I dedicate this book to George W.Bush, my Commander-in-Chief, whose impressive career advancement despite remedial language skills inspired me to believe that I was capable of autoring a book.

—Pedram Amini, "Fuzzing: Brute Force Vulnerability Discovery"

Exercise 1.

We claim that for $u \in (0; \infty)$ inequality

$$(2u + 1)\log(1 + 1/u) - 2 > 0 \tag{1}$$

holds.

To prove (1) we firstly note, that its equivalent to the inequality

$$(\frac{2}{x} + 1)\log(1 + x) - 2 > 0, \tag{2}$$

for $x \in (0; \infty)$ (we make the substitution $x = 1/u$).

Let

$$f(x) = (\frac{2}{x} + 1)\log(1 + x) - 2.$$

To show validity of (2), we prove the following two statements

1. $f(x)$ is continuous map
$$f(x)\colon [0, \infty) \to \mathbb{R},$$
and $f(0) = 0$.
2. $f'(x) > 0$ for $x \in (0, \infty)$.

Since it's well known that

$$\lim_{x \to 0} \frac{\log(x + 1)}{x} = 1,$$

we conclude that $f(x)$ is continuous at the point $x = 0$ and the following equality holds

$$\lim_{x \to 0} f(x) = 0.$$

This proves the first statement.

Let us prove the second statement. After simple calculations we get

$$f'(x) = \frac{x^2 + 2x - 2(x+1)\log(1+x)}{x^2(x+1)}. \tag{3}$$

Let

$$g(x) = x^2 + 2x - 2(x+1)\log(1+x).$$

To prove second statement it is enough to prove that

$$g(x) > 0, \tag{4}$$

for $x > 0$.

To prove (4) we note that $g(0) = 0$ and that

$$g'(x) = 2(x - \log(x+1)) > 0,$$

for $x > 0$ (here we take into the account the well known fact that $x > log(x+1)$ for $x > 0$).

This completes the proof of our initial claim.

Exercise 2.

Let us denote by $L(P, M, N)$ the exact lower bound for the number of neighbors in $M \times N$ table with numbers from 1 to P written in it.

The author doesn't know how to provide exact lower bound for all the cases so it will be shown that

$$L(P, M, N) \geq P - 1, \tag{5}$$

and this estimation is exact when

$$P \leq \left\lceil \frac{MN}{2} \right\rceil + 1,$$

and when M or N equals to 1.

To prove (5) we consider the graph G with the set of vertexes $V = \{1, 2, \ldots, P\}$ and with set of the edges $E = \{(a, b) \mid \text{a,b have common edge}\}$. It is obvious that G is connected, so we have a lower bound on the number of his edges

$$\#E \geq P - 1.$$

Since $\#E = L(P, M, N)$, this completes the proof of (5).

For the case when $M = 1$ or $N = 1$ we just fill the table in a manner

$$1, 2, \ldots, P-1, P \ldots, P.$$

It is obvious that there is exact $P - 1$ neighbors for such filling.

For the case when $M, N > 1$, $P \leq \left\lceil \frac{MN}{2} \right\rceil + 1$, we consider the following construction. Let us divide the table into black and white cells in a checkerboard pattern. Without loss of generality, we will assume that the number of black cells is not bigger than the number of white cells. It is easy to see, that number of white cells equals to $\lceil \frac{MN}{2} \rceil$.

Black cells we will fill by 1's, while the white we will consequentially fill by $2, 3, \dots, P$. If after this process there are still free cells fill them by 1's.

It is easy to see that with this filling there are exactly $P - 1$ neighbors:

$$\{(1, 2), (1, 3), \dots (1, P)\}.$$

Remark. *It seems very likely that for the case when $P > \lceil \frac{MN}{2} \rceil + 1$ the exact lower bound is given via "greedy" filling strategy (fill numbers $1, 2, \dots \lceil \frac{MN}{2} \rceil + 1$ like above and after it we put every new point $K = \lceil \frac{MN}{2} \rceil + 2, \dots, P$ in one of the black cells in such a way that at each step the number of new neighbors is as low as possible).*

And of course we need to note that lower bound given in (5) is not exact. For example if $M = N = 2$, $P = 4$, then $L(P, M, N) = 4$.

Exercise 3.

To begin with, the author should note that the exercise is too vaguely worded.

The most accurate answer to the exercise is "when the driver doesn't fall into the right-handed cycle that doesn't contain the starting point and does not reach a dead end". But it sounds like a tautology.

