Game Theory I (Lecture 1)
Introduction and Overview

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This is a note I took while studying the Coursera course: Game Throry I taught by Prof. Matthew O. Jackson, Prof. Yoav Shoham at Stanford University and Prof. Kevin Leyton-Brown at The University of British Columbia. You can click here for more notes.

Contents

1 Key Notation

We say the agent is **self-interested** means that the agents have different preference and recognize the world in different description. **Utility function** is a mathematical measure that tells you how much an agent like or dislike a given situation.

Utility function describes not only their attitude towards a definite of events, but also the preferences towards a distribution of outcomes. So it captures agents' attitude towards the uncertainty of events. For example, if you have been told that it will be 25°C with probability .7 and 24°C with probability .3, you might have an opinion about how much you like that versus other distributions (Like 0°C with 0.5 and 50°C with 0.5). And you are going to try to act in the way that maximizes your expected utility.

The **Players** in a game are persons who are decision makers, They may be Trader, government, companies and so on. The **Action** means what can players do. Like enter a bid in an auction, decide whether to end a strike, decide when to sell a stock, etc. The **Payoff** represents what motivate players, i.e. the factors that drive the utility function.

There are two standard representations of games. One is the **Normal Form** (a.k.a. the Matrix form). It lists what payoffs players get as a function of their actions. It as if players moved simultaneously. The other alternative representation is known as the **Extension Form**. It includes more explicit timing in the game. The players move sequentially, so it is represented often as a tree.

Now let's define a normal form game. Consider a finite n **Players set** $\mathcal{N} = \{1, 2, \cdots, n\}$, the generic player is indexed by i. The **Action set** for player i is represented by A_i. And $a = (a_1, a_2, \cdots, a_n) \in \mathcal{A} = A_1 \times A_2 \times \cdots \times A_n$ is an **action profile**. The **utility function** or payoff function for player i is $u_i : \mathcal{A} \mapsto \mathbb{R}$ and $u = (u_1, u_2, \cdots, u_n)$ is a **profile of utility functions**. Notice that the others players' actions will affect the utility of player i.

For a 2-players game matrix, the horizontal direction represents player 1, and the vertical direction represents player 2. Each row corresponds to action $a_1 \in A_1$, each column corresponds to action $a_2 \in A_2$. The cells list utility or payoff for each player with the player 1 first and then the 2.

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

Figure 1: TCP Backoff Game

In this TCP Game, we have players set $\mathcal{N} = \{1, 2\}$, action set $A_1 = A_2 = \{C, D\}$ and utility function $u_1(C, C) = -1, u_1(C, D) = -4, u_1(D, C) = 0, u_1(D, D) = -3$. The case of u_2 is corresponding.

2 Examples of Games

Prisoner's dilemma The most famous game is Prisoner's dilemma, which has form like

	C	D
C	a, a	b, c
D	c, b	d, d

Figure 2: Prisoner's dilemma

with $c > a > d > b$. The two prisoners can either Cooperate or Defect. If they both cooperate, they get some payoff a. If they both defect, they get a different payoff d, where a is greater than d. However, if they miscoordinate, then the cooperator gets the lowest possible payoff b and the defector gets the largest possible payoff a.

Pure Competition In a pure competition game, there must be two players, where one player's payoff is exactly the compliment of the another player's payoff. For any action profile $a \in \mathcal{A}$ we have

$$u_1(a) + u_2(a) = Const,$$

if $Const. = 0$, we call it zero sum game. Thus we only need to store a utility function for one player. An example of pure competition game is Matching Pennies: if two players matched then player 1 gets 1 payoff and play 2 gets -1 payoff or vice versa.

	Heads	**Tails**
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

Figure 3: Matching Pennis

Pure Cooperation In this case, all agents have exactly the same interest, or say, the payoffs for every action vector is the same, i.e. for $\forall a \in \mathcal{A}$, $\forall i, j \in \mathcal{N}$,

$$u_i(a) = u_j(a).$$

The expression of this property in matrix is that every cell in the matrix has two same payoff values. An example of pure cooperation game is called walking side game whose rule is Two passers-by walking towards each other, if both pass on their left or right side, each get 1 payoff, otherwise neither get payoff.

	Left	Right
Left	$1,1$	$0,0$
Right	$0,0$	$1,1$

Figure 4: Walking side game

Battle of Sexes In general, games will be neither purely cooperative nor purely competitive. This game says a couple are going to see a movie, and there are two candidate movies "Battle of Armageddon" and "Flower Child".

	B	F
B	$2,1$	$0,0$
F	$0,0$	$1,2$

Figure 5: Battle of Sexes

The non-competition is that a decrease payoff of one player doesn't necessarily leads to an increase payoff of another player, it can reduce that too. The non-cooperation is that two players clearly have conflicting preference.

3 Nash Equilibrium Intro

If you knew what everyone else was going to do, then the best action you can do is called the **Best response**. Technically speaking, let $a_{-i} = \langle a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n \rangle$, then we can say actions profile $a = (a_i, a_{-i})$.

