Introduction to Analysis
Collection

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Abstract

This is the collection of lecture notes for the *Introduction to Analysis* course in Spring 2019. The purpose of this course is to bridge the gap between *Calculus* and *Advanced Calculus*.

Contents

Chapter 1

Completeness of the real numbers

1.1 Real number

Definition 1. Let $S \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, we say that
1. r is an upper (lower) bound of S if $\forall s \in S, r \geq (\leq) s$;
2. r is the greatest (least) element of S if r is an upper (lower) bound of S and $r \in S$, denoted by $r = \max S (\min S)$.
3. r is the least upper (greatest lower) bound of S if r is the least (greatest) element of the set of upper (lower) bound of S, denoted by $r = \sup S (\inf S)$.

Remark 1. r is a least upper bound of S means any element of S which is smaller than r is not an upper bound of S, that is $\forall \epsilon > 0, \exists s \in S,$ s.t. $r - \epsilon < s \leq r$.

We write $\sup S = \infty$ ($\inf S = -\infty$) if and only if S has no upper (lower) bound. If this is the case we say $\sup S (\inf S)$ does not exist. We say S is bounded from above (below) iff S has an upper (lower) bound.

Definition 2 (Dedekind Cut)**.** Let $A, B \subseteq \mathbb{R}$, we say that (A, B) is a Dedekind cut if
1. $A, B \neq \emptyset$;
2. $A \cup B = \mathbb{R}$;
3. $\forall a \in A, b \in B, a < b$.

We usually call $A(B)$ the lower (upper) part of (A, B).

We assume that $\mathbb{R}$ has the **Dedekind's Gapless Property**: If (A, B) is a Dedekind cut of $\mathbb{R}$, then exactly one of the following happens:
1. $\max A$ exists but $\min B$ does not;
2. $\min B$ exists but $\max A$ does not.

We call $\max A$ in 1. (or $\min B$ in 2.) the **cutting** of (A, B).

Exercise 1. *We may define Dedekind cuts on $\mathbb{Q}$ and $\mathbb{Z}$ similarly, does Dedekind Gapless Property hold for $\mathbb{Q}$ and $\mathbb{Z}$?*

Proof. 1. Let $A := \{q \in \mathbb{Q} | q^2 < 2\}, B := \{q \in \mathbb{Q} | q^2 > 2\}$. It is direct to see that $A, B \neq \emptyset$.
If $\exists r \in \mathbb{Q}$, s.t. $r^2 = 2$, then $\exists p, q \in \mathbb{N}$, s.t. $r = p/q$ and p, q are not both even. Then $p^2/q^2 = 2 \Rightarrow p^2 = 2q^2 \Rightarrow p^2$ is even $\Rightarrow p$ is even $\Rightarrow p^2$ can be divided by $4 \Rightarrow q^2$ can be divided by $2 \Rightarrow q^2$ is even $\Rightarrow q$ is even, which leads to a contradiction. Thus $\forall r \in \mathbb{Q}, r^2 \neq 2$. Thus $A \cup B = \mathbb{Q}$.
Finally $\forall q_a \in A, q_b \in B$ one has $q_a^2 < 2 < q_b^2 \Rightarrow q_a < q_b$. Thus (A, B) is a Dedekind cut of $\mathbb{Q}$. It is direct to see that (A, B) has no Dedekind's gapless property:
For example, if $p \in A$, then $p \in \mathbb{Q}$ and $p^2 < 2$, put $\epsilon = 2 - p^2$, then we should find a $q \in \mathbb{Q}$ such that $q^2 < 2$ and $q > p$, which means

$$p^2 < q^2 < 2$$

we consider there exists a function r of p, ϵ, such that $r > 0$ and $r \in \mathbb{Q}$, and put $q = p + r$, thus $q > p$ and $q \in \mathbb{Q}$, we now prove that $q^2 < 2$. Since

$$2 - q^2 = 2 - (p^2 + r^2 + 2pr) = \epsilon - r^2 - 2pr$$

We now try to find a suitable structure of r to make $r^2 + 2pr < \epsilon$. Since $p > 0$ and $\epsilon = 2 - p^2, 0 < \epsilon < 2$. Consider $r = \epsilon/2$ then

$$r^2 = \frac{\epsilon^2}{4} = \frac{\epsilon}{2} \cdot \frac{\epsilon}{2} < \frac{\epsilon}{2}.$$

Consider $r = \epsilon/((2p+1)2) < \epsilon/2$ and

$$2pr = 2p \cdot \frac{\epsilon}{(2p+1)2} < \frac{\epsilon}{2},$$

then we have $r^2 + 2pr < \epsilon$ and

$$q^2 < 2,$$

by defining

$$q = p + \frac{\epsilon}{2(2p+1)} = \frac{3p^2 + 2p + 2}{4p + 2},$$

thus there is no maximal element in A and correspondingly, there is no minimal element in B as well.
2. trivial. □

Theorem 1 (Weierstrass Theorem). *Let $\emptyset \neq S \subseteq \mathbb{R}$, if S has an upper bound, then* $\sup S$ *exists.*

Proof. Let B be the set of all upper bound of S, and define $A := \mathbb{R} \backslash B$.
Claim 1: (A, B) is a Dedekind cut of $\mathbb{R}$:

1. $S \neq \emptyset \Rightarrow \forall s \in S, s-1 \notin B \Rightarrow s-1 \in A \Rightarrow A \neq \emptyset$; And S has an upper bound $\Rightarrow B \neq \emptyset$;
2. $A = \mathbb{R} \backslash R \Rightarrow A \cup B = \mathbb{R}$;
3. If $\exists a \in A, b \in B,$ s.t. $a \geq b$ where b is an upper bound of S while a is not, thus $\exists s' \in S,$ s.t. $a < s' \leq b < a$, which leads to a contradiction. Thus $\forall a \in A, b \in B$ one has $a < b$.

Claim 2: $\min B$ exists:
If $\min B \nexists$, then by Dedekind's gapless property, $\max A \exists$, denoted by a_0. $a_0 \in A \Leftrightarrow a_0 \notin B \Leftrightarrow a_0$ is not an upper bound of $S \Leftrightarrow \exists s_0 \in S,$ s.t. $a_0 < s$. Choose $x \in \mathbb{R}$ such that $a_0 < x < s_0$, thus $\max A < x \Rightarrow x \in B \Rightarrow x$ is an upper bound of S but $x < s_0$ which leads to a contradiction. □

Exercise 2 (Archimedean Property)**.** *Show that* $\forall r \in \mathbb{R}, r > 0 \Rightarrow \exists n \in \mathbb{N},$ s.t. $n > r$. *(or say* $\exists n \in \mathbb{N},$ s.t. $1/n < r$*).*

Proof. Let $r \in \mathbb{R}$, $S := \{n \in \mathbb{N} | n \leq r\}$, since $r > 0, 0 \in S \Rightarrow S \neq \emptyset$. Then $S \subseteq \mathbb{R}$ and S is bounded above (by r), thus S has a least upper bound in $\mathbb{R}$, let $s = \sup S$.
Now consider the number $s-1$. Since s is the supremum of S, $s-1$ cannot be an upper bound of S by definition. Thus $\exists m \in S$ such that

$$m > s-1 \Rightarrow m+1 > s$$

But as $m \in \mathbb{N}$, it follows that $m+1 \in \mathbb{N}$. Because $m+1 > s$, it follows that $m+1 \notin S$ and so $m+1 > r$. Furthermore, for $\forall r > 0, 1/r > 0$ then $\exists n \in \mathbb{N},$ s.t. $n > 1/r \Rightarrow 1/n < r$. □

1.2 Sequence

Definition 3 (Convergence)**.** Let $a_n (n \in \mathbb{N})$ be a sequence in $\mathbb{R}$ and $l \in \mathbb{R}$, we say that a_n converges to l as $n \to \infty$ if $\forall \epsilon > 0, \exists N \in \mathbb{N},$ s.t. $\forall n \geq N, |a_n - l| < \epsilon$, denoted by $a_n \to l$ (as $n \to \infty$).

If such l exists, we call it the limit of $\{a_n\}$ and denote is as $\lim_{n\to\infty} a_n = l$, and call $\{a_n\}$ a convergent sequence; otherwise we call it a **divergent** sequence. Furthermore, we say $\lim_{n\to\infty} a_n = \infty$ if $\forall M > 0, \exists N \in \mathbb{N},$ s.t. $\forall n \geq N, a_n \geq M$.

