Introduction to Analysis
Lecture 6

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Abstract

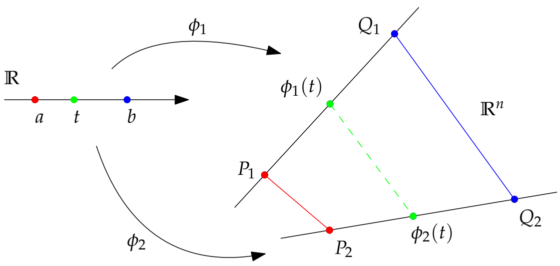
This is the Lecture note for the *Introduction to Analysis* class in Spring 2019.

1 Space filling curves

Lemma 1. *Given $a, b \in \mathbb{R}$ with $a < b$ and $P_1, P_2, Q_1, Q_2 \in \mathbb{R}^n$, let $\mathbb{R} \xrightarrow{\phi_i} \mathbb{R}^n$ be the affine maps (仿射) with $\phi_i(a) = P_i, \phi_i(b) = Q_i, i = 1, 2$. Then*

$$|\phi_1(t) - \phi_2(t)| \leq \max\{|P_1 - P_2|, |Q_1 - Q_2|\}$$

for $t \in [a, b]$.



Proof. Actually,

$$\phi_i(t) = \frac{b-t}{b-a} \cdot P_i + \frac{t-a}{b-a} \cdot Q_i,$$

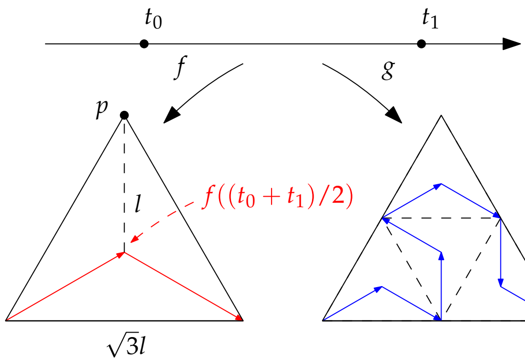
$t \in \mathbb{R}, i = 1, 2$. Then for $t \in [a, b]$, we have that

$$\begin{aligned}
|\phi_1(t) - \phi_2(t)| &= \left| \frac{b-t}{b-a} \cdot (P_1 - P_2) + \frac{t-a}{b-a} \cdot (Q_1 - Q_2) \right| \\
&\leq \frac{b-t}{b-a} \cdot |P_1 - P_2| + \frac{t-a}{b-a} \cdot |Q_1 - Q_2|
\end{aligned}$$

$$\leq \left(\frac{b-t}{b-a} + \frac{t-a}{b-a} \right) \cdot \max\{|P_1 - P_2|, |Q_1 - Q_2|\}$$
$$= \max\{|P_1 - P_2|, |Q_1 - Q_2|\}.$$

□

Lemma 2. *Let $\triangle$ be an equilateral triangle in $\mathbb{R}^n (n \geq 2)$, whose edges all have length $\sqrt{3}l$. Let f and g be maps from $[t_0, t_1]$ to $\triangle$ representing motions with constant speed along the following two given paths respectively from time t_0 to time t_1.*

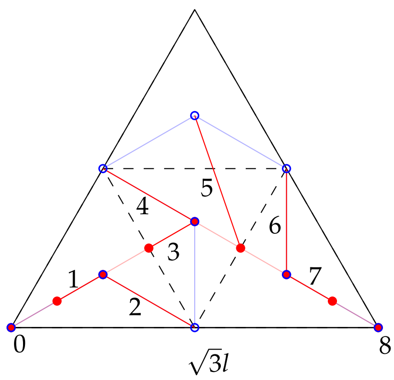


Then
1. $\forall a \in \triangle, \exists t \in [t_0, t_1]$, we have $f(t) \in \overline{B_l(a)}$;
2. $\forall t \in [t_0, t]$, we have $|f(t) - g(t)| \leq \sqrt{7}/4 \cdot l$.

