General Topology

Collection

Haoming Wang

11 July 2020

Abstract

This is the collection of lecture notes for the *General Topology* course in Spring 2020. (containing some materials of *Introduction to Topology* course, This note is recommended, rather than the note of *Introduction to Topology*).

Contents

Chapter 1

Axiom of Choice

1.1 Order and bound

Definition 1 (Partial Order)**.** Given a set X, a relation $\leq$ on X is a partial order if
1. $\forall x \in X \Rightarrow x \leq x$;
2. $\forall x, x' \in X, x \leq x', x' \leq x \Rightarrow x = x'$;
3. $\forall x, x', x'' \in X, x \leq x', x' \leq x'' \Rightarrow x \leq x''$.
We say that $(X, \leq)$ is a partially ordered set (poset).

Remark 1. A relation on X, is a subset of $X \times X$. If $(X, \leq)$ is a poset and $A \subseteq X$, then A has a natural partial order induced by $\leq$.

Example 1. For example, $\leq$ is a partial order on $\mathbb{R}$; given a set X, $\subseteq$ is a partial on $\mathcal{P}(X)$.

Definition 2 (Total Order, Chain)**.** A poset $(X, \leq)$ is a chain (or totally order set) if $\forall x, x' \in X$, then $x \leq x'$ or $x' \leq x$.

Definition 3 (Bound)**.** If $(X, \leq)$ is a poset, $A \subseteq X, b \in X$, we say
1. b is an upper (lower) bound of A (in X w.r.t. $\leq$) if $\forall a \in A, a \leq b (b \leq a)$, denoted the set of upper (lower) bound of A by $U_A(L_A)$.
2. b is a greatest (least) element of A (in X w.r.t. $\leq$), if b is an upper (lower) bound of A and $b \in A$.
3. b is the least upper bound (greatest lower bound) of A, if b is the least (greatest) element of the set of upper bound (lower bound) of A, denoted by lub or sup A (glb or inf A).
4. b is a maximal (minimal) element in X if $b \in X, \forall x \in X, b \leq x \Rightarrow b = x (x \leq b \Rightarrow x = b)$.

Remark 2 (Maximal vs. Greatest). An element $m \in X$ is **maximal** if there does not exist $x \in X$ such that $x > m$. An element $g \in X$ is **greatest** if for all $x \in X$, $g \geq x$.

1. A set may have no greatest elements and no maximal elements: for example, any open interval of real numbers.
2. If a set has a greatest element, that element is also maximal.
3. A set with two maximal elements and no greatest element: $X = \{a, b, c\}$, where $a \leq b, a \leq c$ and b and c are incomparable, then each of b and b are maximal, and none of the elements of this set are greatest.
4. A set can have exactly one maximal element but no greatest element: $X = \{a + q | 0 \leq q < 1\} \cup \{c\}$, where $a \leq c$ and $a + q$ and c are incomparable for any $0 \leq q < 1$. Then only c is maximal, and the set overall has no greatest element.

Example 2. Let $A = [0, 1)$, the set of upper bound of A is $[1, \infty)$, the set of lower bound of A is $(-\infty, 0]$. Thus $\sup A = 1, \inf A = 0$.

Exercise 1. *Suppose $S \subseteq_{close} \mathbb{R}$ and $S \neq \emptyset$, show that S has an upper bound $\Rightarrow \sup S \in S$.*

Proof. Let $s_0 := \sup S$. If $s_0 \notin S \Rightarrow s_0 \in \mathbb{R} \backslash S \subseteq_{open} \mathbb{R}$. Thus $\exists r > 0$, s.t. $B_r(s_0) \in \mathbb{R} \backslash S$, that is $s_0 - r \in \mathbb{R} \backslash S \Rightarrow s_0 - r \geq s (\forall s \in S)$. But s_0 is the smallest upper bound, then $\forall s' < s_0, \exists s \in S$, s.t. $s > s'$, which leads to a contradiction. $\square$

Definition 4 (Well Order)**.** If $(X, \leq)$ is a chain, we say that $(X, \leq)$ is a well-ordered set if $\forall A \subseteq X, A \neq \emptyset \Rightarrow A$ has a least element.

For example, $\mathbb{Z}^+$ is a well-ordered set. If $(X, \leq)$ is a well-ordered set, for any $a \in X$, the **successor** of a is $succ_{(X,\leq)}(a) :=$ the least element of $\{x \in X | a < x\}$. So if $\{x \in X | a < x\} \neq \emptyset$, then $succ_{(X,\leq)}(a)$ exists.

Remark 3. Given a poset X, $a, b \in X$, we say $a < b$ if $a \leq b$ and $a \neq b$.

Definition 5. Given a poset X, $a \in X$, define initial segment as

$$IS_{(X,\leq)}(a) := \{x \in X | x < a\}$$

and weak initial segment as

$$WIS_{(X,\leq)}(a) := \{x \in X | x \leq a\}.$$

1.2 Axiom of Choice

Theorem 1 (Bourbaki's fixed point theorem)**.** *Suppose $(X, \leq)$ is a poset, in which every well-ordered subset has lub. Given a map $X \xrightarrow{f} X$,* s.t. *$x \leq f(x)$ for $\forall x \in X$, then $\exists a \in X$,* s.t. *$f(a) = a$.*

Proof. Pick an element $x_0 \in X$. Let S be the collection of subsets $Y \subseteq X$ such that:

- Y is well ordered with the least element x_0 and successor function $f|_{Y \backslash lub Y}$,

- $x_0 \neq y \in Y \Rightarrow lub_X(IS_Y(y)) \in Y$.

Then we claim:

1. If $Y \in S$ and $Y' \in S$, then Y is an initial segment of Y' or vice versa.
   Let $V = \{x \in Y \cap Y' | WIS_Y(x) = WIS_{Y'}(x)\}$. Suppose first that V has a last element v. If v is not the last element of Y, then $succ_Y(v) = f(v)$; if v is not the last element of Y' then $succ_{Y'}(v) = f(v)$. Hence if neither of Y, Y' is an initial segment of the other, then $succ_Y(v) = succ_{Y'}(v) = f(v) \in V$, thus $f(v) = v$, and v is the fixed point.
   If V has no last element, let $z = lub_X(V)$. If $Y \neq V \neq Y'$, then it follows that $z \in Y \cap Y'$ (because if $y = \inf(Y - V)$ then $V = IS_Y(y)$ and therefore $z = lub_X(IS_Y(y)) \in Y$). Therefore $z \in V$, which is a contradiction.
2. The set $Y_0 = \cup\{Y | Y \in S\} \in S$.
   If $y_0 \in Y \in S$, then it follows from 1. that $\{y \in Y_0 | y < y_0\} = IS_Y(y_0)$ and so this subset is well ordered with successor function f. This implies that Y_0 is well ordered and satisfies first conditions of element in S. Also $lub_X(IS(y_0)) \in Y \subseteq Y_0$ which gives the second condition for Y_0. Thus 2. is proved.

Let $y_0 = lub_X(Y_0)$, if $y_0 \notin Y_0$ then $Y_0 \cup \{y_0\} \in S$ and so $y_0 \in Y_0$ after all. If $f(y_0) > y_0$ then $Y_0 \cup \{f(y_0)\} \in S$ contrary to the definition of Y_0, thus $f(y_0) = y_0$ as desired. □

Remark 4. A map $X \xrightarrow{f} Y$ is a subset $\Gamma \subseteq X \times Y$, s.t. $\forall x \in X, \exists! y \in Y, (x, y) \in \Gamma$.

Theorem 2. *The following statement are equivalent:*

1. *For $\forall$ set X, $\exists$ map $\mathcal{P}_o(X) \xrightarrow{f} X$,* s.t. *$\forall S \in \mathcal{P}_o(X), f(S) \in S$. ($\mathcal{P}_o(X) := \{A | A| \subseteq X, A \neq \emptyset\}$)*
2. *If $(X, \leq)$ is a poset, in which every well-ordered subset has a lub in X, then X has a maximal element.*
3. *(Maximal Chain Theorem) $\forall$ poset $(X, \leq)$ has a maximal chain w.r.t $\subseteq$. i.e. a chain such that there is no other chain in $(X, \leq)$ which has it as a proper subset.*
4. *(Zorn's Lemma) If $(X, \leq)$ is a poset in which every chain has an upper bound in X then X has a maximal element.*
5. *(Zermelo's Well-Ordering Theorem) Every set has a well-order.*
6. *$\forall$ surj. $X \xrightarrow{f} Y$, $\exists$ an injection $Y \xrightarrow{g} X$,* s.t. *$f \circ g = id_Y$.*
7. *(Axiom of Choice) Given non-empty sets $S_\alpha (\alpha \in A)$, there exists a map $A \xrightarrow{f} \cup_{\alpha \in A} S_\alpha$,* s.t. *$f(\alpha) \in S_\alpha$.*

Proof. $7 \Rightarrow 1$: We can number each non-empty subset of X by itself, since any element in a set is unique. That is $\mathcal{P}_o(X) = \{S_\alpha := \alpha | \alpha \in \mathcal{P}_o(X)\}$, here $\mathcal{P}_o(X)$ serves as A. Thus Axiom of Choice means $\exists$ a map $\mathcal{P}_o(X) \xrightarrow{f} X$, s.t. $f(\alpha) \in S_\alpha = \alpha (\alpha \in \mathcal{P}_o(X))$. (we emphasize $\mathcal{P}_o(X)$, rather than $\mathcal{P}(X)$, because there is nothing in $\emptyset$)

Remark 5. Statement 1 claims that given a set X, any non-empty subset of X can be maps to a point inside this subset.

$1 \Rightarrow 2$: Assume that X has no maximal element, i.e. $\forall a \in X, X_a := \{x \in X | a < x\} \neq \emptyset$. $\exists$ map $\mathcal{P}_o(X) \xrightarrow{f} X$, s.t. $f(S) \in S$ for all $S \in \mathcal{P}_o(X)$. Define a map $X \xrightarrow{\pi} \mathcal{P}_o(X) (a \mapsto X_a)$ and $X \xrightarrow{g = f \circ \pi} X$. Thus for any $a \in X$, $g(a) = f(X_a) \in X_a$, thus $a < g(a)$, which leads to a contradiction with Bourbaki's fixed point theorem.

$$\begin{tikzcd} \mathcal{P}_o(X) \arrow[r, "f"] & X \\ X \arrow[u, "\pi"] \arrow[ur, "g"'] & \end{tikzcd}$$

$2 \Rightarrow 3$: Given a poset $(X, \leq)$ consider $S = \{C | C \text{ is a chain in } P \; w.r.t. \leq\}$. Thus $(S, \subseteq)$ is a poset. We claim that any totally ordered set in S has a lub in S. If $T \subseteq S$ is a totally ordered set, (that is T is a chain w.r.t $\subseteq$ of the chains w.r.t. $\leq$), then $\cup_{C \in T} C = lub_S T$. To show this, we need prove 2 things:

1. $\cup_{C \in T} C \in U_T$;
   For any $C \in T, C \subseteq \cup_{C \in T} C$, thus $\cup_{C \in T} C \in U_T$.
2. $\cup_{C \in T} C \in L_{U_T}$.
   For any $v \in \cup_{C \in T} C, O \in U_T$, $\exists C \in T$, s.t. $v \in C \subseteq O$. Thus $\cup_{C \in T} C \subseteq O$, thus $\cup_{C \in T} C \in L_{U_T}$.

Thus every totally ordered subset (including well order subset) of $(S, \subseteq)$ has a lub, and $(S, \subseteq)$ has a maximal element, which implies $(X, \leq)$ has a maximal chain.

Remark 6. $(T, \subseteq)$ is a chain, thus any comparison with the element in T need to use relation $\subseteq$.

$3 \Rightarrow 4$: Given a poset $(X, \leq)$, it has a max. chain C, by assumption, C has an upper bound, say a, in X. Then a is a max. element in X, otherwise $\exists x \in X, a < x$, and hence $C \subsetneq C \cup \{x\}$ and $C \cup \{x\}$ is a chain, which leads to a contradiction to the maximality of C.

$4 \Rightarrow 5$: Let Y be a set, consider $X := \{A | A = (S_A, \leq_A) \text{ where } S_A \subseteq Y \text{ and } \leq_A$ is a well-ordering on $S_A\}$. We define a relation $\preceq$ on X: $A \preceq A' \Leftrightarrow A = A'$ or A is an initial segment of A' (i.e. $\exists a' \in S_{A'}, S_A = \{x \in S_{A'} | x <_{A'} a'\}$) and $\forall x_1, x_2 \in S_A, x_1 \leq_A x_2 \Leftrightarrow x_1 \leq_{A'} x_2$.

It is direct to see that $(X, \preceq)$ is a poset:

1. For any $A \in X, A \preceq A$;
2. If A is initial segment of A' then $A \neq A'$, since if $\exists a' \in S_{A'}, S_A = \{x \in S_{A'} | x <_{A'} a'\}$ then $a' \in A'$ but $a' \notin A$. Thus $A \preceq A', A' \preceq A \Rightarrow A = A'$
3. Suppose that $A \preceq A' \preceq A''$, and A, A' and A'' are not equal. Thus $\exists a'' \in S_{A''}$, s.t. $S_{A'} = IS_{A''}(a'')$, and $\exists a' \in S_{A'}$, s.t. $S_A = IS_{A'}(a')$. Since $a' <_{A''} a''$, any $a \in S_{A''}, a <_{A''}$

$a' \Rightarrow a \in S_{A'}$. Thus $IS_{A''}(a) = \{x \in S_{A''} | x <_{A''} a'\} = \{x \in S_{A'} | x <_{A''} a'\} = \{x \in S_{A'} | x <_{A'} a'\} = IS_{A'}(a') = A$, thus $A \preceq A''$.

Then, we claim:

1. $(X, \preceq)$ has a maximal element:
   Apply Zorn's lemma, let $(C, \preceq)$ be a chain on $(X, \preceq)$. Let $A_0 = (S_{A_0}, \leq_{A_0})$ where $S_{A_0} = \cup_{A \in C} S_A$, and $\leq_{A_0}$: for any $x_1, x_2 \in S_{A_0}$, find $A \in C$, s.t. $x_1, x_2 \in S_A$, we say that $x_1 \leq_{A_0} x_2$ if $x_1 \leq_A x_2$. Then we claim:
   - Such A exists:
     For any $x_1, x_2 \in S_{A_0}, \exists A_1, A_2 \in C$, s.t. $x_1 \in S_{A_1}, x_2 \in S_{A_2}$ and S_{A_1} and S_{A_2} are comparable on X w.r.t. $\preceq$, since C is a chain. Assume that S_{A_1} is an initial segment of S_{A_2}, then $x_1, x_2 \in S_{A_2}$.
   - $x_1 \leq_{A_0} x_2$ is independent of the choice of A, s.t. $x_1, x_2 \in S_A$:
     If $\exists A, A' \in C$, s.t. $x_1, x_2 \in S_A, S_{A'}$, then A, A' are comparable. Assume that $A \preceq A'$, that is A is an initial segment of A', then in S_A, we have $x_1 \leq_A x_2 \Leftrightarrow x_1 \leq_{A'} x_2$.
   - $(S_{A_0}, \leq_{A_0})$ is a total order set :
     Any $x_1, x_2 \in S_{A_0}$ will be covered by a S_A where A is an element of a chain C on X. Thus x_1 and x_2 are comparable by $x_1 \leq_{A_0} x_2 \Leftrightarrow x_1 \leq_A x_2$.
   - $(S_{A_0}, \leq_{A_0})$ is a well order set :
     Let $T \subseteq S_{A_0}$ and $T \neq \emptyset$. Then $T = T \cap S_{A_0} = T \cap \cup_{A \in C} S_A = \cup_{A \in C}(T \cap S_A) \neq \emptyset$. Thus $\exists A \in C$, s.t. $T \cap S_A \neq \emptyset$. Since A is well ordering, $T \cap S_A$ has least element, denoted by t.
     Any $A' \in C$, it is either $A' = A$ or $A' \preceq A$ or $A \preceq A'$. If $A' \preceq A$, then $S_{A'}$ is an initial segment of S_A, that is $\exists a \in S_A$, s.t. $S_{A'} = \{x \in S_A | x <_A a\}$. Thus $S'_A \subseteq S_A$, and $T \cap S_{A'} \subseteq T \cap S_A$, thus t is the least element of $T \cap S_A \Rightarrow t$ is the least element of $T \cap S_{A'}$;
     If $A \preceq A'$, then S_A is an initial segment of $S_{A'}$, thus $\exists a' \in S_{A'}$, s.t. $S_A = \{x \in S_{A'} | x <_{A'} a'\}$ and $T \cap S_A = T \cap \{x \in S_{A'} | x <_{A'} a'\} = \{x \in T \cap S_{A'} | x <_{A'} a'\}$. For any $s \in T \cap S_{A'}$, if $a' \leq_{A'} s$, then $t <_{A'} a' \leq_{A'} s$; if $s <_{A'} a'$, then $s \in T \cap S_A$, and $t \leq_A s \Rightarrow t \leq_{A'} s$. Thus t is the least element of $T \cap S_{A'}$.
     Thus t is the least element of $T \cap S_{A_0} = T$, thus $\leq_{A_0}$ is a well order on S_{A_0}. Furthermore, $(S_{A_0}, \leq_{A_0}) \in X$.
   - S_{A_0} is an upper bound of C on X, w.r.t. $\preceq$:
     Given $A \in C$, since C is a chain, any $A' \in C$ admits 3 cases: $A' = A, A' \preceq A, A \preceq A'$. Define $\Pi := \{A' \in C | A \preceq A'\} \backslash \{A\}$ and $\Gamma := \{A' \in C | A' \preceq A\} \backslash \{A\}$.

     Remark 7. Recall the proof of 2 $\Rightarrow$ 3.

     For any $B \in \Pi$, $\exists b \in S_B$, s.t. $S_A = IS_B(b)$. Define $\Phi := \{A' \in \Pi | A' \preceq B\} \backslash \{B\}$. If $\Phi \neq \emptyset$, then $\exists C \in \Phi, \exists c \in S_C$, s.t. $S_A = IS_C(c)$. Collect all these kind of c and form a set Δ, then Δ is a non-empty subset of S_B. Since S_B is a well ordering set, Δ has a least element μ, and exists the corresponding $D \in \Phi$, s.t. $S_A = IS_D(\mu)$. Thus

$$\begin{aligned} S_A = IS_D(\mu) &= \{x \in S_D | x <_D \mu\} \\ &\overset{x,\mu \in S_{A_0}}{=\!=\!=} \{x \in S_D | x <_{A_0} \mu\} \end{aligned}$$

Since any $A' \in \Pi$, the corresponding $\mu \leq_{A'} a'$, thus

$$\begin{aligned} \{x \in S_{A'} | x <_{A_0} \mu\} &= \{x \in S_{A'} | x <_{A'} \mu\} \\ &\subseteq \{x \in S_{A'} | x <_{A'} a'\} \\ &= IS_{A'}(a') \\ &= S_A = IS_D(\mu) \end{aligned}$$

On the other hand, For any $A'' \in \Gamma$, $A'' \preceq A \Rightarrow S_{A''} \subseteq S_A$, thus $\{x \in S_{A''} | x <_{A_0} \mu\} \subseteq S_A$. Thus

$$\begin{aligned} S_A &= IS_D(\mu) \\ &= \cup_{A' \in \Pi}\{x \in S_{A'} | x <_{A_0} \mu\} \cup (\cup_{A'' \in \Gamma}\{x \in S_{A''} | x <_{A_0} \mu\}) \\ &= \{x \in \cup_{A' \in \Pi \cup \Gamma} S_{A'} | x <_{A_0} \mu\} \\ &= \{x \in \cup_{A' \in C} S_{A'} | x <_{A_0} \mu\} \\ &= IS_{A_0}(\mu) \end{aligned}$$

Thus $A \preceq A_0$ for any $A \in C$, and A_0 is an upper bound of C. $(X, \preceq)$, as a poset, whose any chain C has an upper bound A_0, thus X has a maximal element by Zorn's lemma.

2. A maximal element in $(X, \preceq)$ is $(Y, \leq_Y)$.
   If $(Y_0, \leq_{Y_0})$ is a max. element in X w.r.t. $\preceq$ and $Y_0 \neq Y$, then $\exists y \in Y \backslash Y_0$. Define $Y_1 := Y_0 \cup \{y\}$ and a partial order: $v \leq_{Y_1} y, v_1 \leq_{Y_1} v_2 \Leftrightarrow v_1 \leq_{Y_0} v_2$ for $\forall v, v_1, v_2 \in Y_0$. Then $(Y_1, \leq_{Y_1})$ admits a well-ordering which makes $(Y_0, \leq_{Y_0})$ an initial segment, because any non-empty subset ϕ of Y_1 is either $\{y\}$ or $(\phi \cap Y_0) \cup (\phi \cap \{y\})$, clearly ϕ has least element.
   Thus $(Y_1, \leq_{Y_1}) \in X$ and $(Y_0, \leq_{Y_0}) \preceq (Y_1, \leq_{Y_1})$, which leads to a contradiction to the maximality of $(Y_0, \leq_{Y_0})$.

Since X is the set of well ordering subset on Y, $(Y, \leq_Y) \in X$, thus $(Y, \leq_Y)$ is well ordering.

$5 \Rightarrow 6$: Choose a well ordering $\leq$ on X, For any $y \in Y$, define $g(y) :=$ the least element of $f^{-1}(y)$, then $f \circ g(y) = y$. For any $y_1, y_2 \in Y, y_1 \neq y_2 \Rightarrow f(g(y_1)) \neq f(g(y_2)) \Rightarrow g(y_1) \neq g(y_2) \Rightarrow g$ is injective.

$6 \Rightarrow 7$: Let $S := \cup_{\alpha \in A} S_\alpha$, define $X := \{(s, \alpha) \in S \times A | s \in S_\alpha\}$. Consider two projection

$X \xrightarrow{f} A((s,\alpha) \mapsto \alpha)$ and $X \xrightarrow{p} S((s,\alpha) \mapsto s)$, thus f is a surjection, then $\exists A \xrightarrow{g} X$ such that $f \circ g(\alpha) = \alpha$ for any $\alpha \in A$.
Define s_α is the least element of S_α, then $g(\alpha) = (s_\alpha, \alpha)$ and $p \circ g(\alpha) = p(s_\alpha, \alpha) = s_\alpha \in S_\alpha$. Thus $A \xrightarrow{p \circ g} S = \cup_{\alpha \in A} S_\alpha$ is desired.

$$\begin{tikzcd}
 & & A_{\alpha} \arrow[dll, shift left, "g"] \\
X \arrow[urr, "f"] \arrow[drr, "p"'] & \subseteq & S \times A_{(s,\alpha)} \arrow[u, "\pi"'] \arrow[d, "\pi"] \\
 & & S_{s}
\end{tikzcd}$$

□

1.3 Applications of Zorn's Lemma

1.3.1 Cardinality

Definition 6 (Cardinality)**.** Let X and Y be two sets, we say $|X| = |Y|$ if there exists a bijection $X \to Y$; $|X| \leq |Y|$ if exist an injection $X \to Y$.

Exercise 2. *Let X and Y be two sets, show that $\exists$ an injection $X \to Y \Leftrightarrow \exists$ a surjection $Y \to X$.*

Proof. $\Leftarrow$: If $Y \xrightarrow{f} X$ is a surjection, then $\exists$ an injection $X \xrightarrow{g} Y$ by equivalent statements 6 of AC. $\Rightarrow$: If $X \xrightarrow{f} Y$ is an injection, then $X \xrightarrow{f} f(X)$ is a bijection, and there exists an inverse $f(X) \xrightarrow{f^{-1}} X$. Select $x \in X$, define $g(y) \equiv x, y \in Y \backslash f(X)$, Then $Y \xrightarrow{g} X$ where $y \mapsto f^{-1}(y)$ if $y \in f(X)$ and $y \mapsto x$ if $y \in Y \backslash f(X)$ is as desired. □

Exercise 3. *Let X and Y be two sets, show that there exist an injection from X to Y or from Y to X.*

Proof. Consider $\Pi := \{S_f \xrightarrow{f} Y | f$ is an injection on a subset S_f of $X\}$ and $f \preceq f' \Leftrightarrow S_f \subseteq S_{f'}$ and $f'|_{S_f} = f$. Thus $(\Pi, \preceq)$ is a poset.
If $\Pi = \emptyset$, which implies there is only one element in Y, thus there exists a surjection from X to $Y \Rightarrow$ there exists an injection from Y to X.
If $\Pi \neq \emptyset$:
suppose $(C, \preceq)$ is a chain on $(\Pi, \preceq)$, define $Z = \cup_{S \in C} S$, and for any $z \in Z$, $f_o(z) = f(z)$ if $z \in S_f$. As always: (1) S_f exists by the def. of Z; (2) the def. of f_o is well-defined, that is the value of $f_o(z)$ is independent with the choice of S_f, because any S_f, S'_f that

cover z are in the chain C, thus they are comparable, and one is the extension of the other.
Thus $Z \xrightarrow{f_o} Y$ is an upper bound of $(C, \preceq)$, because for any $S_f \xrightarrow{f} Y \in C$, $S_f \subseteq Z$ by def. and $f_o|_{S_f} = f$ by the independence. Thus any chain on $(\Pi, \preceq)$ has an upper bound, and $(\Pi, \preceq)$ has a maximal element $X_0 \xrightarrow{f_0} Y$. Suppose $X_0 \neq X$:
If f_0 is not surj: Then select $y_0 \in Y \backslash f(X_0)$ and $x \in X \backslash X_0$. Define $X_1 = X_0 \cup \{x\}$, and define $f_1|_{X_0} = f_0$, $f_1(x_0) = y_0$. Then $f_0 \preceq f_0$, which against the maximality of $X_0 \xrightarrow{f_0} Y$.
If f_0 is surj: Then select any $y_0 \in Y$ and define $f_1(x) \equiv y_0$ for any $x \in X \backslash X_0$, thus $X \xrightarrow{f_1} Y$ is a surj. Then there exists an injection $Y \xrightarrow{g} X$, and we are done.

□

Remark 8. A very useful routine:

1. transform the existence of the target to the existence of the maximal element on some poset
2. use Zorn's Lemma (show any chain on the poset has an upper bound, which is usually the union on all elements in the chain)
3. check that the maximal element = target (use contradiction).

Proposition 1 (Bernstein-Schroeder). *$|X| \leq |Y|$ and $|Y| \leq |X| \Rightarrow |X| = |Y|$.*

Proof. The proof of the proposition has been given in *Introduction to Topology, Lecture 2, Proposition 4.* □

1.3.2 Vector Space

1.3.3 Hahn-Banach Theorem

Lemma 1. *Let X be a vector space over $K (= \mathbb{R})$, and $X \xrightarrow{p} \mathbb{R}$ is a func. s.t. $\forall x, x' \in X, t > 0, p(x + x') \leq p(x) + p(x')$ and $p(tx) = tp(x)$.*
For any linear func. $Z \xrightarrow{\Xi_o} \mathbb{R}$ on a vector subspace Z of X s.t. $\Xi_o(z) \leq p(z)$ for any $z \in Z$. If $x_0 \in X \backslash Z$, then there exists a linear func. $Z + \mathbb{R}x_0 \xrightarrow{\Xi} \mathbb{R}$ s.t. $\Xi|_Z = \Xi_o$ and $\Xi(u) \leq p(u)$ for any $u \in Z + \mathbb{R}x_0$.

Proof. All linear func.s $Z + \mathbb{R}x_0 \xrightarrow{\Xi} \mathbb{R}$ such that $\Xi|_Z = \Xi_o$ is of the form $\Xi(z + tx_0) = \Xi_o(z) + t\Xi(x_0)$. It suffices to determine the value of $\Xi(x_0)$ (denoted as a) s.t. $\Xi(u) \leq p(u)$ for any $u \in Z + \mathbb{R}x_0$ holds.
Any $u \in Z + \mathbb{R}x_0$ can be uniquely written as $z + tx_0, z \in Z, t \in \mathbb{R}$. We hope to find $a \in \mathbb{R}$ such that

$$\Xi(u) = \Xi_o(z) + ta \leq p(u) = p(z + tx_0)$$

for all $z \in Z, t \in \mathbb{R}$, or equivalently (if $t < 0$, denote $t = -t'$, $t' > 0$)

$$a \le \frac{p(z + tx_0) - \Xi_o(z)}{t}, \quad z \in Z, t > 0$$
$$a \ge \frac{p(z' - t'x_0) - \Xi_o(z')}{-t'}, \quad z' \in Z, t' > 0$$

Since

$$\begin{aligned}
&\frac{p(z + tx_0) - \Xi_o(z)}{t} - \frac{p(z' - t'x_0) - \Xi_o(z')}{-t'} \\
&= \frac{p(z + tx_0) - \Xi_o(z)}{t} + \frac{p(z' - t'x_0) - \Xi_o(z')}{t'} \\
&= \frac{t'p(z + tx_0) - t'\Xi_o(z) + tp(z' - t'x_0) - t\Xi_o(z')}{tt'} \\
&= \frac{p(t'z + tt'x_0) - \Xi_o(t'x) + p(tz' - tt'x_0) - \Xi_o(tz')}{tt'} \\
&\ge \frac{p(t'z + tt'x_0 + tz' - tt'x_0) - \Xi_o(t'z + tz')}{tt'} \\
&= \frac{p(t'z + tz') - \Xi_o(t'z + tz')}{tt'} \ge 0.
\end{aligned}$$

$\Rightarrow$ such $a\exists$. $\square$

Theorem 3 (Hahn-Banach Theorem)**.** *Let X be a vector space over $K(= \mathbb{R})$, and $X \xrightarrow{p} \mathbb{R}$ is a func. s.t. $\forall x, x' \in X, t > 0, p(x + x') \le p(x) + p(x')$ and $p(tx) = tp(x)$.*
For any linear func. $Y \xrightarrow{\Lambda_o} \mathbb{R}$ on a vector subspace Y of X s.t. $\Lambda_o(y) \le p(y)$ for any $y \in Y$. Then there exists a linear func. $X \xrightarrow{\Lambda} \mathbb{R}$ s.t. $\Lambda|_Y = \Lambda_o$ and $\Lambda(x) \le p(x)$ for any $x \in X$.

Proof. Consider P is the collection of $W_\Theta \xrightarrow{\Theta} \mathbb{R}$ such that Θ is a linear func. on a vec. subspace W_Θ of X containing Y s.t. $\Theta|_Y = \Lambda_o$ and $\Theta(w) \le p(w)$ for all $w \in W_\Theta$. And define $\preceq$: $\Theta \preceq \Theta' \Leftrightarrow W_\Theta \subseteq W_{\Theta'}$ and $\Theta'|_{W_\Theta} = \Theta$. It is direct to see $(P, \preceq)$ is a poset.
If $(P, \preceq)$ has a maximal element $Z \xrightarrow{\Theta} \mathbb{R}$, then $Z = X$ by Lemma 1. otherwise we can extent Z to $Z + \mathbb{R}x_0$ where $x_0 \in X \backslash Z$ which against the maximality of $Z \xrightarrow{\Theta} \mathbb{R}$.
Remark 9. Recall the proof of Well-Ordering Theorem by Zorn's Lemma.
Thus it suffices to show $(P, \preceq)$ has a max. element. Let $(C, \preceq)$ is a chain in $(P, \preceq)$. We take $W = \cup_{\Theta \in C} W_\Theta$ which is a vector subspace of X containing Y. And define $W \xrightarrow{\Pi} \mathbb{R}$ where then $w \mapsto \Theta(w)$ if $w \in W_\theta$. This is well-defined, $\Pi(w)$ is independence of the choice of Θ s.t. $w \in W_\Theta$, since C is a chain, and one of any $W_\Theta, W_{\Theta'}$ that covers w is the extension of the other. Thus for any $\Theta \in C$, $W_\Theta \subseteq W$ and $\Pi|_{W_\Theta} = \Theta$, thus $\Theta \preceq \Pi$. Thus Π is the upper bound of C, and $W = X$ and $X \xrightarrow{\Pi} \mathbb{R}$ is as desired. $\square$

Chapter 2

Metric Space

2.1 Metric space

Definition 7 (Metric Space)**.** Let $X \times X \xrightarrow{d} \mathbb{R}$ be a function, we cay that d is a metric on X or (X, d) is a metric space if for $\forall x, x', x'' \in X$ have
1. Positivity: $d(x, x') \geq 0$ and $d(x, x'') = 0$ iff $x = x'$;
2. Symmetry: $d(x, x') = d(x', x)$;
3. Triangle inequality: $d(x, x') \leq d(x, x'') + d(x'', x')$.

