Introduction to Analysis
Lecture 8

Haoming Wang

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Abstract

This is the Lecture note for the *Introduction to Analysis* class in Spring 2019.

0.1 Properties of Darboux Integral

The Monotonicity (P1), (P5) and (P6) of Darboux integral is trivial, we will show that Darboux integral has linear property (P2):

Proposition 1. *Let f, g be bounded functions on $[a, b]$, then*

$$\overline{\int_a^b} f + g \leq \overline{\int_a^b} f + \overline{\int_a^b} g, \quad \underline{\int_a^b} f + g \geq \underline{\int_a^b} f + \underline{\int_a^b} g.$$

Proof. Since $\sup_X (f + g) \leq \sup_X f + \sup_X g$ (Exercise **??**), then

$$\begin{aligned}
\overline{S}(f + g, \Delta) &= \sum_{j=1}^{k} (\sup_{I_j} f + g) \cdot (x_j - x_{j-1}) \\
&\leq \sum_{j=1}^{k} (\sup_{I_j} f + \sup_{I_j} g) \cdot (x_j - x_{j-1}) \\
&= \sum_{j=1}^{k} \sup_{I_j} f \cdot (x_j - x_{j-1}) + \sum_{j=1}^{k} \sup_{I_j} f \cdot (x_j - x_{j-1}) \\
&= \overline{S}(f, \Delta) + \overline{S}(g, \Delta).
\end{aligned}$$

And for $\forall \epsilon > 0$, $\exists \Delta_1, \Delta_2$ (by Remark **??** (E1)) s.t.

$$\begin{aligned}
\overline{S}(f, \Delta_1 \cup \Delta_2) \leq \overline{S}(f, \Delta_1) < \overline{\int_a^b} f + \epsilon, \\
\overline{S}(g, \Delta_1 \cup \Delta_2) \leq \overline{S}(g, \Delta_2) < \overline{\int_a^b} g + \epsilon.
\end{aligned}$$

and

$$\overline{\int_a^b} f+g \leq \overline{S}(f+g,\Delta_1\cup\Delta_2)$$
$$\leq \overline{S}(f,\Delta_1\cup\Delta_2)+\overline{S}(g,\Delta_1\cup\Delta_2)$$
$$< \overline{\int_a^b} f + \overline{\int_a^b} g + 2\epsilon$$

Thus

$$\overline{\int_a^b} f+g < \overline{\int_a^b} f + \overline{\int_a^b} g + 2\epsilon$$

for $\forall\epsilon>0 \Rightarrow$

$$\overline{\int_a^b} f+g \leq \overline{\int_a^b} f + \overline{\int_a^b} g.$$

$\square$

Therefore if f, g are Darboux integrable on $[a,b]$, then $f+g$ is too, and

$$\int_a^b f+g = \int_a^b f + \int_a^b g.$$

And for $\alpha\in\mathbb{R}$, we have

$$\overline{\int_a^b}\alpha f = \begin{cases} \alpha \overline{\int_a^b} f, & \alpha\geq 0 \\ \alpha \underline{\int_a^b} f, & \alpha\leq 0 \end{cases}, \quad \underline{\int_a^b}\alpha f = \begin{cases} \alpha \underline{\int_a^b} f, & \alpha\geq 0 \\ \alpha \overline{\int_a^b} f, & \alpha\leq 0 \end{cases}$$

Thus Darboux integral has linear property (P2).

Exercise 1 (P7)**.** *If f is Darboux integrable on $[a,b]$, then $|f|$ is too, and*

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Proof. For any subinterval I of $[a,b]$, there are 3 cases:
1. If $\inf_I f\geq 0$, then $f\geq 0$ on I so $\inf_I |f| = \inf_I f$ and $\sup_I |f| = \sup_I f$ and hence

$$\sup_I |f| - \inf_I |f| = \sup_I f - \inf_I f.$$

2. If $\sup_I f\leq 0$, then $f\leq 0$ on I, so $\inf_I |f| = -\sup_I f$ and $\sup_I |f| = -\inf_I f$ and hence

$$\sup_I |f| - \inf_I |f| = \sup_I f - \inf_I f.$$

3. If $\inf_I f < 0 < \sup_I f$, then we have either $\sup_I |f| = \sup_I f$, in which case $\sup_I |f| - \inf_I |f| \leq \sup_I |f| = \sup_I f < \sup_I f - \inf_I f$; or $\sup_I |f| = -\inf_I f$, in which case

$$\sup_I |f| - \inf_I |f| \leq -\inf_I f < \sup_I f - \inf_I f.$$

Then for any $\epsilon > 0, \exists\Delta$ s.t.

