Game Theory I (Lecture 2)

Mixed-Strategy Nash Equilibrium

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This is a note I took while studying the Coursera course: Game Throry I taught by Prof. Matthew O. Jackson, Prof. Yoav Shoham at Stanford University and Prof. Kevin Leyton-Brown at The University of British Columbia. You can click here for more notes.

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1 Mixed Strategies and Nash Equilibrium

It would be a bad idea to play any deterministic strategy in some game, such as Matching pennies game. Since if player 1 declare action H, then the BR for player 2 is T, then the BR for player 1 is T, then the BR for player 2 is H,... This iteration is an infinite loop.

But player can confuse the opponent by playing randomly. For instance, Player 1 could choose action by taking the result of flipping a coin.

Definition 1.1 (Strategy)**.** *strategy s_i for agent i is a probability distribution over the actions A_i. If there exists only one action $a_i' \in A_i$ with probability 1, then call this distribution* ***Pure Strategy****. If there are more than one actions is played with positive probability, then the distribution is called* ***Mixed Strategy****.*

For example, if player choose action C in the prisoner's dilemma, then it is a pure strategy. In this case, the strategy is same with choose an action, as we declare before. If player say he will choose action C with probability .3, and D with probability .7, then it is a mixed strategy.

Define that the set of actions with positive probability in a mixed strategy **support** of the mixed strategy. Denote the probability of action a_i in support of player i's mixed strategy s_i by $s_i(a_i)$. Let the set of all strategies for player i be S_i; and the set of all strategy profile be $\mathcal{S} = S_1 \times S_2 \times \cdots \times S_n$.

If all players follow mixed strategy profile $s \in \mathcal{S}$, then we can not read the payoff of each one from the game matrix anymore. Instead, we use the idea of expected utility to calculate payoff. Review that we define the utility in previous chapter as $u_i(a_i, a_{-i}) = u_i(a)$.

Definition 1.2 (Expected utility)**.** *The expected utility of player i given the strategy profile $s \in \mathcal{S}$ is*

$$u_i(s) = \sum_{a \in \mathcal{A}} u_i(a) \cdot \mathbb{P}\left(a|s\right),$$

where $a = \langle a_1, a_2, \cdots, a_n \rangle$ and $\mathbb{P}\left(a|s\right) = \prod_{j \in N} s_j(a_j)$.

Notice that the expected utility of s calculate all possible action profile $a \in \mathcal{S}$.

Example 1.1. *Consider each player in Matching pennies game chooses H with probability .5 and T with .5. The strategy profile is*

$$s = \langle s_1, s_2 \rangle = \langle (\mathbb{P}_1(H) = 0.5, \mathbb{P}_1(T) = 0.5), (\mathbb{P}_2(H) = 0.5, \mathbb{P}_2(T) = 0.5) \rangle,$$

then the expected utility of player 1 given strategy profile s is

$$\begin{aligned} u_1(s) = &\sum_{a \in \mathcal{A}} u_i(a) \cdot \mathbb{P}\left(a|s\right) \\ = &u_1(H, H) \cdot \mathbb{P}\left(H, H|s\right) + u_1(H, T) \cdot \mathbb{P}\left(H, T|s\right) \\ &+ u_1(T, H) \cdot \mathbb{P}\left(T, H|s\right) + u_1(T, T) \cdot \mathbb{P}\left(T, T|s\right) \end{aligned}$$

where $\mathbb{P}\left(H, H|s\right) = s_1(a_1) \cdot s_2(a_2) = s_1(H) \cdot s_2(H) = 0.25$, others are the same. In summary $u_1(s) = 0$.

Definition 1.3 (Best response)**.** *We say s_i^* is the Best responce for player i given s_{-i}, denote by $s_i^* \in BR_i(s_{-i})$, iff $\forall s_i \in S_i$, $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$.*

Definition 1.4 (Nash equilibrium)**.** *$s = \langle s_1, s_2, \cdots, s_n \rangle$ is a Nash equilibrium iff $\forall i, s_i \in BR_i(s_{-i})$.*

In a mixed strategy Nash equilibrium, Player 1 should select strategy so that Player 2 feels **indifferent** for the actions in his support, which means whatever strategy player 2 chooses, the expected utility is always constant. And player 2 chooses a strategy from his indifferent support so that Player 1 feels indifferent for the actions in his support.

2 Nash Theorem

To prove Nash Theorem, we need to add some mathematical concepts.

Theorem 2.1 (Nash Theorem)**.** *Every finite game with mixed strategy has a Nash equilibrium.*

3 Examples

We now calculate the Nash equilibrium of the Matching pennies game with mixed strategy.