Exercise 4. *Show that the triangle inequality is equivalent with for* $\forall x, x', x'' \in X$

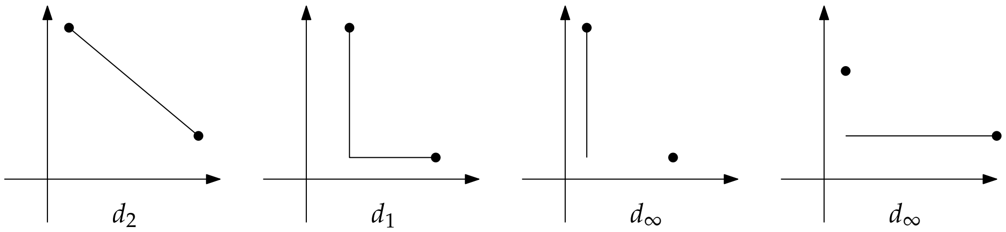
$$d(x, x') \geq |d(x, x'') - d(x', x'')|.$$

Proof. $\geq \Rightarrow \leq$: since $d(x, x') \geq |d(x, x'') - d(x', x'')| \geq d(x, x'') - d(x', x'')$, we have that $d(x, x'') \leq d(x, x') + d(x', x'')$.
$\leq \Rightarrow \geq$: if $\exists x, x', x''$ such that $d(x, x') < |d(x, x'') - d(x', x'')|$, then

$$\begin{aligned} d(x, x') &< |d(x, x'') - d(x', x'')| \\ &\leq |d(x, x') + d(x', x'') - d(x', x'')| \\ &\leq d(x, x') \end{aligned}$$

thus $d(x, x') < d(x, x')$, which leads to a contradiction. □

Example 3. Here are some metric examples:
1. define $d_2(x, y) := \left(\sum_i^m |x_i - y_m|^2\right)^{1/2}$, $x, y \in \mathbb{R}^m$. Then d_2 is a metric on $\mathbb{R}^m$ by cauchy inequality.
2. define $d_1(x, y) := \sum_{i=1}^m |x_i - y_i|$, $x, y \in \mathbb{R}^m$. Then d_1 is a metric on $\mathbb{R}^m$.
3. define $d_\infty(x, y) := \max\{|x_i - y_i|\}$, $i \in \{1, 2, \cdots, m\}$, $x, y \in \mathbb{R}^m$. Then d_∞ is a metric on $\mathbb{R}^m$.



d_2 can be proved to be a metric by Cauchy inequality:

Exercise 5 (Cauchy inequality)**.** *For any* $(x_1, \cdots, x_n), (y_1, \cdots, y_n) \in \mathbb{R}^n$*, show that*

$$(x_1y_1 + \cdots, x_ny_n)^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2)$$

and $=$ *holds iff* $\exists a, b \in \mathbb{R}$ *which are not all* 0*.*

Proof. Consider the polynomial $p(t) = \sum_{i=1}^n (x_i t + y)^2 = t^2 \sum_{i=1}^n x_i^2 + 2\sum_{i=1}^n x_i y_i + \sum_{i=1}^n y_i^2 \geq 0$, thus $\Delta = 4\left(\sum_{i=1}^n x_i y_i\right)^2 - 4\sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2 \leq 0 \Rightarrow \left(\sum_{i=1}^n x_i y_i\right)^2 \leq \sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2$. □

Example 4 (p-adic)**.** If p is a prime number, $x \in \mathbb{Q}$, define

$$|x|_{p-adic} := \begin{cases} p^{-m}, & x = \frac{a}{b} \cdot p^m, x \neq 0, \\ 0, & x = 0, \end{cases}$$

where $a, b, m \in \mathbb{Z}$, $(a, p) = (b, p) = 1$. For $\forall x, y \in \mathbb{Q}$, define $d_{p-adic}(x, y) = |x - y|_{p-adic}$, then d_{p-adic} is a metric on $\mathbb{Q}$.
Assume $x = (a/b)p^m, y = (s/t)p^n \in \mathbb{Q}$ where $a, b, s, t, m, n \in \mathbb{Z}, (a, p) = (b, p) = (s, p) = (t, p) = 1, m > n$, then $|x|_{p-adic} = p^{-m} < |y|_{p-adic} = p^{-n}$, and

$$\begin{aligned} |x - y|_{p-adic} &= |(a/b)p^m - (s/t)p^n|_{p-adic} \\ &= \left| \frac{adp^{m-n} - bc}{bd} p^n \right|_{p-adic}. \end{aligned}$$

it is easy to check $adp^{m-n} - bc, bd \in \mathbb{Z}$ and $(adp^{m-n} - bc, p) = (bd, p) = 1$, thus

$$|x - y|_{p-adic} = p^{-n} = |y|_{p-adic} = \max\{|x|_{p-adic}, |y|_{p-adic}\}.$$

Thus for any $x, y, z \in \mathbb{Q}$, we have that

$$\begin{aligned} d_{p-adic}(x, y) &= \max\{|x|_{p-adic}, |y|_{p-adic}\} \\ &\leq \max\{|x|_{p-adic}, |z|_{p-adic}\} + \max\{|z|_{p-adic}, |y|_{p-adic}\} \\ &= d_{p-adic}(x, z) + d_{p-adic}(y, z), \end{aligned}$$

which follows the triangle inequality, the other two conditions is trivial.

2.2 Open and compact on metric space

Definition 8 (Open Ball)**.** Let (X, d) be a metric space, for $\forall r > 0$ and $x_0 \in X$, we let

$$B_r(x_0) := \{x \in X | d(x, x_0) < r\},$$

and call it the open ball with center x_0 and radius r; let

$$\overline{B_r(x_0)} := \{x \in X | d(x, x_0) \leq r\},$$

and call it the close ball with center x_0 and radius r.

Example 5 (discrete metric)**.** For $\forall x, x' \in \mathbb{R}^2$, define metric $d(x, x') = 0$ if $x = x'$, and $d(x, x') = 1$ if $x \neq x'$, then $B_1(x) = \{x\}$, $\overline{B_1(x)} = \mathbb{R}^2$, $B_{1.1}(x) = \mathbb{R}^2$.

Definition 9 (Open Set)**.** $S(\subseteq X)$ is called an Open Set of X with respect to d, if $\forall x_0 \in S$, $\exists r > 0$ such that $B_r(x_0) \subseteq S$; $F(\subseteq X)$ is Close Set of X w.r.t. d if $X \backslash F$ is open set of X w.r.t. d.

Exercise 6. *Prove that $B_r(x)$ is open set and $\overline{B_r(x)}$ is close.*

Proof. For $\forall x' \in B_r(x)$, we have $d(x, x') < r$, donate $r - d(x, x')$ by s, then for $\forall x'' \in B_{s/2}(x')$ satisfy

$$\begin{aligned} d(x, x'') &\leq d(x, x') + d(x', x'') \\ &\leq d(x, x') + \frac{s}{2} \\ &< r, \end{aligned}$$

thus $x'' \in B_r(x)$ and $B_{s/2}(x') \subseteq B_r(x)$ and $B_r(x)$ is a open set. For $\forall x' \in X \backslash \overline{B_r(x)}$ has $d(x, x') > r$. Denote $d(x, x') - r$ by t, then for $\forall x'' \in B_{t/2}(x')$ satisfy

$$\begin{aligned} d(x, x'') &\geq |d(x, x') - d(x', x'')| \\ &\geq d(x, x') - d(x', x'') \\ &\geq d(x, x;) - \frac{t}{2} \\ &> r. \end{aligned}$$

Thus $B_{t/2}(x') \subseteq X \backslash \overline{B_r}$ and $X \backslash \overline{B_r}$ is an open set, thus $\overline{B_r}$ is a close set. □

Exercise 7. *Let (X, d) be a metric space. show that*
1. $X, \emptyset \subseteq_{open} X$;
2. $O_1, O_2 \subseteq_{open} X \Rightarrow O_1 \cap O_2 \subseteq_{open} X$;
3. $O_\alpha \subseteq_{open} X, (\alpha \in A) \Rightarrow \cup_{\alpha \in A} O_\alpha \subseteq_{open} X$ *(α not necessarily be integral or countable);*
4. *All above corresponding statements for close set are true.*

Proof. 1. Obviously X is an Open set thus $\emptyset$ is a close set. If $\emptyset$ is not an open set, then $\exists x \in \emptyset$, $\forall r > 0$ such that $B_r(x) \nsubseteq \emptyset$, which is impossible. Thus $\emptyset$ is an open set (logically) and X is a close set;
2. $\forall x \in O_1 \cap O_2$, $\exists r_1, r_2 > 0$, s.t. $B_{r_1}(x) \subseteq O_1$ and $B_{r_2}(x) \subseteq O_2$. Thus $\forall x' \in B_{\min\{r_1,r_2\}}(x) = B_{r_1}(x) \cap B_{r_2}(x) \Rightarrow x' \in O_1 \cap O_2 \Rightarrow B_{\min\{r_1,r_2\}}(x) \subseteq O_1 \cap O_2$, thus $O_1 \cap O_2$ is open. Collectively, the intersection of any finite open sets is an open set;
3. For $\forall x \in \cup_{\alpha \in A} O_\alpha, \exists$ at least one $\alpha' \in A$, s.t. $x \in O_{\alpha'}$, then $\exists r > 0$, s.t. $B_r(x) \subseteq O_{\alpha'} \subseteq \cup_{\alpha \in A} O_\alpha$, thus $\cup_{\alpha \in A} O_\alpha$ is an open set;
4. Take complementary set: The union of finite close sets is close; the intersection of any close sets is close.

$\square$

Remark 10. First 3 statements are the essential intuition for the definition of *Topology*.

Exercise 8. *Show that an open set is the union of open balls.*

Proof. Given an open set O, for any $o \in O, \exists r_o > 0$, s.t. $B_{r_o}(o) \subseteq O$, define $O' = \cup_{o \in O} B_{r_o}(o)$. Thus for $\forall x \in O', \exists o'$, s.t. $x \in B_{r'_o}(o') \subseteq O \Rightarrow O' \subseteq O$;
On the other hand, for any $y \in O, \exists r_y > 0$, s.t. $B_{r_y}(y) \subseteq O \Rightarrow y \in B_{r_y}(y) \subseteq O' \Rightarrow O \subseteq O'$. Thus $O = O' = \cup_{o \in O} B_{r_o}(o)$. $\square$

Definition 10 (Convergence)**.** Let (X, d) be a metric space, $a_n \in X, (n \in \mathbb{N})$, $L \in X$, define $\lim_{n\to\infty} a_n = L$ w.r.t. d, if $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N$ s.t. $d(a_n, L) < \epsilon$, that is $a_n \in B_\epsilon(L)$.

Exercise 9. *Show that*
1. $\lim_{n\to\infty} a_n = L \Leftrightarrow \lim_{n\to\infty} d(a_n, L) = 0$;
2. $\lim_{n\to\infty} a_n = L \Leftrightarrow \forall L \in U \subseteq_{open} X, \exists N \in \mathbb{N}, \forall n \geq N$ s.t. $a_n \in U$.

Proof. (1) Trivial; (2) $\Rightarrow$: Suppose that $\lim_{n\to\infty} a_n = L$, for $\forall U$ that $L \in U$, $\exists r > 0$, s.t. $B_r(L) \subseteq U$, and $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, s.t. $a_n \in B_r(L) \subseteq U$; $\Leftarrow$: Suppose $L \in U \subseteq_{open} X$, then $\exists r > 0$ such that $B_r(L) \subseteq U$. Since $B_r(L)$ is also an open set, then $\exists N \in \mathbb{N}$, for $\forall n \geq N$, s.t. $a_n \in B_r(L) \subseteq U$. $\square$

We say $S \subseteq X$ is bounded w.r.t. d, if $\exists r > 0$ and $x_0 \in X$, s.t. $S \subseteq B_r(x_0)$.

Theorem 4 (Bolzano-Weierstrass theorem)**.** *If $a_n \in \mathbb{R}^m (n \in \mathbb{N})$ is bounded w.r.t. d_2, then $\exists$ a subsequence $a_{n_m} (m \in \mathbb{N})$ which converges.*

Proof. We only prove $\mathbb{R}^2$ case. If we want to prove that the vector $a = (a_1, \cdots, a_m) \in \mathbb{R}^m \to L = (l_1, \cdots, l_m) \in \mathbb{R}^m$, all we need to prove is $\lim_{n\to\infty} a_i = l_i, (i = 1, \cdots, m)$.
Choose $M > 0$, s.t. $a_n \in Q = [-M, M] \times [-M, M]$ for all $n \in \mathbb{N}$. Divide Q into 4 squares with equal size and choose one, say Q_1, such that $|\{n | a_n \in Q\}| = \infty$. Select $n_1 \in \mathbb{N}$, such that $a_{n_1} \in Q_1$. Repeat this and we have $\cap_{k=1}^{\infty} Q_k = \{a\}$. By theorem of nested interval we have that $\lim_{k\to\infty} a_{n_k} = a$. $\square$

Remark 11. The conclusion of Bolzano-Weierstrass theorem can be generalize to metric space (Remark 16).

Exercise 10. *Let* (X, d) *be a metric space,* $F \subseteq X$ *show that* $F \subseteq_{close} X \Leftrightarrow \forall a_n \in F(n \in \mathbb{N})$ *and* $\lim_{n\to\infty} a_n = a \in X$ *then* $a \in F$*.*

Proof. $\Rightarrow$: Assume that F is close and $a_n \in F$. If $a_n \to a \in X\backslash F$, then $\exists r > 0$, s.t. $B_r(a) \in X\backslash F$. Since $\lim_{n\to\infty} a_n = a$, for r, there exists $N \in \mathbb{N}, \forall n \geq N$, s.t. $d(a_n, a) < r$, i.e. $a_n \in B_r(a) \subseteq X\backslash F$, which leads to a contradiction. $\Leftarrow$: Suppose that $\forall a_n \in F(n \in \mathbb{N})$ and $\lim_{n\to\infty} a_n = a \in X$ then $a \in F$, and F is not close, which means $X\backslash F$ is not open, and $\exists x \in X\backslash F, \forall r > 0, B_r(x) \cap F \neq \emptyset$. Select $n \in \mathbb{N}$ such that $a_n = B_{\frac{1}{n}}(x) \cap F$. Thus $\lim_{n\to\infty} a_n = x \notin F$, which leads to a contradiction. $\square$

Remark 12. Set family of sets as $\{B_{1/n}(x)\}_{n\in\mathbb{N}}$ is a very useful skill.

Definition 11 (Open cover, Compact set)**.** Let (X, d) be a metric space, $S \subseteq X$, $O_\alpha \in X(\alpha \in A)$, we say that $O_\alpha(\alpha \in A)$ form an open cover of S, if $S \subseteq \cup_{\alpha\in A} O_\alpha$. S is called a compact set if $\forall$ open cover $O_\alpha(\alpha \in A)$ of S, $\exists \alpha_1, \cdots, \alpha_m \in A$, s.t. $S \subseteq \cup_{i=1}^m O_{\alpha_i}$, where $\cup_{i=1}^m O_{\alpha_i}$ is called a finite subcover.

If there exists an open cover of F whose any finite subcover can not cover it, then F is not a compact set. for instance, let $F = (0, 1), O_n = (1/n, 2), n \in \mathbb{N}$, then O_n is an open cover of F, however any finite subcover of O_n can not cover F.

Theorem 5 (Heine-Borel theorem)**.** *Let* $S \subseteq \mathbb{R}^n$*, then* S *is compact* $\Leftrightarrow$ S *is bounded and closed.*

Proof. $\Rightarrow$: Suppose that S is compact, select a point $s \in S$ arbitrarily, define $O_i = B_i(s)$, it is easy to check that $S \subseteq \cup_{i\in\mathbb{N}} O_i$, since for any $s' \in S$, we have that $s' \in B_{2d(s,s')}(s) \subseteq O_{\lceil 2d(s,s') \rceil}$. Since S is compact, there exists a finite subcover, thus S is bounded.
Suppose S is compact, but S is not closed, which means $X\backslash S$ is not open and $\exists x \in X\backslash S$, s.t. $\forall r > 0, B_r(x) \cap S \neq 0$. Since S is bounded, define $\iota = \sup_{s\in S} d(s, x)$, define open cover

$$O_n = B_{\frac{\iota}{n}}(x) - B_{\frac{\iota}{n+1}}(x),$$

thus $O_i \cap O_j = \emptyset(i \neq j)$ and $O_i \cap S \neq \emptyset(\forall i)$. Thus O_n has no finite subcover, which leads to a contradiction and S is closed.
$\Leftarrow$: Suppose that S is bounded and closed, and $\exists$ an open cover $O_\alpha(\alpha \in A)$ of S which admits no finite subcover. Choose a cube Q containing S (S is bounded), divide Q into 4 equal-sized cubes and select one of them denoted by Q_1, s.t. $Q_1 \cap S$ can not be covered by finitely many Q_α, select a point such that $s_1 \in Q_1 \cap S$. Repeat.
Obviously, $\lim_{n\to\infty} s_n = a$, notice that $s_n \in Q_n \cap S \subseteq S$, thus $\lim_{n\to\infty} s_n = a \in S$ for S is closed. Thus there exist O_i such that $a \in O_i$. Since O_i is open, $\exists r > 0$, s.t. $B_r(a) \subseteq O_i$.

Then $\exists N \in \mathbb{N}, \forall n \geq N$, s.t. $Q_n \subseteq B_r(a) \subseteq O_i$. Since $Q_n \cap S$ can not be covered by finitely many O_α, but could be covered by O_i, which leads to a contradiction. $\square$

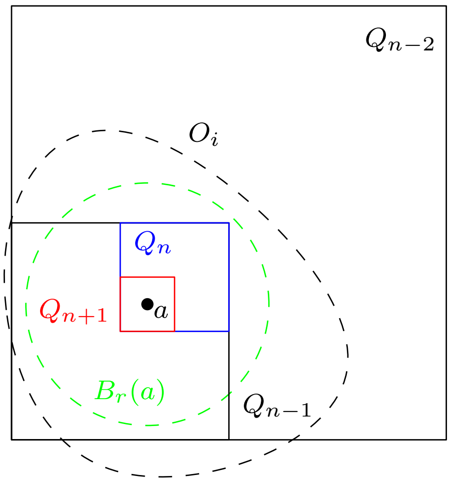


Figure 2.1: Heine-Borel theorem

Theorem 6 (The Lebesgue number of an open cover). *Let (X, d) be a metric space and $K(\subseteq X)$ a compact set. For any given open cover $O_\alpha(\alpha \in A)$ of K, there exists $\delta > 0$,* s.t. *for every $x \in K$ we have $B_\delta(x) \subseteq O'_\alpha$ for some $\alpha' \in A$ (α' depending on x).*

Proof. Since K is compact, for any open cover of K, there exists an finite subcover of K, that is $\exists O_{\alpha_i}, i = 1, \cdots, N$ such that

$$K \subseteq \bigcup_{i=1}^{N} O_{\alpha_i}.$$

For any point $x \in K$ there exist $O_{\alpha_j} \in \{O_{\alpha_i}\}_{i=1}^{N}$, s.t. $x \in O_{\alpha_j}$ and exists $\delta_x > 0$, s.t. $B_{\delta_x/2}(x) \subseteq B_{\delta_x}(x) \subseteq O_{\alpha_j}$ for O_{α_j} is open. Obviously we have that $B_{\delta_x/2}(x)$ is an open cover of K, i.e.

$$K \subseteq \bigcup_{x \in K} B_{\delta_x/2}(x),$$

and $B_{\delta_x/2}(x)$ has an finite subcover of K, donate as $\{B_{\delta_{x_i}/2}(x_i)\}_{i=1}^{M}$. Let $\delta = \min\{\delta_{x_i}\}_{i=1}^{M}$. Then for any $y \in K$, there exists $B_{\delta_{x_j}/2}(x_j) \in \{B_{\delta_{x_i}/2}(x_i)\}_{i=1}^{M}$ such that $y \in B_{\delta_{x_j}/2}(x_j)$ and $d(y, x_j) < \delta_{x_j}/2$. and for any y' where $d(y', y) < \delta/2 < \delta_{x_j}/2$, we have $d(x_j, y') \leq d(x_j, y) + d(y, y') < \delta_{x_j}$, thus $y, y' \in B_{\delta_{x_j}}(x_j) \subseteq O_{\alpha'}$, or say $y, y' \in B_{\delta/2}(y) \subseteq B_{\delta_{x_j}}(x_j) \subseteq O_{\alpha'}$. $\square$

The theorem indicates for any open cover O_α of K, $\exists \delta > 0$, s.t. for $\forall x \in K, x' \in X$, is $d(x, x') < \delta$, then $\exists \alpha \in A$ we have $x, x' \in O_\alpha$. Such a $\delta > 0$ is called a **Lebesgue**

number of the given open cover $O_\alpha(\alpha \in A)$. Notice that the statement could be false if the compactness assumption is dropped.

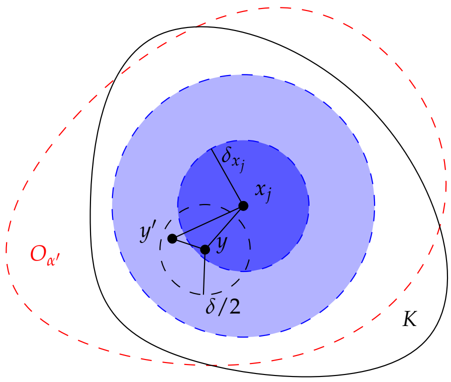
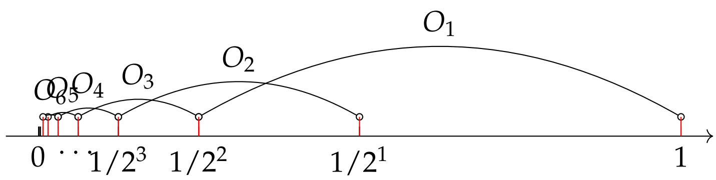
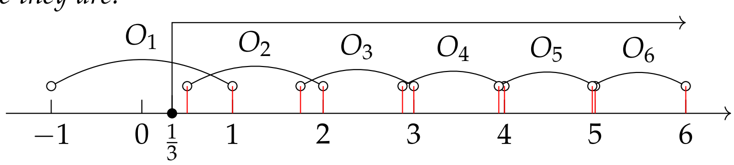


Figure 2.2: The Lebesgue number of an open cover

Exercise 11 (Open set)**.** *Let* $(X,d) = (\mathbb{R}, d_2), K = (0,1), O_\alpha = (1/2^{\alpha+1}, 1/2^{\alpha-1})(\alpha \in \mathbb{N})$*. Thus* $1/2^\alpha \in O_\alpha$ *and* $\notin O_{\alpha'}$ *if* $\alpha' \neq \alpha(\alpha, \alpha' \in \mathbb{N})$*. It is easy to check* O_α *is an open cover of* K*, but* $|1/2^\alpha - 1/2^{\alpha+1}| = 1/2^{\alpha+1}$ *can be arbitrarily small if* $\alpha \uparrow$*. Thus there exists* $x \in K, x' \in X$ *can not be covered one* O_α*, no matter how close they are.*



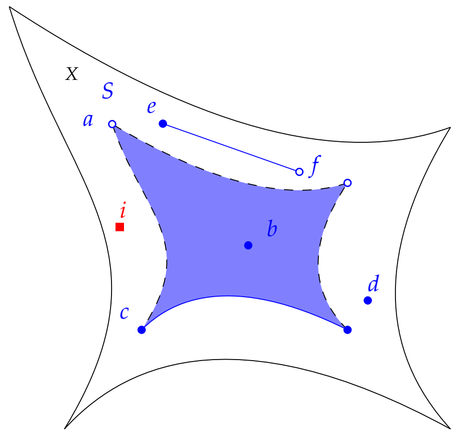
Exercise 12 (Unbounded set)**.** *Let* $(X,d) = (\mathbb{R}, d_2), K = [1/3, \infty), O_\alpha = (\alpha - 1 - 1/2^{\alpha-1}, \alpha)(\alpha \in \mathbb{N})$*. Thus* $x = \alpha - 1/2^\alpha \in O_\alpha$ *and* $x' = \alpha \in O_{\alpha+1}$ *and* $d(x,x')$ *could be arbitrarily small as* $\alpha \uparrow$*. Thus there exists* $x \in K, x' \in X$ *can not be covered one* O_α*, no matter how close they are.*



Definition 12 (Isolated point, limit point and accumulation point)**.** Let (X,d) be a metric space and $S \subseteq X$. A point $x \in X$ is called:

- an **isolated point** of S, if $\exists \epsilon > 0$, s.t. $B_\epsilon(x) \cap S = \{x\}$ ($\Rightarrow x \in S$);
- a **limit point** of S, if $\forall \epsilon > 0$, s.t. $B_\epsilon(x) \cap S \backslash \{x\} \neq \emptyset$;
- an **accumulation point** of S, if $\exists$ seq. $a_n \in S(n \in \mathbb{N})$, s.t. $x = \lim_{n\to\infty} a_n$.

Example 6. $S \subseteq X$ is as the figure, point $i \notin S$:



Then

point	iso. pts. of S	limit pts. of S	acc. pts. of S	$\in S$
i	$\times$	$\times$	$\times$	$\times$
a	$\times$	$\surd$	$\surd$	$\times$
b	$\times$	$\surd$	$\surd$	$\surd$
c	$\times$	$\surd$	$\surd$	$\surd$
d	$\surd$	$\times$	$\surd$	$\surd$
e	$\times$	$\surd$	$\surd$	$\surd$
h	$\times$	$\surd$	$\surd$	$\times$

Notice that x is a isolated point of $S \Rightarrow x \in S$; but x is a limit/accumulate point of $S \not\Rightarrow x \in S$.

Exercise 13. *Let (X, d) be a metric space, $S \subseteq X$,*

1. *Show that x is an isolated/limit point of $S \Rightarrow x$ is a accumulate point of S;*
2. *Donate $\{$iso. pts. of$S\}$, $\{$limit pts. of$S\}$ and $\{$acc. pts. of$S\}$ by I_S, L_S, A_S respectively. Show that $I_S \cup L_S = A_S$;*
3. *Suppose $E \subseteq K \subseteq X$, where E is infinite and K is compact, show that $L_E \neq \emptyset$; (Prove by contradiction)*

Proof. 1. If x is an isolated point of S, thus $x \in S$. Let $a_n \equiv x$, then $\lim_{n\to\infty} a_n = x$, thus x is an accumulate point of S; If x is a limit point of S, then for any $\epsilon > 0, B_\epsilon(x) \cap S \backslash \{x\} \neq \emptyset$. Let $a_n \in B_{1/n}(x)(n \in \mathbb{N})$, then $d(a_n, x) < 1/n$ for $\forall n \in N$, thus $\lim_{n\to\infty} a_n = x$, and x is an accumulate point of S.

2. We have obtained that $I_S, L_S \subseteq A_S$. Suppose $x \in A_S \backslash (I_S \cup L_S) \neq \emptyset$. Which means : (1) there exists seq. $a_n \in S$ such that $\lim_{n\to\infty} a_n = x$; (2) $\forall \epsilon > 0$, s.t. $B_\epsilon(x) \cap S \neq \{x\}$ $(\neg I_S)$;(3) $\exists \epsilon' > 0$, s.t. $B_{\epsilon'}(x) \cap S \backslash \{x\} = \emptyset$ $(\neg L_S)$. Let $B_\epsilon(x) \cap S = Q_\epsilon \neq \{x\}$, if $x \in Q_\epsilon$, then it leads to a contradiction with (3); If $x \notin Q_\epsilon$, then $Q_{\epsilon'} = \emptyset$, that is $B_{\epsilon'}(x) \cap S = \emptyset$ and for $\forall s \in S \Rightarrow d(s, x) \geq \epsilon'$, which leads to a contradiction with (1). Thus $A_S \backslash (I_S \cup L_S) = \emptyset$. Because $I_S, L_S \subseteq A_S$, we have $I_S \cup L_S = A_S$.

3. We claim there exists a limit point s of E in K, i.e. $\exists s \in K$ s.t. $\forall r > 0, B_r(s) \cap E \backslash \{s\} \neq \emptyset$.
   Assume the contrary, that is $\forall s \in K, \exists r_s > 0$ s.t. $B_{r_s}(s) \cap E \backslash \{s\} = \emptyset$, and $B_{r_s}(s)(s \in K)$ form an open cover of K: $K = \cup_{s \in K} B_{r_s}(s)$. Since K is compact, there exists $s_1, \cdots, s_n \in K$ s.t. $K = \cup_{i=1}^{n} B_{r_{s_i}}(s_i)$.
   Define $S = \{s_1, \cdots, s_n\}$, then

$$\begin{aligned} K \cap E \backslash S &= \left(\cup_{i=1}^{n} B_{r_{s_i}}(s_i) \right) \cap E \backslash S \\ &= \cup_{i=1}^{n} B_{r_{s_i}}(s_i E \backslash S) \\ &= \emptyset \end{aligned}$$

   but since E is infinite set, S is finite set and $E \subseteq K \Rightarrow K \cap E \backslash S = E \backslash S \neq \emptyset$, which is contrary.

□

Remark 13. Refer to the proof method of

Exercise 14. *Let* $(X, d) = (\mathbb{R}, d_2), S \subseteq \mathbb{R}$, *show that if* $\sup S$ $(\inf S)$ *exists, then it is an accumulate point.*

Proof. If $\sup S \exists$, then for $\forall x \in S$, s.t. $x \leq \sup S$ and for $\forall \epsilon > 0$, $\exists x' \in S$, s.t. $\sup S - \epsilon < x'$. For any $n \in \mathbb{N}$, there exists $x_n \in S$ s.t. $\sup S - 1/n < x' \leq \sup S$, and $d(x_n, \sup S) < 1/n$, thus $x_n \to \sup S$ as $n \to \infty$. □

Exercise 15. *Show that, if* (X, d) *be a metric space, then*

$$S \subseteq_{close} X \Leftrightarrow A_S = S \Leftrightarrow L_S \subseteq S.$$

Proof. For any $x \in S$, let $a_n = x$, then $\lim_{n \to \infty} a_n = x$, thus $S \subseteq A_S$. Since example (**??**), we have $S \subseteq_{close} X \Leftrightarrow A_S = S$. $\Rightarrow$ Since $I_S \cup L_S = A_S$, we have $L_S \subseteq A_S = S$; $\Leftarrow$, for $L_S \subseteq A_S \subseteq S$, we have $S \subseteq A_S \Rightarrow S = A_S$. □

2.3 Functions on metric space

Definition 13 (Continuous). Let $(X, d_X), (Y, d_Y)$ be metric spaces. $a \in S \subseteq X$, $f : S \mapsto Y$, we say

1. map f is continuous at a if for $\forall \epsilon > 0, \exists \delta > 0$, for $\forall x \in B_\delta(a) \cap S$, s.t. $f(x) \in B_\epsilon(f(a))$, that is $f(B_\delta(a)) \subseteq B_\epsilon(f(a))$.
2. map f is a continuous map if f is continuous at every $a \in S$.

Exercise 16. *Given a map* $X \xrightarrow{f} Y, a \in X$, *Show that*

1. *f is continuous at $a \Leftrightarrow$ for $\forall V \subseteq_{open} Y$, where $f(a) \in V$, $\exists U \subseteq_{open} X$, where $a \in U$, such that $f(U) \subseteq V$.*
2. *f is a continuous map $\Leftrightarrow$ for $\forall V \subseteq_{open} Y, f^{-1}(V) \subseteq_{open} X$.*

Proof. 1. $\Rightarrow$: for $\forall V \subseteq_{open} Y$, where $f(a) \in V$, $\exists \epsilon > 0$, s.t. $B_\epsilon(f(a)) \subseteq V$, thus $\exists U = B_\delta(a)$. $\Leftarrow$: trivial.
2. $\Rightarrow$: Given an open set $V \subseteq_{open} Y$, for $\forall x \in f^{-1}(V)$, have $f(x) \in V$. Since V is open, $\exists r > 0$ s.t. $B_r(f(x)) \subseteq V$. Since $f(x)$ is continuous map, $\exists \epsilon > 0$, s.t. $f(B_\epsilon(x)) \subseteq B_r(f(x)) \subseteq V \Rightarrow B_\epsilon(x) \in f^{-1}(V)$, thus $f^{-1}(V)$ is open.
$\Leftarrow$: Given $x \in X, f(x) \in Y$, given $r > 0$, s.t. $B_r(f(x)) \subseteq Y$, then $f^{-1}(B_r(f(x))) \subseteq_{open} X$, and $x \in f^{-1}(B_r(f(x)))$. Thus $\exists \epsilon > 0$, s.t. $B_\epsilon(x) \subseteq f^{-1}(B_r(f(x)))$ and $f(B_\epsilon(x)) \subseteq B_r(f(x))$. □

Remark 14. It can also be proved that f is cont. $\Leftrightarrow$ for $\forall V \subseteq_{close} Y, f^{-1}(V) \subseteq_{close} X$. Suppose $V \subseteq_{close} Y$, then $X \backslash f^{-1}(V) = f^{-1}(X \backslash V) \subseteq_{open} X$, thus $f^{-1}(V) \subseteq_{close} X$.

