Introduction to Analysis
Lecture 4

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Abstract

This is the Lecture note for the *Introduction to Analysis* class in Spring 2019.

1 Rearrangement theorem

Given a seq. $a_n (n \in \mathbb{N})$, we can separate all terms into two subseq.s:

$$a_{n_1}, a_{n_2}, \cdots \text{ and } a_{n'_1}, a_{n'_2}, \cdots$$

where $n_1 < n_2 < \cdots$ and $n'_1 < n'_2 < \cdots$ and $\{n_1, n_2, \cdots\} \cup \{n'_1, n'_2, \cdots\} = \mathbb{N}$, such that $a_{n_j} \geq 0 (j \in \mathbb{N}), a_{n'_k} \leq 0 (k \in \mathbb{N})$. Let $p_j := a_{n_j} (j \in \mathbb{N})$ and $q_k := a_{n'_k} (k \in \mathbb{N})$.

Exercise 1. *Show that $\sum_n |a_n| < \infty \Leftrightarrow \sum_j p_j < \infty$ and $\sum_k q_k < \infty$. Moreover, if any side holds, then*

$$\sum_n |a_n| = \sum_j p_j + \sum_k q_k$$

and

$$\sum_n a_n = \sum_j p_j - \sum_k q_k.$$

Proof. 1. $\Rightarrow$: since $\sum_n |a_n| < \infty$, any partial sum of a_n has upper bound such as M, then for any $j \in \mathbb{N}$:

$$\begin{aligned} p_1 + \cdots + p_j &= |a_{n_1}| + \cdots + |a_{n_j}| \\ &\leq \sum_{n=1}^{n_j} |a_n| \\ &\leq M, \end{aligned}$$

Thus any partial sum of p_j has upper bound M and hence $\sum_j p_j < \infty$. And $\sum_k q_k < \infty$ in the same way.

2. $\Leftarrow$: The partial sum of $\sum_n |a_n|$ can be decompose by the partial sums of $\sum_n p_n$ and $\sum_n q_n$ which have upper bounds, thus partial sum of $\sum_n |a_n|$ has upper bound, and $\sum_n |a_n| < \infty$.
3. Define the partial sum of $\sum_n |a_n|, \sum_n a_n, \sum_n p_n, \sum_n q_n$ as

$$A_m := \sum_{i=1}^{m} |a_i|, \quad S_m := \sum_{i=1}^{m} a_i$$
$$P_m := \sum_{i=1}^{m} p_i = \sum_{i=1}^{m} |a_{n_i}|, \ Q_m := \sum_{i=1}^{m} q_i = \sum_{i=1}^{m} |a_{n'_i}|$$

Then for any $m \in \mathbb{N}$, we have that

$$A_m \le P_m + Q_m \le A_{\max\{n_m, n'_m\}},$$

thus $\lim_{n\to\infty} A_n = \lim_{n\to\infty}(P_n + Q_n) = \lim_{n\to\infty} P_n + \lim_{n\to\infty} Q_n$ since $\sum_n p_n, \sum_n q_n$ exists, and the squeeze theorem. And hence $\sum_n |a_n| = \sum_j p_j + \sum_k q_k$.
On the contrary, for any $m \in \mathbb{N}$, we can represent the partial sum of $\sum_n a_n$ as

$$s_m = P_l - Q_v$$

where $l, v \to \infty$ as $m \to \infty$, thus $\sum_n a_n = \sum_n p_n - \sum_n q_n$. $\square$

Exercise 2. *If $\sum_n a_n$ converges conditionally, show that*
1. $\sum_j p_j = \infty$ and $\sum_k q_k = \infty$;
2. $\lim_{j\to\infty} p_j = \lim_{k\to\infty} q_k = 0$.

Proof. 1. Denote the partial sum of $\sum_n a_n, \sum_j p_j, \sum_k q_k$ as s_n, P_j, Q_k respectively, then we have that $\lim_{n\to\infty} s_n = \lim_{n\to\infty}(P_j - Q_k)$ exists, then either both $\lim_{n\to\infty} P_j, \lim_{n\to\infty} Q_k$ exist or neither exists, since $\sum_n a_n$ converges conditionally $\Rightarrow \lim_{n\to\infty} P_j = \infty$ and $\lim_{n\to\infty} Q_k = \infty$.
2. Since

$$\lim_{j\to\infty} p_j = \lim_{j\to\infty} a_{n_j} = \lim_{n\to\infty} a_n = 0,$$

and $\lim k \to \infty q_k = 0$ as well in the same way. $\square$

Exercise 3. *If $\sum_n a_n, \sum_n b_n$ converges, show that $\sum_n(a_n \pm b_n) = \sum_n a_n \pm \sum_n b_n$.*

