Adaptation of my equations to a subdivided population. Notation, for a quantity Y that depends on two sites ($Y = e, d, Q$):

$$Y_{\text{self}} = Y_{i,i} \tag{1a}$$

$$Y_{\text{in}} = Y_{i,j}, \quad i \text{ and } j \neq i \text{ in the same deme;} \tag{1b}$$

$$Y_{\text{out}} = Y_{i,j}, \quad i \text{ and } j \text{ in different demes.} \tag{1c}$$

For a site i, G_i denotes the deme it is in, and notation $j \in G_i$ means that sites i and j are in the same deme.

The expected frequency of altruists in the population is given by

$$\mathbb{E}\left[\overline{X}\right] = p + \delta \frac{p(1-p)}{\mu} \left[\mathsf{b}\,(\beta^D - \beta^I) - \mathsf{c}\,(\gamma^D - \gamma^I)\right]. \tag{2}$$

Moran, Birth-Death

$$\begin{aligned}
\beta^D_{\text{BD}} &= \sum_{k,\ell=1}^{N} \frac{1-\mu}{N} e_{kl} Q_{lk} \\
&= \sum_{k=1}^{N} \frac{1-\mu}{N} \Big(e_{\text{self}} + (n-1)\, e_{\text{in}} Q_{\text{in}} + (N-n)\, e_{\text{out}} Q_{\text{out}} \Big) \\
&= (1-\mu) \Big(e_{\text{self}} + (n-1)\, e_{\text{in}} Q_{\text{in}} + (N-n)\, e_{\text{out}} Q_{\text{out}} \Big).
\end{aligned} \tag{3a}$$

$$
\begin{aligned}
\beta^I_{\mathrm{BD}} &= \sum_{j,k,l=1}^{N} \left(\frac{d_{lj}}{N} - \frac{\mu}{N^2}\right) e_{kl} Q_{jk} \\
&= \frac{1}{N} \sum_{j=1}^{N} \left[\left(\sum_{l=1}^{N} d_{lj} e_{jl} - \frac{\mu}{N} \right) + \sum_{\substack{k \in G_j \\ k \neq j}} \left(\sum_{l=1}^{N} d_{lj} e_{kl} Q_{\mathrm{in}} - \frac{\mu}{N} Q_{\mathrm{in}} \right) + \sum_{k \notin G_j} \sum_{l=1}^{N} d_{lj} \left(e_{kl} Q_{\mathrm{out}} - \frac{\mu}{N} Q_{\mathrm{out}} \right) \right] \\
&= \frac{1}{N} \sum_{j=1}^{N} \Bigg[d_{\mathrm{self}} e_{\mathrm{self}} + (n-1) d_{\mathrm{in}} e_{\mathrm{in}} + (N-n) d_{\mathrm{out}} e_{\mathrm{out}} \\
&\quad + \sum_{\substack{k \in G_j \\ k \neq j}} \left(d_{\mathrm{in}} e_{\mathrm{self}} + d_{\mathrm{self}} e_{\mathrm{in}} + (n-2) d_{\mathrm{in}} e_{\mathrm{in}} + (N-n) d_{\mathrm{out}} e_{\mathrm{out}} \right) Q_{\mathrm{in}} \\
&\quad + \sum_{k \notin G_j} \left(d_{\mathrm{self}} e_{\mathrm{out}} + (n-1) d_{\mathrm{in}} e_{\mathrm{out}} + d_{\mathrm{out}} e_{\mathrm{self}} + (n-1) d_{\mathrm{out}} e_{\mathrm{in}} + (N-2n) d_{\mathrm{out}} e_{\mathrm{out}} \right) Q_{\mathrm{out}} \Bigg] \\
&\quad - \frac{\mu}{N} \left(1 + (n-1) Q_{\mathrm{in}} + (N-n) Q_{\mathrm{out}} \right) \\
&= d_{\mathrm{self}} e_{\mathrm{self}} + (n-1) d_{\mathrm{in}} e_{\mathrm{in}} + (N-n) d_{\mathrm{out}} e_{\mathrm{out}} \\
&\quad + (n-1) \left(d_{\mathrm{in}} e_{\mathrm{self}} + d_{\mathrm{self}} e_{\mathrm{in}} + (n-2) d_{\mathrm{in}} e_{\mathrm{in}} + (N-n) d_{\mathrm{out}} e_{\mathrm{out}} \right) Q_{\mathrm{in}} \\
&\quad + (N-n) \left(d_{\mathrm{self}} e_{\mathrm{out}} + (n-1) d_{\mathrm{in}} e_{\mathrm{out}} + d_{\mathrm{out}} e_{\mathrm{self}} + (n-1) d_{\mathrm{out}} e_{\mathrm{in}} + (N-2n) d_{\mathrm{out}} e_{\mathrm{out}} \right) Q_{\mathrm{out}} \\
&\quad - \frac{\mu}{N} \left(1 + (n-1) Q_{\mathrm{in}} + (N-n) Q_{\mathrm{out}} \right). \qquad (3b)
\end{aligned}
$$

