Introduction to Analysis

Lecture 8

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Abstract

This is the Lecture note for the *Introduction to Analysis* class in Spring 2019.

0.1 Properties of Darboux Integral

The Monotonicity (P1), (P5) and (P6) of Darboux integral is trivial, we will show that Darboux integral has linear property (P2):

Proposition 1. *Let f, g be bounded functions on $[a, b]$, then*

$$\overline{\int_a^b} f + g \leq \overline{\int_a^b} f + \overline{\int_a^b} g, \quad \underline{\int_a^b} f + g \geq \underline{\int_a^b} f + \underline{\int_a^b} g.$$

Proof. Since $\sup_X (f+g) \leq \sup_X f + \sup_X g$ (Exercise **??**), then

$$\begin{aligned}
\overline{S}(f+g, \Delta) &= \sum_{j=1}^{k} (\sup_{I_j} f + g) \cdot (x_j - x_{j-1}) \\
&\leq \sum_{j=1}^{k} (\sup_{I_j} f + \sup_{I_j} g) \cdot (x_j - x_{j-1}) \\
&= \sum_{j=1}^{k} \sup_{I_j} f \cdot (x_j - x_{j-1}) + \sum_{j=1}^{k} \sup_{I_j} f \cdot (x_j - x_{j-1}) \\
&= \overline{S}(f, \Delta) + \overline{S}(g, \Delta).
\end{aligned}$$

And for $\forall \epsilon > 0$, $\exists \Delta_1, \Delta_2$ (by Remark **??** (E1)) s.t.

$$\begin{aligned}
\overline{S}(f, \Delta_1 \cup \Delta_2) \leq \overline{S}(f, \Delta_1) < \overline{\int_a^b} f + \epsilon, \\
\overline{S}(g, \Delta_1 \cup \Delta_2) \leq \overline{S}(g, \Delta_2) < \overline{\int_a^b} g + \epsilon.
\end{aligned}$$

and

$$\begin{aligned}
\overline{\int_a^b} f+g &\leq \overline{S}(f+g,\Delta_1\cup\Delta_2)\\
&\leq \overline{S}(f,\Delta_1\cup\Delta_2)+\overline{S}(g,\Delta_1\cup\Delta_2)\\
&< \overline{\int_a^b} f+\overline{\int_a^b} g+2\epsilon
\end{aligned}$$

Thus

$$\overline{\int_a^b} f+g < \overline{\int_a^b} f+\overline{\int_a^b} g+2\epsilon$$

for $\forall\epsilon>0 \Rightarrow$

$$\overline{\int_a^b} f+g \leq \overline{\int_a^b} f+\overline{\int_a^b} g.$$

$\square$

Therefore if f,g are Darboux integrable on $[a,b]$, then $f+g$ is too, and

$$\int_a^b f+g=\int_a^b f+\int_a^b g.$$

And for $\alpha\in\mathbb{R}$, we have

$$\overline{\int_a^b}\alpha f=\begin{cases}\alpha\overline{\int_a^b} f, & \alpha\geq 0\\ \alpha\underline{\int_a^b} f, & \alpha\leq 0\end{cases},\quad \underline{\int_a^b}\alpha f=\begin{cases}\alpha\underline{\int_a^b} f, & \alpha\geq 0\\ \alpha\overline{\int_a^b} f, & \alpha\leq 0\end{cases}$$

Thus Darboux integral has linear property (P2).

Exercise 1 (P7). *If f is Darboux integrable on $[a,b]$, then $|f|$ is too, and*

$$\left|\int_a^b f\right|\leq\int_a^b |f|.$$

Proof. For any subinterval I of $[a,b]$, there are 3 cases:
1. If $\inf_I f\geq 0$, then $f\geq 0$ on I so $\inf_I |f|=\inf_I f$ and $\sup_I |f|=\sup_I f$ and hence

