

Introduction to Analysis

Lecture 4

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Abstract

THIS IS THE LECTURE NOTE FOR THE *Introduction to Analysis* CLASS IN SPRING 2019.

1 Rearrangement theorem

Given a seq. $a_n (n \in \mathbb{N})$, we can separate all terms into two subseq.s:

$$a_{n_1}, a_{n_2}, \dots \text{ and } a_{n'_1}, a_{n'_2}, \dots$$

where $n_1 < n_2 < \dots$ and $n'_1 < n'_2 < \dots$ and $\{n_1, n_2, \dots\} \cup \{n'_1, n'_2, \dots\} = \mathbb{N}$, such that $a_{n_j} \geq 0 (j \in \mathbb{N}), a_{n'_k} \leq 0 (k \in \mathbb{N})$. Let $p_j := a_{n_j} (j \in \mathbb{N})$ and $q_k := a_{n'_k} (k \in \mathbb{N})$.

Exercise 1. Show that $\sum_n |a_n| < \infty \Leftrightarrow \sum_j p_j < \infty$ and $\sum_k q_k < \infty$. Moreover, if any side holds, then

$$\sum_n |a_n| = \sum_j p_j + \sum_k q_k$$

and

$$\sum_n a_n = \sum_j p_j - \sum_k q_k.$$

Proof. 1. \Rightarrow : since $\sum_n |a_n| < \infty$, any partial sum of a_n has upper bound such as M , then for any $j \in \mathbb{N}$:

$$\begin{aligned} p_1 + \dots + p_j &= |a_{n_1}| + \dots + |a_{n_j}| \\ &\leq \sum_{n=1}^{n_j} |a_n| \\ &\leq M, \end{aligned}$$

Thus any partial sum of p_j has upper bound M and hence $\sum_j p_j < \infty$. And $\sum_k q_k < \infty$ in the same way.

2. \Leftarrow : The partial sum of $\sum_n |a_n|$ can be decompose by the partial sums of $\sum_n p_n$ and $\sum_n q_n$ which have upper bounds, thus partial sum of $\sum_n |a_n|$ has upper bound, and $\sum_n |a_n| < \infty$.
3. Define the partial sum of $\sum_n |a_n|, \sum_n a_n, \sum_n p_n, \sum_n q_n$ as

$$A_m := \sum_{i=1}^m |a_i|, \quad S_m := \sum_{i=1}^m a_i$$

$$P_m := \sum_{i=1}^m p_i = \sum_{i=1}^m |a_{n_i}|, \quad Q_m := \sum_{i=1}^m q_i = \sum_{i=1}^m |a_{n'_i}|$$

Then for any $m \in \mathbb{N}$, we have that

$$A_m \leq P_m + Q_m \leq A_{\max\{n_m, n'_m\}},$$

thus $\lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} (P_n + Q_n) = \lim_{n \rightarrow \infty} P_n + \lim_{n \rightarrow \infty} Q_n$ since $\sum_n p_n, \sum_n q_n$ exists, and the squeeze theorem. And hence $\sum_n |a_n| = \sum_j p_j + \sum_k q_k$.

On the contrary, for any $m \in \mathbb{N}$, we can represent the partial sum of $\sum_n a_n$ as

$$s_m = P_l - Q_v$$

where $l, v \rightarrow \infty$ as $m \rightarrow \infty$, thus $\sum_n a_n = \sum_n p_n - \sum_n q_n$. □

Exercise 2. If $\sum_n a_n$ converges conditionally, show that

1. $\sum_j p_j = \infty$ and $\sum_k q_k = \infty$;
2. $\lim_{j \rightarrow \infty} p_j = \lim_{k \rightarrow \infty} q_k = 0$.

Proof. 1. Denote the partial sum of $\sum_n a_n, \sum_j p_j, \sum_k q_k$ as s_n, P_j, Q_k respectively, then we have that $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} (P_j - Q_k)$ exists, then either both $\lim_{n \rightarrow \infty} P_j, \lim_{n \rightarrow \infty} Q_k$ exist or neither exists, since $\sum_n a_n$ converges conditionally $\Rightarrow \lim_{n \rightarrow \infty} P_j = \infty$ and $\lim_{n \rightarrow \infty} Q_k = \infty$.

2. Since

$$\lim_{j \rightarrow \infty} p_j = \lim_{j \rightarrow \infty} a_{n_j} = \lim_{n \rightarrow \infty} a_n = 0,$$

and $\lim_{k \rightarrow \infty} q_k = 0$ as well in the same way. □

Exercise 3. If $\sum_n a_n, \sum_n b_n$ converges, show that $\sum_n (a_n \pm b_n) = \sum_n a_n \pm \sum_n b_n$.

