

Introduction to Logical Relations

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Simply-Typed Lambda Calculus

$A ::= 1 \mid A \rightarrow B \mid A \times B$

$e ::= () \mid (e, e) \mid \pi_i(e) \mid \lambda x. e \mid e e \mid x$

$v ::= () \mid (v, v) \mid \lambda x. e$

$\Gamma ::= \cdot \mid \Gamma, x : A$

$\boxed{\Gamma \vdash e : A}$

Typing

$\boxed{e \rightsquigarrow e'}$

Reduction

Typing

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}$$

$$\frac{}{\Gamma \vdash () : 1}$$

$$\frac{\Gamma \vdash e_1 : A_1 \quad \Gamma \vdash e_2 : A_2}{\Gamma \vdash (e_1, e_2) : A_1 \times A_2}$$

$$\frac{\Gamma \vdash e : A_1 \times A_2}{\Gamma \vdash \pi_1(e) : A_1}$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x. e : A \rightarrow B}$$

$$\frac{\Gamma \vdash e_1 : A \rightarrow B \quad \Gamma \vdash e_2 : A}{\Gamma \vdash e_1 e_2 : B}$$

Evaluation

$$(\lambda x.e) v \rightsquigarrow [v/x]e$$

$$\Pi; (V_1, V_2) \rightsquigarrow V_1;$$

$$P_1 \rightsquigarrow e'_1$$

$$e_2 \rightsquigarrow e'_2$$

$$e_1 e_2 \rightsquigarrow e'_1 e'_2$$

$$V e_2 \rightsquigarrow V e'_2$$

$$P_1 \rightsquigarrow e'_1$$

$$e_2 \rightsquigarrow e'_2$$

$$(e_1, e_2) \rightsquigarrow (e'_1, e'_2)$$

$$(V, e_2) \rightsquigarrow (V, e'_2)$$

Evaluation

$$(\lambda x.e) \vee \rightsquigarrow [\nu/x]e$$

$$\Pi; (\nu_1, \nu_2) \rightsquigarrow \nu_1;$$

$$p_1 \rightsquigarrow e'_1$$

$$e_2 \rightsquigarrow e'_2$$

$$e_1 e_2 \rightsquigarrow e'_1 e'_2$$

$$V e_2 \rightsquigarrow V e'_2$$

$$p_1 \rightsquigarrow e'_1$$

$$e_2 \rightsquigarrow e'_2$$

$$(e_1, e_2) \rightsquigarrow (e'_1, e'_2)$$

$$(V, e_2) \rightsquigarrow (V, e'_2)$$

Note that
evaluation
is deterministic

Type Safety

Progress

If $\vdash e : A$ then $e \rightsquigarrow e'$ or e value

Preservation

If $\vdash e : A$ and $e \rightsquigarrow e'$ then $\vdash e' : A$

Termination

If $e : A$ then $e \rightsquigarrow^* \checkmark$

Termination

If $\cdot : A$ then $e \rightsquigarrow^* \checkmark$

We CANNOT prove
this by induction

Termination

If $\cdot \vdash e : A$ then $e \sim^* \checkmark$

Case
$$\frac{\cdot \vdash e_1 : A \rightarrow B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1 e_2 : B}$$

Termination

If $\cdot \vdash e : A$ then $e \sim^* \checkmark$

Case
$$\frac{\cdot \vdash e_1 : A \rightarrow B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1 e_2 : B}$$

i. $\cdot \vdash e_1 : A \rightarrow B$

Subderivation

Termination

If $\cdot \vdash e : A$ then $e \sim^* \checkmark$

Case
$$\frac{\cdot \vdash e_1 : A \rightarrow B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1 e_2 : B}$$

1. $\cdot \vdash e_1 : A \rightarrow B$

Subderivation

2. $\cdot \vdash e_2 : A$

Subderivation

Termination

If $\cdot \vdash e : A$ then $e \rightsquigarrow^* \checkmark$

Case
$$\frac{\cdot \vdash e_1 : A \rightarrow B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1 e_2 : B}$$

1. $\cdot \vdash e_1 : A \rightarrow B$
2. $\cdot \vdash e_2 : A$
3. $e_1 \rightsquigarrow^* \checkmark_1$