To clarify this problem we formulate it in terms of graph theory. In what follows every graph that we consider is an undirected finite path-connected graph (unless stated otherwise).

Denote by S the road network of the city and by s_0 the center of this network. We associate S with graph $G = (V, E)$ (V and E denotes set of the vertices and edges respectively) by the following rule. For each crossroad $s_i \in S$ we assign a vertex $v_i \in V$, and for each part of the road between two crossroads (s_i, s_j) we assign an edge $e_{ij} \in E$, such that e_{ij} connects vertices v_i, v_j. Such graph $G = (V, E)$ is said to be *an abstract graph of a road network*.

Let us introduce the notion of 3*d-road model* of the finite graph $G = (V, E)$. Let

$$\mathcal{V} \colon V \to \mathbb{R}^3$$

be an injection and consider the family of C^1-smooth injections

$$\gamma_{e_{ij}} \colon [0; 1] \to \mathbb{R}^3,\ e_{ij} \in E,\ e_{ij} \text{ connects vertices } v_i, v_j,$$

such that

$$\gamma_{e_{ij}}(0) = \mathcal{V}(v_i),$$
$$\gamma_{e_{ij}}(1) = \mathcal{V}(v_j),$$
$$\gamma_{e_{ij}}((0; 1)) \cap \gamma_{e_{kl}}((0; 1)) = \varnothing,\ \text{if } e_{ij} \neq e_{kl},$$

and if we denote coordinates of a point $\gamma_{e_{ij}}(t)$ by $(x_{e_{ij}}(t), y_{e_{ij}}(t), z_{e_{ij}}(t))$ then the following condition holds

$$\|(\frac{dx_{e_{ij}}}{dt}(t), \frac{dy_{e_{ij}}}{dt}(t))\| > 0, \tag{6}$$

for $t \in [0;1]$.

We say that $\mathcal{M} = (\mathcal{V}, \{\gamma_e\}_{e\in E})$ is a $3d$*-road model* of graph $G = (V, E)$.

Note that inequality (6) allows us to use geometric notions of left and right in the crossroad (so that the phrase "driver turns right" makes sense).

Let us formulate the so-called Axiom P.

Axiom P. Let $G = (V, E)$ be a finite graph and fix $v_0 \in V$. We say that (G, v_0) satisfies Axiom P if G has no leaf and $G \setminus \{v_0\}$ has no cycles.

Finally let us formulate the main result of this section which in our opinion gives the answer to the question of the exercise.

Proposition 1. *1. Consider road network S and corresponding abstract graph $G = (V, E)$. Denote by v_0 the vertex corresponding to the starting point of road network. If (G, v_0) satisfies* Axiom P *then driver returns to the starting point with any choice of starting direction.*

2. If $G = (E, V))$ is a finite undirected graph with marked vertex v_0 such that (G, v_0) fails to satisfy Axiom P*, then there exists $3d$-road model $\mathcal{M} = (\mathcal{V}, \{\gamma_e\}_{e\in E})$, such that the driver may not return to the staring point $\mathcal{V}(v_0)$.*

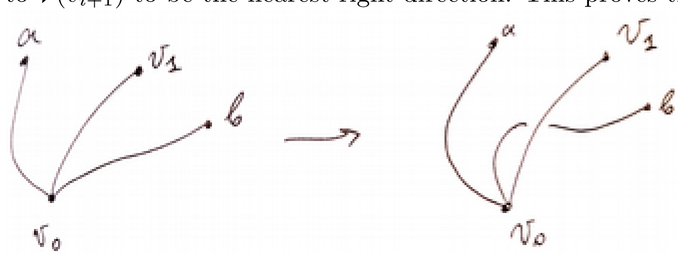
Proof. First statement of the proposition is obvious due to the finiteness of the graph.

Let us prove the second statement. Since (G, v_0) fails to satisfy Axiom P G must contain a leaf or $G \setminus \{v_0\}$ must contain a cycle. We consider the case when $G \setminus \{v_0\}$ contains a cycle (other case is treated similarly).

Let $v_l, v_{l+1}, \dots, v_k = v_l$ be the shortest cycle in $G \setminus \{v_0\}$ and $v_0, v_1, \dots, v_l$ be the shortest path from v_0 to the cycle $(c_1, \dots, c_k)$.

From Morse lemma it follows that "almost every" C^1 injection of graph to $\mathbb{R}^3$ is a $3d$-road model. We fix one of such good injections (denote it by $\mathcal{M} = (\mathcal{V}, \{\gamma\})$) and update it in finite number of steps in such a way that driver doesn't return to the initial point $s_0 = \mathcal{V}(v_0)$ if he starts drive to the direction $s_1 = \mathcal{V}(v_1)$.

