Introduction to Analysis

Lecture 5

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Abstract

This is the Lecture note for the *Introduction to Analysis* class in Spring 2019.

1 Pointwise / uniformly convergent

Definition 1. Let $X \xrightarrow{f_n} Y (n \in \mathbb{N})$ be a seq. of maps, Y is a metric space. We say that $f_n (n \in \mathbb{N})$ converges to a map $X \xrightarrow{f} Y$

- pointwise (逐点收敛): $\forall \epsilon > 0, \forall x \in X, \exists N_x \in \mathbb{N},$ s.t. $\forall n \geq N_x \Rightarrow d(f_n(x), f(x)) < \epsilon$;
- uniformly (均匀收敛): $\forall \epsilon > 0, \exists N \in \mathbb{N},$ s.t. $\forall x \in X, \forall n \geq N \Rightarrow d(f_n(x), f(x)) < \epsilon$.

Denoted as $f_n \to f$ and $f_n \xrightarrow{uni.} f$ as $n \to \infty$ respectively.

Example 1. Given a seq. of maps $X \xrightarrow{f_n} \mathbb{R}$ where $x \in X \in \mathbb{R}$ and $f_n(x) = x^n (n \in \mathbb{N})$. Then f_n converges pointwise if $X \subseteq (-1, 1]$:

$$f_n \to f = \begin{cases} 1, & x = 1 \\ 0, & x \in (-1, 1) \end{cases}$$

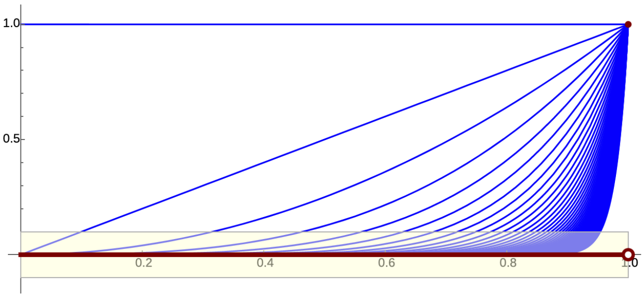


Figure 1: pointwise convergent

However, f_n does not converges to f uniformly, since

$$|f_n(x) - f(x)| = \begin{cases} |1-1| = 0, & x = 1 \\ |x^n - 0| = |x|^n, & x \in (-1,1) \end{cases}$$

For any $\epsilon > 0$, to have $|f_n(x) - f(x)| < \epsilon$, we need $|x|^n < \epsilon$ for $x \in (-1,1)$, that is $n \ln |x| < \ln \epsilon \Leftrightarrow n > \ln \epsilon / \ln |x|$ which has no upper bound, thus there does not exist a $N \in \mathbb{N}$ such that $\forall n \geq N$ has $|f_n - f| < \epsilon$ for $x \in (-1,1)$.

Remark 1. Intuitively, a seq. of maps $f_n \xrightarrow{uni.} f$ means: a pipe with any radius ϵ whose shaft is f can encase all functions after the f_{N_ϵ} of the $f_n(n \in \mathbb{N})$.

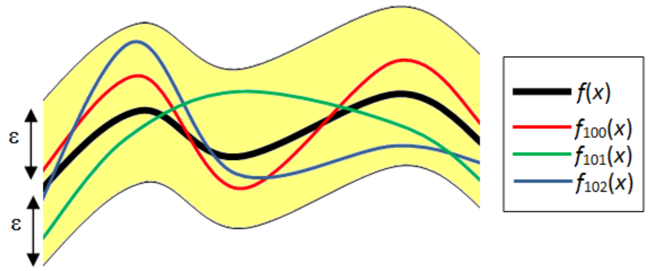


Figure 2: uniformly convergent

Proposition 1. *Let* $X \xrightarrow{f_n} Y(n \in \mathbb{N})$ *is a seq. of maps between metric spaces, which converges to map* $X \xrightarrow{f} Y$ *uniformly, if* f_n *is continuous at* $a \in X$ *for* $\forall n \in \mathbb{N}$*, then* f *is, too.*

Proof. Note that for all $x \in X$ and $n \in \mathbb{N}$, we have that

$$d(f(x), f(a)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(a)) + d(f_n(a), f(a));$$