$$\sup_I|f|-\inf_I|f|=\sup_I f-\inf_I f.$$

2. If $\sup_I f\leq 0$, then $f\leq 0$ on I, so $\inf_I|f|=-\sup_I f$ and $\sup_I|f|=-\inf_I f$ and hence

$$\sup_I|f|-\inf_I|f|=\sup_I f-\inf_I f.$$

3. If $\inf_I f<0<\sup_I f$, then we have either $\sup_I|f|=\sup_I f$, in which case $\sup_I|f|-\inf_I|f|\leq\sup_I|f|=\sup_I f<\sup_I f-\inf_I f$; or $\sup_I|f|=-\inf_I f$, in which case

$$\sup_I|f|-\inf_I|f|\leq-\inf_I f<\sup_I f-\inf_I f.$$

Then for any $\epsilon > 0, \exists \Delta$ s.t.

$$\begin{aligned}
0 &\le \overline{S}(|f|, \Delta) - \underline{S}(|f|, \Delta) \\
&= \sum_{j=1}^{k} (\sup_{I_j} |f| - \inf_{I_j} |f|) \cdot (x_j - x_{j-1}) \\
&\le \sum_{j=1}^{k} (\sup_{I_j} f - \inf_{I_j} f) \cdot (x_j - x_{j-1}) \\
&= \overline{S}(f, \Delta) - \underline{S}(f, \Delta) \\
&< \epsilon
\end{aligned}$$

thus $|f|$ is Darboux integrable. $\square$

Proposition 2. *Let f be Darboux integrable on $[a,b], c \in (a,b)$, then*

$$\overline{\int_a^c} f + \overline{\int_c^b} f \le \overline{\int_a^b} f, \quad \underline{\int_a^c} f + \underline{\int_c^b} f \ge \underline{\int_a^b} f.$$

Proof. Let Δ_1, Δ_2 be partitions of $[a,c], [c,b]$ respectively, then

$$\overline{S}(f, \Delta_1) + \overline{S}(f, \Delta_2) = \overline{S}(f, \Delta_1 \cup \Delta_2),$$

Let Δ be a partition of $[a,b]$, and define $\Delta_c = (\Delta \cap [a,c]) \cup \{c\}$ and ${}_c\Delta = (\Delta \cap [c,b]) \cup \{c\}$, then

$$\overline{S}(f, \Delta_c) + \overline{S}(f, {}_c\Delta) = \overline{S}(f, \Delta \cup \{c\}) \le \overline{S}(f, \Delta)$$

thus $\overline{\int_a^c} f + \overline{\int_c^b} f \le \overline{\int_a^b} f$. $\square$

Thus if f is Darboux integrable on $[a,b], c \in (a,b)$, then it is Darboux on $[a,c]$ and $[c,b]$ and

$$\int_a^b f = \int_c^b f + \int_a^c f. \tag{P3}$$

Proposition 3 (P4)**.** *f is continuous on $[a,b] \Rightarrow f$ is Darboux integrable on $[a,b]$.*

Proof. $[a,b]$ is a compact set in $\mathbb{R}$ (Heine-Borel theorem, Theorem **??**), thus f is continuous on compact $\Rightarrow f$ is uniformly continuous on $[a,b]$ (Theorem **??**). Thus for any $\epsilon > 0, \exists \delta > 0,$ s.t. $\forall |x - x'| < \delta \Rightarrow |f(x) - f(x')| < \epsilon$.
Choose partition Δ s.t. $\max_{1 \le j \le k(\Delta)} (x_j - x_{j-1}) < \delta$, then for any j we have

$$0 \le \sup_{I_j} f - \inf_{I_j} f \le \epsilon \tag{Exercise ??}$$

Thus

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le \overline{S}(f, \Delta) - \underline{S}(f, \Delta) \le \epsilon \cdot (b-a)$$

for $\forall \epsilon > 0 \Rightarrow \overline{\int_a^b} f = \underline{\int_a^b} f \Rightarrow f$ is Darboux integrable by definition. $\square$

Proposition 4. *If $f\nearrow(\searrow)$ on $[a,b] \Rightarrow f$ is Darboux integrable.*

Proof. If $f\nearrow$, then

$$\begin{aligned}\overline{S}(f,\Delta) - \underline{S}(f,\Delta) &= \sum_{j=1}^{k}(f(x_j) - f(x_{j-1})) \cdot (x_j - x_{j-1}) \\ &= (f(b) - f(a)) \cdot \max_{1\le j\le k}(x_j - x_{j-1})\end{aligned}$$

Choose Δ s.t. $\max_{1\le j\le k}(x_j - x_{j-1})$ small enough. □

Remark 1. Furthermore, if f can be represented by $f = f_1 + f_2$, where f_1, f_2 are monotone, then f is Darboux integrable.

