Introduction to Analysis
Lecture 3

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Abstract

This is the Lecture note for the *Introduction to Analysis* class in Fall 2019.

1 Cauchy seq.

Given a seq. $a_n(n \in \mathbb{N})$ in $\mathbb{R}$, can we determine whether a_n converges or not without referring a limit candidate l, but concluding according to the mutual behavior of the terms of $a_n(n \in \mathbb{N})$?

Definition 1 (Cauchy Sequence)**.** A seq. $a_n(n \in \mathbb{N})$ in $\mathbb{R}$ is a Cauchy seq. if $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n, m \geq N \Rightarrow |a_n - a_m| < \epsilon$.

Exercise 1. *Show that*
1. a_n is convergent $\Rightarrow$ a_n is Cauchy seq.
2. a_n is Cauchy seq. $\Rightarrow$ a_n is bounded.

Proof. 1. assume that a_n converges to l, then for any $\epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N$ one has $|a_n - l| < \epsilon/2$, then for any $m, n \geq N$ we have

$$|a_m - a_n| \leq |a_m - l| - |a_n - l| < \epsilon$$

thus $a_n(n \in \mathbb{N})$ is Cauchy seq.
2. For $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n, m \geq N$ one has $|a_m - a_n| \leq \epsilon$, thus for $\forall m \geq N \Rightarrow |a_N - a_m| \leq \epsilon \Rightarrow a_n - \epsilon \leq a_m \leq a_N + \epsilon$, thus $a_n(n \in N)$ has upper and lower bound

$$\max\{a_1, \cdots, a_N, a_N + \epsilon\}, \quad \min\{a_1, \cdots, a_N, a_N - \epsilon\},$$

thus a_n is bounded. □

Theorem 1. *Let $a_n(n \in \mathbb{N})$ be a seq. in $\mathbb{R}$, then a_n is convergent $\Leftrightarrow$ a_n is Cauchy seq.*

Proof. $\Leftarrow$: a_n is Cauchy seq. $\Rightarrow$ a_n is bdd. $\Rightarrow$ the upper/lower seq. u_n, l_n of a_n converges. Thus $\lim_{n\to\infty} u_n - \lim_{n\to\infty} l_n = \lim_{n\to\infty}(u_n - l_n)$. For $\forall \epsilon > 0$, there is some $N \in \mathbb{N}$ s.t. $\forall m, n \geq N, |a_n - a_m| < \epsilon/3$. In particular, $\forall n \geq N \Rightarrow |a_n - a_N| < \epsilon/3$ and hence

$$a_N - \frac{\epsilon}{3} < a_n < a_N + \frac{\epsilon}{3}$$

which means $a_N - \epsilon/3$ is a lower bound of $a_n (n \geq N)$ and is not greater that $\{a_n | n \geq N\}$'s greatest lower bound l_N, and the same to $a_N + \epsilon/3$, thus

$$a_N - \frac{\epsilon}{3} \leq l_N \leq u_N \leq a_N + \frac{\epsilon}{3}$$

and since $l_n \nearrow$ and $u_n \searrow$, we have that for $\forall n \geq N$

$$0 \leq u_n - l_n \leq u_N - l_N \leq \frac{2\epsilon}{3} < \epsilon$$

thus $\lim_{n\to\infty}(u_n - l_n) = 0 \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} l_n \Rightarrow a_n$ converges. $\square$

Exercise 2. *Let $S \subseteq \mathbb{R}$, if $|s - s'| \leq 3$ for $\forall s, s' \in S$, show that*
1. *S is bdd.;*
2. *$\sup S - \inf S \leq 3$;*

Proof. 1. If S has no upper bound, then for any $s \in S$, define $M = s + 4$, then $\exists s' \in S$ s.t. $s' > M = s + 4 \Rightarrow s' - s = |s' - s| > 4$, which is contrary.
2. Let $u = \sup S, l = \inf S$, suppose $u - l > 3$, then let $\epsilon = u - l - 3$, we have that $\exists s \in S$, s.t.

$$u - \frac{\epsilon}{3} < s \leq u,$$

and $\exists s' \in S$ s.t.

$$l \leq s' < l + \frac{\epsilon}{3},$$

then

$$u - l - \frac{2\epsilon}{3} < s - s' \leq u - l$$

thus $3 + \epsilon/3 < s - s' = |s - s'| \leq 3 + \epsilon \Rightarrow |s - s'| > 3$, which is contrary. $\square$

2 Positive series

Definition 2. Let $a_n (n \in \mathbb{N})$ be a seq. in $\mathbb{R}$, we say that the series $\sum_n^\infty a_n$ (or $\sum_n a_n$) converges to a real number s if

$$\lim_{n\to\infty} s_n = s,$$

where $s_n := \sum_{j=1}^n a_j$ is called the n - th partial sum of $\sum_n a_n$.