Then for any $\epsilon > 0$, since $f_n \xrightarrow{uni.} f$ as $n \to \infty, \exists N_\epsilon \in \mathbb{N}$ s.t. $\forall x \in X, n \geq N_\epsilon \Rightarrow d(f_n(x), f(x)) < \epsilon/3$. In particular, $d(f_{N_\epsilon(x)}, f(x)) < \epsilon/3$ for $\forall x \in X$.
On the other hand, since f_{N_ϵ} is continuous at a, then $\exists \delta_{N_\epsilon} > 0$ s.t. $d(x,a) < \delta_{N_\epsilon} \Rightarrow d(f_{N_\epsilon}(x), f_{N_\epsilon}(a)) < \epsilon/3$. Then given $\epsilon > 0, \exists \delta_{N_\epsilon} > 0$, s.t. for $\forall x \in B_{\delta_{N_\epsilon}}(a)$ one has

$$\begin{aligned} d(f(x), f(a)) &\leq d(f(x), f_{N_\epsilon}(x)) + d(f_{N_\epsilon}(x), f_{N_\epsilon}(a)) + d(f_{N_\epsilon}(a), f(a)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus f is continuous at x. $\square$

2 Complete metric space

Definition 2 (Complete, 完备)**.** A metric space (Y, d) is complete if every Cauchy sequence $a_n (n \in \mathbb{N})$ in Y converges. That is $\lim_{n\to\infty} a_n = a \in Y$.

Example 2. $(\mathbb{R}^n, d_2)$ is complete; $(\mathbb{Q}, d_2)$ is incomplete.

Proposition 2 (Uniform Cauchy)**.** *Let $X \xrightarrow{f_n} Y(n \in \mathbb{N})$ be a seq. of maps, and Y be a complete metric space. Then $f_n(n \in \mathbb{N})$ converges uniformly $\Leftrightarrow \forall \epsilon, \exists N,$ s.t. $\forall x \in X, [n, m \geq N \Rightarrow d(f_n(x), f_m(x)) < \epsilon]$ (such $f_n(n \in \mathbb{N})$ is called **uniform Cauchy seq.**).*

Proof. $\Rightarrow$: (The completeness of Y is not need). Since $f_n \xrightarrow{uni.} f$, then for $\forall \epsilon > 0, \exists N \in \mathbb{N},$ s.t. $\forall x \in X, n \geq N \Rightarrow |f - f_n| < \epsilon/2$, then for $\forall x \in X, \forall n, m \geq N$ one has

$$\begin{aligned} |f_n - f_m| &\leq |f_n - f| + |f - f_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$\Leftarrow$: The assumption implies that for every fixed $x \in X$, the seq. $f_n(x)(n \in \mathbb{N})$ is a Cauchy seq. in Y and hence $\lim_{n\to\infty} f_n(x)$ exists, which we denoted as $f(x)$. This define a map $X \xrightarrow{f} Y$. Now we will show that $f_n \xrightarrow{uni.} f$.
Since for $\forall x \in X$ and a fixed $m \in \mathbb{N}$, map $Y \xrightarrow{d} \mathbb{R}$ where $y \mapsto d(y, f_m(x))$ is continuous, then

$$d(f(x), f_m(x)) = \lim_{n\to\infty} d(f_n(x), f_m(x))$$

for all $x \in X$ (Remark **??**). Since for $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall x \in X, [m, n \geq N \Rightarrow d(f_n(x), f_m(x)) < \epsilon/2]$. For every $x \in X, m \geq N$, let $n \to \infty$, we obtain that

$$d(f(x), f_m(x)) = \lim_{n\to\infty} d(f_n(x), f_m(x)) \leq \frac{\epsilon}{2} < \epsilon$$

thus $f_n \xrightarrow{uni.} f$. $\square$

Remark 2. It is direct to see that: $f_n(n \in \mathbb{N})$ converges pointwise $\Leftrightarrow \forall \epsilon, \forall x, \exists N,$ s.t. $\in X, [n, m \geq N \Rightarrow d(f_n(x), f_m(x)) < \epsilon]$.

