Introduction to Analysis
Lecture 2

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Abstract

This is the Lecture note for the *Introduction to Analysis* class in Fall 2019.

Exercise 1 (Squeeze theorem)**.** *If* $\lim_{n\to\infty} a_n = l$ *and* $\lim_{n\to\infty} b_n = m$ *and* $a_n \le c_n \le b_n$*, show that* $l = m \Rightarrow \lim_{n\to\infty} c_n = l$.

Proof. Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = l$, for $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \ge N$ has $|a_n - l| < \epsilon/3$ and $|b_n - l| < \epsilon/3$. And since $a_n \le c_n \le b_n$, we have that $0 \le c_n - a_n \le b_n - a_n$. Thus for $\forall n \ge N$, we have

$$\begin{aligned} |c_n - l| &= |c_n - a_n + a_n - l| \\ &\le |c_n - a_n| + |a_n - l| \\ &\le |b_n - a_n| + |a_n - l| \\ &= |b_n - l + l - a_n| + |a_n - l| \\ &\le |b_n - l| + 2|a_n - l| \\ &< \epsilon. \end{aligned}$$

thus $\lim_{n\to\infty} c_n = l$. □

Exercise 2. *If* $a > 1$ *show that* $\lim_{n\to\infty} 1/a^n = 0$.

Proof. Since $a > 1 \Rightarrow b := a - 1 > 0$, thus

$$0 \le \frac{1}{a^n} = \frac{1}{(1+b)^n} \le \frac{1}{1+nb} \to 0$$

as $n \to \infty$, thus $\lim_{n\to\infty} 1/a^n = 0$ by Squeeze theorem. □

Definition 1. A seq. $a_n (n \in \mathbb{N})$ in $\mathbb{R}$ is

1. nondecreasing monotone/increasing if $a_n \le a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_{n\nearrow}$; nonincreasing monotone/decreasing if $a_n \ge a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_{n\searrow}$.

2. strictly increasing if $a_n < a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n \nearrow\nearrow$; strictly decreasing if $a_n > a_{n+1}$ for $\forall n \in \mathbb{N}$, denoted by $a_n \searrow\searrow$.

Theorem 1 (Monotone Seq. Property)**.** *If $a_n \nearrow$ and $\{a_n | n \in \mathbb{N}\}$ has an upper bound, then $\lim_{n\to\infty} a_n = \sup\{a_n | n \in \mathbb{N}\}$; $a_n \searrow$ and $\{a_n | n \in \mathbb{N}\}$ has an lower bound, then $\lim_{n\to\infty} a_n = \inf\{a_n | n \in \mathbb{N}\}$.*

Proof. $\{a_n | n \in \mathbb{N}\}$ has an upper bound $\Rightarrow l := \sup\{a_n | n \in \mathbb{N}\}$ exists by Weierstrass theorem. Thus for $\forall \epsilon > 0, l - \epsilon$ is not an upper bound of $\{a_n\}$, then $\exists N \in \mathbb{N}$, s.t. $a_N > l$ and since $a_n \nearrow$, we have that $\forall n \geq N, l - \epsilon < a_n \leq l \Rightarrow \lim_{n\to\infty} a_n = l$. □

Example 1 (Decimal expression gives real number)**.** Suppose $d_i \in \mathbb{N}$ and $0 \leq d_i \leq 9$ for $i \in \mathbb{N}$, and define

$$a_n = \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

for $n \in \mathbb{N}$, then it is direct to see that $a_n \nearrow$ and

$$\begin{aligned} a_n &\leq \frac{9}{10} + \frac{9}{10^2} + \cdots \frac{9}{10^n} \\ &= \frac{9}{10}\left(\frac{1 - \frac{1}{10^{n+1}}}{1 - \frac{1}{10}}\right) \\ &< \frac{9}{10}\left(\frac{1}{1 - \frac{1}{10}}\right) \\ &= 1 \end{aligned}$$

and hence $\lim_{n\to\infty} a_n$ exists, and we can define a real number by $\lim_{n\to\infty} a_n =: 0.d_1 d_2 \cdots$