If such s exists (resp. does not exist), we say that the series $\sum_n a_n$ convergent (resp. divergent). For a series $\sum_n a_n$ and $l, m \in \mathbb{N}, l < m$, we let $s_{l,m} := \sum_{j=l}^{m} a_j$ the (l, m) - tail of $\sum_n a_n$.

Exercise 3. *If a series $\sum_n a_n$ converges, show that $\lim_{n\to\infty} a_n = 0$.*

Proof. Trivial. $\square$

$\sum_n a_n$ converges $\Leftrightarrow$ s_n converges by definition and $\Leftrightarrow$ s_n is Cauchy seq., i.e. $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geq N$, (assume that $n > m$)

$$\begin{aligned}
|s_n - s_m| &= |a_{m+1} + \cdots + a_n| \\
&= |a_{m+1} + a_{m+2} + \cdots + a_{m+1+(n-1)}| \\
&\leq \epsilon.
\end{aligned}$$

In particular, for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ and for $\forall k > N, \forall l \geq 0$, then $|a_k| + \cdots + |a_{k+l}| < \epsilon \Leftrightarrow \sum_n |a_n|$ convergent $\Leftrightarrow \exists M > 0, \forall n \in \mathbb{N}, s_n = \sum_{j=1}^{n} a_j \leq M$, since $s_n \nearrow$. Collectively, we have some conclusions:

1. series $\sum_n a_n$ converges $\Leftrightarrow$ for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ and for $\forall k > N, \forall l \geq 0$, $|a_k + \cdots + a_{k+l}| < \epsilon$;
2. series $\sum_n b_n$, where $b_n \geq 0$, converges $\Leftrightarrow \exists M > 0, \forall n \in \mathbb{N}, \sum_{j=1}^{n} b_j \leq M$.
3. series $\sum_n |a_n|$ converges $\Rightarrow \sum_n a_n$ converges.

Example 1. Given series $\sum_n 1/n$. we have that

$$\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2} \\
s_8 &\geq 1 + \frac{2}{2} + \frac{1}{5} + \cdots + \frac{1}{8} \geq 1 + \frac{2}{2} + \frac{4}{8} = 1 + \frac{3}{2}
\end{aligned}$$

In general, for $\forall m \in \mathbb{N}, s_{2^m} \geq 1 + m/2$ which has no upper bound $\Leftrightarrow \sum_n 1/n$ diverges.

Example 2. Given series $\sum_n 1/n^2$. we have that $1/n^2 < 1/((n-1)n) = 1/(n-1) - 1/n$. Then

$$\begin{aligned}
s_n &= 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \\
&< 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} \\
&= 2 - \frac{1}{n} \\
&< 2.
\end{aligned}$$

thus s_n has upper bound 2 $\Leftrightarrow \sum_n 1/n^2$ converges.

Definition 3. Given a seq. $a_n(n \in \mathbb{N})$, we say that
1. $\sum_n a_n$ converges absolutely if $\sum_n |a_n|$ converges;
2. $\sum_n a_n$ converges conditionally if $\sum_n |a_n|$ diverges but $\sum_n a_n$ converges.

Theorem 2 (Comparison Test). *If $a_n, b_n \geq 0 (n \in \mathbb{N})$, then $\exists C > 0$ and $N \in \mathbb{N}, n \geq N \Rightarrow a_n \leq Cb_n \Rightarrow [\sum_n b_n$ converges $\Rightarrow \sum_n a_n$ converges$]$.*

Proof. If $\sum_n b_n$ converges, then for $\forall n \geq N$,

$$\begin{aligned} a_1 + \cdots + a_n &= a_1 + \cdots + a_N + a_{N+1} + \cdots + a_n \\ &\leq a_1 + \cdots + a_N + C \cdot (b_{N+1} + \cdots + b_n) \\ &\leq a_1 + \cdots + a_N + C \cdot M =: H, \end{aligned}$$

where M is an upper bound of $\sum_{j=1}^n b_j$, thus $\sum_j^n a_j$ as upper bound $H \Leftrightarrow \sum_n a_n$ converges. □

Theorem 3 (Limit Form of Comparison Test). *If $a_n, b_n \geq 0 (n \in \mathbb{N})$, and if $\lim_{n\to\infty} a_n/b_n$ exists $\Rightarrow [\sum_n b_n$ converges $\Rightarrow \sum_n a_n$ converges$]$.*

