$ and $x \in X$, consider the set

$$\Delta_x := \{\delta > 0 | f(B_\delta(x) \subseteq B_\epsilon(f(x))\}$$

Then if f is continuous at $x \Leftrightarrow \Delta_x \neq \emptyset$. And if f is continuous at x, define ϵ **- threshold** of f at x as

$$\delta_{f,\epsilon}(x) := \sup \Delta_x$$

Consider a map $(0, 1] \to \mathbb{R}$ where $x \mapsto 1/x$, if any δ works for the given ϵ and x, then

$$\frac{1}{x-\delta} - \frac{1}{x} = \frac{\delta}{(x-\delta)x} < \epsilon$$

thus $\delta < \epsilon(x-\delta)x < x^2\epsilon \Rightarrow \delta_{f,\epsilon}(x) \leq x^2\epsilon \to 0$ as $x \to 0$, thus there does not exist a δ for given ϵ such that works for all $x \in X$.

Theorem 14. *If $X \xrightarrow{f} Y$ is a continuous map between metric space and X is compact, then f is uniformly continuous.*

Proof 1. Given $\epsilon > 0$, for every $a \in X$, choose a number $\delta_a > 0$ s.t. $\forall x \in X, f(B_{\delta_a}(a)) \subseteq B_{\epsilon/2}(f(a))$. Then $B_{\delta_a}(a)(a \in X)$ is an open cover of X, then let $\delta > 0$ be a Lebesgue number of this cover.
Thus for $\forall x, x' \in X, d(x, x') < \delta \Rightarrow \exists a \in X$, s.t. $x, x' \in B_{\delta_a}(a) \Rightarrow f(x), f(x') \in B_{\epsilon/2}(f(a)) \Rightarrow d(f(x), f(x')) \leq d(f(x), f(a)) + d(f(a), f(x')) < \epsilon/2 + \epsilon/2 = \epsilon$. □

Proof 2. Assume the contrary, that is there exists $\epsilon > 0$, $\forall \delta = 1/n (n \in \mathbb{N})$, exists $x_n, x'_n \in X$, s.t. $d(x_n, x'_n) < \delta$ but $d(f(x_n), f(x'_n)) > \epsilon$. And then we can form two sequence: $x_1, x_2, \cdots$ and $x'_1, x'_2, \cdots$.

Since X is compact, and $x_n(n \in \mathbb{N})$ is a infinite subsets of $X \Rightarrow x_n(n \in \mathbb{N})$ has a limit point $a \in X$. And x_n has a subseq. $x_{n_k}(k \in \mathbb{N})$, s.t. $\lim_{k\to\infty} x_{n_k} = a$. The correspond subseq. x'_{n_k} is a infinite subset of compact set $X \Rightarrow x'_{n_k}$ has a limit point $b \in X$, and has a subseq. $x'_{n_{k_j}}(j \in \mathbb{N})$ s.t. $\lim_{j\to\infty} x'_{n_{k_j}} = b$. (Remark 13)
Since $x_{n_k} \to a$, then $x_{n_{k_j}} \to a$ as well (Exercise 6). Thus we have that

$$\lim_{j\to\infty} x_{n_{k_j}} = a, \quad \lim_{j\to\infty} x'_{n_{k_j}} = b,$$

and $d(x_{n_{k_j}}, x'_{n_{k_j}}) < 1/n_{k_j}$. Thus for any $\epsilon_1 > 0$, $\exists J$, s.t. $\forall j \geq J$ has $d(a, x_{n_{k_j}}) < \epsilon_1/3$ and $d(b, x'_{n_{k_j}}) < \epsilon_1/3$ and $d(x_{n_{k_j}}, x'_{n_{k_j}}) < \epsilon_1/3$ (Archimedean Property), thus

$$\begin{aligned} d(a,b) &\leq d(a, x_{n_{k_j}}) + d(x_{n_{k_j}}, x'_{n_{k_j}}) + d(x'_{n_{k_j}}, b) \\ &< \frac{\epsilon_1}{3} + \frac{\epsilon_1}{3} + \frac{\epsilon_1}{3} = \epsilon_1 \end{aligned}$$

thus $d(a,b) = 0 \Leftrightarrow a = b$. Since f is continuous, then (Exercise 35)

$$\lim_{j\to\infty} f(x_{n_{k_j}}) = f(a) = f(b) = \lim_{j\to\infty} f(x'_{n_{k_j}})$$