Subderivation
Subderivation
Induction

Termination

If $\cdot \vdash e : A$ then $e \rightsquigarrow^* \checkmark$

Case
$$\frac{\cdot \vdash e_1 : A \rightarrow B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1 e_2 : B}$$

1. $\cdot \vdash e_1 : A \rightarrow B$
2. $\cdot \vdash e_2 : A$
3. $e_1 \rightsquigarrow^* v_1$
4. $\cdot \vdash v_1 : A \rightarrow B$

Subderivation
Subderivation
Induction
Type Safety

Termination

If $\cdot \vdash e : A$ then $e \rightsquigarrow^* \checkmark$

Case
$$\frac{\cdot \vdash e_1 : A \rightarrow B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1 e_2 : B}$$

1. $\cdot \vdash e_1 : A \rightarrow B$ Subderivation
2. $\cdot \vdash e_2 : A$ Subderivation
3. $e_1 \rightsquigarrow^* v_1$ Induction
4. $\cdot \vdash v_1 : A \rightarrow B$ Type Safety
5. $v_1 = \lambda x. e$ Inversion

Termination

If $\cdot \vdash e : A$ then $e \rightsquigarrow^* \checkmark$

Case
$$\frac{\cdot \vdash e_1 : A \rightarrow B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1 e_2 : B}$$

- | | | |
|----|--------------------------------------|---------------|
| 1. | $\cdot \vdash e_1 : A \rightarrow B$ | Subderivation |
| 2. | $\cdot \vdash e_2 : A$ | Subderivation |
| 3. | $e_1 \rightsquigarrow^* v_1$ | Induction |
| 4. | $\cdot \vdash v_1 : A \rightarrow B$ | Type Safety |
| 5. | $v_1 = \lambda x. e$ | Inversion |
| 6. | $x : A \vdash e : B$ | Inversion |

Termination

If $\cdot \vdash e : A$ then $e \rightsquigarrow^* \checkmark$

Case
$$\frac{\cdot \vdash e_1 : A \rightarrow B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1 e_2 : B}$$

1. $\cdot \vdash e_1 : A \rightarrow B$
2. $\cdot \vdash e_2 : A$
3. $e_1 \rightsquigarrow^* v_1$
4. $\cdot \vdash v_1 : A \rightarrow B$
5. $v_1 = \lambda x. e$
6. $x : A \vdash e : B$
7. $e_2 \rightsquigarrow^* v_2$

Subderivation
Subderivation
Induction
Type Safety
Inversion
Inversion
Induction

Termination

If $\cdot \vdash e : A$ then $e \rightsquigarrow^* \checkmark$

Case
$$\frac{\cdot \vdash e_1 : A \rightarrow B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1 e_2 : B}$$

1. $\cdot \vdash e_1 : A \rightarrow B$ Subderivation
2. $\cdot \vdash e_2 : A$ Subderivation
3. $e_1 \rightsquigarrow^* v_1$ Induction
4. $\cdot \vdash v_1 : A \rightarrow B$ Type Safety
5. $v_1 = \lambda x. e$ Inversion
6. $x : A \vdash e : B$ Inversion
7. $e_2 \rightsquigarrow^* v_2$ Induction
8. $\cdot \vdash v_2 : A$ Type Safety

Termination

If $\cdot \vdash e : A$ then $e \rightsquigarrow^* \checkmark$

Case
$$\frac{\cdot \vdash e_1 : A \rightarrow B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1 e_2 : B}$$

1. $\cdot \vdash e_1 : A \rightarrow B$ Subderivation
2. $\cdot \vdash e_2 : A$ Subderivation
3. $e_1 \rightsquigarrow^* v_1$ Induction
4. $\cdot \vdash v_1 : A \rightarrow B$ Type Safety
5. $v_1 = \lambda x. e$ Inversion
6. $x : A \vdash e : B$ Inversion
7. $e_2 \rightsquigarrow^* v_2$ Induction
8. $\cdot \vdash v_2 : A$ Type Safety
9. $\cdot \vdash [v_2/x]e : B$ Substitution

Termination

If $\cdot \vdash e : A$ then $e \rightsquigarrow^* \checkmark$

Case
$$\frac{\cdot \vdash e_1 : A \rightarrow B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1 e_2 : B}$$