Proof. Let $l = \lim_{n\to\infty} a_n/b_n$, then for $\epsilon = 1, \exists N \in \mathbb{N},$ s.t. $\forall n \geq N, a_n/b_m < l + 1 \Rightarrow a_n < (l+1)b_n$, which follows the proof by Comparison test. Furthermore if $l \neq 0$, then for $\epsilon = l/2, \exists N_l \in \mathbb{N},$ s.t. $\forall n \geq N_l$, s.t.

$$\frac{l}{2} \leq \frac{a_n}{b_n} \leq \frac{3l}{2},$$

and hence $b_n \leq a_n \cdot 2/l$ and $a_n \leq b_n \cdot 3l/2$, therefore $\sum_n b_n$ converges $\Leftrightarrow \sum_n a_n$ converges. □

Exercise 4. *If $a_n, b_n \geq 0 (n \in \mathbb{N})$, show that if $\overline{\lim}_{n\to\infty} a_n/b_n$ exists $\Rightarrow [\sum_n b_n$ converges $\Rightarrow \sum_n a_n$ converges$]$.*

Proof. Let $u_n = a_n/b_n$ and $\lim_{n\to\infty} u_n = l$, then $l = \inf_{n\in\mathbb{N}} u_n$ and for $\epsilon = 1, \exists n' \in \mathbb{N}$ s.t.

$$l \leq u_{n'} < l + 1$$

and hence for $\forall n \geq n'$ we have that

$$\frac{a_n}{b_n} \leq u_{n'} < l + 1$$

thus $a_n < (l+1) \cdot b_n$ for $\forall n \geq n'$ and finish the proof by comparison test. □

Exercise 5 (Ratio and Root test). *If $a_n, b_n \geq 0 (n \in \mathbb{N})$, show that*
1. *$\lim_{n\to\infty} a_{n+1}/a_n < 1 \Rightarrow \sum_n a_n$ converges; $\lim_{n\to\infty} a_{n+1}/a_n > 1 \Rightarrow \sum_n a_n$ diverges.*
2. *$\lim_{n\to\infty} (a_n)^{1/n} < 1 \Rightarrow \sum_n a_n$ converges; $\lim_{n\to\infty} (a_n)^{1/n} > 1 \Rightarrow \sum_n a_n$ diverges.*

Proof. Trivial. □

3 Alternating series

Definition 4. A series $\sum_n a_n$ is called alternating series, if $\exists b_n > 0 (n \in \mathbb{N})$ s.t. $a_n = (-1)^{n-1} b_n (n \in \mathbb{N})$.

Theorem 4 (Leibniz's Criterion). *Let $\sum_n a_n$ be an alternating series, and $b_n = |a_n|_{\searrow 0}$ as $n \to \infty$, then $\sum_n a_n$ converges.*

Proof. Since $b_n = (-1)^{n-1} a_n$, for any $k, l \in \mathbb{N}$ the tail of $\sum_n a_n$ is

$$\begin{aligned} |a_k + \cdots + a_{k+l}| &= (-1)^{k-1} |b_k - b_{k+1} + \cdots + (-1)^l b_{k+l}| \\ &= |b_k - b_{k+1} + \cdots + (-1)^l b_{k+l}| \end{aligned}$$

define $\lambda_{k,l} = b_k - b_{k+1} + \cdots + (-1)^l b_{k+l}$. Then if l is even, then

$$\lambda_{k,l} = (b_k - b_{k+1}) + \cdots + (b_{k+l-2} - b_{k+l-1}) + b_{k+l} \geq 0,$$

and if l is odd, then

$$\lambda_{k,l} = (b_k - b_{k+1}) + \cdots + (b_{k+l-1} - b_{k+l}) \geq 0,$$

thus $\lambda_{k,l} \geq 0$ for $\forall k, l \in \mathbb{N}$. And hence

$$\begin{aligned} |a_k + \cdots + a_{k+l}| = |\lambda_{k,l}| = \lambda_{k,l} \\ &= \begin{cases} b_k - (b_{k+1} - b_{k+2}) - \cdots - (b_{k+l-1} - b_{k+l}), & l \text{ is even} \\ b_k - (b_{k+1} - b_{k+2}) - \cdots - (b_{k+l-2} - b_{k+k-1}) - b_{k+l}, & l \text{ is odd} \end{cases} \\ &\leq b_k \end{aligned}$$

Since $\lim_{n\to\infty} b_n = 0 \Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N},$ s.t. $n \geq N \Rightarrow |b_n| < \epsilon \Rightarrow |a_n + \cdots + a_{n+l}| \leq b_n < \epsilon$ for $\forall l \in \mathbb{N}$, thus $\sum_n a_n$ converges. $\square$