$$
\gamma^D_{\mathrm{BD}} = 1 - \mu. \qquad (3c)
$$

$$
\begin{aligned}
\gamma^I_{\mathrm{BD}} &= \frac{1}{N} \sum_{j,k=1}^{N} \left(d_{kj} - \frac{\mu}{N} \right) Q_{jk} \\
&= \frac{1}{N} \sum_{j=1}^{N} \left[d_{\mathrm{self}} - \frac{\mu}{N} + (n-1) \left(d_{\mathrm{in}} - \frac{\mu}{N} \right) Q_{\mathrm{in}} + (N-n) \left(d_{\mathrm{out}} - \frac{\mu}{N} \right) Q_{\mathrm{out}} \right] \\
&= d_{\mathrm{self}} + (n-1) d_{\mathrm{in}} Q_{\mathrm{in}} + (N-n) d_{\mathrm{out}} Q_{\mathrm{out}} \\
&\quad - \frac{\mu}{N} \left(1 + (n-1) Q_{\mathrm{in}} + (N-n) Q_{\mathrm{out}} \right) \qquad (3d)
\end{aligned}
$$

Moran, Death-Birth

$$\begin{aligned}\beta^D_{\mathrm{DB}} &= \frac{1-\mu}{N}\sum_{j,k=1}^{N} Q_{jk}e_{jk} = \beta^D_{\mathrm{BD}} \\ &= (1-\mu)\Big(e_{\mathrm{self}} + (n-1)e_{\mathrm{in}}Q_{\mathrm{in}} + (N-n)e_{\mathrm{out}}Q_{\mathrm{out}}\Big). \end{aligned} \tag{4a}$$

$$\begin{aligned}\beta^I_{\mathrm{DB}} &= \frac{1-\mu}{N}\sum_{i,j,k,l=1}^{N} d_{ji}d_{li}e_{kl}Q_{jk} \\ &= \frac{1-\mu}{N}\sum_{i,j,l=1}^{N}\Bigg[d_{ji}d_{li}e_{jl} + \sum_{\substack{k\in G_j \\ k\neq j}} d_{ji}d_{li}e_{kl}Q_{\mathrm{in}} + \sum_{k\notin G_j} d_{ji}d_{li}e_{kl}Q_{\mathrm{out}}\Bigg] \\ &= \frac{1-\mu}{N}\sum_{i,j,l=1}^{N}\Bigg[d_{ji}d_{li}e_{jl} \\ &\quad + \sum_{\substack{k\in G_j \\ k\neq j}} d_{ji}d_{li}e_{kl}Q_{\mathrm{in}} \\ &\quad + \sum_{k\notin G_j} d_{ji}d_{li}e_{kl}Q_{\mathrm{out}}\Bigg] \end{aligned} \tag{4b}$$

Appendix

All combinations for i, j, k, l. Notation: (i, j) means that i and j are in the same deme, but are different; G_i refers to the deme containing site i.

