Introduction to Analysis

Collection

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Contents

Chapter 1

Completeness of the real numbers

1.1 Real number

Definition 1. Let $S \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, we say that
1. r is an upper (lower) bound of S if $\forall s \in S, r \geq (\leq) s$;
2. r is the greatest (least) element of S if r is an upper (lower) bound of S and $r \in S$, denoted by $r = \max S (\min S)$.
3. r is the least upper (greatest lower) bound of S if r is the least (greatest) element of the set of upper (lower) bound of S, denoted by $r = \sup S (\inf S)$.

Remark 1. r is a least upper bound of S means any element of S which is smaller than r is not an upper bound of S, that is $\forall \epsilon > 0, \exists s \in S,$ s.t. $r - \epsilon < s \leq r$.

We write $\sup S = \infty$ ($\inf S = -\infty$) if and only if S has no upper (lower) bound. If this is the case we say $\sup S (\inf S)$ does not exist. We say S is bounded from above (below) iff S has an upper (lower) bound.

Definition 2 (Dedekind Cut)**.** Let $A, B \subseteq \mathbb{R}$, we say that (A, B) is a Dedekind cut if
1. $A, B \neq \emptyset$;
2. $A \cup B = \mathbb{R}$;
3. $\forall a \in A, b \in B, a < b$.

We usually call $A(B)$ the lower (upper) part of (A, B).

We assume that $\mathbb{R}$ has the **Dedekind's Gapless Property**: If (A, B) is a Dedekind cut of $\mathbb{R}$, then exactly one of the following happens:
1. $\max A$ exists but $\min B$ does not;
2. $\min B$ exists but $\max A$ does not.

We call $\max A$ in 1. (or $\min B$ in 2.) the **cutting** of (A, B).

Exercise 1. *We may define Dedekind cuts on $\mathbb{Q}$ and $\mathbb{Z}$ similarly, does Dedekind Gapless Property hold for $\mathbb{Q}$ and $\mathbb{Z}$?*

Proof. 1. Let $A := \{q \in \mathbb{Q} | q^2 < 2\}, B := \{q \in \mathbb{Q} | q^2 > 2\}$. It is direct to see that $A, B \neq \emptyset$.
If $\exists r \in \mathbb{Q}$, s.t. $r^2 = 2$, then $\exists p, q \in \mathbb{N}$, s.t. $r = p/q$ and p, q are not both even. Then $p^2/q^2 = 2 \Rightarrow p^2 = 2q^2 \Rightarrow p^2$ is even $\Rightarrow p$ is even $\Rightarrow p^2$ can be divided by $4 \Rightarrow q^2$ can be divided by $2 \Rightarrow q^2$ is even $\Rightarrow q$ is even, which leads to a contradiction. Thus $\forall r \in \mathbb{Q}, r^2 \neq 2$. Thus $A \cup B = \mathbb{Q}$.
Finally $\forall q_a \in A, q_b \in B$ one has $q_a^2 < 2 < q_b^2 \Rightarrow q_a < q_b$. Thus (A, B) is a Dedekind cut of $\mathbb{Q}$. It is direct to see that (A, B) has no Dedekind's gapless property:
For example, if $p \in A$, then $p \in \mathbb{Q}$ and $p^2 < 2$, put $\epsilon = 2 - p^2$, then we should find a $q \in \mathbb{Q}$ such that $q^2 < 2$ and $q > p$, which means

$$p^2 < q^2 < 2$$

we consider there exists a function r of p, ϵ, such that $r > 0$ and $r \in \mathbb{Q}$, and put $q = p + r$, thus $q > p$ and $q \in \mathbb{Q}$, we now prove that $q^2 < 2$. Since

$$2 - q^2 = 2 - (p^2 + r^2 + 2pr) = \epsilon - r^2 - 2pr$$

We now try to find a suitable structure of r to make $r^2 + 2pr < \epsilon$. Since $p > 0$ and $\epsilon = 2 - p^2, 0 < \epsilon < 2$. Consider $r = \epsilon/2$ then

$$r^2 = \frac{\epsilon^2}{4} = \frac{\epsilon}{2} \cdot \frac{\epsilon}{2} < \frac{\epsilon}{2}.$$

Consider $r = \epsilon/((2p+1)2) < \epsilon/2$ and

$$2pr = 2p \cdot \frac{\epsilon}{(2p+1)2} < \frac{\epsilon}{2},$$

then we have $r^2 + 2pr < \epsilon$ and

$$q^2 < 2,$$

by defining

$$q = p + \frac{\epsilon}{2(2p+1)} = \frac{3p^2 + 2p + 2}{4p + 2},$$

thus there is no maximal element in A and correspondingly, there is no minimal element in B as well.
2. trivial. □

Theorem 1 (Weierstrass Theorem). *Let $\emptyset \neq S \subseteq \mathbb{R}$, if S has an upper bound, then* $\sup S$ *exists.*

Proof. Let B be the set of all upper bound of S, and define $A := \mathbb{R} \backslash B$.
Claim 1: (A, B) is a Dedekind cut of $\mathbb{R}$:

1. $S \neq \emptyset \Rightarrow \forall s \in S, s-1 \notin B \Rightarrow s-1 \in A \Rightarrow A \neq \emptyset$; And S has an upper bound $\Rightarrow B \neq \emptyset$;
2. $A = \mathbb{R} \backslash R \Rightarrow A \cup B = \mathbb{R}$;
3. If $\exists a \in A, b \in B,$ s.t. $a \geq b$ where b is an upper bound of S while a is not, thus $\exists s' \in S,$ s.t. $a < s' \leq b < a$, which leads to a contradiction. Thus $\forall a \in A, b \in B$ one has $a < b$.

