Introduction to Topology

Naive Set Theory, Lecture 2

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THIS IS THE LECTURE NOTE FOR THE *Introduction to Topology*. The course covers the following topics: Naive Set Theory, Elementary Number Theory, Group Theory, Topological Spaces and Continuous Maps, Introduction to Algebraic Topology.

CONTENT:

1. Maps
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Maps

Definition 1 (injection, surjection and bijection)**.** We say a map $X \xrightarrow{f} Y$ is an injection (1-1) if for $\forall x, x' \in X, f(x) = f(x')$ then $x = x'$; a surjection (onto) if $\forall y \in Y, \exists x \in X,$ s.t. $f(x) = y$; a bijection (1-1 correspondence) if it is an injection and also a surjection.

Note 1. When we say a map $X \xrightarrow{f} Y$, we want say $\forall x \in X, \exists! y \in Y,$ s.t. $y = f(x)$. When we try to think the occasion that from Y to X, the conception of *injection* preserve the "$\exists!$" of a map, and the *surjection* guarantees the "$\forall$" of a map.

If $X \xrightarrow{f} Y$ is a bijection, it has an inverse map $X \xleftarrow{f^{-1}} Y$. Notice that the inverse map f^{-1} is not the same as the pre-image f^{-1}.

For a bijection, the relationship between these is: for $y \in Y$ then

$$\{f^{-1}(y)\} = f^{-1}(\{y\}).$$

For the others cases, there does not exist an inverse map.

Exercise 1. Given maps $X \xrightarrow{f} Y, Y \xrightarrow{g} Z$, show that:

1. $g \circ f$ is an injective $\Rightarrow$ f is an injective;
2. $g \circ f$ is a surjective $\Rightarrow$ g is a surjective.

Proof. 1. Since $g \circ f$ is injection, thus for any different $x_1, x_2 \in X$, we have $g(f(x_1)) \neq g(f(x_2))$, thus $f(x_1) \neq f(x_2)$, and f is injection.

2. Since $g \circ f$ is surjection, thus for any $z \in Z$ there exists $x \in X,$ s.t. $g(f(x)) = z$, which means $\exists y = f(x),$ s.t. $z = g(y)$, thus g is surjection.

□

Exercise 2. Given maps $X \xrightarrow{f_1} Y, X \xrightarrow{f_2} Y, Y \xrightarrow{g} Z$, if g is an injection, and $g \circ f_1 = g \circ f_2$ show that $f_1 = f_2$. Correspondingly, Given maps $X \xrightarrow{f} Y, Y \xrightarrow{g_1} Z, Y \xrightarrow{g_2} Z$, if f is a surjection, and $g_1 \circ f = g_2 \circ f$ show that $g_1 = g_2$.

Proof. 1. For $\forall x \in X$, we have $g(f_1(x)) = g(f_2(x))$, since g is injection, thus $f_1(x) = f_2(x)$, and $f_1 = f_2$;

2. Since f is surjection, thus $f(X) = Y$, and $g_1(f(x)) = g_2(f(x))$ for any $x \in X$, thus $g_1(y) = g_2(y)$ for any $y \in Y$, and $g_1 = g_2$.

□

Cardinality

Def.

Definition 2. Two sets X, Y have the same cardinality, if $\exists$ bijection $X \xrightarrow{f} Y$, denote as $|X| = |Y|$.

Definition 3. A set X has its cardinality smaller or equal to that of a set Y if $\exists$ an injection $X \xrightarrow{f} Y$, denote as $|X| \leq |Y|$.

Note 2. The subset of a set could have the same cardinality with it. For example, just as mentioned last lecture, $|\mathbb{N}| = |\mathbb{Z}|$.

$\mathbb{N}$ *and* $\mathbb{Q}$

We will show that the natural number set $\mathbb{N}$ could 1-1 correspond to rational number set $\mathbb{Q}$. List the rational number as a matrix, we can encode them from southwest to northeast line by line, and skip the rational number that has been encoded. We can see that specify any natural number n, there is a definite law to query the corresponding rational number in $\mathbb{Q}$ or vice versa. Thus $|\mathbb{N}| = |\mathbb{Q}|$.