Proof. Denote the partial sum of $\sum_n(a_n + b_n), \sum_n a_n, \sum_n b_n$ as S_n, A_n, B_n respectively, then for any $n \in \mathbb{N}$

$$S_n = A_n + B_n$$

and hence

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(A_n + B_n) = \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n$$

since $\lim_{n\to\infty} A_n, \lim_{n\to\infty} B_n$ exists, thus $\sum_n(a_n + b_n) = \sum_n a_n + \sum_n b_n$, and $\sum_n(a_n - b_n) = \sum_n a_n - \sum_n b_n$ in the same way. $\square$

Exercise 4. *Inserting* 0*s to a series will not affect its convergence / divergence and its sum (if the sum exists).*

Proof. Consider the tail of series. Trivial. □

Recall that a sequence a_n is a map $\mathbb{N} \xrightarrow{a.} \mathbb{R}$ where $n \mapsto a(n)$ denoted by a_n. A subsequence a_{n_m} is a composite map

$$\mathbb{N} \xrightarrow{n.} \mathbb{N} \xrightarrow{a.} \mathbb{R}$$

where $n.$ is a strictly monotone injection and $m \mapsto n(m)$ denoted by n_m. A **rearrangement** is also a composite map

$$\mathbb{N} \xrightarrow{n(\cdot)} \mathbb{N} \xrightarrow{a.} \mathbb{R}$$

where $n(\cdot)$ is a bijection. (To distinguish it from the notion of subseq., we don't use subscripts here)
Now we gonna explore two questions: if a series $\sum_n$ converges, $a_{n(m)}(m \in \mathbb{N})$ is a rearrangement of $a_n(n \in \mathbb{N})$, then
1. whether $\sum_m a_{n(m)}$ converges ?
2. whether $\sum_n a_n = \sum_m a_{n(m)}$?

Exercise 5. *Let $\sum_n a_n$ be a positive series, show that*

$$\sum_n a_n = \sup \Lambda$$

including the case $\sum_n a_n = \infty$. Here $\Lambda = \{a_{n_1} + \cdots + a_{n_k} | n_1 < \cdots < n_k, k \in \mathbb{N}\}$ represents the set of every sum of finite terms of $a_n(n \in \mathbb{N})$.

Proof. 1. $\leq$: since $\sum_n a_n$ is the limit of the partial sum s_n (which is the sum of finite terms, i.e. $s_n \in \Lambda$ for any $n \in \mathbb{N}$), and since $a_n \geq 0$, s_n monotone, then

$$\sum_n a_n = \lim_{n\to\infty} s_n = \sup_{n\in\mathbb{N}} s_n \leq \sup \Lambda$$

2. $\geq$: If $\sup \Lambda > \sup s_n$, let $\epsilon := \sup \Lambda - \sup s_n$, then $\exists \lambda = a_{n_1} + \cdots + a_{n_{k_\lambda}} \in \Lambda$ such that

$$s_m \leq \sup_{n\in\mathbb{N}} s_n < \sup \Lambda - \frac{\epsilon}{2} < \lambda \leq \sup \Lambda$$

for $\forall m \in \mathbb{N}$, but

$$\lambda = a_{n_1} + \cdots + a_{n_{k_\lambda}} \leq s_{n_{k_\lambda}}$$

which leads to a contradiction.
3. If $\sum_n a_n = \infty$, it is direct to see that $\sup \Lambda = \infty$ as well by 1. □

Exercise 6. *If $\sum_n a_n$ is a convergent positive series, show that for every rearrangement $a_{n(m)}(m \in \mathbb{N})$ of $a_n(n \in \mathbb{N})$, we have that $\sum_n a_n = \sum_m a_{n(m)}$.*

Proof. If $\sum_n a_n$ is positive series, then $\sum_m a_{n(m)}$ is positive series as well.

$$\sum_n a_n = \sup \Lambda_{a_n} = \sup \Lambda_{a_{n(m)}} = \sum_m a_{n(m)}$$

where Λ_{a_n} and $\Lambda_{a_{n(m)}}$ are the set of every sum of finite terms of a_n and $a_{n(m)}$ respectively. That is the proof follows by the $\Lambda_{a_n} = \Lambda_{a_{n(m)}}$. □