Example 2 (The natural base e)**.** Define a seq. $a_n = (1 + 1/n)^n (n \in \mathbb{N})$, then we have

$$\begin{aligned} a_n &= \left(1 + \frac{1}{n}\right)^n \\ &= \binom{n}{0} + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{n}\frac{1}{n^n} \\ &= \sum_{j=0}^{n} \binom{n}{j}\frac{1}{n^j} = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \cdot \frac{1}{n^j} \\ &= \sum_{j=0}^{n} \frac{1}{j!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{j-1}{n}\right) \\ &< \sum_{j=0}^{n+1} \frac{1}{j!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{j-1}{n+1}\right) \end{aligned}$$

Thus $a_n \nearrow\nearrow$. On the other hand, for $\forall n \in \mathbb{N}$, we have

$$
\begin{aligned}
a_n &< \sum_{j=0}^{n+1} \frac{1}{j!} = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} \\
&< 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} \\
&= 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} \\
&< 3
\end{aligned}
$$

Thus a_n has an upper bound and hence a_n converges, and we define $\lim_{n\to\infty} a_n =: e$.

1 Nested Intervals

Definition 2 (Nested). A seq. of intervals $I_n (n \in \mathbb{N})$ is nested if $I_n \neq \emptyset$ and $I_{n+1} \subseteq I_n$ for $\forall n \in \mathbb{N}$.

Example 3. If we have a seq. of nested intervals $I_n (n \in \mathbb{N})$, do we have $\cap_{n\in\mathbb{N}} I_n \neq \emptyset$? The answer is not sure. For example,

1. $I_n = (0, 1/n), n \in \mathbb{N}$, then for any $r \in \mathbb{R}, \exists N \in \mathbb{N},$ s.t. $1/N < r$ by Archimedean Property, thus $r \notin I_N$, and hence $\cap_{n\in\mathbb{N}} I_n = \emptyset$;
2. $I_n = [n, \infty), n \in \mathbb{N}$, then for any $r \in \mathbb{R}, \exists N \in \mathbb{N},$ s.t. $r < N$ by Archimedean Property, thus $r \notin I_N$, and hence $\cap_{n\in\mathbb{N}} I_n = \emptyset$;

Theorem 2 (Theorem of Nested Interval). *If $I_n (n \in \mathbb{N})$ is a seq. of bounded closed nested intervals, then $\cap_{n\in\mathbb{N}} I_n \neq \emptyset$. (In the other word, there exists a real number $c \in \mathbb{R}$ such that $c \in \cap_{n\in\mathbb{N}} I_n$)*

Proof. Write $I_n = [a_n, b_n] (n \in \mathbb{N})$, then $I_n (n \in \mathbb{N})$ is nested $\Leftrightarrow$ $a_n \leq b_n$ and $a_n \nearrow$ and $b_n \searrow$. And furthermore, for $\forall n, m \in \mathbb{N}$,

$$a_n \leq a_{\max\{m,n\}} \leq b_{\max\{m,n\}} \leq b_m,$$

in the other word, for $\forall m \in \mathbb{N}$, b_m is an upper bound of $\{a_n | n \in \mathbb{N}\}$, thus seq. a_n converges. Let $c = \lim_{n\to\infty} a_n$, then given $m \in \mathbb{N}$, for $\forall n \in \mathbb{N}, a_n \leq b_m$ thus

$$c = \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_m = b_m.$$

On the other hand, $c = \sup\{a_n | n \in \mathbb{N}\}$, thus for all $m \in \mathbb{N}$, we have

$$a_m \leq c \leq b_m$$

thus $c \in I_m$ for $\forall m \in \mathbb{N} \Rightarrow c \in \cap_{n\in\mathbb{N}} I_n$. $\square$

Exercise 3. *Show that* $\cap_{n\in\mathbb{N}} I_n \neq \emptyset$, *if*
1. $I_n = (a_n, b_n)$, *nested and* $a_n \nearrow\nearrow$ *and* $b_n \searrow\searrow$*?*
2. $I_n = (a_n, \infty)$, *nested and* $\{a_n | n \in \mathbb{N}\}$ *is bounded from above.*