Exercise 17. *Given maps $X \xrightarrow{f} Y, Y \xrightarrow{g} Z$, show that*
1. *If f is continuous at x_0, g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.*
2. *If f, g are continuous maps, then $g \circ f$ is a continuous map.*

Proof. 1. For any V, s.t. $g(f(x_0)) \in V \subseteq_{open} Z, \exists U$, s.t. $f(x_0) \in U \subseteq_{open} Y, \exists W$, s.t. $x_0 \in W \subseteq X$, thus $g \circ f$ is continuous at x_0.
2. For any $V \subseteq_{open} Z, \exists U \subseteq_{open} Y, \exists W \subseteq_{open} X$, thus $g \circ f$ is continuous. □

Remark 15. Note that we prove this exercise using general sets instead of metrics. We replaced open ball with open set in last Exercise, this is a meaningful operation, which means we could **substitute the metric with set** (here is open set), which drives the concept of topology. Generally, topology is a family of sets which have the basic properties, using these sets, we can define open set no longer rely on metric d.

Theorem 7. *Let $X \xrightarrow{f} \mathbb{R}$ be a continuous map between metric space, X is compact, then $\max_{x \in X} f(x), \min_{x \in X} f(x)$ exists.*

Proof. 1. f is bdd. and hence $\sup_{x \in X} f(x)$ exists (l.u.b. property):
Assume the contrary. Then $\forall n \in \mathbb{N}, \exists x_n \in X$ s.t. $f(x_n) > n$ and we can form a seq. $x_n (n \in \mathbb{N})$ which is a infinite subset of a compact set, thus there exists $a \in X$ and a convergent subseq. $x_{n_k} (k \in \mathbb{N}) \to a$ as $k \to \infty$ (see Remark 16). And hence $\lim_{k \to \infty} f(x_{n_k}) = f(a)$ since f is continuous, which leads to a contradiction with $f(x_{n_k}) \geq n_k$. Thus f is bdd. (Continuous map on compact set is bounded)
2. Let $M = \sup_{x \in X} f(x)$, then $\exists x \in X$, s.t. $f(x) = M$:
Assume the contrary, i.e. $\forall x \in X, f(x) < M$. Then the map $X \xrightarrow{\phi} \mathbb{R}$ where $x \mapsto$

$1/(M - f(x))$ is well-defined continuous map, and hence ϕ is bounded by 1. Then for any $R \in \mathbb{R}_+, 1/R > 0$ and $\exists x \in X$ s.t.

$$M - \frac{1}{R} < f(x) \leq M$$

thus $\phi(x) = 1/(M - f(x)) > R$ which leads to a contradiction with ϕ is bdd. □

Remark 16. Two facts:

1. Any infinite set of a compact set K has a limit point in K (Exercise 13);
2. x is a limit point of $A \subseteq X$, where X is a metric space $\Leftrightarrow$ $\exists$ *seq.* $a_n \in A \backslash \{x\} (n \in \mathbb{N})$, s.t. $a_n \to x$ as $n \to \infty$ (Exercise 63).

These two facts generalize the Bolzano-Weierstrass theorem (Theorem 4) from $\mathbb{R}^n$ space to general metric space.

2.4 Uniformly continuous function

Recall that the concept of continuous map: let $X \xrightarrow{f} Y$ be a map between metric space,

- f is continuous
- $\Leftrightarrow$ f is continuous at every $x \in X$
- $\Leftrightarrow \forall \epsilon > 0, \forall x \in X, \exists \delta > 0$ s.t. $\forall x' \in X, d(x, x') < \delta \Rightarrow d(f(x), f'(x)) < \epsilon$

(or say $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$). **Note that here the order of x and ϵ does not matter, and δ relies on the choice of x and ϵ.**

Definition 14 (Uniformly continuous, 均匀连续)**.** Let $X \xrightarrow{f} Y$ be a map between metric space, we say f is uniformly continuous if

- $\forall \epsilon > 0, \exists \delta > 0$ s.t. for $\forall x, x' \in X, d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \epsilon$.

Remark 17. Now, δ only relies on the choice of ϵ. If f is uniformly continuous $\Rightarrow$ f is continuous.

For a given $\epsilon > 0$ and $x \in X$, consider the set

$$\Delta_x := \{\delta > 0 | f(B_\delta(x) \subseteq B_\epsilon(f(x))\}$$

Then if f is continuous at $x \Leftrightarrow \Delta_x \neq \emptyset$. And if f is continuous at x, define ϵ **- threshold** of f at x as

$$\delta_{f,\epsilon}(x) := \sup \Delta_x$$

Consider a map $(0, 1] \to \mathbb{R}$ where $x \mapsto 1/x$, if any δ works for the given ϵ and x, then

$$\frac{1}{x - \delta} - \frac{1}{x} = \frac{\delta}{(x - \delta)x} < \epsilon$$

thus $\delta < \epsilon(x - \delta)x < x^2\epsilon \Rightarrow \delta_{f,\epsilon}(x) \leq x^2\epsilon \to 0$ as $x \to 0$, thus there does not exist a δ for given ϵ such that works for all $x \in X$.

Theorem 8. *If $X \xrightarrow{f} Y$ is a continuous map between metric space and X is compact, then f is uniformly continuous.*

Proof 1. Given $\epsilon > 0$, for every $a \in X$, choose a number $\delta_a > 0$ s.t. $\forall x \in X, f(B_{\delta_a}(a)) \subseteq B_{\epsilon/2}(f(a))$. Then $B_{\delta_a}(a)(a \in X)$ is an open cover of X, then let $\delta > 0$ be a Lebesgue number of this cover.
Thus for $\forall x, x' \in X, d(x,x') < \delta \Rightarrow \exists a \in X$, s.t. $x, x' \in B_{\delta_a}(a) \Rightarrow f(x), f(x') \in B_{\epsilon/2}(f(a)) \Rightarrow d(f(x), f(x')) \leq d(f(x), f(a)) + d(f(a), f(x')) < \epsilon/2 + \epsilon/2 = \epsilon$. □

Proof 2. □

Exercise 18. *Let $a < 1$, show that $[0, \infty) \to \mathbb{R}$ where $x \mapsto a^x$ is uniformly continuous.*

2.5 Limit superior / inferior for function

Let X be metric space, $S \subseteq X$ and $S \xrightarrow{f} \mathbb{R}$ be a map, for $a \in X$, we define

$$\overline{f}^*(\delta) := \sup_{x \in B_\delta(a)\setminus\{a\}} (a) = \sup\{f(x) | 0 < d(x,a) < \delta\}$$

$$\underline{f}^*(\delta) := \inf_{x \in B_\delta(a)\setminus\{a\}} (a) = \inf\{f(x) | 0 < d(x,a) < \delta\}$$

It is direct to see that $\overline{f}^*_{\searrow}$ as $\delta \to 0$: Assume that if $\exists \delta < \delta'$ and $\overline{f}^*(\delta) > \overline{f}^*(\delta')$, let

$$\epsilon = \overline{f}^*(\delta) - \overline{f}^*(\delta')$$

then $\exists x \in B_\delta(a)\setminus\{a\} \subseteq B_{\delta'}(a)\setminus\{a\}$ such that

$$\overline{f}^*(\delta) \geq f(x) > \overline{f}^*(\delta) - \epsilon/2 > \overline{f}^*(\delta') = \sup_{x \in B_{\delta'}(a)\setminus\{a\}} f(x),$$

which is contrary. Thus similarly, $\underline{f}^*_{\nearrow}$ as $\delta \to 0$. For any $\delta, \delta' \in \mathbb{R}$, we have that

$$\underline{f}^*(\delta) \leq \underline{f}^*(\min\{\delta, \delta'\}) \leq \overline{f}^*(\min\{\delta, \delta'\}) \leq \overline{f}^*(\delta')$$

thus $\underline{f}^*(\delta)$ has upper bound and $\overline{f}^*(\delta)$ has lower bound when $\delta \to 0$. And hence $\overline{f}^*(\delta)$ converges to its infimum: assume the contrary, if $\lim_{\delta \to 0} \overline{f}^*(\delta) > \inf_{\delta > 0} \overline{f}^*(\delta)$, then $\exists \epsilon > 0$ and $\delta' > 0$ s.t.

$$\inf_{\delta > 0} \overline{f}^*(\delta) \leq \overline{f}^*(\delta') < \inf_{\delta > 0} \overline{f}^*(\delta) + \epsilon < \lim_{\delta \to 0} \overline{f}^*(\delta)$$

and hence $\forall \delta < \delta'$ has

$$\overline{f}^*(\delta) \leq \overline{f}^*(\delta') < \lim_{\delta \to 0} \overline{f}^*(\delta)$$

since $\overline{f}^*(\delta)\searrow$ as $\delta \to 0$. And it is contrary.
Thus $\overline{f}^*(\delta)$ converges to its infimum, $\underline{f}^*(\delta)$ converges to its supremum, and we can define

$$\limsup_{x\to a}^{*} f(x) := \overline{\lim_{x\to a}}^{*} f(x) := \inf_{\delta>0} \overline{f}^*(\delta) = \inf_{\delta>0} \sup_{x\in B_\delta(a)\setminus\{a\}} f(x)$$

$$\liminf_{x\to a}^{*} f(x) := \underline{\lim_{x\to a}}^{*} f(x) := \inf_{\delta>0} \underline{f}^*(\delta) = \sup_{\delta>0} \inf_{x\in B_\delta(a)\setminus\{a\}} f(x)$$

Corresponding, we can define the 'non - $*$' conception by containing the $\{a\}$:

$$\overline{f}(\delta) := \sup_{x\in B_\delta(a)}(a) = \sup\{f(x)|0 \le d(x,a) < \delta\}$$

$$\underline{f}(\delta) := \inf_{x\in B_\delta(a)}(a) = \inf\{f(x)|0 \le d(x,a) < \delta\}$$

and

$$\limsup_{x\to a} f(x) := \overline{\lim_{x\to a}} f(x) := \inf_{\delta\ge 0} \overline{f}(\delta) = \inf_{\delta\ge 0} \sup_{x\in B_\delta(a)} f(x)$$

$$\liminf_{x\to a} f(x) := \underline{\lim_{x\to a}} f(x) := \inf_{\delta\ge 0} \underline{f}(\delta) = \sup_{\delta\ge 0} \inf_{x\in B_\delta(a)} f(x)$$

Then it is direct to see that

$$\underline{\lim_{x\to a}} f(x) \le \underline{\lim_{x\to a}}^{*} f(x) \le \overline{\lim_{x\to a}}^{*} f(x) \le \overline{\lim_{x\to a}} f(x)$$

Example 7. Consider a map $\mathbb{R} \xrightarrow{f} \mathbb{R}$ where $x \mapsto 1$ if $x \neq 0$ and $0 \mapsto 0$, then

$$\begin{aligned} \overline{\lim_{x\to 0}}^{*} f(x) &= 1, & \underline{\lim_{x\to 0}}^{*} f(x) &= 1 \\ \overline{\lim_{x\to 0}} f(x) &= 1, & \underline{\lim_{x\to 0}} f(x) &= 0 \end{aligned}$$

Exercise 19. *Let X be metric space, $a \in S \subseteq X$ and $S \xrightarrow{f} \mathbb{R}$ be a map, show that*
1. $\lim_{x\to a} f(x)$ exists $\Leftrightarrow$ $\overline{\lim}^*_{x\to a} f(x)$ and $\underline{\lim}^*_{x\to a} f(x)$ exists and equal to each other.
2. $f(x)$ is continuous at a exists $\Leftrightarrow$ $\overline{\lim}_{x\to a} f(x)$ and $\underline{\lim}_{x\to a} f(x)$ exists and equal to each other.

Chapter 3

Topology Space and Basis

3.1 Topology Space

Definition 15 (Topology Space). A topology space $X = (\underline{X}, \mathscr{T}_X)$ consists of a set $\underline{X}$, called the underlying space of X and a family $\mathscr{T}_X$ of subset of $\underline{X}$ (i.e. $\mathscr{T}_X \subseteq \mathcal{P}(X)$) s.t.
1. $\underline{X}, \emptyset \in \mathscr{T}_X$;
2. $U_\alpha \in \mathscr{T}_X (\alpha \in A) \Rightarrow \cup_{\alpha \in A} U_\alpha \in \mathscr{T}_X$;
3. $U, U' \in \mathscr{T}_X \Rightarrow U \cap U' \in \mathscr{T}_X$.
$\mathscr{T}_X$ is called a topology on $\underline{X}$, the element in $\mathscr{T}_X$ is called the open set on $\underline{X}$ w.r.t. $\mathscr{T}_X$.

Remark 18. Conventionally, we usually use X to indicate the set $\underline{X}$ and omit the subscript X in $\mathscr{T}_X$ by saying 'a topology space $(X, \mathscr{T})$'.

Exercise 20. *Let X be a topology space, $U \subseteq X$, show that U is open $\Leftrightarrow$ for any $u \in U$, $\exists O_u \subseteq U$,* s.t. *$u \in O_u \subseteq_{open} X$.*

Proof. $\Rightarrow$: define $O_u := U$ for $\forall u \in U$; $\Leftarrow$: since $O_u \subseteq U$, $\cup_{u \in U} O_u \subseteq U$; on the other hand, for any $v \in U$, $v \in O_v \subseteq \cup_{u \in U} O_u \Rightarrow U \subseteq \cup_{u \in U} O_u$. Thus $U = \cup_{u \in U} O_u \subseteq_{open} X$. $\square$

Definition 16 (Continuous). Let X and Y are top. spaces and $\underline{X} \xrightarrow{f} \underline{Y}$ is a map. We say f is conti. at a point $x_0 \in X$ (from X to Y), if for $\forall f(x_0) \in V \in \mathscr{T}_Y, \exists x \in U \in \mathscr{T}_X$, s.t. $f(U) \subseteq V$.
We say f is continuous (from X to Y) if it is continuous at every point of $\underline{X}$.

Remark 19. We will denote $U \in \mathscr{T}_X$ as $U \subseteq_{open} X$, and denote $X \backslash A \subseteq_{open} X$ as $A \subseteq_{close} X$.

Definition 17 (Compact). X is a top. sp. $K \subseteq \underline{X}$. We say K is compact in X if $\forall U_\alpha \subseteq_{open} X (\alpha \in A), K \subseteq \cup_{\alpha \in A} U_\alpha \Rightarrow \exists$ finite set $S \subseteq A$, s.t. $K \subseteq \cup_{\alpha \in S} U_\alpha$, and denote by $K \subseteq_{cpt.} X$. We say X is a compact space if $\underline{X}$ is compact in X.

Definition 18 (Neighborhood)**.** Let X be a top. sp. and $x \in X$. A subset N of X is called a neighborhood of x if $\exists U \subseteq N$, s.t. $x \in U \subseteq_{open} X$. (That is $x \in N^o$.)

Exercise 21. *$X \xrightarrow{f} Y$ is a map between top. sp., $x_0 \in X$, show that f is conti. at $x_0 \Leftrightarrow \forall$ nbd. V of $f(x_0), \exists$ nbd. U of x_0,* s.t. *$f(U) \subseteq V \Leftrightarrow \forall$ nbd. V of $f(x_0)$, $f^{-1}(V)$ is a nbd. of x_0.*

Proof. 1. $\Rightarrow$: Suppose $V \subseteq Y$ is a nbd. of $f(x_0)$, then $\exists V_0 \subseteq V$, s.t. $f(x_0) \in V_0 \subseteq_{open} Y \Rightarrow \exists U_0 \subseteq_{open} X$, s.t. $x \in U_0$ and $f(U_0) \subseteq V_0$, since f is conti. at x_0. Thus U_0 is the nbd. that we desire.
$\Leftarrow$: For any open set $V_0 \subseteq_{open} Y$ and $f(x_0) \in V_0$, V_0 is a nbd. of $f(x_0)$. Thus $\exists$ a nbd. U of x_0 such that $\exists U_0 \subseteq U, x_0 \in U_0 \subseteq_{open} X$. And $f(U_0) \subseteq f(U) \subseteq V_0$. Thus f is conti.
2. $\Rightarrow$: For any nbd. V of $f(x_0)$, $\exists$ nbd. U of x_0 and $\exists U_0 \subseteq U$, s.t. $x_0 \in U_0 \subseteq_{open} X$ and $f(U) \subseteq V$. Thus $x_0 \in U_0 \subseteq U \subseteq f^{-1}(V)$, that is $U \in f^{-1}(V)$ and $x_0 \in U_0 \subseteq_{open} X$, thus $f^{-1}(V)$ is a nbd. of x_0.
$\Leftarrow$: Trivial. $\square$

Definition 19 (Separation Axioms)**.** Let X be a top. space:

(T_0 or Kolmogorov Space) For any distinct $x, y \in X, \exists U \subseteq_{open} X$, s.t. $x \in U \not\ni y$ or $y \in U \not\ni x$.

(T_1 or Fréchet Space) For any distinct $x, y \in X, \exists U, V \subseteq_{open} X, x \in U \not\ni y$ and $y \in V \not\ni x$.

(T_2 or Hausdorff Space) For any distinct $x, y \in X, \exists U, V \subseteq_{open} X$, s.t. $x \in U, y \in V$ and $U \cap V = \emptyset$.

(T_3 or Regular Space) If X is a T_1 space, and $\forall x \in X, C \subseteq_{close} X, x \notin C \Rightarrow \exists U, V \subseteq_{open} X$, s.t. $x \in U, C \in V$ and $U \cap V = \emptyset$.

(T_4 or Normal Space) If X is a T_1 space, and $\forall C_1, C_2 \subseteq_{close} X, C_1 \cap C_2 = \emptyset \Rightarrow \exists U, V \subseteq_{open} X$, s.t. $C_1 \subseteq U, C_2 \subseteq V$ and $U \cap V = \emptyset$.

Exercise 22. *Show that X is a T_1 space $\Leftrightarrow \forall x \in X, \{x\} \subseteq_{close} X$.*

Proof. $\Rightarrow$: Given $x \in X$, for any $y \in X \backslash \{x\}$, there exists $U_y \subseteq_{open} X$, s.t. $y \in U_y \not\ni x$. Thus $\cup_{y \in X\backslash\{x\}} U_y \subseteq_{open} X$. If $z \in \cup_{y \in X\backslash\{x\}} U_y$, $\exists y' \in X$, s.t. $z \in U_{y'} \subseteq_{open} X$ and $x \notin U_{y'} \Rightarrow z \neq x \Rightarrow z \in X \backslash \{x\}$. For any $z \in X\backslash\{x\} \Rightarrow \exists U_z \subseteq_{open} X$, s.t. $z \in U_z \not\ni x \Rightarrow z \in \cup_{y \in X\backslash\{x\}} U_y$. Thus $X\backslash\{x\} = \cup_{y \in X\backslash\{x\}} U_y \subseteq_{open} X \Rightarrow \{x\} \subseteq_{close} X$.
$\Leftarrow$: For any distinct $x, y \in X$, $x \in X\backslash\{y\} \subseteq_{open} X$ and $y \in X\backslash\{x\} \subseteq_{open} X$ where $x \notin X\backslash\{x\}$ and $y \notin X\backslash\{y\}$. $\square$

There are some examples of topologies:

Example 8. X is a set, $\mathcal{P}(X)$ is called discrete topology; $(\emptyset, X)$ is called trivial topology. Note that discrete topology is defined by discrete metric:

$$d(x, x') = \begin{cases} 1, & x = x' \\ 0, & x \neq x' \end{cases}$$

Thus $\{x\} \subseteq B_{1/2}(x)$ for any $x \in X$, and any $S \in \mathcal{P}(X)$ is the union of these balls, i.e. $S = \cup_{x \in S} B_{1/2}(x)$, and holds an open set in discrete topology.
But trivial topology can not be defined by metric. If it can, then $\forall x \in X, \exists r_x > 0$, s.t. $B_{r_x}(x) \subseteq X$, which implies $B_{r_x}(x) \in (\emptyset, X)$ and leads to a contradiction.

Example 9. X is an uncountable set. $\mathscr{T}_c := \{U \subseteq X | U = \emptyset \vee X \backslash U \text{ is countable}\}$ is called **co-countable** topology. Thus any countable set in X is the close set on topology space $(X, \mathscr{T}_c)$.
Similarly, $\mathscr{T}_f := \{U \subseteq X | U = \emptyset \vee X \backslash U \text{ is finite}\}$ is called **co-finite** topology. Thus any finite set in X is the close set on topology space $(X, \mathscr{T}_f)$.
It is direct to see $\mathscr{T}_c$ and $\mathscr{T}_f$ are topology:
1. $\emptyset \in \mathscr{T}_c, X \in \mathscr{T}_c$ for $X \backslash X = \emptyset$ is countable;
2. Any $U_\alpha \in \mathscr{T}_c (\alpha \in A) \Rightarrow X \backslash U_\alpha$ is countable $\Rightarrow X \backslash \cup_{\alpha \in A} U_\alpha = \cap_{\alpha \in A} (X \backslash U_\alpha)$ is the intersection of countable sets, thus be countable $\Rightarrow \cup_{\alpha \in A} U_\alpha \in \mathscr{T}_c$.
3. $U, V \in \mathscr{T}_c \Rightarrow X \backslash (U \cap V) = (X \backslash U) \cup (X \backslash V)$ is countable, thus $U \cap V \in \mathscr{T}_c$.

3.2 Interior & Closure

Definition 20. X is a top. sp., $p \in X$, $A \subseteq X$:
1. p is an interior point of A in X, if $\exists$ nbd. U of p, s.t. $U \subseteq A$;
2. p is an exterior point of A in X, if $\exists$ nbd. U of p, s.t. $U \subseteq X \backslash A$, i.e. $U \cap A = \emptyset$;
3. p is a boundary point of A in X, if $\forall$ nbd. U of p, s.t. $U \cap A \neq \emptyset \neq U \cap (X \backslash A)$;

Correspondingly, define
1. $int_X A = A^o :=$ {all interior point of A in X},
2. $ext_X A = A^e :=$ {all exterior point of A in X},
3. $bd_X A = \partial A :=$ {all boundary point of A in X}

Example 10. Given a top. space $(\mathbb{R}, \mathscr{T}_d)$, where $d = |x - y|, \forall x, y \in \mathbb{R}$. Let $A = [0, 1)$. Then $A^\circ = (0, 1), A^e = (-\infty, 0) \cup (1, \infty), \partial A = \{0, 1\}$.

Exercise 23. *Let $(X, \mathscr{T})$ be a top. sp., show that A°, A^e are open sets (on X w.r.t $\mathscr{T}$); ∂A is close set.*

Proof. 1. $\forall x \in A^\circ, \exists U_x \in \mathscr{T}$, s.t. $x \in U_x$, thus $A^\circ = \cup_{x \in A^\circ} U_x \in \mathscr{T}$, thus A° is open on X w.r.t. $\mathscr{T}$.
2. A^e is the interior of $X \backslash A$ by definition, thus A^e is open.
3. $A^\circ, A^e \in \mathscr{T} \Rightarrow A^\circ \cup A^e \in \mathscr{T}$, thus $\partial A = X \backslash (A^\circ \cup A^e) \in \mathscr{T}$. □

Exercise 24. *Given a topology space $(X, \mathscr{T}), A \subseteq X$, show that*

$$A^\circ = \cup \{U | U \subseteq_{open} A\}.$$

Proof. $\subseteq$: for $\forall x \in A^\circ, \exists U \in \mathscr{T}$, s.t. $x \in U \subseteq A \Rightarrow x \in \cup\{U|U \subseteq_{open} A\}$; $\supseteq$: for $\forall x \in \cup\{U|U \subseteq_{open} A\}, \exists U_x \subseteq_{open} A$, s.t. $x \in U_x$, thus x is an interior point, and $x \in A^\circ$. □

Definition 21 (Closure)**.** Given a topology space $(X, \mathscr{T}), A \subseteq X$, the set

$$\overline{A} = cls_x A := \cap\{C|A \subseteq C \subseteq_{close} X\}$$

is called the closure of A in X w.r.t. $\mathscr{T}$.

Remark 20. A° is the **largest open set** in X contained in A. Thus,

$$A = A^\circ \Leftrightarrow A \subseteq_{open} X \Leftrightarrow \partial A \cap A = \emptyset$$

for $\partial A \cap A = \partial A \cap A^\circ = \emptyset$. And furthermore $(A^\circ)^\circ = A^\circ$. $\overline{A}$ is the **smallest close set** in X containing in A. Thus,

$$A = \overline{A} \Leftrightarrow A \subseteq_{close} X \Leftrightarrow \partial A \subseteq A$$

for $\partial A \subseteq A^\circ \cup \partial A = \overline{A} = A$. And furthermore $\overline{\overline{A}} = \overline{A}$.

Exercise 25. *Given a topology space $(X, \mathscr{T}), A \subseteq X$, show that $\overline{A} = A^\circ \cup \partial A$.*

Proof.

$$\begin{aligned} A^\circ \cup \partial A &= X \backslash A^e \\ &= X \backslash (X \backslash A)^\circ \\ &= X \backslash \cup \{U|U \subseteq_{open} X \backslash A\} \\ &= \cap\{X \backslash U|U \subseteq_{open} X \backslash A\} \\ &= \cap\{C|A \subseteq C \subseteq_{close} X\} \\ &= \overline{A}. \end{aligned}$$

□

Remark 21.

$$\begin{aligned} &U \subseteq X \backslash A \\ &\Rightarrow \forall x \in U \Rightarrow x \in X \backslash A \\ &\Rightarrow \forall x \notin X \backslash A \Rightarrow x \notin U \\ &\Rightarrow \forall x \in A \Rightarrow x \in X \backslash U \\ &\Rightarrow A \subseteq X \backslash U, \end{aligned}$$

U is open $\Rightarrow X \backslash U$ is close, hence $C = X \backslash U \subseteq_{close} A$.

Exercise 26. *Show that $X \backslash \overline{A} = (X \backslash A)^\circ$ and $X \backslash A^\circ = \overline{(X \backslash A)}$.*

Proof. 1.

$$\begin{aligned}\overline{A} &= A^\circ \cup \partial A \\ &= X \backslash A^e \\ &= X \backslash (X \backslash A)^\circ \\ X \backslash \overline{A} &= (X \backslash A)^\circ.\end{aligned}$$

2.

$$\begin{aligned}X \backslash A^\circ &= A^e \cup \partial A \\ &= (X \backslash A)^c \cup \partial (X \backslash A) \\ &= \overline{(X \backslash A)}.\end{aligned}$$

□

Remark 22. We denote $X \backslash A$ as A^c if X is clearly given. Thus

$$\begin{aligned}(\overline{A})^c &= (A^c)^\circ \\ (A^\circ)^c &= \overline{A^c}\end{aligned}$$

Exercise 27. *If $A \subseteq B$, show that $A^\circ \subseteq B^\circ$, $\overline{A} \subseteq \overline{B}$.*

Proof. 1. Given $x \in A^\circ = \cup\{U | U \subseteq_{open} A\}$, $\exists U_x \subseteq_{open} A$, s.t. $x \in U_x \subseteq_{open} A \subseteq B$, thus $x \in \cup\{V | V \subseteq_{open} B\}$, and $x \in B^\circ$. 2. the same way with 1. □

Exercise 28. *Given a set U, (denote $\overline{U}$ as U^-,) show that $U \subseteq_{open} X \Rightarrow U^- = U^{-c-c-}$.*

Proof.

$$\begin{aligned}U^{-c-c-} &= (U^-)^{c-c-} \\ &= (U^-)^{\circ cc-} \\ &= U^{-\circ-}\end{aligned}$$

$U \subseteq U^- \Rightarrow U = U^\circ \subseteq U^{-\circ} \Rightarrow U^- \subseteq U^{-\circ-}$. Let $C = U^- \subseteq_{close} X$, thus $C^\circ \subseteq C \Rightarrow C^{\circ-} \subseteq C^- = C \Rightarrow U^{-\circ-} \subseteq U^-$, thus $U^- = U^{-\circ-} = U^{-c-c-}$. □

Exercise 29 (Kuratowski's 14 sets)**.** *Given a top. sp. X, $A \subseteq X$, Show that among*

$$\begin{aligned}&A\ , A^-, A^{-c}, A^{-c-}, A^{-c-c} \cdots \\ &A^c, A^{c-}, A^{c-c}, A^{c-c-} \cdots\end{aligned}$$

there are at most 14 different subsets of A.

Proof. On the one hand,

$$A\ , A^-, \underbrace{A^{-c}}_{open}, A^{-c-}, A^{-c-c}, A^{-c-c-}, A^{-c-c-c}, \underbrace{A^{-c-c-c-}}_{=(A^{-c})^{-c-c-}}, \cdots$$

On the other hand,

$$A^c, A^{c-}, \underbrace{A^{c-c}}_{open}, A^{c-c-}, A^{c-c-c}, A^{c-c-c-}, \underbrace{A^{c-c-c-c-}}_{=(A^{c-c})^{-c-c-}}, \cdots$$

thus there are at most 14 different subsets of A. □

Definition 22. X is a top. sp., $p \in X$, $A \subseteq X$:
1. p is an isolated point of A in X, if $\exists$ nbd. U of p, s.t. $U \cap A = \{p\}$;
2. p is a limit point of A in X, if $\forall$ nbd. U of $p, U \cap (A \backslash \{p\}) \neq \emptyset$.

Correspondingly, define $L_A := \{\text{all limit point of } A \text{ in } X\}$.

Exercise 30. *Show that* $\partial A \backslash A \subseteq L_A$.

Proof. $x \in \partial A \backslash A \Rightarrow x \in \partial A$ and $x \notin A \Rightarrow$ for any nbd. U of $x, U \cap A = U \cap (A \backslash \{x\}) = U \cap A \backslash \{x\} \neq \emptyset \Rightarrow x \in L_A$. □

Remark 23. In general, $\partial A \nsubseteq L_A$. For example, if x is an isolate point of A, then it is a boundary point of A, but not be the limit point of A.

Exercise 31. *Show that* $\overline{A} = A \cup L_A$.

Proof 1. 1. $\overline{A} \subseteq A \cup L_A$: If $x \in A \Rightarrow x \in A \cup L_A$; If $x \in \overline{A} \backslash A$: since $x \in \overline{A} = A^o \cup \partial A = X \backslash A^e$, any nbd. U of x has $U \nsubseteq X \backslash A \Rightarrow U \cap A \neq \emptyset$. Since $x \notin A$, $U \cap A = U \cap (A \backslash \{x\}) \neq \emptyset \Rightarrow x \in L_A$.
2. $A \cup L_A \subseteq \overline{A}$: If $x \in A \Rightarrow x \in \overline{A}$; If $x \in L_A \Rightarrow$ any nbd. U of x has $U \cap A \neq \emptyset \Rightarrow x \notin A^e \Rightarrow x \in X \backslash A^e = \overline{A}$. □

Proof 2. 1. $\overline{A} = A^o \cup \partial A = A^o \cup (\partial A \cap A) \cup (\partial A \backslash A)$. If $x \in A^o \cup (\partial A \cap A) \Rightarrow x \in A$; if $x \in \partial A \backslash A \Rightarrow x \in L_A$. Thus $\overline{A} \subseteq A \cup L_A$.
2. If $x \in X \backslash \overline{A} = (X \backslash A)^o$, then $\exists$ a nbd. U of x, s.t. $U \subseteq X \backslash A \Rightarrow U \cap A = \emptyset \Rightarrow x$ is not a limit point of A in $X \Rightarrow x \in X \backslash L_A \Rightarrow X \backslash \overline{A} \subseteq X \backslash L_A \Rightarrow L_A \subseteq \overline{A} \Rightarrow A \cup L_A \subseteq A \cup \overline{A} = \overline{A}$.

□

Remark 24. Useful routines:
1. $A \subseteq B \Leftrightarrow X \backslash A \supseteq X \backslash B$
2. $x \notin \overline{A} \Leftrightarrow \exists$ nbd. U of x, s.t. $U \cap A = \emptyset$.

Exercise 32. *Show that* $\overline{A} = \{x \in X | \forall \text{ open nbd. } U_x \text{ of } x, U_x \cap A \neq \emptyset\}$.