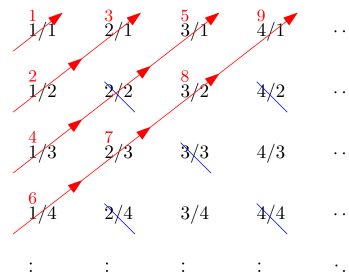

Figure 1: $\mathbb{Q} \leftrightarrow \mathbb{N}_0$

$\mathbb{N}$ *and* $\mathbb{R}$

Thus we can see that the natural number set $\mathbb{N}$ can correspond with rational number set $\mathbb{Q}$ 1 by 1, although it is density. But how about the real number set $\mathbb{R}$? Before we answer this question, we need to recall the definition of real number in Decimal notation.

Given a real number in decimal notation, like $r = 0.112123123412345\cdots$, what does it mean? Define a family of close intervals $I_{i,j}(i \in \mathbb{N}, j \in \{0, 1, \cdots, 9\})$, where $I_{0,0} = [0, 1]$ and $I_{i,j}$ is the $j+1$-th part of tenth division of $I_{i-1,*}$. For example, $I_{1,3}$ is the 4-th of ten division of $I_{0,0}$, thus $I_{1,3} = [0.3, 0.4]$. On this base, $I_{2,2} = [0.32, 0.33]$, and $I_{3,9} = [0.329, 0.330]$ and so on. Thus we have that

$$I_{0,0} \supseteq I_{1,*} \supseteq I_{2,*} \supseteq I_{3,*} \supseteq \cdots$$

Thus the definition of real number in decimal notation is the intersection of thus a family of interval, for example,

$$r = I_{0,0} \cap I_{1,1} \cap I_{2,1} \cap I_{3,2} \cap I_{4,1} \cap I_{5,2} \cap \cdots ;$$

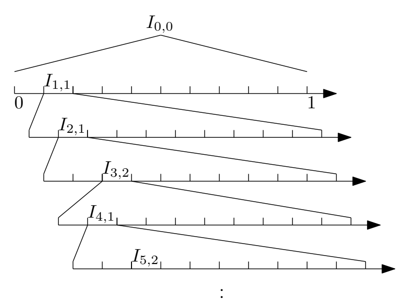

Figure 2: real number in decimal notation

Since the length of $I_{i,*}$ is the one tenth of the $I_{i-1,*}$, the length of interval will trend to 0 as i approaches to ∞. Thus any given decimal notation only represents one real number. If there is a decimal notation $\{I_{i,j}\}$ that denotes two different real number r, r', where $d(r, r') > 0$. then there exist N for any $i > N$, the length of $I_{i,*}$ is small than $d(r, r')$, thus $I_{i,*}$ can not cover r, r' at the same time, which leads to a contradiction.

But please note that, although a given decimal notation only represents one real number, some real number could be represented in two kind of decimal notations. This kind of real number is so called *finite decimal*, that is it locates on the bounds of some intervals. Like $r' = 0.113$ falls on the right boundary of $I_{3,2} = [0.112, 0.113]$ and the left boundary of $I_{3,3} = [0.113, 0.114]$, thus

$$r' = I_{0,0} \cap I_{1,1} \cap I_{2,1} \cap I_{3,3} \cap I_{4,0} \cap I_{5,0} \cdots$$

and could be written as $r' = 0.113000 \cdots$; but as we said, r' can also be covered by another family of intervals:

$$r' = I_{0,0} \cap I_{1,1} \cap I_{2,1} \cap I_{3,2} \cap I_{4,9} \cap I_{5,9} \cdots$$

thus it could be also written as $r' = 0.112999 \cdots$, and these two forms are equivalent. We call the latter form of expression as *infinite expression*.

