Introduction to Analysis
Lecture 7

Haoming Wang

14 May 2019

Abstract

This is the Lecture note for the *Introduction to Analysis* class in Spring 2019.

1 Signed area and indefinite integral

For $a, b \in \mathbb{R}$, we let

$$\underline{ab} := \begin{cases} [a,b], & a \leq b \\ [b,a], & a \geq b \end{cases}$$

We hope to define and study the properties of 'signed area' $S(f;a,b)$ of region in the xy - plane enclosed by $x = a, x = b, y = 0$ and $y = f(x)$ where $D \xrightarrow{f} \mathbb{R}$ is a (reasonally well behavior) function such that $\underline{ab} \subseteq D$.

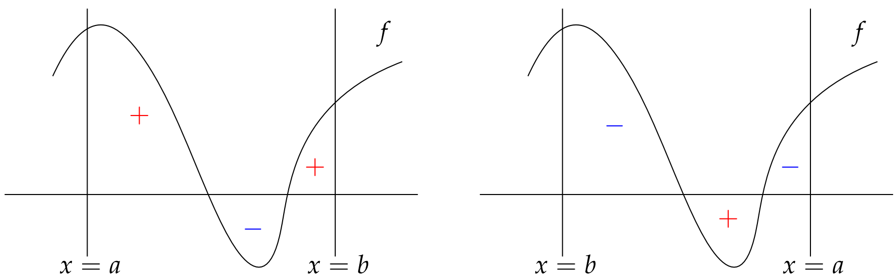

Figure 1: $S(f;a,b)$

If $S(f;a,b)$ is defined, we call f integrable on $\underline{ab}$ w.r.t. the specific definition of $S(\cdot;\cdot,\cdot)$. Assuming we have known the definition of $S(\cdot;\cdot,\cdot)$, we expect it to satisfy several properties (P):

(P1) [Monotonicity]: f is integrable and ≥ 0 on $[a,b]$, then $S(f;a,b) \geq 0$.

(P2) [Linearity]: f, g are integrable on $[a,b], \alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is integrable on $[a,b]$

and

$$S(\alpha f + \beta g; a, b) = \alpha S(f; a, b) + \beta S(g; a, b).$$

(P3) f is integrable on $\underline{ab}$ and $c \in \underline{ab}$, then f is integrable on $\underline{ac}$ and $\underline{cb}$ and

$$S(f; a, b) = S(f; a, c) + S(f; c, b);$$
$$S(f; a, c) = S(f; a, b) + S(f; b, c).$$

(P4) f is conti. on $\underline{ab}$, then f is integrable on $\underline{ab}$.
(P5) Then constant function 1 is integrable on $\underline{ab}$ for all $a, b \in \mathbb{R}$ and $S(1; a, b) = b - a$.
(P6) All $D \xrightarrow{f} \mathbb{R}$ are integrable on $[a, a]$ if $a \in D$.
(P7) f is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$.

Exercise 1. *Using the above properties, show that*
1. *f, g are integrable on $[a, b]$, $f(x) \le g(x)$ for all $x \in [a, b]$, then $S(f; a, b) \le S(g; a, b)$.*
2. *f are integrable on $[a, b]$ then $|S(f; a, b)| \le S(|f|; a, b)$.*
3. *f is integrable on $\underline{ab}$, $\underline{cd} \subseteq \underline{ab}$, then f is integrable on $\underline{cd}$.*
4. *$D \xrightarrow{f} \mathbb{R}, a \in D$, then $S(f; a, a) = 0$.*
5. *f is integrable on $\underline{ab}$, then $S(f; a, b) = -S(f; b, a)$.*

Proof. 1. $f(x) \le g(x) \Rightarrow g(x) - f(x) \ge 0$ for $\forall x \in [a, b]$, thus

$$\begin{aligned} S(g; a, b) - S(f; a.b) &= S(g - f; a, b) && \text{(P2)} \\ &\ge 0 && \text{(P1)} \end{aligned}$$

2. f is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and since $-|f| \le f \le |f|$, we have that

$$-S(|f|; a, b) = S(-|f|; a, b) \le S(f; a, b) \le S(|f|; a, b) \qquad \text{(P2, 1)}$$

and hence

$$|S(f; a, b)| \le S(|f|; a, b).$$

3. trivial by P3.
4. since $a \in [a, a]$, then

$$S(f; a, a) = S(f; a, a) + S(f; a, a) \qquad \text{(P3,P6)}$$

and hence $S(f; a, a) = 0$.
5. $S(f; a, b) + S(f; b, a) = S(f; a, a) = 0$. □

Theorem 1 (Fundamental theorem of calculus, FTC[1]). *Let $a \le b$ and f is integrable on $[a, b]$. Let $F(x) := S(f; a, x), x \in [a, b]$. If $c \in [a, b]$ and f is conti. at c, then*

[1] Assume that we have known the definition of $S(\cdot; \cdot, \cdot)$. And if we can actually define $S(\cdot; \cdot, \cdot)$ that satisfies P1 - P7, then all properties we discuss will work for such $S(\cdot; \cdot, \cdot)$.