Exercise 3. *Show that*
1. $\lim_{n\to\infty} a_n = l$ *and* $\lim_{n\to\infty} a_n = m \Rightarrow l = m$;
2. $a_n (n \in \mathbb{N})$ *is convergent* $\Rightarrow \{a_n | n \in \mathbb{N}\}$ *is bounded;*
3. if $a_n < b_n$ *for all* $n \in \mathbb{N}, \lim_{n\to\infty} a_n = l, \lim_{n\to\infty} b_n = m \Rightarrow l \leq m$.

Proof. 1. $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} a_n = m \Rightarrow$ for $\forall \epsilon > 0$, $\exists N, M \in \mathbb{N},$ s.t. $\forall n \geq N$

one has $|a_n - l| < \epsilon/2$ and $\forall n \geq M$ has $|a_n - m| < \epsilon/2$, thus for $\forall n \geq \max\{N, M\}$, has

$$|l - m| = |l - a_n + a_n - m| \leq |a_n - l| + |a_n - m| < \epsilon$$

holds for $\forall \epsilon > 0 \Rightarrow l = m$.

2. Suppose $a_n \to l$ as $n \to \infty$, then given an $\epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ we have $|a_n - l| < \epsilon \Rightarrow l - \epsilon < a_n < l + \epsilon$, thus a_n has upper bound

$$\max\{a_1, \cdots, a_{n-1}, l + \epsilon\},$$

and lower bound

$$\min\{a_1, \cdots, a_{n-1}, l - \epsilon\}.$$

3. if $l > m$, let $\epsilon = l - m$, then $\exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ has $|a_n - l| < \epsilon/2$ and $|b_n - m| < \epsilon/2$ thus

$$a_n < \frac{l + m}{2} < b_n,$$

which leads to a contradiction, thus $l \leq m$. □

Remark 2. Changing or removing finitely many terms in $a_n (n \in \mathbb{N})$ does not effect a_n's being convergent (and its limit)/ divergent.

Proposition 1. *If* $\lim_{n\to\infty} a_n = l$ *and* $\lim_{n\to\infty} b_n = m$ *then*
1. $\lim_{n\to\infty}(a_n \pm b_n) = l \pm m$*;*
2. $\lim_{n\to\infty} a_n b_n = lm$*;*
3. if $m \neq 0$ *and* $b_n \neq 0$ *for all but finitely many* n *then* $\lim_{n\to\infty} a_n/b_n = l/m$*.*

Proof. 1. For $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N, |a_n - l| \leq \epsilon/2$ and $\exists M \in \mathbb{N}$, s.t. $\forall n \geq M, |b_n - m| \leq \epsilon/2$, thus $\forall n \geq \max\{N, M\}$, one has

$$\begin{aligned}|(a_n \pm b_n) - (l \pm m)| &= |(a_n - l) \pm (b_m - m)| \\ &\leq |a_n - l| + |b_n - m| \\ &\leq \epsilon,\end{aligned}$$

thus $(a_n \pm b_n) \to l \pm m$ as $n \to \infty$.

2. Since a_n, b_n are convergent, thus they are bounded. Choose $C > 0$ such that $|b_n| \leq C$ for all $n \in \mathbb{N}$ and $|l| \leq C$, then for $\forall \epsilon > 0, \exists N, M \in \mathbb{N}$, s.t. $\forall n \geq N$ one has $|a_n - l| \leq \epsilon/(2C)$ and $\forall n \geq M$ has $|b_n - m| \leq \epsilon/(2C)$, thus $\forall n \geq \max\{N, M\}$ one has

$$\begin{aligned}|a_n b_n - lm| &= |a_n b_n - l b_n + l b_n - lm| \\ &\leq |a_n - l| \cdot |b_n| + |b_n - m| \cdot |l| \\ &\leq (|a_n - l| + |b_n - m|) \cdot |C| \\ &\leq \epsilon\end{aligned}$$

thus $a_n b_n \to lm$.

3. all we need to show is $\lim_{n\to\infty} 1/b_n = 1/m$ which is trivial. □

Exercise 4 (Squeeze theorem). *If* $\lim_{n\to\infty} a_n = l$ *and* $\lim_{n\to\infty} b_n = m$ *and* $a_n \le c_n \le b_n$, *show that* $l = m \Rightarrow \lim_{n\to\infty} c_n = l$.

Proof. Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = l$, for $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \ge N$ has $|a_n - l| < \epsilon/3$ and $|b_n - l| < \epsilon/3$. And since $a_n \le c_n \le b_n$, we have that $0 \le c_n - a_n \le b_n - a_n$. Thus for $\forall n \ge N$, we have

$$\begin{aligned}
|c_n - l| &= |c_n - a_n + a_n - l| \\
&\le |c_n - a_n| + |a_n - l| \\
&\le |b_n - a_n| + |a_n - l| \\
&= |b_n - l + l - a_n| + |a_n - l| \\
&\le |b_n - l| + 2|a_n - l| \\
&< \epsilon.
\end{aligned}$$

thus $\lim_{n\to\infty} c_n = l$. □

Exercise 5. *If* $a > 1$ *show that* $\lim_{n\to\infty} 1/a^n = 0$.

Proof. Since $a > 1 \Rightarrow b := a - 1 > 0$, thus

$$0 \le \frac{1}{a^n} = \frac{1}{(1+b)^n} \le \frac{1}{1+nb} \to 0$$

as $n \to \infty$, thus $\lim_{n\to\infty} 1/a^n = 0$ by Squeeze theorem. □

Definition 4. A seq. $a_n (n \in \mathbb{N})$ in $\mathbb{R}$ is

1. nondecreasing monotone/increasing if $a_n \le a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n\nearrow$; nonincreasing monotone/decreasing if $a_n \ge a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n\searrow$.
2. strictly increasing if $a_n < a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n\nearrow\nearrow$; strictly decreasing if $a_n > a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n\searrow\searrow$.

Theorem 2 (Monotone Seq. Property). *If* $a_n\nearrow$ *and* $\{a_n | n \in \mathbb{N}\}$ *has an upper bound, then* $\lim_{n\to\infty} a_n = \sup\{a_n | n \in \mathbb{N}\}$; $a_n\searrow$ *and* $\{a_n | n \in \mathbb{N}\}$ *has an lower bound, then* $\lim_{n\to\infty} a_n = \inf\{a_n | n \in \mathbb{N}\}$.

Proof. $\{a_n | n \in \mathbb{N}\}$ has an upper bound $\Rightarrow l := \sup\{a_n | n \in \mathbb{N}\}$ exists by Weierstrass theorem. Thus for $\forall \epsilon > 0, l - \epsilon$ is not an upper bound of $\{a_n\}$, then $\exists N \in \mathbb{N}$, s.t. $a_N > l$ and since $a_n\nearrow$, we have that $\forall n \ge N, l - \epsilon < a_n \le l \Rightarrow \lim_{n\to\infty} a_n = l$. □

Example 1 (Decimal expression gives real number). Suppose $d_i \in \mathbb{N}$ and $0 \le d_i \le 9$ for $i \in \mathbb{N}$, and define

$$a_n = \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

for $n \in \mathbb{N}$, then it is direct to see that $a_n \nearrow$ and

$$\begin{aligned} a_n &\le \frac{9}{10} + \frac{9}{10^2} + \cdots \frac{9}{10^n} \\ &= \frac{9}{10}\left(\frac{1-\frac{1}{10^{n+1}}}{1-\frac{1}{10}}\right) \\ &< \frac{9}{10}\left(\frac{1}{1-\frac{1}{10}}\right) \\ &= 1 \end{aligned}$$

and hence $\lim_{n\to\infty} a_n$ exists, and we can define a real number by $\lim_{n\to\infty} a_n =: 0.d_1 d_2 \cdots$

Example 2 (The natural base e)**.** Define a seq. $a_n = (1+1/n)^n (n \in \mathbb{N})$, then we have

$$\begin{aligned} a_n &= \left(1+\frac{1}{n}\right)^n \\ &= \binom{n}{0} + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{n}\frac{1}{n^n} \\ &= \sum_{j=0}^{n} \binom{n}{j}\frac{1}{n^j} = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \cdot \frac{1}{n^j} \\ &= \sum_{j=0}^{n} \frac{1}{j!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{j-1}{n}\right) \\ &< \sum_{j=0}^{n+1} \frac{1}{j!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{j-1}{n+1}\right) \end{aligned}$$

Thus $a_n \nearrow\nearrow$. On the other hand, for $\forall n \in \mathbb{N}$, we have

$$\begin{aligned} a_n &< \sum_{j=0}^{n+1} \frac{1}{j!} = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} \\ &< 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} \\ &= 1 + \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} \\ &< 3 \end{aligned}$$

Thus a_n has an upper bound and hence a_n converges, and we define $\lim_{n\to\infty} a_n =: e$.