Proposition 5. *Let $[a,b] \xrightarrow{f_n} \mathbb{R}$ be integrable on $[a,b]$ and $f_n \xrightarrow{uni.} f$, then f is integrable on $[a,b]$.*

Proof. Since $f_n \xrightarrow{uni.} f$, then for $\forall\epsilon > 0, \exists N \in \mathbb{N}, \forall x \in [a,b], n \ge N \Rightarrow |f(x) - f_n(x)| < \epsilon$, and then

$$\left|\sup_S f(x) - \sup_S f_n(x)\right| \le \epsilon$$

for any $S \subseteq [a,b]$. Assume the contrary, that is, $|\sup_S f(x) - sup_S f_n(x)| > \epsilon$. w.l.o.g. assume that $\sup_S f(x) > \sup_S f_n(x) + \epsilon$, then $\exists x' \in S$, s.t.

$$\begin{aligned}f(x') &> \sup_S f_n(x) + \epsilon \\ &\ge \sup_S f_n(x') + \epsilon\end{aligned}$$

thus $|f(x') - f_n(x')| > \epsilon \to \perp$. Then for any $\mu > 0$, let $\forall\epsilon = \mu/4(b-a)$, then $\exists N_\mu \in \mathbb{N}, \forall x \in S \subseteq [a,b], n \ge N_\mu$, we have

$$\begin{aligned}\sup_S f - \inf_S f &\le \sup_S f_n + \epsilon - (\inf_S f_n - \epsilon) \\ &= \sup_S f_n - \inf_S f_n + 2\epsilon.\end{aligned}$$

and since f_n is integrable, then for $\forall\mu > 0, \exists\Delta_{n,\mu}$ s.t. $\overline{S}(f_n, \Delta_{n,\mu}) - \underline{S}(f_n, \Delta_{n,\mu}) < \mu/2$, and hence

$$\begin{aligned}\overline{S}(f, \Delta_{n,\mu}) - \underline{S}(f, \Delta_{n,\mu}) &= \sum_{j=1}^{k(\Delta_{n,\mu})} \left(\sup_{I_j} f - \inf_{I_j} f\right) \cdot vol(I_j) \\ &\le \sum_{j=1}^{k(\Delta_{n,\mu})} \left(\sup_{I_j} f_n - \inf_{I_j} f_n + 2\epsilon\right) \cdot vol(I_j)\end{aligned}$$

$$
\begin{aligned}
&= \overline{S}(f_n, \Delta_{n,\mu}) - \underline{S}(f_n, \Delta_{n,\mu}) + 2\epsilon \cdot \sum_{j=1}^{k(\Delta_{n,\mu})} vol(I_j) \\
&< \frac{\mu}{2} + \frac{\mu}{2} = \mu.
\end{aligned}
$$

Thus for any $\mu > 0$, $\exists$ such $\Delta := \Delta_{n,\mu}$ s.t. $\overline{S}(f, \Delta_{n,\mu}) - \underline{S}(f, \Delta_{n,\mu}) < \mu \Rightarrow f$ is integrable on $[a, b]$. $\square$

Collectively Darboux integral satisfies P1 - P7 we claimed before, and hence we can define

$$S(f; a, b) := \int_a^b f \, \mathrm{d}x.$$

if f is Darboux integrable on $[a, b]$. And by FTC, let $F(x) := \int_a^x f(t) \, \mathrm{d}t$, and if f is continuous at $c \in (a, b)$, then $F'(c) = f(c)$.
And by FTC' if f is continuous on (a, b) and $x_0 \in (a, b)$, then $F(x) := \int_{x_0}^x f(t) \, \mathrm{d}t (x \in (a, b))$ is a primitive function of f on (a, b). Thus **function which is continuous on an open interval has (theoretical) primitive functions**. And if f is continuous on (a, b) and F is a primitive function of f on (a, b), then

$$F(d) - F(c) = \int_c^d f(t) \, \mathrm{d}t$$

for $a < c < d < b$.