Definition 3.1 (Best response). $a_i^* \in BR_i(a_{-i}) \iff \forall a_i \in A_i, u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$.

Notice that we say a_i^* is a Best response for player i with a_{-i}, which means a Best response a_i^* is a notation corresponds to a specific action vector a_{-i}.

Definition 3.2 (Nash equilibrium)**.** *$a = \langle a_1, a_2, \cdots, a_n \rangle$ is a (pure strategy) Nash equilibrium $\iff \forall i, a_i \in BR_i(a_{-i})$.*

Note 3.1. *Notice that $BR_i : \mathcal{A} \mapsto 2^{A_i}$ is a set-value function, which means Best response of a_{-i} maybe more than 1. For the sake of analysis, we let $BR_i(a_{-i})$ denote some specific best response action, although this is not rigorous.*

The definition is abstract, and we will clarify it from a simple Nash equilibrium with only two players. For player 2, if $\exists a_2 \in A_2$ s.t. (such that)

$$BR_2(BR_1(a_2)) = a_2,$$

then we say $\langle BR_1(a_2), a_2 \rangle$ is a Nash equilibrium of the game. The equation means that the special actions profile $\langle BR_1(a_2), a_2 \rangle$ reaches a stable status where everybody's action matches Best response given by the others Best response action. Obviously, if equation $BR_2(BR_1(a_2)) = a_2$ holds, define $a_1 = BR_1(a_2)$, then

$$\begin{aligned} BR_1(BR_2(a_1)) &= BR_1(BR_2(BR_1(a_2))) \\ &= BR_1(a_2) \\ &= a_1. \end{aligned}$$

Now, extend it to the general n-Players case. If there is an action profile $a = \langle a_1, a_2, \cdots, a_n \rangle$, define $a_{-i} = \langle a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, \cdots, a_n \rangle$, such that

$$\langle BR_1(a_{-1}), BR_1(a_{-2}), \cdots, BR_1(a_{-n}) \rangle = a,$$

then we say a is a Nash equilibrium.

Let's see some Nash equilibrium cases.

Example 3.1 (The prisoner's dilemma)**.** *For prisoner's dilemma:*

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

Figure 6: The prisoner's dilemma

we have that

$$\begin{aligned} BR_2(BR_1(C)) &= D \neq C; \\ BR_1(BR_2(C)) &= D \neq C; \\ BR_2(BR_1(D)) &= D, \end{aligned}$$

Thus $\langle C, D \rangle, \langle D, C \rangle, \langle C, C \rangle$ are neither Nash equilibrium, whereas $\langle D, D \rangle$ is the Nash equilibrium.

Example 3.2 (Walking side game). *This is a game of pure coordination.*

	Left	Right
Left	$1,1$	$0,0$
Right	$0,0$	$1,1$

Figure 7: Walking side game

We have that

$$BR_2(BR_1(L)) = L$$
$$BR_1(BR_2(R)) = R,$$

Thus $\langle R, R\rangle, \langle F, F\rangle$ *are Nash equilibriums, while* $\langle R, L\rangle, \langle L, R\rangle$ *are not.*

Example 3.3 (Matching pennies game). *Matching pennies game is a special one*

	Heads	Tails
Heads	$1,-1$	$-1,1$
Tails	$-1,1$	$1,-1$

Figure 8: Matching pennies

we can see

$$BR_2(BR_1(T)) = H \neq T$$
$$BR_2(BR_1(H)) = T \neq H$$
$$BR_1(BR_2(T)) = H \neq T$$
$$BR_1(BR_2(H)) = T \neq H,$$

for any action profile we can not match the equation $BR_2(BR_1(a_2)) = a_2$*, that means this game doesn't have Nash equilibrium.*

Example 3.4. *Two firms produce identical goods, with a production cost of* $c > 0$ *per unit. Each firm sets a nonnegative price (*p_1 *and* p_2*). All consumers buy from the firm with the lower price, if* $p_1 \neq p_2$*. Half of the consumers buy from each firm if* $p_1 = p_2$*. D is the total demand. Profit of firm i is:*

- 0*, if* $p_i > p_j$*;*
- $D \cdot (p_i - c)$*, if* $p_i < p_j$*;*
- $D \cdot (p_i - c)/2$*, if* $p_i = p_j$*.*

For any price p_j from firm j, the BR of firm i is the price p_i that is a little bit lower than p_j. And for this p_i, the BR of firm j is the price p'_j that is a little bit lower than p_i. Iterate this process until converging at $p_i = p_j = c$.

Easy to know, if $p_j = c$, then $BR_i(c)$ is any price that greater than or equal to c, or vice versa. This means equation

$$BR_j(BR_i(c)) = c$$

holds, thus $\langle c, c\rangle$ is Nash equilibrium.

4 Dominant Strategies

Let s_i and s'_i be two strategies for player i, and let S_{-i} be the set of all possible strategy profiles for the other players. For now, let's comprehend strategy as an action.