1.3 Nested Intervals

Definition 5 (Nested)**.** A seq. of intervals $I_n (n \in \mathbb{N})$ is nested if $I_n \neq \emptyset$ and $I_{n+1} \subseteq I_n$ for $\forall n \in \mathbb{N}$.

Example 3. If we have a seq. of nested intervals $I_n(n \in \mathbb{N})$, do we have $\cap_{n\in\mathbb{N}} I_n \neq \emptyset$? The answer is not sure. For example,

1. $I_n = (0, 1/n), n \in \mathbb{N}$, then for any $r \in \mathbb{R}, \exists N \in \mathbb{N}$, s.t. $1/N < r$ by Archimedean Property, thus $r \notin I_N$, and hence $\cap_{n\in\mathbb{N}} I_n = \emptyset$;
2. $I_n = [n, \infty), n \in \mathbb{N}$, then for any $r \in \mathbb{R}, \exists N \in \mathbb{N}$, s.t. $r < N$ by Archimedean Property, thus $r \notin I_N$, and hence $\cap_{n\in\mathbb{N}} I_n = \emptyset$;

Theorem 3 (Theorem of Nested Interval)**.** *If $I_n(n \in \mathbb{N})$ is a seq. of bounded closed nested intervals, then $\cap_{n\in\mathbb{N}} I_n \neq \emptyset$. (In the other word, there exists a real number $c \in \mathbb{R}$ such that $c \in \cap_{n\in\mathbb{N}} I_n$)*

Proof. Write $I_n = [a_n, b_n](n \in \mathbb{N})$, then $I_n(n \in \mathbb{N})$ is nested $\Leftrightarrow a_n \leq b_n$ and $a_n \nearrow$ and $b_n \searrow$. And furthermore, for $\forall n, m \in \mathbb{N}$,

$$a_n \leq a_{\max\{m,n\}} \leq b_{\max\{m,n\}} \leq b_m,$$

in the other word, for $\forall m \in \mathbb{N}$, b_m is an upper bound of $\{a_n | n \in \mathbb{N}\}$, thus seq. a_n converges. Let $c = \lim_{n\to\infty} a_n$, then given $m \in \mathbb{N}$, for $\forall n \in \mathbb{N}, a_n \leq b_m$ thus

$$c = \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_m = b_m.$$

On the other hand, $c = \sup\{a_n | n \in \mathbb{N}\}$, thus for all $m \in \mathbb{N}$, we have

$$a_m \leq c \leq b_m$$

thus $c \in I_m$ for $\forall m \in \mathbb{N} \Rightarrow c \in \cap_{n\in\mathbb{N}} I_n$. □

Exercise 6. *Show that $\cap_{n\in\mathbb{N}} I_n \neq \emptyset$, if*

1. *$I_n = (a_n, b_n)$, nested and $a_n \nearrow\nearrow$ and $b_n \searrow\searrow$?*
2. *$I_n = (a_n, \infty)$, nested and $\{a_n | n \in \mathbb{N}\}$ is bounded from above.*

Proof. 1. Just as analyzed before, there exist $c \in \mathbb{R}$ such that $c = \lim_{n\to\infty} a_n$, and $c = \sup\{a_n | n \in \mathbb{N}\}$ and hence $a_n \leq c \leq b_m$ for $\forall n, m \in \mathbb{N}$. Note that $a_n \leq c$ implies that $a_n < c$ for $\forall n \in N$, otherwise if $\exists n' \in \mathbb{N}$, s.t. $a_{n'} = c$ then

$$a_{n'+1} \geq a_{n'} = c,$$

which leads to the contradiction. In the same way $c \leq b_m$ implies that $c < b_m$ for $\forall m \in \mathbb{N}$. Thus there $\exists c \in \mathbb{R}$ such that

$$a_n < c < b_m$$

for $\forall n, m \in \mathbb{N} \Rightarrow c \in \cap_{n\in\mathbb{N}} I_n$.

2. Since $I_n = (a_n, \infty)$ is a nested interval, $a_n \nearrow \Rightarrow a_n$ converges since a_n is upper bounded. That is $\exists c \in \mathbb{R}$, s.t. $c = \lim_{n\to\infty} a_n = \sup\{a_n\}$, thus for $\forall n \in \mathbb{N}, c \geq a_n$, that is

$$c + 1 > c \geq a_n$$

for $\forall n \in \mathbb{N} \Rightarrow c + 1 \in \cap_{n\in\mathbb{N}} I_n$. □

Exercise 7. *Show that Theorem of Nested Interval implies Dedekind's Gapless Property.*

Proof. Let (A, B) be a Dedekind cut of $\mathbb{R}$, pick a from A and b from B, and form an interval $I_0 = [a, b]$. Then $(a + b)/2$ lies in the middle of I_0 and must belong to A or B. If $(a + b)/2$ belongs to A, we let

$$a_1 = \frac{a+b}{2}, \quad b_1 = b$$

and if $(a + b)/2$ belongs to B, let

$$a_1 = a, \quad b_1 = \frac{a+b}{2}$$

and hence we can form a new interval $I_1 = [a_1, b_1]$ whose length is half of the former I_0. Repeat this process, we obtain a seq. of nested intervals

$$I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots,$$

where $I_n = [a_n, b_n], b_n - a_n = (b_{n-1} - a_{n-1})/2$. Thus there exists $s \in \mathbb{R}$ lies in the $\cap_{n\in\mathbb{N}} I_n$ by the theorem of nested intervals, and either $s \in A$ or $s \in B$.
Assume that $s \in A$, for any $s' \in \mathbb{R}, s < s'$, exists b_n such that $s < b_n < s'$ since $b_n \to s$, thus $s' \in B$. That is $s \in A$ and for any $s' > s, s' \in B$. In the other word, s is the maximal element of A and B has no minimal element in this case, since assume s' is the minimal element of B then $\exists b_n$, s.t. $b_n < s'$ and $b_n \in B$, which is a contradiction. □

Remark 3. Summary, we have discussed
1) Dedekind's Gapless Property;
2) Weierstrass Theorem;
3) Monotone Seq. Property;
4) Theorem of Nested Interval.
which have the relationship:

$$\begin{array}{ccc} 1) & \Longrightarrow & 2) \\ \Uparrow & & \Downarrow \\ 4) & \Longleftarrow & 3) \end{array}$$

These 5 properties are equivalent and we call the these the **Completeness of the real numbers**.

1.4 Limit superior / inferior

Let $a_n (n \in \mathbb{N})$ be a bounded (upper bdd. and lower bdd.) seq. in $\mathbb{R}$, we define **upper seq. of** a_n as

$$u_n := \sup\{a_m | m \geq n\},$$

and **lower seq. of** a_n as

$$l_n := \inf\{a_m | m \geq n\},$$

for $n \in \mathbb{N}$. Thus give $n \in \mathbb{N}$, we have that for $\forall m \geq n$

$$l_n \leq a_m \leq u_n,$$

We now show that l_n and u_n is monotone. Assume that $\exists n \in \mathbb{N}$, s.t. $u_n < u_{n+1}$, let $\epsilon = (u_{n+1} - u_n)/2$, then

$$u_{n+1} - \epsilon > u_n = \sup\{a_m | m \geq n\},$$

thus for $\forall m \geq n$, $u_{n+1} - \epsilon > a_m$ and hence $u_{n+1} - \epsilon$ is an upper bound of $\{a_m | m \geq n+1\}$, which leads to a contradiction. Thus for $\forall n \in \mathbb{N}, u_n \geq u_{n+1} \Rightarrow u_n \searrow$, and $l_n \nearrow$ in the same way.
Thus we have that for any $n, m \in \mathbb{N}$,

$$l_m \leq l_{\max\{m,n\}} \leq u_{\max\{m,n\}} \leq u_n,$$

thus l_1 is a lower bound for $\{u_n | n \in \mathbb{N}\}$ and u_1 is an upper bound of $\{l_n | n \in \mathbb{N}\}$ and hence $u_n, l_n (n \in \mathbb{N})$ are convergent by Monotone seq. property. We define the **limit superior** of a_n as the limit of u_n:

$$\overline{\lim_{n\to\infty}} a_n := \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sup_{m\geq n} a_m = \inf_{n\in\mathbb{N}} \sup_{m\geq n} a_m$$