0.2 Improper integral

Define improper integral (瑕积分)

$$\int_0^\infty \frac{\sin x}{x} \, \mathrm{d}x := \lim_{a \to 0} \int_a^c \frac{\sin x}{x} \, \mathrm{d}x + \lim_{b \to \infty} \int_c^b \frac{\sin x}{x} \, \mathrm{d}x$$

where $\sin x / x$ is integrable on $[a, c]$ and $[c, b]$. If the both limitations exists, we say the improper integral convergent.
It is direct to see that $\lim_{a \to 0} \int_a^c \sin x / x \, \mathrm{d}x$ exists, and we will show that $\lim_{b \to \infty} \int_c^b \sin x / x \, \mathrm{d}x$ exists by Cauchy criterion (Exercise **??**).
Let $f(b) = \int_c^b \sin x / x \, \mathrm{d}x$, then for $\forall \epsilon$, select $b, b' > 1/\epsilon$, then

$$
\begin{aligned}
f(b') - f(b) &= \int_c^{b'} \frac{\sin x}{x} \, \mathrm{d}x - \int_c^b \frac{\sin x}{x} \, \mathrm{d}x \\
&= \int_b^{b'} \frac{\sin x}{x} \, \mathrm{d}x \\
&= \frac{1}{b} \cdot \int_b^{\xi} \sin x \, \mathrm{d}x \qquad (\star)
\end{aligned}
$$

$$\leq \frac{1}{b} < \epsilon$$

$(\star)$ is since Second mean value theorem for definite integrals. Then by Cauchy criterion, $\lim_{b\to\infty} f(b)$ exists, and hence the improper integral

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x$$

convergent.

Example 1 (Gamma function, 伽马函数)**.** For $\forall s > 0$, Gamma function

$$\Gamma(s) := \int_0^\infty x^{s-1} e^{-x}\,\mathrm{d}x,$$

convergent.

0.3 Substitution

Proposition 6. *Assume functions*

$$D' \xrightarrow{\phi} D \xrightarrow{f} \mathbb{R}$$

where $[\alpha, \beta] \subseteq D' \subseteq_{open} \mathbb{R}$, $\phi([\alpha, \beta]) \subseteq [a, b] \subseteq D$ *and* $\phi(\alpha) = a, \phi(\beta) = b$, f *is continuous on* $[a, b]$ *and* $\phi \in C^1$, *then*

$$\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f(\phi(t))\phi'(t)\,\mathrm{d}t$$

without requiring ϕ *is a* ***bijection****.*

Proof. Since f is conti. on $[a, b] \Rightarrow f$ is integrable on $[a, b]$. Let $F(y) := \int_a^y f(x)\,\mathrm{d}x$, then $F'(y) = f(y)$ by FTC. And

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} F(\phi(t)) &= F'(\phi(t)) \cdot \phi'(t) \\ &= f(\phi(t)) \cdot \phi'(t) \end{aligned}$$

that is, $F(\phi(t))$ is a primitive function, since $\phi \in C^1, f$ is conti. $\Rightarrow f(\phi(t)) \cdot \phi'(t)$ is conti. $\Rightarrow$

$$\begin{aligned} \int_a^b f(x)\,\mathrm{d}x &= F(b) - F(a) && \text{(FTC')} \\ &= F(\phi(\beta)) - F(\phi(\alpha)) \\ &= \int_\alpha^\beta f(\phi(t)) \cdot \phi'(t)\,\mathrm{d}t && \text{(FTC')} \end{aligned}$$

□

1 Riemann integral

Definition 1 (Riemann integrable, 黎曼可积)**.** Let $D \xrightarrow{f} \mathbb{R}$ be a bounded function and $[a,b] \subseteq D$, we say f is Riemann integrable on $[a,b]$, if $\exists L \in \mathbb{R}$, $\forall \epsilon > 0, \exists \delta > 0,$ s.t. $\forall \Delta$ of $[a,b]$ and $\forall c_j \in I_j$, if $\max_{1\leq j\leq k}(x_j - x_{j-1}) < \delta \Rightarrow$

$$\left| \sum_{j=1}^{k} f(c_j) \cdot (x_j - x_{j-1}) - L \right| < \epsilon.$$

If this is the case, such L must be unique, and be called the Riemann integral of f on $[a,b]$.