Proof. Denote the partial sum of $\sum_n (a_n + b_n), \sum_n a_n, \sum_n b_n$ as S_n, A_n, B_n respectively, then for any $n \in \mathbb{N}$

$$S_n = A_n + B_n$$

and hence

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} (A_n + B_n) = \lim_{n \rightarrow \infty} A_n + \lim_{n \rightarrow \infty} B_n$$

since $\lim_{n \rightarrow \infty} A_n, \lim_{n \rightarrow \infty} B_n$ exists, thus $\sum_n (a_n + b_n) = \sum_n a_n + \sum_n b_n$, and $\sum_n (a_n - b_n) = \sum_n a_n - \sum_n b_n$ in the same way. □

Exercise 4. Inserting 0s to a series will not affect its convergence / divergence and its sum (if the sum exists).

Proof. Consider the tail of series. Trivial. \square

Recall that a sequence a_n is a map $\mathbb{N} \xrightarrow{a_\cdot} \mathbb{R}$ where $n \mapsto a(n)$ denoted by a_n . A subsequence a_{n_m} is a composite map

$$\mathbb{N} \xrightarrow{n_\cdot} \mathbb{N} \xrightarrow{a_\cdot} \mathbb{R}$$

where n_\cdot is a strictly monotone injection and $m \mapsto n(m)$ denoted by n_m . A **rearrangement** is also a composite map

$$\mathbb{N} \xrightarrow{n(\cdot)} \mathbb{N} \xrightarrow{a_\cdot} \mathbb{R}$$

where $n(\cdot)$ is a bijection. (To distinguish it from the notion of subseq., we don't use subscripts here)

Now we gonna explore two questions: if a series \sum_n converges, $a_{n(m)} (m \in \mathbb{N})$ is a rearrangement of $a_n (n \in \mathbb{N})$, then

1. whether $\sum_m a_{n(m)}$ converges ?
2. whether $\sum_n a_n = \sum_m a_{n(m)}$?

Exercise 5. Let $\sum_n a_n$ be a positive series, show that

$$\sum_n a_n = \sup \Lambda$$

including the case $\sum_n a_n = \infty$. Here $\Lambda = \{a_{n_1} + \dots + a_{n_k} \mid n_1 < \dots < n_k, k \in \mathbb{N}\}$ represents the set of every sum of finite terms of $a_n (n \in \mathbb{N})$.

Proof. 1. \leq : since $\sum_n a_n$ is the limit of the partial sum s_n (which is the sum of finite terms, i.e. $s_n \in \Lambda$ for any $n \in \mathbb{N}$), and since $a_n \geq 0$, s_n monotone, then

$$\sum_n a_n = \lim_{n \rightarrow \infty} s_n = \sup_{n \in \mathbb{N}} s_n \leq \sup \Lambda$$

2. \geq : If $\sup \Lambda > \sup s_n$, let $\epsilon := \sup \Lambda - \sup s_n$, then $\exists \lambda = a_{n_1} + \dots + a_{n_{k_\lambda}} \in \Lambda$ such that

$$s_m \leq \sup_{n \in \mathbb{N}} s_n < \sup \Lambda - \frac{\epsilon}{2} < \lambda \leq \sup \Lambda$$

for $\forall m \in \mathbb{N}$, but

$$\lambda = a_{n_1} + \dots + a_{n_{k_\lambda}} \leq s_{n_{k_\lambda}}$$

which leads to a contradiction.

3. If $\sum_n a_n = \infty$, it is direct to see that $\sup \Lambda = \infty$ as well by 1. \square

Exercise 6. If $\sum_n a_n$ is a convergent positive series, show that for every rearrangement $a_{n(m)}$ ($m \in \mathbb{N}$) of a_n ($n \in \mathbb{N}$), we have that $\sum_n a_n = \sum_m a_{n(m)}$.

Proof. If $\sum_n a_n$ is positive series, then $\sum_m a_{n(m)}$ is positive series as well.

$$\sum_n a_n = \sup \Lambda_{a_n} = \sup \Lambda_{a_{n(m)}} = \sum_m a_{n(m)}$$

where Λ_{a_n} and $\Lambda_{a_{n(m)}}$ are the set of every sum of finite terms of a_n and $a_{n(m)}$ respectively. That is the proof follows by the $\Lambda_{a_n} = \Lambda_{a_{n(m)}}$. \square

Exercise 7 (Dirichlet's Rearrangement Theorem (1829)). If $\sum_n a_n$ converges absolutely, show that for every rearrangement $a_{n(m)}$ ($m \in \mathbb{N}$) of a_n ($n \in \mathbb{N}$), we have that $\sum_n a_n = \sum_m a_{n(m)}$.