The last equals sign is because $u_n \searrow$ and hence converges to its greatest lower bound by Monotone seq. property. And the **limit inferior** of a_n as the limit of l_n:

$$\underline{\lim_{n\to\infty}} a_n := \lim_{n\to\infty} l_n = \lim_{n\to\infty} \inf_{m\geq n} a_m = \sup_{n\in\mathbb{N}} \inf_{m\geq n} a_m$$

Exercise 8. *Let $a_n (n \in \mathbb{N})$, show that*

$$a_n \textit{ converges} \Leftrightarrow \overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n$$

and if any of both sides holds, then

$$\lim_{n\to\infty} a_n = \overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n$$

Proof. $\Rightarrow$: Suppose that $\lim_{n\to\infty} a_n = s$. Then for any $\epsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $\forall n \geq N$, $|a_n - s| < \epsilon/2$, thus $s - \epsilon/2 < a_n < s + \epsilon/2$ for $\forall n \geq N$. Thus the upper seq. u_n of a_n has

$$s - \frac{\epsilon}{2} < a_n \leq u_n \leq s + \frac{\epsilon}{2},$$

for $\forall n \geq N$. The third inequality symbol is because if $\exists n' \geq N$ such that $u_{n'} > s + \epsilon/2$, then there exist a real number q such that $s + \epsilon/2 < q < u_{n'}$ and $q > s + \epsilon/2 > a_n$ for $\forall n \geq N$ and hence $q > a_n$ for $\forall n \geq n'$, and then $u_{n'}$ is not the least upper bound of $\{a_n | n \geq n'\}$ which is contrary. Thus $|u_n - s| \leq \epsilon/2 < \epsilon$, thus

$$\lim_{n\to\infty} u_n = \overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} a_n = s,$$

and $\lim_{n\to\infty} l_n = \underline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty} a_n = s$ in the same way.
$\Leftarrow$: Suppose $\lim_{n\to\infty} u_n = \lim_{n\to\infty} l_n = s$, then for $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ one has $|u_n - s| < \epsilon/3$ and $|l_n - s| < \epsilon/3$ and $|u_n - l_n| \leq |u_n - s| + |l_n - s| < 2\epsilon/3$, since $l_n \leq a_n \leq u_n$ then $0 \leq a_n - l_n \leq u_n - l_n$. Then we have that

$$\begin{aligned}
|a_n - s| &= |a_n - l_n + l_n - s| \\
&\leq |a_n - l_n| + |l_n - s| \\
&\leq |u_n - l_n| + |l_n - s| \\
&< \epsilon
\end{aligned}$$

for $\forall n \geq N \Rightarrow \lim_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n = \underline{\lim}_{n\to\infty} a_n = s$. $\square$

Exercise 9. *Let $a_n, b_n (n \in \mathbb{N})$ be two bdd. seq. show that*
1. $\overline{\lim}_{n\to\infty}(a_n + b_n) \leq \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n$;
2. $\underline{\lim}_{n\to\infty} a_n + \underline{\lim}_{n\to\infty} b_n \leq \underline{\lim}_{n\to\infty}(a_n + b_n)$.

Proof. 1. Let $u_n = \sup_{m\geq n} a_m, v_n = \sup_{m \geq n} b_m, w_n = \sup_{m\geq n}(a_m + b_m)$. If $\exists n' \in \mathbb{N}$ such that $w_{n'} > u_{n'} + v_{n'}$, then $\exists r \in \mathbb{R}$ s.t. $u_{n'} + v_{n'} < r < w_{n'}$ and hence for any $m \geq n'$, $a_m \leq u_{n'}, b_m \leq v_{n'}$ and

$$a_m + b_m \leq u_{n'} + v_{n'} < r$$

which means r is an upper bound of $\{a_m | m \geq n'\}$ which leads to a contradiction with $w_{n'}$ is the least upper bound of $\{a_m | m \geq n'\}$. Thus for $\forall n \in \mathbb{N}, u_n + v_n \leq w_n$, and since $\lim_{n\to\infty} u_n, \lim_{n\to\infty} v_n$ exists, we have that

$$\lim_{n\to\infty}(u_n + v_n) = \lim_{n\to\infty} u_n + \lim_{n\to\infty} v_n \leq \lim_{n\to\infty} w_n$$

that is

$$\overline{\lim_{n\to\infty}}\, a_n + \overline{\lim_{n\to\infty}}\, b_n \leq \overline{\lim_{n\to\infty}}\,(a_n + b_n).$$

2. The same as 1. $\square$

And in the same way, we can prove that
1. $\overline{\lim}_{n\to\infty}(a_n \cdot b_n) \leq \overline{\lim}_{n\to\infty} a_n \cdot \overline{\lim}_{n\to\infty} b_n$;
2. $\underline{\lim}_{n\to\infty} a_n \cdot \underline{\lim}_{n\to\infty} b_n \leq \underline{\lim}_{n\to\infty}(a_n \cdot b_n)$.
In general, the properties does not hold for subtraction.

1.5 Cauchy seq.

Given a seq. $a_n(n \in \mathbb{N})$ in $\mathbb{R}$, can we determine whether a_n converges or not without referring a limit candidate l, but concluding according to the mutual behavior of the terms of $a_n(n \in \mathbb{N})$?

Definition 6 (Cauchy Sequence). A seq. $a_n(n \in \mathbb{N})$ in $\mathbb{R}$ is a Cauchy seq. if $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n, m \geq N \Rightarrow |a_n - a_m| < \epsilon$.

Exercise 10. *Show that*
1. a_n is convergent $\Rightarrow$ a_n is Cauchy seq.
2. a_n is Cauchy seq. $\Rightarrow$ a_n is bounded.

Proof. 1. assume that a_n converges to l, then for any $\epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N$ one has $|a_n - l| < \epsilon/2$, then for any $m, n \geq N$ we have

$$|a_m - a_n| \leq |a_m - l| - |a_n - l| < \epsilon$$

thus $a_n(n \in \mathbb{N})$ is Cauchy seq.
2. For $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n, m \geq N$ one has $|a_m - a_n| \leq \epsilon$, thus for $\forall m \geq N \Rightarrow |a_N - a_m| \leq \epsilon \Rightarrow a_n - \epsilon \leq a_m \leq a_N + \epsilon$, thus $a_n(n \in N)$ has upper and lower bound

$$\max\{a_1, \cdots, a_N, a_N + \epsilon\}, \quad \min\{a_1, \cdots, a_N, a_N - \epsilon\},$$

thus a_n is bounded. $\square$

Theorem 4. *Let $a_n(n \in \mathbb{N})$ be a seq. in $\mathbb{R}$, then a_n is convergent $\Leftrightarrow$ a_n is Cauchy seq.*

Proof. $\Leftarrow$: a_n is Cauchy seq. $\Rightarrow$ a_n is bdd. $\Rightarrow$ the upper/lower seq. u_n, l_n of a_n converges. Thus $\lim_{n\to\infty} u_n - \lim_{n\to\infty} l_n = \lim_{n\to\infty}(u_n - l_n)$. For $\forall \epsilon > 0$, there is some $N \in \mathbb{N}$ s.t. $\forall m, n \geq N, |a_n - a_m| < \epsilon/3$. In particular, $\forall n \geq N \Rightarrow |a_n - a_N| < \epsilon/3$ and hence

$$a_N - \frac{\epsilon}{3} < a_n < a_N + \frac{\epsilon}{3}$$

which means $a_N - \epsilon/3$ is a lower bound of $a_n(n \geq N)$ and is not greater that $\{a_n | n \geq N\}$'s greatest lower bound l_N, and the same to $a_N + \epsilon/3$, thus

$$a_N - \frac{\epsilon}{3} \leq l_N \leq u_N \leq a_N + \frac{\epsilon}{3}$$

and since $l_n \nearrow$ and $u_n \searrow$, we have that for $\forall n \geq N$

$$0 \leq u_n - l_n \leq u_N - l_N \leq \frac{2\epsilon}{3} < \epsilon$$

thus $\lim_{n\to\infty}(u_n - l_n) = 0 \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} l_n \Rightarrow a_n$ converges. $\square$