Proof. $\subseteq$: if $x \in \overline{A} \Rightarrow X \in A \cup L_A$. If $x \in A, \forall$ open nbd. U_x of x has $x \in U_x \cap A \neq \emptyset$; If $x \in L_A \backslash A, \forall$ open nbd. U_x of x has $U_x \cap A \backslash \{x\} \neq \emptyset \Rightarrow U_x \cap A \neq \emptyset$.
$\supseteq$: If $x \notin \overline{A} \Rightarrow x \in X \backslash (A^o \cup \partial A) = A^e = (X \backslash A)^o$, then $\exists$ an open ubd. U_x of x s.t. $U_x \subseteq X \backslash A \Rightarrow U_x \cap A = \emptyset$. Thus $\forall$ open ubd. U_x of x if $U_x \cap A \neq \emptyset \Rightarrow x \in \overline{A}$. □

Exercise 33. *Let* $X \xrightarrow[conti.]{f} Y, A \subseteq X, B \subseteq Y$, *show that:*

1. $f^{-1}(\overline{B}) \supseteq \overline{f^{-1}(B)}$; $f(\overline{A}) \subseteq \overline{f(A)}$
2. $f^{-1}(B^o) \subseteq f^{-1}(B)^o$; $f(A^o) \supseteq f(A)^o$.
3. $f^{-1}(B^e) \subseteq f^{-1}(B)^e$; *if f is a surjection,* $f(A^e) \supseteq f(A)^e$.
4. $f^{-1}(\partial B) \supseteq \partial f^{-1}(B)$; $f(\partial A) \subseteq \partial f(A)$.

Proof. 1. $B \subseteq \overline{B} \Rightarrow f^{-1}(B) \subseteq f^{-1}(\overline{B}) \subseteq_{close} X \Rightarrow \overline{f^{-1}(B)} \subseteq \overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B})$;
$f(A) \subseteq \overline{f(A)} \Rightarrow A \subseteq f^{-1}(\overline{f(A)}) \subseteq_{close} X \Rightarrow \overline{A} \subseteq \overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)}) \Rightarrow f(\overline{A}) \subseteq \overline{f(A)}$.

2. $B^o \subseteq B \Rightarrow X\ {}_{open} \supseteq f^{-1}(B^o) \subseteq f^{-1}(B) \Rightarrow f^{-1}(B^o) = f^{-1}(B^o)^o \subseteq f^{-1}(B)^o$;
$f(A)^o \subseteq f(A) \Rightarrow f^{-1}(f(A)^o) \subseteq A \Rightarrow f^{-1}(f(A)^o) = f^{-1}(f(A)^o)^o \subseteq A^o \Rightarrow f(A)^o \subseteq f(A^o)$.

3. Since $B^e = (Y \backslash B)^e$,

$$\begin{aligned} f^{-1}(B^e) &= f^{-1}((Y\backslash B)^o) \\ &\subseteq f^{-1}(Y\backslash B)^o \\ &= [f^{-1}(Y)\backslash f^{-1}(B)]^o \\ &= [X\backslash f^{-1}(B)]^o \\ &= f^{-1}(B)^e. \end{aligned}$$

and

$$\begin{aligned} f(A^e) &= f((X\backslash A)^o) \\ &\supseteq f(X\backslash A)^o \\ &\supseteq [f(X)\backslash f(A)]^o \\ &\overset{f \text{ is surj.}}{=\!=\!=} [Y\backslash f(A)]^o \\ &= f(A)^e. \end{aligned}$$

4. Since $\overline{B} = B^o \cup \partial B$,

$$\begin{aligned} &\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \\ &\Rightarrow f^{-1}(B)^o \cup \partial f^{-1}(B) \subseteq f^{-1}(B^o \cup \partial B) \\ &\Rightarrow f^{-1}(B)^o \cup \partial f^{-1}(B) \subseteq f^{-1}(B^o) \cup f^{-1}(\partial B). \end{aligned}$$

since $f^{-1}(B)^o \supseteq f^{-1}(B^o)$, $\partial f^{-1}(B) \subseteq f^{-1}(\partial B)$.

and

$$\begin{aligned} &f(\overline{A}) \subseteq \overline{f(A)} \\ &\Rightarrow f(\partial A) \cup f(A^o) = f(\partial A \cup A^o) \\ &\subseteq \partial f(A) \cup f(A)^o \end{aligned}$$

since $f(A^o) \supseteq f(A)^o$, $f(\partial A) \subseteq \partial f(A)$. $\square$

Remark 25.1: Recall that:

(a) $X \xrightarrow{f} Y$ is conti. $\Leftrightarrow$ for any $B \subseteq_{open} Y (\subseteq_{close} Y)$, $f^{-1}(B) \subseteq_{open} X (\subseteq_{close} X)$.

(b) $A^o \subseteq A \subseteq \overline{A}$.

(c) $A \subseteq_{close} X \Rightarrow \overline{A} = A$; $A \subseteq_{open} X \Rightarrow A^o = A$.

4: $A \subseteq B, A \cup C \supseteq B \cup D \Rightarrow C \supseteq D$.

Proof. $A \cup C \supseteq B \cup D \supseteq A \cup D \Rightarrow A^c \cup A \cup C \supseteq A^c \cup A \cup D \Rightarrow X \cup C \supseteq X \cup D \Rightarrow C \supseteq D$. $\square$

Exercise 34. *X is a top. sp., $A_i \subseteq X (i \in I)$, show that*

$$\cup_{i\in I}\overline{A_i} \subseteq \overline{\cup_{i\in I}A_i}$$

and

$$\overline{\cap_{i\in I}A_i} \subseteq \cap_{i\in I}\overline{A_i}.$$

Proof. 1. For any $i \in I, A_i \subseteq \cup_{i\in I}A_i \Rightarrow \overline{A_i} \subseteq \overline{\cup_{i\in I}A_i} \Rightarrow \cup_{i\in I}\overline{A_i} \subseteq \overline{\cup_{i\in I}A_i}$.
2. For any $i \in I$, $A_i \subseteq \overline{A_i} \subseteq_{close} X \Rightarrow \cap_{i\in I}A_i \subseteq \cap_{i\in I}\overline{A_i} \subseteq_{close} X \Rightarrow \overline{\cap_{i\in I}A_i} \subseteq \overline{\cap_{i\in I}\overline{A_i}} = \cap_{i\in I}\overline{A_i}$. $\square$

Note that the '=' doer not necessary hold. For example, let $A_r = (1/r, 1-1/r), r > 2$, then $\cup_{r>2}A_r = \cup_{r>2}\overline{A_r} = (0,1) \subsetneq \overline{\cup_{r>2}A_r} = [0,1]$.
Let $B_1 = (0,1/2), B_2 = (1/2,1)$, then $\overline{B_1 \cap B_2} = B_1 \cap B_2 = \emptyset$, but $\overline{B_1} \cap \overline{B_2} = [0,1/2] \cap [1/2,1] = 1/2$.

Remark 26. If I is finite, then $\cup_{i\in I}\overline{A_i} = \overline{\cup_{i\in I}A_i}$.
Since $A_i \subseteq \overline{A_i} \Rightarrow \cup_{i\in I}A_i \subseteq \cup_{i\in I}\overline{A_i} \Rightarrow \overline{\cup_{i\in I}A_i} \subseteq \overline{\cup_{i\in I}\overline{A_i}}$, and since I is finite, $\cup_{i\in I}\overline{A_i}$ is closed, thus $\overline{\cup_{i\in I}A_i} \subseteq \cup_{i\in I}\overline{A_i}$.

3.3 Locally Finite

Definition 23 (Locally Finite). A family $\mathcal{S}$ of some subsets of a top. space X is locally finite if $\forall p \in X, \exists$ nbd. U of p s.t. $\{S \in \mathcal{S} | U \cap S \neq \emptyset\}$ is a finite set.

Exercise 35. *If $\mathcal{S}$ is locally finite family, show that*

$$\overline{\cup_{S\in\mathcal{S}}S} = \cup_{S\in\mathcal{S}}\overline{S}.$$

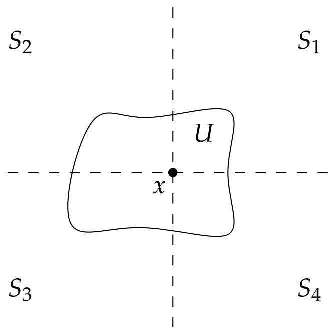
Proof 1. We claim $\overline{\cup_{S\in\mathcal{S}}S} \subseteq \cup_{S\in\mathcal{S}}\overline{S}$, i.e. $\cap_{S\in\mathcal{S}}(X\backslash\overline{S}) = X\backslash \cup_{S\in\mathcal{S}}\overline{S} \subseteq X\backslash\overline{\cup_{S\in\mathcal{S}}S}$. Note that $x \in X\backslash\overline{\cup_{S\in\mathcal{S}}S} \Leftrightarrow \exists$ a nbd. W of x, s.t. $W \cap S = \emptyset$ for $\forall S \in \mathcal{S}$. That is, we want to find a nbd of x such that has no intersection with any S in $\mathcal{S}$, the locally finiteness of $\mathcal{S}$ tells us there exists a nbd. U of x that intersects with only finite sets $S_1, \cdots, S_k \in \mathcal{S}$. Thus all we need to do is eliminate these intersected part from U.

$x \in \cap_{S\in\mathcal{S}}(X\backslash\overline{S}) \Rightarrow x \in X\backslash\overline{S}$ for any $S \in \mathcal{S}$. Thus for any $S \in \mathcal{S}$, $\exists$ a nbd V of x, s.t. $V \cap S = \emptyset$. And $\exists$ a nbd U of x, s.t. U only intersects with finite set $S_1, \cdots, S_k \in \mathcal{S}$. Note that $W := U \cap V_1 \cap \cdots \cap V_k$ is still a nbd. of x, since the finite union of open set is open. And $W \cap S = \emptyset$ for any $S \in \mathcal{S}$, thus for $\exists$ a nbd. W of x, s.t. $W \cap \cup_{S\in\mathcal{S}} S = \emptyset \Rightarrow x \in X\backslash\overline{\cup_{S\in\mathcal{S}} S} \Rightarrow \overline{\cup_{S\in\mathcal{S}} S} \subseteq \cup_{S\in\mathcal{S}}\overline{S}$. □

Proof 2. Pick $x \notin \bigcup_{S\in\mathcal{S}} \overline{S}$. Due to local finiteness, there is an (open) neighborhood U of x, such that U intersects only finitely many of S: let's say $S_1, S_2, \ldots, S_n$. Now create a new neighborhood $U' = U \setminus (\overline{S_1} \cup \overline{S_2} \cup \cdots \cup \overline{S_n})$, which is an open set containing x, and U' does not intersect any of $S \in \mathcal{S}$. Thus for any $S \in \mathcal{S}, S \subseteq X\backslash U' \Rightarrow \overline{S} \subseteq \overline{X\backslash U'} \overset{X\backslash U' \subseteq_{close} X}{=\!=\!=} X\backslash U'$. Thus U' also does not intersect any of $\overline{S}$.
Thus, for any $x \in X\backslash \bigcup_{S\in\mathcal{S}} \overline{S}, \exists$ an open nbd. U' of x, such that $U' \cap \bigcup_{S\in\mathcal{S}} \overline{S} = \emptyset$. Thus $X\backslash \bigcup_{S\in\mathcal{S}} \overline{S}$ is open, i.e. $\bigcup_{S\in\mathcal{S}} \overline{S}$ is closed. Thus $\bigcup_{S\in\mathcal{S}} S \subseteq \bigcup_{S\in\mathcal{S}} \overline{S} \Rightarrow \overline{\bigcup_{S\in\mathcal{S}} S} \subseteq \overline{\bigcup_{S\in\mathcal{S}} \overline{S}} = \bigcup_{S\in\mathcal{S}} \overline{S}$. □

Remark 27. There is no similar feature for the intersection, for example, $S_1 = (0, 1)$ and $S_2 = (1, 2)$.

If $\mathcal{S}$ is locally finite, given a $x \in X$, then $\exists$ a nbd. U of x, s.t. U intersects only finite, such as k, Ss in $\mathcal{S}$. Clearly k has a minimal number, such as 3. Note that it does not imply x is covered by 3 Ss in $\mathcal{S}$.



3.4 Basis

Definition 24 (Coarser Topology)**.** Let X be a set, and $\mathscr{T}$ and $\mathscr{T}'$ be two topologies on X. We say that $\mathscr{T}$ is coarser/weaker than $\mathscr{T}'$ if $\mathscr{T} \subseteq \mathscr{T}'$ (or say $\mathscr{T}'$ is finer/stronger than $\mathscr{T}$).

Remark 28. In other words, $\mathscr{T}$ is weaker than $\mathscr{T}'$ iff $X \xrightarrow{id_X} X$, where the former and later X are equipped with $\mathscr{T}'$ and $\mathscr{T}$ respectively, is continuous.

Let X be a set and $\mathcal{S} \subseteq \mathcal{P}(X)$ be a family of subsets of X. Are there a smallest topology

$\mathscr{T}'$ on X s.t. all $\mathcal{S} \subseteq \mathscr{T}'$? It is direct to check that if $\mathscr{T}_\alpha(\alpha \in A)$ is a family of topologies on X, then $\cap_{\alpha \in A}\mathscr{T}_\alpha$ is also a topology on X. For any $\alpha \in A$:
1. $\emptyset, X \in \mathscr{T}_\alpha \Rightarrow \emptyset, X \in \cap_{\alpha \in A}\mathscr{T}_\alpha$;
2. $U_\beta \in \mathscr{T}_\alpha(\beta \in B) \Rightarrow \cup_{\beta \in B}U_\beta \in \mathscr{T}_\alpha \Rightarrow \cup_{\beta \in B}U_\beta \in \cap_{\alpha \in A}\mathscr{T}_\alpha$.
3. $U_1, U_2 \in \mathscr{T}_\alpha \Rightarrow U_1 \cap U_2 \in \mathscr{T}_\alpha \Rightarrow U_1 \cap U_2 \in \cap_{\alpha \in A}\mathscr{T}_\alpha$.
Define $\mathcal{T}$ be the family of all topologies on X containing the elements in $\mathcal{S}$, that is for $\forall \mathscr{T} \in \mathcal{T}, \mathcal{S} \subseteq \mathscr{T}$. We call

$$\mathscr{T}(\mathcal{S}) := \cap_{\mathscr{T} \in \mathcal{T}}\mathscr{T}$$

the topology induced by $\mathcal{S}$, which is clearly the coarsest topology containing $\mathcal{S}$.
Let Π be the family of any finite intersection of the element in $\mathcal{S}$, then for $\forall \mathscr{T} \in \mathcal{T}, \Pi \subseteq \mathscr{T}$ by def. Furthermore, for $\forall \mathscr{T} \in \mathcal{T}$, the arbitrary union of the elements in Π must in $\mathscr{T}$, that is $\{\cup_{V \in \mathcal{F}}V | \mathcal{F} \subseteq \Pi\} \subseteq \mathscr{T}$. Thus $\{\cup_{V \in \mathcal{F}}V | \mathcal{F} \subseteq \Pi\} \subseteq \cap_{\mathscr{T} \in \mathcal{T}}\mathscr{T} = \mathscr{T}(\mathcal{S})$.

Proposition 2. *Let X be a set and $\mathcal{S} \subseteq \mathcal{P}(X)$ be a family of subsets of X. Then*

$$\mathscr{T}(\mathcal{S}) = \{\cup_{V \in \mathcal{F}}V | \mathcal{F} \subseteq \Pi\},$$

where Π is the family of any finite intersection of elements in $\mathcal{S}$, that is

$$\Pi := \{S_1 \cup \cdots \cup S_k | S_1, \cdots, S_k \in \mathcal{S}, k \in \mathbb{N}\} \cup \{X\}.$$

Proof. We have proved that $\{\cup_{V \in \mathcal{F}}V | \mathcal{F} \subseteq \Pi\} \subseteq \mathscr{T}(\mathcal{S})$. Note that $\mathscr{T}(\mathcal{S})$ is the coarsest topology containing $\mathcal{S}$, Thus if $\{\cup_{V \in \mathcal{F}}V | \mathcal{F} \subseteq \Pi\}$ is a topology containing $\mathcal{S}$, we are done.
1. $\{X\}, \emptyset \subseteq \Pi$, thus $X = \cup_{V \in \{X\}}V \in \{\cup_{V \in \mathcal{F}}V | \mathcal{F} \subseteq \Pi\}$, $\emptyset = \cup_{V \in \{\emptyset\}}V \in \{\cup_{V \in \mathcal{F}}V | \mathcal{F} \subseteq \Pi\}$.
2. For any $U_\alpha = \{\cup_{V \in \mathcal{F}_\alpha}V | \mathcal{F}_\alpha \subseteq \Pi\} \in \{\cup_{V \in \mathcal{F}}V | \mathcal{F} \subseteq \Pi\}(\alpha \in A)$, we have $\mathcal{F}_\alpha \subseteq \Pi(\alpha \in A) \Rightarrow \cup_{\alpha \in A}\mathcal{F}_\alpha \subseteq \Pi \Rightarrow \cup_{\alpha \in A}U_\alpha = \{\cup_{V \in \cup_{\alpha \in A}\mathcal{F}_\alpha}V | \cup_{\alpha \in A}\mathcal{F}_\alpha \subseteq \Pi\} \in \{\cup_{V \in \mathcal{F}}V | \mathcal{F} \subseteq \Pi\}$
3. If $\cup_{V \in \mathcal{F}_1}V, \cup_{W \in \mathcal{F}_2}W \in \{\cup_{V \in \mathcal{F}}V | \mathcal{F} \subseteq \Pi\}$, then $(\cup_{V \in \mathcal{F}_1}V) \cap (\cup_{W \in \mathcal{F}_2}W) = \cup_{V \in \mathcal{F}_1, W \in \mathcal{F}_2}(V \cap W)$ where $V, W \in \Pi$. Since Π is the family of finite intersection, $V \cap W$ is the finite intersection of elements of $\mathcal{S}$ or X, i.e. $V \cap W \in \Pi$. Let $\mathcal{F}_3 := \{V \cap W | V \in \mathcal{F}_1, W \in \mathcal{F}_2\}$, thus $\mathcal{F}_3 \subseteq \Pi$. Then $\cup_{V \in \mathcal{F}_1, W \in \mathcal{F}_2}(V \cap W) = \cup_{Z \in \mathcal{F}_3}Z \in \{\cup_{V \in \mathcal{F}_3}V | \mathcal{F}_3 \subseteq \Pi\}$.

Thus $\{\cup_{V \in \mathcal{F}}V | \mathcal{F} \subseteq \Pi\}$ is a topology containing $\mathcal{S}$, and $\mathscr{T}(S) \subseteq \{\cup_{V \in \mathcal{F}}V | \mathcal{F} \subseteq \Pi\} \Rightarrow \mathscr{T}(S) = \{\cup_{V \in \mathcal{F}}V | \mathcal{F} \subseteq \Pi\}$. □

Remark 29. Orally, $\mathscr{T}(\mathcal{S})$ consists of arbitrary unions of finite intersection of elements of $\mathcal{S}$.
Conventionally, when we talking about the subsets of X, we define $\cap\emptyset := X$.

Definition 25 (Sub-basis)**.** Given a set X, $\mathcal{S} \subseteq \mathcal{P}(X)$, $\mathcal{S}$ is called a sub-basis of a topology $\mathscr{T}$ on X if $\mathscr{T} = \mathscr{T}(S)$.

To obtain $\mathscr{T}(\mathcal{S})$ from $\mathcal{S}$, we need two steps: first, perform the finite intersection of elements in $\mathcal{S}$; then perform arbitrary union of the these intersection. But can we construct a topology that contains $\mathcal{S}$ only by union?

Definition 26 (Basis)**.** Given a set X, let $\mathcal{B} \subseteq \mathcal{P}(X)$ and $\mathscr{T}$ is a topology on X. We say that $\mathcal{B}$ is a basis of $\mathscr{T}$ if $\mathcal{B} \subseteq \mathscr{T}$ and for any $U \in \mathscr{T}, \exists \mathcal{F} \subseteq \mathcal{B}$, s.t. $U = \cup \mathcal{F} (:= \cup_{B \in \mathcal{F}} B)$.

Remark 30. Thus given a sub-basis $\mathcal{S}$, we can induce the basis Π, and then perform the union on basis to obtain the topology $\mathscr{T}(\mathcal{S})$.

Note that if $\mathcal{B}$ is a basis of $\mathscr{T}$, then $B \in \mathscr{T}$ for any $B \in \mathcal{B}$, thus any union of elements of $\mathcal{B}$ is in $\mathscr{T}$. Thus we can define the $\mathcal{B}$ is a basis of $\mathscr{T}$ directly:

$$\mathscr{T} = \{\cup \mathcal{F} | \mathcal{F} \subseteq \mathcal{B}\}.$$

In general, a topological space $(X, \mathscr{T})$ can have many bases. The whole topology $\mathscr{T}$ is always a base for itself (that is, $\mathscr{T}$ is a base for $\mathscr{T}$).

Definition 27 (Local Basis)**.** For a given $x \in X$, we say that $\mathcal{B}_x$ is a local basis of $\mathscr{T}$ at x, if
1. for $\forall V \in \mathcal{B}_x, x \in V \in \mathscr{T}$ and
2. for $\forall U \in \mathscr{T}$ where $x \in U$, $\exists V \in \mathcal{B}_x$, s.t. $x \in V \subseteq U$.

Example 11. Let X be a metric space and $\mathscr{T}$ is the topology defined by metric. Then $\mathcal{B} = \{B_r(x) | r > 0\}$ is a local basis of $\mathscr{T}$ at x.

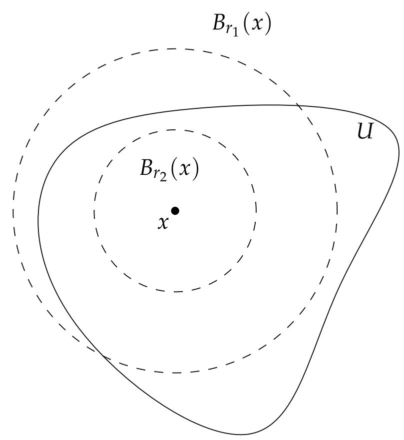


Figure 3.1: Local Basis

Exercise 36. *Let $(X, \mathscr{T})$ be a topology space and $\mathcal{B} \subseteq \mathcal{P}(X)$. For $x \in X$, define $\mathcal{B}_x := \{U \in \mathcal{B} | x \in U\}$. Show that $\mathcal{B}$ is a basis of $\mathscr{T}$ on $X \Leftrightarrow \forall x \in X, \mathcal{B}_x$ is a local basis of $\mathscr{T}$ on X at x.*

Proof. $\Rightarrow$: pick a $x \in X$ and $U \in \mathscr{T}$ where $x \in U$, then $\exists \mathcal{F} \subseteq \mathcal{B}$, s.t. $x \in U = \cup \mathcal{F}$, since $\mathcal{B}$ is a basis of $\mathscr{T}$. Then $\exists B \in \mathcal{F}$ such that $x \in B \subseteq \cup \mathcal{F} = U$, it is clear to see $B \in \mathcal{B}_x$. Since $\mathcal{B}$ is a basis of $\mathscr{T}$, $B \in \mathscr{T}$ for $\forall B \in \mathcal{B}$, Thus $\mathcal{B}_x$ is a local basis of $\mathscr{T}$ at x for any $x \in X$.

$\Leftarrow$: On the one hand, given a $x \in X, \mathcal{B}_x \subseteq \mathcal{B} \Rightarrow \cup_{x \in X} \mathcal{B}_x \subseteq \mathcal{B}$. For any $B \in \mathcal{B}$, if $B \neq \emptyset$, there exists $x' \in B$, thus $B \in \mathcal{B}_{x'} \subseteq \cup_{x \in X} \mathcal{B}_c$. Thus $\mathcal{B} = \cup_{x \in X} \mathcal{B}_x$. $\mathcal{B}_x$ is a local basis of $\mathscr{T}$ at any $x \in X \Rightarrow \mathcal{B}_x \subseteq \mathscr{T}$ for any $x \in X$. Thus $\mathcal{B} \subseteq \mathscr{T}$.

On the other hand, given a non-empty $U \in \mathscr{T}$, for any $x \in U$, $\exists B_x \in \mathcal{B}_x$, such that $x \in B_x \subseteq U$. Thus $\cup_{x \in U} B_x \subseteq U$. For any $x' \in U$, $\exists B_{x'} \in \mathcal{B}_{x'}$, s.t. $x' \in B_{x'} \subseteq U \Rightarrow x' \in \cup_{x \in U} B_x \Rightarrow \cup_{x \in U} B_x = U$, where $B_x \in \mathcal{B}$. Thus $\mathcal{B}$ is a basis of $\mathscr{T}$. $\square$

Remark 31. Very useful routine. We use it to prove the open set, in metric space, is the union of open balls as well.

Exercise 37. *Let X be a set and $\mathcal{B} \subseteq \mathcal{P}(X)$. Show that there exists a topology $\mathscr{T}$ such that $\mathcal{B}$ is a basis of $\mathscr{T}$ $\Leftrightarrow$*

1. *$\cup \mathcal{B} = X$ and*
2. *$\forall U, V \in \mathcal{B}$ and $x \in U \cap V, \exists W \in \mathcal{B}$,* s.t. *$x \in W \subseteq U \cap V$.*

(Hint: if such $\mathscr{T}$ exists, it must be $\{\cup \mathcal{F} | \mathcal{F} \subseteq \mathcal{B}\}$.)

Proof. $\Rightarrow$: 1) $X \in \mathscr{T} \Rightarrow \exists \mathcal{F} \subseteq \mathcal{B}$, s.t. $X = \cup \mathcal{F} \subseteq \cup \mathcal{B} \subseteq X \Rightarrow X = \cup \mathcal{B}$; 2) $\mathcal{B}$ is a basis of $\mathscr{T} \Rightarrow \forall U, V \in \mathcal{B}, U, V \in \mathscr{T}$, thus $U \cap V \in \mathscr{T}$. Pick $x \in U \cap V$, $\mathcal{B}_x$ is a local basis of $\mathscr{T}$ at x. Thus $\exists B \in \mathcal{B}_x \subseteq \mathcal{B}$, s.t. $x \in B \subseteq U \cap V$.

$\Leftarrow$: Define $\mathscr{T} = \{\cup \mathcal{F} | \mathcal{F} \subseteq \mathcal{B}\}$, all we need to de is show $\mathscr{T}$ is a topology:

1. $\emptyset \subseteq \mathcal{B} \Rightarrow \emptyset = \cup \emptyset \in \mathscr{T}$; $\mathcal{B} \subseteq \mathcal{B} \Rightarrow X = \cup \mathcal{B} \in \mathscr{T}$.
2. for any $\mathcal{F}_\alpha \subseteq \mathcal{B}(\alpha \in A)$,

$$\begin{aligned}
\cup_{\alpha \in A}(\cup \mathcal{F}_\alpha) &= \cup_{\alpha \in A}(\cup_{B \in \mathcal{F}_\alpha} B) \\
&= \cup_{B \in \cup_{\alpha \in A} \mathcal{F}_\alpha} B \\
&= \cup(\cup_{\alpha \in A} \mathcal{F}_\alpha) \\
&\in \mathscr{T},
\end{aligned}$$

   since $\cup_{\alpha \in A} \mathcal{F}_\alpha \subseteq \mathcal{B}$.

3. for any $U = \cup \mathcal{F}_1, V = \cup \mathcal{F}_2 \in \mathscr{T}$,

$$\begin{aligned}
U \cap V &= (\cup \mathcal{F}_1) \cap (\cup \mathcal{F}_2) \\
&= \cup_{B \in \mathcal{F}_1, C \in \mathcal{F}_2}(B \cap C)
\end{aligned}$$

   where $B, C \in \mathcal{B}$, thus for any $x \in B \cap C, \exists D_x \in \mathcal{B}$ such that $x \in D_x \subseteq B \cap C$. Thus it is direct to see that $B \cap C = \cup_{x \in B \cap C} D_x$. Thus

$$\begin{aligned} D_x \in \mathcal{B} \Rightarrow D_x &\in \mathscr{T} \\ &\Rightarrow \cup_{x \in B \cap C} D_x \in \mathscr{T} \\ &\Rightarrow U \cap V = \cup_{B \in \mathcal{F}_1, C \in \mathcal{F}_2} (B \cap C) \\ &= \cup_{B \in \mathcal{F}_1, C \in \mathcal{F}_2} (\cup_{x \in B \cap C} D_x) \in \mathscr{T}. \end{aligned}$$

Thus $\mathscr{T}$ is such topology as desired. □

Recall that when we check whether a map $X \xrightarrow{f} Y$ is conti., we need show that for $\forall V \subseteq_{open} Y$, $f^{-1}(V) \subseteq_{open} X$. But if Y is equipped with a topology induced by some sub-basis, we can only check some subset of Y, instead of any subset of Y.

Exercise 38. *Let Z be a topology space and $Z \xrightarrow{f} X$ is a map. Show that f is continuous when X is topologized by $\mathscr{T}(\mathcal{S}) \Leftrightarrow \forall S \in \mathcal{S}, f^{-1}(S) \subseteq_{open} Z$.*

Proof. $\Rightarrow$: $\forall S \in \mathcal{S} \Rightarrow S \in \mathscr{T}(\mathcal{S})$, that is $S \subseteq_{open} X \Rightarrow f^{-1}(S) \subseteq_{open} Z$.
$\Leftarrow$: for any $U \in \mathscr{T}(\mathcal{S})$, it can be represented by the union of some finite intersections of elements of $\mathcal{S}$, that is $U = \cup_{F \in \mathcal{F}} F$, where $\mathcal{F} \subseteq \Pi$, and $F = \cap_{i=1}^{k_F} S_i, S_i \in \mathcal{S}$. Thus

$$\begin{aligned} f^{-1}(U) &= f^{-1}(\cup_{F \in \mathcal{F}} F) \\ &= \cup_{F \in \mathcal{F}} f^{-1}(\cap_{i=1}^{k_F} S_i) \\ &= \cup_{F \in \mathcal{F}} \left(\cap_{i=1}^{k_F} f^{-1}(S_i) \right) \\ &\subseteq_{open} Z. \end{aligned}$$

Thus $Z \xrightarrow{f} X$ is continuous. □

3.5 Countable, Separable and Lindelof Compact

Definition 28. A topology space $(X, \mathscr{T})$ is

1. 1st-countable if $\forall x \in X, \exists$ countable local basis of $\mathscr{T}$ at x;
2. 2nd-countable if $\exists$ countable basis of $\mathscr{T}$. (That is $\exists$ countable open set in X such that any element in $\mathscr{T}$ is the union of these open set.)

Remark 32. $\mathcal{B}$ is a basis of $\mathscr{T}$ $\Rightarrow$ $\mathcal{B}_x$ is a local basis of $\mathscr{T}$ at x. Thus $(X, \mathscr{T})$ is 2nd-countable $\Rightarrow$ $(X, \mathscr{T})$ is 1st-countable.

Example 12. 1. Let X be a metric space and $\mathscr{T}$ is the topology defined by metric. Then $\mathcal{B} = \{B_r(x) | r > 0, r \in \mathbb{Q}\}$ is a countable local basis of $\mathscr{T}$ at x, Thus metric space is 1st-countable.

2. Note that the open set in $\mathbb{R}$ is the union of disjoined open intervals in $\mathbb{R}$. Any open interval can be represented by the union of countable open intervals that start and end at rational number. Thus any open set in $\mathbb{R}$ is the union of countable open intervals. Thus $\mathbb{R}$ is 2nd-countable.

Definition 29 (Dense)**.** Given a topology space X, we say a subset $A \subseteq X$ is dense if $\overline{A} = X$.

Exercise 39. *X is a topology space, $A \subseteq X$, show that A is dense $\Leftrightarrow \forall U \subseteq_{open} X, U \neq \emptyset$, then $U \cap A \neq \emptyset$.*

Proof. $\Rightarrow$: $\overline{A} = A^o \cup \partial A = X$, thus $X \backslash A^o = \partial A$ as A^o and ∂A are disjoined. For any $U \subseteq_{open} X$, if $U \neq \emptyset$, pick $x \in U$, then either $x \in A^o$ or $x \in X \backslash A^o = \partial A$.
If $x \in A^o \Rightarrow x \in U \cap A \neq \emptyset$; If $x \in \partial A$, U is a nbd. of $x \Rightarrow U \cap A \neq \emptyset$.
$\Leftarrow$: If $\overline{A} \neq X \Rightarrow W := X \backslash \overline{A} \neq \emptyset$, and $W \subseteq_{open} X, W \cap \overline{A} = (X \backslash \overline{A}) \cap \overline{A} = \emptyset$, which leads to a contradiction. □

Definition 30 (Separable)**.** A topology space $(X, \mathscr{T})$ is separable if X has a countable dense subset.