Claim 2: $\min B$ exists:
If $\min B \nexists$, then by Dedekind's gapless property, $\max A \exists$, denoted by a_0. $a_0 \in A \Leftrightarrow a_0 \notin B \Leftrightarrow a_0$ is not an upper bound of $S \Leftrightarrow \exists s_0 \in S,$ s.t. $a_0 < s$. Choose $x \in \mathbb{R}$ such that $a_0 < x < s_0$, thus $\max A < x \Rightarrow x \in B \Rightarrow x$ is an upper bound of S but $x < s_0$ which leads to a contradiction. □

Exercise 2 (Archimedean Property)**.** *Show that* $\forall r \in \mathbb{R}, r > 0 \Rightarrow \exists n \in \mathbb{N},$ s.t. $n > r$. *(or say* $\exists n \in \mathbb{N},$ s.t. $1/n < r$*).*

Proof. Let $r \in \mathbb{R}$, $S := \{n \in \mathbb{N} | n \leq r\}$, since $r > 0, 0 \in S \Rightarrow S \neq \emptyset$. Then $S \subseteq \mathbb{R}$ and S is bounded above (by r), thus S has a least upper bound in $\mathbb{R}$, let $s = \sup S$.
Now consider the number $s-1$. Since s is the supremum of S, $s-1$ cannot be an upper bound of S by definition. Thus $\exists m \in S$ such that

$$m > s-1 \Rightarrow m+1 > s$$

But as $m \in \mathbb{N}$, it follows that $m+1 \in \mathbb{N}$. Because $m+1 > s$, it follows that $m+1 \notin S$ and so $m+1 > r$. Furthermore, for $\forall r > 0, 1/r > 0$ then $\exists n \in \mathbb{N},$ s.t. $n > 1/r \Rightarrow 1/n < r$. □

1.2 Sequence

Definition 3 (Convergence)**.** Let $a_n (n \in \mathbb{N})$ be a sequence in $\mathbb{R}$ and $l \in \mathbb{R}$, we say that a_n converges to l as $n \to \infty$ if $\forall \epsilon > 0, \exists N \in \mathbb{N},$ s.t. $\forall n \geq N, |a_n - l| < \epsilon$, denoted by $a_n \to l$ (as $n \to \infty$).

If such l exists, we call it the limit of $\{a_n\}$ and denote is as $\lim_{n\to\infty} a_n = l$, and call $\{a_n\}$ a convergent sequence; otherwise we call it a **divergent** sequence. Furthermore, we say $\lim_{n\to\infty} a_n = \infty$ if $\forall M > 0, \exists N \in \mathbb{N},$ s.t. $\forall n \geq N, a_n \geq M$.

Exercise 3. *Show that*
1. $\lim_{n\to\infty} a_n = l$ *and* $\lim_{n\to\infty} a_n = m \Rightarrow l = m$*;*
2. $a_n (n \in \mathbb{N})$ *is convergent* $\Rightarrow \{a_n | n \in \mathbb{N}\}$ *is bounded;*
3. *if* $a_n < b_n$ *for all* $n \in \mathbb{N}, \lim_{n\to\infty} a_n = l, \lim_{n\to\infty} b_n = m \Rightarrow l \leq m$*.*

Proof. 1. $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} a_n = m \Rightarrow$ for $\forall \epsilon > 0$, $\exists N, M \in \mathbb{N},$ s.t. $\forall n \geq N$

one has $|a_n - l| < \epsilon/2$ and $\forall n \geq M$ has $|a_n - m| < \epsilon/2$, thus for $\forall n \geq \max\{N, M\}$, has

$$|l - m| = |l - a_n + a_n - m| \leq |a_n - l| + |a_n - m| < \epsilon$$

holds for $\forall \epsilon > 0 \Rightarrow l = m$.
2. Suppose $a_n \to l$ as $n \to \infty$, then given an $\epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ we have $|a_n - l| < \epsilon \Rightarrow l - \epsilon < a_n < l + \epsilon$, thus a_n has upper bound

$$\max\{a_1, \cdots, a_{n-1}, l + \epsilon\},$$

and lower bound

$$\min\{a_1, \cdots, a_{n-1}, l - \epsilon\}.$$

3. if $l > m$, let $\epsilon = l - m$, then $\exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ has $|a_n - l| < \epsilon/2$ and $|b_n - m| < \epsilon/2$ thus

$$a_n < \frac{l+m}{2} < b_n,$$

which leads to a contradiction, thus $l \leq m$. □

Remark 2. Changing or removing finitely many terms in $a_n (n \in \mathbb{N})$ does not effect a_n's being convergent (and its limit)/ divergent.