Exercise 11. *Let $S \subseteq \mathbb{R}$, if $|s - s'| \leq 3$ for $\forall s, s' \in S$, show that*
1. S is bdd.;
2. $\sup S - \inf S \leq 3$;

Proof. 1. If S has no upper bound, then for any $s \in S$, define $M = s + 4$, then $\exists s' \in S$ s.t. $s' > M = s + 4 \Rightarrow s' - s = |s' - s| > 4$, which is contrary.
2. Let $u = \sup S, l = \inf S$, suppose $u - l > 3$, then let $\epsilon = u - l - 3$, we have that $\exists s \in S$, s.t.

$$u - \frac{\epsilon}{3} < s \leq u,$$

and $\exists s' \in S$ s.t.

$$l \leq s' < l + \frac{\epsilon}{3},$$

then

$$u - l - \frac{2\epsilon}{3} < s - s' \leq u - l$$

thus $3 + \epsilon/3 < s - s' = |s - s'| \leq 3 + \epsilon \Rightarrow |s - s'| > 3$, which is contrary. $\square$

Chapter 2

Series

2.1 Positive series

Definition 7. Let $a_n (n \in \mathbb{N})$ be a seq. in $\mathbb{R}$, we say that the series $\sum_n^\infty a_n$ (or $\sum_n a_n$) converges to a real number s if

$$\lim_{n\to\infty} s_n = s,$$

where $s_n := \sum_{j=1}^n a_j$ is called the n - th partial sum of $\sum_n a_n$.

If such s exists (resp. does not exist), we say that the series $\sum_n a_n$ convergent (resp. divergent). For a series $\sum_n a_n$ and $l, m \in \mathbb{N}, l < m$, we let $s_{l,m} := \sum_{j=l}^m a_j$ the (l, m) - tail of $\sum_n a_n$. If a series $\sum_n a_n$ converges, we denote it as $\sum_n a_n < \infty$.

Exercise 12. *If a series $\sum_n a_n < \infty$, show that $\lim_{n\to\infty} a_n = 0$.*

Proof. Trivial. $\square$

$\sum_n a_n$ converges $\Leftrightarrow$ s_n converges by definition and $\Leftrightarrow$ s_n is Cauchy seq., i.e. $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geq N$, (assume that $n > m$)

$$\begin{aligned} |s_n - s_m| &= |a_{m+1} + \cdots + a_n| \\ &= |a_{m+1} + a_{m+2} + \cdots + a_{m+1+(n-1)}| \\ &\leq \epsilon. \end{aligned}$$

In particular, for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ and for $\forall k > N, \forall l \geq 0$, then $|a_k| + \cdots + |a_{k+l}| < \epsilon \Leftrightarrow \sum_n |a_n|$ convergent $\Leftrightarrow \exists M > 0, \forall n \in \mathbb{N}, s_n = \sum_{j=1}^n a_j \leq M$, since $s_n \nearrow$. Collectively, we have some conclusions:

1. series $\sum_n a_n$ converges $\Leftrightarrow$ for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ and for $\forall k > N, \forall l \geq 0$, $|a_k + \cdots + a_{k+l}| < \epsilon$;
2. series $\sum_n b_n$, where $b_n \geq 0$, converges $\Leftrightarrow \exists M > 0, \forall n \in \mathbb{N}, \sum_{j=1}^n b_j \leq M$.
3. series $\sum_n |a_n|$ converges $\Rightarrow \sum_n a_n$ converges.

Example 4. Given series $\sum_n 1/n$. we have that

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2} \\
s_8 &\geq 1 + \frac{2}{2} + \frac{1}{5} + \cdots + \frac{1}{8} \geq 1 + \frac{2}{2} + \frac{4}{8} = 1 + \frac{3}{2}
\end{aligned}
$$

In general, for $\forall m \in \mathbb{N}, s_{2^m} \geq 1 + m/2$ which has no upper bound $\Leftrightarrow \sum_n 1/n$ diverges.

Example 5. Given series $\sum_n 1/n^2$. we have that $1/n^2 < 1/((n-1)n) = 1/(n-1) - 1/n$. Then

$$
\begin{aligned}
s_n &= 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \\
&< 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} \\
&= 2 - \frac{1}{n} \\
&< 2.
\end{aligned}
$$

thus s_n has upper bound 2 $\Leftrightarrow \sum_n 1/n^2$ converges.

Definition 8. Given a seq. $a_n (n \in \mathbb{N})$, we say that
1. $\sum_n a_n$ converges absolutely if $\sum_n |a_n|$ converges;
2. $\sum_n a_n$ converges conditionally if $\sum_n |a_n|$ diverges but $\sum_n a_n$ converges.

Theorem 5 (Comparison Test). *If $a_n, b_n \geq 0 (n \in \mathbb{N})$, then $\exists C > 0$ and $N \in \mathbb{N}, n \geq N \Rightarrow a_n \leq C b_n \Rightarrow [\sum_n b_n < \infty \Rightarrow \sum_n a_n < \infty]$.*

Proof. If $\sum_n b_n$ converges, then for $\forall n \geq N$,

$$
\begin{aligned}
a_1 + \cdots + a_n &= a_1 + \cdots + a_N + a_{N+1} + \cdots + a_n \\
&\leq a_1 + \cdots + a_N + C \cdot (b_{N+1} + \cdots + b_n) \\
&\leq a_1 + \cdots + a_N + C \cdot M =: H,
\end{aligned}
$$

where M is an upper bound of $\sum_{j=1}^n b_j$, thus $\sum_j^n a_j$ as upper bound $H \Leftrightarrow \sum_n a_n$ converges. □

Theorem 6 (Limit Form of Comparison Test). *If $a_n, b_n \geq 0 (n \in \mathbb{N})$, and if $\lim_{n\to\infty} a_n/b_n$ exists $\Rightarrow [\sum_n b_n < \infty \Rightarrow \sum_n a_n < \infty]$.*

Proof. Let $l = \lim_{n\to\infty} a_n/b_n$, then for $\epsilon = 1, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N, a_n/b_m < l + 1 \Rightarrow a_n < (l+1)b_n$, which follows the proof by Comparison test. Furthermore if $l \neq 0$, then for $\epsilon = l/2, \exists N_l \in \mathbb{N}$, s.t. $\forall n \geq N_l$, s.t.

$$\frac{l}{2} \leq \frac{a_n}{b_n} \leq \frac{3l}{2},$$

and hence $b_n \leq a_n \cdot 2/l$ and $a_n \leq b_n \cdot 3l/2$, therefore $\sum_n b_n$ converges $\Leftrightarrow \sum_n a_n$ converges. $\square$

Exercise 13. *If $a_n, b_n \geq 0 (n \in \mathbb{N})$, show that if $\overline{\lim}_{n\to\infty} a_n/b_n$ exists $\Rightarrow [\sum_n b_n < \infty \Rightarrow \sum_n a_n < \infty]$.*

Proof. Let $u_n = a_n/b_n$ and $\lim_{n\to\infty} u_n = l$, then $l = \inf_{n\in\mathbb{N}} u_n$ and for $\epsilon = 1, \exists n' \in \mathbb{N}$ s.t.

$$l \leq u_{n'} < l + 1$$

and hence for $\forall n \geq n'$ we have that

$$\frac{a_n}{b_n} \leq u_{n'} < l + 1$$

thus $a_n < (l+1) \cdot b_n$ for $\forall n \geq n'$ and finish the proof by comparison test. $\square$

Exercise 14 (Ratio and Root test). *If $a_n, b_n \geq 0 (n \in \mathbb{N})$, show that*
1. $\lim_{n\to\infty} a_{n+1}/a_n < 1 \Rightarrow \sum_n a_n < \infty$; $\lim_{n\to\infty} a_{n+1}/a_n > 1 \Rightarrow \sum_n a_n$ *diverges.*
2. $\lim_{n\to\infty}(a_n)^{1/n} < 1 \Rightarrow \sum_n a_n < \infty$; $\lim_{n\to\infty}(a_n)^{1/n} > 1 \Rightarrow \sum_n a_n$ *diverges.*

Proof. Trivial. $\square$

2.2 Alternating series

Definition 9. A series $\sum_n a_n$ is called alternating series, if $\exists b_n > 0 (n \in \mathbb{N})$ s.t. $a_n = (-1)^{n-1} b_n (n \in \mathbb{N})$.