Exercise 40. *If $\mathcal{B}$ is a basis of a topology space X and pick a point x_B in B for any non-empty set $B \in \mathcal{B}$. Show that $\{x_B \in B | B \in \mathcal{B}, B \neq \emptyset\} \subseteq_{dense} X$.*

Proof. If $U \subseteq_{open} X$ and $U \neq \emptyset$, then $\exists \mathcal{F} \subseteq \mathcal{S}$, s.t. $U = \cup \mathcal{F}$. Then $x_F \in F \in \mathcal{F} \subseteq \cup \mathcal{F} = U \Rightarrow x_F \in U \cap \{x_B \in B | B \in \mathcal{B}\} \neq \emptyset \Rightarrow \{x_B \in B | B \in \mathcal{B}\} \subseteq_{dense} X$. □

Remark 33. Thus if $\mathcal{B}$ is a countable basis of $\mathscr{T}$ on X, then $\{x_B \in B | B \in \mathcal{B}\}$ is a countable dense subset of X, and $(X, \mathscr{T})$ is a separable topology space.

Definition 31 (Lindelof Compact)**.** A topology space $(X, \mathscr{T})$ is Lindelof compact if $\forall U_\alpha \subseteq_{open} X (\alpha \in A), \cup_{\alpha \in A} U_\alpha = X \Rightarrow \exists$ countable set $A_0 \subseteq A$, s.t. $\cup_{\alpha \in A_0} U_\alpha = X$.

It is direct to see that 2nd-countable $\Rightarrow$ Lindelof Compact, since if $\mathcal{B}$ is a basis of $\mathscr{T}$ on X, then $X = \cup \mathcal{B}$. Collectively, we have

$$\begin{tikzcd}
 & \text{1st-Countable} \\
\text{2nd-Countable} \arrow[ur, Rightarrow] \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{Separable} \\
 & \text{Lindelof Compact}
\end{tikzcd}$$

Exercise 41. *If X is topologized by a metric (a.k.a. X is a metrizable topology space) then 2nd-Countable $\Leftrightarrow$ Separable $\Leftrightarrow$ Lindelof Compact.*

Proof. 1. Separable $\Rightarrow$ 2nd-Countable: To prove this statement, we need to track back to the $\Leftarrow$ case: If D is the countable dense subset of X, we claim that $\mathcal{B} := \{B_{\frac{1}{n}}(s) | s \in D, n \in \mathbb{N}\}$ is the basis of metric topology on X.
Given a $U \subseteq_{open} X$ and $U \neq \emptyset$, we have $U \cap D \neq \emptyset$. For any $u \in U \cap D$, exists $n_u \in \mathbb{N}$, s.t. $B_{\frac{1}{n_u}}(u) \subseteq U$. Obviously,

$$W := \bigcup_{u \in U \cap D} B_{\frac{1}{n_u}}(u) \subseteq U.$$

For any $v \in U$, if $v \in U \cap D \Rightarrow v \in W$; if $v \notin D \Rightarrow v \in L_D$, since $X = D \cup L_D$. Thus $\exists n_v \in \mathbb{N}$, s.t. $\exists u \in B_{\frac{1}{n_v}}(v) \cap D \backslash \{v\}$, where $B_{\frac{1}{n_v}}(v) \subseteq U$ and $u \in U \cap D$ whose $1/n_u > 1/n_v$. Thus $v \in B_{\frac{1}{n_u}}(u) \subseteq W \Rightarrow U = W = \bigcup_{u \in U \cap D} B_{\frac{1}{n_u}}(u)$, where $\{B_{\frac{1}{n_u}}(u) | u \in U \cap D, B_{\frac{1}{n_u}}(u) \subseteq U\} \subseteq \mathcal{B}$. Thus $\mathcal{B}$ is a basis of metric topology on X.
2. Lindelof Compact $\Rightarrow$ Separable: For any $x \in X, \exists r_x > 0$, s.t. $B_{r_x}(x) \subseteq X$, it is direct to see that $X = \cup_{x \in X} B_{r_x}(x)$. X is Lindelof Compact, thus exist countable subset D of X such that $X = \cup_{x \in D} B_{r_x}(x)$. For any non-empty $U \subseteq_{open} X$, any $u \in U \subseteq X = \cup_{x \in D} B_{r_x}(x)$, thus $U \cap D \neq \emptyset \Rightarrow D$ is dense $\Rightarrow X$ is separable. $\square$

Chapter 4

Initial / Final Topology

4.1 Initial Topology

Given maps $X \xrightarrow{f_\alpha} Y_\alpha(\alpha \in A)$ from a set X to topology spaces $Y_\alpha(\alpha \in A)$. It is direct to see that if X is topoloized by discrete topology, the f_α are all continuous. Now the question is how coarse the topology $\mathscr{T}$ on X could be to ensure $f_\alpha(\alpha \in A)$ to be continuous.
Let $\mathcal{S} := \{f_\alpha^{-1}(V) | V \subseteq_{open} Y_\alpha, \alpha \in A\}$, then $\mathscr{T}(\mathcal{S})$ is the expected coarsest topology, called the **initial topology** induced by the family of maps $\{f_\alpha | \alpha \in A\}$.

4.1.1 Subspace Topology

Let $(Y, \mathscr{T}_Y)$ be a topology space, for a subset $X \subseteq Y$. We want to define an natural topology $\mathscr{T}_X$ on X from Y, such that keep **inclusion map** $X \xrightarrow{id_X} Y(x \mapsto x)$ be continuous.
As we said, $\mathscr{T}_X$ is the arbitrary union of finite intersection of the pre-image of the open set in Y. We call this initial topology induced by the inclusion map the **subspace topology** on X inherited from Y.
Note that the arbitrary union of finite intersection of the pre-image of the open set in Y is just the pre-image of arbitrary union of finite intersection of the open set in Y, which is just the pre-image of the open set in Y. Thus $\mathscr{T}_X = \{id_X^{-1}(V) | V \subseteq_{open} Y\} = \{V \cap X | V \subseteq_{open} Y\}$.

Exercise 42 (The universal property of subspace topologies)**.** *Suppose Y is a topology space, X is a subspace (i.e. a subset equipped with the subspace topology from Y). Given a topology space Z, for $\forall$ map $Z \xrightarrow{g} Y$, if $g(Z) \subseteq X$, show that $Z \xrightarrow{g} Y$ is conti. $\Leftrightarrow$ $Z \xrightarrow{g|^X} X$ is conti.*

Proof. $\Rightarrow$: any open set in X can be represented by $U \cap X$ where $U \subseteq_{open} Y$, thus $g^{-1}(U \cap X) = g^{-1}(U) \cap g^{-1}(X) = g^{-1}(U) \cap Z \subseteq_{open} Z \Rightarrow Z \xrightarrow{g|^X} X$ is conti. $\Leftarrow$: Trivial. $\square$

Exercise 43. *Let X be a topology space, $Z \subseteq Y \subseteq Z$, where Z, Y are equipped with subspace topology, show that*
1. $Z \subseteq_{open} Y \subseteq_{open} X \Rightarrow Z \subseteq_{open} X$;
2. $Z \subseteq_{close} Y \subseteq_{close} X \Rightarrow Z \subseteq_{close} X$.

Proof. 1. $Z \subseteq_{open} Y \subseteq_{open} X \Rightarrow \exists U \subseteq_{open} X$, s.t. $Z = U \cap Y$, since $Y \subseteq_{open} X \Rightarrow Z = U \cap Y \subseteq_{open} X$.
2. $Y \subseteq_{close} X \Rightarrow \exists U \subseteq_{open} X$, s.t. $Y = X \backslash U; Z \subseteq_{close} Y \Rightarrow \exists V \subseteq_{open} Y$, s.t. $Z = Y \backslash V$ and $W \subseteq_{open} X$, s.t. $V = Y \cap W$, thus

$$\begin{aligned} Z &= Y \backslash V \\ &= (X \backslash U) \backslash (Y \cap W) \\ &= (X \backslash U) \backslash ((X \backslash U) \cap W) \\ &= (X \cap U^c) \cap (X \cap U^c \cap W)^c \\ &= U^c \cap (U \cup W^c) \\ &= U^c \cap W^c \\ &= X \backslash (U \cup W) \\ &\subseteq_{close} X \end{aligned}$$

$\square$

Exercise 44. *Let X be a topology space, $A \subseteq B \subseteq X$, show that*
1. $A \subseteq_{open} X \Rightarrow A \subseteq_{open} B$;
2. $A \subseteq_{close} X \Rightarrow A \subseteq_{close} B$.

Proof. 1. Trivial; 2. $A \subseteq_{close} X \Rightarrow X \backslash A \subseteq_{open} X$, since

$$\begin{aligned} B \backslash A &= B \cap A^c \\ &= B \cap (X \cap A^c) \\ &= B \cap (X \backslash A) \\ &\subseteq_{open} B \end{aligned}$$

and hence $A \subseteq_{close} B$. $\square$

4.1.2 **Product Space**

Let $(Y_1, \mathscr{T}_1)$ and $(Y_2, \mathscr{T}_2)$ be topology spaces, we want to create a natural topology $\mathscr{T}_{Y_1 \times Y_2}$ on $Y_1 \times Y_2$ which makes the projections $Y_1 \times Y_2 \xrightarrow{p_i} Y_i (i = 1, 2)$ be continuous. Suppose $U_i (i = 1, \cdots, k_U) \subseteq_{open} Y_1$ and $V_j (j = 1, \cdots, k_V) \subseteq_{open} Y_2$, then

$$\left(\cap_{i=1}^{k_U} f^{-1}(U_i)\right) \cap \left(\cap_{i=1}^{k_V} f^{-1}(V_i)\right) = f^{-1}\left(\cap_{i=1}^{k_U} U_i\right) \cap f^{-1}\left(\cap_{i=1}^{k_V} V_i\right)$$

where $\cap_{i=1}^{k_U} U_i \subseteq_{open} Y_1$ and $\cap_{i=1}^{k_V} V_i \subseteq_{open} Y_2$. Thus the desired initial topology can be represented as the arbitrary union of the intersection of the pre-image of an open set in Y_1 and the pre-image of an open set in Y_2. (instead of the finite intersection of pre-image of open sets in Y_1 and Y_2, it is subtle) Thus the basis of the expected initial topology is

$$\begin{aligned}
\Pi &= \{p_1^{-1}(W_1) \cap p_2^{-1}(W_2) | W_1 \subseteq_{open} Y_1, W_2 \subseteq_{open} Y_2\} \\
&= \{(W_1 \times Y_2) \cap (Y_1 \times W_2) | W_1 \subseteq_{open} Y_1, W_2 \subseteq_{open} Y_2\} \\
&= \{W_1 \times W_2 | W_1 \subseteq_{open} Y_1, W_2 \subseteq_{open} Y_2\}
\end{aligned}$$

Thus the topology desired is all unions of rectangle:

$$\mathscr{T}_{Y_1 \times Y_2} = \{\cup \mathcal{F} | \mathcal{F} \subseteq \Pi\}.$$

We call such initial topology **product topology** of Y_1 and Y_2, denote as $\mathscr{T}_1 \times \mathscr{T}_2$.
In particular, the open set O in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ can be written by $O = \cup U \times V$ where $U, V \subseteq_{open} \mathbb{R}$.

Remark 34. We call such a $W_1 \times W_2$ a rectangle.

4.1.3 **Cartesian Product**

Let's recall the definition of Cartesian product. Given two sets Y_1, Y_2, there exists a **bijection** between $Y_1 \times Y_2$ and the family of maps $\{\{1, 2\} \xrightarrow{s} Y_1 \cup Y_2 | s(1) \in Y_1, s(2) \in Y_2\} =: \mathcal{M}_{Y_1 \times Y_2}$. First, there is an injection from left to right: for any $(s_1, s_2) \in Y_1 \times Y_2$, define s as $s(1) = s_1, s(2) = s_2$. Thus different points in $Y_1 \times Y_2$ reflect to different maps in $\mathcal{M}_{Y_1 \times Y_2}$.
On the other hand, there exists an injection from right to left as well: for any $s', s \in \mathcal{M}_{Y_1 \times Y_2}$, correspond to $(s(1), s(2)), (s'(1), s'(2)) \in Y_1 \times Y_2$, and $(s(1), s(2)) \neq (s'(1), s'(2))$ if $s \neq s'$.
Furthermore, when we project a point $(y_1, y_2) \in Y_1 \times Y_2$ to $y_1 \in Y_1$ (using projection

$Y_1 \times Y_1 \xrightarrow{p_1} Y_1$), it is equivalent with mapping the corresponding map s to $s(1)$.

$$\begin{tikzcd}
Y_1 \times Y_2 \arrow[rrrrr, bend left, "p_1"] & \ni & (y_1, y_2) \arrow[r, mapsto] \arrow[d, Leftrightarrow] & y_1 \arrow[d, equal] & \in & Y_1 \\
\mathcal{M}_{Y_1 \times Y_2} \arrow[rrrrr, bend right] & \ni & s \arrow[r, mapsto] & s(1) & \in & Y_1
\end{tikzcd}$$

Similarly, we can define infinite dimension Cartesian product as

$$\prod_{\alpha \in A} Y_\alpha := \{A \xrightarrow{s} \cup_{\alpha \in A} Y_\alpha | \forall \alpha \in A, s(\alpha) \in Y_\alpha\} =: \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha},$$

according to the axiom of choice, if $Y_\alpha \neq \emptyset$ for any $\alpha \in A$, then such map s exists, then $\prod_{\alpha \in A} Y_\alpha \neq \emptyset$. For $\alpha \in A$, we often denote the value of s at α by s_α rather than $s(\alpha)$; we call it the α-th **coordinate** of s. And we often denote the function s itself by the symbol

$$(s_\alpha)_{\alpha \in A},$$

which is as close as we can come to a `tuple notation` for an arbitrary index set A. Corresponding, we can define the projection on infinite dimension cartesian product: for any $\beta \in A$,

$$\prod_{\alpha \in A} Y_\alpha \xrightarrow{p_\beta} Y_\beta$$

as a map

$$\mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} \longrightarrow Y_\beta$$

with $s \mapsto s_\beta$.

4.1.4 Infinite Dimension Product Topology

Now we can define the product topology on infinite dimension. As we discussed, the topology is arbitrary union of finite intersection of pre-image of the open set in $Y_\alpha (\alpha \in A)$. Since the intersection is finite, we can still exchange the order of pre-image and intersection, and then represent the open sets from the same $Y_\alpha (\alpha \in A)$ as one open set. Note that the pre-image of $U_\beta \subseteq_{open} Y_\beta$ can be represented by

$$\{s \in \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} | s_\beta \in U_\beta\}.$$

Thus finite intersection of the pre-image of open sets, i.e. the basis of the infinite dimension product topology is

$$\Pi_{\prod_{\alpha \in A} Y_\alpha} = \{s \in \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} | s_{\beta_1} \in U_{\beta_1}, \cdots, s_{\beta_k} \in U_{\beta_k}, k \in \mathbb{N}\}.$$

That the basis of infinite product topology is set of maps that only map **finite** points in domain to the open sets of codomain. Alternatively, we can represent it as

$$\Pi_{\prod_{\alpha\in A} Y_\alpha} = \left\{ \prod_{\alpha\in A} V_\alpha | \forall \alpha \in A, V_\alpha \subseteq_{open} Y_\alpha \wedge \{\alpha \in A | V_\alpha \neq Y_\alpha\} \text{ is finite} \right\}.$$

And the topology is

$$\mathscr{T}_{\prod_{\alpha\in A} Y_\alpha} = \{\cup \mathcal{F} | \mathcal{F} \subseteq \Pi_{\prod_{\alpha\in A} Y_\alpha}\}$$

Exercise 45 (The universal property of product topology)**.** *Let $Z, Y_\alpha(\alpha \in A)$ are topology spaces, $\prod_{\alpha\in A} Y_\alpha$ is equipped with product topology, show that for any group of maps*

$$Z \xrightarrow[conti.]{g_\alpha} Y_\alpha(\alpha \in A)$$

$\exists! Z \xrightarrow[conti.]{g} \prod_{\alpha\in A} Y_\alpha$, s.t. $p_\alpha \circ g = g_\alpha$ *for* $\forall \alpha \in A$. *That is, such commutative diagram holds*

$$\begin{tikzcd} Z \arrow[r, "g", "conti."'] \arrow[dr, bend right, "g_\alpha"', "conti."] & \prod_{\alpha\in A} Y_\alpha \arrow[d, "p_\alpha"] \\ & Y_\alpha \end{tikzcd}$$

Proof. Existence: Select a group of $g_\alpha(\alpha \in A)$ such that for a given $z \in Z$ has

$$g_\alpha(z) = y_\alpha \in Y_\alpha.$$

Define a map $Z \xrightarrow{g} \mathcal{M}_{\prod_{\alpha\in A} Y_\alpha}$ with $z \mapsto s$ where $s_\alpha = y_\alpha(\alpha \in A)$. Thus for any $\beta \in A$, we have

$$p_\beta \circ g(z) = p_\beta(s) = s_\beta = y_\beta = g_\beta(z)$$

Thus $p_\alpha \circ g = g_\alpha$ for any $\alpha \in A$. We now show g is continuous.
Any open set U in $\mathcal{M}_{\prod_{\alpha\in A} Y_\alpha}$ can be written as $U = \cup\mathcal{F} = \cup_{V\in\mathcal{F}} V$, where $\mathcal{F} \subseteq \Pi_{\prod_{\alpha\in A} Y_\alpha}$. Thus

$$g^{-1}(U) = g^{-1}(\cup_{V\in\mathcal{F}} V) = \cup_{V\in\mathcal{F}} g^{-1}(V).$$

Here V is the element in the basis, and can be represented as

$$V = \{s \in \mathcal{M}_{\prod_{\alpha\in A} Y_\alpha} | s_{\alpha_1} \in U_{\alpha_1}, \cdots, s_{\alpha_k} \in U_{\alpha_k}\},$$

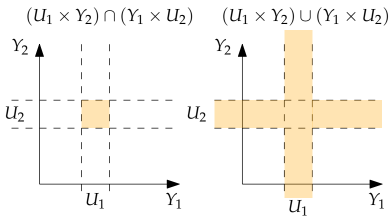
where $U_{\alpha_i} \subseteq_{open} Y_{\alpha_i}(i = 1, \cdot, k)$, thus

$$\begin{aligned} g^{-1}(V) &= \{z \in Z | g_{\alpha_1}(z) \in U_{\alpha_1}, \cdots, g_{\alpha_k}(z) \in U_{\alpha_k}\} \\ &= \cap_{i=1}^k g_{\alpha_i}^{-1}(U_{\alpha_i}) \\ &\subseteq_{open} Z \end{aligned}$$

Thus $g^{-1}(U) = \cup_{V\in\mathcal{F}} g^{-1}(V) \subseteq_{open} Z \Rightarrow g$ is continuous.

Remark 35. There is a trap:

- $(U_1 \times Y_2) \cap (Y_1 \times U_2) = U_1 \times U_2$;
- $(U_1 \times Y_2) \cup (Y_1 \times U_2) \neq Y_1 \times Y_2$;

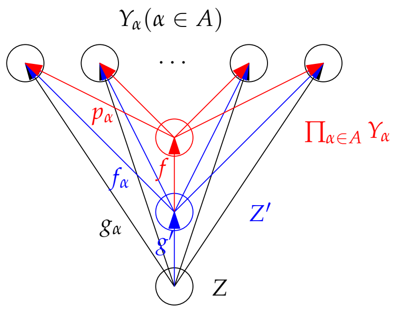


Uniqueness: for any h such that $p_\alpha \circ h = g_\alpha$, given a $z \in Z$, we have $p_\alpha(h(z)) = g_\alpha(z)$ for $\forall \alpha \in A$. Thus

$$\begin{aligned} h(z) &\in \cap_{\alpha \in A} p_\alpha^{-1}(g_\alpha(z)) \\ &= \cap_{\alpha \in A} \{s \in \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} | s_\alpha = g_\alpha(z)\} \\ &= \{s \in \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} | s_\alpha = g_\alpha(z), \alpha \in A\} \end{aligned}$$

Thus $h(z) = s$ where $s_\alpha = g_\alpha(z), \alpha \in A \Rightarrow h = g$. $\square$

The conclusion of the universal property of product topology is : for any group of maps $Z \xrightarrow{g_\alpha} Y_\alpha(\alpha \in A)$, if they can be substitute by another group of map $f_\alpha \circ g'$ where $Z \xrightarrow{g'} Z'$ and $Z' \xrightarrow{f_\alpha} Y_\alpha$, we say Z' is **closer** to $Y_\alpha(\alpha \in A)$ than Z.
Then $\prod_{\alpha \in A} Y_\alpha$ is the **closest** set to $Y_\alpha(\alpha \in A)$.



Exercise 46. *Let $Z, Y_\alpha(\alpha \in A)$ are top. spaces. Show that $Z \xrightarrow{g} \prod_{\alpha \in A} Y_\alpha$ is continuous $\Leftrightarrow p_\alpha \circ g(\alpha \in A)$ are continuous.*

Proof. $\Rightarrow$: Since $p_\alpha \circ g = g_\alpha$, we need to prove g is continuous $\Rightarrow g_\alpha$ is continuous.
For any open set $U_\alpha \subseteq_{open} Y_\alpha$. $p_\alpha^{-1}(U_\alpha) = \{s \in \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} | s_\alpha \in U_\alpha\} \subseteq_{open} \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha} = \prod_{\alpha \in A} Y_\alpha$. And $g^{-1}(\{s \in \mathcal{M}_{\prod_{\alpha \in A} Y_\alpha}) = \{z \in Z | g_\alpha \in U_\alpha\} = g_\alpha^{-1}(U_\alpha) \subseteq_{open} Z$, since g is continuous, thus g_α is continuous. $\Leftarrow$: has been given in Ex2. $\square$

4.2 Final Topology

Given topology spaces $X_\alpha(\alpha \in A)$ and maps $X_\alpha \xrightarrow{f_\alpha} Y(\alpha \in A)$, does there exist a finest topology on Y, such that f_α is continuous for every $\alpha \in A$? Define

$$\mathscr{T}_Y := \{V \subseteq Y | f_\alpha^{-1}(V) \subseteq_{open} X_\alpha, \forall \alpha \in A\}.$$

It is direct to see $\mathscr{T}_Y$ is a topology: Given an $\alpha \in A$, define $\mathscr{T}_\alpha := \{V \subseteq Y | f_\alpha^{-1}(V) \subseteq_{open} X_\alpha\}$, we have
1. $f_\alpha^{-1}(\emptyset) = \emptyset \subseteq_{open} X_\alpha$; $f_\alpha^{-1}(Y) = X_\alpha \subseteq_{open} X_\alpha$, thus $\emptyset, Y \in \mathscr{T}_\alpha$.
2. $\forall V_\beta \in \mathscr{T}_\alpha(\beta \in B)$, $f_\alpha^{-1}(\cup_{\beta \in B} V_\beta) = \cup_{\beta \in B} f^{-1}(V_\beta) \subseteq_{open} X_\alpha$, thus $\cup_{\beta \in B} V_\beta \in \mathscr{T}_\alpha$;
3. $\forall V_1, V_2 \in \mathscr{T}_\alpha$, $f_\alpha^{-1}(V_1 \cap V_2) = f_\alpha^{-1}(V_1) \cap f_\alpha^{-1}(V_2) \subseteq_{open} X_\alpha$, thus $V_1 \cap V_2 \in \mathscr{T}_\alpha$.
Thus $\mathscr{T}_\alpha$ is a topology. On the other hand, $\mathscr{T}_Y = \cap_{\alpha \in A} \mathscr{T}_\alpha$, thus $\mathscr{T}_Y$ is a topology.
Suppose $\mathscr{T}'$ is a topology makes maps $X_\alpha \xrightarrow{f_\alpha} Y(\alpha \in A)$ be continuous. Then $\forall U \in \mathscr{T}'$, $f_\alpha^{-1}(U) \subseteq_{open} X_\alpha$ for all $\alpha \in A$, thus $U \in \mathscr{T}_Y \Rightarrow \mathscr{T}' \subseteq \mathscr{T}_Y$.
Thus $\mathscr{T}_Y$ is the expected finest topology such that f_α is continuous for any $\alpha \in A$.

4.2.1 Equivalence Relation

Definition 32 (Equivalence Relation)**.** Let X be a set. A relation R on X (i.e. $R \subseteq X \times X$) is equivalence relation, if
1. $\forall x \in X \Rightarrow xRx$;
2. $\forall x, x' \in X, xRx' \Rightarrow x'Rx$;
3. $\forall x, x', x'' \in X, xRx', x'Rx'' \Rightarrow xRx''$.

For an equivalence relation R on X, and every $x \in X$, we call

$$R(x) := \{x' \in X | x'Rx\}$$

the **equivalence class** of x w.r.t. R on X. Obviously $R(x) \neq \emptyset$ for $\forall x \in X$, since $x \in R(x)$ for any $x \in X$.

Exercise 47. *For $\forall x_1, x_2 \in X$, either $R(x_1) = R(x_2)$ or $R(x_1) \cap R(x_2) = \emptyset$.*

Proof. If $\exists x \in R(x_1) \cap R(x_2) \neq \emptyset$, then for any $x_3 \in R(x_2)$, we have x_3Rx_2, x_2Rx and $xRx_1 \Rightarrow x_3Rx_1 \Rightarrow x_3 \in R(x_1) \Rightarrow R(x_2) \subseteq R(x_1)$. And $R(x_1) \subseteq R(x_2)$ in the same way, thus $R(x_1) = R(x_2)$. □

In summary, R provides a decomposition of X into disjoint union of nonempty subsets. On the contrary, if we have a decomposition of X into disjoint union of nonempty subsets, we can define an equivalence relation where the element in the same subsets are equivalent. Thus an equivalence relation and a decomposition are the same thing.

4.2.2 Quotient Space

We call $\{R(x)|x \in X\}$ the **quotient set** of X by the relation R, denoted as X/R. And we can define a **natural projection** on X: $X \xrightarrow{\pi} X/R$ where $x \mapsto R(x)$. It is direct to see that π is a surjection.

Exercise 48 (The universal property of $X \xrightarrow{\pi} X/R$). *Given a map $X \xrightarrow{g} Z$ such that $\forall x, x' \in X, xRx' \Rightarrow g(x) = g(x')$, show that $\exists!$ map $X/R \xrightarrow{\overline{g}} Z$ s.t. $\overline{g} \circ \pi = g$.*

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "\pi"'] & & Z \\ & X/R \arrow[ur, "\overline{g}"'] & \end{tikzcd}$$

Proof. Given a $R(x) \in X/R$, define $\overline{g}(R(x)) = g(x)$. Since for any $x' \in R(x)$, $g(x') = g(x)$, the map $\overline{g}$: $X/R \ni R(x) = S \mapsto g(x) \in Z$ is well defined, i.e. independent of the choice of x s.t. $S = R(x)$.
For $\forall x \in X$, $\overline{g} \circ \pi(x) = \overline{g}(R(x)) = g(x)$, thus $\overline{g} \circ \pi = g$. If $\exists h$, s.t. $h \circ \pi = g = \overline{g} \circ \pi$, then $h = \overline{g}$ since π is a surjection. ☐

Remark 36. Recall that
1. g is an injection, $g \circ f = g \circ f' \Rightarrow f = f'$;
2. f is a surjection, $g \circ f = g' \circ f \Rightarrow g = g'$.

Now we consider a topology space X on which an equivalence relation R is specified. We aim at defining a topology space obtained by gluing mutually R - equivalent points in X to a point.

Definition 33 (Quotient Topology). Let $X \xrightarrow{\pi} X/R$ where $x \mapsto R(x)$ be the natural projection. The finial topology on X/R induced by $\{\pi\}$ (i.e. the finest topology on X/R s.t. π is continuous) is called the quotient topology on X/R induced by R, denoted by $\mathscr{T}_{(X,R)}$.

More explicitly,
$$\mathscr{T}_{(X,R)} = \{S \subseteq X/R | \pi^{-1}(S) \subseteq_{open} X\},$$
that is, $S \subseteq_{open} X/R$ w.r.t $\mathscr{T}_{(X,R)} \Leftrightarrow \pi^{-1}(S) \subseteq_{open} X$.

Definition 34 (Saturated). Let X is a set, R is an equivalence relation on X. $A(\subseteq X)$ is a R - saturated if $\forall x \in X, a \in A, xRa \Rightarrow x \in A$.

Exercise 49. *A is R - saturated $\Leftrightarrow$ A is a union of some R - equivalence class $\Leftrightarrow$ $\exists S \subseteq X/R$,* s.t. *$A = \pi^{-1}(S)$.*

Proof. 1. $\Rightarrow$: If A is R- saturated, then for $\forall a \in A$, $R(a) \subseteq A$ by definition. Thus $\cup_{a\in A} R(a) \subseteq_{open} A$. On the other hand, for any $a' \in A, a' \in R(a') \subseteq \cup_{a\in A} R(a)$, thus $A = \cup_{a\in A} R(a)$.
$\Leftarrow$: If $R_\beta(\beta \in B)$ are some R - equivalence class in X/R, then for any $r \in \cup_{\beta\in B} R_\beta$, $\exists \gamma \in B$, s.t. $r \in R_\gamma$, thus $R(r) = R_\gamma$, thus $R(r) \subseteq \cup_{\beta\in B} R_\beta$.
For any $x \in X$, if xRr, then $x \in R(r) \subseteq \cup_{\beta\in B} R_\beta \Rightarrow x \in \cup_{\beta\in B} R_\beta \Rightarrow \cup_{\beta\in B} R_\beta$ is R - saturated.
2. $\Rightarrow$: Note that for $R(a) \in X/R, \pi^{-1}(R(a)) = R(a) \subseteq X$. Thus

$$\begin{aligned} A &= \cup_{\alpha\in A} R(a) \\ &= \cup_{a\in A} \pi^{-1}(R(a)) \\ &= \pi^{-1}(\cup_{a\in A} R(a)) \end{aligned}$$

where $\cup_{a\in A} R(a) \subseteq X/R$ is the expected S.
$\Leftarrow$: we will show that for $\forall S \subseteq X/R, \pi^{-1}(S)$ is R-saturated on X. For any $s \in \pi^{-1}(S)$, $\pi(s) = R(s) \subseteq S$. For any $x \in X$, if xRs, then $R(x) = R(s) \subseteq S$, thus $x \in \pi^{-1}(S)$, thus $\pi^{-1}(S)$ is R-saturated.

$\square$

Definition 35 (Quotient Map)**.** Let $X \xrightarrow{p} Y$ be a map between topology spaces. We say p is a quotient map if:
1. p is a surjection;
2. for any $V \subseteq Y$, we have $V \subseteq_{open} Y \Leftrightarrow p^{-1}(V) \subseteq_{open} X$.

Remark 37*.* The second statement is equivalent with

$$V \subseteq_{close} Y \Leftrightarrow p^{-1}(V) \subseteq_{close} X$$

since $p^{-1}(V) \subseteq_{close} X \Leftrightarrow \left(p^{-1}(V)\right)^c = p^{-1}(V^c) \subseteq_{open} X \Leftrightarrow V^c \subseteq_{open} X \Leftrightarrow V \subseteq_{close} X$.
Thus the topology on Y is the final topology induced by $\{p\}$, since the second statement.
For a topology space X with an equivalence relation R, a topology $\mathscr{T}_{X/R}$ on X/R makes the natural projection $X \xrightarrow{\pi} X/R$ a quotient map iff $\mathscr{T}_{X/R} = \mathscr{T}_{(X,R)}$. And we call $(X/R, \mathscr{T}_{X,R})$ the **quotient space** on X w.r.t. R.

Exercise 50 (The universal property of quotient topology/map)**.** *Let $X \xrightarrow{p} Y$ be a quotient map. Show that for $\forall X \xrightarrow[conti.]{g} Z$ s.t. $\forall x, x' \in X, p(x) = p(x') \Rightarrow g(x) = g(x')$, $\exists ! Y \xrightarrow[conti.]{h} Z$ s.t. $h \circ p = g$.*

$$\begin{tikzcd} X \arrow[rr, "g", "conti."'] \arrow[dr, "p"'] & & Z \\ & Y \arrow[ur, "h", "conti."'] & \end{tikzcd}$$

Proof. Existence: for any $y \in Y$, $p^{-1}(y)\exists$ for p is a surjection. Define $h(y) = g(p^{-1}(y))$. Since $g(p^{-1}(y))$ is a constant, h is well defined. And $h \circ p(x) = h(p(x)) = g(p^{-1}(p(x)))$. Since $x \in p^{-1}(p(x))$ and $g(p^{-1}(p(x)))$ is a constant, thus $h \circ p(x) = g(x)$.
Uniqueness: since p is surjection, h is unique.
Continuousness: for any $U \subseteq_{open} Z$, $h^{-1}(U) \subseteq_{open} Y \Leftrightarrow p^{-1}(h^{-1}(U)) \subseteq_{open} X$. Since $p^{-1}(h^{-1}(U)) = g^{-1}(U) \subseteq_{open} X$ since g is conti. and $g = h \circ p$. Thus h is continuous. $\square$

Remark 38. $p(x) = p(x') \Rightarrow g(x) = g(x')$ means that given a $y \in Y$, g is a constant on $p^{-1}(y)$.