Proposition 1. *If* $\lim_{n\to\infty} a_n = l$ *and* $\lim_{n\to\infty} b_n = m$ *then*
1. $\lim_{n\to\infty}(a_n \pm b_n) = l \pm m$;
2. $\lim_{n\to\infty} a_n b_n = lm$;
3. if $m \neq 0$ *and* $b_n \neq 0$ *for all but finitely many* n *then* $\lim_{n\to\infty} a_n/b_n = l/m$.

Proof. 1. For $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N, |a_n - l| \leq \epsilon/2$ and $\exists M \in \mathbb{N}$, s.t. $\forall n \geq M, |b_n - m| \leq \epsilon/2$, thus $\forall n \geq \max\{N, M\}$, one has

$$\begin{aligned} |(a_n \pm b_n) - (l \pm m)| &= |(a_n - l) \pm (b_m - m)| \\ &\leq |a_n - l| + |b_n - m| \\ &\leq \epsilon, \end{aligned}$$

thus $(a_n \pm b_n) \to l \pm m$ as $n \to \infty$.
2. Since a_n, b_n are convergent, thus they are bounded. Choose $C > 0$ such that $|b_n| \leq C$ for all $n \in \mathbb{N}$ and $|l| \leq C$, then for $\forall \epsilon > 0, \exists N, M \in \mathbb{N}$, s.t. $\forall n \geq N$ one has $|a_n - l| \leq \epsilon/(2C)$ and $\forall n \geq M$ has $|b_n - m| \leq \epsilon/(2C)$, thus $\forall n \geq \max\{N, M\}$ one has

$$\begin{aligned} |a_n b_n - lm| &= |a_n b_n - l b_n + l b_n - lm| \\ &\leq |a_n - l| \cdot |b_n| + |b_n - m| \cdot |l| \\ &\leq (|a_n - l| + |b_n - m|) \cdot |C| \\ &\leq \epsilon \end{aligned}$$

thus $a_n b_n \to lm$.
3. all we need to show is $\lim_{n\to\infty} 1/b_n = 1/m$ which is trivial. □

Exercise 4 (Squeeze theorem). *If* $\lim_{n\to\infty} a_n = l$ *and* $\lim_{n\to\infty} b_n = m$ *and* $a_n \le c_n \le b_n$, *show that* $l = m \Rightarrow \lim_{n\to\infty} c_n = l$.

Proof. Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = l$, for $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \ge N$ has $|a_n - l| < \epsilon/3$ and $|b_n - l| < \epsilon/3$. And since $a_n \le c_n \le b_n$, we have that $0 \le c_n - a_n \le b_n - a_n$. Thus for $\forall n \ge N$, we have

$$\begin{aligned}
|c_n - l| &= |c_n - a_n + a_n - l| \\
&\le |c_n - a_n| + |a_n - l| \\
&\le |b_n - a_n| + |a_n - l| \\
&= |b_n - l + l - a_n| + |a_n - l| \\
&\le |b_n - l| + 2|a_n - l| \\
&< \epsilon.
\end{aligned}$$

thus $\lim_{n\to\infty} c_n = l$. □

Exercise 5. *If* $a > 1$ *show that* $\lim_{n\to\infty} 1/a^n = 0$.

Proof. Since $a > 1 \Rightarrow b := a - 1 > 0$, thus

$$0 \le \frac{1}{a^n} = \frac{1}{(1+b)^n} \le \frac{1}{1+nb} \to 0$$

as $n \to \infty$, thus $\lim_{n\to\infty} 1/a^n = 0$ by Squeeze theorem. □

Definition 4. A seq. $a_n (n \in \mathbb{N})$ in $\mathbb{R}$ is

1. nondecreasing monotone/increasing if $a_n \le a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n\nearrow$; nonincreasing monotone/decreasing if $a_n \ge a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n\searrow$.
2. strictly increasing if $a_n < a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n\nearrow\nearrow$; strictly decreasing if $a_n > a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n\searrow\searrow$.

Theorem 2 (Monotone Seq. Property). *If* $a_n\nearrow$ *and* $\{a_n | n \in \mathbb{N}\}$ *has an upper bound, then* $\lim_{n\to\infty} a_n = \sup\{a_n | n \in \mathbb{N}\}$; $a_n\searrow$ *and* $\{a_n | n \in \mathbb{N}\}$ *has an lower bound, then* $\lim_{n\to\infty} a_n = \inf\{a_n | n \in \mathbb{N}\}$.