1. $\cdot \vdash e_1 : A \rightarrow B$ Subderivation
2. $\cdot \vdash e_2 : A$ Subderivation
3. $e_1 \rightsquigarrow^* v_1$ Induction
4. $\cdot \vdash v_1 : A \rightarrow B$ Type Safety
5. $v_1 = \lambda x. e$ Inversion
6. $x : A \vdash e : B$ Inversion
7. $e_2 \rightsquigarrow^* v_2$ Induction
8. $\cdot \vdash v_2 : A$ Type Safety
9. $\cdot \vdash [v_2/x]e : B$ Substitution
10. $[v_2/x]e \rightsquigarrow^* \checkmark$

Termination

If $\cdot \vdash e : A$ then $e \rightsquigarrow^* \checkmark$

Case
$$\frac{\cdot \vdash e_1 : A \rightarrow B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1 e_2 : B}$$

1. $\cdot \vdash e_1 : A \rightarrow B$ Subderivation
2. $\cdot \vdash e_2 : A$ Subderivation
3. $e_1 \rightsquigarrow^* v_1$ Induction
4. $\cdot \vdash v_1 : A \rightarrow B$ Type Safety
5. $v_1 = \lambda x. e$ Inversion
6. $x : A \vdash e : B$ Inversion
7. $e_2 \rightsquigarrow^* v_2$ Induction
8. $\cdot \vdash v_2 : A$ Type Safety
9. $\cdot \vdash [v_2/x]e : B$ Substitution
10. $[v_2/x]e \rightsquigarrow^* \checkmark$ NOT BY INDUCTION

Termination

If $\cdot \vdash e : A$ then $e \rightsquigarrow^* \checkmark$

Case
$$\frac{\cdot \vdash e_1 : A \rightarrow B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1 e_2 : B}$$

1. $\cdot \vdash e_1 : A \rightarrow B$

Subderivation

2. $\cdot \vdash e_2 : A$

Subderivation

3. $e_1 \rightsquigarrow^* v_1$

Induction

4. $\cdot \vdash v_1 : A \rightarrow B$

Type Safety

5. $v_1 = \lambda x. e$

Inversion

6. $x : A \vdash e : B$

Inversion

7. $e_2 \rightsquigarrow^* v_2$

Induction

8. $\cdot \vdash v_2 : A$

Type Safety

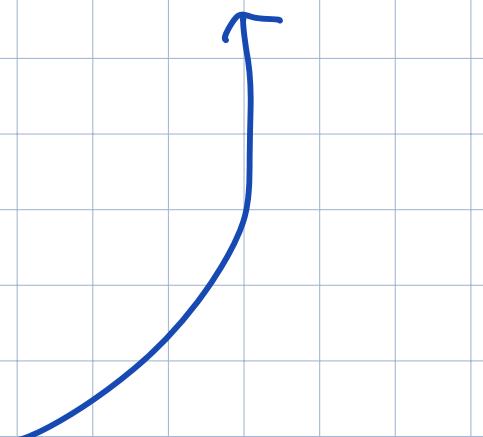
9. $\cdot \vdash [v_2/x]e : B$

Substitution

10. $[v_2/x]e \rightsquigarrow^* \checkmark$

NOT BY INDUCTION

1. $[v_2/x]e$
is not a
subterm
of $e_1 e_2$!



Termination

If $\cdot \vdash e : A$ then $e \rightsquigarrow^* \checkmark$

Case
$$\frac{\cdot \vdash e_1 : A \rightarrow B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1 e_2 : B}$$

1. $\cdot \vdash e_1 : A \rightarrow B$
2. $\cdot \vdash e_2 : A$
3. $e_1 \rightsquigarrow^* v_1$
4. $\cdot \vdash v_1 : A \rightarrow B$
5. $v_1 = \lambda x. e$
6. $x : A \vdash e : B$
7. $e_2 \rightsquigarrow^* v_2$
8. $\cdot \vdash v_2 : A$
9. $\cdot \vdash [v_2/x]e : B$
10. $[v_2/x]e \rightsquigarrow^* \checkmark$

Subderivation
Subderivation
Induction
Type Safety
Inversion
Inversion
Induction
Type Safety
Substitution

NOT BY INDUCTION

1. $[v_2/x]e$
is not a
subterm
of e_1, e_2 !