The power of this proposition is to convert the seq. of functions $f_n(n \in \infty)$. to a series of functions $\sum_{n=1}^{\infty} g_n$, where we define $f_0 \equiv 0$ and

$$g_n = f_n - f_{n-1}, \quad n \in \mathbb{N},$$

then partial sum $s_n = g_1 + \cdots + g_n = f_n(n \in \mathbb{N})$, and hence $\sum_{n=1}^{\infty} g_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} f_n$.

Definition 3. Let $X \xrightarrow{g_n} \mathbb{R}(n \in \mathbb{N})$ be a seq. of functions, we say that $\sum_{n=1}^{\infty} g_n$ converges pointwise / uniformly the partial sum $s_n = g_1 + \cdots + g_n (n \in \mathbb{N})$ does.

Proposition 3 (Weierstrass's M - test)**.** *Let $X \xrightarrow{g_n} \mathbb{R}(n \in \mathbb{N})$ be a seq. of functions, if there exists a positive seq. $M_n (n \in \mathbb{N})$ in $\mathbb{R}$ s.t.*
1. $|g_n(x)| \leq M_n$ for all $x \in X, n \in \mathbb{N}$, and
2. $\sum_{n=1}^{\infty} M_n < \infty$,
then $\sum_{n=1}^{\infty} g_n$ converges uniformly.

Proof. Let partial sum $s_n(x) = g_1(x) + \cdots + g_n(x)(x, \in X, n \in \mathbb{N})$, it is sufficient to show that $s_n(n \in \mathbb{N})$ is uniformly Cauchy seq. (since $\mathbb{R}$ is complete metric space.)
Simce series $\sum_n M_n < \infty$, for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > m \geq N \Rightarrow$ the tail $M_{m+1} + \cdots + M_n =< \epsilon$, then for any such n, m, for $\forall x \in X$ we have that

$$\begin{aligned} |s_n(x) - s_m(x)| &= |g_{m+1}(x) + \cdots + g_n(x)| \\ &\leq |g_{m+1}(x)| + \cdots + |g_n(x)| \\ &\leq M_{m+1} + \cdots + M_n \\ &< \epsilon \end{aligned}$$

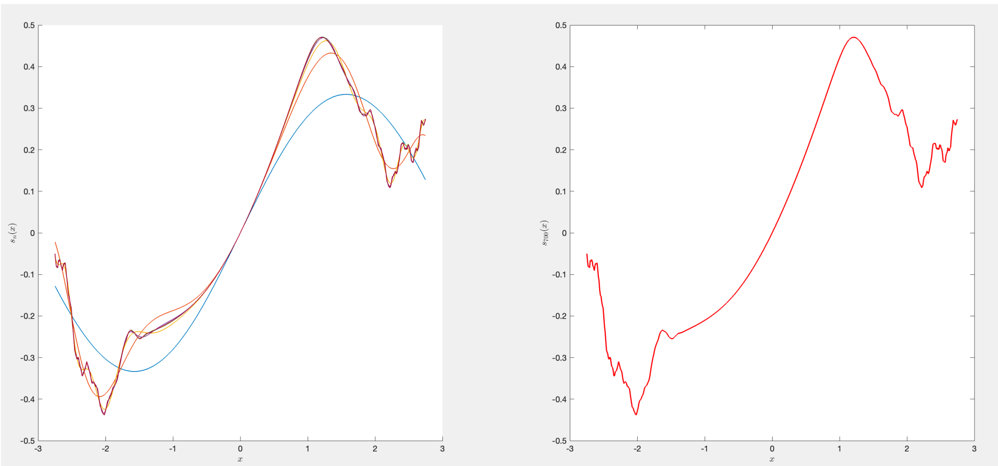
Thus $s_n(n \in \mathbb{N})$ converges uniformly and hence $\sum_{n=1}^{\infty} g_n$ converges uniformly. □

Remark 3. The above conclusion still holds if modify $\mathbb{R}$ to $\mathbb{R}^k$ for some $k \in \mathbb{N}$.