Proof. $\{a_n | n \in \mathbb{N}\}$ has an upper bound $\Rightarrow l := \sup\{a_n | n \in \mathbb{N}\}$ exists by Weierstrass theorem. Thus for $\forall \epsilon > 0, l - \epsilon$ is not an upper bound of $\{a_n\}$, then $\exists N \in \mathbb{N}$, s.t. $a_N > l$ and since $a_n\nearrow$, we have that $\forall n \ge N, l - \epsilon < a_n \le l \Rightarrow \lim_{n\to\infty} a_n = l$. □

Example 1 (Decimal expression gives real number). Suppose $d_i \in \mathbb{N}$ and $0 \le d_i \le 9$ for $i \in \mathbb{N}$, and define

$$a_n = \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

for $n \in \mathbb{N}$, then it is direct to see that $a_n \nearrow$ and

$$\begin{aligned} a_n &\leq \frac{9}{10} + \frac{9}{10^2} + \cdots \frac{9}{10^n} \\ &= \frac{9}{10}\left(\frac{1-\frac{1}{10^{n+1}}}{1-\frac{1}{10}}\right) \\ &< \frac{9}{10}\left(\frac{1}{1-\frac{1}{10}}\right) \\ &= 1 \end{aligned}$$

and hence $\lim_{n\to\infty} a_n$ exists, and we can define a real number by $\lim_{n\to\infty} a_n =: 0.d_1 d_2 \cdots$

Example 2 (The natural base e)**.** Define a seq. $a_n = (1+1/n)^n (n \in \mathbb{N})$, then we have

$$\begin{aligned} a_n &= \left(1+\frac{1}{n}\right)^n \\ &= \binom{n}{0} + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{n}\frac{1}{n^n} \\ &= \sum_{j=0}^{n} \binom{n}{j}\frac{1}{n^j} = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \cdot \frac{1}{n^j} \\ &= \sum_{j=0}^{n} \frac{1}{j!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{j-1}{n}\right) \\ &< \sum_{j=0}^{n+1} \frac{1}{j!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{j-1}{n+1}\right) \end{aligned}$$

Thus $a_n \nearrow\nearrow$. On the other hand, for $\forall n \in \mathbb{N}$, we have

$$\begin{aligned} a_n &< \sum_{j=0}^{n+1} \frac{1}{j!} = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} \\ &< 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} \\ &= 1 + \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} \\ &< 3 \end{aligned}$$

Thus a_n has an upper bound and hence a_n converges, and we define $\lim_{n\to\infty} a_n =: e$.

1.3 Nested Intervals

Definition 5 (Nested)**.** A seq. of intervals $I_n (n \in \mathbb{N})$ is nested if $I_n \neq \emptyset$ and $I_{n+1} \subseteq I_n$ for $\forall n \in \mathbb{N}$.

Example 3. If we have a seq. of nested intervals $I_n(n \in \mathbb{N})$, do we have $\cap_{n\in\mathbb{N}} I_n \neq \emptyset$? The answer is not sure. For example,

1. $I_n = (0, 1/n), n \in \mathbb{N}$, then for any $r \in \mathbb{R}, \exists N \in \mathbb{N}$, s.t. $1/N < r$ by Archimedean Property, thus $r \notin I_N$, and hence $\cap_{n\in\mathbb{N}} I_n = \emptyset$;
2. $I_n = [n, \infty), n \in \mathbb{N}$, then for any $r \in \mathbb{R}, \exists N \in \mathbb{N}$, s.t. $r < N$ by Archimedean Property, thus $r \notin I_N$, and hence $\cap_{n\in\mathbb{N}} I_n = \emptyset$;

Theorem 3 (Theorem of Nested Interval). *If $I_n(n \in \mathbb{N})$ is a seq. of bounded closed nested intervals, then $\cap_{n\in\mathbb{N}} I_n \neq \emptyset$. (In the other word, there exists a real number $c \in \mathbb{R}$ such that $c \in \cap_{n\in\mathbb{N}} I_n$)*

Proof. Write $I_n = [a_n, b_n](n \in \mathbb{N})$, then $I_n(n \in \mathbb{N})$ is nested $\Leftrightarrow a_n \leq b_n$ and $a_n \nearrow$ and $b_n \searrow$. And furthermore, for $\forall n, m \in \mathbb{N}$,

$$a_n \leq a_{\max\{m,n\}} \leq b_{\max\{m,n\}} \leq b_m,$$

in the other word, for $\forall m \in \mathbb{N}$, b_m is an upper bound of $\{a_n | n \in \mathbb{N}\}$, thus seq. a_n converges. Let $c = \lim_{n\to\infty} a_n$, then given $m \in \mathbb{N}$, for $\forall n \in \mathbb{N}, a_n \leq b_m$ thus

$$c = \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_m = b_m.$$

On the other hand, $c = \sup\{a_n | n \in \mathbb{N}\}$, thus for all $m \in \mathbb{N}$, we have

$$a_m \leq c \leq b_m$$

thus $c \in I_m$ for $\forall m \in \mathbb{N} \Rightarrow c \in \cap_{n\in\mathbb{N}} I_n$. □

Exercise 6. *Show that $\cap_{n\in\mathbb{N}} I_n \neq \emptyset$, if*

1. *$I_n = (a_n, b_n)$, nested and $a_n \nearrow\nearrow$ and $b_n \searrow\searrow$?*
2. *$I_n = (a_n, \infty)$, nested and $\{a_n | n \in \mathbb{N}\}$ is bounded from above.*

Proof. 1. Just as analyzed before, there exist $c \in \mathbb{R}$ such that $c = \lim_{n\to\infty} a_n$, and $c = \sup\{a_n | n \in \mathbb{N}\}$ and hence $a_n \leq c \leq b_m$ for $\forall n, m \in \mathbb{N}$. Note that $a_n \leq c$ implies that $a_n < c$ for $\forall n \in N$, otherwise if $\exists n' \in \mathbb{N}$, s.t. $a_{n'} = c$ then

$$a_{n'+1} \geq a_{n'} = c,$$

which leads to the contradiction. In the same way $c \leq b_m$ implies that $c < b_m$ for $\forall m \in \mathbb{N}$. Thus there $\exists c \in \mathbb{R}$ such that

$$a_n < c < b_m$$

for $\forall n, m \in \mathbb{N} \Rightarrow c \in \cap_{n\in\mathbb{N}} I_n$.