Proof. $\sum_n a_n$ converges absolutely $\Rightarrow \sum_m a_{n(m)}$ converges absolutely. Furthermore

$$\begin{aligned} \sum_n a_n &= \sum_j p_j - \sum_k q_k \\ &= \sum_{\mu} p_{j_{\mu}} - \sum_{\nu} q_{k_{\nu}} \\ &= \sum_m a_{n_m}. \end{aligned}$$

\square

Theorem 1 (Riemann's Rearrangement Theorem(1852)). If $\sum_n a_n$ converges conditionally, then for $\forall r \in \mathbb{R}$, there exists a rearrangement $a_{n(m)}$ ($m \in \mathbb{N}$) of a_n ($n \in \mathbb{N}$) such that $\sum_m a_{n(m)} = r$.

Proof. We will only use two known fact:

1. $\sum_j p_j = \infty$ and $\sum_k q_k = \infty$;
2. $\lim_{j \rightarrow \infty} p_j = \lim_{k \rightarrow \infty} q_k = 0$.

Given a $L \in \mathbb{R}$, start with p_1 , plus by p_2 and so on till p_{m_1-1} where

$$\sum_i^{m_1-1} p_i \leq L \quad \text{but} \quad \sum_i^{m_1} p_i > L.$$

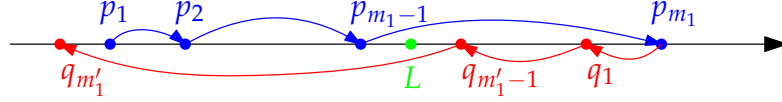
Then minus by q_1, q_2 and so on till $q_{m'_1-1}$ where

$$\sum_i^{m_1} p_i - \sum_{j=1}^{m'_1-1} q_j > L \quad \text{but} \quad \sum_i^{m_1} p_i - \sum_{j=1}^{m'_1} q_j \leq L.$$

This process can be repeat since $\sum_j p_j = \infty$ and $\sum_k q_k = \infty$ and hence any tail of $\sum_j p_j, \sum_k q_k$ has no upper bound, therefore the cross action can always happen, in the other word, $m_i, m'_i (i \in \mathbb{N})$ exists.

Thus we can form a rearrangement χ_n of $\sum_n a_n$ as

$$p_1, \dots, p_{m_1}, -q_1, \dots, -q_{m'_1}, \dots$$



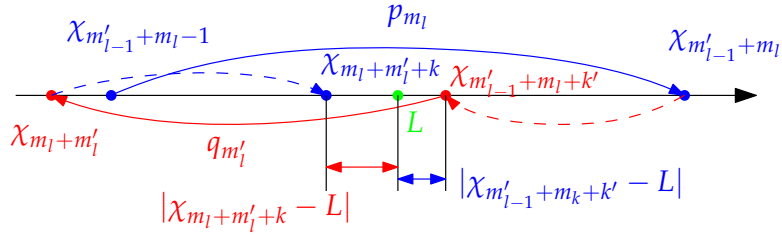
Now we will show that this rearrangement converges to L , i.e. $\lim_{n \rightarrow \infty} \chi_n = L$. Consider $\chi_{\dots+m'_{l-1}+m_l-1}$ which implies the point lies in the left of L and will cross the L in next jump, and we denote it by $\chi_{m'_{l-1}+m_l-1}$ for simply. And hence we have that

$$|\chi_{m'_{l-1}+m_l+k'} - L| < p_{m_l}$$

if $0 \leq k' < m'_l - m'_{l-1}$. And similarly

$$|\chi_{m_l+m'_l+k} - L| < q_{m'_l}$$

if $0 \leq k < m_{m+1}-m_l$.



And since $\lim_{l \rightarrow \infty} p_{m_l} = \lim_{l \rightarrow \infty} q_{m'_l} = 0$, for $\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ s.t. $l \geq N_0 \Rightarrow p_{m_l}$ and $q_{m'_l} < \epsilon$. Let $N = m'_{N_0-1} + m_{N_0}$, then $n \geq N \Rightarrow |\chi_n - L| < \epsilon$. \square

Remark 1 ($2S = S$). Dirichelet made the following observation in 1827: Consider a alternating series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

which is convergent (conditionally) by Leibniz's Criterion, and hence

$$\begin{aligned} 2S &= 2 - 1 + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \frac{2}{7} - \frac{2}{8} + \frac{2}{9} - \frac{2}{10} + \dots \\ &= (2 - 1) - \frac{2}{4} + \left(\frac{2}{3} - \frac{2}{6} \right) - \frac{2}{8} + \left(\frac{2}{5} - \frac{2}{10} \right) + \dots \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \\ &= S \end{aligned}$$