Proposition 1 (Cantor). *$\nexists$ a surjection such that $\mathbb{N} \xrightarrow{f} \mathbb{R}$.*

Proof. Assume that f is a surjection from $\mathbb{N}$ to $\mathbb{R}$. Write down the maps relationship in infinite expression:

$$\begin{aligned} f(1) &= a_1 + 0.a_{11}a_{12}a_{13} \cdots \\ f(2) &= a_2 + 0.a_{21}a_{22}a_{23} \cdots \\ f(3) &= a_3 + 0.a_{31}a_{32}a_{33} \cdots \\ f(4) &= a_4 + 0.a_{41}a_{42}a_{43} \cdots \\ &\vdots \end{aligned}$$

Where $a_i \in \mathbb{Z}, a_{ij} \in \mathbb{N}(i, j \in \mathbb{N})$. Define a real number $r = b + 0.b_1b_2b_3 \cdots$, such that $b \in \mathbb{Z}$ and b_i is the smallest number among $\{1, 2, \cdots, 9\}$ which is not a_{ii}. Thus r is not equal to any of the numbers on the right-hand side of the above equations, which represent $\mathbb{R}$ since f is surjection. Thus it leads to a contradiction. $\square$

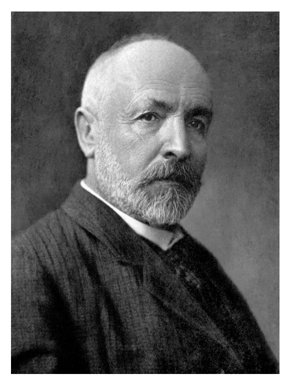
Figure 3: Georg Cantor (1845-1918)

This proof method is called *Cantor's diagonal argument*, it is a powerful weapon.

S and $\mathcal{P}(S)$

If S is a finite set, then the number of elements in S and $\mathcal{P}(S)$ are n and 2^n respectively. It is easy to check that there is no 1 to 1 correspondence between S and $\mathcal{P}(S)$ since $n < 2^n$ for any $n \in \mathbb{N}$. But what if S is infinite? We will elaborate it beginning with the case $S = \mathbb{N}$

Proposition 2. *$\nexists$ a surjection such that $\mathbb{N} \xrightarrow{f} \mathcal{P}(\mathbb{N})$.*

Proof. Suppose there exists a surjection such that $\mathbb{N} \xrightarrow{f} \mathcal{P}(\mathbb{N})$, then for any natural number n, $f(n) \subseteq \mathbb{N}$. Denote $f(n)$ as $a_{n1}a_{n2}a_{n3}\cdots$ where if $i \in f(n)$ then set $a_{ni} = 1$, otherwise set $a_{ni} = 0$. Thus we have:

$$\begin{aligned}
f(1) &= a_{11}a_{12}a_{13}a_{14}\cdots \\
f(2) &= a_{21}a_{22}a_{23}a_{24}\cdots \\
f(3) &= a_{31}a_{32}a_{33}a_{34}\cdots \\
f(4) &= a_{41}a_{42}a_{43}a_{44}\cdots \\
&\vdots
\end{aligned}$$

Define a series $b = b_1b_2b_3b_4\cdots$ where $b_i \in \{0,1\}$ and $b_i \neq a_{ii}$, thus the subset of $\mathbb{N}$, which is in $\mathcal{P}(N)$, represented by b is not in the $f(\mathbb{N})$, thus f is not a surjection. $\square$

Note 3. That is, for example, if $6 \notin f(6)$ then select 6 in b otherwise the opposite. Clarify this will help to understand the proof in the general case.

Proposition 3. *$\nexists$ a surjection such that $S \xrightarrow{f} \mathcal{P}(S)$ for any set S.*

Proof. Suppose f is a surjection such that $S \xrightarrow{f} \mathcal{P}(S)$. Then for any $x \in S$, we have $f(x) \in \mathcal{P}(S)$ is a subset of S. Define a subset of S: $A := \{x \in S | x \notin f(x)\}$ (which is just the series $b_1b_2b_3b_4\cdots$ in the last case), we will show that $A \notin f(S)$.