Any maps between sets $X \xrightarrow{f} Y$ induces an equivalence relation R_f on X: for $x, x' \in X, xR_f x' \Leftrightarrow f(x) = f(x')$. And the equivalence classes is the $f^{-1}(\{y\})$, for $y \in f(X)$.

Exercise 51. *Given a continuous surjection $X \xrightarrow{f} Y$, show that f is a quotient map $\Leftrightarrow$ the image of every f - saturated open/close subset of X is open/close in Y.*

Proof. $\Rightarrow$: If A is a f - saturated , then $A = f^{-1}(f(A))$: if $\exists b \in f^{-1}(f(A))\backslash A$, then $f(b) \in f(A) \Rightarrow \exists a \in A$, s.t. $f(b) = f(a) \Rightarrow aR_f b \Rightarrow b \in A$, which leads to a contradiction. Thus $A = f^{-1}(f(A))$.
Thus if A is an open f - saturated set on X then $f^{-1}(f(A)) \subseteq_{open} X \Leftrightarrow f(A) \subseteq_{open} Y$ since f is a quotient map.
$\Leftarrow$: all we need to show is for any $V \subseteq Y$, $f^{-1}(V) \subseteq_{open} X \Rightarrow V \subseteq_{open} Y$. For any $V \subseteq Y, f^{-1}(V)$ is f - saturate: for any $r \in f^{-1}(V) \Rightarrow f(r) \in V$. If $\exists x \in X$ s.t. $xR_f r \Rightarrow f(x) = f(r) \in V \Rightarrow x \in f^{-1}(V)$.
If $f^{-1}(V) \subseteq_{open} X$, then $f(f^{-1}(V)) \subseteq_{open} X$. Since f is a surjection, $V = f(f^{-1}(V)) \subseteq_{open} X \Rightarrow f$ is quotient map. $\square$

Remark 39. If A is a f - saturated , then $A = f^{-1}(f(A))$.

Chapter 5

Compact Space and HLC Space

5.1 Compactness

Definition 36 (Compact Subset)**.** Let $(X, \mathscr{T})$ be a topology space and $K \subseteq X$, we call K is compact subset of X if $\forall \mathcal{U} \subseteq \mathscr{T}, K \subseteq \cup \mathcal{U} \Rightarrow \exists$ finite $\mathcal{S} \subseteq U,$ s.t. $K \subseteq \cup \mathcal{S}$.

We say $(X, \mathscr{T})$ is a compact space if X is a compact subset of itself.

Exercise 52. *Let $(X, \mathscr{T})$ be a topology space and $K \subseteq X$, show that K is a compact subset of $X \Leftrightarrow (K, \mathscr{T}_K)$ is a compact space, where $\mathscr{T}_K$ is subspace topology.*

Proof. $\Rightarrow$: For any $V_\alpha \subseteq_{open} K, \exists U_\alpha \subseteq_{open} X,$ s.t. $V_\alpha = U_\alpha \cap K$. For any

$$\begin{aligned} K &= \cup_{\alpha \in A} V_\alpha \\ &= \cup_{\alpha \in A} (U_\alpha \cap K) \\ &= K \cap \cup_{\alpha \in A} U_\alpha \\ &= K \cap (U_{\alpha_1} \cup \cdots U_{\alpha_k}) \\ &= V_{\alpha_1} \cup \cdots V_{\alpha_k} \end{aligned}$$

Thus K is compact. $\Leftarrow$: for any $K \subseteq \cup_{\alpha \in A} U_\alpha$, we have $\cup_{\alpha \in A}(U_\alpha \cap K) \subseteq K$ and

$$\begin{aligned} K &= K \cap K \\ &\subseteq K \cap \cup_{\alpha \in A} U_\alpha \\ &= \cup_{\alpha \in A} (K \cap U_\alpha) \end{aligned}$$

Thus $K = \cup_{\alpha \in A}(K \cap U_\alpha) = \cup_{\alpha \in A} V_\alpha$, where $V_\alpha \subseteq_{open} K$. And $\exists V_{\alpha_1}, \cdots, V_{\alpha_k} \subseteq_{open} K,$ s.t.

$$\begin{aligned} K &= V_{\alpha_1} \cup \cdots V_{\alpha_k} \\ &= K \cap (U_{\alpha_1} \cup \cdots U_{\alpha_k}) \\ &\subseteq U_{\alpha_1} \cup \cdots U_{\alpha_k} \end{aligned}$$

Thus K is a compact subset in X. $\square$

Suppose we admit that the Dedekind gapless property of real number: if $\emptyset \neq S \subseteq \mathbb{R}$ has upper bound (lower bound), then $\sup S (\inf S) \exists$.

Theorem 9. $[0,1]$ *(as a subspace of* $\mathbb{R}$*) is compact.*

Proof. Suppose that $V_\alpha \subseteq_{open} \mathbb{R} (\alpha \in A)$ cover $[0,1]$. Consider

$$S := \{s \in [0,1] | [0,s] \text{ can be covered by finitely many } V_\alpha\}$$

Thus $0 \in S$, $S \neq \emptyset$. $S \subseteq [0,1]$, thus S has an upper bound $\Rightarrow \sup S \exists$. Let $s_0 := \sup S$. Since 1 is an upper bound of S, $s_0 \leq 1$. For $\forall t \leq s_0$, t is not an upper bound of S, $\exists s' \in S$, s.t. $t < s'$, thus $[0,t]$ could be covered by finitely many V_α.
Since $s_0 \leq 1$, $\exists \alpha_0$, s.t. $s_0 \subseteq V_{\alpha_0}$, $\exists r > 0$, s.t. $B_r(s_0) \subseteq V_{\alpha_0}$. Thus $[0, s_0 - r]$ can be covered by finitely many of V_α, and $(s_0 - r, s_0 + r)$ can be covered by V_{α_0}. Thus $[0, s_0 + r)$ can be covered by finitely many V_α. Thus $s_0 = 1$ and $s_0 \in S \Rightarrow S = [0,1]$. □

Thus $[0,1] \times [0,1]$, as a subspace of $\mathbb{R}^2$, which coincides with the product space of $[0,1]$ and $[0,1]$, is compact.
More generally, we can reprove the **Heine–Borel theorem**: for $K \subseteq \mathbb{R}^n$, then $K \subseteq_{cpt.} \mathbb{R}^n \Leftrightarrow K \subseteq_{close} \mathbb{R}^n$ and K is bdd.

Proof. $\Rightarrow$: $\mathbb{R}^n$ is metric space $\Rightarrow \mathbb{R}^n$ is Hausdorff $\Rightarrow K \subseteq_{close} \mathbb{R}^n$. Since $K \subseteq \cup_{n \in \mathbb{N}} B_n(0) \Rightarrow \exists r_1, \cdots, r_k$, s.t. $K \subseteq \cup_{i=1}^k B_{n_{r_i}}(0) \Rightarrow K$ is bdd.
$\Leftarrow$: K is bdd. $\Rightarrow, \exists r > 0$, s.t. $K \subseteq B_r(0), \Rightarrow \exists [a_1, b_1], \cdots, [a_n, b_n] \in \mathbb{R}$, s.t. $K \subseteq B_r(0) \subseteq \times_{i=1}^n [a_i, b_i]$. Since $K \subseteq_{close} \mathbb{R}^n \Rightarrow K \subseteq_{close} \times_{i=1}^n [a_i, b_i] \subseteq_{cpt.} \mathbb{R}^n \Rightarrow K \subseteq_{cpt.} \times_{i=1}^n [a_i, b_i] \Rightarrow$ K is cpt. □

Remark 40. Actually, In any metric space X, $K \subseteq_{cpt.} X \Rightarrow K \subseteq_{close} X$ and be bdd.

Definition 37 (Finite Intersection Property, FIP)**.** Let S be a set and $\mathcal{F} \subseteq \mathcal{P}(S)$ is a family of subsets of S. We say that $\mathcal{F}$ has the finite intersection property (FIP) if $\forall \mathcal{F}_0 \subseteq \mathcal{F}, \mathcal{F}_0$ is finite $\Rightarrow \cap \mathcal{F}_0 \neq \emptyset$.

Exercise 53. *For a set X and a family of subsets* $\mathcal{U} \subseteq \mathcal{P}(X)$*, let* $\mathcal{F} := \{X \backslash U | U \in \mathcal{U}\}$*, then* $X = \cup \mathcal{U} \Leftrightarrow \cap \mathcal{F} = \emptyset$*.*

Proof. $\Rightarrow$: if $\cap \mathcal{F} \neq \emptyset$, then $\exists x \in \cap \mathcal{F}$, that is for $\forall U \in \mathcal{U}, x \in X \backslash U \Rightarrow x \notin U$ but $x \in X$. Thus $\cup \mathcal{U} \neq X$ which leads to a contradiction.
$\Leftarrow$: $\cap \mathcal{F} = \emptyset$, thus for any $x \in X$, $\exists F \in \mathcal{F}$, s.t. $x \notin F$, that is $\exists U \in \mathcal{U}$, s.t. $x \notin X \backslash U \Rightarrow x \in U$. Thus $X \subseteq \cup \mathcal{U} \subseteq X \Rightarrow X = \cup \mathcal{U}$. □

Exercise 54. *Let* $(X, \mathscr{T})$ *be a topology space, show that X is compact space* $\Leftrightarrow \forall$ *family* $\mathcal{F} (\subseteq \mathcal{P}(X))$ *of closed subsets of X,* $\mathcal{F}$ *has FIP* $\Rightarrow \cap \mathcal{F} \neq \emptyset$*.*

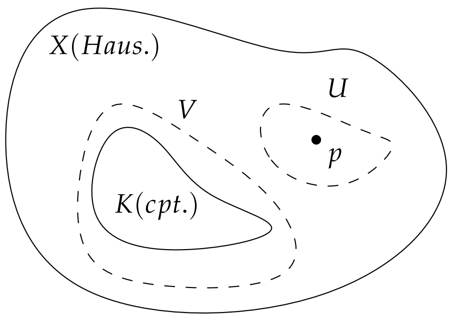
Proof. $\Rightarrow$: For any family $\mathcal{F}$ of closed subset of X, define $\mathcal{U} := \{X \backslash F | F \in \mathcal{F}\}$, thus $\mathcal{U}$ is a family of open subsets of X. If $\cup\mathcal{U} = X$, since X is compact, $\exists$ a finite $\mathcal{U}_0 \subseteq \mathcal{U}$, s.t. $X = \cup\mathcal{U}_0$.
Define $\mathcal{F}_0 := \{X \backslash U | U \in \mathcal{U}_0\}$, thus $\mathcal{F}_0$ is finite and $\cap\mathcal{F}_0 = \varnothing$, which leads to the FIP of X. Thus $\cup\mathcal{U} \neq X \Leftrightarrow \cap\mathcal{F} \neq \varnothing$.
$\Leftarrow$: If X is not a compact set, we will show the statement in the right side is wrong. If X is not a compact set then $\exists$ s family $\mathcal{U}$ of open subsets of X such that any finite $\mathcal{U}_0 \subseteq \mathcal{U}$ has $X \neq \cup\mathcal{U}_0$.
Define $\mathcal{F} := \{X \backslash U | U \in \mathcal{U}\}; \mathcal{F}_0 := \{X \backslash U | U \in \mathcal{U}_0\}$ for any finite $\mathcal{U}_0 \subseteq \mathcal{U}$. Thus $\mathcal{F}$ has FIP, but $\cap\mathcal{F} = \varnothing$. □

Proposition 3. *Suppose X is Hausdorff, $K(\subseteq X)$ is compact, $p \in X \backslash K \Rightarrow \exists U, V \subseteq_{open} X$, s.t. $K \subseteq V, p \in U$, and $U \cap V = \varnothing$.*

Proof. X is Hausdorff $\Rightarrow \forall q \in K, \exists U_q, V_q \subseteq_{open} X$ s.t. $q \in V_q, p \in U_q, U_q \cap V_q = \varnothing$. Thus $K \subseteq \cup_{q\in K} V_q = \cup_{i=1}^k V_{q_i}$, Let $V = \cup_{i=1}^k V_{q_i}, U = \cap_{i=1}^k U_{q_i}$, then

$$\begin{aligned} U \cap V &= \left(\cap_{j=1}^k U_{q_j}\right) \cap \left(\cup_{i=1}^k V_{q_i}\right) \\ &= \cup_{i=1}^k \left[\cap_{j=1}^k \left(U_{q_j} \cap V_{q_i}\right)\right] \\ &= \cup_{i=1}^k \varnothing = \varnothing. \end{aligned}$$

□



Proposition 4. *Let $(X, \mathscr{T})$ be a topology space, $K \subseteq X$, then*
1. *X is Hausdorff space, K is compact $\Rightarrow K \subseteq_{close} X$;*
2. *X is compact space, $K \subseteq_{close} X \Rightarrow K$ is compact.*

Proof. 1 For $\forall p \in X \backslash K, \exists W_p \subseteq_{open} X$, s.t. $x \in W_p$ and $W_p \cap K = \varnothing$, by Proposition 4, that is $W_p \subseteq X \backslash K$. And because

$$X \backslash K = \cup_{p\in X\backslash K}\{p\} \subseteq \cup_{p \in X\backslash K} W_p \subseteq X \backslash K$$

we have that $X \backslash K = \cup_{p\in X\backslash K} W_p \subseteq_{open} X$, and then $K \subseteq_{close} X$.

2 Suppose $\exists U_\alpha \subseteq_{open} X(\alpha \in A)$, s.t. $K \subseteq \cup_{\alpha\in A} U_\alpha$, thus $X = K \cup X\backslash K = (X\backslash K) \cup \cup_{\alpha\in A} U_\alpha$. Since X is compact thus $\exists$ finite $A_0 \subseteq A$, s.t. $K \subseteq X = (X\backslash K) \cup \cup_{\alpha\in A_0} U_\alpha \Rightarrow$ K is compact.

□

Proposition 5 (Continuous Maps Preserve Compactness)**.** *Suppose X, Y are top. sp. $X \xrightarrow{f} Y$ is continuous. $K \subseteq_{cpt.} X \Rightarrow f(K) \subseteq_{cpt.} Y$.*

Proof. Suppose $\exists U_\alpha \subseteq_{open} Y(\alpha \in A)$, s.t. $f(K) \subseteq \cup_{\alpha\in A} U_\alpha \Rightarrow K \subseteq f^{-1}(\cup_{\alpha\in A} U_\alpha) = \cup_{\alpha\in A} f^{-1}(U_\alpha)$. Since K is compact, $\exists$ finite $A_0 \subseteq A$, s.t. $K \subseteq \cup_{\alpha\in A_0} U_\alpha \Rightarrow f(K) \subseteq \cup_{\alpha\in A_0} U_\alpha \Rightarrow f(K)$ is compact. □

Proposition 6. *Let $X \xrightarrow{f} Y$ be a continuous map with X is compact and Y is Hausdorff, then*
1. *f is a close map (i.e. $\forall C \subseteq_{close} X, f(C) \subseteq_{close} Y$);*
2. *f is a surjection $\Rightarrow f$ is a quotient map;*
3. *f is a bijection $\Rightarrow f$ is a homeomorphism (同胚) (i.e. a bijection which and whose inverse are both continuous).*

Proof. 1. Any close set C in X is compact, thus $f(C)$ is compact, since Y is Hausdorff, $f(C)$ is close;
2. For any $V \subseteq Y$, if $f^{-1}(V)$ is closed $\Rightarrow f(f^{-1}(V))$ is closed, and $V = f(f^{-1}(V))$ is closed since f is surjection.
   On the other hand, if V is closed, since f is continuous, $f^{-1}(V)$ is closed. Thus f is quotient map.
3. All we need to prove is the inverse of f, denoted by $Y \xrightarrow{\overline{f}} X$ is continuous.
   Note that for any $y \in f(U)$, $\exists x \in U$, s.t. $y = f(x)$ and $x = \overline{f}(y)$, thus $y \in \overline{f}^{-1}(x) \subseteq \overline{f}^{-1}(U)$, thus $f(U) \subseteq \overline{f}^{-1}(U)$. On the other hand, for any $y \in \overline{f}^{-1}(U)$, $\overline{f}(y) \in U \Rightarrow \exists x \in U$, s.t. $x = \overline{f}(y)$ and $y = f(x) \in f(U)$. Thus $\overline{f}^{-1}(U) \subseteq f(U)$. Thus we have for any $U \in X$,
$$f(U) = \overline{f}^{-1}(U),$$
   For any $V \subseteq_{close} X$, $\overline{f}^{-1}(V) = f(V) \subseteq_{close} Y$, since f is a close map, thus $\overline{f}$ is continuous and f is a homeomorphism.

□

Remark 41. Given a map $X \xrightarrow{f} Y$, for any $A \subseteq X, B \subseteq Y$:
1. f is injection $\Rightarrow f^{-1}(f(A)) = A$;
2. f is surjection $\Rightarrow f(f^{-1}(B)) = B$;

Exercise 55. *Given a conti. map $K \xrightarrow{f} \mathbb{R}$, K is cpt. $\Rightarrow f$ has a max. and min.*

Proof. K is cpt., f is conti. $\Rightarrow f(K) \subseteq_{cpt.} \mathbb{R} \Rightarrow f(K) \subseteq_{close} \mathbb{R}$ and be bdd. Thus $f(K)$ has an upper bound and lower bound, thus $\max f(K) = \sup f(K) \in f(K)$ and $\min f(K) = \inf f(K) \in f(K)$. $\square$

Remark 42. Two facts:
1. $K \subseteq_{cpt.} \mathbb{R} \Leftrightarrow K \subseteq_{close} \mathbb{R}$ and be bounded (Heine-Borel theorem, Theorem 5);
2. $K \subseteq_{close} \mathbb{R}$ and be bounded $\Rightarrow \sup K \in K$ and $\inf K \in K$ (Exercise 1);

Exercise 56. *Let R be an equiv. rel. on $[0,1] \times [0,1]$ whose equiv. classes are exactly*

$$\begin{aligned}&\{(x,y)\}, \quad \text{if } (x,y) \in (0,1) \times [0,1]\\&\{(0,y),(1,1-y)\}, \quad \text{if } y \in [0,1]\end{aligned}$$

Define

$$\begin{aligned}Y := \{&(2+t\cos(\theta/2))\cos(\theta),\\&(2+t\cos(\theta/2))\sin(\theta),\\&t\sin(\theta/2)\\&|(\theta,t) \in [0,2\pi] \times [-0.5,0.5]\}\end{aligned}$$

as a subspace of $\mathbb{R}^3$. Show that there exists a homeomorphism from $X := [0,1] \times [0,1]/R$ to Y.

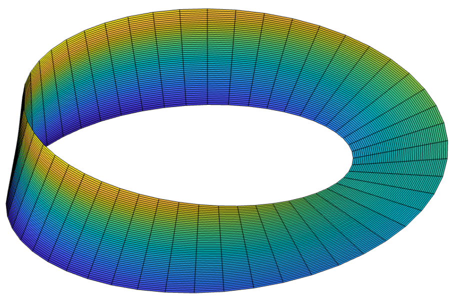


Figure 5.1: Y: a subspace of $\mathbb{R}^3$

Proof. 1. Y, equipped with subspace topology, is a Hausdorff space:
   For any $y_1, y_2 \in Y, \exists U_1, U_2 \subseteq_{open} \mathbb{R}^3$, s.t. $y_1 \in U_1, y_2 \in U_2$ and $U_1 \cap U_2$. Thus $y_1 \in Y \cap U_1 \subseteq_{open} Y$ and $y_2 \in Y \cap U_2 \subseteq_{open} Y$ and $(Y \cap U_1) \cap (Y \cap U_2) = Y \cap (U_1 \cap U_2) = \emptyset \Rightarrow Y$ is Hausdorff space.
2. X, equipped with quotient topology, is a compact space:
   Since X is equipped with the quotient topology, thus the natural projection $[0,1] \times [0,1] \xrightarrow{\pi} [0,1] \times [0,1]/R$ is continuous. Since $[0,1] \times [0,1]$ is a compact subset of $\mathbb{R}^2 \Leftrightarrow [0,1] \times [0,1]$ is a compact set, thus $X = \pi([0,1] \times [0,1])$ is a compact set.

3. $\exists$ a bijection $X \xrightarrow{m} Y$:
   For any $(x,y) \in (0,1) \times [0,1]$, define a map $h : \{(x,y)\} \mapsto (\theta, t)$ where $\theta = 2\pi x, t = y - 0.5$;
   For any $x = 0$ (or 1), define $h : \{(0,y),(1,1-y)\} \mapsto (2\pi, t)$ (or $(0,-t)$) where $t = y - 0.5$; It is direct to see $X \xrightarrow{h} \{(\theta,t)|\theta \in [0,2\pi], t \in [-0.5, 0.5]\}$ is a bijection.
   Finally, define $\{(\theta,t)|\theta \in [0,2\pi], t \in [-0.5,0.5]\} \xrightarrow{g} Y$ which is a bijection as well, Thus $m = g \circ h$ is a bijection.

Collectively, $X \xrightarrow{m} Y$ is a bijection from compact space to Hausdorff space, thus m is a homeomorphism. $\square$

Definition 38 (Proper Map)**.** A map $X \xrightarrow{f} Y$ between topology spaces is called a proper map if $f^{-1}(K) \subseteq_{cpt.} X$ for $\forall K \subseteq_{cpt.} Y$.

Proposition 7. *X, Y are compact spaces $\Rightarrow$ $X \times Y$ equipped with the product topology is compact.*

Thus if Y is compact, X is topology space, then the projection $X \times Y \xrightarrow{\pi_X} X$ is a proper map.

Exercise 57. *Let $X \xrightarrow{f} Y$ is a map between topology spaces, $\mathcal{B}$ is a basis of the topology of X, show that f is an open map $\Leftrightarrow \forall B \in \mathcal{B}, f(B) \subseteq_{open} Y$.*

Proof. $\Rightarrow$: $\forall B \in \mathcal{B}, B \subseteq_{open} X \Rightarrow f(B) \subseteq_{open} Y$. $\Leftarrow$: $\forall U \subseteq_{open} X$ can be represented as $U = \cup_{F \in \mathcal{F}} F$ where $\mathcal{F} \subseteq \mathcal{B}$. Thus $f(U) = f(\cup_{F \in \mathcal{F}} F) = \cup_{F \in \mathcal{F}} f(F) \subseteq_{open} Y$. $\square$

Thus if X, Y are topology, then map $X \times Y \xrightarrow{\pi} X$ is an open map.

5.2 HLC Space

Definition 39 (Locally Compact)**.** X is a locally compact space if $\forall x \in X$ has a compact nbd. (i.e. $\forall x \in X, \exists K \subseteq_{cpt.} X$, s.t. $x \in K^o$, or equivalently, $\forall x \in X, \exists U \subseteq_{open} X, x \in U \subseteq \overline{U} \subseteq_{cpt.} X$)

Exercise 58. *X is locally compact Hausdorff (LCH) space, $C \subseteq_{close} X$, show that $\forall c \in C, \exists T_c \subseteq_{cpt.} C$,* s.t. *$c \in T_c$.*

Proof. For $\forall c \in C, \exists S_c \subseteq_{cpt.} X$, s.t. $c \in S_c$ and $c \in S_c \cap C$. Since $S_c \subseteq_{cpt.} X \Rightarrow S_c \subseteq_{close}$

$X \Rightarrow S_c \cap C \subseteq_{close} X$

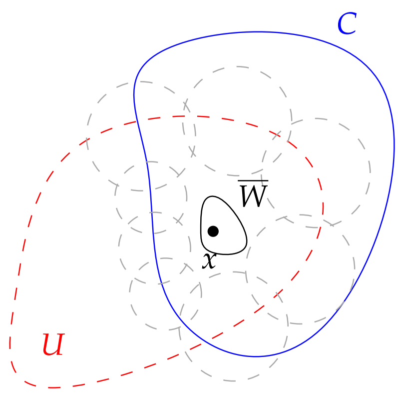
$$\begin{aligned} &X \backslash (S_c \cap C) \subseteq_{open} X \\ &\Rightarrow S_c \cap (X \backslash (S_c \cap C)) \subseteq_{open} S_c \\ &\Rightarrow S_c \backslash [S_c \cap (X \backslash (S_c \cap C))] \subseteq_{close} S_c \\ &\Rightarrow (S_c \backslash S_c) \cup [S_c \backslash X \backslash (S_c \cap C)] \subseteq_{close} S_c \\ &\Rightarrow S_c \backslash X \backslash (S_c \cap C) \\ &= S_c \cap C \subseteq_{close} S_c. \end{aligned}$$

Since $S_c \cap C \subseteq_{close} S_c, S_c$ is cpt. $\Rightarrow S_c \cap C \subseteq_{cpt.} S_c \Rightarrow S_c \cap C$ is cpt. $\Rightarrow S_c \cap C \subseteq_{cpt.} C$. □

Remark 43. $A \subseteq_{close} X$, $A \subseteq B \subseteq X$, then $A \subseteq_{close} B$ (Exercise 44).

Exercise 59. *If X is a locally compact Hausdorff (LCH) space and $x \in X$ has an open nbd. U, show that, there is a compact nbd. of x which is a subset of U. (That is $x \in U \subseteq_{open} X$, then $\exists W \subseteq_{open} X$,* s.t. *$x \in W \subseteq \overline{W} \subseteq U$ where $\overline{W} \subseteq_{cpt.} X$).*

Proof. Given a $x \in X$ and an open nbd. U of x. Since X is locally compact, $\exists C \subseteq_{cpt.} X$, s.t. $x \in C$. Since X is Hausdorff $\Rightarrow C$ is closed $\Rightarrow x \in U \cap C^o \subseteq_{open} X$.
Denote $\partial[U \cap C^o]$ as ∂, since $\partial = X \backslash ((U \cap C^o)^o \cup (U \cap C^o)^e)$, ∂ is closed. Since $\partial \subseteq \partial[U \cap C^o] \cup (U \cap C^o)^o = \overline{U \cap C^o} \subseteq \overline{C} = C$, ∂ is a closed subset of compact set C, thus ∂ is compact.
Since $x \in U \cap C^o$, thus $x \notin \partial$. Since X is Hausdorff, for any $s \in \partial, \exists V_s, W_s \subseteq_{open} X$, s.t. $s \in V_s$ and $x \in W_s$ and $V_s \cap W_s = \emptyset$. Thus $\partial \subseteq \cup_{s \in \partial} V_s \Rightarrow \exists$ finite $\partial_0 \subseteq \partial$, s.t. $\partial \subseteq \cup_{s \in \partial_0} V_s \subseteq_{open} X$ and $x \in \cap_{s \in \partial_0} W_s \subseteq_{open} X$.
Denote $\cap_{s \in \partial_0} W_s =: W$ and $\cup_{s \in \partial_0} V_s =: V$, thus $W \cap V = \emptyset \Rightarrow W \subseteq X \backslash V \Rightarrow \overline{W} \subseteq \overline{X \backslash V} = X \backslash V \Rightarrow \overline{W} \cap V = \emptyset \Rightarrow \overline{W} \cap \partial = \emptyset$. Since $\overline{W} \subseteq \overline{U \cap C^o} = (U \cap C^o) \cup \partial \Rightarrow \overline{W} \subseteq U \cap C^o \subseteq U$ and $\overline{W} \subseteq C$.
Finally, since C is compact, $\overline{W}$ is closed $\Rightarrow \overline{W}$ is compact. Thus $x \in W \subseteq \overline{W} \subseteq U$ and $\overline{W} \subseteq_{cpt.} X$. □



Exercise 60. *More generally, we can replace the point x with a compact set, i.e. X is HLC space, $\forall K \subseteq_{cpt.} X$ if $\exists U \subseteq_{open} X$,* s.t. *$K \subseteq U$ show that $\exists W \subseteq_{open} X$,* s.t. *$K \subseteq W \subseteq \overline{W} \subseteq U$ where $\overline{W} \subseteq_{cpt.} X$.*

Proof. For any $k \in K, k \in U$, thus $\exists W^{(k)} \subseteq_{open} X$, s.t. $k \in W^{(k)} \subseteq \overline{W^{(k)}} \subseteq U$ where $\overline{W^{(k)}} \subseteq_{cpt.} X$. Thus $K \subseteq \cup_{k \in K} W^{(k)}$ and since K is compact, there exists a finite $K_0 \subseteq K$, s.t.

$$K \subseteq \cup_{k \in K_0} W^{(k)} \subseteq \cup_{k \in K_0} \overline{W^{(k)}} \subseteq U.$$

Since $\overline{W^{(k)}}$ is compact for $\forall k \in K \Rightarrow \cup_{k \in K_0} \overline{W^{(k)}}$ is compact. And since K_0 is finite, $\cup_{k \in K_0} \overline{W^{(k)}} = \overline{\cup_{k \in K_0} W^{(k)}}$. Thus $W := \cup_{k \in K_0} W^{(k)}$ and

$$K \subseteq W \subseteq \overline{W} \subseteq U$$

where $W \subseteq_{open} X$ and $\overline{W} \subseteq_{cpt.} X$. $\square$

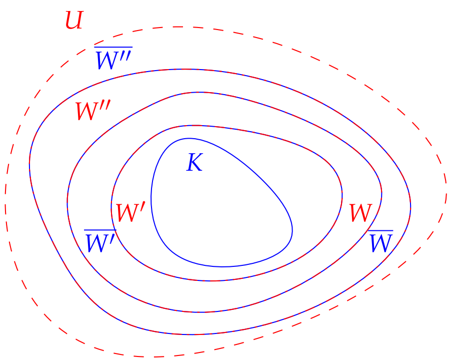
Note that there is an iteration process, that is if $K \subseteq_{cpt.} X, and K \subseteq U \subseteq_{open} X$, and then $\exists W \subseteq_{open} X$ and $\overline{W} \subseteq_{cpt.} X$, s.t. $K \subseteq W \subseteq \overline{W} \subseteq U$. Then $\exists W', W'' \subseteq_{open} X$ and $\overline{W'}, \overline{W''} \subseteq_{cpt.} X$, s.t.

$$K \subseteq W' \subseteq \overline{W'} \subseteq W,$$

and

$$\overline{W} \subseteq W'' \subseteq \overline{W''} \subseteq U$$

and so on.



5.3 Continuous $\mathbb{R}$ - value maps

Let X be a topology space, consider a $\mathbb{R}$ - value map $X \xrightarrow{f} \mathbb{R}$ on it. Now we want to explore the relationship between the continuity of f and the topology structure of X.

Exercise 61. *Given a trivial topology space X, show that $X \xrightarrow{f} \mathbb{R}$ is constant $\Leftrightarrow X \xrightarrow{f} \mathbb{R}$ is continuous.*

Proof. $\Rightarrow$: Suppose for $\forall x \in X, f(x) \equiv r \in \mathbb{R}$. For any $U \subseteq_{open} \mathbb{R}$ containing r, $f^{-1}(U) = X \subseteq_{open} X$; and for any $V \subseteq_{open} \mathbb{R}$ that do not contain r, $f^{-1}(V) = \emptyset \subseteq_{open} X$, thus f is continuous.
$\Leftarrow$: If f is not a constant map, then $\exists x_1, x_2 \in X$, s.t. $f(x_1) = r_1, f(x_2) = r_2$ and $r_1 \neq r_2$. Since $r_1, r_2 \in \mathbb{R}$, and $\mathbb{R}$ is Hausdorff space, then $\exists U, V \subseteq_{open} \mathbb{R}$, s.t. $r_1 \in U, r_2 \in V$ and $U \cap V = \emptyset$. Thus $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset$. And $x_1 \in f^{-1}(U) \neq \emptyset$ and $x_2 \in f^{-1}(V) \neq \emptyset$.
If f is continuous, then $f^{-1}(U) \subseteq_{open} X \Rightarrow f^{-1}(U) = X$ which leads to a contradiction with $f^{-1}(V) = f^{-1}(U \cap V) = \emptyset$, thus f is not continuous. □

As we can see that if X is a trivial topology space, then the $\mathbb{R}$ - value map f on it is continuous iff f is constant map. Now if we add some sets into the trivial topology, can we obtain more continuous $\mathbb{R}$ - value maps that are not constant?