Remark 2. In summary, given a series $\sum_n a_n$, and its any rearrangement $\sum_m a_{n(m)}$, then

1. If $a_n \geq 0$ for $\forall n \in \mathbb{N} \Rightarrow \sum_n a_n = \sum_m a_{n(m)}$;
2. If $\sum_n |a_n| < \infty \Rightarrow \sum_n a_n = \sum_m a_{n(m)}$;
3. If $\sum_n |a_n| = \infty$ but $\sum_n a_n < \infty \Rightarrow \sum_m a_{n(m)}$ could be anything.

2 Multiplying absolutely convergent series

Proposition 1. Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely, let

$$c_n = a_n b_0 + \cdots + a_0 b_n = \sum_{j+k=n; j,k \geq 0} a_j b_k,$$

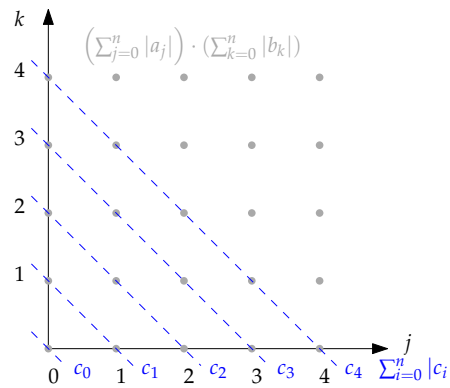
then $\sum_n |c_n| < \infty$ and $\sum_{n=0}^{\infty} c_n = (\sum_{n=0}^{\infty} a_n) \cdot (\sum_{n=0}^{\infty} b_n)$.

Proof. 1. $\sum_n |c_n| < \infty$

For all n ,

$$\begin{aligned} \sum_{m=0}^n |c_m| &= \sum_{m=0}^n \left| \sum_{\substack{j+k=m \\ j,k \geq 0}} a_j b_k \right| \leq \sum_{m=0}^n \sum_{\substack{j+k=m \\ j,k \geq 0}} |a_j| |b_k| \\ &\leq \left(\sum_{j=0}^n |a_j| \right) \cdot \left(\sum_{k=0}^n |b_k| \right). \end{aligned}$$

Since $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converges absolutely, the partial sums of $|a_n|, |b_n|$ have upper bounds, denoted by M, N respectively, then $\sum_{m=0}^n |c_m|$ has a upper bound $M \cdot N$ and hence $\sum_{n=0}^{\infty} c_n$ converges absolutely.



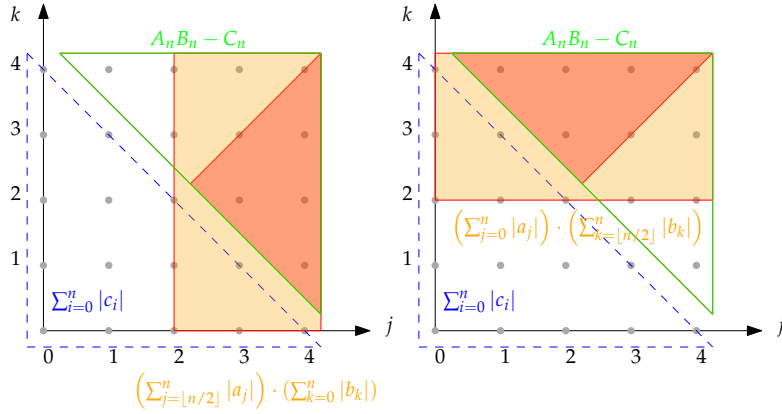
$$2. \sum_{n=0}^{\infty} c_n = (\sum_{n=0}^{\infty} a_n) \cdot (\sum_{n=0}^{\infty} b_n).$$

Let $A_n := a_0 + \cdots + a_n$, $B_n := b_0 + \cdots + b_n$ and $C_n := c_0 + \cdots + c_n$, we claim that $\lim_{n \rightarrow \infty} (A_n B_n - C_n) = 0$. Then

$$\begin{aligned}
|A_n B_n - C_n| &= \sum_{\substack{j+k > n \\ 0 \leq j, k \leq n}} |a_j b_k| \\
&\leq \left(\sum_{j=\lfloor n/2 \rfloor}^n |a_j| \right) \cdot \left(\sum_{k=0}^n |b_k| \right) + \left(\sum_{j=0}^n |a_j| \right) \cdot \left(\sum_{k=\lfloor n/2 \rfloor}^n |b_k| \right)
\end{aligned}$$