Proof. 1. It is direct to see that the farthest point in $\triangle$ to the path $f(t)(t \in [t_0, t_1])$ is p, and $p \in \overline{B_l(f((t_0 + t_1)/2))}$.
2. We cut interval $[t_0, t_1]$ into 8 parts equally. And on each part, f and g are affine maps. Thus we have that

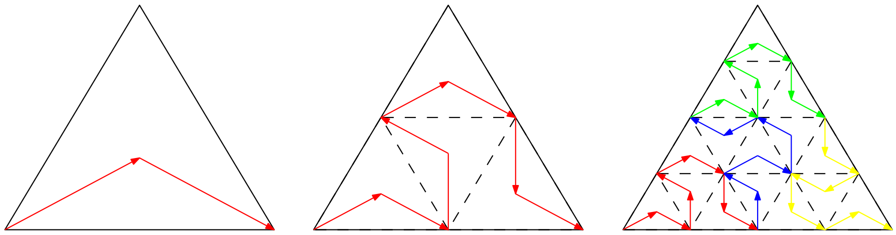
t	t_0	$t_{1/8}$	$t_{2/8}$	$t_{3/8}$	$t_{4/8}$	$t_{5/8}$	$t_{6/8}$	$t_{7/8}$	t_1
$\lvert f(t) - g(t)\rvert$	0	$l/4$	$l/2$	$l/4$	$l/2$	$l\sqrt{7}/4$	$l/2$	$l/4$	0

Then by lemma 1, we obtain 2.



$\square$

Let $l = 1$, we can define a sequence of functions $[0,1] \xrightarrow{f_i} \triangle, i = 0,1,2,\cdots$ like



$i = 0$ $i = 1$ $i = 2$

Then

$$|f_n(t) - f_{n-1}(t)| \leq \frac{\sqrt{7}}{4} \cdot \frac{1}{2^{n-1}}$$

for all $t \in [0,1], n \in \mathbb{N}$. And $\forall a \in \triangle, \exists t \in [t_0, t_1]$, we have $f_n(t) \in \overline{B_{1/2^n}(a)}$ for $\forall n \in \mathbb{N}_0$. In particular, for all $t \in [0,1]$, define $f_{-1}(t) = 0$, then for any $m \in \mathbb{N}_0$:

$$f_m(t) = \sum_{n=0}^{m} (f_n(t) - f_{n-1}(t)) \leq \frac{\sqrt{7}}{4} \cdot \frac{1}{2^{n-1}},$$

thus f_m converges uniformly to a map $[0,1] \xrightarrow{f} \triangle$ by Weierstrasse's M - test. And for all $t \in [0,1]$:

$$\begin{aligned}
|f(t) - f_m(t)| &= \left| \sum_{n=0}^{\infty} (f_n(t) - f_{n-1}(t)) - \sum_{n=0}^{m} (f_n(t) - f_{n-1}(t)) \right| \\
&= \left| \sum_{n=m+1}^{\infty} (f_n(t) - f_{n-1}(t)) \right| \\
&\leq \sum_{n=m+1}^{\infty} |f_n(t) - f_{n-1}(t)| \\
&\leq \sum_{n=m+1}^{\infty} \frac{\sqrt{7}}{4} \cdot \frac{1}{2^{n-1}} \\
&= \frac{\sqrt{7}}{2} \cdot \frac{1}{2^m}.
\end{aligned}$$

Since f_m is continuous, and hence f is continuous. Furthermore, since for any $t \in [0,1], m \in \mathbb{N}_0, f_m(t) \in \triangle$, thus $\lim_{m\to\infty} f_m(t) \in \triangle$ since $\triangle$ is close, thus $\forall t \in [0,1] \Rightarrow f(t) \in \triangle \Rightarrow f([0,1]) \subseteq \triangle$.

Theorem 1. $f([0,1]) = \triangle$.

Proof. $[0,1]$ is compact $\Rightarrow f([0,1])$ is compact subset of $\mathbb{R}^n$ and hence $f([0,1])$ is closed. We will show that $\forall a \in \triangle, \forall r > 0, \exists t \in [0,1]$, s.t. $f(t) \in B_r(a) \Rightarrow a$ **is limit of a seq. in the closed set** $f([0,1])$**, and hence** $a \in f([0,1]) \Rightarrow \triangle \subseteq f([0,1])$.
For any $a \in \triangle$, and $r > 0$, choose $m \in \mathbb{N}$ so large that

$$\frac{1}{2^m} + \frac{\sqrt{7}}{2} \cdot \frac{1}{2^m} < r,$$

Then by lemma 2 (1), $\exists t \in [0,1]$, s.t. $f_m(t) \in \overline{B_{1/2^m}(a)}$, i.e.