If $A \in f(S)$, then $\exists s \in S$, such that $A = f(s)$. If $s \in A = f(s)$, then $s \notin A$; If $s \notin A = f(s)$ then $s \in A$, which all lead to contradiction, thus $A \notin f(S)$, and f is not a surjection. $\square$

$\mathbb{R}$ *and* $\mathbb{C}$

Proposition 4. *Given sets S, T. If exist two injections f, g such that $S \xrightarrow{f} T$ and $T \xrightarrow{g} S$, then exist a bijection h such that $S \xrightarrow{h} T$. Briefly, $|S| \leq |T| \wedge |T| \leq |S| \Rightarrow |T| = |S|$.*

Proof. For any point $s \in S$, We do two operations: Inferring and tracing, that is what is the point $t \in T$ such that $t = f(s)$; and whether there exists a point $t' \in T$ such that $s = g(t')$. And repeat the operations above in S and T alternatively.

Since f, g are injection, thus we can always infer next step infinitely, that is for $\forall s \in S$, there exist a t such that $t = f(s)$, and then $\exists s'$, s.t. $s' = g(t)$, and then $\exists t'$, s.t. $t' = f(s')$, and so on.

But when tracing the point s (or t), there would be two occasions, (1) there is no t' (or s'), such that $t' = f(s)$ (or $s' = f(t)$). (2) There is one and only one to correspond. Thus when we infer and trace for all elements in S and T, there would be only 4 kinds of occasions:

1. Infer infinity and trace end at T:

$$T \xrightarrow{g} S \xrightarrow{f} T \xrightarrow{g} S \xrightarrow{f} T \xrightarrow{g} S \xrightarrow{f} \cdots$$

2. Infer infinity and trace end at S:

$$S \xrightarrow{f} T \xrightarrow{g} S \xrightarrow{f} T \xrightarrow{g} S \xrightarrow{f} T \xrightarrow{g} \cdots$$

3. Infer and trace construct a loop:

$$\begin{tikzcd} S \arrow[r, "f"] & T \arrow[r, "g"] & S \arrow[r, "f"] & \cdots \arrow[r, "g"] & S \arrow[r, "f"] & T \arrow[lllll, bend right, "g"'] \end{tikzcd}$$

4. Infer and trace infinity without repeat:

$$\cdots \xrightarrow{f} T \xrightarrow{g} S \xrightarrow{f} T \xrightarrow{g} S \xrightarrow{f} T \xrightarrow{g} \cdots$$

These 4 occasion consist of all element of S and T, and there is nothing in common between any two occasions. Thus we can define a bijection h from S to T: for any $s \in S$, if s belongs to the last 3 occasions, then $h(s) = f(s)$; if s belongs to the first occasion, then $h(s) = \arg_t\{s = g(t)\}$. Thus for any $t \in T$ there exists a $s \in S$, such that $t = h(s)$, and for any s_1, s_2 ($s_1 \neq s_2$), we have $h(s_1) \neq h(s_2)$, since f, g are injections. Thus $S \xrightarrow{h} T$ is a bijection, and $|S| = |T|$. □

Proposition 5. *$\exists$ a bijection f such that $\mathbb{R} \xrightarrow{f} \mathbb{C}$.*

Proof. Only thing we need to do is construct two injection between $\mathbb{R}$ and $\mathbb{C}$. Define for any $r \in \mathbb{R}$, $f(r) = (r, r)$, then $\mathbb{R} \xrightarrow{f} \mathbb{C}$ is an injection. For any $(a, b) \in \mathbb{C}$, we could write them as infinite expression decimal notation:

$$a = a_0 + 0.a_1a_2a_3\cdots$$
$$b = b_0 + 0.b_1b_2b_3\cdots$$

where $a_i, b_i (i \in \mathbb{N}_0) \in \mathbb{N}_0$. Define $g(a, b) = 0.a_0b_0a_1b_1a_2b_2a_3b_3\cdots \in \mathbb{R}$, thus $\mathbb{C} \xrightarrow{g} \mathbb{R}$ is a injection. □