Introduction to Topology

General Topology, Lecture 10,11

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13 July 2019

Content:

This is the Lecture note for the *Introduction to Topology*. The course covers the following topics: Naive Set Theory, Elementary Number Theory, Group Theory, Topological Spaces and Continuous Maps, Introduction to Algebraic Topology.

Open set on metric space

Theorem 1 (The Lebesgue number of an open cover)**.** *Let (X, d) be a metric space and $K(\subseteq X)$ a compact set. For any given open cover $O_\alpha(\alpha \in A)$ of K, there exists $\delta > 0$,* s.t. *for every $x \in K$ we have $B_\delta(x) \subseteq O'_\alpha$ for some $\alpha' \in A$ (α' depending on x).*

Proof. Since K is compact, for any open cover of K, there exists an finite subcover of K, that is $\exists O_{\alpha_i}, i = 1, \cdots, N$ such that

$$K \subseteq \bigcup_{i=1}^{N} O_{\alpha_i}.$$

For any point $x \in K$ there exist $O_{\alpha_j} \in \{O_{\alpha_i}\}_{i=1}^N$, s.t. $x \in O_{\alpha_j}$ and exists $\delta_x > 0$, s.t. $B_{\delta_x/2}(x) \subseteq B_{\delta_x}(x) \subseteq O_{\alpha_j}$ for O_{α_j} is open. Obviously we have that $B_{\delta_x/2}(x)$ is an open cover of K, i.e.

$$K \subseteq \bigcup_{x \in K} B_{\delta_x/2}(x),$$

and $B_{\delta_x/2}(x)$ has an finite subcover of K, donate as $\{B_{\delta_{x_i}/2}(x_i)\}_{i=1}^M$. Let $\delta = \min\{\delta_{x_i}\}_{i=1}^M$. Then for any $y \in K$, there exists $B_{\delta_{x_j}/2}(x_j) \in \{B_{\delta_{x_i}/2}(x_i)\}_{i=1}^M$ such that $y \in B_{\delta_{x_j}/2}(x_j)$ and $d(y, x_j) < \delta_{x_j}/2$. and for any y' where $d(y', y) < \delta/2 < \delta_{x_j}/2$, we have $d(x_j, y') \leq d(x_j, y) + d(y, y') < \delta_{x_j}$, thus $y, y' \in B_{\delta_{x_j}}(x_j) \subseteq O_{\alpha'}$, or say $y, y' \in B_{\delta/2}(y) \subseteq B_{\delta_{x_j}}(x_j) \subseteq O_{\alpha'}$.

$\square$

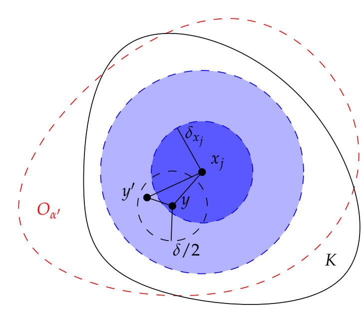
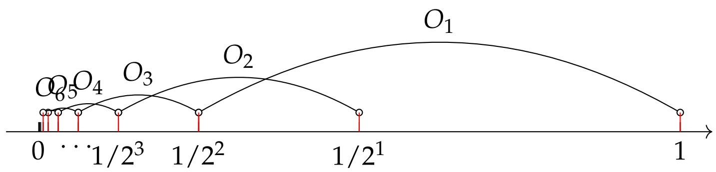


Figure 1: The Lebesgue number of an open cover

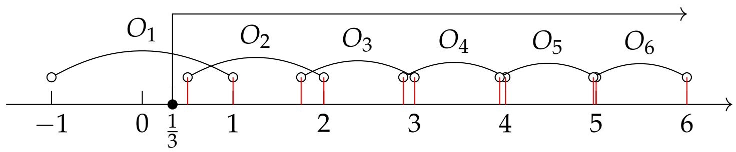
The theorem indicates for any open cover O_α of K, $\exists \delta > 0$, s.t. for $\forall x \in K, x' \in X$, is $d(x, x') < \delta$, then $\exists \alpha \in A$ we have $x, x' \in O_\alpha$. Such a $\delta > 0$ is called a **Lebesgue number** of the given open cover $O_\alpha(\alpha \in A)$. Notice that the statement could be false if the compactness assumption is dropped.

Exercise 1 (Open set)**.** Let $(X, d) = (\mathbb{R}, d_2), K = (0, 1), O_\alpha = (1/2^{\alpha+1}, 1/2^{\alpha-1})(\alpha \in \mathbb{N})$. Thus $1/2^\alpha \in O_\alpha$ and $\notin O_{\alpha'}$ if $\alpha' \neq$

$\alpha(\alpha, \alpha' \in \mathbb{N})$. It is easy to check O_α is an open cover of K, but $|1/2^\alpha - 1/2^{\alpha+1}| = 1/2^{\alpha+1}$ can be arbitrarily small if $\alpha \uparrow$. Thus there exists $x \in K, x' \in X$ can not be covered one O_α, no matter how close they are.