$$|f_m(t) - a| \leq \frac{1}{2^m},$$

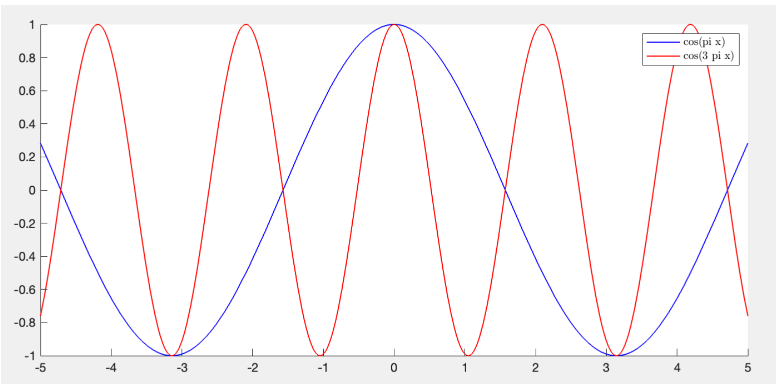
and hence

$$\begin{aligned} |f(t) - a| &\leq |f(t) - f_m(t)| + |f_m(t) - a| \\ &\leq \frac{\sqrt{7}}{2} \cdot \frac{1}{2^m} + \frac{1}{2^m} \\ &< r. \end{aligned}$$

Thus $f(t) \in B_r(a)$. $\square$

2 Weierstrass's function

Consider a cosine function $\cos(\pi x)$. The slope of its peaks and trough at $x = 0$ is 2, we can steepen it by 'squeezing' the function, such as $\cos(3\pi x)$.



Following this method, we can construct a function

$$f_n(x) = b^n \cos(a^n \pi x), \quad F(x) = \sum_{n=0}^{\infty} f_n(x),$$

where

- $0 < b < 1$, to satisfy the Weierstrass's M - test, and hence $\sum_{n=0}^{m} f_n(x)$ uniformly converges to $F(x)$;
- $a(> 1)$ is an odd number, to ensure for any $n_1 < n_2$, The peaks and troughs of the $b^{n_1}\cos(a^{n_1}\pi x)$ remain the peaks and troughs of the $b^{n_2}\cos(a^{n_2}\pi x)$.

The main idea of this construction if to superpose a seq. of squeezed (by a^n) maps to increase the slope at some point. And control the amplitudes (by b^n) of these maps to make them cvg. uni.
But the problem is the slop decreases as the amplitudes decreases, thus we need to find a balance between a and b, so that the slope at any point is infinitely large when the sequence of functions converges uniformly.

Theorem 2 (Weierstrass). *If $ab > 1 + 3\pi/2$, then F is nowhere differentiable.*

Proof. We will estimate $\left|\frac{F(x)-F(c)}{x-c}\right|$ for every $c \in \mathbb{R}$ and x near c. For any $m \in \mathbb{N}$, define

$$F_m(x) := \sum_{n=0}^{m-1} b^n \cos(a^n \pi x), \quad F'_m(x) = \sum_{n=m}^{\infty} b^n \cos(a^n \pi x).$$

Then for any $c \in \mathbb{R}, m \in \mathbb{N}, x$ near c, we have that

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} b^n \cos(a^n \pi x) \\ &= \sum_{n=0}^{m-1} b^n \cos(a^n \pi x) + \sum_{n=m}^{\infty} b^n \cos(a^n \pi x) \\ &= F_m(x) + F'_m(x) \end{aligned}$$

and

$$\begin{aligned} |F(x) - F(c)| &= |F_m(x) - F_m(c) + F'_m(x) - F'_m(c)| \\ &\geq -|F_m(x) - F_m(c)| + |F'_m(x) - F'_m(c)| \qquad \text{(triangle inequality)} \end{aligned}$$

and hence
$$\left|\frac{F(x) - F(c)}{x - c}\right| \geq -\left|\frac{F_m(x) - F_m(c)}{x - c}\right| + \left|\frac{F'_m(x) - F'_m(c)}{x - c}\right|.$$

Now we will focus on $\left|\frac{F_m(x)-F_m(c)}{x-c}\right|$ and $\left|\frac{F'_m(x)-F'_m(c)}{x-c}\right|$ respectively.
1.

$$\begin{aligned} \left|\frac{F_m(x) - F_m(c)}{x - c}\right| &= \left|\frac{\sum_{n=0}^{m-1} b^n \cos(a^n \pi x) - \sum_{n=0}^{m-1} b^n \cos(a^n \pi c)}{x - c}\right| \\ &= \left|b^n \cdot \sum_{n=0}^{m-1} \frac{[\cos(a^n \pi x) - \cos(a^n \pi c)]}{x - c}\right| \end{aligned}$$