Then for any $j \in \mathbb{N}$, we have that

$$\begin{aligned} d(f(x_{n_{k_j}}), b) &= d(f(x_{n_{k_j}}), \lim_{j'\to\infty} f(x'_{n_{k'_j}})) \\ &= \lim_{j'\to\infty} d(f(x_{n_{k_j}}), f(x'_{n_{k'_j}})) && \text{(Remark 10)} \\ &\geq \epsilon \end{aligned}$$

and hence

$$\begin{aligned} d(a,b) &= d(\lim_{j\to\infty} f(x_{n_{k_j}}), b) \\ &= \lim_{j\to\infty} d(f(x_{n_{k_j}}), b) \\ &\geq \epsilon \end{aligned}$$

which leads to a contradiction. □

3.5 Limit superior / inferior for function

Let X be metric space, $S \subseteq X$ and $S \xrightarrow{f} \mathbb{R}$ be a map, for $a \in X$, we define

$$\overline{f}^*(\delta) := \sup_{x \in B_\delta(a)\setminus\{a\}} f(x) = \sup\{f(x) | 0 < d(x,a) < \delta\}$$

$$\underline{f}^*(\delta) := \inf_{x\in B_\delta(a)\backslash\{a\}} f(x) = \inf\{f(x)|0 < d(x,a) < \delta\}$$

It is direct to see that $\overline{f}^*{}_{\searrow}$ as $\delta \to 0$: Assume that if $\exists\delta < \delta'$ and $\overline{f}^*(\delta) > \overline{f}^*(\delta')$, let

$$\epsilon = \overline{f}^*(\delta) - \overline{f}^*(\delta')$$

then $\exists x \in B_\delta(a)\backslash\{a\} \subseteq B_{\delta'}(a)\backslash\{a\}$ such that

$$\overline{f}^*(\delta) \geq f(x) > \overline{f}^*(\delta) - \epsilon/2 > \overline{f}^*(\delta') = \sup_{x\in B_{\delta'}(a)\backslash\{a\}} f(x),$$

which is contrary. Thus similarly, $\underline{f}^*{}_{\nearrow}$ as $\delta \to 0$. For any $\delta, \delta' \in \mathbb{R}$, we have that

$$\underline{f}^*(\delta) \leq \underline{f}^*(\min\{\delta,\delta'\}) \leq \overline{f}^*(\min\{\delta,\delta'\}) \leq \overline{f}^*(\delta')$$

thus $\underline{f}^*(\delta)$ has upper bound and $\overline{f}^*(\delta)$ has lower bound when $\delta \to 0$. And hence $\overline{f}^*(\delta)$ converges to its infimum: assume the contrary, if $\lim_{\delta\to 0} \overline{f}^*(\delta) > \inf_{\delta>0} \overline{f}^*(\delta)$[1] , then $\exists\epsilon > 0$ and $\delta' > 0$ s.t.

$$\inf_{\delta>0} \overline{f}^*(\delta) \leq \overline{f}^*(\delta') < \inf_{\delta>0} \overline{f}^*(\delta) + \epsilon < \lim_{\delta\to 0} \overline{f}^*(\delta)$$

and hence $\forall\delta < \delta'$ has

$$\overline{f}^*(\delta) \leq \overline{f}^*(\delta') < \lim_{\delta\to 0} \overline{f}^*(\delta)$$

since $\overline{f}^*(\delta)_{\searrow}$ as $\delta \to 0$. And it is contrary.
Thus $\overline{f}^*(\delta)$ converges to its infimum, $\underline{f}^*(\delta)$ converges to its supremum, and we can define

$$\overset{*}{\limsup_{x\to a}} f(x) = \overset{*}{\overline{\lim_{x\to a}}} f(x) := \inf_{\delta>0} \overline{f}^*(\delta) = \inf_{\delta>0} \sup_{x\in B_\delta(a)\backslash\{a\}} f(x) = \lim_{\delta\to 0} \overline{f}^*(\delta)$$

$$\overset{*}{\liminf_{x\to a}} f(x) = \overset{*}{\underline{\lim}}_{x\to a} f(x) := \inf_{\delta>0} \underline{f}^*(\delta) = \sup_{\delta>0} \inf_{x\in B_\delta(a)\backslash\{a\}} f(x) = \lim_{\delta\to 0} \underline{f}^*(\delta)$$