Example 3. Consider series $\sum_{n=1}^{\infty} \sin(x^n)/3^n$ since

$$|g_n| = \left| \frac{\sin(x^n)}{3^n} \right| \leq \frac{1}{3^n} =: M_n$$

thus $\sum_{n=1}^{\infty} \sin(x^n)/3^n$ converges uniformly. We can plot them out, define $s_n = \sum_{i=1}^{n} g_i$, then



with the MATLAB code:

```
gn = 1000; % grid number
fn = 700;  % func number
X = linspace(-5,5,gn);
Y = zeros(gn,fn);
for n = 1:fn
    F = @(x) sin(x.^n)./(3.^n);
    Y(:,n) = F(X)';
end
T = triu(ones(fn,fn));
YY = Y*T;

clf;
subplot(1,2,1);
hold on;
for n = 1:fn
    plot(X,YY(:,n), LineWidth=1);
end
xlabel('$x$','Interpreter','latex');
ylabel('$s_n(x)$','Interpreter','latex');
hold off;

subplot(1,2,2);
plot(X,YY(:,end), LineWidth=1.5, Color='r');
xlabel('$x$','Interpreter','latex');
ylabel('$s_{700}(x)$','Interpreter','latex');
```

Exercise 1. *Let X be a metric space, and define*

$$C_b(X) := \{x \xrightarrow{f} \mathbb{R} | f \text{ is bounded continuous}\}.$$

For any $f \in C_b(X)$, we let

$$\|f\|_{\sup} := \sup_{x \in X} |f(x)|$$

For $f, g \in C_b(X)$, define

$$d(f,g) := \|f - g\|_{\sup}$$

show that

1. *(1.a) $\|f\|_{\sup} \geq 0$ and equality holds iff $f(x) \equiv 0$ for $\forall x \in X$;*
   (1.b) $\|f + g\|_{\sup} \leq \|f\|_{\sup} + \|g\|_{\sup}$ for all $f, g \in C_b(X)$;
   (1.c) $\|cf\|_{\sup} = |c| \cdot \|f\|_{\sup}$ for all $f \in C_b(X), c \in \mathbb{R}$;

2. *d is a metric on $C_b(X)$;*
3. *$(C_b(X), d)$ is complete;*
4. *if $f_n \in C_b(X)(n \in \mathbb{N})$ and $f \in C_b(X)$, $[f_n \xrightarrow{uni.} f \Leftrightarrow f_n \xrightarrow{w.r.t.\ d} f]$ as $n \to \infty$.*

Proof. Since $\forall f \in C_b(X)$ is bounded, then any $\|f\|_{\sup}$ exists.
1. (1.a) trivial; (1.b) Assume that exists $f, g \in C_b(X)$ s.t. $\sup_{x\in X}(|f|+|g|) > \sup_{x\in X}|f| + \sup_{x\in X}|g|$. Then exists $x \in X$, s.t.

$$\sup_{x\in X}|f| + \sup_{x\in X}|g| < |f(x)| + |g(x)| \le \sup_{x\in X}(|f|+|g|)$$

which is contrary, thus

$$\begin{aligned}\|f+g\|_{\sup} &= \sup_{x\in X}|f+g| \le \sup_{x\in X}(|f|+|g|)\\ &\le \sup_{x\in X}|f| + \sup_{x\in X}|g|\\ &= \|f\|_{\sup} + \|g\|_{\sup}\end{aligned}$$

(1.c) $\|cf\|_{\sup} = \sup_{x\in X}|c\cdot f(x)| = \sup_{x\in X}|c|\cdot|f(x)| = |c|\cdot\sup_{x\in X}|f(x)| = |c|\cdot\|f\|_{\sup}$.
2. We only prove the triangle inequality: for any $f, g \in C_b(X)$, we have

$$\begin{aligned}d(f,g) &= \|f-g\|_{\sup} = \|f+(-g)\|_{\sup}\\ &\le \|f\|_{\sup} + \|-g\|_{\sup}\\ &=\le \|f\|_{\sup} + \|g\|_{\sup}.\end{aligned}$$