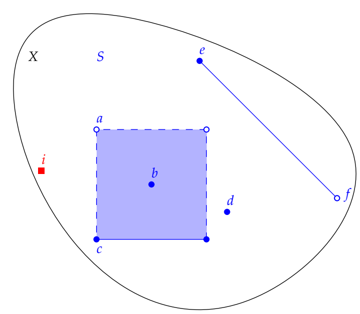
Exercise 2 (Unbounded set)**.** Let $(X, d) = (\mathbb{R}, d_2), K = [1/3, \infty), O_\alpha = (\alpha - 1 - 1/2^{\alpha-1}, \alpha)(\alpha \in \mathbb{N})$. Thus $x = \alpha - 1/2^\alpha \in O_\alpha$ and $x' = \alpha \in O_{\alpha+1}$ and $d(x, x')$ could be arbitrarily small as $\alpha \uparrow$. Thus there exists $x \in K, x' \in X$ can not be covered one O_α, no matter how close they are.



Definition 1 (Isolated point, limit point and accumulation point)**.** Let (X, d) be a metric space and $S \subseteq X$. A point $x \in X$ is called:

- an **isolated point** of S, if $\exists \epsilon > 0$, s.t. $B_\epsilon(x) \cap S = \{x\}$ ($\Rightarrow x \in S$);
- a **limit point** of S, if $\forall \epsilon > 0$, s.t. $B_\epsilon(x) \cap S \backslash \{x\} \neq \emptyset$;
- an **accumulation point** of S, if $\exists$ seq. $a_n \in S (n \in \mathbb{N})$, s.t. $x = \lim_{n\to\infty} a_n$.

Example 1. $S \subseteq X$ is as the margin figure, point $i \notin S$:



point	iso. pts. of S	limit pts. of S	acc. pts. of S	$\in S$
i	$\times$	$\times$	$\times$	$\times$
a	$\times$	$\surd$	$\surd$	$\times$
b	$\times$	$\surd$	$\surd$	$\surd$
c	$\times$	$\surd$	$\surd$	$\surd$
d	$\surd$	$\times$	$\surd$	$\surd$
e	$\times$	$\surd$	$\surd$	$\surd$
h	$\times$	$\surd$	$\surd$	$\times$

Notice that x is a isolated point of $S \Rightarrow x \in S$; but x is a limit/accumulate point of $S \not\Rightarrow x \in S$.

Exercise 3. Let (X, d) be a metric space, $S \subseteq X$,

1. Show that x is an isolated/limit point of $S \Rightarrow x$ is a accumulate point of S;
2. Donate $\{\text{iso. pts. of} S\}, \{\text{limit pts. of} S\}$ and $\{\text{acc. pts. of} S\}$ by I_S, L_S, A_S respectively. Show that $I_S \cup L_S = A_S$;

3. Suppose $S \subseteq K \subseteq X$, where S is infinite and K is compact, show that $\{\text{limit pts. of} S\} \neq \emptyset$; (Prove by contradiction)

Proof. 1. If x is an isolated point of S, thus $x \in S$. Let $a_n \equiv x$, then $\lim_{n\to\infty} a_n = x$, thus x is an accumulate point of S; If x is a limit point of S, then for any $\epsilon > 0, B_\epsilon(x) \cap S\backslash\{x\} \neq \emptyset$. Let $a_n \in B_{1/n}(x)(n \in \mathbb{N})$, then $d(a_n, x) < 1/n$ for $\forall n \in N$, thus $\lim_{n\to\infty} a_n = x$, and x is an accumulate point of S.
2. We have obtained that $I_S, L_S \subseteq A_S$. Suppose $x \in A_S\backslash(I_S \cup L_S) \neq \emptyset$. Which means : (1) there exists seq. $a_n \in S$ such that $\lim_{n\to\infty} a_n = x$; (2) $\forall \epsilon > 0$, s.t. $B_\epsilon(x) \cap S \neq \{x\}$ $(\neg I_S)$;(3) $\exists \epsilon' > 0$, s.t. $B_{\epsilon'}(x) \cap S\backslash\{x\} = \emptyset$ $(\neg L_S)$. Let $B_\epsilon(x) \cap S = Q_\epsilon \neq \{x\}$, if $x \in Q_\epsilon$, then it leads to a contradiction with (3); If $x \notin Q_\epsilon$, then $Q_{\epsilon'} = \emptyset$, that is $B_{\epsilon'}(x) \cap S = \emptyset$ and for $\forall s \in S \Rightarrow d(s, x) \geq \epsilon'$, which leads to a contradiction with (1). Thus $A_S\backslash(I_S \cup L_S) = \emptyset$. Because $I_S, L_S \subseteq A_S$, we have $I_S \cup L_S = A_S$.
3. Since S is infinite, there exists an infinite seq. $a_n \in S$. By Bolzano-Weierstrass theorem, there exists a subseq. $a_{n_i} \in S$ such that $\lim_{i\to\infty} a_{n_i} = a$. Suppose $L_S = \emptyset$, which means for $\forall x, \exists \epsilon' > 0$, s.t. $B_{\epsilon'}(x) \cap S\backslash\{x\} = \emptyset$, thus there exists ϵ_a, s.t. $B_{\epsilon_a}(a) \cap S\backslash\{a\} = \emptyset$, which means $\forall s \in S, d(s, a) \geq \epsilon_a$ and leads to a contradiction.