Corresponding, we can define the 'non - $*$' conception by containing the $\{a\}$:

$$\overline{f}(\delta) := \sup_{x\in B_\delta(a)} f(x) = \sup\{f(x)|0 \leq d(x,a) < \delta\}$$

$$\underline{f}(\delta) := \inf_{x\in B_\delta(a)} f(x) = \inf\{f(x)|0 \leq d(x,a) < \delta\}$$

and

[1]In this section, $\delta \to 0$ is regarded as $\delta \to 0^+$ by default.

$$\limsup_{x\to a} f(x) = \overline{\lim_{x\to a}} f(x) := \inf_{\delta>0} \overline{f}(\delta) = \inf_{\delta>0} \sup_{x\in B_\delta(a)} f(x) = \lim_{\delta\to 0} \overline{f}(\delta)$$

$$\liminf_{x\to a} f(x) = \underline{\lim_{x\to a}} f(x) := \inf_{\delta>0} \underline{f}(\delta) = \sup_{\delta>0} \inf_{x\in B_\delta(a)} f(x) = \lim_{\delta\to 0} \underline{f}(\delta)$$

Then it is direct to see that

$$\underline{\lim_{x\to a}} f(x) \le \underline{\lim_{x\to a}}^{*} f(x) \le \overline{\lim_{x\to a}}^{*} f(x) \le \overline{\lim_{x\to a}} f(x)$$

Example 10. Consider a map $\mathbb{R} \xrightarrow{f} \mathbb{R}$ where $x \mapsto 1$ if $x \neq 0$ and $0 \mapsto 0$, then

$$\begin{aligned} \overline{\lim_{x\to 0}}^{*} f(x) &= 1, & \underline{\lim_{x\to 0}}^{*} f(x) &= 1 \\ \overline{\lim_{x\to 0}} f(x) &= 1, & \underline{\lim_{x\to 0}} f(x) &= 0 \end{aligned}$$

Exercise 39. *Let X be metric space, $a \in S \subseteq X$ and $S \xrightarrow{f} \mathbb{R}$ be a map, show that*

1. $\lim_{x\to a} f(x)$ *exists* $\Leftrightarrow$ $\overline{\lim}^{*}_{x\to a} f(x)$ *and* $\underline{\lim}^{*}_{x\to a} f(x)$ *exists and equal to each other.*
2. $f(x)$ *is continuous at a exists* $\Leftrightarrow$ $\overline{\lim}_{x\to a} f(x)$ *and* $\underline{\lim}_{x\to a} f(x)$ *exists and equal to each other.*

Proof. Define $B^*_\delta(a) := B_\delta(a)\backslash\{a\}$.
1. $\Rightarrow$: $\exists l \in \mathbb{R}$ s.t. $\lim_{x\to a} f(x) = l$, then for any $\epsilon > 0, \exists \delta > 0$ s.t. $\forall x \in B^*_\delta(a) \Rightarrow l - \epsilon/2 < f(x) < l + \epsilon/2$. Then for any $x \in B^*_\delta(a)$, one has

$$\begin{aligned} l - \epsilon n < l - \frac{\epsilon}{2} &\le \inf_{x\in B^*_\delta(a)} f(x) = \underline{f}^*(\delta) \\ &\le f(x) \le \sup_{x\in B^*_\delta(a)} f(x) = \overline{f}^*(\delta) \\ &\le l + \frac{\epsilon}{2} < l + \epsilon. \end{aligned}$$

Since $\overline{f}^*(\delta)_{\searrow}$ as $\delta \to 0$, then for any $\mu \le \delta \Rightarrow$

$$l - \epsilon < \underline{f}^*(\delta) \le \overline{f}^*(\mu) \le \overline{f}^*(\delta) < l + \epsilon.$$

Thus for any $\epsilon > 0, \exists \delta > 0,$ s.t. $\mu \in B^*_\delta(0) \Rightarrow \overline{f}^*(\mu) \in B_\epsilon(l)$, thus

$$\overline{\lim_{x\to a}}^{*} f(x) = \lim_{\delta\to 0} \overline{f}^*(\delta) = l.$$

and $\underline{\lim}^{*}_{x\to a} f(x) = l$ in the same way.

$\Leftarrow$: Assume that $\lim_{\delta\to 0}\overline{f}^*(\delta) = \lim_{\delta\to 0}\underline{f}^*(\delta) = r$. Then for any $\epsilon > 0, \exists\delta > 0$ s.t. $\forall\mu \in B^*_\delta(0)$ has

$$r - \epsilon < \underline{f}^*(\mu) \leq f(x) \leq \overline{f}^*(\mu) < r + \epsilon$$

for any $x \in B^*_\mu(a)$. Thus for $\forall\epsilon > 0, \exists\mu > 0,$ s.t. $\forall x \in B^*_\mu(a) \Rightarrow |f(x) - r| < \epsilon$, thus $\lim_{x\to a} f(x) = r$.