2. We know
nothing
about e !



Termination

- knowing $e_1 \rightsquigarrow^* \lambda x.e$
doesn't tell us anything about
 $[v/x]e$
- We need to know applying v to
 $\lambda x.e$ terminates!

Defining a Logical Relation

We will define a type-indexed family of sets of terms

$$V_1 = \{()\}$$

$$V_{A \times B} = \{ (v_1, v_2) \mid v_1 \in V_A \text{ and } v_2 \in V_B \}$$

$$V_{A \rightarrow B} = \{ v \mid \forall v' \in V_A . \quad v v' \in E_B \}$$

$$E_A = \{ e \mid e \sim^* v \text{ and } v \in V_A \}$$

Defining a Logical Relation

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$$E_A = \{ e \mid e \sim^* v \text{ and } v \in V_A \}$$

} one set
 V_A for
each A

} one set
 E_A for
each A

$$\mathcal{E}_1 = \{e \mid e \rightsquigarrow^* v, v \in V_1\}$$

$$= \{e \mid e \rightsquigarrow^* ()\}$$

$$\mathcal{E}_{1 \rightarrow 1} = \{e \mid e \rightsquigarrow^* f \text{ and } f \in V_{1 \rightarrow 1}\}$$

$$= \{e \mid e \rightsquigarrow^* f \text{ and } \forall v' \in V_1. f_{v'} \in \mathcal{E}_1\}$$

$$= \{e \mid e \rightsquigarrow^* f \text{ and } f() \in \mathcal{E}_1\}$$

$$= \{e \mid e \rightsquigarrow^* f \text{ and } f() \rightsquigarrow^* ()\}$$

$$\mathcal{E}_{(1 \rightarrow 1) \rightarrow 1}$$

Properties

I. for all A. $V_A \subseteq \mathcal{E}_A$

Properties

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Proof: $\mathcal{E}_A = \{e \mid e \rightsquigarrow^* v \wedge v \in V_A\}$

Since $v \rightsquigarrow^* v$ in 0 steps,

if $v \in V_A$ then $v \in \mathcal{E}_A$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in E_A$ iff $e' \in E_A$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in E_A$ iff $e' \in E_A$

Proof : Assume $e \rightsquigarrow e'$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \ell_A$ iff $e' \in \ell_A$

Proof : Assume $e \rightsquigarrow e'$

\Leftarrow : Assume $e' \in \ell_A$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof : Assume $e \rightsquigarrow e'$

\Leftarrow : Assume $e' \in \mathcal{E}_A$

$$\mathcal{E}_A = \{e \mid e \rightsquigarrow^* \checkmark, v \in V_A\}$$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof : Assume $e \rightsquigarrow e'$

\Leftarrow : Assume $e' \in \mathcal{E}_A$

$$\mathcal{E}_A = \{e \mid e \rightsquigarrow^*_v, v \in V_A\}$$

Hence $e' \rightsquigarrow^*_v$ and $v \in V_A$

Properties: Closure

If $e \sim e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof : Assume $e \sim e'$

\Leftarrow : Assume $e' \in \mathcal{E}_A$

$$\mathcal{E}_A = \{e \mid e \sim^* v, v \in V_A\}$$

Hence $e' \sim^* v$ and $v \in V_A$

Since $e \sim e'$ and $e' \sim^* v$, $e \sim^* v$

Properties: Closure

If $e \sim e'$ then $e \in E_A$ iff $e' \in E_A$

Proof : Assume $e \sim e'$

\Leftarrow : Assume $e' \in E_A$

$$E_A = \{e \mid e \sim^* v, v \in V_A\}$$

Hence $e' \sim^* v$ and $v \in V_A$

Since $e \sim e'$ and $e' \sim^* v$, $e \sim^* v$

So $e \sim^* v$ and $v \in V_A$

Properties: Closure

If $e \sim e'$ then $e \in E_A$ iff $e' \in E_A$

Proof : Assume $e \sim e'$

\Leftarrow : Assume $e' \in E_A$

$$E_A = \{e \mid e \sim^* v, v \in V_A\}$$

Hence $e' \sim^* v$ and $v \in V_A$

Since $e \sim e'$ and $e' \sim^* v$, $e \sim^* v$

So $e \sim^* v$ and $v \in V_A$

So $e \in E_A$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in E_A$ iff $e' \in E_A$