2. Since $I_n = (a_n, \infty)$ is a nested interval, $a_n \nearrow \Rightarrow a_n$ converges since a_n is upper bounded. That is $\exists c \in \mathbb{R}$, s.t. $c = \lim_{n\to\infty} a_n = \sup\{a_n\}$, thus for $\forall n \in \mathbb{N}, c \geq a_n$, that is

$$c + 1 > c \geq a_n$$

for $\forall n \in \mathbb{N} \Rightarrow c + 1 \in \cap_{n\in\mathbb{N}} I_n$. $\square$

Exercise 7. *Show that Theorem of Nested Interval implies Dedekind's Gapless Property.*

Proof. Let (A, B) be a Dedekind cut of $\mathbb{R}$, pick a from A and b from B, and form an interval $I_0 = [a, b]$. Then $(a + b)/2$ lies in the middle of I_0 and must belong to A or B. If $(a + b)/2$ belongs to A, we let

$$a_1 = \frac{a+b}{2}, \quad b_1 = b$$

and if $(a + b)/2$ belongs to B, let

$$a_1 = a, \quad b_1 = \frac{a+b}{2}$$

and hence we can form a new interval $I_1 = [a_1, b_1]$ whose length is half of the former I_0. Repeat this process, we obtain a seq. of nested intervals

$$I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots,$$

where $I_n = [a_n, b_n], b_n - a_n = (b_{n-1} - a_{n-1})/2$. Thus there exists $s \in \mathbb{R}$ lies in the $\cap_{n\in\mathbb{N}} I_n$ by the theorem of nested intervals, and either $s \in A$ or $s \in B$.
Assume that $s \in A$, for any $s' \in \mathbb{R}, s < s'$, exists b_n such that $s < b_n < s'$ since $b_n \to s$, thus $s' \in B$. That is $s \in A$ and for any $s' > s, s' \in B$. In the other word, s is the maximal element of A and B has no minimal element in this case, since assume s' is the minimal element of B then $\exists b_n$, s.t. $b_n < s'$ and $b_n \in B$, which is a contradiction. $\square$

Remark 3. Summary, we have discussed
1) Dedekind's Gapless Property;
2) Weierstrass Theorem;
3) Monotone Seq. Property;
4) Theorem of Nested Interval.
which have the relationship:

$$\begin{array}{ccc} 1) & \Longrightarrow & 2) \\ \Uparrow & & \Downarrow \\ 4) & \Longleftarrow & 3) \end{array}$$

These 5 properties are equivalent and we call the these the **Completeness of the real numbers**.

1.4 Limit superior / inferior

Let $a_n (n \in \mathbb{N})$ be a bounded (upper bdd. and lower bdd.) seq. in $\mathbb{R}$, we define **upper seq. of** a_n as

$$u_n := \sup\{a_m | m \geq n\},$$

and **lower seq. of** a_n as

$$l_n := \inf\{a_m | m \geq n\},$$

for $n \in \mathbb{N}$. Thus give $n \in \mathbb{N}$, we have that for $\forall m \geq n$

$$l_n \leq a_m \leq u_n,$$

We now show that l_n and u_n is monotone. Assume that $\exists n \in \mathbb{N}$, s.t. $u_n < u_{n+1}$, let $\epsilon = (u_{n+1} - u_n)/2$, then

$$u_{n+1} - \epsilon > u_n = \sup\{a_m | m \geq n\},$$

thus for $\forall m \geq n$, $u_{n+1} - \epsilon > a_m$ and hence $u_{n+1} - \epsilon$ is an upper bound of $\{a_m | m \geq n+1\}$, which leads to a contradiction. Thus for $\forall n \in \mathbb{N}, u_n \geq u_{n+1} \Rightarrow u_n \searrow$, and $l_n \nearrow$ in the same way.
Thus we have that for any $n, m \in \mathbb{N}$,

$$l_m \leq l_{\max\{m,n\}} \leq u_{\max\{m,n\}} \leq u_n,$$

thus l_1 is a lower bound for $\{u_n | n \in \mathbb{N}\}$ and u_1 is an upper bound of $\{l_n | n \in \mathbb{N}\}$ and hence $u_n, l_n (n \in \mathbb{N})$ are convergent by Monotone seq. property. We define the **limit superior** of a_n as the limit of u_n:

$$\overline{\lim_{n\to\infty}} a_n := \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sup_{m\geq n} a_m = \inf_{n\in\mathbb{N}} \sup_{m\geq n} a_m$$

