Introduction to Topology

General Topology, Lecture 8,9

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This is the Lecture note for the *Introduction to Topology*. The course covers the following topics: Naive Set Theory, Elementary Number Theory, Group Theory, Topological Spaces and Continuous Maps, Introduction to Algebraic Topology.

Content:

Metric space

Definition 1 (Metric Space)**.** Let $X \times X \xrightarrow{d} \mathbb{R}$ be a function, we cay that d is a metric on X or (X, d) is a metric space if for $\forall x, x', x'' \in X$ have

1. Positivity: $d(x, x') \geq 0$ and $d(x, x'') = 0$ iff $x = x'$;
2. Symmetry: $d(x, x') = d(x', x)$;
3. Triangle inequality: $d(x, x') \leq d(x, x'') + d(x'', x')$.

Exercise 1. Show that the triangle inequality is equivalent with for $\forall x, x', x'' \in X$

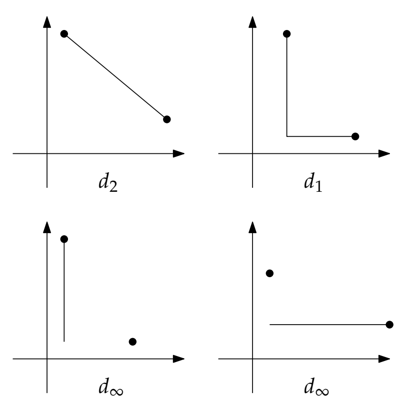
$$d(x, x') \geq |d(x, x'') - d(x', x'')|.$$

Proof. $\geq \Rightarrow \leq$: since $d(x, x') \geq |d(x, x'') - d(x', x'')| \geq d(x, x'') - d(x', x'')$, we have that $d(x, x'') \leq d(x, x') + d(x', x'')$.

$\leq \Rightarrow \geq$: if $\exists x, x', x''$ such that $d(x, x') < |d(x, x'') - d(x', x'')|$, then

$$\begin{aligned} d(x, x') &< |d(x, x'') - d(x', x'')| \\ &\leq |d(x, x') + d(x', x'') - d(x', x'')| \\ &\leq d(x, x') \end{aligned}$$

thus $d(x, x') < d(x, x')$, which leads to a contradiction. □



Example 1. Here are some metric examples:

1. define $d_2(x, y) := \left(\sum_i^m |x_i - y_m|^2\right)^{1/2}$, $x, y \in \mathbb{R}^m$. Then d_2 is a metric on $\mathbb{R}^m$ by cauchy inequality.
2. define $d_1(x, y) := \sum_{i=1}^m |x_i - y_i|$, $x, y \in \mathbb{R}^m$. Then d_1 is a metric on $\mathbb{R}^m$.
3. define $d_\infty(x, y) := \max\{|x_i - y_i|\}$, $i \in \{1, 2, \cdots, m\}$, $x, y \in \mathbb{R}^m$. Then d_∞ is a metric on $\mathbb{R}^m$.

d_2 can be proved to be a metric by Cauchy inequality:

Exercise 2 (Cauchy inequality)**.** For any $(x_1, \cdots, x_n), (y_1, \cdots, y_n) \in \mathbb{R}^n$, show that

$$(x_1 y_1 + \cdots, x_n y_n)^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2)$$

and "=" holds iff $\exists a, b \in \mathbb{R}$ which are not all 0.

Proof. Consider the polynomial $p(t) = \sum_{i=1}^{n}(x_i t + y)^2 = t^2 \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 \geq 0$, thus $\Delta = 4\left(\sum_{i=1}^{n} x_i y_i\right)^2 - 4\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 \leq 0 \Rightarrow \left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2$. $\square$

Example 2 (p-adic)**.** If p is a prime number, $x \in \mathbb{Q}$, define

$$|x|_{p-adic} := \begin{cases} p^{-m}, & x = \dfrac{a}{b} \cdot p^m, x \neq 0, \\ 0, & x = 0, \end{cases}$$

where $a, b, m \in \mathbb{Z}$, $(a, p) = (b, p) = 1$. For $\forall x, y \in \mathbb{Q}$, define $d_{p-adic}(x, y) = |x - y|_{p-adic}$, then d_{p-adic} is a metric on $\mathbb{Q}$.