Exercise 7 (Dirichlet's Rearrangement Theorem (1829))**.** *If $\sum_n a_n$ converges absolutely, show that for every rearrangement $a_{n(m)}(m \in \mathbb{N})$ of $a_n(n \in \mathbb{N})$, we have that $\sum_n a_n = \sum_m a_{n(m)}$.*

Proof. $\sum_n a_n$ converges absolutely $\Rightarrow \sum_m a_{n(m)}$ converges absolutely. Furthermore

$$\begin{aligned}\sum_n a_n &= \sum_j p_j - \sum_k q_k \\ &= \sum_\mu p_{j_\mu} - \sum_\nu q_{k_\nu} \\ &= \sum_m a_{n_m}.\end{aligned}$$

□

Theorem 1 (Riemann's Rearrangement Theorem(1852))**.** *If $\sum_n a_n$ converges conditionally, then for $\forall r \in \mathbb{R}$, there exists a rearrangement $a_{n(m)}(m \in \mathbb{N})$ of $a_n(n \in \mathbb{N})$ such that $\sum_m a_{n(m)} = r$.*

Proof. We will only use two known fact:
1. $\sum_j p_j = \infty$ and $\sum_k q_k = \infty$;
2. $\lim_{j\to\infty} p_j = \lim_{k\to\infty} q_k = 0$.

Given a $L \in \mathbb{R}$, start with p_1, plus by p_2 and so on till p_{m_1-1} where

$$\sum_i^{m_1-1} p_i \leq L \quad \text{but} \quad \sum_i^{m_1} p_i > L.$$

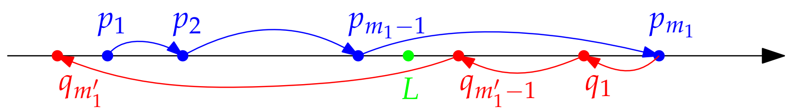
Then minus by q_1, q_2 and so on till $q_{m_1'-1}$ where

$$\sum_i^{m_1} p_1 - \sum_{j=1}^{m_1'-1} q_j > L \quad \text{but} \quad \sum_i^{m_1} p_1 - \sum_{j=1}^{m_1'} q_j \leq L.$$

This process can be repeat since $\sum_j p_j = \infty$ and $\sum_k q_k = \infty$ and hence any tail of $\sum_j p_j, \sum_k q_k$ has no upper bound, therefore the *cross* action can always happen, in the other word, $m_i, m_i'(i \in \mathbb{N})$ exists.

Thus we can form a rearrangement χ_n of $\sum_n a_n$ as

$$p_1, \cdots, p_{m_1}, -q_1, \cdots, -q_{m_1'}, \cdots$$



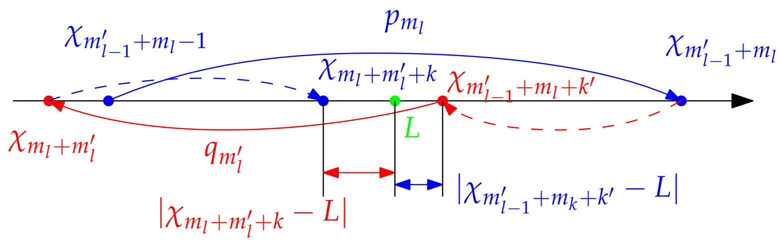
Now we will show that this rearrangement converges to L, i.e. $\lim_{n\to\infty} \chi_n = L$. Consider $\chi_{\cdots+m_{l-1}'+m_l-1}$ which implies the point lies in the left of L and will cross the l in next jump, and we denote it by $\chi_{m_{l-1}'+m_l-1}$ for simply. And hence we have that

$$|\chi_{m_{l-1}'+m_k+k'} - L| < p_{m_l}$$

if $0 \le k' < m_l' - m_{l-1}'$. And similarly

$$|\chi_{m_l+m_l'+k} - L| < q_{m_l'}$$

if $0 \le k < m_{m+1-m_l}$.



And since $\lim_{l\to\infty} p_{m_l} = \lim_{l\to\infty} q_{m_l'} = 0$, for $\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ s.t. $l \ge N_0 \Rightarrow p_{m_l}$ and $q_{m_l'} < \epsilon$. Let $N = m_{N_0-1}' + m_{N_0}$, then $n \ge N \Rightarrow |\chi_n - L| < \epsilon$. $\square$