The last equals sign is because $u_n \searrow$ and hence converges to its greatest lower bound by Monotone seq. property. And the **limit inferior** of a_n as the limit of l_n:

$$\underline{\lim_{n\to\infty}} a_n := \lim_{n\to\infty} l_n = \lim_{n\to\infty} \inf_{m\geq n} a_m = \sup_{n\in\mathbb{N}} \inf_{m\geq n} a_m$$

Exercise 8. *Let $a_n (n \in \mathbb{N})$, show that*

$$a_n \text{ converges} \Leftrightarrow \overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n$$

and if any of both sides holds, then

$$\lim_{n\to\infty} a_n = \overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n$$

Proof. $\Rightarrow$: Suppose that $\lim_{n\to\infty} a_n = s$. Then for any $\epsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $\forall n \geq N$, $|a_n - s| < \epsilon/2$, thus $s - \epsilon/2 < a_n < s + \epsilon/2$ for $\forall n \geq N$. Thus the upper seq. u_n of a_n has
$$s - \frac{\epsilon}{2} < a_n \leq u_n \leq s + \frac{\epsilon}{2},$$
for $\forall n \geq N$. The third inequality symbol is because if $\exists n' \geq N$ such that $u_{n'} > s + \epsilon/2$, then there exist a real number q such that $s + \epsilon/2 < q < u_{n'}$ and $q > s + \epsilon/2 > a_n$ for $\forall n \geq N$ and hence $q > a_n$ for $\forall n \geq n'$, and then $u_{n'}$ is not the least upper bound of $\{a_n | n \geq n'\}$ which is contrary. Thus $|u_n - s| \leq \epsilon/2 < \epsilon$, thus
$$\lim_{n\to\infty} u_n = \overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} a_n = s,$$
and $\lim_{n\to\infty} l_n = \underline{\lim}_{n\to\infty}\, a_n = \lim_{n\to\infty} a_n = s$ in the same way.
$\Leftarrow$: Suppose $\lim_{n\to\infty} u_n = \lim_{n\to\infty} l_n = s$, then for $\forall\epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ one has $|u_n - s| < \epsilon/3$ and $|l_n - s| < \epsilon/3$ and $|u_n - l_n| \leq |u_n - s| + |l_n - s| < 2\epsilon/3$, since $l_n \leq a_n \leq u_n$ then $0 \leq a_n - l_n \leq u_n - l_n$. Then we have that
$$\begin{aligned} |a_n - s| &= |a_n - l_n + l_n - s| \\ &\leq |a_n - l_n| + |l_n - s| \\ &\leq |u_n - l_n| + |l_n - s| \\ &< \epsilon \end{aligned}$$
for $\forall n \geq N \Rightarrow \lim_{n\to\infty} a_n = \overline{\lim}_{n\to\infty}\, a_n = \underline{\lim}_{n\to\infty}\, a_n = s$. $\square$

Exercise 9. *Let $a_n, b_n (n \in \mathbb{N})$ be two bdd. seq. show that*
1. $\overline{\lim}_{n\to\infty}(a_n + b_n) \leq \overline{\lim}_{n\to\infty}\, a_n + \overline{\lim}_{n\to\infty}\, b_n$;
2. $\underline{\lim}_{n\to\infty}\, a_n + \underline{\lim}_{n\to\infty}\, b_n \leq \underline{\lim}_{n\to\infty}(a_n + b_n)$.

1.5 Cauchy seq.

Given a seq. $a_n (n \in \mathbb{N})$ in $\mathbb{R}$, can we determine whether a_n converges or not without referring a limit candidate l, but concluding according to the mutual behavior of the terms of $a_n (n \in \mathbb{N})$?

Definition 6 (Cauchy Sequence)**.** A seq. $a_n (n \in \mathbb{N})$ in $\mathbb{R}$ is a Cauchy seq. if $\forall\epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n, m \geq N \Rightarrow |a_n - a_m| < \epsilon$.

Exercise 10. *Show that*
1. *a_n is convergent $\Rightarrow$ a_n is Cauchy seq.*
2. *a_n is Cauchy seq. $\Rightarrow$ a_n is bounded.*

Proof. 1. assume that a_n converges to l, then for any $\epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N$ one has $|a_n - l| < \epsilon/2$, then for any $m, n \geq N$ we have

$$|a_m - a_n| \leq |a_m - l| - |a_n - l| < \epsilon$$

thus $a_n (n \in \mathbb{N})$ is Cauchy seq.
2. For $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n, m \geq N$ one has $|a_m - a_n| \leq \epsilon$, thus for $\forall m \geq N \Rightarrow |a_N - a_m| \leq \epsilon \Rightarrow a_n - \epsilon \leq a_m \leq a_N + \epsilon$, thus $a_n (n \in N)$ has upper and lower bound

$$\max\{a_1, \cdots, a_N, a_N + \epsilon\}, \quad \min\{a_1, \cdots, a_N, a_N - \epsilon\},$$

thus a_n is bounded. □

Theorem 4. *Let $a_n (n \in \mathbb{N})$ be a seq. in $\mathbb{R}$, then a_n is convergent $\Leftrightarrow$ a_n is Cauchy seq.*