□

Exercise 4. Let $(X, d) = (\mathbb{R}, d_2), S \subseteq \mathbb{R}$, show that if $\sup S$ ($\inf S$) exists, then it is an accumulate point.

Proof. If $\sup S \exists$, then for $\forall x \in S$, s.t. $x \leq \sup S$ and for $\forall \epsilon > 0$, $\exists x' \in S$, s.t. $\sup S - \epsilon < x'$. For any $n \in \mathbb{N}$, there exists $x_n \in S$ s.t. $\sup S - 1/n < x' \leq \sup S$, and $d(x_n, \sup S) < 1/n$, thus $x_n \to \sup S$ as $n \to \infty$. □

Exercise 5. Show that, if (X, d) be a metric space, then

$$S \subseteq_{close} X \Leftrightarrow A_S = S \Leftrightarrow L_S \subseteq S.$$

Proof. For any $x \in S$, let $a_n = x$, then $\lim_{n\to\infty} a_n = x$, thus $S \subseteq A_S$. Since example (**??**), we have $S \subseteq_{close} X \Leftrightarrow A_S = S$. $\Rightarrow$ Since $I_S \cup L_S = A_S$, we have $L_S \subseteq A_S = S$; $\Leftarrow$, for $L_S \subseteq A_S \subseteq S$, we have $S \subseteq A_S \Rightarrow S = A_S$. □

Limits of maps

Let (X, d_X) and (Y, d_Y) be metric spaces and $S \subseteq X$. We consider a map $f : S \mapsto Y$. e.g. $X = \mathbb{R}^2, Y = \mathbb{R}, S = \mathbb{R}^2\backslash\{(0, 0)\}, f : (x, y) \mapsto 1/x^2 + y^2$. (the reason why shrink X)

Definition 2. Limit Let $a \in X$ (not necessarily $\in S$) and $b \in Y$. We say that $\lim_{x \to a} f(x) = b$ if $\forall \epsilon > 0, \exists \delta > 0, \forall x \in S,$ s.t. $0 < d_X(x,a) < \delta \Rightarrow d_Y(f(x), b) < \epsilon$.

Note 1. There are 3 points deserved mention.

1. 3 conditions of x: 1. $x \in B_\delta(a)$; 2. $x \neq a$; 3. $x \in S$. Collectively, $x \in B_\delta(a) \cap S \backslash \{a\}$.
2. We require $d_X(x,a) > 0$, since $f(a)$ could be totally unconnective with $f(B_\delta(a) \cap S \backslash \{a\})$.
3. If $\exists r > 0$, s.t. $B_r(a) \cap S = \emptyset$, then $\lim_{x \to a} f(x) = b$ (logically) holds for every $b \in Y$. Otherwise $\exists \epsilon > 0, \forall \delta > 0, \exists x \in S, 0 < d_X(x,a) < \delta, \cdots$, but if let $\delta < r$, then any $x \in S$ commits $d(x,a) > r > \delta$, which leads to a contradiction.

Exercise 6. Show that

1. If a is a limit point of S and $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} f(x) = b'$ then $b = b'$;
2. Let $(Y, d_Y) = (\mathbb{R}^m, d_2)$ and $f : S \mapsto Y, g : S \mapsto Y$, where $S \subseteq X, a \in X$. If $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$, then $\lim_{x \to a}(f(x) \pm g(x)) = b \pm c$. If furthermore $(Y, d_2) = (\mathbb{R}, d_2)$, then $\lim_{x \to a} f(x)g(x) = bc$; if $c \neq 0$, then $\exists \delta > 0,$ s.t. $g(x) \neq 0$ for all $x \in B_\delta(a) \backslash \{a\}$ and $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{b}{c}$.