Assume $x = (a/b)p^m, y = (s/t)p^n \in \mathbb{Q}$ where $a, b, s, t, m, n \in \mathbb{Z}, (a, p) = (b, p) = (s, p) = (t, p) = 1, m > n$, then $|x|_{p-adic} = p^{-m} < |y|_{p-adic} = p^{-n}$, and

$$\begin{aligned} |x - y|_{p-adic} &= |(a/b)p^m - (s/t)p^n|_{p-adic} \\ &= \left|\frac{adp^{m-n} - bc}{bd} p^n\right|_{p-adic}. \end{aligned}$$

it is easy to check $adp^{m-n} - bc, bd \in \mathbb{Z}$ and $(adp^{m-n} - bc, p) = (bd, p) = 1$, thus

$$|x - y|_{p-adic} = p^{-n} = |y|_{p-adic} = \max\{|x|_{p-adic}, |y|_{p-adic}\}.$$

Thus for any $x, y, z \in \mathbb{Q}$, we have that

$$\begin{aligned} d_{p-adic}(x, y) &= \max\{|x|_{p-adic}, |y|_{p-adic}\} \\ &\leq \max\{|x|_{p-adic}, |z|_{p-adic}\} + \max\{|z|_{p-adic}, |y|_{p-adic}\} \\ &= d_{p-adic}(x, z) + d_{p-adic}(y, z), \end{aligned}$$

which follows the triangle inequality, the other two conditions is trivial.

Open set on metric space

Definition 2 (Open Ball)**.** Let (X, d) be a metric space, for $\forall r > 0$ and $x_0 \in X$, we let

$$B_r(x_0) := \{x \in X | d(x, x_0) < r\},$$

and call it the open ball with center x_0 and radius r; let

$$\overline{B_r(x_0)} := \{x \in X | d(x, x_0) \leq r\},$$

and call it the close ball with center x_0 and radius r.

Example 3 (discrete metric)**.** For $\forall x, x' \in \mathbb{R}^2$, define metric $d(x, x') = 0$ if $x = x'$, and $d(x, x') = 1$ if $x \neq x'$, then $B_1(x) = \{x\}$, $\overline{B_1(x)} = \mathbb{R}^2$, $B_{1.1}(x) = \mathbb{R}^2$.

Definition 3 (Open Set). $S(\subseteq X)$ is called an Open Set of X with respect to d, if $\forall x_0 \in S$, $\exists r > 0$ such that $B_r(x_0) \subseteq S$; $F(\subseteq X)$ is Close Set of X w.r.t. d if $X \backslash F$ is open set of X w.r.t. d.

Exercise 3. Prove that $B_r(x)$ is open set and $\overline{B_r(x)}$ is close.

Proof. For $\forall x' \in B_r(x)$, we have $d(x, x') < r$, donate $r - d(x, x')$ by s, then for $\forall x'' \in B_{s/2}(x')$ satisfy

$$\begin{aligned} d(x, x'') &\leq d(x, x') + d(x', x'') \\ &\leq d(x, x') + \frac{s}{2} \\ &< r, \end{aligned}$$

thus $x'' \in B_r(x)$ and $B_{s/2}(x') \subseteq B_r(x)$ and $B_r(x)$ is a open set. For $\forall x' \in X \backslash \overline{B_r(x)}$ has $d(x, x') > r$. Denote $d(x, x') - r$ by t, then for $\forall x'' \in B_{t/2}(x')$ satisfy

$$\begin{aligned} d(x, x'') &\geq |d(x, x') - d(x', x'')| \\ &\geq d(x, x') - d(x', x'') \\ &\geq d(x, x;) - \frac{t}{2} \\ &> r. \end{aligned}$$

Thus $B_{t/2}(x') \subseteq X \backslash \overline{B_r}$ and $X \backslash \overline{B_r}$ is an open set, thus $\overline{B_r}$ is a close set. □

Exercise 4. Let (X, d) be a metric space. show that

1. $X, \emptyset \subseteq_{open} X$;
2. $O_1, O_2 \subseteq_{open} X \Rightarrow O_1 \cap O_2 \subseteq_{open} X$;
3. $O_\alpha \subseteq_{open} X, (\alpha \in A) \Rightarrow \cup_{\alpha \in A} O_\alpha \subseteq_{open} X$ (α not necessarily be integral or countable);
4. All above corresponding statements for close set are true.

Note 1. First 3 statements are the essential intuition for the definition of *Topology*.