Proof. $\Leftarrow$: a_n is Cauchy seq. $\Rightarrow$ a_n is bdd. $\Rightarrow$ the upper/lower seq. u_n, l_n of a_n converges. Thus $\lim_{n\to\infty} u_n - \lim_{n\to\infty} l_n = \lim_{n\to\infty}(u_n - l_n)$. For $\forall \epsilon > 0$, there is some $N \in \mathbb{N}$ s.t. $\forall m, n \geq N, |a_n - a_m| < \epsilon/3$. In particular, $\forall n \geq N \Rightarrow |a_n - a_N| < \epsilon/3$ and hence

$$a_N - \frac{\epsilon}{3} < a_n < a_N + \frac{\epsilon}{3}$$

which means $a_N - \epsilon/3$ is a lower bound of $a_n (n \geq N)$ and is not greater that $\{a_n | n \geq N\}$'s greatest lower bound l_N, and the same to $a_N + \epsilon/3$, thus

$$a_N - \frac{\epsilon}{3} \leq l_N \leq u_N \leq a_N + \frac{\epsilon}{3}$$

and since $l_n \nearrow$ and $u_n \searrow$, we have that for $\forall n \geq N$

$$0 \leq u_n - l_n \leq u_N - l_N \leq \frac{2\epsilon}{3} < \epsilon$$

thus $\lim_{n\to\infty}(u_n - l_n) = 0 \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} l_n \Rightarrow a_n$ converges. □

Exercise 11. *Let $S \subseteq \mathbb{R}$, if $|s - s'| \leq 3$ for $\forall s, s' \in S$, show that*
1. *S is bdd.;*
2. $\sup S - \inf S \leq 3$;
3. *and how will* $\sup S - \inf S$ *be if* $|s - s'| < 3$?

Chapter 2

Series

2.1 Positive series

Definition 7. Let $a_n (n \in \mathbb{N})$ be a seq. in $\mathbb{R}$, we say that the series $\sum_n^\infty a_n$ (or $\sum_n a_n$) converges to a real number s if

$$\lim_{n\to\infty} s_n = s,$$

where $s_n := \sum_{j=1}^n a_j$ is called the n - th partial sum of $\sum_n a_n$.

If such s exists (resp. does not exist), we say that the series $\sum_n a_n$ convergent (resp. divergent). For a series $\sum_n a_n$ and $l, m \in \mathbb{N}, l < m$, we let $s_{l,m} := \sum_{j=l}^m a_j$ the (l, m) - tail of $\sum_n a_n$.

Exercise 12. *If a series $\sum_n a_n$ converges, show that $\lim_{n\to\infty} a_n = 0$.*

$\sum_n a_n$ converges $\Leftrightarrow$ s_n converges by definition and $\Leftrightarrow$ s_n is Cauchy seq., i.e. $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geq N$, (assume that $n > m$)

$$\begin{aligned}
|s_n - s_m| &= |a_{m+1} + \cdots + a_n| \\
&= |a_{m+1} + a_{m+2} + \cdots + a_{m+1+(n-1)}| \\
&\leq \epsilon.
\end{aligned}$$

In particular, for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ and for $\forall k > N, \forall l \geq 0$, then $|a_k| + \cdots + |a_{k+l}| < \epsilon \Leftrightarrow \sum_n |a_n|$ convergent $\Leftrightarrow \exists M > 0, \forall n \in \mathbb{N}, s_n = \sum_{j=1}^n a_j \leq M$, since $s_n \nearrow$. Collectively, we have some conclusions:

1. series $\sum_n a_n$ converges $\Leftrightarrow$ for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ and for $\forall k > N, \forall l \geq 0$, $|a_k + \cdots + a_{k+l}| < \epsilon$;
2. series $\sum_n b_n$, where $b_n \geq 0$, converges $\Leftrightarrow \exists M > 0, \forall n \in \mathbb{N}, \sum_{j=1}^n b_j \leq M$.
3. series $\sum_n |a_n|$ converges $\Rightarrow \sum_n a_n$ converges.

Example 4. Given series $\sum_n 1/n$. we have that

$$\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2} \\
s_8 &\geq 1 + \frac{2}{2} + \frac{1}{5} + \cdots + \frac{1}{8} \geq 1 + \frac{2}{2} + \frac{4}{8} = 1 + \frac{3}{2}
\end{aligned}$$

In general, for $\forall m \in \mathbb{N}, s_{2^m} \geq 1 + m/2$ which has no upper bound $\Leftrightarrow \sum_n 1/n$ diverges.

Example 5. Given series $\sum_n 1/n^2$. we have that $1/n^2 < 1/((n-1)n) = 1/(n-1) - 1/n$. Then

$$\begin{aligned}
s_n &= 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \\
&< 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} \\
&= 2 - \frac{1}{n} \\
&< 2.
\end{aligned}$$

thus s_n has upper bound 2 $\Leftrightarrow \sum_n 1/n^2$ converges.

Definition 8. Given a seq. $a_n (n \in \mathbb{N})$, we say that
1. $\sum_n a_n$ converges absolutely if $\sum_n |a_n|$ converges;
2. $\sum_n a_n$ converges conditionally if $\sum_n |a_n|$ diverges but $\sum_n a_n$ converges.

