
Problem Set 6 — *Due Friday, December 14, before class starts*
For the Exercise Sessions on Nov 30 and Dec 7

Last name	First name	SCIPER Nr	Points

Note: For full Homework Credit, it is sufficient to complete 6 out of the 8 problems.

Problem 1: Eckart–Young Theorem

In class, we proved the converse part of the Eckart–Young theorem for the spectral norm. Here, you do the same for the case of the Frobenius norm.

(a) For any matrix A of dimension $m \times n$ and an arbitrary orthonormal basis $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ of \mathbb{C}^n , prove that

$$\|A\|_F^2 = \sum_{k=1}^n \|A\mathbf{x}_k\|^2. \quad (1)$$

(b) Consider any $m \times n$ matrix B with $\text{rank}(B) \leq p$. Clearly, its null space has dimension no smaller than $n - p$. Therefore, we can find an orthonormal set $\{\mathbf{x}_1, \dots, \mathbf{x}_{n-p}\}$ in the null space of B . Prove that for such vectors, we have

$$\|A - B\|_F^2 \geq \sum_{k=1}^{n-p} \|A\mathbf{x}_k\|^2. \quad (2)$$

(c) (*This requires slightly more subtle manipulations.*) For any matrix A of dimension $m \times n$ and any orthonormal set of $n - p$ vectors in \mathbb{C}^n , denoted by $\{\mathbf{x}_1, \dots, \mathbf{x}_{n-p}\}$, prove that

$$\sum_{k=1}^{n-p} \|A\mathbf{x}_k\|^2 \geq \sum_{j=p+1}^r \sigma_j^2. \quad (3)$$

Hint: Consider the case $m \geq n$ and the set of vectors $\{\mathbf{z}_1, \dots, \mathbf{z}_{n-p}\}$, where $\mathbf{z}_k = V^H \mathbf{x}_k$. Express your formulas in terms of these and the SVD representation $A = U\Sigma V^H$.

(d) Briefly explain how (a)–(c) imply the desired statement.

Problem 2: Fourier (Review problem in view of our discussion of wavelets)

Suppose that a signal $x(t)$ satisfies

$$\int_{-\infty}^{\infty} x(t-n)x^*(t-m)dt = \begin{cases} 1, & \text{if } n=m \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

(In other words, the set of functions $\{x(t-n)\}_{n \in \mathbb{Z}}$ is an orthonormal set.) Show that then, its Fourier transform $X(\omega)$ must satisfy

$$\sum_{k \in \mathbb{Z}} |X(\omega + 2\pi k)|^2 = 1. \quad (5)$$

Problem 3: Hilbert Space Projection Theorem

Given a Hilbert space H and a Hilbert subspace G , the Hilbert space projection theorem states that for every $x \in H$, there exists a unique $y \in G$ such that

$$(i) \quad x - y \in G^\perp$$

$$(ii) \quad \|x - y\| = \inf_{u \in G} \|x - u\|$$

Recall that $G^\perp = \{z \in H : \langle z, x \rangle = 0 \text{ for all } x \in G\}$.

Just like in class, prove that if y is indeed the minimizer of $\|x - u\|$ over all $u \in G$, then it must be true that $x - y \in G^\perp$, — except this time, justify every step as you “unpack” the norms into inner products, and use the properties of the inner product.

Problem 4: Dual Basis

In class, we have mostly discussed orthonormal bases. Let $\{\varphi_n\}_{n \in \mathbb{Z}}$ be a basis for the Hilbert space H . Then, for any vector $x \in H$, we have

$$x = \sum_n \langle x, \varphi_n \rangle \varphi_n \quad (6)$$

Now, suppose that $\{\varphi'_n\}_{n \in \mathbb{Z}}$ is also a basis for H , but it is *not* orthonormal. Show that if we can find a so-called *dual* basis $\{\varphi''_n\}_{n \in \mathbb{Z}}$ satisfying $\langle \varphi'_n, \varphi''_m \rangle = \delta(n - m)$ then for any vector $x \in H$, we have

$$x = \sum_n \langle x, \varphi''_n \rangle \varphi'_n. \quad (7)$$

Problem 5: Minimum-norm Solutions

In this problem, we consider an *underdetermined* system of linear equations, i.e., $A\mathbf{x} = \mathbf{b}$, where A is a “fat” matrix ($m < n$) and \mathbf{b} is chosen such that a solution exists. As you know, in this case, there exist infinitely many solutions. Prove that the one solution \mathbf{x} that has the minimum 2-norm can be expressed as

$$\mathbf{x}_{MN} = V\Sigma^{-1}U^H\mathbf{b}, \quad (8)$$

where, as usual, the SVD of $A = U\Sigma V^H$, and Σ^{-1} is the matrix Σ where all non-zero diagonal entries are inverted.

