# General Topology
## Lecture 4

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## 1 Initial Topology

Given maps $X \xrightarrow{f_\alpha} Y_\alpha(\alpha \in A)$ from a set $X$ to topology spaces $Y_\alpha(\alpha \in A)$. It is direct to see that if $X$ is topoloized by discrete topology, the $f_\alpha$ are all continuous. Now the question is how coarse the topology $\mathscr{T}$ on $X$ could be to ensure $f_\alpha(\alpha \in A)$ to be continuous.
Let $\mathcal{S} := \{f_\alpha^{-1}(V) | V \subseteq_{open} Y_\alpha, \alpha \in A\}$, then $\mathscr{T}(\mathcal{S})$ is the expected coarsest topology, called the **initial topology** induced by the family of maps $\{f_\alpha | \alpha \in A\}$.

### 1.1 Subspace Topology

Let $(Y, \mathscr{T}_Y)$ be a topology space, for a subset $X \subseteq Y$. We want to define an natural topology $\mathscr{T}_X$ on $X$ from $Y$, such that keep **inclusion map** $X \xrightarrow{id_X} Y(x \mapsto x)$ be continuous.
As we said, $\mathscr{T}_X$ is the arbitrary union of finite intersection of the pre-image of the open set in $Y$. We call this initial topology induced by the inclusion map the **subspace topology** on $X$ inherited from $Y$.
Note that the arbitrary union of finite intersection of the pre-image of the open set in $Y$ is just the pre-image of arbitrary union of finite intersection of the open set in $Y$, which is just the pre-image of the open set in $Y$. Thus $\mathscr{T}_X = \{id_X^{-1}(V) | V \subseteq_{open} Y\} = \{V \cap X | V \subseteq_{open} Y\}$.

**Exercise 1** (The universal property of subspace topologies)**.** *Suppose $Y$ is a topology space, $X$ is a subspace (i.e. a subset equipped with the subspace topology from $Y$). Given a topology space $Z$, for $\forall$ map $Z \xrightarrow{g} Y$, if $g(Z) \subseteq X$, show that $Z \xrightarrow{g} Y$ is conti. $\Leftrightarrow Z \xrightarrow{g|^X} X$ is conti.*

*Proof.* $\Rightarrow$: any open set in $X$ can be represented by $U \cap X$ where $U \subseteq_{open} Y$, thus $g^{-1}(U \cap X) = g^{-1}(U) \cap g^{-1}(X) = g^{-1}(U) \cap Z \subseteq_{open} Z \Rightarrow Z \xrightarrow{g|^X} X$ is conti. $\Leftarrow$: Trivial. $\square$

**Exercise 2.** *Let $X$ be a topology space, $Z \subseteq Y \subseteq Z$, where $Z, Y$ are equipped with subspace topology, show that*

1. *$Z \subseteq_{open} Y \subseteq_{open} X \Rightarrow Z \subseteq_{open} X$;*
2. *$Z \subseteq_{close} Y \subseteq_{close} X \Rightarrow Z \subseteq_{close} X$.*

*Proof.* 1. $Z \subseteq_{open} Y \subseteq_{open} X \Rightarrow \exists U \subseteq_{open} X$, s.t. $Z = U \cap Y$, since $Y \subseteq_{open} X \Rightarrow Z = U \cap Y \subseteq_{open} X$.
2. $Y \subseteq_{close} X \Rightarrow \exists U \subseteq_{open} X$, s.t. $Y = X \backslash U; Z \subseteq_{close} Y \Rightarrow \exists V \subseteq_{open} Y$, s.t. $Z = Y \backslash V$ and $W \subseteq_{open} X$, s.t. $V = Y \cap W$, thus

$$\begin{aligned}
Z &= Y \backslash V \\
&= (X \backslash U) \backslash (Y \cap W) \\
&= (X \backslash U) \backslash ((X \backslash U) \cap W) \\
&= (X \cap U^c) \cap (X \cap U^c \cap W)^c \\
&= U^c \cap (U \cup W^c) \\
&= U^c \cap W^c \\
&= X \backslash (U \cup W) \\
&\subseteq_{close} X
\end{aligned}$$