Theorem 7 (Leibniz's Criterion). *Let $\sum_n a_n$ be an alternating series, and $b_n = |a_n|_{\searrow 0}$ as $n \to \infty$, then $\sum_n a_n < \infty$.*

Proof. Since $b_n = (-1)^{n-1} a_n$, for any $k, l \in \mathbb{N}$ the tail of $\sum_n a_n$ is

$$\begin{aligned} |a_k + \cdots + a_{k+l}| &= (-1)^{k-1} |b_k - b_{k+1} + \cdots + (-1)^l b_{k+l}| \\ &= |b_k - b_{k+1} + \cdots + (-1)^l b_{k+l}| \end{aligned}$$

define $\lambda_{k,l} = b_k - b_{k+1} + \cdots + (-1)^l b_{k+l}$. Then if l is even, then

$$\lambda_{k,l} = (b_k - b_{k+1}) + \cdots + (b_{k+l-2} - b_{k+l-1}) + b_{k+l} \geq 0,$$

and if l is odd, then

$$\lambda_{k,l} = (b_k - b_{k+1}) + \cdots + (b_{k+l-1} - b_{k+l}) \geq 0,$$

thus $\lambda_{k,l} \geq 0$ for $\forall k, l \in \mathbb{N}$. And hence

$$\begin{aligned}
|a_k + \cdots + a_{k+l}| = |\lambda_{k,l}| = \lambda_{k,l} \\
&= \begin{cases} b_k - (b_{k+1} - b_{k+2}) - \cdots - (b_{k+l-1} - b_{k+l}), \ l \text{ is even} \\ b_k - (b_{k+1} - b_{k+2}) - \cdots - (b_{k+l-2} - b_{k+k-1}) - b_{k+l}, \ l \text{ is odd} \end{cases} \\
&\leq b_k
\end{aligned}$$

Since $\lim_{n\to\infty} b_n = 0 \Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N},$ s.t. $n \geq N \Rightarrow |b_n| < \epsilon \Rightarrow |a_n + \cdots + a_{n+l}| \leq b_n < \epsilon$ for $\forall l \in \mathbb{N}$, thus $\sum_n a_n$ converges. $\square$

2.3 Rearrangement theorem

Given a seq. $a_n (n \in \mathbb{N})$, we can separate all terms into two subseq.s:

$$a_{n_1}, a_{n_2}, \cdots \text{ and } a_{n'_1}, a_{n'_2}, \cdots$$

where $n_1 < n_2 < \cdots$ and $n'_1 < n'_2 < \cdots$ and $\{n_1, n_2, \cdots\} \cup \{n'_1, n'_2, \cdots\} = \mathbb{N}$, such that $a_{n_j} \geq 0 (j \in \mathbb{N}), a_{n'_k} \leq 0 (k \in \mathbb{N})$. Let $p_j := a_{n_j} (j \in \mathbb{N})$ and $q_k := a_{n'_k} (k \in \mathbb{N})$.

Exercise 15. *Show that $\sum_n |a_n| < \infty \Leftrightarrow \sum_j p_j < \infty$ and $\sum_k q_k < \infty$. Moreover, if any side holds, then*

$$\sum_n |a_n| = \sum_j p_j + \sum_k q_k$$

and

$$\sum_n a_n = \sum_j p_j - \sum_k q_k.$$

Proof. 1. $\Rightarrow$: since $\sum_n |a_n| < \infty$, any partial sum of a_n has upper bound such as M, then for any $j \in \mathbb{N}$:

$$\begin{aligned}
p_1 + \cdots + p_j &= |a_{n_1}| + \cdots + |a_{n_j}| \\
&\leq \sum_{n=1}^{n_j} |a_n| \\
&\leq M,
\end{aligned}$$

Thus any partial sum of p_j has upper bound M and hence $\sum_j p_j < \infty$. And $\sum_k q_k < \infty$ in the same way.

2. $\Leftarrow$: The partial sum of $\sum_n |a_n|$ can be decompose by the partial sums of $\sum_n p_n$ and $\sum_n q_n$ which have upper bounds, thus partial sum of $\sum_n |a_n|$ has upper bound, and $\sum_n |a_n| < \infty$.
3. Define the partial sum of $\sum_n |a_n|, \sum_n a_n, \sum_n p_n, \sum_n q_n$ as

$$A_m := \sum_{i=1}^{m} |a_i|, \quad S_m := \sum_{i=1}^{m} a_i$$
$$P_m := \sum_{i=1}^{m} p_i = \sum_{i=1}^{m} |a_{n_i}|, \; Q_m := \sum_{i=1}^{m} q_i = \sum_{i=1}^{m} |a_{n_i'}|$$

Then for any $m \in \mathbb{N}$, we have that

$$A_m \le P_m + Q_m \le A_{\max\{n_m, n_m'\}},$$

thus $\lim_{n\to\infty} A_n = \lim_{n\to\infty}(P_n + Q_n) = \lim_{n\to\infty} P_n + \lim_{n\to\infty} Q_n$ since $\sum_n p_n, \sum_n q_n$ exists, and the squeeze theorem. And hence $\sum_n |a_n| = \sum_j p_j + \sum_k q_k$.
On the contrary, for any $m \in \mathbb{N}$, we can represent the partial sum of $\sum_n a_n$ as

$$s_m = P_l - Q_v$$

where $l, v \to \infty$ as $m \to \infty$, thus $\sum_n a_n = \sum_n p_n - \sum_n q_n$. □

Exercise 16. *If $\sum_n a_n$ converges conditionally, show that*
1. $\sum_j p_j = \infty$ and $\sum_k q_k = \infty$;
2. $\lim_{j\to\infty} p_j = \lim_{k\to\infty} q_k = 0$.

Proof. 1. Denote the partial sum of $\sum_n a_n, \sum_j p_j, \sum_k q_k$ as s_n, P_j, Q_k respectively, then we have that $\lim_{n\to\infty} s_n = \lim_{n\to\infty}(P_j - Q_k)$ exists, then either both $\lim_{n\to\infty} P_j, \lim_{n\to\infty} Q_k$ exist or neither exists, since $\sum_n a_n$ converges conditionally $\Rightarrow \lim_{n\to\infty} P_j = \infty$ and $\lim_{n\to\infty} Q_k = \infty$.
2. Since

$$\lim_{j\to\infty} p_j = \lim_{j\to\infty} a_{n_j} = \lim_{n\to\infty} a_n = 0,$$

and $\lim k \to \infty q_k = 0$ as well in the same way. □

Exercise 17. *If $\sum_n a_n, \sum_n b_n$ converges, show that $\sum_n (a_n \pm b_n) = \sum_n a_n \pm \sum_n b_n$.*

Proof. Denote the partial sum of $\sum_n (a_n + b_n), \sum_n a_n, \sum_n b_n$ as S_n, A_n, B_n respectively, then for any $n \in \mathbb{N}$

$$S_n = A_n + B_n$$

and hence

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(A_n + B_n) = \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n$$

since $\lim_{n\to\infty} A_n, \lim_{n\to\infty} B_n$ exists, thus $\sum_n (a_n + b_n) = \sum_n a_n + \sum_n b_n$, and $\sum_n (a_n - b_n) = \sum_n a_n - \sum_n b_n$ in the same way. □

Exercise 18. *Inserting* 0s *to a series will not affect its convergence / divergence and its sum (if the sum exists).*

Proof. Consider the tail of series. Trivial. $\square$

Recall that a sequence a_n is a map $\mathbb{N} \xrightarrow{a.} \mathbb{R}$ where $n \mapsto a(n)$ denoted by a_n. A subsequence a_{n_m} is a composite map

$$\mathbb{N} \xrightarrow{n.} \mathbb{N} \xrightarrow{a.} \mathbb{R}$$

where $n.$ is a strictly monotone injection and $m \mapsto n(m)$ denoted by n_m. A **rearrangement** is also a composite map

$$\mathbb{N} \xrightarrow{n(\cdot)} \mathbb{N} \xrightarrow{a.} \mathbb{R}$$

where $n(\cdot)$ is a bijection. (To distinguish it from the notion of subseq., we don't use subscripts here)
Now we gonna explore two questions: if a series $\sum_n$ converges, $a_{n(m)}(m \in \mathbb{N})$ is a rearrangement of $a_n(n \in \mathbb{N})$, then
1. whether $\sum_m a_{n(m)}$ converges ?
2. whether $\sum_n a_n = \sum_m a_{n(m)}$?

