Point Set Topology

Lecture 5

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Content:

This is the Lecture note for the *Point Set Topology*.

Final Topology

Given topology spaces $X_\alpha(\alpha \in A)$ and maps $X_\alpha \xrightarrow{f_\alpha} Y(\alpha \in A)$, does there exist a finest topology on Y, such that f_α is continuous for every $\alpha \in A$? Define

$$\mathscr{T}_Y := \{V \subseteq Y | f_\alpha^{-1}(V) \subseteq_{open} X_\alpha, \forall \alpha \in A\}.$$

It is direct to see $\mathscr{T}_Y$ is a topology: Given an $\alpha \in A$, define $\mathscr{T}_\alpha := \{V \subseteq Y | f_\alpha^{-1}(V) \subseteq_{open} X_\alpha\}$, we have

1. $f_\alpha^{-1}(\emptyset) = \emptyset \subseteq_{open} X_\alpha$; $f_\alpha^{-1}(Y) = X_\alpha \subseteq_{open} X_\alpha$, thus $\emptyset, Y \in \mathscr{T}_\alpha$.
2. $\forall V_\beta \in \mathscr{T}_\alpha(\beta \in B)$, $f_\alpha^{-1}(\cup_{\beta\in B} V_\beta) = \cup_{\beta \in B} f^{-1}(V_\beta) \subseteq_{open} X_\alpha$, thus $\cup_{\beta\in B} V_\beta \in \mathscr{T}_\alpha$;
3. $\forall V_1, V_2 \in \mathscr{T}_\alpha$, $f_\alpha^{-1}(V_1 \cap V_2) = f_\alpha^{-1}(V_1) \cap f_\alpha^{-1}(V_2) \subseteq_{open} X_\alpha$, thus $V_1 \cap V_2 \in \mathscr{T}_\alpha$.

Thus $\mathscr{T}_\alpha$ is a topology. On the other hand, $\mathscr{T}_Y = \cap_{\alpha \in A} \mathscr{T}_\alpha$, thus $\mathscr{T}_Y$ is a topology.

Suppose $\mathscr{T}'$ is a topology makes maps $X_\alpha \xrightarrow{f_\alpha} Y(\alpha \in A)$ be continuous. Then $\forall U \in \mathscr{T}'$, $f_\alpha^{-1}(U) \subseteq_{open} X_\alpha$ for all $\alpha \in A$, thus $U \in \mathscr{T}_Y \Rightarrow \mathscr{T}' \subseteq \mathscr{T}_Y$.

Thus $\mathscr{T}_Y$ is the expected finest topology such that f_α is continuous for any $\alpha \in A$.

Equivalence Relation

Definition 1 (Equivalence Relation)**.** Let X be a set. A relation R on X (i.e. $R \subseteq X \times X$) is equivalence relation, if

1. $\forall x \in X \Rightarrow xRx$;
2. $\forall x, x' \in X, xRx' \Rightarrow x'Rx$;
3. $\forall x, x', x'' \in X, xRx', x'Rx'' \Rightarrow xRx''$.

For an equivalence relation R on X, and every $x \in X$, we call

$$R(x) := \{x' \in X | x'Rx\}$$

the **equivalence class** of x w.r.t. R on X. Obviously $R(x) \neq \emptyset$ for $\forall x \in X$, since $x \in R(x)$ for any $x \in X$.

Exercise 1. For $\forall x_1, x_2 \in X$, either $R(x_1) = R(x_2)$ or $R(x_1) \cap R(x_2) = \emptyset$.

Proof. If $\exists x \in R(x_1) \cap R(x_2) \neq \emptyset$, then for any $x_3 \in R(x_2)$, we have $x_3 R x_2$, $x_2 R x$ and $x R x_1 \Rightarrow x_3 R x_1 \Rightarrow x_3 \in R(x_1) \Rightarrow R(x_2) \subseteq R(x_1)$. And $R(x_1) \subseteq R(x_2)$ in the same way, thus $R(x_1) = R(x_2)$. $\square$

In summary, R provides a decomposition of X into disjoint union of nonempty subsets. On the contrary, if we have a decomposition of X into disjoint union of nonempty subsets, we can define an equivalence relation where the element in the same subsets are equivalent. Thus an equivalence relation and a decomposition are the same thing.

Quotient Space

We call $\{R(x) | x \in X\}$ the **quotient set** of X by the relation R, denoted as X/R. And we can define a **natural projection** on X: $X \xrightarrow{\pi} X/R$ where $x \mapsto R(x)$. It is direct to see that π is a surjection.