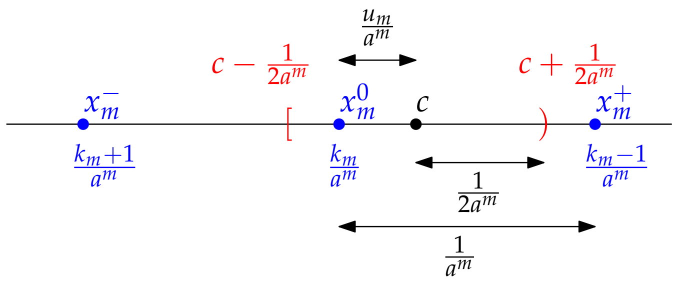
$$
\begin{aligned}
&\le b^n \cdot \sum_{n=0}^{m-1} \left| \frac{\cos(a^n \pi x) - \cos(a^n \pi c)}{x-c} \right| \\
&= \le b^n \cdot \sum_{n=0}^{m-1} a^n \pi \, |\sin \xi| && \text{(mean-value thm)} \\
&\le \sum_{n=0}^{m-1} (ab)^n \pi \\
&= \frac{(ab)^m - 1}{ab-1} \pi.
\end{aligned}
$$

2. For any given $c \in \mathbb{R}$ and $m \in \mathbb{N}$, the wavelength of $f_m = b^m \cos(a^m \pi x)$ is $2/a^m$, and hence f_m achieve peaks or troughs at $k/a^m (k \in \mathbb{Z})$. And there exists a unique $k_m \in \mathbb{Z}$ s.t.

$$
c - \frac{1}{2a^m} \le \frac{k_m}{a^m} < c + \frac{1}{2a^m}.
$$

Let $x_m^0 := k_m/a^m, x_m^+ := (k_m+1)/a^m$ and $x_m^- := (k_m-1)/a^m$. (Thus if x_m^0 is peak, then x_m^+, x_m^- is trough, otherwise the vice.) And $\exists u_m \in \mathbb{R}$ s.t. $c = (k_m + u_m)/a^m$. And since $x_m^0 \in [c - 1/2a^m, c + 1/2a^m) \Rightarrow u_m \in [-1/2, 1/2)$. And then

$$
a^m \pi x_m^{\pm} = (k_m \pm 1)\pi, \quad a^m \pi c = (u_m + k_m)\pi
$$



Then

$$
\begin{aligned}
\frac{F_m'(x_m^{\pm}) - F_m'(c)}{x-c} &= \sum_{n=m}^{\infty} \frac{f_n(x_m^{\pm}) - f_n(c)}{x_m^{\pm} - c} \\
&= \frac{f_m(x_m^{\pm}) - f_m(c)}{x_m^{\pm} - c} + \sum_{n=m+1}^{\infty} \frac{f_n(x_m^{\pm}) - f_n(c)}{x_m^{\pm} - c}
\end{aligned}
$$

(2.a) $l = 0$, substitute $a^m \pi x_m^{\pm} = (k_m \pm 1)\pi, a^m \pi c = (u_m + k_m)\pi$, we have that

$$
\begin{aligned}
\frac{f_m(x_m^{\pm}) - f_m(c)}{x_m^{\pm} - c} &= (ab)^m \cdot \frac{\cos((k_m \pm 1)\pi) - \cos((u_m + k_m)\pi)}{-u_m \pm 1} \\
&= (ab)^m \cdot \frac{(-1)^{k_m + 1} - (-1)^{k_m} \cos(u_m \pi)}{\pm 1 - u_m}
\end{aligned}
$$

$$= (-1)^{k_m+1}(\pm 1)(ab)^m \cdot \frac{1+\cos(u_m\pi)}{1 \mp u_m}$$

where $\frac{1+\cos(u_m\pi)}{1\mp u_m} \geq 0$, thus $(-1)^{K_m+1}(\pm 1)$ is the sign of $\frac{f_m(x)-f_m(c)}{x-c}$. Since $u_m \in [-1/2, 1/2) \Rightarrow \cos(u_m\pi) \geq 0 \Rightarrow \frac{1+\cos(u_m\pi)}{1\mp u_m} \geq \frac{2}{3}$. Thus

$$\left| \frac{f_m(x_m^\pm) - f_m(c)}{x_m^\pm - c} \right| \geq \frac{(ab)^m 2}{3}.$$

(2.b) $l > 0$, for any $l \in \mathbb{N}$:

$$\begin{aligned} \frac{f_{m+l}(x_m^\pm) - f_{m+l}(c)}{x_m^\pm - c} &= b^{m+l} \cdot \frac{\cos(a^l a^m \pi x_m^\pm) - \cos(a^l a^m \pi c)}{x_m^\pm - c} \\ &= a^m b^{m+l} \cdot \frac{\cos(a^l(k_m \pm 1)\pi) - \cos(a^l(k_m + u_m)\pi)}{-u \pm 1} \end{aligned}$$