Proof. 1. Obviously X is an Open set thus $\emptyset$ is a close set. If $\emptyset$ is not an open set, then $\exists x \in \emptyset$, $\forall r > 0$ such that $B_r(x) \nsubseteq \emptyset$, which is impossible. Thus $\emptyset$ is an open set (logically) and X is a close set;
2. $\forall x \in O_1 \cap O_2$, $\exists r_1, r_2 > 0$, s.t. $B_{r_1}(x) \subseteq O_1$ and $B_{r_2}(x) \subseteq O_2$. Thus $\forall x' \in B_{\min\{r_1,r_2\}}(x) = B_{r_1}(x) \cap B_{r_2}(x) \Rightarrow x' \in O_1 \cap O_2 \Rightarrow B_{\min\{r_1,r_2\}}(x) \subseteq O_1 \cap O_2$, thus $O_1 \cap O_2$ is open. Collectively, the intersection of any finite open sets is an open set;
3. For $\forall x \in \cup_{\alpha \in A} O_\alpha, \exists$ at least one $\alpha' \in A$, s.t. $x \in O_{\alpha'}$, then $\exists r > 0$, s.t. $B_r(x) \subseteq O_{\alpha'} \subseteq \cup_{\alpha \in A} O_\alpha$, thus $\cup_{\alpha \in A} O_\alpha$ is an open set;
4. Take complementary set: The union of finite close sets is close; the intersection of any close sets is close.

□

Exercise 5. Show that an open set is the union of open balls.

Proof. Given an open set O, for any $o \in O, \exists r_o > 0$, s.t. $B_{r_o}(o) \subseteq O$, define $O' = \cup_{o \in O} B_{r_o}(o)$. Thus for $\forall x \in O', \exists o'$, s.t. $x \in B_{r'_o}(o') \subseteq O \Rightarrow O' \subseteq O$;

On the other hand, for any $y \in O, \exists r_y > 0$, s.t. $B_{r_y}(y) \subseteq O \Rightarrow y \in B_{r_y}(y) \subseteq O' \Rightarrow O \subseteq O'$. Thus $O = O' = \cup_{o \in O} B_{r_o}(o)$. $\square$

Definition 4 (Convergence)**.** Let (X, d) be a metric space, $a_n \in X, (n \in \mathbb{N})$, $L \in X$, define $\lim_{n\to\infty} a_n = L$ w.r.t. d, if $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N$ s.t. $d(a_n, L) < \epsilon$, that is $a_n \in B_\epsilon(L)$.

Exercise 6. Show that

1. $\lim_{n\to\infty} a_n = L \Leftrightarrow \lim_{n\to\infty} d(a_n, L) = 0$;
2. $\lim_{n\to\infty} a_n = L \Leftrightarrow \forall L \in U \subseteq_{open} X, \exists N \in \mathbb{N}, \forall n \geq N$ s.t. $a_n \in U$.

Proof. (1) Trivial; (2) $\Rightarrow$: Suppose that $\lim_{n\to\infty} a_n = L$, for $\forall U$ that $L \in U, \exists r > 0$, s.t. $B_r(L) \subseteq U$, and $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, s.t. $a_n \in B_r(L) \subseteq U$; $\Leftarrow$: Suppose $L \in U \subseteq_{open} X$, then $\exists r > 0$ such that $B_r(L) \subseteq U$. Since $B_r(L)$ is also an open set, then $\exists N \in \mathbb{N}$, for $\forall n \geq N$, s.t. $a_n \in B_r(L) \subseteq U$. $\square$

We say $S \subseteq X$ is bounded w.r.t. d, if $\exists r > 0$ and $x_0 \in X$, s.t. $S \subseteq B_r(x_0)$.

Theorem 1 (Bolzano-Weierstrass theorem)**.** *If $a_n \in \mathbb{R}^m (n \in \mathbb{N})$ is bounded w.r.t. d_2, then $\exists$ a subsequence $a_{n_m} (m \in \mathbb{N})$ which converges.*

Proof. We only prove $\mathbb{R}^2$ case. If we want to prove that the vector $a = (a_1, \cdots, a_m) \in \mathbb{R}^m \to L = (l_1, \cdots, l_m) \in \mathbb{R}^m$, all we need to prove is $\lim_{n\to\infty} a_i = l_i, (i = 1, \cdots, m)$.

Choose $M > 0$, s.t. $a_n \in Q = [-M, M] \times [-M, M]$ for all $n \in \mathbb{N}$. Divide Q into 4 squares with equal size and choose one, say Q_1, such that $|\{n | a_n \in Q\}| = \infty$. Select $n_1 \in \mathbb{N}$, such that $a_{n_1} \in Q_1$. Repeat this and we have $\cap_{k=1}^{\infty} Q_k = \{a\}$. By theorem of nested interval we have that $\lim_{k\to\infty} a_{n_k} = a$. $\square$

Exercise 7. Let (X, d) be a metric space, $F \subseteq X$ show that $F \subseteq_{close} X \Leftrightarrow \forall a_n \in F (n \in \mathbb{N})$ and $\lim_{n\to\infty} a_n = a \in X$ then $a \in F$.