Proof. 1. Since $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} f(x) = b'$, for $\forall \epsilon > 0$, $\exists \delta_1, \delta_2 > 0,$ s.t. $\forall x \in B_{\delta_1}(a) \cap S \backslash \{a\} \Rightarrow d(f(x), b) < \epsilon$, and the same thing for δ_2. Let $\delta = \min\{\delta_1, \delta_2\}$, then for $\forall x \in B_\delta(a) \cap S \backslash \{a\}$, we have $d(f(x), b) < \epsilon$ and $d(f(x), b') < \epsilon$ simultaneously , thus $d(b, b') < \epsilon$ for $\forall \epsilon > 0$, thus $b = b'$.

2. for $\forall \epsilon > 0, \exists \delta > 0, \forall x \in B_\delta(a) \cap S \backslash \{a\} \Rightarrow d_2(f(x), b) < \epsilon$ and $d_2(g(x), c) < \epsilon$. Thus

$$\begin{aligned} d_2(f(x) + g(x), b + c) &= [(f(x) + g(x) - b - c)^T (f(x) + g(x) - b - c)]^{1/2} \\ &= [(f(x) - b)^T (f(x) - b) + (g(x) - c)^T (g(x) - c) + 2(f(x) - b)^T (g(x) - c)]^{1/2} \\ &< [2\epsilon^2 + 2(f(x) - b)^T (g(x) - c)]^{1/2}. \end{aligned}$$

Notice that $(f(x) - b)^T (g(x) - c) = (g(x) - c)^T (f(x) - b)$, thus $(f(x) - b)^T (g(x) - c) = [(g(x) - c)^T (f(x) - b)(f(x) - b)^T (g(x) - c)]^{1/2} = \epsilon^2$. and $d_2(f(x) + g(x), b + c) < (4\epsilon^2)^{1/2} = 2\epsilon$, thus $\lim_{x \to a}(f(x) + g(x)) = b + c$. Others are trivial.

□

Exercise 7 (Composite maps)**.** Let X, Y, Z be metric space and $f : S \mapsto T, g : T \mapsto Z$, where $S \subseteq X, T \subseteq Y$. Show that if $\lim_{x \to a} f(x) = b, \lim_{y \to b} g(y) = c$ and $b \notin f(S)$, then $\lim_{x \to a}(g \circ f)(x) = c$. If condition $b \notin f(S)$ is dropped, find an example s.t. $\lim_{x \to a}(g \circ f)(x) \neq c$.

Proof. 1. Since $\lim_{y \to b} g(y) = c$, then for $\forall \epsilon > 0, \exists \delta_y > 0, \forall y \in B_{\delta_y}(b) \cap T \backslash \{b\} \Rightarrow d(c, g(y)) < \epsilon$. And because $\lim_{x \to a} f(x) = b$, then $\exists \delta_x > 0$, for $\forall x \in B_{\delta_x}(a) \cap S \backslash \{a\}$, s.t. $d(b, f(x)) < \delta_y \Rightarrow f(x) \in B_{\delta_y}(b) \cap T$ (Since $f : S \mapsto T$). If $\forall x \in S$ has $f(x) \neq b$, that is $b \notin f(S)$, then $f(x) \in B_{\delta_y}(b) \cap T \backslash \{b\}$, and then $d(g(f(x)), c) < \epsilon$, i.e. $\lim_{x \to a}(g \circ f)(x) \neq c$.

2. Intuitively, If $g(y)$ is un-continuous as $y = b$, and $f(x)$ touches b with an extremely frequence as $x \to a$ then $g \circ f$ would be

oscillating as $x \to a$. For example, let $f(x) = \sin(1/x), g(y) = y$ for $y \neq 0$ and 1 for $y = 0$, then $g \circ f(x)$ has no limit as $x \to 0$.

$\square$

Exercise 8 (Example of nonexistence of limit)**.** Let $f : \mathbb{R}^2 \backslash \{(0,0)\} \mapsto \mathbb{R}$ where $f(x,y) = \frac{xy}{x^2+y^2}$. Show that $\lim_{(x,y)\to(0,0)} f(x,y) \nexists$, using property of composite maps.

Proof. Consider map $g : \mathbb{R}\backslash\{0\} \mapsto \mathbb{R}^2\backslash\{(0,0)\}$ thus $(0,0) \notin g(\mathbb{R}\backslash\{0\})$. Let $g(t) = (at, bt)$ then $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} f(g(t)) = \frac{ab}{a^2+b^2}$ depending on g. Thus if you set different g, that is different parameters a, b then you get different limit of composite maps $f \circ g$ which is equal to the limit of f, thus $\lim f \nexists$. $\square$