Proof. 1. Just as analyzed before, there exist $c \in \mathbb{R}$ such that $c = \lim_{n\to\infty} a_n$, and $c = \sup\{a_n | n \in \mathbb{N}\}$ and hence $a_n \leq c \leq b_m$ for $\forall n, m \in \mathbb{N}$. Note that $a_n \leq c$ implies that $a_n < c$ for $\forall n \in N$, otherwise if $\exists n' \in \mathbb{N}$, s.t. $a_{n'} = c$ then

$$a_{n'+1} \geq a_{n'} = c,$$

which leads to the contradiction. In the same way $c \leq b_m$ implies that $c < b_m$ for $\forall m \in \mathbb{N}$. Thus there $\exists c \in \mathbb{R}$ such that

$$a_n < c < b_m$$

for $\forall n, m \in \mathbb{N} \Rightarrow c \in \cap_{n\in\mathbb{N}} I_n$.
2. Since $I_n = (a_n, \infty)$ is a nested interval, $a_n \nearrow \Rightarrow a_n$ converges since a_n is upper bounded. That is $\exists c \in \mathbb{R}$, s.t. $c = \lim_{n\to\infty} a_n = \sup\{a_n\}$, thus for $\forall n \in \mathbb{N}, c \geq a_n$, that is

$$c + 1 > c \geq a_n$$

for $\forall n \in \mathbb{N} \Rightarrow c + 1 \in \cap_{n\in\mathbb{N}} I_n$. □

Exercise 4. *Show that Theorem of Nested Interval implies Dedekind's Gapless Property.*

Proof. Let (A, B) be a Dedekind cut of $\mathbb{R}$, pick a from A and b from B, and form an interval $I_0 = [a, b]$. Then $(a+b)/2$ lies in the middle of I_0 and must belong to A or B. If $(a+b)/2$ belongs to A, we let

$$a_1 = \frac{a+b}{2}, \quad b_1 = b$$

and if $(a+b)/2$ belongs to B, let

$$a_1 = a, \quad b_1 = \frac{a+b}{2}$$

and hence we can form a new interval $I_1 = [a_1, b_1]$ whose length is half of the former I_0. Repeat this process, we obtain a seq. of nested intervals

$$I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots,$$

where $I_n = [a_n, b_n], b_n - a_n = (b_{n-1} - a_{n-1})/2$. Thus there exists $s \in \mathbb{R}$ lies in the $\cap_{n\in\mathbb{N}} I_n$ by the theorem of nested intervals, and either $s \in A$ or $s \in B$.
Assume that $s \in A$, for any $s' \in \mathbb{R}, s < s'$, exists b_n such that $s < b_n < s'$ since $b_n \to s$, thus $s' \in B$. That is $s \in A$ and for any $s' > s, s' \in B$. In the other word, s is the maximal element of A and B has no minimal element in this case, since assume s' is the minimal element of B then $\exists b_n$, s.t. $b_n < s'$ and $b_n \in B$, which is a contradiction. □

Remark 1. Summary, we have discussed
1) Dedekind's Gapless Property;
2) Weierstrass Theorem;
3) Monotone Seq. Property;
4) Theorem of Nested Interval.
which have the relationship:

$$\begin{array}{ccc} 1) & \Longrightarrow & 2) \\ \Uparrow & & \Downarrow \\ 4) & \Longleftarrow & 3) \end{array}$$

These 5 properties are equivalent and we call the these the **Completeness of the real numbers**.

2 Limit Superior / Inferior

Let $a_n (n \in \mathbb{N})$ be a bounded (upper bdd. and lower bdd.) seq. in $\mathbb{R}$, we define **upper seq. of** a_n as

$$u_n := \sup\{a_m | m \geq n\},$$

and **lower seq. of** a_n as

$$l_n := \inf\{a_m | m \geq n\},$$

for $n \in \mathbb{N}$. Thus give $n \in \mathbb{N}$, we have that for $\forall m \geq n$

$$l_n \leq a_m \leq u_n,$$

We now show that l_n and u_n is monotone. Assume that $\exists n \in \mathbb{N}$, s.t. $u_n < u_{n+1}$, let $\epsilon = (u_{n+1} - u_n)/2$, then

$$u_{n+1} - \epsilon > u_n = \sup\{a_m | m \geq n\},$$

thus for $\forall m \geq n$, $u_{n+1} - \epsilon > a_m$ and hence $u_{n+1} - \epsilon$ is an upper bound of $\{a_m | m \geq n+1\}$, which leads to a contradiction. Thus for $\forall n \in \mathbb{N}, u_n \geq u_{n+1} \Rightarrow u_n \searrow$, and $l_n \nearrow$ in the same way.
Thus we have that for any $n, m \in \mathbb{N}$,