Example 3.1 (Matching pennies game)**.** *Assume the general strategy profile s has following structure*

	H	T
Player 1	α	$1-\alpha$
Player 2	β	$1-\beta$

Then the expected utility of Player 1 is

$$\begin{aligned} u_1(s) &= \sum_{a \in \mathcal{A}} u_1(a) \cdot \mathbb{P}(a|s) \\ &= 4\alpha\beta - 2\alpha - 2\beta + 1 \end{aligned}$$

And correspondingly,

$$u_2(s) = -4\alpha\beta + 2\alpha + 2\beta - 1,$$

For any β player 2 decided, player 1 will choose α that maximize $u_1(s)$ and minimize $u_2(s)$ at the same time since $u_1(s) = -u_2(s)$. Thus player 2 will change β if player 1 chooses α, unless $\exists \beta'$ such that u_1 is constant whatever α is. and let $\alpha = \beta'$, then u_2 is constant. thus (β', β') is Nash equilibrium. Solve the equation, we have $\beta' = 0.5$, thus the strategy profile

	H	T
Player 1	0.5	0.5
Player 2	0.5	0.5

is Nash equilibrium.

Example 3.2 (BoS)**.** *Review the Battle of sexes game*

		β B	$1-\beta$ F	
α	B	2,1	0,0	$\mathbb{E}_{1B} = \beta \cdot 2 + (1-\beta) \cdot 0$
$1-\alpha$	F	0,0	1,2	$\mathbb{E}_{1F} = \beta \cdot 0 + (1-\beta) \cdot 1$
		$\mathbb{E}_{2F} = \alpha \cdot 1 + (1-\alpha) \cdot 0$	$\mathbb{E}_{2F} = \alpha \cdot 0 + (1-\alpha) \cdot 2$	

If player 1 best-responds with a mixed strategy, player 2 must choose β to make player 1 be indifferent between F and B a.k.a. $\mathbb{E}_{1B} = \mathbb{E}_{1F}$, otherwise, assume $\mathbb{E}_{1B} > \mathbb{E}_{1F}$, player 1 will prefer B than F and will add more probability from F to B until the mixed strategy degenerate to pure strategy. Correspondingly, player 1 should make $\mathbb{E}_{2B} = \mathbb{E}_{2F}$. Thus $\alpha = 2/3$ and $\beta = 1/3$. according to the definition,

$$
\begin{aligned}
u_1(s) &= \sum_{a \in \mathcal{A}} u_1(a) \cdot s_1(a_1) \cdot s_2(a_2) \\
&= \sum_{a_1 \in A_1} \mathbb{E}_{1a_1} \cdot s_1(a_1) \\
&= s_1(B) \cdot \mathbb{E}_{1B} + s_1(F) \cdot \mathbb{E}_{1F} \\
&\overset{\mathbb{E}_{1B}+\mathbb{E}_{1F}}{=\!=\!=\!=} \mathbb{E}_{1B} \cdot (s_1(B) + s_1(F)) \\
&= \mathbb{E}_{1B}.
\end{aligned}
$$

Easy to check, if $\alpha = 2/3$, then $u_2(s)$ is constant, then $\beta = 1/3$ is one of the BR, or vice versa. thus $s = \langle \alpha = 2/3, \beta = 1/3 \rangle$ is Nash equilibrium.

Example 3.3 (Soccer Penalty Kicks)**.** *Let's consider an example of Mixed strategies in sports and competitive games, Soccer Penalty Kicks. Assume player 1 is kicker and player 2 is goalie, who form the following game*

	Left	Right
Left	0, 1	1, 0
Right	1, 0	0,1

This means if kicker and goalie choose the same direction, then the goalie will keep the door, otherwise the kicker will shoot successfully. But what if the kicker kicks worse to the right than left, for example

	Left	Right
Left	0, 1	1, 0
Right	.75, .25	0,1

The Nash equilibrium is easy to calculate:

	Left	Right
Kicker	4/7	3/7
Goalie	3/7	4/7

We can see by adjusting the strategy to keep the opponent indifferent, the Goalie takes advantage of the kicker's weak right kick and wins more often.

Example 3.4. *There are 2 firms, each advertising an available job opening. Firms offer different wages: Firm 1 offers* $w_1 = 4$ *and 2 offers* $w_2 = 6$. *There are two unemployed workers looking for jobs. They simultaneously apply to either of the firms. If only one worker applies to a firm, then he/she gets the job. If both workers apply to the same firm, the firm 1 hires a worker 1 with probability* α *(worker 2 with* $1-\alpha$*)and the firm 2 hires a worker 1 with probability* β *(worker 2 with* $1-\beta$*), the other worker remains unemployed (and receives a payoff of 0).*

Find a mixed strategy Nash Equilibrium where p is the probability that worker 1 applies to firm 1 and q is the probability that worker 2 applies to firm 1.

If worker 1 and 2 both apply firm 1, then the expected wage for worker 1 is $4\alpha + 0(1-\alpha) = 4\alpha$*; the expected wage for worker 2 is* $0\alpha + 4(1-\alpha) = 4(1-\alpha)$. *The situation that both workers choose firm 2 is similar. Thus we have game matrix*

	Firm 1	Firm 2
Firm 1	4α, $4(1-\alpha)$	4,6
Firm 2	6,4	6β, $6(1-\beta)$

thus we can get the Nash Equilibrium is $p = \frac{3\beta-1}{2\alpha+3\beta}$ *and* $q = \frac{3\beta-2}{2\alpha+3\beta-5}$. *If* $\alpha = \beta = .5$, *then the Nash Equilibrium is* $p = .5$ *and* $q = .5$.