2. $\Rightarrow$: assume that f is continuous at a and $f(a) = l$, then for any $\epsilon > 0, \exists\delta > 0$ and for any $0 < \mu < \delta$ one has for any $x \in B_\mu(a) \subseteq B_\delta(a)$

$$l - \frac{\epsilon}{2} \leq \underline{f}(\mu) \leq f(x) \leq \overline{f}(\mu) \leq l + \frac{\epsilon}{2}$$

Thus for any $\epsilon > 0, \exists\delta > 0,$ s.t. $\forall\mu \in B^*_\delta(0)$ has $\underline{f}(\mu), \overline{f}(\mu) \in B_\epsilon(l) \Rightarrow \lim_{\delta\to 0}\underline{f}(\delta) = \lim_{\delta\to 0}\overline{f}(\delta) = l$.

$\Leftarrow$: assume that $\lim_{\delta\to 0}\underline{f}(\delta) = \lim_{\delta\to 0}\overline{f}(\delta) = r$, then for any $\epsilon > 0, \exists\delta > 0,$ s.t. $\forall 0 < \mu < \delta$ has

$$r - \epsilon < \underline{f}(\mu) \leq f(x) \leq \overline{f}(\mu) < r + \epsilon$$

for $\forall x \in B_\mu(a)$. That is for any $\epsilon > 0, \exists\mu > 0, \forall x \in B_\mu(a)$ has $|f(x) - r| < \epsilon \Rightarrow f$ is continuous at a and $f(a) = r$. $\square$

Chapter 4

Convergence of sequence / series of functions

4.1 Pointwise / uniformly convergent

Definition 21. Let $X \xrightarrow{f_n} Y (n \in \mathbb{N})$ be a seq. of maps, Y is a metric space. We say that $f_n (n \in \mathbb{N})$ converges to a map $X \xrightarrow{f} Y$
- pointwise (逐点收敛): $\forall \epsilon > 0, \forall x \in X, \exists N_x \in \mathbb{N},$ s.t. $\forall n \geq N_x \Rightarrow d(f_n(x), f(x)) < \epsilon$;
- uniformly (均匀收敛): $\forall \epsilon > 0, \exists N \in \mathbb{N},$ s.t. $\forall x \in X, \forall n \geq N \Rightarrow d(f_n(x), f(x)) < \epsilon$.

Denoted as $f_n \to f$ and $f_n \xrightarrow{uni.} f$ as $n \to \infty$ respectively.

Example 11. Given a seq. of maps $X \xrightarrow{f_n} \mathbb{R}$ where $x \in X \in \mathbb{R}$ and $f_n(x) = x^n (n \in \mathbb{N})$. Then f_n converges pointwise if $X \subseteq (-1, 1]$:

$$f_n \to f = \begin{cases} 1, & x = 1 \\ 0, & x \in (-1, 1) \end{cases}$$

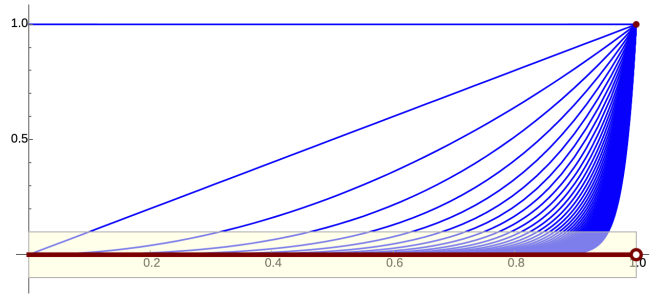


Figure 4.1: pointwise convergent

However, f_n does not converges to f uniformly, since

$$|f_n(x) - f(x)| = \begin{cases} |1-1| = 0, & x = 1 \\ |x^n - 0| = |x|^n, & x \in (-1,1) \end{cases}$$

For any $\epsilon > 0$, to have $|f_n(x) - f(x)| < \epsilon$, we need $|x|^n < \epsilon$ for $x \in (-1,1)$, that is $n \ln |x| < \ln \epsilon \Leftrightarrow n > \ln \epsilon / \ln |x|$ which has no upper bound, thus there does not exist a $N \in \mathbb{N}$ such that $\forall n \geq N$ has $|f_n - f| < \epsilon$ for $x \in (-1,1)$.