Remark 1 ($2S = S$). Dirichelet made the following observation in 1827: Consider a alternating series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$$

which is convergent (conditionally) by Leibniz's Criterion, and hence

$$\begin{aligned} 2S &= 2 - 1 + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \frac{2}{7} - \frac{2}{8} + \frac{2}{9} - \frac{2}{10} + \cdots \\ &=' (2-1) - \frac{2}{4} + \left(\frac{2}{3} - \frac{2}{6}\right) - \frac{2}{8} + \left(\frac{2}{5} - \frac{2}{10}\right) + \cdots \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots \\ &= S \end{aligned}$$

Remark 2. In summary, given a series $\sum_n a_n$, and its any rearrangement $\sum_m a_{n(m)}$, then
1. If $a_n \geq 0$ for $\forall n \in \mathbb{N} \Rightarrow \sum_n a_n = \sum_m a_{n(m)}$;
2. If $\sum_n |a_n| < \infty \Rightarrow \sum_n a_n = \sum_m a_{n(m)}$;
3. If $\sum_n |a_n| = \infty$ but $\sum_n a_n < \infty \Rightarrow \sum_m a_{n(m)}$ could be anything.

2 Multiplying absolutely convergent series

Proposition 1. *Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely, let*

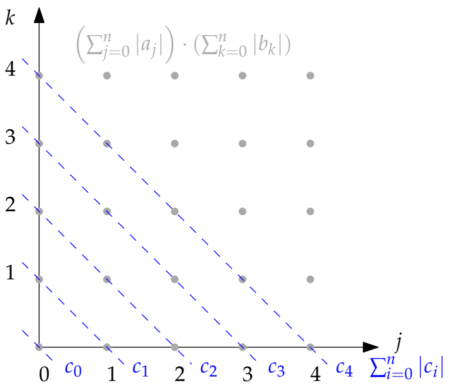
$$c_n = a_n b_0 + \cdots + a_0 b_n = \sum_{j+k=n;\ j,k\geq 0} a_j b_k,$$

then $\sum_n |c_n| < \infty$ and $\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right)$.

Proof. 1. $\sum_n |c_n| < \infty$
For all n,

$$\begin{aligned}\sum_{m=0}^{n} |c_m| = \sum_{m=0}^{n} \left| \sum_{\substack{j+k=m \\ j,k\geq 0}} a_j b_k \right| &\leq \sum_{m=0}^{n} \sum_{\substack{j+k=m \\ j,k\geq 0}} |a_j||b_k| \\ &\leq \left(\sum_{j=0}^{n} |a_j|\right) \cdot \left(\sum_{k=0}^{n} |b_k|\right).\end{aligned}$$

Since $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converges absolutely, the partial sums of $|a_n|, |b_n|$ have upper bounds, denoted by M, N respectively, then $\sum_{m=0}^{n} |c_m|$ has a upper bound $M \cdot N$ and hence $\sum_{n=0}^{\infty} c_n$ converges absolutely.

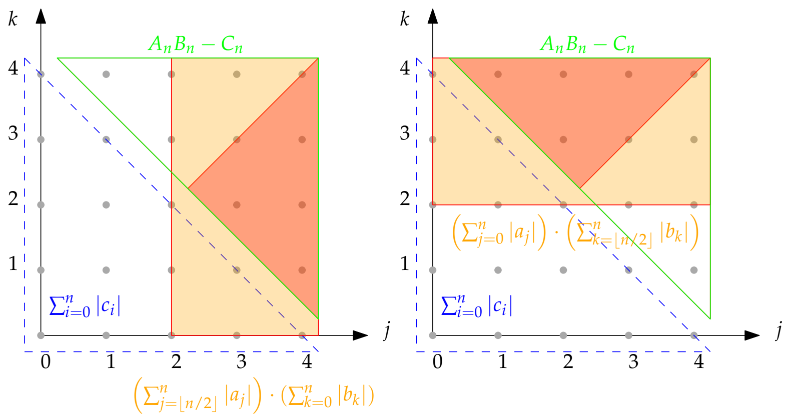


2. $\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right)$.
Let $A_n := a_0 + \cdots + a_n$; $B_n := b_0 + \cdots + b_n$ and $C_n := c_0 + \cdots + c_n$, we claim that $\lim_{n\to\infty}(A_n B_n - C_n) = 0$. Then

$$
\begin{aligned}
|A_nB_n - C_n| &= \sum_{\substack{j+k>n \\ 0\le j,k\le n}} |a_jb_k| \\
&\le \left(\sum_{j=\lfloor n/2\rfloor}^{n} |a_j|\right)\cdot\left(\sum_{k=0}^{n}|b_k|\right) + \left(\sum_{j=0}^{n}|a_j|\right)\cdot\left(\sum_{k=\lfloor n/2\rfloor}^{n}|b_k|\right)
\end{aligned}
$$