- $c \in (a,b) \Rightarrow F'(c) = f(c)$;
- $c = a \Rightarrow F'_+(a) = f(a)$;
- $c = b \Rightarrow F'_-(b) = f(b)$.

Proof. For $\forall x \in [a,b]$,

$$\begin{aligned} F(x) - F(c) - f(c)(x-c) &= S(f;a,x) - S(f;a,c) - f(c)\cdot S(1;c,x) && \text{(P5)} \\ &= S(f;a,x) + S(f;c,a) - S(f(c);c,x) && \text{(P2)} \\ &= S(f;c,x) - S(f(c);c,x) && \text{(P3)} \\ &= S(f - f(c);c,x) && \text{(P2)} \end{aligned}$$

And hence

$$\begin{aligned} |F(x) - F(c) - f(c)(x-c)| &= |S(f-f(c);c,x)| \\ &\leq S(|f-f(c)|;\min\{c,x\},\max\{c,x\}) \end{aligned}$$

Since f is continuous at c, then for any $\epsilon > 0, \exists \delta > 0$, s.t. $\forall |x-c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$. So if $|x-c| < \delta$, then

$$|F(x) - F(c) - f(c)(x-c)| \leq S(\epsilon;\min\{c,x\},\max\{c,x\}) = \epsilon \cdot \delta$$

thus

$$\left| \frac{F(x)-F(c)}{x-c} - f(c) \right| < \epsilon \cdot \delta$$

if $0 < |x-c| < \delta$. □

Corollary 1 (FTC′)**.** *Let $D \xrightarrow{F} \mathbb{R}$ be continuously differentiable on $[a,b]$ where $[a,b] \subseteq D \subseteq \mathbb{R}$, i.e. $f(x) = F'(x)$ exists for all $x \in [a,b]$ and $f(x)$ is continuous on $[a,b]$. Then*

$$F(b) - F(a) = S(f;a,b).$$

Proof. f is continuous on $[a,b] \Rightarrow f$ is integrable on $[a,b]$. Let $G(x) = S(f;a,x)$. Then FTC $\Rightarrow G'(x) = f(x)$ for $\forall x \in (a,b)$. And since $F'(x) = f(x)$, then $0 = f(x) = f(x) = F'(x) - G'(x) = (F(x) - G(x))'$ and $F(x) - G(x)$ is continuous on $[a,b] \Rightarrow F - G$ is const., thus

$$F(b) - F(a) = G(b) - G(a) = S(f;a,b).$$

□

The FTC′ motivates the following definition:

Definition 1 (Indefinite integral)**.** Given two functions $D \xrightarrow{F} \mathbb{R}$ and $D \xrightarrow{f} \mathbb{R}$, where $D \subseteq \mathbb{R}$, we say that F is a primitive (function) on D of f if $F' = f$ on D. We also say that

$$\int f(x)\,\mathrm{d}x = F(x) + C,$$

(where $C \in \mathbb{R}$ is const.) the indefinite integral of f on D.

Remark 1 (integration by part). Recall Leibniz's rule: if f, g is diff. then

$$(fg)' = f'g + fg'$$

and hence

$$\int f'g = fg - \int fg'$$

this is called **integration by part**.

Remark 2 (substitution). Recall chain rule: if F, h are diff. on the relevant domain, then

$$(F \circ h)'(t) = F'(h(t)) \cdot h'(t).$$

if $F'(x) = f(x)$, then

$$\int f(x)\,\mathrm{d}x \bigg|_{x=h(t)} = \int f(h(t))h'(t)\,\mathrm{d}t.$$

this is called **substitution**.

2 Darboux integral

2.1 Definitions

Definition 2 (Partition)**.** A finite subset $\Delta \subseteq [a, b]$ is a partition of $[a, b]$ if $a, b \in \Delta$. We usually list the elements of Δ in order as

$$\Delta: \quad a = x_0^\Delta < x_1^\Delta < \cdots < x_{k(\Delta)}^\Delta = b,$$

and let $I_j^\Delta := [x_{j-1}^\Delta, x_j^\Delta], j = 1, \cdots, k(\Delta)$. We may write x_j^Δ, I_j^Δ and $k(\Delta)$ as x_j, I_j and k if no confusion will be caused.