$\square$

## 1.2 Product Space

Let $(Y_1, \mathscr{T}_1)$ and $(Y_2, \mathscr{T}_2)$ be topology spaces, we want to create a natural topology $\mathscr{T}_{Y_1 \times Y_2}$ on $Y_1 \times Y_2$ which makes the projections $Y_1 \times Y_2 \xrightarrow{p_i} Y_i (i = 1, 2)$ be continuous. Suppose $U_i (i = 1, \cdots, k_U) \subseteq_{open} Y_1$ and $V_j (j = 1, \cdots, k_V) \subseteq_{open} Y_2$, then

$$\left(\cap_{i=1}^{k_U} f^{-1}(U_i)\right) \cap \left(\cap_{i=1}^{k_V} f^{-1}(V_i)\right) = f^{-1}\left(\cap_{i=1}^{k_U} U_i\right) \cap f^{-1}\left(\cap_{i=1}^{k_V} V_i\right)$$

where $\cap_{i=1}^{k_U} U_i \subseteq_{open} Y_1$ and $\cap_{i=1}^{k_V} V_i \subseteq_{open} Y_2$. Thus the desired initial topology can be represented as the arbitrary union of the intersection of the pre-image of an open set in $Y_1$ and the pre-image of an open set in $Y_2$. (instead of the finite intersection of pre-image of open sets in $Y_1$ and $Y_2$, it is subtle) Thus the basis of the expected initial topology is

$$\begin{aligned}
\Pi &= \{p_1^{-1}(W_1) \cap p_2^{-1}(W_2) | W_1 \subseteq_{open} Y_1, W_2 \subseteq_{open} Y_2\} \\
&= \{(W_1 \times Y_2) \cap (Y_1 \times W_2) | W_1 \subseteq_{open} Y_1, W_2 \subseteq_{open} Y_2\} \\
&= \{W_1 \times W_2 | W_1 \subseteq_{open} Y_1, W_2 \subseteq_{open} Y_2\}
\end{aligned}$$

Thus the topology desired is all unions of rectangle:

$$\mathscr{T}_{Y_1\times Y_2} = \{\cup\mathcal{F} | \mathcal{F} \subseteq \Pi\}.$$

We call such initial topology **product topology** of $Y_1$ and $Y_2$, denote as $\mathscr{T}_1 \times \mathscr{T}_2$.
In particular, the open set $O$ in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ can be written by $O = \cup U \times V$ where $U, V \subseteq_{open} \mathbb{R}$.

*Remark* 1. We call such a $W_1 \times W_2$ a rectangle.

### 1.3 Cartesian Product

Let's recall the definition of Cartesian product. Given two sets $Y_1, Y_2$, there exists a **bijection** between $Y_1 \times Y_2$ and the family of maps $\{\{1,2\} \xrightarrow{s} Y_1 \cup Y_2 | s(1) \in Y_1, s(2) \in Y_2\} =: \mathcal{M}_{Y_1\times Y_2}$. First, there is an injection from left to right: for any $(s_1, s_2) \in Y_1 \times Y_2$, define $s$ as $s(1) = s_1, s(2) = s_2$. Thus different points in $Y_1 \times Y_2$ reflect to different maps in $\mathcal{M}_{Y_1\times Y_2}$.
On the other hand, there exists an injection from right to left as well: for any $s', s \in \mathcal{M}_{Y_1\times Y_2}$, correspond to $(s(1), s(2)), (s'(1), s'(2)) \in Y_1 \times Y_2$, and $(s(1), s(2)) \neq (s'(1), s'(2))$ if $s \neq s'$.
Furthermore, when we project a point $(y_1, y_2) \in Y_1 \times Y_2$ to $y_1 \in Y_1$ (using projection $Y_1 \times Y_1 \xrightarrow{p_1} Y_1$), it is equivalent with mapping the corresponding map $s$ to $s(1)$.

$$\begin{tikzcd}
Y_1 \times Y_2 \arrow[rrrrr, bend left, "p_1"] & \ni & (y_1, y_2) \arrow[r, mapsto] \arrow[d, Leftrightarrow] & y_1 \arrow[d, equal] & \in & Y_1 \\
\mathcal{M}_{Y_1\times Y_2} \arrow[rrrrr, bend right] & \ni & s \arrow[r, mapsto] & s(1) & \in & Y_1
\end{tikzcd}$$

Similarly, we can define infinite dimension Cartesian product as

$$\prod_{\alpha\in A} Y_\alpha := \{A \xrightarrow{s} \cup_{\alpha\in A} Y_\alpha | \forall \alpha \in A, s(\alpha) \in Y_\alpha\} =: \mathcal{M}_{\prod_{\alpha\in A} Y_\alpha},$$

according to the axiom of choice, if $Y_\alpha \neq \emptyset$ for any $\alpha \in A$, then such map $s$ exists, then $\prod_{\alpha\in A} Y_\alpha \neq \emptyset$. For $\alpha \in A$, we often denote the value of $s$ at $\alpha$ by $s_\alpha$ rather than $s(\alpha)$; we call it the $\alpha$-th **coordinate** of $s$. And we often denote the function $s$ itself by the symbol

$$(s_\alpha)_{\alpha\in A},$$

which is as close as we can come to a `tuple notation` for an arbitrary index set $A$.