Exercise 2 (The universal property of $X \xrightarrow{\pi} X/R$)**.** Given a map $X \xrightarrow{g} Z$ such that $\forall x, x' \in X, xRx' \Rightarrow g(x) = g(x')$, show that $\exists!$ map $X/R \xrightarrow{\overline{g}} Z$ s.t. $\overline{g} \circ \pi = g$.

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "\pi"'] & & Z \\
& X/R \arrow[ur, "\overline{g}"'] &
\end{tikzcd}$$

Proof. Given a $R(x) \in X/R$, define $\overline{g}(R(x)) = g(x)$. Since for any $x' \in R(x)$, $g(x') = g(x)$, the map $\overline{g}$: $X/R \ni R(x) = S \mapsto g(x) \in Z$ is well defined, i.e. independent of the choice of x s.t. $S = R(x)$.

For $\forall x \in X$, $\overline{g} \circ \pi(x) = \overline{g}(R(x)) = g(x)$, thus $\overline{g} \circ \pi = g$. If $\exists h$, s.t. $h \circ \pi = g = \overline{g} \circ \pi$, then $h = \overline{g}$ since π is a surjection. $\square$

Note 1. Recall that
1. g is an injection, $g \circ f = g \circ f' \Rightarrow f = f'$;
2. f is a surjection, $g \circ f = g' \circ f \Rightarrow g = g'$.

Now we consider a topology space X on which an equivalence relation R is specified. We aim at defining a topology space obtained by gluing mutually R - equivalent points in X to a point.

Definition 2 (Quotient Topology)**.** Let $X \xrightarrow{\pi} X/R$ where $x \mapsto R(x)$ be the natural projection. The finial topology on X/R induced by $\{\pi\}$ (i.e. the finest topology on X/R s.t. π is continuous) is called the quotient topology on X/R induced by R, denoted by $\mathscr{T}_{(X,R)}$.

More explicitly,

$$\mathscr{T}_{(X,R)} = \{S \subseteq X/R | \pi^{-1}(S) \subseteq_{open} X\},$$

that is, $S \subseteq_{open} X/R$ w.r.t $\mathscr{T}_{(X,R)} \Leftrightarrow \pi^{-1}(S) \subseteq_{open} X$.

Definition 3 (Saturated)**.** Let X is a set, R is an equivalence relation on X. $A(\subseteq X)$ is a R - saturated if $\forall x \in X, a \in A, xRa \Rightarrow x \in A$.

Exercise 3. A is R - saturated $\Leftrightarrow$ A is a union of some R - equivalence class $\Leftrightarrow \exists S \subseteq X/R$, s.t. $A = \pi^{-1}(S)$.

Proof. 1. $\Rightarrow$: If A is R- saturated, then for $\forall a \in A$, $R(a) \subseteq A$ by definition. Thus $\cup_{a\in A} R(a) \subseteq_{open} A$. On the other hand, for any $a' \in A, a' \in R(a') \subseteq \cup_{a\in A} R(a)$, thus $A = \cup_{a\in A} R(a)$.
$\Leftarrow$: If $R_\beta(\beta \in B)$ are some R - equivalence class in X/R, then for any $r \in \cup_{\beta\in B} R_\beta$, $\exists \gamma \in B$, s.t. $r \in R_\gamma$, thus $R(r) = R_\gamma$, thus $R(r) \subseteq \cup_{\beta\in B} R_\beta$.
For any $x \in X$, if xRr, then $x \in R(r) \subseteq \cup_{\beta\in B} R_\beta \Rightarrow x \in \cup_{\beta\in B} R_\beta \Rightarrow \cup_{\beta\in B} R_\beta$ is R - saturated.
2. $\Rightarrow$: Note that for $R(a) \in X/R, \pi^{-1}(R(a)) = R(a) \subseteq X$. Thus

$$\begin{aligned} A &= \cup_{\alpha\in A} R(a) \\ &= \cup_{a\in A} \pi^{-1}(R(a)) \\ &= \pi^{-1}(\cup_{a\in A} R(a)) \end{aligned}$$

where $\cup_{a\in A} R(a) \subseteq X/R$ is the expected S.
$\Leftarrow$: we will show that for $\forall S \subseteq X/R, \pi^{-1}(S)$ is R-saturated on X. For any $s \in \pi^{-1}(S)$, $\pi(s) = R(s) \subseteq S$. For any $x \in X$, if xRs, then $R(x) = R(s) \subseteq S$, thus $x \in \pi^{-1}(S)$, thus $\pi^{-1}(S)$ is R-saturated.