This update steps is as follows. If direction from $\mathcal{V}(v_1)$ to $\mathcal{V}(v_2)$ is not the nearest right direction then by small perturbation of $\mathcal{M}$ we can make it so (see the picture below). Arguing by induction we can make every direction from $\mathcal{V}(v_i)$ to $\mathcal{V}(v_{i+1})$ to be the nearest right direction. This proves the second



statement. $\square$

Exercise 4.

The task of this exercise is to maximize the function

$$F(x_1, \dots, x_n) = \sum_{i=1}^{n} \log(x_i + a_i), \tag{7}$$

where $a_i > 0$, subject to the constraints

$$x_1 + \dots x_n = 1,$$

$$x_i \geq 0.$$

Let

$$\mathcal{C} = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1, x_i \geq 0, i = 1, \dots, n\}.$$

First let us note that since $\mathcal{C}$ is compact it follows that F attains the maximum at some point $p = (p_1, p_2, \dots, p_n)$. Explicit view of this point p is rather complicated, so instead of writing it out explicitly we describe the process of finding it (this process contains not more than n steps).

To do this we need the following well known proposition (we omit the proof since it's a log form of inequality of arithmetic and geometric means).

Proposition 2. *Consider function $G(x_1, \dots, x_n) = \sum_{i=1}^{n} \log(x_i)$ for $x_i > 0$, $i = 1, 2, \dots, n$. Let $S > 0$.*

G attains maximum on the set

$$\{x \in \mathbb{R}^n \mid x_1, \dots, x_n > 0, \sum_{i=1}^{n} x_i = S\}$$

at the point $x = (\frac{S}{n}, \frac{S}{n}, \dots, \frac{S}{n})$.

Let us make changing of variable in (7):

$$y_i = x_i + a_i.$$

Then maximization problem reformulates as follows: maximize the function

$$G(y_1, y_2, \dots, y_n) = \sum_{i=1}^{n} \log(y_i) \tag{8}$$

subject to constraints

$$y_1 + \dots + y_n = S_n, \tag{9}$$

$$y_1 \geq a_1, \dots, y_n \geq a_n, \tag{10}$$

where

$$S_n = 1 + \sum_{i=1}^{n} a_i.$$

If the inequalities

$$a_i \le \frac{S_n}{n}$$

hold, then by proposition 2 the maximum is attained at the point

$$q = (\frac{S_n}{n}, \frac{S_n}{n}, \dots, \frac{S_n}{n}).$$

If it is not the case, we can reorder the coordinates of $\mathbb{R}^n$ if needed in such a way, that there exists $k \in \mathbb{N}$, such that the following inequalities hold

$$a_1 \le \frac{S_n}{n}, a_2 \le \frac{S_n}{n}, \dots, a_k \le \frac{S_n}{n}, \tag{11}$$

$$a_{k+1} > \frac{S_n}{n}, \dots, a_n > \frac{S_n}{n}. \tag{12}$$

We show now, that for the maximum point q the following condition holds:

$$q_{k+1} = a_{k+1}, \dots, q_n = a_n. \tag{13}$$

To get a contradiction, let us assume that for some $l \in \mathbb{N}$

$$q_{k+l} > a_{k+l}.$$

Consider the vector $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, such that $v_1 = 1$, $v_{k+l} = -1$, and $v_i = 0$ for all other i. Let $t > 0$. It is obvious that point $(q + t \cdot v)$ satisfies (9)-(10) if t is small enough. At the same time, from (11)-(12) and from our assumption it follows that

$$\frac{dG(q + t \cdot v)}{dt} \mid_{t=0} = \frac{1}{q_1} - \frac{1}{q_{k+l}} > 0,$$

so q is not the maximum point. This finishes proof of (13).

Now our optimization problem (7) takes the following form: maximize the function

$$H(y_1, \dots, y_k) = \sum_{i=1}^{k} \log(y_i) + \text{const} \tag{14}$$

subject to constraints

$$y_1 + \dots + y_k = S_k, \tag{15}$$

$$y_1 \ge a_1, \dots, y_k \ge a_k, \tag{16}$$

where

$$S_k = 1 + \sum_{i=1}^{k} a_i,$$

$$\text{const} = \sum_{i=k+1}^{n} a_i.$$

Note that we can omit the const term in (14).

As we can see, optimization problem (14)-(16) coincides with (8)-(10). Arguing by induction we can find maximum for the function (7). This completes the exercise.