Theorem 5 (Comparison Test). *If $a_n, b_n \geq 0 (n \in \mathbb{N})$, then $\exists C > 0$ and $N \in \mathbb{N}, n \geq N \Rightarrow a_n \leq C b_n \Rightarrow [\sum_n b_n$ converges $\Rightarrow \sum_n a_n$ converges].*

Proof. If $\sum_n b_n$ converges, then for $\forall n \geq N$,

$$\begin{aligned}
a_1 + \cdots + a_n &= a_1 + \cdots + a_N + a_{N+1} + \cdots + a_n \\
&\leq a_1 + \cdots + a_N + C \cdot (b_{N+1} + \cdots + b_n) \\
&\leq a_1 + \cdots + a_N + C \cdot M =: H,
\end{aligned}$$

where M is an upper bound of $\sum_{j=1}^n b_j$, thus $\sum_j^n a_j$ as upper bound $H \Leftrightarrow \sum_n a_n$ converges. □

Theorem 6 (Limit Form of Comparison Test). *If $a_n, b_n \geq 0 (n \in \mathbb{N})$, and if $\lim_{n \to \infty} a_n / b_n$ exists $\Rightarrow [\sum_n b_n$ converges $\Rightarrow \sum_n a_n$ converges].*

Proof. Let $l = \lim_{n \to \infty} a_n / b_n$, then for $\epsilon = 1, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N, a_n / b_m < l + 1 \Rightarrow a_n < (l+1) b_n$, which follows the proof by Comparison test. Furthermore if $l \neq 0$, then for $\epsilon = l/2, \exists N_l \in \mathbb{N}$, s.t. $\forall n \geq N_l$, s.t.

$$\frac{l}{2} \leq \frac{a_n}{b_n} \leq \frac{3l}{2},$$

and hence $b_n \leq a_n \cdot 2/l$ and $a_n \leq b_n \cdot 3l/2$, therefore $\sum_n b_n$ converges $\Leftrightarrow \sum_n a_n$ converges. □

Exercise 13. *If $a_n, b_n \geq 0 (n \in \mathbb{N})$, show that if $\overline{\lim}_{n\to\infty} a_n/b_n$ exists $\Rightarrow$ [$\sum_n b_n$ converges $\Rightarrow \sum_n a_n$ converges].*

Exercise 14 (Ratio and Root test). *If $a_n, b_n \geq 0 (n \in \mathbb{N})$, show that*
1. $\lim_{n\to\infty} a_{n+1}/a_n < 1 \Rightarrow \sum_n a_n$ *converges;* $\lim_{n\to\infty} a_{n+1}/a_n > 1 \Rightarrow \sum_n a_n$ *diverges.*
2. $\lim_{n\to\infty} (a_n)^{1/n} < 1 \Rightarrow \sum_n a_n$ *converges;* $\lim_{n\to\infty} (a_n)^{1/n} > 1 \Rightarrow \sum_n a_n$ *diverges.*

2.2 Alternating series

Definition 9. A series $\sum_n a_n$ is called alternating series, if $\exists b_n > 0 (n \in \mathbb{N})$ s.t. $a_n = (-1)^{n-1} b_n (n \in \mathbb{N})$.

Theorem 7 (Leibniz's Criterion). *Let $\sum_n a_n$ be an alternating series, and $b_n = |a_n|_{\searrow 0}$ as $n \to \infty$, then $\sum_n a_n$ converges.*

Proof. Since $b_n = (-1)^{n-1} a_n$, for any $k, l \in \mathbb{N}$ the tail of $\sum_n a_n$ is

$$\begin{aligned} |a_k + \cdots + a_{k+l}| &= (-1)^{k-1} |b_k - b_{k+1} + \cdots + (-1)^l b_{k+l}| \\ &= |b_k - b_{k+1} + \cdots + (-1)^l b_{k+l}| \end{aligned}$$

define $\lambda_{k,l} = b_k - b_{k+1} + \cdots + (-1)^l b_{k+l}$. Then if l is even, then

$$\lambda_{k,l} = (b_k - b_{k+1}) + \cdots + (b_{k+l-2} - b_{k+l-1}) + b_{k+l} \geq 0,$$

and if l is odd, then

$$\lambda_{k,l} = (b_k - b_{k+1}) + \cdots + (b_{k+l-1} - b_{k+l}) \geq 0,$$

thus $\lambda_{k,l} \geq 0$ for $\forall k, l \in \mathbb{N}$. And hence

$$\begin{aligned} |a_k + \cdots + a_{k+l}| = |\lambda_{k,l}| &= \lambda_{k,l} \\ &= \begin{cases} b_k - (b_{k+1} - b_{k+2}) - \cdots - (b_{k+l-1} - b_{k+l}), & l \text{ is even} \\ b_k - (b_{k+1} - b_{k+2}) - \cdots - (b_{k+l-2} - b_{k+k-1}) - b_{k+l}, & l \text{ is odd} \end{cases} \\ &\leq b_k \end{aligned}$$

Since $\lim_{n\to\infty} b_n = 0 \Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $n \geq N \Rightarrow |b_n| < \epsilon \Rightarrow |a_n + \cdots + a_{n+l}| \leq b_n < \epsilon$ for $\forall l \in \mathbb{N}$, thus $\sum_n a_n$ converges. □