Exercise 62. *Let X be an infinite set, define $\mathscr{T} := \{U \subseteq X | U = \emptyset \vee X \backslash U$ is finite$\}$ which is called **Cofinite topology**. Show that The only $\mathbb{R}$ - valued continuous maps on $(X, \mathscr{T})$ are constant maps.*

Proof. We have proved that any $\mathbb{R}$ - valued constants map on X is continuous, we will show that any $\mathbb{R}$ - valued un-constants maps on X is not continuous.
Just as we shown before, If f is not a constant map, then $\exists x_1, x_2 \in X$, s.t. $f(x_1) = r_1, f(x_2) = r_2$ and $r_1 \neq r_2$. Since $r_1, r_2 \in \mathbb{R}$, and $\mathbb{R}$ is Hausdorff space, then $\exists U, V \subseteq_{open} \mathbb{R}$, s.t. $r_1 \in U, r_2 \in V$ and $U \cap V = \emptyset$. Thus $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset$. And $x_1 \in f^{-1}(U) \neq \emptyset$ and $x_2 \in f^{-1}(V) \neq \emptyset$.
Then if f is continuous, then $f^{-1}(U) \in \mathscr{T}$, since $f^{-1}(U) \neq \emptyset \Rightarrow X \backslash f^{-1}(U)$ is finite. Since $f^{-1}(U) \cap f^{-1}(V) = \emptyset \Rightarrow f^{-1}(V) \subseteq X \backslash f^{-1}(U)$ and thus $f^{-1}(V)$ is finite. Since X is infinite, $X \backslash f^{-1}(V)$ is infinite $\Rightarrow f^{-1}(V) \notin \mathscr{T} \Rightarrow f$ is not continuous. □

As we can see that, even though we add some sets into the topology of X, we can not construct some 'nontrivial' $\mathbb{R}$ - valued maps. Actually, if X is uncountable, even if we add sets into $\mathscr{T}$ again, such as define $\mathscr{T}' := \{U \subseteq X | U = \emptyset \vee X \backslash U \text{ is countable}\}$ which is called **Cocountable topology**, the only $\mathbb{R}$ - valued continuous maps on $(X, \mathscr{T}')$ are still constant maps.
These examples tell us, there is a 'threshold' of the topology, if the topology is too coarse to achieve the threshold, then the only $\mathbb{R}$ - valued continuous maps on X are constant maps.
Let X be a topology space and $A, B \subseteq X$ be disjoint. We say a **Chain** $\mathcal{C}$ from A to B consists of a sequence of subsets C_k of $X (k = 0, 1, \cdots, r)$, s.t.

$$A = C_0 \subseteq \overline{C_0} \subseteq C_1^o \subseteq \overline{C_1} \subseteq \cdots \subseteq \overline{C_{r-1}} \subseteq C_r^o \subseteq \overline{C_r} \subseteq X \backslash B.$$

For a chain $\mathcal{C}: C_k(k=0,\cdots,r)$, we let $C_0 := \emptyset$ and $C_{r+1} := X$ and define

$$f_{\mathcal{C}}(x) := \begin{cases} 1-\dfrac{k}{r}, & \text{if } x \in C_k \backslash C_{k-1}, k=0,\cdots,r \\ 0, & \text{if } x \notin C_r, \end{cases}$$

It is direct to see that $|f_{\mathcal{C}}(x) - f_{\mathcal{C}}(x')| \leq \frac{1}{r}$ if $x, x' \in C^o_{k+1} \backslash \overline{C_{k-1}} =: \Omega_k$ for any $k = 0, \cdots, r$. And $\Omega_k \subseteq_{open} X$ and $\cup_{i=0}^{r} \Omega_k = X$.

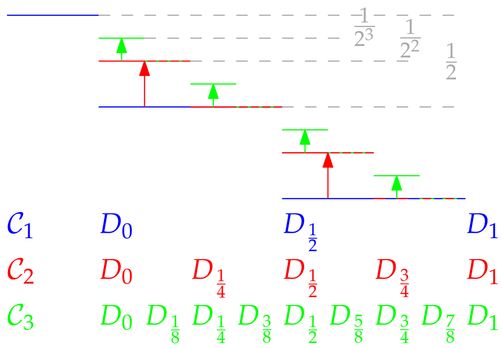
Lemma 2. *Suppose X is a topology space, $A, B \subseteq X$ are disjoint. $D_q \subseteq X$ where*

$$q \in \left\{ \frac{l}{2^m} \middle| l, m \in \mathbb{N}_0, l \leq 2^m \right\} =: Q,$$

s.t. $q \leq q' \Rightarrow \overline{D_q} \subseteq D^o_{q'}$ and $A = D_0, D_1 \subseteq X \backslash B$. Then $\exists$ a continuous map $X \xrightarrow{f} [0,1]$ s.t. $f(A) = \{1\}$ and $f(B) = \{0\}$.

Proof. Let $\mathcal{C}_m$ be the chain $D_0, D_{\frac{1}{2^m}}, \cdots, D_{\frac{2^m-1}{2^m}}, D_1$ from A to B. Thus

$$\begin{aligned} \mathcal{C}_0 &= D_0 (= A), D_1 \\ \mathcal{C}_1 &= D_0, D_{\frac{1}{2}}, D_1 \\ \mathcal{C}_2 &= D_0, D_{\frac{1}{4}}, D_{\frac{1}{2}}, D_{\frac{3}{4}}, D_1 \\ &\cdots \end{aligned}$$



Define $f_m := f_{\mathcal{C}_m} : X \to \mathbb{R} (m \in \mathbb{N}_0)$. Since for any $x \in X, m, m' \in \mathbb{N}_0$, $f_m(x) \leq 1$, and if $m \leq m' \Rightarrow f_m(x) \leq f_{m'}(x)$. Thus $f_m \to f$ as $m \to \infty$. And

$$f(x) - f_m(x) = \lim_{k \to \infty} \sum_{n=m}^{k} (f_{n+1}(x) - f_n(x))$$

where $f_{n+1}(x) - f_n(x) \leq \frac{1}{2^{n+1}}$ for $\forall x \in X$. Thus

$$f(x) - f_m(x) \leq \sum_{n=m}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2^m}$$

for any $x \in X$ and $m \in \mathbb{N}_0$. Thus for a given $x_0 \in X$ and any $x \in X$, we have

$$\begin{aligned}|f(x) - f(x_0)| &\leq |f(x) - f_m(x)| + |f_m(x) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\ &\leq \frac{1}{2^m} + \frac{1}{2^m} + |f_m(x) - f_m(x_0)|\end{aligned}$$

For any $\epsilon > 0$, we can choose and fix a large enough m such that $\frac{1}{2^m} < \frac{\epsilon}{3}$. Assume that $x_0 \in \Omega_s$ of $\mathcal{C}_m$ (that is $x_0 \in C^o_{\frac{s+1}{2^m}} \backslash \overline{C_{\frac{s-1}{2^m}}}$), then for any $x \in \Omega_s \subseteq_{open} X, wehavethat |f_m(x) - f_m(x_0)| \leq \frac{1}{2^m}$ and

$$|f(x) - f(x_0)| \leq \frac{1}{2^m} + \frac{1}{2^m} + \frac{1}{2^m} \leq \epsilon,$$

thus f is continuous, and $f(A) = \{1\}, f(B) = \{0\}$. □

Thus if X is a HLC space, $A, B \subseteq_{cpt.} X$ are disjoint, then there exists a continuous $\mathbb{R}$ - valued map $X \xrightarrow{f} \mathbb{R}$ such that $f(A) = \{1\}$ and $f(B) = \{0\}$.

Chapter 6

Sequence & Net

6.1 Seq. description of metric space

Definition 40 (Convergence)**.** Let $(X, \mathscr{T})$ be a topology space, $x \in X$ and $x_n \in X(n \in \mathbb{N})$, we say $x_n \to x$ as $n \to \infty$ if for any open nbd. U_x of x, $\exists N,$ s.t. $\forall n \in \mathbb{N}, n \geq N \Rightarrow x_n \in U$.

We define

$$\overline{A}' := \{x \in X | \exists \; seq. \; a_n \in A(n \in \mathbb{N}), \text{ s.t. } a_n \to x \; as \; n \to \infty\}$$

and

$$L'_A := \{x \in X | \exists \; seq. \; a_n \in A \backslash \{x\}(n \in \mathbb{N}), \text{ s.t. } a_n \to x \; as \; n \to \infty\}.$$

Exercise 63. *Let (X, d) be a metric space, $A \subseteq X$, show that*
1. $\overline{A} = \overline{A}'$*;*
2. $L_A = L'_A$

Proof. 1. $\subseteq$: if $x \in \overline{A}$, then $B_{\frac{1}{n}}(x) \cap A \neq \emptyset$ for $\forall n \in \mathbb{N}$. Then we can form a seq. $x_n(n \in \mathbb{N})$ such that $x_n \in B_{\frac{1}{n}}(x) \cap A$ for $\forall n \in \mathbb{N}$. Thus for any open nbd. U_x of x, since X is metric space, $\exists r > 0$, s.t. $B_r(x) \subseteq U_x$. Let $N = \lceil \frac{1}{r} \rceil$, then for any $n \in \mathbb{N}, n \geq N \Rightarrow x_n \in B_{\frac{1}{n}}(x) \subseteq B_r(x) \subseteq U_x \Rightarrow x_n \to x$ as $n \to \infty \Rightarrow x \in \overline{A}'$.
$\supseteq$: If $x \in \overline{A}' \Rightarrow \exists$ a seq. $x_n(n \in \mathbb{R})$, s.t. $x_n \to x$ as $n \to \infty$. Thus $\forall$ open nbd. U_x of x, $\exists N$, s.t. $\forall n \in \mathbb{N}, n \geq N \Rightarrow x_n \in U_x \Rightarrow$ such $x_n \in U_x \cap A \neq \emptyset \Rightarrow x \in \overline{A}$.
2. The same as above. □

Exercise 64. *Let X $\xrightarrow{f}$ Y is a map between metric spaces and $x_0 \in X$, show that f is continuous at $x_0 \Leftrightarrow \forall$ seq. $x_n \in X(n \in \mathbb{N}), x_n \to x$ as $n \to \infty \Rightarrow f(x_n) \to f(x)$ as $n \to \infty$.*

Proof. $\Rightarrow$: For any open nbd. V of $f(x_0), f^{-1}(V) \subseteq_{open} X$ is an open nbd. of x_0, since

$x_n \to x$ as $n \to \infty$, $\exists N$ s.t. $\forall n \in \mathbb{N}, n \geq N \Rightarrow x_n \in f^{-1}(V) \Rightarrow f(x_n) \in V \Rightarrow f(x_n) \to f(x_0)$ as $n \to \infty$.
$\Leftarrow$: Form a seq. $x_n (n \in \mathbb{N})$ such that $x_n \in B_{\frac{1}{n}}(x_0)$ for any $n \in \mathbb{N}$, then $x_n \to x_0$ and $f(x_n) \to f(x_0)$ as $n \to \infty$. Thus for any open nbd. V of $f(x_0), \exists N$, s.t. $\forall n \in \mathbb{N}, n \geq N \Rightarrow f(x_n) \in V$, which means for any $x \in B_{\frac{1}{n}}(x_0), f(x) \in V \Rightarrow B_{\frac{1}{n}}(x_0) \subseteq V \Rightarrow f$ is continuous at x_0. $\square$

As we shown, given metric spaces, then we can re-define the concept of *closure, limit points* and *continuity of the map* with sequential description. But if given topology spaces, instead of metric spaces, we only have
1. $\overline{A}' \subseteq \overline{A}$;
2. $L'_A \subseteq L_A$;
3. f is continuous at $x_0 \Rightarrow \forall$ seq. $x_n \in X (n \in \mathbb{N}), x_n \to x$ as $n \to \infty$ then $f(x_n) \to f(x)$ as $n \to \infty$

since the proofs of these properties are independent with the features of metric space. And in general, the converses do not hold.

Exercise 65. *If X are 1-st countable topology space, $A \subseteq X$, show that $\overline{A}' = \overline{A}$ and $L'_A = L_A$.*

Proof. All we need to prove is $\overline{A} \subseteq \overline{A}'$ and $L_A \subseteq L'_A$:
1. For any $x \in X, \exists$ a countable local basis $\mathcal{B}_x$ of x such as $\mathcal{B}_x = \{V_1, V_2, \cdots\}$, thus we can form a seq. $x_n (n \in \mathbb{N})$ such that $x_n \in A \cap (\cap_{i=1}^n V_i)$ for any $n \in \mathbb{N}$. Note that $x \in \overline{A} \Rightarrow A \cap (\cap_{i=1}^n V_i) \neq \emptyset$, thus x_n exists and $x_n \in A$.
Thus for any open nbd. U of x, $\exists V_m \in \mathcal{B}_x$ such that $x \in V_m \subseteq U$, and for any $n \geq m, x_n \subseteq V_m \subseteq U \Rightarrow x_n \to x$ as $n \to \infty$. Thus $x \in \overline{A}'$.
2. The same as 1. $\square$

6.2 Sequentially Compact, Totally Bounded

Definition 41. Let (X, d) be a metric space, we say
1. (X, d) is a sequentially compact if every sequence in X has a convergent subsequence.
2. (X, d) is a totally bounded if $\forall \epsilon > 0, \exists$ finite set $S \subseteq X$, s.t. $X = \cup_{s \in S} B_\epsilon(s)$.

Exercise 66. *Let (X, d) be a totally bounded metric space, that is for any $n \in \mathbb{N}$, there exist a finite set $S_n \subseteq X$,* s.t. *$X = \cup_{s \in S} B_{\frac{1}{n}}(s)$, show that $S := \cup_{n \in \mathbb{N}} S_n$ is a countable dense subset in X w.r.t. d.*

Proof. S is countable is trivial, we will show that S is dense. If U is an un-empty open set in X, then $\exists x \in U$ and $\exists r > 0$, s.t. $B_r(x) \subseteq U$, define $N = \lceil \frac{1}{r} \rceil$ then for any given $n \geq N, x \in U \subseteq \cup_{s \in S_n} B_{\frac{1}{n}}(s)$. And $\exists s' \in S_n$, s.t.

$$x \in B_{\frac{1}{n}}(s') \Rightarrow s' \in B_{\frac{1}{n}}(x) \subseteq B_r(x) \subseteq U$$

since $s' \in S_n \subseteq S$, $s' \in U \cap S \Rightarrow U \cap S \neq \emptyset \Rightarrow S$ is dense. $\square$

Thus Total boundedness $\Rightarrow$ separability (and hence 2-nd countability and Lindelof since X is a metric space).

Proposition 8. *Let (X, d) be metric space, the following are equivalent:*
1. X is compact (w.r.t $\mathscr{T}_d$);
2. X is sequentially compact (w.r.t. d);
3. X is complete and totally bounded (w.r.t. d).

Proof. $1 \Rightarrow 2$: Assume that $\exists$ seq. $x_n \in X (n \in \mathbb{N})$ such that any subseq. of it is not convergent, that is $\forall x \in X, x$ is not the limit of any subseq. of $x_n (n \in \mathbb{N})$. Thus for any $x \in X, \exists$ open nbd. U_x, s.t. $\{n \in \mathbb{N} | x_n \in U_x\}$ is finite.

Remark 44. We highlight that the index number $\{n \in \mathbb{N} | x_n \in U_x\}$ is finite instead of the point. Since the same point can be encoded by multi index number, and we want to expile that case.

Since X is compact, $X = \cup_{x \in X} U_x \Rightarrow \exists$ finite $X_0 \subseteq X$, s.t. $X = \cup_{x \in X_0} U_x$. Thus $\mathbb{N} = \{n \in \mathbb{N} | x_n \in X\} = \cup_{x \in X_0} \{n \in \mathbb{N} | x_n \in U_x\}$ which leads to a contradiction since $\cup_{x \in X_0} \{n \in \mathbb{N} | x_n \in U_x\}$ is finite.
$2 \Rightarrow 3$: Let $x_n (n \in \mathbb{N})$ be a Cauchy seq. in X, it is suffices to show that $x_n (n \in \mathbb{N})$ has a convergent subseq. and this is implied by 2.
Suppose (X, d) is not totally bounded, then $\exists \epsilon > 0$, such that pick any $x_1 \in X$ we have that

$$B_\epsilon(x_1) \subsetneq X \Rightarrow X \backslash B_\epsilon(x_1) \neq \emptyset,$$

and pick $x_2 \in X \backslash B_\epsilon(x_1)$ have

$$B_\epsilon(x_1) \cup B_\epsilon(x_2) \subsetneq X \Rightarrow X \backslash (B_\epsilon(x_1) \cup B_\epsilon(x_2)) \neq \emptyset,$$

and pick $x_3 \in X \backslash (B_\epsilon(x_1) \cup B_\epsilon(x_2))$, and so on.
Thus we can find a seq. $x_1, x_2, \cdots$ such that $d(x_i, x_j) \geq \epsilon$ for $i \neq j$ (since $x_i \in X \backslash B_\epsilon(x_j)$). Thus any subseq. of $x_n (n \in \mathbb{N})$ is not Cauchy seq. and hence is not convergent, which leads to a contradiction with 2.
$3 \Rightarrow 2$: Let $x_n (n \in \mathbb{N})$ be a seq. in X, since (X, d) is totally bounded $\Rightarrow$ For any given $n \in \mathbb{N}$, X can be covered by finitely many $\frac{1}{n}$ balls.
Thus X can be covered by finite many 1-balls, $x_n \in X (n \in \mathbb{N}) \Rightarrow \exists$ a 1-ball B_1, s.t.

$$\{n \in \mathbb{N} | x_n \in B_1\} \text{ is infinite;}$$

X can be covered by finite many 1/2-balls, and so do B_1, thus $\exists$ a 1/2-ball B_2, s.t.

$$\{n \in \mathbb{N} | x_n \in B_1 \cap B_2\} \text{ is infinite.}$$

And if $\exists$ 1/m-ball B_m, s.t. $\{n \in \mathbb{N} | x_n \in \cap_{i=1}^m B_i\}$ is infinite, then since $\cap_{i=1}^m B_i$, which covers infinite points of the seq., can be covered by finitely many 1/(m+1) balls, there $\exists$ a 1/(m+1) ball B_{m+1} s.t.

$$\{n \in \mathbb{N} | x_n \in \cap_{i=1}^{m+1} B_i\} \text{ is infinite.}$$

Thus $\exists$ subseq. $x_{n_k}(k \in \mathbb{N})$, s.t. $x_{n_k} \in B_1 \cap \cdots \cap B_k$ for every $k \in \mathbb{N}$. And for every $l, l' \geq k, x_{n_l}, x_{n'_l} \in B_k$ and hence $d(x_{n_l}, x_{n'_l}) \leq \frac{1}{k}$. Thus $x_{n_k}(k \in \mathbb{N})$ is a Cauchy seq., and since X is complete, $x_{n_k}(k \in \mathbb{N})$ is convergent.

Remark 45. Refer to the proof of *Bolzano-Weierstrass theorem* in *Introduction to Topology, Lecture 8,9*.

2 $\Rightarrow$ 1: Let $\mathcal{F}$ be a family of closed subsets of X which satisfies the FIP, we need to show that $\cap\mathcal{F} \neq \emptyset$. Suppose that $\cap\mathcal{F} = \emptyset$. Then $\{X \backslash C | C \in \mathcal{F}\}$ is an open cover of X, since X is sequentially compact, then X is totally bounded, and hence X is Lindelof countable.
Thus $\exists$ a countable $\mathcal{F}_0 = \{C_1, C_2, \cdots\} \subseteq \mathcal{F}$ s.t. $\{X \backslash C | C \in \mathcal{F}_0\}$ still cover X, and hence $\cap_{C \in \mathcal{F}_0} C = \emptyset$. Note that $\mathcal{F}$ satisfies FIP, thus $\mathcal{F}_0$ satisfies FIP as well. Thus any finite intersection of the elements in $\mathcal{F}_0$ is not empty, thus exists

$$\begin{aligned} &x_1 \in C_1, \\ &x_2 \in C_1 \cap C_2, \\ &\quad \cdots \\ &x_n \in \cap_{i=1}^n C_i, \\ &\quad \cdots \end{aligned}$$

which forms a seq. $x_n(n \in \mathbb{N})$ in X, and since X is seq. cpt., there exists a convergent subseq. $x_{n_k}(k \in \mathbb{N})$. And $x_{n_k} \to x \in X$ as $k \to \infty$.
Note that since $C_n(n \in \mathbb{N})$ are closed, then for any given $N \in \mathbb{N}, \cap_{i=1}^N C_i$ is still closed. Since $x_{n_k} \in \cap_{i=1}^{n_k} C_i$ and for any $k \geq$ given $K \in \mathbb{N}$ have that $x_{n_k} \in \cap_{i=1}^{n_K} C_i$ and $\cap_{i=1}^{n_K} C_i$ is closed $\Rightarrow x \in \cap_{i=1}^{n_K} C_i$ for any $K \in \mathbb{N}$. Since $n_k \to \infty$ as $k \to \infty$, thus $x \in \cap_{i=1}^N C_i$ for any $N \in \mathbb{N} \Rightarrow x \in \lim_{n\to\infty} \cap_{i=1}^n C_i = \cap_{C \in \mathcal{F}_0} C \Rightarrow \cap_{C \in \mathcal{F}_0} C \neq \emptyset$ which leads to the contradiction with the assumption. $\square$

Exercise 67. *Let (X, d) be a complete metric space, $K \subseteq X$, show that*
1. *(K, d) is complete $\Leftrightarrow K \subseteq_{close} X$;*
2. *(K, d) is compact $\Leftrightarrow K \subseteq_{close} X$ and (K, d) is totally bounded;*
3. *(K, d) is totally bounded $\Leftrightarrow \forall \epsilon > 0, \exists$ finite set $S \subseteq X$,* s.t. $K \subseteq \cup_{s \in S} B_\epsilon(s)$.

Proof. 1. This will be proved by demonstrating the contrapositive: K is not complete if and only if K is not closed.

$\Rightarrow$: Suppose that K is not complete. Then there exists a Cauchy sequence x_n in K such that the limit $x = \lim_{n\to\infty} x_n$, which exists in the complete metric space X, is not a member of K.
For all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ has $d(x, x_n) < \epsilon$, and hence $X \backslash K$ is not open (if $X \backslash K$ is open then $\exists r > 0$, s.t. $B_r(x) \subseteq X \backslash K \Rightarrow d(x, x_n) > r$ for all $n \in \mathbb{N}$). Therefore, K is not closed.
$\Leftarrow$: Suppose that K is not closed. Then $X \backslash K$ is not open. Therefore, there exists a $x \in X \backslash K$ such that for all $\epsilon > 0$, there exists a $y \in K$ such that $d(x, y) < \epsilon$. Thus we can form a seq. $y_n (n \in \mathbb{N})$ in K such that $y_n \in K \cap B_{\frac{1}{n}}(x)$ for all $n \in \mathbb{N}$ and hence $d(x, y_n) < \frac{1}{n}$.
Now, we show that y_n is a Cauchy sequence. Given an $\epsilon > 0$, let $N \in \mathbb{N}$ be such that for all $n \geq N$ has $d(x, y_n) < \frac{\epsilon}{2}$. Let $m, n \geq N$, then by the triangle inequality:

$$d(y_n, y_m) \leq d(x, y_m) + d(x, y_n) \leq \epsilon,$$

Hence y_n is a Cauchy sequence. Because (X, d) is a complete metric space by assumption, the limit $\lim_{n\to\infty} y_n$ exists and is in X. Denote this limit by y. By the definition of y_n we have that $\lim_{n\to\infty} d(x, y_n) = 0$. From Distance Function of Metric Space is Continuous and Composite of Continuous Mappings is Continuous we have $d(x, y) = 0 \Rightarrow x = y$, since $x \notin K \Rightarrow y \notin K \Rightarrow K$ is not complete.
2. trivial
3. $\Rightarrow$ is trivial; $\Leftarrow$: Since given any $\epsilon > 0, \exists$ finite $S \subseteq X$ s.t. $K \subseteq \cup_{s\in S} B_\epsilon(s)$. Define $S_0 = \{s_1, \cdots, s_n\} \subseteq S$ where $B_\epsilon(s) \cap K \neq \emptyset$ for any $s \in S_0$. Then pick $k_i \in K \cap B_\epsilon(s_i)$ for $i = 1, \cdots, 2$, then we have that

$$k_i \in B_\epsilon(s_i) \Rightarrow d(s_i, k_i) < \epsilon,$$

thus for any $k \in K$, $\exists s_i \in S_0$, s.t. $k \in B_\epsilon(s_i) \Rightarrow d(k, s_i) < \epsilon$, thus

$$d(k, k_i) \leq d(k, s_i) + d(s_i + k_i) \leq 2\epsilon$$

thus $k \in B_{2\epsilon}(k_i) \Rightarrow K \subseteq \cup_{i=1}^n B_{2\epsilon}(k_i) \Rightarrow K$ is totally bounded. □

Remark 46. Let (X, d) be a metric space, define $d'(x_1, x_2) := \min\{1, d(x_1, x_2)\}$, then d' is still a metric. And

- $\{$the Cauchy seq.s in $(X, d)\} = \{$the Cauchy seq.s in $(X, d')\}$
- $\mathscr{T}_d = \mathscr{T}_{d'}$
- (X, d') is always a **bounded** metric space.

6.3 Net

Let X be set, then a sequence $x_n (n \in \mathbb{N})$ in X is such a map $\mathbb{N} \xrightarrow{x.} X$ (denote $x(n)$ by x_n). Now we gonna generalize this concept.

Definition 42 (Directed Set)**.** A directed set $(D, \geq)$ consists of a non-empty set D and a relation $\geq$ on D s.t.
1. $\forall d \in D, d \geq d$;
2. $\forall d, d', d'' \in D, d \geq d', d' \geq d'' \Rightarrow d \geq d''$,
i.e. $(D, \geq)$ is pre-order. And $\forall d, d' \in D, \exists d'' \in D,$ s.t. $d'' \geq d, d'' \geq d'$.

Remark 47. Note that the pre-order is not total order, which means there could exist $d_1, d_2 \in D$ which are not comparable. On the other hand, the pre-order is not partial order yet, which means it does not require $d \geq d' \wedge d' \geq d \Rightarrow d = d'$. Thus the following statement in a directed set may hold: $\exists d_1, d_2, d_3, d_4 \in D$ such that

$$d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_1,$$

but $d_1 \neq d_2 \neq d_3 \neq d_4$.

Example 13. Let X be a topology space, $x \in X$, $D = \{\text{all open nbd.s of } x\}$ and for any $U, V \in D$ define $U \geq V \Leftrightarrow U \subseteq V$, then $(D, \geq)$ is a directed set. (Since for any $U, V \in D, \exists W := U \cap V \in D,$ s.t. $W \geq U, W \geq V$)

Definition 43 (Net)**.** Let X be a set, a net $(D, \geq) \xrightarrow{x.} X$, ($x_\alpha(\alpha \in D)$ for short,) in X consists of a directed set $(D, \geq)$ and a map $D \xrightarrow{x.} X$.

Suppose that a net $x.$ ($x_\alpha(\alpha \in D)$) is a net in a set X, and $S \subseteq X$, we say that $x.$ lies in S
- **eventually** if $\exists \delta \in D, \forall \alpha \in D, \alpha \geq \delta \Rightarrow x_\alpha \in S$;
- **frequently** if $\forall \delta \in D, \exists \alpha \in D,$ s.t. $\alpha \geq \delta$ and $x_\alpha \in S$.

Remark 48. $\neg$($x.$ lies in S eventually) $\Leftrightarrow$ $x.$ lies in $X \backslash S$ frequently.

Definition 44 (Convergence)**.** Let X be a topology space, $x_\alpha(\alpha \in D)$ is a net in X, $x \in X$. We say that $x.$ converges x (or say x is a limit of $x.$) if $\forall$ open nbd. U of x in X, $x.$ lies in U eventually.

Exercise 68. *Show that X is a Hausdorff space $\Leftrightarrow$ every net has at most one limit.*

Proof. $\Rightarrow$: Suppose a net $D \xrightarrow{x.} X$ converges to x and y in X and $x \neq y$, then $\exists$ open nbd.s U of x and V of y, s.t. $x \in U, y \in V$ and $U \cap V = \emptyset$. Since $x. \to x$ then $\exists \delta_x \in D,$ s.t. $\forall \alpha \in D, \alpha \geq \delta_x \Rightarrow x_\alpha \in U$. And since $x. \to y, \exists \delta_y \in D,$ s.t. $\forall \alpha \in D, \alpha \geq \delta_y \Rightarrow x_\alpha \in V$. Then $\exists \delta \in D,$ s.t. $\delta \geq \delta_x \wedge \delta \geq \delta_y$, thus for $\forall \alpha \in D, \alpha \geq \delta$ has $x_\alpha \in U \subseteq X \backslash V$ and $x_\alpha \in V$ which leads to a contradiction.
$\Leftarrow$: Suppose X is not a Hausdorff space, then $\exists x, y \in X,$ s.t. $\forall$ open nbd.s U of x, V of y has $U \cap V \neq \emptyset$. Thus we can form a net in X.
Define $D = \{U \cap V | x \in U \subseteq_{open} X, y \in V \subseteq_{open} X\}$ and $\forall d_1, d_2 \in D, d_1 \geq d_2 \Leftrightarrow d1 \subseteq d_2$, it is direct to see $(D, \geq)$ is a directed set. And then $D \xrightarrow{x.} X$ where $d \mapsto x_d \in d$ is a net (since $\forall d \in D, d \neq \emptyset$, and hence $x_d \exists$).

Thus given any open nbd. W of x, $W \cap V \in D$ where D is a open nbd. of y, then $\forall \alpha \in D, \alpha \geq W \cap V$ we have

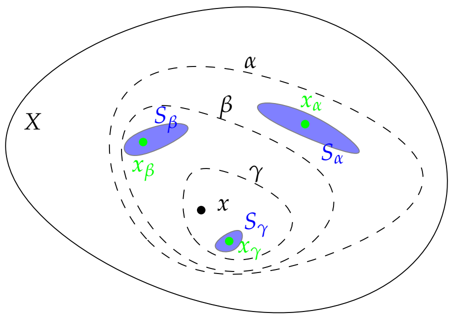
$$x_\alpha \in \alpha \subseteq W \cap V \subseteq W$$

thus $x.$ lies in any open nbd. W of x eventually, and hence $x.$ converges to x. Thus $x.$ converges to y as well in the same way, which leads to a contradiction. $\square$

Remark 49 (A naturally convergent net). If X is a set, $x \in X$, if we define the directed set as $D = \{U | x \in U \subseteq_{open} X\}$ and $\geq \Leftrightarrow \subseteq$, then $(D, \geq)$ is a directed set. And define the net $D \xrightarrow{x.} X$, where $\alpha \mapsto x_\alpha \in S_\alpha \subseteq \alpha$. Then for any open nbd. U of x, $U \in D$ and $\forall \alpha \in D, \alpha \geq U$ has

$$x_\alpha \in S_\alpha \subseteq \alpha \subseteq U.$$

Thus such $x.$ converges to x naturally.



Exercise 69. *Let X be a topology space, $A \subseteq X$, define*

$$\overline{A}'' := \{x \in X | \exists \textit{ net } a. \textit{ in } A \textit{ converging to } x\}$$

and

$$L''_A := \{x \in X | \exists \textit{ net } a. \textit{ in } A \backslash \{x\} \textit{ converging to } x\}$$

show that $\overline{A} = \overline{A}''$ and $L_A = L''_A$.