Exercise 19. *Let* $\sum_n a_n$ *be a positive series, show that*

$$\sum_n a_n = \sup \Lambda$$

including the case $\sum_n a_n = \infty$. *Here* $\Lambda = \{a_{n_1} + \cdots + a_{n_k} | n_1 < \cdots < n_k, k \in \mathbb{N}\}$ *represents the set of every sum of finite terms of* $a_n(n \in \mathbb{N})$.

Proof. 1. $\leq$: since $\sum_n a_n$ is the limit of the partial sum s_n (which is the sum of finite terms, i.e. $s_n \in \Lambda$ for any $n \in \mathbb{N}$), and since $a_n \geq 0$, s_n monotone, then

$$\sum_n a_n = \lim_{n\to\infty} s_n = \sup_{n\in\mathbb{N}} s_n \leq \sup \Lambda$$

2. $\geq$: If $\sup \Lambda > \sup s_n$, let $\epsilon := \sup \Lambda - \sup s_n$, then $\exists \lambda = a_{n_1} + \cdots + a_{n_{k_\lambda}} \in \Lambda$ such that

$$s_m \leq \sup_{n\in\mathbb{N}} s_n < \sup \Lambda - \frac{\epsilon}{2} < \lambda \leq \sup \Lambda$$

for $\forall m \in \mathbb{N}$, but

$$\lambda = a_{n_1} + \cdots + a_{n_{k_\lambda}} \leq s_{n_{k_\lambda}}$$

which leads to a contradiction.
3. If $\sum_n a_n = \infty$, it is direct to see that $\sup \Lambda = \infty$ as well by 1. $\square$

Exercise 20. *If $\sum_n a_n$ is a convergent positive series, show that for every rearrangement $a_{n(m)}(m \in \mathbb{N})$ of $a_n(n \in \mathbb{N})$, we have that $\sum_n a_n = \sum_m a_{n(m)}$.*

Proof. If $\sum_n a_n$ is positive series, then $\sum_m a_{n(m)}$ is positive series as well.

$$\sum_n a_n = \sup \Lambda_{a_n} = \sup \Lambda_{a_{n(m)}} = \sum_m a_{n(m)}$$

where Λ_{a_n} and $\Lambda_{a_{n(m)}}$ are the set of every sum of finite terms of a_n and $a_{n(m)}$ respectively. That is the proof follows by the $\Lambda_{a_n} = \Lambda_{a_{n(m)}}$. $\square$

Exercise 21 (Dirichlet's Rearrangement Theorem (1829))**.** *If $\sum_n a_n$ converges absolutely, show that for every rearrangement $a_{n(m)}(m \in \mathbb{N})$ of $a_n(n \in \mathbb{N})$, we have that $\sum_n a_n = \sum_m a_{n(m)}$.*

Proof. $\sum_n a_n$ converges absolutely $\Rightarrow \sum_m a_{n(m)}$ converges absolutely. Furthermore

$$\begin{aligned}\sum_n a_n &= \sum_j p_j - \sum_k q_k \\ &= \sum_\mu p_{j_\mu} - \sum_\nu q_{k_\nu} \\ &= \sum_m a_{n_m}.\end{aligned}$$

$\square$

Theorem 8 (Riemann's Rearrangement Theorem(1852))**.** *If $\sum_n a_n$ converges conditionally, then for $\forall r \in \mathbb{R}$, there exists a rearrangement $a_{n(m)}(m \in \mathbb{N})$ of $a_n(n \in \mathbb{N})$ such that $\sum_m a_{n(m)} = r$.*

Proof. We will only use two known fact:
1. $\sum_j p_j = \infty$ and $\sum_k q_k = \infty$;
2. $\lim_{j\to\infty} p_j = \lim_{k\to\infty} q_k = 0$.

Given a $L \in \mathbb{R}$, start with p_1, plus by p_2 and so on till p_{m_1-1} where

$$\sum_i^{m_1-1} p_i \leq L \quad \text{but} \quad \sum_i^{m_1} p_i > L.$$

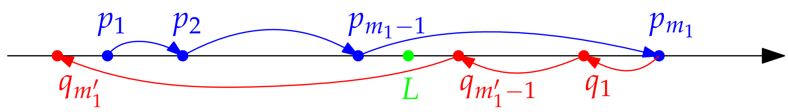
Then minus by q_1, q_2 and so on till $q_{m_1'-1}$ where

$$\sum_i^{m_1} p_1 - \sum_{j=1}^{m_1'-1} q_j > L \quad \text{but} \quad \sum_i^{m_1} p_1 - \sum_{j=1}^{m_1'} q_j \leq L.$$

This process can be repeat since $\sum_j p_j = \infty$ and $\sum_k q_k = \infty$ and hence any tail of $\sum_j p_j, \sum_k q_k$ has no upper bound, therefore the *cross* action can always happen, in the other word, $m_i, m_i'(i \in \mathbb{N})$ exists.

Thus we can form a rearrangement χ_n of $\sum_n a_n$ as

$$p_1, \cdots, p_{m_1}, -q_1, \cdots, -q_{m'_1}, \cdots$$



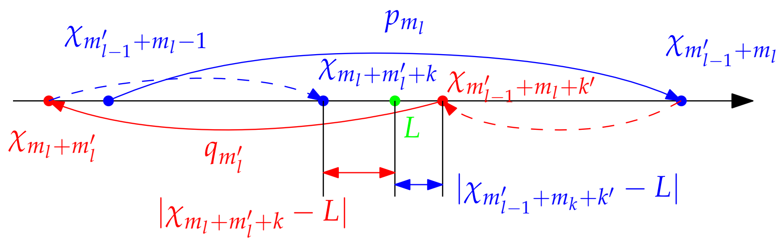
Now we will show that this rearrangement converges to L, i.e. $\lim_{n\to\infty} \chi_n = L$. Consider $\chi_{\cdots+m'_{l-1}+m_l-1}$ which implies the point lies in the left of L and will cross the l in next jump, and we denote it by $\chi_{m'_{l-1}+m_l-1}$ for simply. And hence we have that

$$|\chi_{m'_{l-1}+m_k+k'} - L| < p_{m_l}$$

if $0 \le k' < m'_l - m'_{l-1}$. And similarly

$$|\chi_{m_l+m'_l+k} - L| < q_{m'_l}$$

if $0 \le k < m_{m+1-m_l}$.



And since $\lim_{l\to\infty} p_{m_l} = \lim_{l\to\infty} q_{m'_l} = 0$, for $\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ s.t. $l \ge N_0 \Rightarrow p_{m_l}$ and $q_{m'_l} < \epsilon$. Let $N = m'_{N_0-1} + m_{N_0}$, then $n \ge N \Rightarrow |\chi_n - L| < \epsilon$. □

Remark 4 ($2S = S$). Dirichelet made the following observation in 1827: Consider a alternating series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$$

which is convergent (conditionally) by Leibniz's Criterion, and hence

$$\begin{aligned}
2S &= 2 - 1 + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \frac{2}{7} - \frac{2}{8} + \frac{2}{9} - \frac{2}{10} + \cdots \\
&=' (2-1) - \frac{2}{4} + \left(\frac{2}{3} - \frac{2}{6}\right) - \frac{2}{8} + \left(\frac{2}{5} - \frac{2}{10}\right) + \cdots \\
&= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots \\
&= S
\end{aligned}$$

Remark 5. In summary, given a series $\sum_n a_n$, and its any rearrangement $\sum_m a_{n(m)}$, then
1. If $a_n \geq 0$ for $\forall n \in \mathbb{N} \Rightarrow \sum_n a_n = \sum_m a_{n(m)}$;
2. If $\sum_n |a_n| < \infty \Rightarrow \sum_n a_n = \sum_m a_{n(m)}$;
3. If $\sum_n |a_n| = \infty$ but $\sum_n a_n < \infty \Rightarrow \sum_m a_{n(m)}$ could be anything.