$$l_m \leq l_{\max\{m,n\}} \leq u_{\max\{m,n\}} \leq u_n,$$

thus l_1 is a lower bound for $\{u_n | n \in \mathbb{N}\}$ and u_1 is an upper bound of $\{l_n | n \in \mathbb{N}\}$ and hence $u_n, l_n (n \in \mathbb{N})$ are convergent by Monotone seq. property. We define the **limit superior** of a_n as the limit of u_n:

$$\overline{\lim_{n\to\infty}}\, a_n := \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sup_{m \geq n} a_m = \inf_{n \in \mathbb{N}} \sup_{m \geq n} a_m$$

The last equals sign is because $u_n \searrow$ and hence converges to its greatest lower bound by Monotone seq. property. And the **limit inferior** of a_n as the limit of l_n:

$$\varliminf_{n\to\infty} a_n := \lim_{n\to\infty} l_n = \lim_{n\to\infty} \inf_{m\geq n} a_m = \sup_{n\in\mathbb{N}} \inf_{m\geq n} a_m$$

Exercise 5. *Let $a_n (n \in \mathbb{N})$, show that*

$$a_n \text{ converges} \Leftrightarrow \varlimsup_{n\to\infty} a_n = \varliminf_{n\to\infty} a_n$$

and if any of both sides holds, then

$$\lim_{n\to\infty} a_n = \varlimsup_{n\to\infty} a_n = \varliminf_{n\to\infty} a_n$$

Proof. $\Rightarrow$: Suppose that $\lim_{n\to\infty} a_n = s$. Then for any $\epsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $\forall n \geq N$, $|a_n - s| < \epsilon/2$, thus $s - \epsilon/2 < a_n < s + \epsilon/2$ for $\forall n \geq N$. Thus the upper seq. u_n of a_n has

$$s - \frac{\epsilon}{2} < a_n \leq u_n \leq s + \frac{\epsilon}{2},$$

for $\forall n \geq N$. The third inequality symbol is because if $\exists n' \geq N$ such that $u_{n'} > s + \epsilon/2$, then there exist a real number q such that $s + \epsilon/2 < q < u_{n'}$ and $q > s + \epsilon/2 > a_n$ for $\forall n \geq N$ and hence $q > a_n$ for $\forall n \geq n'$, and then $u_{n'}$ is not the least upper bound of $\{a_n | n \geq n'\}$ which is contrary. Thus $|u_n - s| \leq \epsilon/2 < \epsilon$, thus

$$\lim_{n\to\infty} u_n = \varlimsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = s,$$

and $\lim_{n\to\infty} l_n = \varliminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = s$ in the same way.
$\Leftarrow$: Suppose $\lim_{n\to\infty} u_n = \lim_{n\to\infty} l_n = s$, then for $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall n \geq N$ one has $|u_n - s| < \epsilon/3$ and $|l_n - s| < \epsilon/3$ and $|u_n - l_n| \leq |u_n - s| + |l_n - s| < 2\epsilon/3$, since $l_n \leq a_n \leq u_n$ then $0 \leq a_n - l_n \leq u_n - l_n$. Then we have that

$$\begin{aligned} |a_n - s| &= |a_n - l_n + l_n - s| \\ &\leq |a_n - l_n| + |l_n - s| \\ &\leq |u_n - l_n| + |l_n - s| \\ &< \epsilon \end{aligned}$$

for $\forall n \geq N \Rightarrow \lim_{n\to\infty} a_n = \varlimsup_{n\to\infty} a_n = \varliminf_{n\to\infty} a_n = s$. $\square$

Exercise 6. *Let $a_n, b_n (n \in \mathbb{N})$ be two bdd. seq. show that*
1. $\varlimsup_{n\to\infty}(a_n + b_n) \leq \varlimsup_{n\to\infty} a_n + \varlimsup_{n\to\infty} b_n$;
2. $\varliminf_{n\to\infty} a_n + \varliminf_{n\to\infty} b_n \leq \varliminf_{n\to\infty}(a_n + b_n)$.