Proposition 7. *Let $D \xrightarrow{f} \mathbb{R}$ be Riemann integrable on $[a,b]$ where $[a,b] \subseteq D \Rightarrow f$ is Darboux integrable on $[a,b]$.*

Proof. $\exists L \in \mathbb{R}$, s.t. for any $\epsilon > 0$, we can find $\delta > 0$ as in the definition such that if $\max_{1\leq j\leq k}(x_j - x_{j-1}) < \delta$, then

$$L - \epsilon < \sum_{j=1}^{k} f(c_j) \cdot (x_j - x_{j-1}) < L + \epsilon$$

for $\forall c_j \in I_j$. Then we have that

$$\overline{S}(f,\Delta) = \sum_{j=1}^{k} \sup_{I_j} f \cdot (x_j - x_{j-1}) \leq L + \epsilon$$
$$\underline{S}(f,\Delta) = \sum_{j=1}^{k} \inf_{I_j} f \cdot (x_j - x_{j-1}) \geq L - \epsilon$$

and hence

$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq \overline{S}(f,\Delta) - \underline{S}(f,\Delta) \leq 2\epsilon.$$

Thus f is Darboux integrable, and $\int_a^b f = L$. ☐

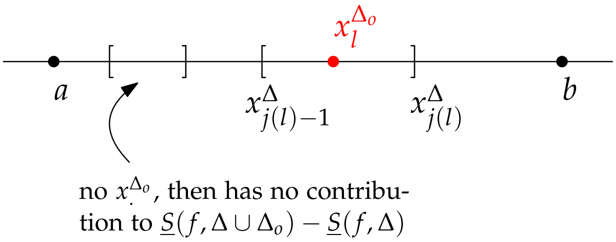
Theorem 1 (Darboux Theorem)**.** *Let $D \xrightarrow{f} \mathbb{R}$ be Darboux integrable on $[a,b]$ where $[a,b] \subseteq D \Rightarrow f$ is Riemann integrable on $[a,b]$.*

Proof. Let $L := \int_a^b f(x)\,\mathrm{d}x$. For any given $\epsilon > 0$ there exists a partition Δ_o of $[a,b]$ s.t.

$$\overline{S}(f,\Delta_o) - \underline{S}(f,\Delta_o) < \epsilon,$$

and in particular, $L < \underline{S}(f,\Delta_o) + \epsilon$. Let $\delta_o := \min_{1\leq l\leq k(\Delta_o)}(x_l^{\Delta_o} - x_{l-1}^{\Delta_o})$. Then choose partition Δ of $[a,b]$ such that $mesh(\Delta) := \max_{1\le j\le k(\Delta)}(x_j^\Delta - x_{j-1}^\Delta) < \delta_o$. Then $I_j^\Delta \cap \Delta_o$ has at most one element for $j = 1, \cdots, k(\Delta)$. Thus