$$\begin{aligned}
0 &\leq \overline{S}(|f|,\Delta) - \underline{S}(|f|,\Delta) \\
&= \sum_{j=1}^{k} (\sup_{I_j} |f| - \inf_{I_j} |f|) \cdot (x_j - x_{j-1}) \\
&\leq \sum_{j=1}^{k} (\sup_{I_j} f - \inf_{I_j} f) \cdot (x_j - x_{j-1}) \\
&= \overline{S}(f,\Delta) - \underline{S}(f,\Delta) \\
&< \epsilon
\end{aligned}$$

thus $|f|$ is Darboux integrable. □

Proposition 2. *Let f be Darboux integrable on $[a,b], c \in (a,b)$, then*

$$\overline{\int_a^c} f + \overline{\int_c^b} f \leq \overline{\int_a^b} f, \quad \underline{\int_a^c} f + \underline{\int_c^b} f \geq \underline{\int_a^b} f.$$

Proof. Let Δ_1, Δ_2 be partitions of $[a,c], [c,b]$ respectively, then

$$\overline{S}(f,\Delta_1) + \overline{S}(f,\Delta_2) = \overline{S}(f,\Delta_1 \cup \Delta_2),$$

Let Δ be a partition of $[a,b]$, and define $\Delta_c = (\Delta \cap [a,c]) \cup \{c\}$ and ${}_c\Delta = (\Delta \cap [c,b]) \cup \{c\}$, then

$$\overline{S}(f,\Delta_c) + \overline{S}(f,{}_c\Delta) = \overline{S}(f,\Delta \cup \{c\}) \leq \overline{S}(f,\Delta)$$

thus $\overline{\int_a^c} f + \overline{\int_c^b} f \leq \overline{\int_a^b} f$. □

Thus if f is Darboux integrable on $[a,b], c \in (a,b)$, then it is Darboux on $[a,c]$ and $[c,b]$ and

$$\int_a^b f = \int_c^b f + \int_a^c f. \tag{P3}$$

Proposition 3 (P4)**.** *f is continuous on $[a,b] \Rightarrow f$ is Darboux integrable on $[a,b]$.*

Proof. $[a,b]$ is a compact set in $\mathbb{R}$ (Heine-Borel theorem, Theorem **??**), thus f is continuous on compact $\Rightarrow f$ is uniformly continuous on $[a,b]$ (Theorem **??**). Thus for any $\epsilon > 0, \exists\delta > 0,$ s.t. $\forall |x - x'| < \delta \Rightarrow |f(x) - f(x')| < \epsilon$.
Choose partition Δ s.t. $\max_{1 \leq j \leq k(\Delta)}(x_j - x_{j-1}) < \delta$, then for any j we have

$$0 \leq \sup_{I_j} f - \inf_{I_j} f \leq \epsilon \tag{Exercise ??}$$

Thus

$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq \overline{S}(f,\Delta) - \underline{S}(f,\Delta) \leq \epsilon \cdot (b-a)$$

for $\forall\epsilon > 0 \Rightarrow \overline{\int_a^b} f = \underline{\int_a^b} f \Rightarrow f$ is Darboux integrable by definition. □

Proposition 4. *If $f\nearrow(\searrow)$ on $[a,b] \Rightarrow f$ is Darboux integrable.*

Proof. If $f\nearrow$, then

$$\begin{aligned}\overline{S}(f,\Delta)-\underline{S}(f,\Delta) &= \sum_{j=1}^{k}(f(x_j)-f(x_{j-1}))\cdot(x_j-x_{j-1}) \\ &= (f(b)-f(a))\cdot\max_{1\le j\le k}(x_j-x_{j-1})\end{aligned}$$

Choose Δ s.t. $\max_{1\le j\le k}(x_j-x_{j-1})$ small enough. □

Remark 1. Furthermore, if f can be represented by $f=f_1+f_2$, where f_1,f_2 are monotone, then f is Darboux integrable.