j	k	l	Notation	Count
$j = i$	$k = i$	$l = i$	$(l = k = j = i)$	1
$j = i$	$k = i$	$l \neq i; l \in G_i$	$(i = j = k, l)$	$n-1$
$j = i$	$k = i$	$l \notin G_i$	$(i = j = k), (l)$	$N-n$
$j = i$	$k \neq i; k \in G_i$	$l = i$	$(i = j = l, k)$	$n-1$
$j = i$	$k \neq i; k \in G_i$	$l = k$	$(i = j, k = l)$	$n-1$
$j = i$	$k \neq i; k \in G_i$	$l \neq i, k; l \in G_i$	$(i = j, k, l)$	$(n-1)(n-2)$
$j = i$	$k \neq i; k \in G_i$	$l \notin G_i$	$(i = j, k), (l)$	$(n-1)(N-n)$
$j = i$	$k \notin G_i$	$l = i = j$	$(i = j = l), (k)$	$(N-n)$
$j = i$	$k \notin G_i$	$l \neq i, l \in G_i$	$(i = j, l), (k)$	$(N-n)(n-1)$
$j = i$	$k \notin G_i$	$l = k$	$(i = j), (k = l)$	$(N-n)$
$j = i$	$k \notin G_i$	$l \neq k; l \in G_k$	$(i = j), (k, l)$	$(N-n)(n-1)$
$j = i$	$k \notin G_i$	$l \notin G_i, G_k$	$(i = j), (k), (l)$	$(N-n)(N-2n)$
$j \neq i, j \in G_i$	$k = i$	$l = i$	$(i = k = l, j)$	$(n-1)$
$j \neq i, j \in G_i$	$k = i$	$l = j$	$(i = k, j = l)$	$(n-1)$
$j \neq i, j \in G_i$	$k = i$	$l \neq i, j; l \in G_i$	$(i = k, j, l)$	$(n-1)(n-2)$
$j \neq i, j \in G_i$	$k = i$	$l \notin G_i$	$(i = k, j), (l)$	$(n-1)(N-n)$
$j \neq i, j \in G_i$	$k = j$	$l = i$	$(i = l, j = k)$	$(n-1)$
$j \neq i, j \in G_i$	$k = j$	$l = j$	$(i, j = k = l)$	$(n-1)$
$j \neq i, j \in G_i$	$k = j$	$l \neq i, j; l \in G_i$	$(i, j = k, l)$	$(n-1)(n-2)$
$j \neq i, j \in G_i$	$k = j$	$l \notin G_i$	$(i, j = k), (l)$	$(N-n)$
$j \neq i, j \in G_i$	$k \neq i, j; k \in G_i$	$l = i$	$(i = l, j, k)$	$(n-1)(n-2)$
$j \neq i, j \in G_i$	$k \neq i, j; k \in G_i$	$l = j$	$(i, j = l, k)$	$(n-1)(n-2)$
$j \neq i, j \in G_i$	$k \neq i, j; k \in G_i$	$l = k$	$(i, j, k = l)$	$(n-1)(n-2)$
$j \neq i, j \in G_i$	$k \neq i, j; k \in G_i$	$l \neq i, j, k; l \in G_i$	(i, j, k, l)	$(n-1)(n-2)(n-3)$
$j \neq i, j \in G_i$	$k \neq i, j; k \in G_i$	$l \notin G_i$	$(i, j, k), l$	$(n-1)(n-2)(N-n)$
$j \neq i; j \in G_i$	$k \notin G_i$	$l = i$	$(i = l, j), (k)$	$(n-1)(N-n)$
$j \neq i; j \in G_i$	$k \notin G_i$	$l = j$	$(i, j = l), (k)$	$(n-1)(N-n)$
$j \neq i; j \in G_i$	$k \notin G_i$	$l \neq i, j; l \in G_i$	$(i, j, l), (k)$	$(n-1)(N-n)(n-2)$
$j \neq i; j \in G_i$	$k \notin G_i$	$l = k$	$(i, j), (k = l)$	$(n-1)(N-n)$
$j \neq i; j \in G_i$	$k \notin G_i$	$l \neq k; l \in G_k$	$(i, j), (k, l)$	$(n-1)(N-n)(n-1)$
$j \neq i; j \in G_i$	$k \notin G_i$	$l \notin G_i, G_k$	$(i, j), (k), (l)$	$(n-1)(N-n)(N-2n)$