2.4 Multiplying absolutely convergent series

Proposition 2. *Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely, let*

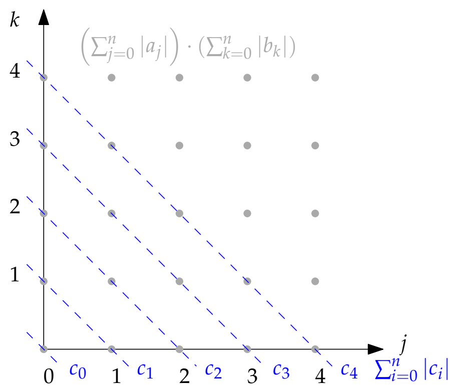
$$c_n = a_n b_0 + \cdots + a_0 b_n = \sum_{j+k=n;\ j,k\geq 0} a_j b_k,$$

then $\sum_n |c_n| < \infty$ and $\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right)$.

Proof. 1. $\sum_n |c_n| < \infty$
For all n,

$$\sum_{m=0}^{n} |c_m| = \sum_{m=0}^{n} \left| \sum_{\substack{j+k=m \\ j,k\geq 0}} a_j b_k \right| \leq \sum_{m=0}^{n} \sum_{\substack{j+k=m \\ j,k\geq 0}} |a_j||b_k|$$
$$\leq \left(\sum_{j=0}^{n} |a_j|\right) \cdot \left(\sum_{k=0}^{n} |b_k|\right).$$

Since $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converges absolutely, the partial sums of $|a_n|, |b_n|$ have upper bounds, denoted by M, N respectively, then $\sum_{m=0}^{n} |c_m|$ has a upper bound $M \cdot N$ and hence $\sum_{n=0}^{\infty} c_n$ converges absolutely.

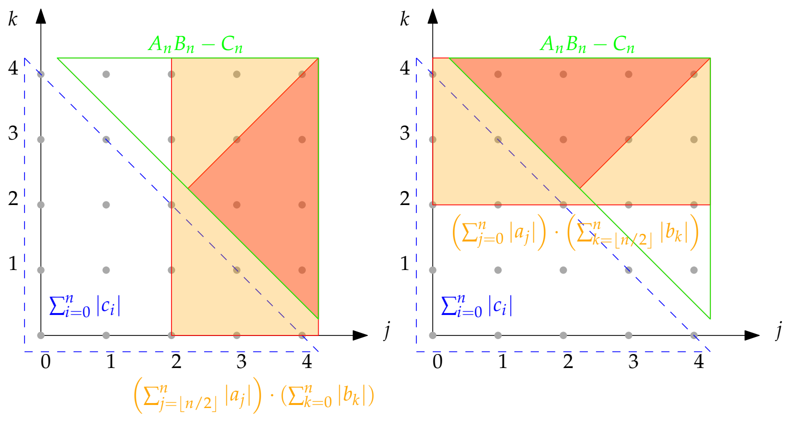


2. $\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right)$.
Let $A_n := a_0 + \cdots + a_n; B_n := b_0 + \cdots + b_n$ and $C_n := c_0 + \cdots + c_n$, we claim that $\lim_{n\to\infty}(A_n B_n - C_n) = 0$. Then

$$
\begin{aligned}
|A_nB_n - C_n| &= \sum_{\substack{j+k>n \\ 0\le j,k\le n}} |a_jb_k| \\
&\le \left(\sum_{j=\lfloor n/2\rfloor}^{n} |a_j|\right)\cdot\left(\sum_{k=0}^{n}|b_k|\right) + \left(\sum_{j=0}^{n}|a_j|\right)\cdot\left(\sum_{k=\lfloor n/2\rfloor}^{n}|b_k|\right)
\end{aligned}
$$

where $\sum_{k=0}^{n}|b_k|, \sum_{j=0}^{n}|a_j|$ are bounded, and tails $\sum_{j=\lfloor n/2\rfloor}^{n}|a_j|, \sum_{k=\lfloor n/2\rfloor}^{n}|b_k| \to 0$ as $n\to\infty$ since $\sum_n a_n, \sum_n b_n$ are converges abs. Thus $\lim_{n\to\infty}|A_nB_n - C_n| = 0$ and since $\lim_{n\to\infty} A_n, \lim_{n\to\infty} B_n, \lim_{n\to\infty} C_n$ exists, we have that

$$
\begin{aligned}
\sum_{n=0}^{\infty} c_n &= \lim_{n\to\infty} C_n \\
&= \lim_{n\to\infty} A_nB_b = \lim_{n\to\infty} A_n \cdot \lim_{n\to\infty} B_n \\
&= \left(\sum_{n=0}^{\infty} a_n\right)\cdot\left(\sum_{n=0}^{\infty} b_n\right)
\end{aligned}
$$



$\square$

Theorem 9. *If $\sum_n a_n, \sum_n b_n$ cvg. abs., $\mathbb{N} \xrightarrow{(j(\cdot),k(\cdot))} \mathbb{N}\times\mathbb{N}$ is bijection where $n \mapsto (j(n), k(n))$, let $c_n := a_{j(n)}b_{k(n)} (n\in\mathbb{N})$, then $\sum_n |c_n| < \infty$ (cvg. abs.) and $\sum_n c_n = (\sum_n a_n)(\sum_n b_n)$.*

Proof. 1. $\sum_n c_n$ cvg. abs.
For $\forall n\in\mathbb{N}$, let $l = \max\{j(1),\cdots,j(n),k(1),\cdots,k(n)\}$. Then

$$
\begin{aligned}
|c_1| + \cdots + |c_n| &= |a_{j(1)}b_{k(1)}| + \cdots + |a_{j(n)}b_{k(n)}| \\
&\le \left(\sum_{j=1}^{l}|a_j|\right)\cdot\left(\sum_{k=1}^{l}|b_k|\right) \\
&\le M\cdot N
\end{aligned}
$$

Thus $\sum_n c_n$ cvg. abs.
2. $\sum_n c_n = (\sum_n a_n)(\sum_n b_n)$.
Let $A_n = a_1 + \cdots + a_n$, $B_n = b_1 + \cdots + b_n$ and $C_n = c_1 + \cdots + c_n (n \in \mathbb{N})$. And define the bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by the second one in Figure 2.1. Then

$$\begin{aligned} A_n B_n &= (a_1 + \cdots + a_n)(b_1 + \cdots + b_n) \\ &= \sum_{1 \le j,k \le n} a_j b_k \\ &= C_{n^2} \end{aligned}$$

Thus $\lim_{n\to\infty} A_n B_n = \lim_{n\to\infty} C_{n^2} = \lim_{n\to\infty} C_n \Rightarrow \sum_n c_n = (\sum_n a_n)(\sum_n b_n)$. □

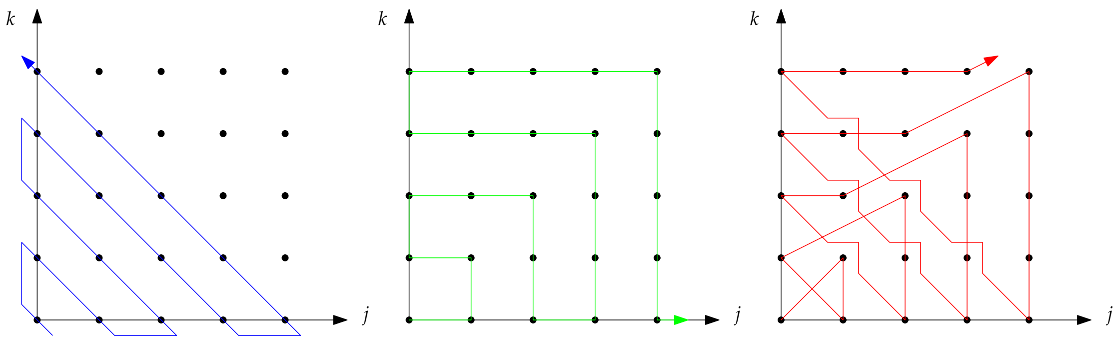


Figure 2.1: 3 kinds of bijections $(j(\cdot), k(\cdot))$

Chapter 3

$\mathbb{R}$ - valued functions