Remark 15. Intuitively, a seq. of maps $f_n \xrightarrow{uni.} f$ means: a pipe with any radius ϵ whose shaft is f can encase all functions after the f_{N_ϵ} of the $f_n(n \in \mathbb{N})$.

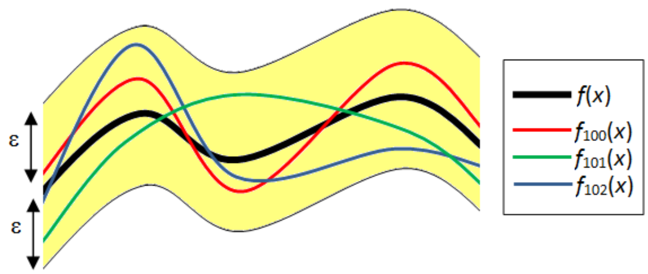


Figure 4.2: uniformly convergent

Proposition 3. *Let $X \xrightarrow{f_n} Y(n \in \mathbb{N})$ is a seq. of maps between metric spaces, which converges to map $X \xrightarrow{f} Y$ uniformly, if f_n is continuous at $a \in X$ for $\forall n \in \mathbb{N}$, then f is, too.*

Proof. Note that for all $x \in X$ and $n \in \mathbb{N}$, we have that

$$d(f(x), f(a)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(a)) + d(f_n(a), f(a));$$

Then for any $\epsilon > 0$, since $f_n \xrightarrow{uni.} f$ as $n \to \infty, \exists N_\epsilon \in \mathbb{N}$ s.t. $\forall x \in X, n \geq N_\epsilon \Rightarrow d(f_n(x), f(x)) < \epsilon/3$. In particular, $d(f_{N_\epsilon(x)}, f(x)) < \epsilon/3$ for $\forall x \in X$.
On the other hand, since f_{N_ϵ} is continuous at a, then $\exists \delta_{N_\epsilon} > 0$ s.t. $d(x,a) < \delta_{N_\epsilon} \Rightarrow d(f_{N_\epsilon}(x), f_{N_\epsilon}(a)) < \epsilon/3$. Then given $\epsilon > 0, \exists \delta_{N_\epsilon} > 0$, s.t. for $\forall x \in B_{\delta_{N_\epsilon}}(a)$ one has

$$\begin{aligned} d(f(x), f(a)) &\leq d(f(x), f_{N_\epsilon}(x)) + d(f_{N_\epsilon}(x), f_{N_\epsilon}(a)) + d(f_{N_\epsilon}(a), f(a)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus f is continuous at x. $\square$

4.2 Complete metric space

Definition 22 (Complete, 完备)**.** A metric space (Y, d) is complete if every Cauchy sequence $a_n (n \in \mathbb{N})$ in Y converges. That is $\lim_{n\to\infty} a_n = a \in Y$.

Example 12. $(\mathbb{R}^n, d_2)$ is complete; $(\mathbb{Q}, d_2)$ is incomplete.

Proposition 4 (Uniform Cauchy)**.** *Let $X \xrightarrow{f_n} Y (n \in \mathbb{N})$ be a seq. of maps, and Y be a complete metric space. Then $f_n (n \in \mathbb{N})$ converges uniformly $\Leftrightarrow \forall \epsilon, \exists N$,* s.t. $\forall x \in X, [n, m \geq N \Rightarrow d(f_n(x), f_m(x)) < \epsilon]$ *(such $f_n (n \in \mathbb{N})$ is called **uniform Cauchy seq.**).*

Proof. $\Rightarrow$: (The completeness of Y is not need). Since $f_n \xrightarrow{uni.} f$, then for $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall x \in X, n \geq N \Rightarrow |f - f_n| < \epsilon/2$, then for $\forall x \in X, \forall n, m \geq N$ one has

$$\begin{aligned} |f_n - f_m| &\leq |f_n - f| + |f - f_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$\Leftarrow$: The assumption implies that for every fixed $x \in X$, the seq. $f_n(x)(n \in \mathbb{N})$ is a Cauchy seq. in Y and hence $\lim_{n\to\infty} f_n(x)$ exists, which we denoted as $f(x)$. This define a map $X \xrightarrow{f} Y$. Now we will show that $f_n \xrightarrow{uni.} f$.
Since for $\forall x \in X$ and a fixed $m \in \mathbb{N}$, map $Y \xrightarrow{d} \mathbb{R}$ where $y \mapsto d(y, f_m(x))$ is continuous, then