Corresponding, we can define the projection on infinite dimension cartesian product: for any $\beta \in A$,

$$\prod_{\alpha \in A} Y_\alpha \xrightarrow{p_\beta} Y_\beta$$

as a map

$$\mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} \longrightarrow Y_\beta$$

with $s \mapsto s_\beta$.

## 1.4 Infinite Dimension Product Topology

Now we can define the product topology on infinite dimension. As we discussed, the topology is arbitrary union of finite intersection of pre-image of the open set in $Y_\alpha(\alpha \in A)$. Since the intersection is finite, we can still exchange the order of pre-image and intersection, and then represent the open sets from the same $Y_\alpha(\alpha \in A)$ as one open set. Note that the pre-image of $U_\beta \subseteq_{open} Y_\beta$ can be represented by

$$\{s \in \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} | s_\beta \in U_\beta\}.$$

Thus finite intersection of the pre-image of open sets, i.e. the basis of the infinite dimension product topology is

$$\Pi_{\prod_{\alpha \in A} Y_\alpha} = \{s \in \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} | s_{\beta_1} \in U_{\beta_1}, \cdots, s_{\beta_k} \in U_{\beta_k}, k \in \mathbb{N}\}.$$

That the basis of infinite product topology is set of maps that only map **finite** points in domain to the open sets of codomain. Alternatively, we can represent it as

$$\Pi_{\prod_{\alpha \in A} Y_\alpha} = \left\{ \prod_{\alpha \in A} V_\alpha | \forall \alpha \in A, V_\alpha \subseteq_{open} Y_\alpha \wedge \{\alpha \in A | V_\alpha \neq Y_\alpha\} \text{ is finite} \right\}.$$

And the topology is

$$\mathscr{T}_{\prod_{\alpha \in A} Y_\alpha} = \{\cup \mathcal{F} | \mathcal{F} \subseteq \Pi_{\prod_{\alpha \in A} Y_\alpha}\}$$

**Exercise 3** (The universal property of product topology)**.** *Let $Z, Y_\alpha(\alpha \in A)$ are topology spaces, $\prod_{\alpha \in A} Y_\alpha$ is equipped with product topology, show that for any group of maps*

$$Z \xrightarrow[conti.]{g_\alpha} Y_\alpha(\alpha \in A)$$

$\exists ! Z \xrightarrow[conti.]{g} \prod_{\alpha \in A} Y_\alpha$, s.t. $p_\alpha \circ g = g_\alpha$ *for* $\forall \alpha \in A$. *That is, such commutative diagram holds*

$$\begin{tikzcd} Z \arrow[r, "g", "conti."'] \arrow[dr, bend right, "g_\alpha"', "conti."] & \prod_{\alpha \in A} Y_\alpha \arrow[d, "p_\alpha"] \\ & Y_\alpha \end{tikzcd}$$

*Proof.* Existence: Select a group of $g_\alpha(\alpha \in A)$ such that for a given $z \in Z$ has

$$g_\alpha(z) = y_\alpha \in Y_\alpha.$$

Define a map $Z \xrightarrow{g} \mathcal{M}_{\prod_{\alpha\in A} Y_\alpha}$ with $z \mapsto s$ where $s_\alpha = y_\alpha(\alpha \in A)$. Thus for any $\beta \in A$, we have

$$p_\beta \circ g(z) = p_\beta(s) = s_\beta = y_\beta = g_\beta(z)$$

Thus $p_\alpha \circ g = g_\alpha$ for any $\alpha \in A$. We now show $g$ is continuous.
Any open set $U$ in $\mathcal{M}_{\prod_{\alpha\in A} Y_\alpha}$ can be written as $U = \cup\mathcal{F} = \cup_{V\in\mathcal{F}} V$, where $\mathcal{F} \subseteq \Pi_{\prod_{\alpha\in A} Y_\alpha}$. Thus

$$g^{-1}(U) = g^{-1}(\cup_{V\in\mathcal{F}} V) = \cup_{V\in\mathcal{F}} g^{-1}(V).$$

Here $V$ is the element in the basis, and can be represented as

$$V = \{s \in \mathcal{M}_{\prod_{\alpha\in A} Y_\alpha} | s_{\alpha_1} \in U_{\alpha_1}, \cdots, s_{\alpha_k} \in U_{\alpha_k}\},$$