Since a is odd, then a^l is odd $\Rightarrow \cos(a^l(k_m+1)\pi) = \cos((k_m+1)\pi) = -1^{k_m+1}$ and $\cos(a^l(k_m+u_m)\pi) = \cos(a^l k_m \pi + a^l u_m \pi) = -1^{k_m}\cos(a^l u_m \pi)$. Thus

$$\frac{f_{m+l}(x) - f_{m+l}(c)}{x - c} = a^m b^{m+l}(-1)^{k_m+1}(\pm 1)\frac{1 + (-1)^{k_m}\cos(a^l u_m \pi)}{1 \mp u_m}$$

since $\frac{1+(-1)^{k_m}\cos(a^l u_m\pi)}{1\mp u_m} \geq 0$, $\frac{f_m(x)-f_m(c)}{x-c}$ has the same sign with $\frac{f_{m+l}(x)-f_{m+l}(c)}{x-c}$ for any $l \in \mathbb{N}$. Therefore

$$\left| \frac{F_m'(x_m^\pm) - F_m'(c)}{x_m^\pm - c} \right| = \left| \frac{f_m(x_m^\pm) - f_m(c)}{x_m^\pm - c} + \sum_{n=m+1}^{\infty} \frac{f_n(x_m^\pm) - f_n(c)}{x_m^\pm - c} \right| \geq \frac{2}{3}(ab)^m.$$

In summary,

$$\begin{aligned} \left| \frac{F(x_m^\pm) - F(c)}{x_m^\pm - c} \right| &\geq - \left| \frac{F_m(x_m^\pm) - F_m(c)}{x_m^\pm - c} \right| + \left| \frac{F_m'(x_m^\pm) - F_m'(c)}{x_m^\pm - c} \right| \\ &\geq \frac{2}{3}(ab)^m - \frac{(ab)^m - 1}{ab - 1}\pi \\ &> \frac{2}{3}(ab)^m - \frac{(ab)^m}{ab-1} \qquad \text{(let } ab > 1) \\ &= (ab)^m \cdot \left[\frac{2}{3} - \frac{\pi}{ab-1} \right]. \end{aligned}$$

Let $\frac{2}{3} - \frac{\pi}{ab-1} > 0 \Rightarrow ab > 1 + 3\pi/2$. Then

$$\left| \frac{F(x_m^\pm) - F(c)}{x_m^\pm - c} \right| > \lambda \cdot (ab)^m$$

where $\lambda > 0$. Note that $x_m^\pm \to c$ and $\lambda \cdot (ab)^m \to \infty$ as $m \to \infty$. Thus $\lim_{x\to c} \left| \frac{F(x)-F(c)}{x-c} \right| = \infty$. □

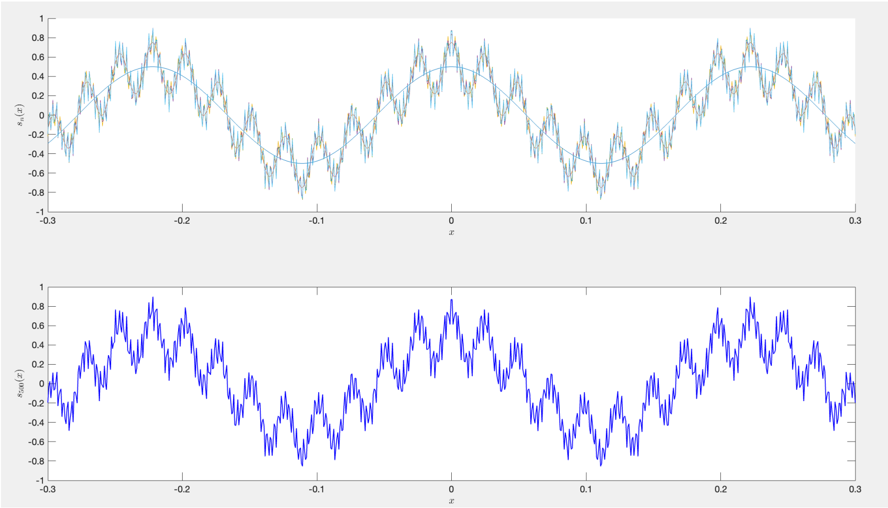


Figure 1: Weierstrass's function