j	k	l	Notation	Count
$j \notin G_i$	$k = i$	$l = i$	$(i = k = l), (j)$	$(N-n)$
$j \notin G_i$	$k = i$	$l \neq i; l \in G_i$	$(i = k, l), (j)$	$(N-n)(n-1)$
$j \notin G_i$	$k = i$	$l = j$	$(i = k), (j = l)$	$(N-n)$
$j \notin G_i$	$k = i$	$l \neq j; l \in G_j$	$(i = k), (j, l)$	$(N-n)(n-1)$
$j \notin G_i$	$k = i$	$l \notin G_i, G_j$	$(i = k), (j), (l)$	$(N-n)(N-2n)$
$j \notin G_i$	$k \neq i; k \in G_i$	$l = i$	$(i = l, k), (j)$	$(N-n)(n-1)$
$j \notin G_i$	$k \neq i; k \in G_i$	$l = k$	$(i, k = l), (j)$	$(N-n)(n-1)$
$j \notin G_i$	$k \neq i; k \in G_i$	$l \neq i, k; l \in G_i$	$(i, k, l), (j)$	$(N-n)(n-1)(n-2)$
$j \notin G_i$	$k \neq i; k \in G_i$	$l = j$	$(i, k), (j = l)$	$(N-n)(n-1)$
$j \notin G_i$	$k \neq i; k \in G_i$	$l \neq j; l \in G_j$	$(i, k), (j, l)$	$(N-n)(n-1)(n-1)$
$j \notin G_i$	$k \neq i; k \in G_i$	$l \notin G_i, G_j$	$(i, k), (j), (l)$	$(N-n)(n-1)(N-2n)$
$j \notin G_i$	$k = j$	$l = i$	$(i = l), (j = k)$	$(N-n)$
$j \notin G_i$	$k = j$	$l \neq i; l \in G_i$	$(i, l), (j = k)$	$(N-n)(n-1)$
$j \notin G_i$	$k = j$	$l = j$	$(i), (j = k = l)$	$(N-n)$
$j \notin G_i$	$k = j$	$l \neq j; l \in G_j$	$(i), (j = k, l)$	$(N-n)(n-1)$
$j \notin G_i$	$k = j$	$l \notin G_i, G_j$	$(i), (j = k)$	$(N-n)(N-2n)$
$j \notin G_i$	$k \neq j; k \in G_j$	$l = i$	$(i = l), (j, k)$	$(N-n)(n-1)$
$j \notin G_i$	$k \neq j; k \in G_j$	$l \neq i; l \in G_i$	$(i, l), (j, k)$	$(N-n)(n-1)(n-1)$
$j \notin G_i$	$k \neq j; k \in G_j$	$l = j$	$(i), (j = l, k)$	$(N-n)(n-1)$
$j \notin G_i$	$k \neq j; k \in G_j$	$l = k$	$(i), (j, k = l)$	$(N-n)(n-1)$
$j \notin G_i$	$k \neq j; k \in G_j$	$l \neq j, k; l \in G_j$	$(i), (j, k, l)$	$(N-n)(n-1)(n-2)$
$j \notin G_i$	$k \neq j; k \in G_j$	$l \notin G_i, G_j$	$(i), (j, k), (l)$	$(N-n)(n-1)(N-2n)$
$j \notin G_i$	$k \notin G_j$	$l = i$	$(i = l), (j), (k)$	$(N-n)(N-2n)$
$j \notin G_i$	$k \notin G_j$	$l \neq i; l \in G_i$	$(i, l), (j), (k)$	$(N-n)(N-2n)(n-1)$
$j \notin G_i$	$k \notin G_j$	$l = j$	$(i), (j = l), (k)$	$(N-n)(N-2n)$
$j \notin G_i$	$k \notin G_j$	$l \neq j; l \in G_j$	$(i), (j, l), (k)$	$(N-n)(N-2n)(n-1)$
$j \notin G_i$	$k \notin G_j$	$l = k$	$(i), (j), (k = l)$	$(N-n)(N-2n)$
$j \notin G_i$	$k \notin G_j$	$l \neq k; l \in G_k$	$(i), (j), (k, l)$	$(N-n)(N-2n)(n-1)$
$j \notin G_i$	$k \notin G_j$	$l \notin G_i, G_j, G_k$	$(i), (j), (k), (l)$	$(N-n)(N-2n)(N-3n)$