$$d(f(x), f_m(x)) = \lim_{n\to\infty} d(f_n(x), f_m(x))$$

for all $x \in X$ (Remark 10). Since for $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall x \in X, [m, n \geq N \Rightarrow d(f_n(x), f_m(x)) < \epsilon/2]$. For every $x \in X, m \geq N$, let $n \to \infty$, we obtain that

$$d(f(x), f_m(x)) = \lim_{n\to\infty} d(f_n(x), f_m(x)) \leq \frac{\epsilon}{2} < \epsilon$$

thus $f_n \xrightarrow{uni.} f$. □

Remark 16. It is direct to see that: $f_n(n \in \mathbb{N})$ converges pointwise $\Leftrightarrow \forall \epsilon, \forall x, \exists N$, s.t. $\in X, [n, m \geq N \Rightarrow d(f_n(x), f_m(x)) < \epsilon]$.

The power of this proposition is to convert the seq. of functions $f_n(n \in \infty)$. to a series of functions $\sum_{n=1}^{\infty} g_n$, where we define $f_0 \equiv 0$ and

$$g_n = f_n - f_{n-1}, \quad n \in \mathbb{N},$$

then partial sum $s_n = g_1 + \cdots + g_n = f_n(n \in \mathbb{N})$, and hence $\sum_{n=1}^{\infty} g_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} f_n$.

Definition 23. Let $X \xrightarrow{g_n} \mathbb{R} (n \in \mathbb{N})$ be a seq. of functions, we say that $\sum_{n=1}^{\infty} g_n$ converges pointwise / uniformly the partial sum $s_n = g_1 + \cdots + g_n (n \in \mathbb{N})$ does.

Proposition 5 (Weierstrass's M - test)**.** *Let $X \xrightarrow{g_n} \mathbb{R} (n \in \mathbb{N})$ be a seq. of functions, if there exists a positive seq. $M_n (n \in \mathbb{N})$ in $\mathbb{R}$ s.t.*
1. $|g_n(x)| \leq M_n$ for all $x \in X, n \in \mathbb{N}$, and
2. $\sum_{n=1}^{\infty} M_n < \infty$,
then $\sum_{n=1}^{\infty} g_n$ converges uniformly.

Proof. Let partial sum $s_n(x) = g_1(x) + \cdots + g_n(x) (x, \in X, n \in \mathbb{N})$, it is sufficient to show that $s_n (n \in \mathbb{N})$ is uniformly Cauchy seq. (since $\mathbb{R}$ is complete metric space.) Simce series $\sum_n M_n < \infty$, for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > m \geq N \Rightarrow$ the tail $M_{m+1} + \cdots + M_n =< \epsilon$, then for any such n, m, for $\forall x \in X$ we have that

$$\begin{aligned} |s_n(x) - s_m(x)| &= |g_{m+1}(x) + \cdots + g_n(x)| \\ &\leq |g_{m+1}(x)| + \cdots + |g_n(x)| \\ &\leq M_{m+1} + \cdots + M_n \\ &< \epsilon \end{aligned}$$

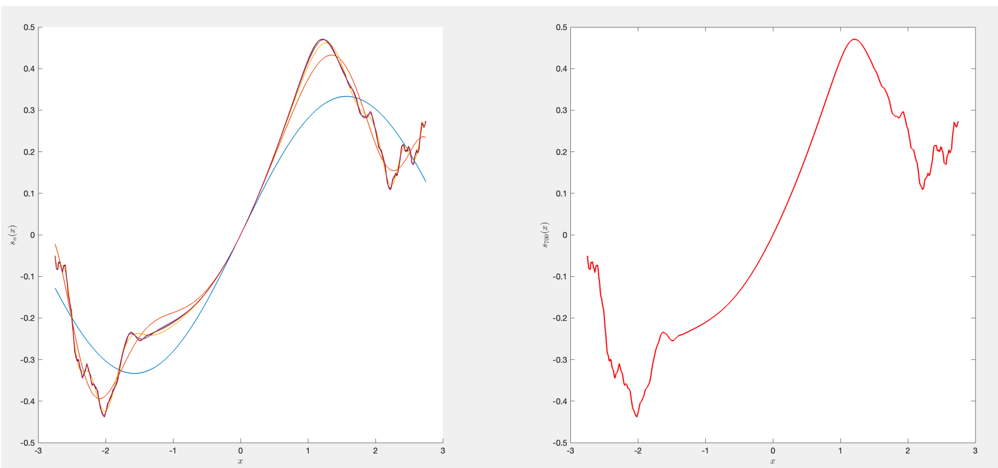
Thus $s_n (n \in \mathbb{N})$ converges uniformly and hence $\sum_{n=1}^{\infty} g_n$ converges uniformly. $\square$

Remark 17. The above conclusion still holds if modify $\mathbb{R}$ to $\mathbb{R}^k$ for some $k \in \mathbb{N}$.