where $\sum_{k=0}^{n}|b_k|, \sum_{j=0}^{n}|a_j|$ are bounded, and tails $\sum_{j=\lfloor n/2\rfloor}^{n}|a_j|, \sum_{k=\lfloor n/2\rfloor}^{n}|b_k| \to 0$ as $n\to\infty$ since $\sum_n a_n, \sum_n b_n$ are converges abs. Thus $\lim_{n\to\infty}|A_nB_n - C_n| = 0$ and since $\lim_{n\to\infty}A_n, \lim_{n\to\infty}B_n, \lim_{n\to\infty}C_n$ exists, we have that

$$
\begin{aligned}
\sum_{n=0}^{\infty} c_n &= \lim_{n\to\infty} C_n \\
&= \lim_{n\to\infty} A_nB_b = \lim_{n\to\infty}A_n\cdot\lim_{n\to\infty}B_n \\
&= \left(\sum_{n=0}^{\infty}a_n\right)\cdot\left(\sum_{n=0}^{\infty}b_n\right)
\end{aligned}
$$



$\square$

Theorem 2. *If $\sum_n a_n, \sum_n b_n$ cvg. abs., $\mathbb{N} \xrightarrow{(j(\cdot),k(\cdot))} \mathbb{N}\times\mathbb{N}$ is bijection where $n \mapsto (j(n),k(n))$, let $c_n := a_{j(n)}b_{k(n)} (n\in\mathbb{N})$, then $\sum_n|c_n| < \infty$ (cvg. abs.) and $\sum_n c_n = (\sum_n a_n)(\sum_n b_n)$.*

Proof. 1. $\sum_n c_n$ cvg. abs.
For $\forall n\in\mathbb{N}$, let $l = \max\{j(1),\cdots,j(n),k(1),\cdots,k(n)\}$. Then

$$
\begin{aligned}
|c_1| + \cdots + |c_n| &= |a_{j(1)}b_{k(1)}| + \cdots + |a_{j(n)}b_{k(n)}| \\
&\le \left(\sum_{j=1}^{l}|a_j|\right)\cdot\left(\sum_{k=1}^{l}|b_k|\right) \\
&\le M\cdot N
\end{aligned}
$$

Thus $\sum_n c_n$ cvg. abs.
2. $\sum_n c_n = (\sum_n a_n)(\sum_n b_n)$.
Let $A_n = a_1 + \cdots + a_n$, $B_n = b_1 + \cdots + b_n$ and $C_n = c_1 + \cdots + c_n (n \in \mathbb{N})$. And define the bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by the second one in Figure 1. Then

$$\begin{aligned}
A_n B_n &= (a_1 + \cdots + a_n)(b_1 + \cdots + b_n) \\
&= \sum_{1 \le j,k \le n} a_j b_k \\
&= C_{n^2}
\end{aligned}$$

Thus $\lim_{n\to\infty} A_n B_n = \lim_{n\to\infty} C_{n^2} = \lim_{n\to\infty} C_n \Rightarrow \sum_n c_n = (\sum_n a_n)(\sum_n b_n)$. $\square$

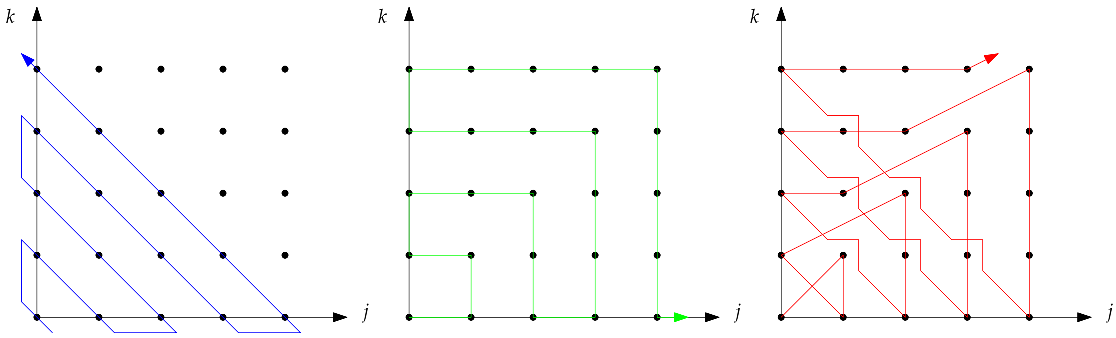


Figure 1: 3 kinds of bijections $(j(\cdot), k(\cdot))$