Proof. 1. $\subseteq$: if $x \in \overline{A}$, then any open nbd. U of x has $U \cap A \neq \emptyset$, thus we can form a net. Define $D = \{U | x \in U \subseteq_{open} X\}$ and $\geq \Leftrightarrow \subseteq$ then $(D, \geq)$ is a directed set and $D \xrightarrow{x.} A$ where $d \mapsto x_d \in d \cap A$ is a net. And $x.$ converges to $x \Rightarrow x \in \overline{A}''$ by *Remark* 2.
$\supseteq$: if $x \in \overline{A}''$, then $\exists$ a net $D \xrightarrow{x.} A$ s.t. for $\forall$ open nbd. U of x, $\exists \delta \in D$ s.t. $\forall \alpha \in D, \alpha \geq \delta \Rightarrow \alpha \in U$, then $U \cap A \neq \emptyset \Rightarrow x \in \overline{A}$.
2. the same as above. $\square$

Exercise 70. *Let $X \xrightarrow{f} Y$ be a map between topology spaces, $x_0 \in X$, show that f is continuous at $x_0 \Leftrightarrow$ for $\forall$ net $D \xrightarrow{x.} X$ in X that converges to x_0, $f(x.)$ is a net in Y converges to $f(x_0)$.*

Proof. $\Rightarrow$: if V is an open nbd. of $f(x_0)$, since f is continuous, $f^{-1}(V)$ is an open nbd. of x_0, then $\exists \delta \in D,$ s.t. $\forall \alpha \in D, \alpha \geq \delta \Rightarrow x_\alpha \in f^{-1}(V) \Rightarrow f(x_\alpha) \in V \Rightarrow f(x.)$ converges to $f(x_0)$.
$\Leftarrow$: suppose f is not continuous at x_0, then $\exists$ an open nbd. V of $f(x_0)$, $f^{-1}(V)$ is not an open nbd. of x_0, that is $x_0 \notin (f^{-1}(V))^o$, since $x_0 \in f^{-1}(V)$, $x_0 \in f^{-1}(V) \backslash (f^{-1}(V))^o = \partial f^{-1}(V)$. Thus any open nbd. U of x has $U \cap f^{-1}(V) \neq \emptyset$ and $U \cap X \backslash f^{-1}(V) \neq \emptyset$, and hence we can form a net.
Define $D = \{U | x \in U \subseteq_{open} X\}$ and $\geq \Leftrightarrow \subseteq$ then $(D, \geq)$ is a directed set, and define a net $D \xrightarrow{x.} X \backslash f^{-1}(V)$ where $\alpha \mapsto x_\alpha \in \alpha \cap X \backslash f^{-1}(V)$, then $x.$ converges to x by *Remark 2*, and hence $f(x.)$ converges to $f(x_0)$ by assumptions, which means $\exists \delta \in D,$ s.t. $\forall \alpha \in D, \alpha \geq \delta \Rightarrow f(x_\alpha) \in V$ which leads to a contradiction with $f(x_\alpha) \in X \backslash f^{-1}(V)$. ☐

Remark 50. $f(x.)$ is a net in Y:

$$D \xrightarrow{x.} X \xrightarrow{f} Y$$

6.4 Subnet

Recall that given a sequence $x_n (n \in \mathbb{N})$ in a set X, a subsequence $x_{n_k} (k \in \mathbb{N})$ is composite map

$$\mathbb{N} \xrightarrow{n.} \mathbb{N} \xrightarrow{x.} X$$

(denote $x(n(k))$ as x_{n_k}), where $\mathbb{N} \xrightarrow{n.} \mathbb{N}$ is a monotone injection. We now want to generalize this conception.

Definition 45 (Final Map)**.** Let $(D, \geq)$ and $(D', \geq')$ be directed sets, a map $D' \xrightarrow{h} D$ is a final map (w.r.t. $\geq$ and $\geq'$) if $\forall \delta \in D, \exists \delta' \in D',$ s.t. $\forall \alpha \in D', \alpha \geq \delta' \Rightarrow h(\alpha) \geq \delta$.

Remark 51. Final map analogizes the monotones of $\mathbb{N} \xrightarrow{n.} \mathbb{N}$. Final map require the `tail` of the map is `monotones`.

Definition 46. Let $D' \xrightarrow{h} D$ is a final map between directed sets, net $x_{h(\cdot)}$:

$$\begin{tikzcd} D' \arrow[r, "h"'] \arrow[rr, bend left, "x_{h(\cdot)}"] & D \arrow[r, "x."'] & X \end{tikzcd}$$

is called a subnet of x.

Exercise 71. *If a net $x.$ converges to x_0 show that the subnet $x_{h(\cdot)}$ converges to x_0 as well.*

Proof. For any open nbd. U of x_0, $\exists \delta \in D,$ s.t. $\forall \alpha \in D, \alpha \geq \delta \Rightarrow x_\alpha \in U \Rightarrow \exists \delta' \in D', \forall \alpha' \geq \delta', h(\alpha') \geq \delta \Rightarrow x_{h(\alpha')} \in U \Rightarrow x_{h(\cdot)}$ converges to x_0. ☐

Exercise 72. *Let X be a set, $x.$ is a net in X, $S \subseteq X$. Show that $x.$ lies in S frequently $\Leftrightarrow \exists$ subnet of $x.$ lies in S eventually.*

Proof. $\Rightarrow$: $D \xrightarrow{x.} X$ lies in S frequently, then $\forall \delta \in D, \exists \alpha_\delta \in D,$ s.t. $\alpha_\delta \geq \delta$ and $x_{\alpha_\delta} \in S$. Then we can for a final map $D \xrightarrow{h} D$ where $\delta \mapsto \alpha_\delta$. Thus for any $\alpha_\delta \in D, \exists \alpha_\delta \in D,$ s.t. $\forall \alpha \in D, \alpha \geq \alpha_\delta \Rightarrow \alpha \geq \alpha_\delta$, thus h is a final map, and $x_{h(\cdot)}$ is a subnet of $x.$ and for any $\alpha \in D, x_{h(\alpha)} = x_{\alpha_\delta} \in S \Rightarrow x_{h(\cdot)}$ lies in S eventually.
$\Leftarrow$: if $D \xrightarrow{x.} X$ has an subnet $D' \xrightarrow{x_{h(\cdot)}} X$ which lies in S eventually. Then $\exists \beta \in D',$ s.t. $\forall \alpha' \in D', \alpha' \geq \beta \Rightarrow x_{h(\alpha')} \in S$. On the other hand, $\forall \delta \in D, \exists \delta' \in D',$ s.t. $\forall \alpha' \in D', \alpha' \geq \delta' \Rightarrow h(\alpha') \geq \delta$. Since D' is directed set, $\exists \gamma \in D',$ s.t. $\gamma \geq \beta$ and $\gamma \geq \delta'$, then $h(\gamma) \geq \delta$ and $x_{h(\gamma)} \in S$.
Collectively, $\forall \delta \in D, \exists h(\gamma) \in D,$ s.t. $h(\gamma) \geq \delta$ and $x_{h(\gamma)} \in S \Rightarrow x.$ lies in S frequently. □

Definition 47 (Universal Net)**.** A net $x.$ in a set X is universal if $\forall A \subseteq X$ either $x.$ lies in A eventually or $x.$ lies in $X \backslash A$ eventually.

Exercise 73. *$X \xrightarrow{f} Y$ is a map, show that $x.$ is a universal net in $X \Rightarrow f(x.)$ is universal net in Y.*

Proof. For any $B \subseteq Y, f^{-1}(B) \subseteq X$, since $D \xrightarrow{x.} X$ is a universal net, $x.$ lies in $f^{-1}(B)$ eventually or $X \backslash f^{-1}(B)$.
If $x.$ lies in $f^{-1}(B)$ eventually, $\exists \delta \in D,$ s.t. $\forall \alpha \in D, \alpha \geq \delta, \Rightarrow x_\alpha \in f^{-1}(B) \Rightarrow f(x_\alpha) \in B$;
If $x.$ lies in $X \backslash f^{-1}(B)$ eventually, $\exists \delta \in D,$ s.t. $\forall \alpha \in D, \alpha \geq \delta, \Rightarrow x_\alpha \in X \backslash f^{-1}(B) = f^{-1}(Y) \backslash f^{-1}(B) = f^{-1}(Y \backslash B) \Rightarrow f(x_\alpha) \in Y \backslash B$. Thus $f(x.)$ is a universal net in Y. □

Exercise 74. *Show that every subnet of a universal net is universal.*

Proof. Suppose $D \xrightarrow{x.} X$ is a universal net in X which has a subnet $D' \xrightarrow{x_{h(\cdot)}} X$. And for any $A \subseteq X$, $x.$ lies in A or $X \backslash A$ eventually. Suppose $x.$ lies in A, then $\exists \delta \in D,$ s.t. $\forall \alpha \in D, \alpha \geq \delta \Rightarrow x_\alpha \in A$. On the other hand, $\exists \delta' \in D',$ s.t. $\forall \alpha' \in D', \alpha' \geq \delta' \Rightarrow h(\alpha') \geq \delta \Rightarrow x_{h(\alpha')} \in A \Rightarrow x_{h(\cdot)}$ lies in A eventually as well $\Rightarrow x_{h(\cdot)}$ is universal. □

Theorem 10. *Every net has a universal subnet.*

Proof. Let $(D, \geq_D) \xrightarrow{x.} X$ be a net in a set X, where $(D, \geq_D)$ is a directed set.
1. Define Y as the family of some families $\mathcal{A}(\subseteq \mathcal{P}(X))$ of subsets of X such that
   (a) $\forall A \in \mathcal{A}$, $x.$ lies in A frequently;
   (b) $\forall A_1, A_2 \in \mathcal{A}, A_1 \cap A_2 \in \mathcal{A}$.

   That is the element of Y is the family of subsets of X that satisfies the above conditions. Thus $Y \neq \emptyset$ (since $\{X\} \in Y$) and $(Y, \subseteq)$ is a poset. We now apply Zorn's lemma to get a maximal element of Y.

Let C be a chain in Y w.r.t. $\subseteq$. Then we claim that $\cup_{\mathcal{A}\in C}\mathcal{A} \in Y$ and is an upper bound of C.

(a) For any $A \in \cup_{\mathcal{A}\in C}\mathcal{A}$ there $\exists \mathcal{A}' \in C$, s.t. $A \in \mathcal{A}'$, thus $x.$ lies in A eventually;

(b) For any $A_1, A_2 \in \mathcal{A}$ there $\exists \mathcal{A}_1, \mathcal{A}_2 \in C$, s.t. $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$ and $\mathcal{A}_1$ is comparable with $\mathcal{A}_2$ w.r.t. $\subseteq$, for example $\mathcal{A}_1 \subseteq \mathcal{A}_2$, then $A_1, A_2 \in \mathcal{A}_2 \Rightarrow A_1 \cap A_2 \in \mathcal{A}_2 \subseteq \cup_{\mathcal{A}\in C}\mathcal{A}$.

Thus $\exists$ maximal element $\mathcal{A}_0$ of Y.

2. Let $D_0 := \{(A,\alpha) \in \mathcal{A}_0 \times D | x_\alpha \in A\}$ with the pre-order $\geq_0$ on D_0: $(A',\alpha') \geq_0 (A,\alpha) \Leftrightarrow A' \subseteq A$ and $\alpha' \geq_D \alpha$. Since
   (a) For any $A \in \mathcal{A}_{\prime}, \alpha \in D, A \subseteq A$ and $\alpha \geq_D \alpha \Rightarrow (A,\alpha) \geq_0 (A,\alpha)$;
   (b) For any $(A_1,\alpha_1),(A_2,\alpha_2),(A_3,\alpha_3) \in D_0$, $(A_1,\alpha_1) \geq_0 (A_2,\alpha_2)$ and $(A_2,\alpha_2) \geq_0 (A_3,\alpha_3)$ means
   $$\alpha_1 \geq_D \alpha_2 \geq_D \alpha_3$$
   and
   $$A_1 \subseteq A_2 \subseteq A_3$$
   thus $(A_1,\alpha_1) \geq_0 (A_3,\alpha_3)$
   (c) For any $(A_1,\alpha_1),(A_2,\alpha_2) \in D_0, \mathcal{A}_0 \ni A_1 \cap A_2 \subseteq A_1$ and A_2; and $\exists \alpha' \geq_D \alpha_1$ and $\alpha_2 \Rightarrow D_0 \ni (A_1 \cap A_2, \alpha') \geq_0 (A_1,\alpha_1)$ and (A_2,α_2).

   Thus $(D_0, \geq_0)$ is a directed set.
3. And then we can define a final map $D_0 \xrightarrow{h} D$ where $(A,\alpha) \mapsto \alpha$. Given $\delta \in D$, for any $A \in \mathcal{A}_0$, since $x.$ lies in A frequently, $\exists \alpha \in D$, s.t. $\alpha \geq \delta$ and $x_\alpha \in A$, and hence $(A,\alpha) \in D_0$. For any $(A',\alpha') \geq_0 (A,\alpha)$, we have that $h((A',\alpha')) = \alpha' \geq \alpha \geq \delta$, thus h is a final map.

   In particular, we donate the subnet of $x.$, i.e. the composite of $D_0 \xrightarrow{h} D \xrightarrow{x.} X$ as $D_0 \xrightarrow{y.:=x.\circ h} X$ where $(A,\alpha) \mapsto x_\alpha = y_{(A,\alpha)}$.
4. Let $S \subseteq X$, we will show that the subnet $y.$ is universal: if $\neg$ ($y.$ lies in $X \backslash S$ eventually) $\Leftrightarrow$ ($y.$ lies in S frequently) then we will show that it implies $y.$ lies in S eventually.

   For any $A \in \mathcal{A}_0$, $x.$ lies in A frequently $\Rightarrow$ for any $\delta \in D$, there exists $\alpha \in D$, s.t. $\alpha \geq_D \delta$ and $x_\alpha \in A$ and hence $(A,\alpha) \in D_0$. And since $y.$ lies in S frequently, $\exists (A_1,\alpha_1) \in D_0$, s.t. $(A_1,\alpha_1) \geq_0 (A,\alpha)$, (i.e. $A_1 \subseteq A$ and $\alpha_1 \geq_D \alpha_0$) and $y_{(A_1,\alpha_1)} \in S$. And $y_{(A_1,\alpha_1)} = x_{\alpha_1} \in A_1$ since $(A_1,\alpha_1) \in D_0$. Thus
   $$x_{\alpha_1} = y_{(A_1,\alpha_1)} \in S \cap A_1 \subseteq S \cap A$$
   thus $x.$ lies in $S \cap A$ frequently for any $A \in \mathcal{A}_0$ and thus $x.$ lies in S frequently, thus we have that
   $$\mathcal{A}_0 \cup \{S \cap A | A \in \mathcal{A}_0\} \cup \{S\} \in Y$$
   by the definition of Y, and since $\mathcal{A}_0$ is the maximal element of $Y \Rightarrow S \in \mathcal{A}_0$.

If $\neg$ ($y.$ lies in S eventually) holds, then $y.$ lies in $X\backslash S$ frequently holds $\Rightarrow X\backslash S \in \mathcal{A}_0$, thus $S, X\backslash S \in \mathcal{A}_0 \Rightarrow \emptyset = S \cap (X\backslash S) \in \mathcal{A}_0$, which leads to a contradiction with $x.$ lies in it frequently.

□

Remark 52. Thus we have a corollary: if $x.$ is a universal net in X, $S \subseteq X$, then $\neg$ ($x.$ lies in S eventually) $\Rightarrow x.$ lies in $X\backslash S$ eventually.

6.5 Net and Compactness

Proposition 9. *Let X be a topology space, the following are equivalent:*
1. X is a compact space;
2. $\forall$ family $\mathcal{F}$ of closed subsets of X, $\mathcal{F}$ has FIP $\Leftrightarrow \cap\mathcal{F} \neq \emptyset$;
3. $\forall$ universal net in X converges;
4. $\forall$ net in X has a convergent subnet.

Proof. We will prove this in order $1 \Rightarrow 3 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$.
$1 \Rightarrow 3$: Suppose that $x_\alpha(\alpha \in D)$ is a universal net in X which does not converge to any $x \in X$, thus $\exists$ open nbd. U_x of x in X s.t. $\neg$ ($x.$ lies in U_x eventually) $\Rightarrow x.$ lies in $X\backslash U_x$ frequently. Since $X = \cup_{x\in X} U_x$ and X is compact, there $\exists$ finite $X_0 \subseteq X$, s.t. $X = \cup_{x\in X_0} U_x \Rightarrow \emptyset = \cup_{x\in X_0}(X\backslash U_x)$ which leads to a contradiction with $x.$ lines in $X\backslash U_x$ frequently.
$3 \Rightarrow 4$: $\forall$ net in X has a universal subnet and it is convergent by 3.
$4 \Rightarrow 2$: Let $\mathcal{F}$ be a family of closed subsets of X which has FIP, we can **expand** it as $\mathcal{F}' := \{\cap_{i=1}^m F_i | m \in \mathbb{N}, F_i \in \mathcal{F}, i = 1, \cdots, m\}$. Note that there are 3 facts for $\mathcal{F}'$:
1. $\mathcal{F}'$ also has FIP;
   since finite intersection of $\mathcal{F}'$ is a finite intersection of $\mathcal{F}$;
2. $\cap\mathcal{F}' = \cap\mathcal{F}$;
   since for any $c \in \cap\mathcal{F}' \Rightarrow c \in$ every finite intersection of $\mathcal{F} \Rightarrow c \in \cap_{F\in\{F\}} F = F$ for $\forall F \in \mathcal{F} \Rightarrow c \in \cap\mathcal{F}$. On the contrary, for any $c \in \cap\mathcal{F} \Rightarrow c \in F$ for any $F \in \mathcal{F} \Rightarrow c \in \mathcal{F}'$.
3. $\mathcal{F}'$ is closed under $\cap$.

It is direct to see that $(\mathcal{F}', \geq')$ with $\geq' := \subseteq$ is a directed set. For any $C \in \mathcal{F}'$, (it is finite intersection of $\mathcal{F}$ and hence $C \neq \emptyset$,) choose $x_C \in C$ and form a net $\mathcal{F}' \xrightarrow{x.} X$ where $C \mapsto x_C$.
By 4, net $x.$ has a convergent subnet, that is $\exists$ a final map $D \xrightarrow{h} \mathcal{F}'$ for some directed set $(D, \geq_D)$, s.t. subnet $D \xrightarrow{y.} X$ (where $\alpha \mapsto x_{h(\alpha)} = y_\alpha$) converges to some point $x \in X$.
Since h is finial, $\forall C \in \mathcal{F}', \exists \alpha \in D, \forall \beta \in D, \beta \geq_D \alpha \Rightarrow h(\beta) \geq C \Leftrightarrow h(\beta) \subseteq C$ and thus

$$y_\beta = x_{h(\beta)} \in h(\beta) \subseteq C$$

thus $y.$ lies in C eventually. For any $C \in \mathcal{F}', y.$ converges to $x \Rightarrow x \in C$ since C is closed, thus $x \in \cap_{C \in \mathcal{F}'} C = \cap \mathcal{F} \Rightarrow \mathcal{F} \neq \emptyset$.
$2 \Rightarrow 1$: has been given in *Point Set Topology Lecture 6*. $\square$

Remark 53. $1 \Rightarrow 3$: A common routine to utilize the compactness of X: find an open nbd. U_x for any $x \in X$, and then $X = \cup_{x \in X} U_x$.
$4 \Rightarrow 2$: The key to form a net is to find some sets $\neq \emptyset$.

Lemma 3. *Let $X_j (j \in J)$ be a family of topology spaces and $D \xrightarrow{x.} \prod_{j \in J} X_j$ where $\alpha \mapsto x_\alpha = (x_{\alpha_j})_{j \in J}$ be a net. There are groups of corresponding projective nets $D \xrightarrow{x_{\cdot j}} X_j$ where $\alpha \mapsto x_{\alpha_j}$ for $j \in J$.*
Then $x.$ converges to x in $\prod_{j \in J} X_j$ (equipped with the product topology) $\Leftrightarrow \forall j \in J, x_{\cdot j}$ converges x_j in X_j where $x_j = \pi_j(x)$.

Proof. $\Rightarrow$: Since $\prod_{j \in J} X_j \xrightarrow{\pi_k} X_k$ where $(x_j)_{j \in J} \mapsto x_k$ is continuous and $x_{\cdot k} = \pi_k(x.)$, then $x. \to x \in \prod_{j \in J} X_j \Rightarrow x_{\cdot k} = \pi_k(x.) \to \pi_k(x) = x_k$.
$\Leftarrow$: Recall that $\mathcal{B} := \{\prod_{j \in J} Y_j | Y_j \subseteq_{open} X_j (j \in J) \wedge \{j \in J | Y_j \neq X_j\}$ is finite$\}$ is a basis of the product space $\prod_{j \in J} X_j$. For any open nbd. U of x, there exists $\prod_{j \in J} Y_j \in \mathcal{B}$ s.t.

$$x \in \prod_{j \in J} Y_j \subseteq U$$

Let $J_0 = \{j \in J | Y_j \subsetneq X_j\}$, which is a finite set. $x_{\cdot j}$ converges to $x_j \in X_j \Rightarrow x_{\cdot j}$ lies in Y_j eventually i.e. $\exists \alpha_j \in D$, s.t. $\forall \alpha \in D, \alpha \geq \alpha_j \Rightarrow x_{\alpha_j} \in Y_j$ for all $j \in J_0$.
Choose $\tilde{\alpha} \in D$, s.t. $\tilde{\alpha} \geq \alpha_j$ for all $j \in J_0$, then for $D \ni \alpha \geq \tilde{\alpha}, x_{\alpha_j} \in Y_j$ for all $j \in J_0$ and hence for all $j \in J$. $\square$

Theorem 11 (Tychonoff Theorem)**.** *For compact space $X_j (j \in J)$ the product space $\prod_{j \in J} X_j =: X$ is also compact.*

Proof. Let $x.$ be a universal net in X, then $x_{\cdot j} = \pi_j(x.)$ is a universal net in X_j, for every $j \in J \Rightarrow x_{\cdot j}$ converges in X_j since X_j is compact $\Rightarrow x.$ converges by Lemma $\Rightarrow X$ is compact. $\square$

Chapter 7

Revisit Compactness

7.1 Generalization of Ascoli's Theorem

Recall that for metric spaces X and Y, a family $\mathcal{F}$ of maps from X to Y (i.e. Y - valued functions on X) is **equicontinuous** at a point $x_0 \in X$ if $\forall \epsilon > 0, \exists \delta > 0, \forall f \in \mathcal{F}$ and $x \in X$, $d(x_0, x) < \delta \Rightarrow d(f(x_0), f(x)) < \epsilon$. We now generalize this concept.

Definition 48 (Equicontinuous)**.** Let X be a topology space and Y be a metric space, a family $\mathcal{F}$ of maps from X to Y is equicontinuous at a point $x_0 \in X$ if $\forall \epsilon > 0, \exists$ open nbd. U of x_0 s.t. $\forall f \in \mathcal{F}$ and $x \in X$, $x \in U \Rightarrow d(f(x_0), f(x)) < \epsilon$.

Definition 49 (Point-wise convergence)**.** Let X, Y be metric spaces, and $X \xrightarrow{f_n} Y (n \in \mathbb{N})$ is a sequence of functions, then f_n converges point-wise to $X \xrightarrow{f} Y$ if for $\forall x \in X$ one has $f_n(x) \to f(x)$ as $n \to \infty$.

Definition 50 (Uniform convergence)**.** Let X, Y be metric spaces, a sequence of functions $X \xrightarrow{f_n} Y (n \in \mathbb{N})$ converges uniformly to $X \xrightarrow{f} Y$ if for $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that for $\forall n \geq N$ and $\forall x \in X$ one has $d(f_n(x), f(x)) < \epsilon$.

Definition 51 (Compact convergence)**.** Let $(X, \mathscr{T})$ be a topological space and (Y, d_Y) be a metric space. A sequence of functions $X \xrightarrow{f_n} Y (n \in \mathbb{N})$ is said to converge compactly to some function $X \xrightarrow{f} Y$ if, for every compact set $K \subseteq X$, $f_n|_K \to f|_K$ uniformly.

Theorem 12 (A generalization of Ascoli's theorem)**.** *Let X be a topology space and $\mathcal{F}$ be a family of $\mathbb{R}$ - valued functions on X, if*
1. *X is separable;*
2. *$\mathcal{F}$ is equicontinuous for $\forall x \in X$;*
3. *for $\forall x \in X$, $\{f(x) | f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{R}$,*

then every seq. in $\mathcal{F}$ has a subseq. which converges compactly, i.e. uniformly on every compact subset of X.

Proof. Let $A = \{a_1, a_2, \cdots\}$ be a countable dense subset, suppose $f_n (n \in \mathbb{N})$ is a seq. in $\mathcal{F}$.

Claim 1: $\exists$ subseq. $f_{n_m}(m \in \mathbb{N})$ which converges point-wise on A:

For $a_1 \in A$, we have that $\{f_n(a_1) | n \in \mathbb{N}\} \subseteq \{f(a_1) | f \in \mathcal{F}\} \subseteq_{bdd.} \mathbb{R}$. Then by Bolzano-Weierstrass theorem, there exists a $n_m^{(1)}(m \in \mathbb{N})$, which is strictly monotone, such that $f_{n_\cdot^{(1)}}(a_1)$ converges. Inductively, we can construct $n_m^{(j)}(m \in \mathbb{N})(j \in \mathbb{N}_0)$, and let $n_m^{(0)} = m$, such that

1. $n_\cdot^{(j)}$ monotone strictly;
2. $\{n_m^{(j)} | m \in \mathbb{N}\} \subseteq \{n_m^{(j-1)} | m \in \mathbb{N}\}$;
3. $f_{n_m^{(j)}}(a_j)$ converges as $m \to \infty$.

Let $n_m := n_m^{(m)}(m \in \mathbb{N})$, then $f_{n_m}(m = k, k+1, \cdots)$ is a subseq. of $f_{n_m^{(k)}}(m \in \mathbb{N})$ and hence $f_{n_m}(a_k)$ converges as $m \to \infty$ for every $k \in \mathbb{N}$.

Remark 54. For instance, $f_{n_m^{(2)}}$ is a subseq. of $f_{n_m^{(1)}}$ and $f_{n_m^{(1)}}(a_1)$ converges hence $f_{n_m^{(2)}}(a_1)$ converges as well. Thus $f_{n_m^{(2)}}(a_1)$ and $f_{n_m^{(2)}}(a_2)$ both converge.

Since given $j \in \mathbb{N}$, the `tail` of seq. $f_{n_m} = f_{n_m^{(m)}}$ is subseq. of $f_{n_m^{(j)}}$, for example, $f_{n_m}(m = 3, 4, \cdots)$ is subseq. of $f_{n_m^{(3)}}(m \in \mathbb{N})$, thus $f_{n_m}(a_j)$ converges for all $j \in \mathbb{N}$.

$$
\begin{array}{cccccc}
 & a_1 & a_2 & a_3 & a_4 & \cdots \\
f_1 & f_{n_1^{(1)}} & f_{n_1^{(2)}} & f_{n_1^{(3)}} & f_{n_1^{(4)}} & \cdots \\
f_2 & f_{n_2^{(1)}} & f_{n_2^{(2)}} & f_{n_2^{(3)}} & f_{n_2^{(4)}} & \cdots \\
f_3 & f_{n_3^{(1)}} & f_{n_3^{(2)}} & f_{n_3^{(3)}} & f_{n_3^{(4)}} & \cdots \\
f_4 & f_{n_4^{(1)}} & f_{n_4^{(2)}} & f_{n_4^{(3)}} & f_{n_4^{(4)}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Figure 7.1: $f_m(m \in \mathbb{N}), a_j(j \in \mathbb{N})$

Claim 2: $\forall \epsilon > 0$ and $x \in X, \exists$ (open) nbd. U_x of x in X and a number $N_x > 0$ s.t. if $x' \in U_x$ and $k, l \geq N_x \Rightarrow |f_{n_k}(x') - f_{n_l}(x')| < \epsilon$.

Since $\mathcal{F}$ is equicontinuous at x, thus for $\forall \epsilon > 0, \exists$ (open) nbd. U_x of x, s.t. $|f(z) - f(x)| < \epsilon/6$ for $f \in \mathcal{F}, z \in U_x$. Since $A \subseteq_{dense} X, \exists a \in U_x \cap A$. For any $x' \in U_x$ we have that

$$\begin{aligned}|f_{n_k}(x') - f_{n_l}(x')| &\le |f_{n_k}(x') - f_{n_k}(x)| \\ &+ |f_{n_k}(x) - f_{n_k}(a)| \\ &+ |f_{n_k}(a) - f_{n_l}(a)| \\ &+ |f_{n_l}(a) - f_{n_l}(x)| \\ &+ |f_{n_l}(x) - f_{n_l}(x')| \\ &< |f_{n_k}(a) - f_{n_l}(a)| + \frac{2}{3}\epsilon.\end{aligned}$$

since $f_{n.}(a)$ converges $\Rightarrow \exists N_x > 0$, s.t. $\forall k, l \ge N \Rightarrow |f_{n_k}(a) - f_{n_l}(a)| < \epsilon/3 \Rightarrow |f_{n_k}(x') - f_{n_l}(x')| < \epsilon$.
Claim 3: $\forall K \subseteq_{cpt.} X, f_{n_m}|_K (m \in \mathbb{N})$ converges uniformly.
For any given $\epsilon > 0$, we have found U_x and N_x as in Claim 2, $K = \cup_{x \in K}\{x\} \subseteq \cup_{x \in K} U_x, K \subseteq_{cpt.} X \Rightarrow \exists x_1, \cdots, x_p$, s.t. $K \subseteq U_{x_1} \cup \cdots \cup U_{x_p}$. Let $N = \max\{N_{x_1}, \cdots, N_{x_p}\}$, then for any $q \in K$ and $k, l \ge N$ we have $|f_{n_k}(q) - f_{n_l}(q)| < \epsilon$. Thus $f_{n_m}(q) (m \in \mathbb{N})$ is a Cauchy seq. in compact set $K \Rightarrow f_{n_m}(q) \to f(q)$ as $m \to \infty$, thus $f_{n.}$ converges uniformly in K. $\square$

Remark 55. The condition (3) is equivalent to that $\mathcal{F}$ is uniformly bounded when X is assumed to be compact and $\mathcal{F}$ is equicontinuous everywhere.

7.2 Relatively Compact

Definition 52. Let X be a topology space and $A \subseteq X$, A is relatively compact if $\overline{A}$ is compact.

Example 14. Every subset of a compact subset of a Hausdorff space is relatively compact: Suppose X is Hausdorff, $Y \subseteq_{cpt.} X \Rightarrow Y \subseteq_{close} X$. For $\forall Z \subseteq Y, \overline{Z} \subseteq \overline{Y} = Y$. And since $\overline{Z} \subseteq_{close} Y$, Z is compact $\Rightarrow \overline{Z}$ is compact.

Exercise 75. *Let (X, d) is a metric space, $A \subseteq X$, show that A is rel. cpt. $\Leftrightarrow$ any seq. in A has a subseq. which converges in X.*

Proof. $\Rightarrow$: A is rel. cpt. $\Rightarrow \overline{A}$ is cpt. $\Leftrightarrow \overline{A}$ is sequential compact $\Rightarrow$ every seq. in $\overline{A}$ converges $\Rightarrow$ every seq. in A converges in $\overline{A}$ (or in X).
$\Leftarrow$: Suppose that $\overline{A}$ is not compact then there is a seq. $\{a_n\}$ in $\overline{A}$ which is not convergent. So then for each $n \in \mathbb{N}$ define $A_n := A \cap B_{\frac{1}{n}}(a_n) \ne \emptyset$. Then pick a b_n from each A_n so that $\{b_n\}$ is a sequence in A, where for any $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that for any $n > N$,

$$d(a_n, b_n) < 1/n < \epsilon.$$

Then $\{b_n\}$ has a convergent subseq. $\{b_{n_k}\}$ with limit b by assumption. Thus for any $\epsilon > 0$, there exists a $K \in \mathbb{N}$ such that for all $k > K$,

$$d(a_{n_k}, b) \leq d(a_{n_k}, b_{n_k}) + d(b_{n_k, b}) < \epsilon/2 + \epsilon/2,$$

where $\{a_{n_k}\}$ is the corresponding subseq. of $\{a_n\}$, thus $a_{n_k} \to b$ as $k \to \infty$, implying $\overline{A}$ is seq. cpt. and hence cpt. and contradicting the supposition. $\square$

Remark 56. At first glance, the definition of A being **rel. cpt.** appears to be the same as **seq. cpt.** (which is equivalent to cpt. in metric space), but there is a difference:
the subseq. are required to converge in X (or $\overline{A}$ since it is closed), not necessarily in A, while actual seq. cpt. does require it to be in A. (more)

Example 15. Let X be a compact topology space, $C(X, \mathbb{R}) := \{X \xrightarrow{f} \mathbb{R} | f \text{ if continuous}\}$ and $d_{\text{sup}} := \sup_{x \in X} |f(x) - g(x)|$ ($= \max_{x \in X} |f(x) - g(x)|$ since X is compact)
Then $(C(X, \mathbb{R}), d_{\text{sup}})$ is a complete metric space, and $f_n (n \in \mathbb{N})$ converges w.r.t. $d_{\text{sup}} \Leftrightarrow$ f_n converges uniformly on $X \Leftrightarrow f_n$ is uniformly Cauchy seq.
By the generalization of Ascoli's theorem, when X is compact and separable, $\mathcal{F} \subseteq C(X, \mathbb{R})$ is equiconti. and uniformly bdd. (or satisfies condition 3.) $\Rightarrow \mathcal{F}$ is rel. cpt. by Ex1.