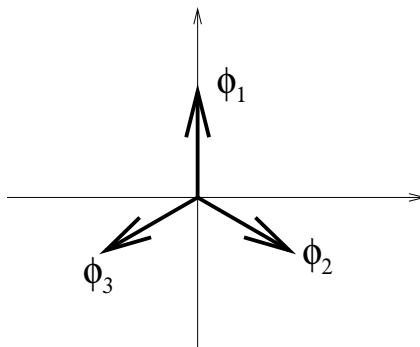


Figure 1: The three vectors ϕ_1, ϕ_2 , and ϕ_3 are at 120 degrees of each other and are of unit length each.

Problem 6: Frames

(a) We now turn to *overcomplete* expansions. The classic picture is given in Figure 1. In this picture, it is clear that every two-dimensional vector \mathbf{x} can be written as

$$\mathbf{x} = a_1\phi_1 + a_2\phi_2 + a_3\phi_3 \quad (9)$$

in many different ways. Explicitly and for every two-dimensional vector \mathbf{x} , find the solution $\mathbf{a} = (a_1, a_2, a_3)^t$ with minimum energy,¹ i.e., minimizing $a_1^2 + a_2^2 + a_3^2$. Then, give a general formula for any finite-dimensional overcomplete expansion $\{\phi_n\}_{n=1}^N$ in k -dimensional space.

(b) There is obviously no Parseval theorem, i.e., $x_1^2 + x_2^2 \neq a_1^2 + a_2^2 + a_3^2$. An overcomplete expansion is called a *frame* if there exist constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_n |\langle x, \phi_n \rangle|^2 \leq B\|x\|^2. \quad (10)$$

Find the frame bounds A and B for the “Mercedes” frame above. *Note:* Because frames satisfy such a Parseval-like property, they are the most common overcomplete expansions. *Another note:* If $A = B$, the frame is called *tight*.

Problem 7: Time–Frequency Representations

The elementary B-spline of degree 0 is the function $\beta^{(0)}(t) = 1$, for $-\frac{1}{2} \leq t < \frac{1}{2}$, and $\beta^{(0)}(t) = 0$ otherwise. The elementary B-spline of degree K is defined recursively as $\beta^{(K)} = \beta^{(K-1)} * \beta^{(0)}$. Find the Heisenberg box of the elementary B-splines of orders 0 and 1 (and 2, if you like). For each case, compare the size of the Heisenberg box to the lower bound (the uncertainty principle from class).

Problem 8: Haar Wavelet

This problem is taken from Vetterli/Kovacevic, p. 295.

¹It should also be pointed out that in some Data Science applications, we don’t want the minimum-energy solution, but the *sparsest* one, i.e., the one that has the fewest non-zero coefficients. In two dimensions, this is a trivial problem, but in N dimensions, there is no general simple solution, unfortunately...

Consider the wavelet series expansion of continuous-time signals $f(t)$ and assume that $\psi(t)$ is the Haar wavelet.

(a) Give the expansion coefficients for $f(t) = 1, t \in [0, 1]$, and 0 otherwise.

(b) Verify that for $f(t)$ as in Part (a), $\sum_m \sum_n \|\langle \psi_{m,n}, f \rangle\|^2 = 1$ (i.e., Parseval's identity).

(c) Consider $f_1(t) = f(t - 2^{-i})$, where i is a positive integer. Give the range of scales over which expansion coefficients are non-zero. (Take $f(t)$ as in Part (a).)

(d) Same as above, but now for $f_2(t) = f(t - 1/\sqrt{2})$. (Take $f(t)$ as in Part (a).)