1 Riemann integral

Definition 1 (Riemann integrable, 黎曼可积)**.** Let $D\xrightarrow{f}\mathbb{R}$ be a bounded function and $[a,b]\subseteq D$, we say f is Riemann integrable on $[a,b]$, if $\exists L\in\mathbb{R}, \forall\epsilon>0, \exists\delta>0,$ s.t. $\forall\Delta$ of $[a,b]$ and $\forall c_j\in I_j$, if $\max_{1\le j\le k}(x_j-x_{j-1})<\delta\Rightarrow$

$$\left|\sum_{j=1}^{k}f(c_j)\cdot(x_j-x_{j-1})-L\right|<\epsilon.$$

If this is the case, such L must be unique, and be called the Riemann integral of f on $[a,b]$.

Proposition 5. *Let $D\xrightarrow{f}\mathbb{R}$ be Riemann integrable on $[a,b]$ where $[a,b]\subseteq D\Rightarrow f$ is Darboux integrable on $[a,b]$.*

Proof. $\exists L\in\mathbb{R}$, s.t. for any $\epsilon>0$, we can find $\delta>0$ as in the definition such that if $\max_{1\le j\le k}(x_j-x_{j-1})<\delta$, then

$$L-\epsilon<\sum_{j=1}^{k}f(c_j)\cdot(x_j-x_{j-1})<L+\epsilon$$

for $\forall c_j\in I_j$. Then we have that

$$\overline{S}(f,\Delta)=\sum_{j=1}^{k}\sup_{I_j}f\cdot(x_j-x_{j-1})\le L+\epsilon$$

$$\underline{S}(f,\Delta)=\sum_{j=1}^{k}\inf_{I_j}f\cdot(x_j-x_{j-1})\ge L-\epsilon$$

and hence

$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq \overline{S}(f,\Delta) - \underline{S}(f,\Delta) \leq 2\epsilon.$$

Thus f is Darboux integrable, and $\int_a^b f = L$. $\square$

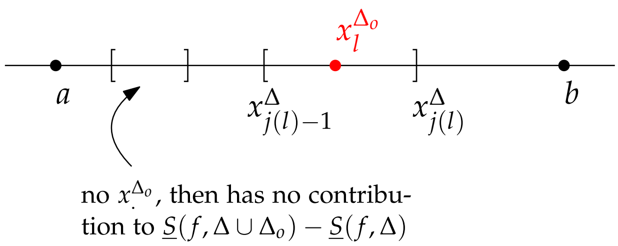
Theorem 1 (Darboux Theorem)**.** *Let* $D \xrightarrow{f} \mathbb{R}$ *be Darboux integrable on* $[a,b]$ *where* $[a,b] \subseteq D \Rightarrow f$ *is Riemann integrable on* $[a,b]$.

Proof. Let $L := \int_a^b f(x)\,\mathrm{d}x$. For any given $\epsilon > 0$ there exists a partition Δ_o of $[a,b]$ s.t.

$$\overline{S}(f,\Delta_o) - \underline{S}(f,\Delta_o) < \epsilon,$$

and in particular, $L < \underline{S}(f,\Delta_o) + \epsilon$. Let $\delta_o := \min_{1\leq l\leq k(\Delta_o)}(x_l^{\Delta_o} - x_{l-1}^{\Delta_o})$. Then choose partition Δ of $[a,b]$ such that $mesh(\Delta) := \max_{1\leq j\leq k(\Delta)}(x_j^{\Delta} - x_{j-1}^{\Delta}) < \delta_o$. Then $I_j^{\Delta} \cap \Delta_o$ has at most one element for $j = 1,\cdots,k(\Delta)$. Thus