where $\sum_{k=0}^n |b_k|, \sum_{j=0}^n |a_j|$ are bounded, and tails $\sum_{j=\lfloor n/2 \rfloor}^n |a_j|, \sum_{k=\lfloor n/2 \rfloor}^n |b_k| \rightarrow 0$ as $n \rightarrow \infty$ since $\sum_n a_n, \sum_n b_n$ are converges abs. Thus $\lim_{n \rightarrow \infty} |A_n B_n - C_n| = 0$ and since $\lim_{n \rightarrow \infty} A_n, \lim_{n \rightarrow \infty} B_n, \lim_{n \rightarrow \infty} C_n$ exists, we have that

$$\begin{aligned}
\sum_{n=0}^{\infty} c_n &= \lim_{n \rightarrow \infty} C_n \\
&= \lim_{n \rightarrow \infty} A_n B_n = \lim_{n \rightarrow \infty} A_n \cdot \lim_{n \rightarrow \infty} B_n \\
&= \left(\sum_{n=0}^{\infty} a_n \right) \cdot \left(\sum_{n=0}^{\infty} b_n \right)
\end{aligned}$$



□

Theorem 2. If $\sum_n a_n, \sum_n b_n$ cvg. abs., $\mathbb{N} \xrightarrow{(j(\cdot), k(\cdot))} \mathbb{N} \times \mathbb{N}$ is bijection where $n \mapsto (j(n), k(n))$, let $c_n := a_{j(n)} b_{k(n)}$ ($n \in \mathbb{N}$), then $\sum_n |c_n| < \infty$ (cvg. abs.) and $\sum_n c_n = (\sum_n a_n)(\sum_n b_n)$.

Proof. 1. $\sum_n c_n$ cvg. abs.

For $\forall n \in \mathbb{N}$, let $l = \max\{j(1), \dots, j(n), k(1), \dots, k(n)\}$. Then

$$\begin{aligned}
|c_1| + \dots + |c_n| &= |a_{j(1)} b_{k(1)}| + \dots + |a_{j(n)} b_{k(n)}| \\
&\leq \left(\sum_{j=1}^l |a_j| \right) \cdot \left(\sum_{k=1}^l |b_k| \right) \\
&\leq M \cdot N
\end{aligned}$$

Thus $\sum_n c_n$ cvg. abs.

2. $\sum_n c_n = (\sum_n a_n)(\sum_n b_n)$.

Let $A_n = a_1 + \cdots + a_n$, $B_n = b_1 + \cdots + b_n$ and $C_n = c_1 + \cdots + c_n$ ($n \in \mathbb{N}$). And define the bijection $\mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ by the second one in Figure 1. Then

$$\begin{aligned} A_n B_n &= (a_1 + \cdots + a_n)(b_1 + \cdots + b_n) \\ &= \sum_{1 \leq j, k \leq n} a_j b_k \\ &= C_{n^2} \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} A_n B_n = \lim_{n \rightarrow \infty} C_{n^2} = \lim_{n \rightarrow \infty} C_n \Rightarrow \sum_n c_n = (\sum_n a_n)(\sum_n b_n)$. \square

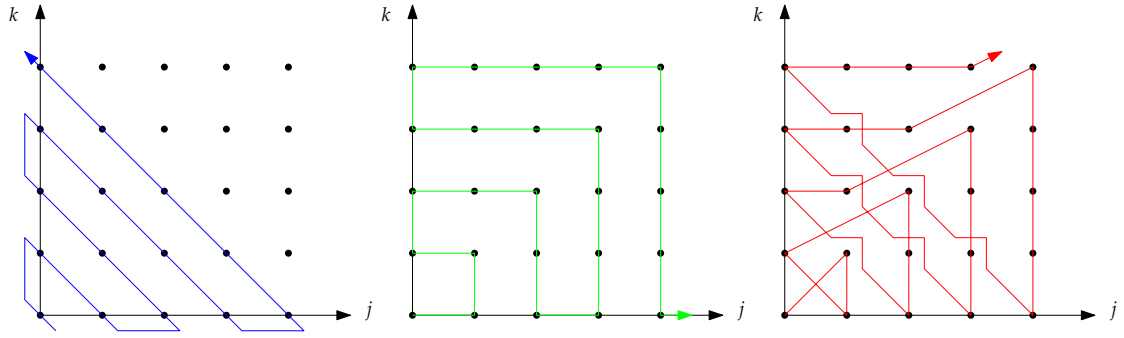


Figure 1: 3 kinds of bijections $(j(\cdot), k(\cdot))$