If $D \xrightarrow{f} \mathbb{R}$ is bounded on $[a, b]$, where $[a, b] \subseteq D$, and the signed area $S(f; a, b)$ is defined (with properties P1 - P7). Since f is bdd. $\Rightarrow$ $\sup f$ exists, and thus for any partition of $[a, b]$:

$$\begin{aligned} S(f; a, b) &= \sum_{j=1}^{k} S(f; x_{j-1}, x_j) \\ &\leq \sum_{j=1}^{k} S(\sup_{I_j} f; x_{j-1}, x_j) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{k} (\sup_{I_j} f) \cdot (x_j - x_{j-1}) \\
&:= \overline{S}(f, \Delta)
\end{aligned}
$$

We call $\overline{S}(f, \Delta)$ the **upper sum** (上和) of f w.r.t. Δ. Similarly, we define

$$\underline{S}(f, \Delta) := \sum_{j=1}^{k} (\inf_{I_j} f) \cdot (x_j - x_{j-1})$$

and call it the **lower sum** (下和) of f w.r.t. Δ. Then for any partition Δ of $[a, b]$, we have

$$\underline{S}(f, \Delta) \leq S(f; a, b) \leq \overline{S}(f, \Delta)$$

Definition 3 (Refine)**.** Let Δ', Δ are partitions of $[a, b]$, we say that
1. Δ' refines Δ, if $\Delta \subseteq \Delta'$;
2. $\Delta \cup \Delta'$ the common refinement of Δ and Δ'.

Proposition 1. *Let Δ_1, Δ_2 are partitions of $[a, b]$ and $\Delta_1 \subseteq \Delta_2$, then*

$$\underline{S}(f, \Delta_1) \leq \underline{S}(f, \Delta_2) \leq \overline{S}(f, \Delta_2) \leq \overline{S}(f, \Delta_1).$$

Proof. w.l.o.g. let $\Delta_1 = \{a, b\}$ and $\Delta_2 = \{a, c, b\}$ where $c \in (a, b)$. Then

$$
\begin{aligned}
\overline{S}(f, \Delta_2) &= \sup_{[a,c]} f \cdot (c - a) + \sup_{[c,b]} f \cdot (b - c) \\
&\leq \sup_{[a,b]} f \cdot (c - a) + \sup_{[a,b]} f \cdot (b - c) && \text{(Exercise ??)} \\
&= \sup_{[a,b]} f \cdot (b - a) \\
&= \overline{S}(f, \Delta_1).
\end{aligned}
$$

and $\underline{S}(f, \Delta_2) \geq \underline{S}(f, \Delta_1)$ in the same way. □

Remark 3. In particular, for $\forall$ partitions Δ, Δ' of $[a, b]$, we have that

$$
\begin{aligned}
\underline{S}(f, \Delta) &\leq \underline{S}(f, \Delta \cup \Delta') \\
&\leq \overline{S}(f, \Delta \cup \Delta') \\
&\leq \overline{S}(f, \Delta'),
\end{aligned}
$$

that is any lower sum is smaller than any upper sum, thus the set of all lower/upper sum has upper/lower bound, and hence has l.u.b/g.l.b. which is called **upper/lower integral**.

Definition 4 (Upper/lower integral, 上/下积分)**.** For a function $D \xrightarrow{f} \mathbb{R}$ which is bounded on $[a,b]$, where $[a,b] \subseteq D \subseteq \mathbb{R}$, we define upper/lower integral of f on $[a,b]$ as

$$\overline{\int_a^b} f(x)\,\mathrm{d}x := \inf_{\Delta} \overline{S}(f,\Delta)$$
$$\underline{\int_a^b} f(x)\,\mathrm{d}x := \sup_{\Delta} \underline{S}(f,\Delta)$$

It is direct to see that

$$\underline{\int_a^b} f(x)\,\mathrm{d}x \leq \overline{\int_a^b} f(x)\,\mathrm{d}x.$$

Definition 5 (Darboux integrable, 达布可积)**.** For a function $D \xrightarrow{f} \mathbb{R}$ which is bounded on $[a,b]$, where $[a,b] \subseteq D \subseteq \mathbb{R}$, we say f is Darboux integrable on $[a,b]$ is

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x = S,$$

if this is the case, we call the S the (definite) integral of f on $[a,b]$, denoted as

$$\int_a^b f(x)\,\mathrm{d}x.$$

And if f is Darboux integrable on $[a,b]$, we define

$$\int_b^a f(x)\,\mathrm{d}x := -\int_a^b f(x)\,\mathrm{d}x.$$

Example 1 (Dirichlet function, 狄利克雷函数)**.** Consider the Dirichlet function $\mathbb{R} \xrightarrow{f} \mathbb{R}$ where

$$x \mapsto \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Then for any $[a,b] \subseteq \mathbb{R}$ and any partition Δ of $[a,b]$, we have that

$$\overline{S}(f,\Delta) = \sum_{j=1}^{k} 1 \cdot (x_j - x_{j-1}) = b - a$$
$$\underline{S}(f,\Delta) = \sum_{j=1}^{k} 0 \cdot (x_j - x_{j-1}) = 0$$

and hence

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = b - a > 0 = \underline{\int_a^b} f(x)\,\mathrm{d}x,$$

thus Dirichlet function f is non - Darboux integral on any interval $[a,b] \subseteq \mathbb{R}$.