$$\begin{aligned}
\underline{S}(f,\Delta\cup\Delta_o) - \underline{S}(f,\Delta) &= \sum_{l=1}^{k(\Delta_o)} \Bigg[\inf_{[x_{j(l)-1}^{\Delta}, x_l^{\Delta_o}]} f \cdot (x_l^{\Delta_o} - x_{j(l)-1}^{\Delta}) + \inf_{[x_l^{\Delta_o}, x_{j(l)}^{\Delta},]} f \cdot (x_{j(l)}^{\Delta} - x_l^{\Delta_o}) \\
&\qquad - \inf_{[x_{j(l)-1}^{\Delta}, x_{j(l)}^{\Delta}]} f \cdot (x_{j(l)}^{\Delta} - x_{j(l)-1}^{\Delta}) \Bigg] \\
&= \sum_{l=1}^{k(\Delta_o)} \Bigg[\inf_{[x_{j(l)-1}^{\Delta}, x_l^{\Delta_o}]} f \cdot (x_l^{\Delta_o} - x_{j(l)-1}^{\Delta}) - \inf_{[x_{j(l)-1}^{\Delta}, x_{j(l)}^{\Delta}]} f \cdot (x_l^{\Delta_o} - x_{j(l)-1}^{\Delta}) \\
&\qquad + \inf_{[x_l^{\Delta_o}, x_{j(l)}^{\Delta},]} f \cdot (x_{j(l)}^{\Delta} - x_l^{\Delta_o}) - \inf_{[x_{j(l)-1}^{\Delta}, x_{j(l)}^{\Delta},]} f \cdot (x_{j(l)}^{\Delta} - x_l^{\Delta_o}) \Bigg] \\
&= \sum_{l=1}^{k(\Delta_o)} \Bigg[\left(\inf_{[x_{j(l)-1}^{\Delta}, x_l^{\Delta_o}]} f - \inf_{[x_{j(l)-1}^{\Delta}, x_{j(l)}^{\Delta}]} f \right) \cdot (x_l^{\Delta_o} - x_{j(l)-1}^{\Delta}) \\
&\qquad + \left(\inf_{[x_l^{\Delta_o}, x_{j(l)}^{\Delta},]} f - \inf_{[x_{j(l)-1}^{\Delta}, x_{j(l)}^{\Delta},]} f \right) \cdot (x_{j(l)}^{\Delta} - x_l^{\Delta_o}) \Bigg] \\
&\leq (M-m) \cdot \sum_{l=1}^{k(\Delta_o)} (x_{j(l)}^{\Delta} - x_{j(l)-1}^{\Delta}) \\
&\leq (M-m) \cdot k(\Delta_o) \cdot mesh(\Delta).
\end{aligned}$$



where $m \leq f(x) \leq M$ for $\forall x \in [a, b]$. Since $\underline{S}(f, \Delta \cup \Delta_o) \geq \underline{S}(f, \Delta_o) > L - \epsilon$, then

$$\begin{aligned}\underline{S}(f, \Delta) &\geq \underline{S}(f, \Delta \cup \Delta_o) - (M - m) \cdot k(\Delta_o) \cdot mesh(\Delta)\\ &> L - \epsilon - (M - m) \cdot k(\Delta_o) \cdot mesh(\Delta)\end{aligned}$$

Choose Δ, such that $mesh(\Delta) < \max\{\delta_0, \epsilon/(M - m)k(\Delta_o)\}$, then

$$\sum_{j=1}^{k(\Delta)} f(c_j) \cdot (x_j^{\Delta} - x_{j-1}^{\Delta}) \geq \underline{S}(f, \Delta) > L - 2\epsilon.$$

for any $c_j \in I_j^{\Delta}$, and in the same way,

$$\sum_{j=1}^{k(\Delta)} f(c_j) \cdot (x_j^{\Delta} - x_{j-1}^{\Delta}) \leq \overline{S}(f, \Delta) < L + 2\epsilon.$$

Thus $\left|\sum_{j=1}^{k(\Delta)} f(c_j) \cdot (x_j^{\Delta} - x_{j-1}^{\Delta}) - L\right| < 2\epsilon$. □

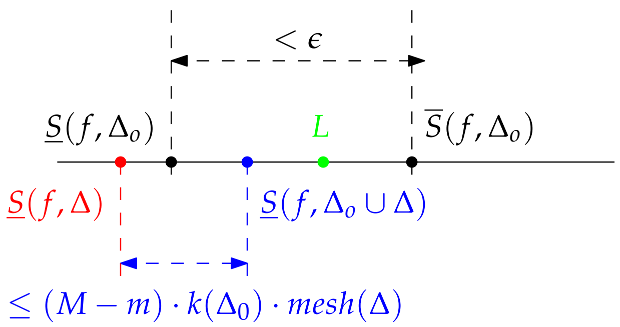


Figure 1: Darboux Theorem