Example 13. Consider series $\sum_{n=1}^{\infty} \sin(x^n)/3^n$ since

$$|g_n| = \left| \frac{\sin(x^n)}{3^n} \right| \leq \frac{1}{3^n} =: M_n$$

thus $\sum_{n=1}^{\infty} \sin(x^n)/3^n$ converges uniformly. We can plot them out, define $s_n = \sum_{i=1}^{n} g_i$, then



with the MATLAB code:

```matlab
gn = 1000; % grid number
fn = 700;  % func number
X = linspace(-5,5,gn);
Y = zeros(gn,fn);
for n = 1:fn
    F = @(x) sin(x.^n)./(3.^n);
    Y(:,n) = F(X)';
end
T = triu(ones(fn,fn));
YY = Y*T;

clf;
subplot(1,2,1);
hold on;
for n = 1:fn
    plot(X,YY(:,n), LineWidth=1);
end
xlabel('$x$','Interpreter','latex');
ylabel('$s_n(x)$','Interpreter','latex');
hold off;

subplot(1,2,2);
plot(X,YY(:,end), LineWidth=1.5, Color='r');
xlabel('$x$','Interpreter','latex');
ylabel('$s_{700}(x)$','Interpreter','latex');
```

Exercise 40. *Let X be a metric space, and define*

$$C_b(X) := \{x \xrightarrow{f} \mathbb{R} | f \text{ is bounded continuous}\}.$$

For any $f \in C_b(X)$, we let

$$\|f\|_{\sup} := \sup_{x \in X} |f(x)|$$

For $f, g \in C_b(X)$, define

$$d(f,g) := \|f - g\|_{\sup}$$

show that

1. *(1.a) $\|f\|_{\sup} \geq 0$ and equality holds iff $f(x) \equiv 0$ for $\forall x \in X$;*
   (1.b) $\|f + g\|_{\sup} \leq \|f\|_{\sup} + \|g\|_{\sup}$ for all $f, g \in C_b(X)$;
   (1.c) $\|cf\|_{\sup} = |c| \cdot \|f\|_{\sup}$ for all $f \in C_b(X), c \in \mathbb{R}$;

2. *d is a metric on $C_b(X)$;*
3. *$(C_b(X), d)$ is complete;*
4. *if $f_n \in C_b(X)(n \in \mathbb{N})$ and $f \in C_b(X)$, $[f_n \xrightarrow{uni.} f \Leftrightarrow f_n \xrightarrow{w.r.t.\ d} f]$ as $n \to \infty$.*

Proof. Since $\forall f \in C_b(X)$ is bounded, then any $\|f\|_{\sup}$ exists.
1. (1.a) trivial; (1.b) Assume that exists $f, g \in C_b(X)$ s.t. $\sup_{x\in X}(|f| + |g|) > \sup_{x\in X}|f| + \sup_{x\in X}|g|$. Then exists $x \in X$, s.t.

$$\sup_{x\in X}|f| + \sup_{x\in X}|g| < |f(x)| + |g(x)| \le \sup_{x\in X}(|f| + |g|)$$

which is contrary, thus

$$\begin{aligned} \|f+g\|_{\sup} &= \sup_{x\in X}|f+g| \le \sup_{x\in X}(|f|+|g|) \\ &\le \sup_{x\in X}|f| + \sup_{x\in X}|g| \\ &= \|f\|_{\sup} + \|g\|_{\sup} \end{aligned}$$

(1.c) $\|cf\|_{\sup} = \sup_{x\in X}|c \cdot f(x)| = \sup_{x\in X}|c| \cdot |f(x)| = |c| \cdot \sup_{x\in X}|f(x)| = |c| \cdot \|f\|_{\sup}$.
2. We only prove the triangle inequality: for any $f, g \in C_b(X)$, we have

$$\begin{aligned} d(f,g) &= \|f-g\|_{\sup} = \|f + (-g)\|_{\sup} \\ &\le \|f\|_{\sup} + \|-g\|_{\sup} \\ &=\le \|f\|_{\sup} + \|g\|_{\sup}. \end{aligned}$$