$$
\begin{aligned}
\underline{S}(f,\Delta\cup\Delta_o) - \underline{S}(f,\Delta) &= \sum_{l=1}^{k(\Delta_o)} \left[\inf_{[x^\Delta_{j(l)-1}, x_l^{\Delta_o}]} f\cdot(x_l^{\Delta_o} - x^\Delta_{j(l)-1}) + \inf_{[x_l^{\Delta_o}, x^\Delta_{j(l)},]} f\cdot(x^\Delta_{j(l)} - x_l^{\Delta_o}) \right. \\
&\qquad \left. - \inf_{[x^\Delta_{j(l)-1}, x^\Delta_{j(l)}]} f\cdot(x^\Delta_{j(l)} - x^\Delta_{j(l)-1}) \right] \\
&= \sum_{l=1}^{k(\Delta_o)} \left[\inf_{[x^\Delta_{j(l)-1}, x_l^{\Delta_o}]} f\cdot(x_l^{\Delta_o} - x^\Delta_{j(l)-1}) - \inf_{[x^\Delta_{j(l)-1}, x^\Delta_{j(l)}]} f\cdot(x_l^{\Delta_o} - x^\Delta_{j(l)-1}) \right. \\
&\qquad \left. + \inf_{[x_l^{\Delta_o}, x^\Delta_{j(l)},]} f\cdot(x^\Delta_{j(l)} - x_l^{\Delta_o}) - \inf_{[x^\Delta_{j(l)-1}, x^\Delta_{j(l)},]} f\cdot(x^\Delta_{j(l)} - x_l^{\Delta_o}) \right] \\
&= \sum_{l=1}^{k(\Delta_o)} \left[\left(\inf_{[x^\Delta_{j(l)-1}, x_l^{\Delta_o}]} f - \inf_{[x^\Delta_{j(l)-1}, x^\Delta_{j(l)}]} f \right)\cdot(x_l^{\Delta_o} - x^\Delta_{j(l)-1}) \right. \\
&\qquad \left. + \left(\inf_{[x_l^{\Delta_o}, x^\Delta_{j(l)},]} f - \inf_{[x^\Delta_{j(l)-1}, x^\Delta_{j(l)},]} f \right)\cdot(x^\Delta_{j(l)} - x_l^{\Delta_o}) \right] \\
&\le (M-m)\cdot \sum_{l=1}^{k(\Delta_o)} (x^\Delta_{j(l)} - x^\Delta_{j(l)-1}) \\
&\le (M-m)\cdot k(\Delta_o)\cdot mesh(\Delta).
\end{aligned}
$$



where $m \le f(x) \le M$ for $\forall x \in [a,b]$. Since $\underline{S}(f,\Delta\cup\Delta_o) \ge \underline{S}(f,\Delta_o) > L - \epsilon$, then

$$
\begin{aligned}
\underline{S}(f,\Delta) &\ge \underline{S}(f,\Delta\cup\Delta_o) - (M-m)\cdot k(\Delta_o)\cdot mesh(\Delta) \\
&> L - \epsilon - (M-m)\cdot k(\Delta_o)\cdot mesh(\Delta)
\end{aligned}
$$

Choose Δ, such that $mesh(\Delta) < \max\{\delta_0, \epsilon/(M-m)k(\Delta_o)\}$, then

$$
\sum_{j=1}^{k(\Delta)} f(c_j)\cdot(x_j^\Delta - x_{j-1}^\Delta) \ge \underline{S}(f,\Delta) > L - 2\epsilon.
$$

for any $c_j \in I_j^{\Delta}$, and in the same way,

$$\sum_{j=1}^{k(\Delta)} f(c_j) \cdot (x_j^{\Delta} - x_{j-1}^{\Delta}) \leq \overline{S}(f, \Delta) < L + 2\epsilon.$$

Thus $\left|\sum_{j=1}^{k(\Delta)} f(c_j) \cdot (x_j^{\Delta} - x_{j-1}^{\Delta}) - L\right| < 2\epsilon$. □

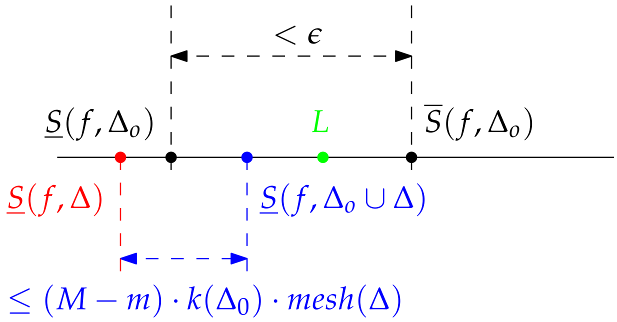


Figure 1: Darboux Theorem