3. Suppose $f_n(n \in \mathbb{N})$ is a Cauchy seq. in $(C_b(X), d)$, thus for any $\epsilon > 0, \exists N \in \mathbb{N}$, s.t. for $\forall n, m \ge N$, one has

$$d(f_n, f_m) = \|f_n - f_m\|_{\sup} = \sup_{x\in X}|f_n - f_m| < \epsilon$$

thus for $\forall x \in X$, $|f_n(x) - f_m(x)| \le \sup_{x\in X}|f_n - f_m| < \epsilon$. Thus fix any $x' \in X$, then $f_n(x')(n \in \mathbb{N})$ is a Cauchy seq. in $\mathbb{R}$, and converges since $\mathbb{R}$ is complete metric space, denote the limit as $f(x')$. It is direct to see that f is bounded, and we will show that f is continuous on X as well.
Since for any $n \in \mathbb{N}$, $f_n \in C_b(X) \Rightarrow f_n$ is continuous on X, thus for any $x \in X, \epsilon > 0, \exists\delta > 0$ s.t. for any $x' \in B_\delta(x)$ (w.r.t. d_2), we have that $d_2(f_n(x'), f_n(x)) < \epsilon/3$. And since for any $x \in X$, $f_n(x)$, as a Cauchy seq. in $\mathbb{R}$, converges to $f(x)$, and hence $\exists N \in \mathbb{N}$, s.t. for $n \ge N$, $d_2(f(x), f_n(x)) < \epsilon/3$. Thus for any $n \ge N, x' \in B_\delta(x)$ (w.r.t. d_2), we have

$$\begin{aligned}d(f(x), f(x')) &\le d(f(x), f_n(x)) + d(f_n(x), f_n(x')) + d(f_n(x'), f(x'))\\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Thus f is continuous on $X \Rightarrow f \in C_b(X)$. Now we show that $f_n \to f$ w.r.t. d. Assume that f_n does not converges to f w.r.t. d, that is $\exists \epsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N,$ s.t.

$$d(f, f_n) = \|f - f_m\|_{\sup} = \sup_{x \in X} |f - f_n| \geq \epsilon > \frac{\epsilon}{2},$$

and hence $\exists x \in X$ s.t.

$$\frac{\epsilon}{2} < |f(x) - f_n(x)| \leq \sup_{x \in X} |f - f_n|$$

which leads to a contradiction with $f_n(x)$ is Cauchy in $\mathbb{R}$ and converges to $f(x)$. Thus $f_n \to f \in C_b(X)$ w.r.t. d.

4. It is sufficient to show that **bounded continuous** $f_n (n \in \mathbb{N})$ **is a uniform Cauchy seq. of functions** $\Leftrightarrow$ $f_n (n \in \mathbb{N})$ **is a Cauchy seq. in** $(C_b(X), d)$.

$\Rightarrow$: $f_n (n \in \mathbb{N})$ are bounded continuous $\Rightarrow f_n \in C_b(X) (n \in \mathbb{N})$. And for any $\epsilon > 0, \exists N \in \mathbb{N},$ s.t. $\forall n > m \geq M$, has $|f_n(x) - f_m(x)| < \epsilon/2$ for $\forall x \in X$, thus $\sup_{x \in X} |f_n - f_m| \leq \epsilon/2 < \epsilon \Rightarrow d(f_m, f_n) < \epsilon \Rightarrow f_n (n \in \mathbb{N})$ are Cauchy seq. in $(C_b(X), d)$.

$\Leftarrow$: $f_n (n \in \mathbb{N})$ are Cauchy seq. in $(C_b(X), d)$, then for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s,t, $\forall n, m \geq N$ has $d(f_m, f_n) = \sup_{x \in X} |f_m - f_n| < \epsilon \Rightarrow \forall x \in X$ has $|f_m(x) - f_n(x)| < \epsilon \Rightarrow f_n (n \in \mathbb{N})$ are uniform Cauchy seq.

Since $(C_b(X), d)$ is complete, then

$$\begin{aligned} f_n \xrightarrow{w.r.t.\ d} f &\Leftrightarrow f_n \text{ are Cauchy seq. in } (C_b(X), d) \\ &\Leftrightarrow f_n \text{ are uniform Cauchy seq.} \\ &\Leftrightarrow f_n \xrightarrow{uni.} f. \end{aligned}$$

$\square$