3. Suppose $f_n(n \in \mathbb{N})$ is a Cauchy seq. in $(C_b(X), d)$, thus for any $\epsilon > 0, \exists N \in \mathbb{N}$, s.t. for $\forall n, m \ge N$, one has

$$d(f_n, f_m) = \|f_n - f_m\|_{\sup} = \sup_{x\in X}|f_n - f_m| < \epsilon$$

thus for $\forall x \in X$, $|f_n(x) - f_m(x)| \le \sup_{x\in X}|f_n - f_m| < \epsilon$. Thus fix any $x' \in X$, then $f_n(x')(n \in \mathbb{N})$ is a Cauchy seq. in $\mathbb{R}$, and converges since $\mathbb{R}$ is complete metric space, denote the limit as $f(x')$. It is direct to see that f is bounded, and we will show that f is continuous on X as well.
Since for any $n \in \mathbb{N}$, $f_n \in C_b(X) \Rightarrow f_n$ is continuous on X, thus for any $x \in X, \epsilon > 0, \exists\delta > 0$ s.t. for any $x' \in B_\delta(x)$ (w.r.t. d_2), we have that $d_2(f_n(x'), f_n(x)) < \epsilon/3$. And since for any $x \in X$, $f_n(x)$, as a Cauchy seq. in $\mathbb{R}$, converges to $f(x)$, and hence $\exists N \in \mathbb{N}$, s.t. for $n \ge N$, $d_2(f(x), f_n(x)) < \epsilon/3$. Thus for any $n \ge N, x' \in B_\delta(x)$ (w.r.t. d_2), we have

$$\begin{aligned} d(f(x), f(x')) &\le d(f(x), f_n(x)) + d(f_n(x), f_n(x')) + d(f_n(x'), f(x')) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus f is continuous on $X \Rightarrow f \in C_b(X)$. Now we show that $f_n \to f$ w.r.t. d. Assume that f_n does not converges to f w.r.t. d, that is $\exists \epsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N,$ s.t.

$$d(f, f_n) = \|f - f_m\|_{\sup} = \sup_{x \in X} |f - f_n| \geq \epsilon > \frac{\epsilon}{2},$$

and hence $\exists x \in X$ s.t.

$$\frac{\epsilon}{2} < |f(x) - f_n(x)| \leq \sup_{x \in X} |f - f_n|$$

which leads to a contradiction with $f_n(x)$ is Cauchy in $\mathbb{R}$ and converges to $f(x)$. Thus $f_n \to f \in C_b(X)$ w.r.t. d.

4. It is sufficient to show that **bounded continuous** $f_n(n \in \mathbb{N})$ **is a uniform Cauchy seq. of functions** $\Leftrightarrow$ $f_n(n \in \mathbb{N})$ **is a Cauchy seq. in** $(C_b(X), d)$.

$\Rightarrow$: $f_n(n \in \mathbb{N})$ are bounded continuous $\Rightarrow f_n \in C_b(X)(n \in \mathbb{N})$. And for any $\epsilon > 0, \exists N \in \mathbb{N},$ s.t. $\forall n > m \geq M$, has $|f_n(x) - f_m(x)| < \epsilon/2$ for $\forall x \in X$, thus $\sup_{x \in X} |f_n - f_m| \leq \epsilon/2 < \epsilon \Rightarrow d(f_m, f_n) < \epsilon \Rightarrow f_n(n \in \mathbb{N})$ are Cauchy seq. in $(C_b(X), d)$.

$\Leftarrow$: $f_n(n \in \mathbb{N})$ are Cauchy seq. in $(C_b(X), d)$, then for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s,t, $\forall n, m \geq N$ has $d(f_m, f_n) = \sup_{x \in X} |f_m - f_n| < \epsilon \Rightarrow \forall x \in X$ has $|f_m(x) - f_n(x)| < \epsilon \Rightarrow f_n(n \in \mathbb{N})$ are uniform Cauchy seq.

Since $(C_b(X), d)$ is complete, then

$$\begin{aligned} f_n \xrightarrow{w.r.t.\ d} f &\Leftrightarrow f_n \text{ are Cauchy seq. in } (C_b(X), d) \\ &\Leftrightarrow f_n \text{ are uniform Cauchy seq.} \\ &\Leftrightarrow f_n \xrightarrow{uni.} f. \end{aligned}$$

□