□

Definition 4 (Quotient Map)**.** Let $X \xrightarrow{p} Y$ be a map between topology spaces. We say p is a quotient map if:
1. p is a surjection;
2. for any $V \subseteq Y$, we have $V \subseteq_{open} Y \Leftrightarrow p^{-1}(V) \subseteq_{open} X$.

Note 2. The second statement is equivalent with

$$V \subseteq_{close} Y \Leftrightarrow p^{-1}(V) \subseteq_{close} X$$

since $p^{-1}(V) \subseteq_{close} X \Leftrightarrow \left(p^{-1}(V)\right)^c = p^{-1}(V^c) \subseteq_{open} X \Leftrightarrow V^c \subseteq_{open} X \Leftrightarrow V \subseteq_{close} X$.

Thus the topology on Y is the final topology induced by $\{p\}$, since the second statement.

For a topology space X with an equivalence relation R, a topology $\mathscr{T}_{X/R}$ on X/R makes the natural projection $X \xrightarrow{\pi} X/R$ a quotient map iff $\mathscr{T}_{X/R} = \mathscr{T}_{(X,R)}$.

Exercise 4 (The universal property of quotient topology/map)**.** Let $X \xrightarrow{p} Y$ be a quotient map. Show that for $\forall X \xrightarrow[conti.]{g} Z$ s.t. $\forall x, x' \in X, p(x) = p(x') \Rightarrow g(x) = g(x'), \exists! Y \xrightarrow[conti.]{h} Z$ s.t. $h \circ p = g$.

Note 3. $p(x) = p(x') \Rightarrow g(x) = g(x')$ means that given a $y \in Y$, g is a constant on $p^{-1}(y)$.

$$\begin{tikzcd} X \arrow[rr, "g", "conti."'] \arrow[dr, "p"'] & & Z \\ & Y \arrow[ur, "h", "conti."'] & \end{tikzcd}$$

Proof. Existence: for any $y \in Y$, $p^{-1}(y)\exists$ for p is a surjection. Define $h(y) = g(p^{-1}(y))$. Since $g(p^{-1}(y))$ is a constant, h is well defined. And $h \circ p(x) = h(p(x)) = g(p^{-1}(p(x)))$. Since $x \in p^{-1}(p(x))$ and $g(p^{-1}(p(x)))$ is a constant, thus $h \circ p(x) = g(x)$.

Uniqueness: since p is surjection, h is unique.

Continuousness: for any $U \subseteq_{open} Z$, $h^{-1}(U) \subseteq_{open} Y \Leftrightarrow p^{-1}(h^{-1}(U)) \subseteq_{open} X$. Since $p^{-1}(h^{-1}(U)) = g^{-1}(U) \subseteq_{open} X$ since g is conti. and $g = h \circ p$. Thus h is continuous. $\square$

Any maps between sets $X \xrightarrow{f} Y$ induces an equivalence relation R_f on X: for $x, x' \in X, xR_f x' \Leftrightarrow f(x) = f(x')$. And the equivalence classes is the $f^{-1}(\{y\})$, for $y \in f(X)$.

Exercise 5. Given a continuous surjection $X \xrightarrow{f} Y$, show that f is a quotient map $\Leftrightarrow$ the image of every f - saturated open/close subset of X is open/close in Y.

Proof. $\Rightarrow$: If A is a f - saturated , then $A = f^{-1}(f(A))$: if $\exists b \in f^{-1}(f(A))\backslash A$, then $f(b) \in f(A) \Rightarrow \exists a \in A,$ s.t. $f(b) = f(a) \Rightarrow aR_f b \Rightarrow b \in A$, which leads to a contradiction. Thus $A = f^{-1}(f(A))$.

Thus if A is an open f - saturated set on X then $f^{-1}(f(A)) \subseteq_{open} X \Leftrightarrow f(A) \subseteq_{open} Y$ since f is a quotient map.

$\Leftarrow$: all we need to show is for any $V \subseteq Y$, $f^{-1}(V) \subseteq_{open} X \Rightarrow V \subseteq_{open} Y$. For any $V \subseteq Y, f^{-1}(V)$ is f - saturate: for any $r \in f^{-1}(V) \Rightarrow f(r) \in V$. If $\exists x \in$ X s.t. $xR_f r \Rightarrow f(x) = f(r) \in V \Rightarrow x \in f^{-1}(V)$.

If $f^{-1}(V) \subseteq_{open} X$, then $f(f^{-1}(V)) \subseteq_{open} X$. Since f is a surjection, $V = f(f^{-1}(V)) \subseteq_{open} X \Rightarrow f$ is quotient map. $\square$

Note 4. If A is a f - saturated , then $A = f^{-1}(f(A))$.