Proof. $\Rightarrow$: Assume that F is close and $a_n \in F$. If $a_n \to a \in X \backslash F$, then $\exists r > 0$, s.t. $B_r(a) \in X \backslash F$. Since $\lim_{n\to\infty} a_n = a$, for r, there exists $N \in \mathbb{N}, \forall n \geq N$, s.t. $d(a_n, a) < r$, i.e. $a_n \in B_r(a) \subseteq X \backslash F$, which leads to a contradiction. $\Leftarrow$: Suppose that $\forall a_n \in F (n \in \mathbb{N})$ and $\lim_{n\to\infty} a_n = a \in X$ then $a \in F$, and F is not close, which means $X \backslash F$ is not open, and $\exists x \in X \backslash F, \forall r > 0, B_r(x) \cap F \neq \emptyset$. Select $n \in \mathbb{N}$ such that $a_n = B_{\frac{1}{n}}(x) \cap F$. Thus $\lim_{n\to\infty} a_n = x \notin F$, which leads to a contradiction. $\square$

Note 2. Set family of sets as $\{B_{1/n}(x)\}_{n \in \mathbb{N}}$ is a very useful skill.

Definition 5 (Open cover, Compact set)**.** Let (X, d) be a metric space, $S \subseteq X$, $O_\alpha \in X(\alpha \in A)$, we say that $O_\alpha(\alpha \in A)$ form an open cover of S, if $S \subseteq \cup_{\alpha\in A} O_\alpha$. S is called a compact set if $\forall$ open cover $O_\alpha(\alpha \in A)$ of S, $\exists \alpha_1, \cdots, \alpha_m \in A$, s.t. $S \subseteq \cup_{i=1}^m O_{\alpha_i}$, where $\cup_{i=1}^m O_{\alpha_i}$ is called a finite subcover.

If there exists an open cover of F whose any finite subcover can not cover it, then F is not a compact set. for instance, let $F = (0, 1), O_n = (1/n, 2), n \in \mathbb{N}$, then O_n is an open cover of F, however any finite subcover of O_n can not cover F.

Theorem 2 (Heine-Borel theorem)**.** *Let $S \subseteq \mathbb{R}^n$, then S is compact $\Leftrightarrow$ S is bounded and closed.*

Proof. $\Rightarrow$: Suppose that S is compact, select a point $s \in S$ arbitrarily, define $O_i = B_i(s)$, it is easy to check that $S \subseteq \cup_{i\in\mathbb{N}} O_i$, since for any $s' \in S$, we have that $s' \in B_{2d(s,s')}(s) \subseteq O_{\lceil 2d(s,s')\rceil}$. Since S is compact, there exists a finite subcover, thus S is bounded.

Suppose S is compact, but S is not closed, which means $X\backslash S$ is not open and $\exists x \in X\backslash S$, s.t. $\forall r > 0, B_r(x) \cap S \neq 0$. Since S is bounded, define $\iota = \sup_{s\in S} d(s, x)$, define open cover

$$O_n = B_{\frac{\iota}{n}}(x) - B_{\frac{\iota}{n+1}}(x),$$

thus $O_i \cap O_j = \emptyset(i \neq j)$ and $O_i \cap S \neq \emptyset(\forall i)$. Thus O_n has no finite subcover, which leads to a contradiction and S is closed.

$\Leftarrow$: Suppose that S is bounded and closed, and $\exists$ an open cover $O_\alpha(\alpha \in A)$ of S which admits no finite subcover. Choose a cube Q containing S (S is bounded), divide Q into 4 equal-sized cubes and select one of them denoted by Q_1, s.t. $Q_1 \cap S$ can not be covered by finitely many Q_α, select a point such that $s_1 \in Q_1 \cap S$. Repeat.

Obviously, $\lim_{n\to\infty} s_n = a$, notice that $s_n \in Q_n \cap S \subseteq S$, thus $\lim_{n\to\infty} s_n = a \in S$ for S is closed. Thus there exist O_i such that $a \in O_i$. Since O_i is open, $\exists r > 0$, s.t. $B_r(a) \subseteq O_i$. Then $\exists N \in \mathbb{N}, \forall n \geq N$, s.t. $Q_n \subseteq B_r(a) \subseteq O_i$. Since $Q_n \cap S$ can not be covered by finitely many O_α, but could be covered by O_i, which leads to a contradiction. $\square$

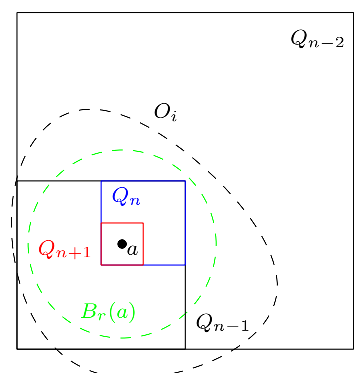


Figure 1: Heine-Borel theorem