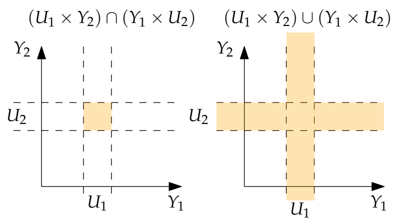
where $U_{\alpha_i} \subseteq_{open} Y_{\alpha_i}(i = 1, \cdot, k)$, thus

$$\begin{aligned} g^{-1}(V) &= \{z \in Z | g_{\alpha_1}(z) \in U_{\alpha_1}, \cdots, g_{\alpha_k}(z) \in U_{\alpha_k}\} \\ &= \cap_{i=1}^k g_{\alpha_i}^{-1}(U_{\alpha_i}) \\ &\subseteq_{open} Z \end{aligned}$$

Thus $g^{-1}(U) = \cup_{V\in\mathcal{F}} g^{-1}(V) \subseteq_{open} Z \Rightarrow g$ is continuous.

*Remark* 2. There is a trap:

- $(U_1 \times Y_2) \cap (Y_1 \times U_2) = U_1 \times U_2$;
- $(U_1 \times Y_2) \cup (Y_1 \times U_2) \neq Y_1 \times Y_2$;

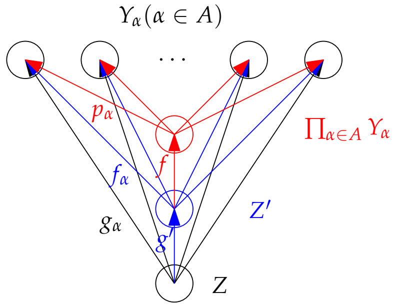



Uniqueness: for any $h$ such that $p_\alpha \circ h = g_\alpha$, given a $z \in Z$, we have $p_\alpha(h(z)) = g_\alpha(z)$ for $\forall \alpha \in A$. Thus

$$\begin{aligned} h(z) &\in \cap_{\alpha\in A} p_\alpha^{-1}(g_\alpha(z)) \\ &= \cap_{\alpha\in A}\{s \in \mathcal{M}_{\prod_{\alpha\in A} Y_\alpha} | s_\alpha = g_\alpha(z)\} \\ &= \{s \in \mathcal{M}_{\prod_{\alpha\in A} Y_\alpha} | s_\alpha = g_\alpha(z), \alpha \in A\} \end{aligned}$$

Thus $h(z) = s$ where $s_\alpha = g_\alpha(z), \alpha \in A \Rightarrow h = g$. □

The conclusion of the universal property of product topology is : for any group of maps $Z \xrightarrow{g_\alpha} Y_\alpha(\alpha \in A)$, if they can be substitute by another group of map $f_\alpha \circ g'$ where $Z \xrightarrow{g'} Z'$ and $Z' \xrightarrow{f_\alpha} Y_\alpha$, we say $Z'$ is **closer** to $Y_\alpha(\alpha \in A)$ than $Z$.
Then $\prod_{\alpha\in A} Y_\alpha$ is the **closest** set to $Y_\alpha(\alpha \in A)$.




**Exercise 4.** *Let $Z, Y_\alpha(\alpha \in A)$ are top. spaces. Show that $Z \xrightarrow{g} \prod_{\alpha\in A} Y_\alpha$ is continuous $\Leftrightarrow p_\alpha \circ g(\alpha \in A)$ are continuous.*

*Proof.* $\Rightarrow$: Since $p_\alpha \circ g = g_\alpha$, we need to prove $g$ is continuous $\Rightarrow g_\alpha$ is continuous.
For any open set $U_\alpha \subseteq_{open} Y_\alpha$. $p_\alpha^{-1}(U_\alpha) = \{s \in \mathcal{M}_{\prod_{\alpha\in A} Y_\alpha} | s_\alpha \in U_\alpha\} \subseteq_{open} \mathcal{M}_{\prod_{\alpha\in A} Y_\alpha} = \prod_{\alpha\in A} Y_\alpha$. And $g^{-1}(\{s \in \mathcal{M}_{\prod_{\alpha\in A} Y_\alpha}) = \{z \in Z | g_\alpha \in U_\alpha\} = g_\alpha^{-1}(U_\alpha) \subseteq_{open} Z$, since $g$ is continuous, thus $g_\alpha$ is continuous. $\Leftarrow$: has been given in Ex2. $\square$