Definition 4.1 (Strictly dominates). *s_i strictly dominates s'_i if $\forall s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.*

Definition 4.2 (weakly dominates). *s_i weakly dominates s'_i if $\forall s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.*

If one strategy dominates all others, we say it is **dominant**. In that case s_i is better than everything else. if you have a dominant strategy then basically you don't have to worry about what the other agents are doing in the game at all, you can just play your dominant strategy and that's going to be the best thing for you to do.

Note 4.1. *Best response is your max utility strategy for a certain action vector s_{-i}; And dominant is your max utility strategy for any action vector $\forall s_{-i} \in S_{-i}$.*

Dominant means that the Best response strategy doesn't change no matter what the other players does. If Best response changes with the other party's decision, then there is no dominant.

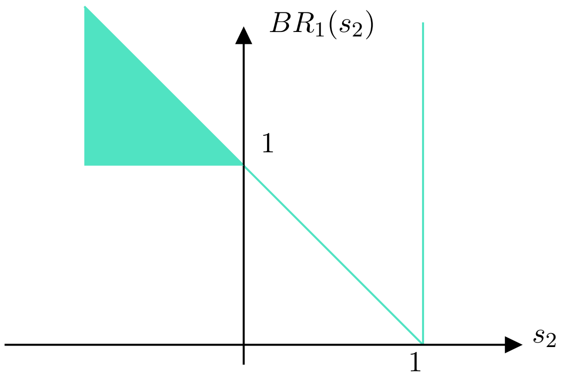
If the BR does not change as other players strategy changes, but when the other players choose some particular strategies, there are two (or more) BR, then there is a weekly dominant.

Example 4.1. *Consider prisoner's dilemma again, we have*

$$BR_1(C) = D;$$
$$BR_1(D) = D,$$

i.e. $\langle D\rangle$ is the BR for player 1 all the time whatever player 2 decides, thus $\langle D\rangle$ is the dominant for player 1 and the same-thing for player 2.

Example 4.2 (Nash Bargaining). *There are 2 players who have to decide how to split one dollar. The bargaining process works as follows. Players simultaneously announce the share they would like to receive s_1 and s_2, with $s_1 \geq 0, s_2 \leq 1$. If $s_1 + s_2 \leq 1$, the* *players receive the shares they named and if $s_1 + s_2 > 1$, then both players fail to achieve an agreement and receive zero.*



Player 1's BR as is shown in the figure, BR is not an consistent strategy, thus player 1 has no dominant in the game and same thing with player 2.

However if player 2 bids $\alpha \in (0,1)$, then BR for player 1 would be $1-\alpha$; and if player 1 bid $1-\alpha$ then BR for player 2 is α, that means

$$BR_2(BR_1(\alpha)) = \alpha, \forall \alpha \in (0,1)$$

i.e. strategy profile $\langle \alpha, 1-\alpha \rangle$, $(\alpha \in (0,1))$ is Nash equilibrium.

Furthermore, if player bids $\alpha \leq 0$, then BR for player 1 is any number $\beta \in [1, 1+|\alpha|]$; and if player 1 bids any number $\beta \in [1, 1+|\alpha|]$, the BR for player 2 is any number $s_2 \in (-\infty, 1]$, and α matches this condition. Thus

$$BR_2(BR_1(\alpha)) = \alpha, \forall \alpha \leq 0$$

i.e. $\langle \alpha, \beta \rangle$, $(\alpha \leq 0, \beta \in [1, 1+|\alpha|])$ is Nash equilibrium, like $\langle 0, 1 \rangle$ or $\langle -5, 2 \rangle$.

A strategy profile in which everybody is playing a dominant strategy has to be in Nash equilibrium . So, if everyone is playing a dominant strategy, then we've just got a Nash equilibrium, because everyone reach the best response. Furthermore, if we all have a strictly dominant strategies then this equilibrium has got to be unique, because there can't be two equilibriums strictly dominant strategies.

Note 4.2. *The existence of Nash equilibrium depends only on BR (dominant is only sufficient condition), so for any (maybe only one) player 2's strategy s_2, if $BR_2(BR_1(s_2)) = s_2$, then $(BR_1(s_2), s_2)$ is a Nash equilibrium.*

5 Pareto Optimality

From the perspective of an outside observer, can we say some outcomes of a game is better than others?

Definition 5.1 (Pareto-dominates)**.** *We say outcome o Pareto-dominates o', if o is at least as good for every agent as another outcome o', and there is some agent who strictly prefers o to o'.*

Definition 5.2 (Pareto-optimal)**.** *An outcome o^* is Pareto-optimal if there is no other outcome that Pareto-dominates o^*.*

According to the definition of Pareto dominance, it is easy to know that there may be more than one Pareto optimality in a game and every game have at least one Pareto-optimal outcome.

Now review the prisoner's dilemma game :

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

the outcomes $(-1, -1), (-4, 0), (0, -4)$ are all Pareto-optimal, however the Nash equilibrium $(-3, -3)$ is Pareto-dominated by $(-1, -1)$. So, almost everything in this game is kind of good from a social perspective (Pareto-optimal) and the only other thing in the game is the thing that we strongly predict ought to happen. So, that's why we think the prisoner's dilemma is such a dilemma.