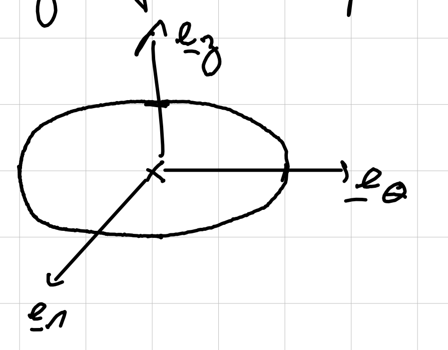
# Verification of anti-plane shear elasticity

We consider a 2D elastic cylinder submitted to gravity and clamped on its border:




The displacement field is: $\underline{u} = u_z(r) \cdot e_r$

$$\Rightarrow \underline{\underline{\nabla u}} = u_{z,r}(r) \cdot \underline{e_z} \otimes \underline{e_r}$$

$$\Rightarrow \underline{\underline{\varepsilon}} = \frac{1}{2} u_{z,r}(r) \cdot \underline{e_z} \otimes \underline{e_r} + \frac{1}{2} u_{z,r}\, \underline{e_r} \otimes \underline{e_z}$$

$$\Rightarrow \underline{\underline{\sigma}} = \mu\, u_{z,r}(r) \cdot (\underline{e_z} \otimes \underline{e_r} + \underline{e_r} \otimes \underline{e_z})$$

The equilibrium is then:

$$\mu \left( u_{z,rr}(r) + \frac{u_{z,r}(r)}{r} \right) \cdot \underline{e_z} - \varrho g \cdot \underline{e_z} = \underline{0}$$

$$u_{z,r}(r=0) = 0 \qquad (\text{sym})$$
$$u_z(r=R) = 0 \qquad (\text{clamp})$$

We obtain:

$$u_{z,rr}(r) + \frac{u_{z,r}(r)}{r} - \frac{\varrho g}{\mu} = 0,$$

Yet, we also have:

$$(r u_{z,r})_{,r} = r u_{z,rr} + u_{z,r}$$

$$\Rightarrow u_{z,rr} + \frac{u_{z,r}}{r} = \frac{(r u_{z,r})_{,r}}{r} .$$

Replacing in the equilibrium gives:

$$\frac{(r u_{z,r})_{,r}}{r} = \frac{\varrho g}{\mu}$$

$$\Rightarrow r u_{z,r}(r) = \frac{1}{2} \frac{\varrho g}{\mu} r^2 + A$$

$$\Rightarrow u_{z,r}(r) = \frac{1}{2} \frac{\varrho g}{\mu} r + \frac{A}{r}$$

Yet $u_{z,r}(r)(r=0) = 0 \Rightarrow A = 0$.

The last integration gives:

$$u_z(r) = \frac{1}{4} \frac{\varrho g}{\mu} r^2 + B$$

The (clamp) BC gives: $u_z(r=R) = 0$

$$\Rightarrow B = -\frac{1}{4} \frac{\varrho g}{\mu} R^2,$$

leading to:

$$u_z(r) = \frac{1}{4} \frac{\varrho g}{\mu} (r^2 - R^2).$$