Proof : Assume $e \rightsquigarrow e'$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof : Assume $e \rightsquigarrow e'$

\Rightarrow : Assume $e \in \mathcal{E}_A$ $\mathcal{E}_A = \{e \mid e \rightsquigarrow^* v, v \in V_A\}$

Properties: Closure

If $e \sim e'$ then $e \in E_A$ iff $e' \in E_A$

Proof : Assume $e \sim e'$

\Rightarrow : Assume $e \in E_A$ $E_A = \{e \mid e \sim^* v, v \in V_A\}$

Hence $e \sim^* v$ and $v \in V_A$

Properties: Closure

If $e \sim e'$ then $e \in E_A$ iff $e' \in E_A$

Proof : Assume $e \sim e'$

\Rightarrow : Assume $e \in E_A$ $E_A = \{e \mid e \sim^* v, v \in V_A\}$

Hence $e \sim^* v$ and $v \in V_A$

Note $e \sim e'$ and $e \sim^* v$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof : Assume $e \rightsquigarrow e'$

\Rightarrow : Assume $e \in \mathcal{E}_A$ $\mathcal{E}_A = \{e \mid e \rightsquigarrow^* \checkmark, v \in V_A\}$

Hence $e \rightsquigarrow^* \checkmark$ and $v \in V_A$

Note $e \rightsquigarrow e'$ and $e \rightsquigarrow^* \checkmark$

Since evaluation is deterministic , $e' \rightsquigarrow^* \checkmark$

Properties: Closure

If $e \sim e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof : Assume $e \sim e'$

\Rightarrow : Assume $e \in \mathcal{E}_A$ $\mathcal{E}_A = \{e \mid e \sim^* v, v \in V_A\}$

Hence $e \sim^* v$ and $v \in V_A$

Note $e \sim e'$ and $e \sim^* v$

Since evaluation is deterministic , $e' \sim^* v$

So $e' \sim^* v$ and $v \in V_A$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in E_A$ iff $e' \in E_A$

Proof : Assume $e \rightsquigarrow e'$

\Rightarrow : Assume $e \in E_A$ $E_A = \{e \mid e \rightsquigarrow^* \checkmark, \checkmark \in V_A\}$

Hence $e \rightsquigarrow^* \checkmark$ and $\checkmark \in V_A$

Note $e \rightsquigarrow e'$ and $e \rightsquigarrow^* \checkmark$

Since evaluation is deterministic , $e' \rightsquigarrow^* \checkmark$

So $e' \rightsquigarrow^* \checkmark$ and $\checkmark \in V_A$

So $e' \in E_A$

Fundamental Lemma

Define V_Γ as follows:

$$V_0 = \{ [] \}$$

$$V_{(\Gamma, x:A)} = \{ (\gamma, v/x) \mid \gamma \in V_\Gamma \text{ and } v \in V_A \}$$

$$\begin{aligned} \Gamma &= x: A, y: A \rightarrow A, z: A \times A \\ V_\Gamma &= \left\{ \left(\frac{v_1/x}{x}, \frac{v_2/y}{y}, \frac{v_3/z}{z} \right) \mid \right. \\ &\quad \left. v_1 \in V_A, v_2 \in V_{A \rightarrow A}, \right. \\ &\quad \left. v_3 \in V_{A \times A} \right\}. \end{aligned}$$

Fundamental Lemma

Define V_Γ as follows:

$$V_\cdot = \{ [] \}$$

$$V_{(\Gamma, x : A)} = \{ (\gamma, v/x) \mid \gamma \in V_\Gamma \text{ and } v \in V_A \}$$

Fundamental Lemma:

If $\Gamma \vdash e : A$ and $\gamma \in V_\Gamma$ then $[\gamma]e \in E_A$

Fundamental Lemma

Define V_Γ as follows:

$$V_\cdot = \{ [] \}$$

$$V_{(\Gamma, x:A)} = \{ (\gamma, v/x) \mid \gamma \in V_\Gamma \text{ and } v \in V_A \}$$

Fundamental Lemma:

If $\Gamma \vdash e : A$ and $\gamma \in V_\Gamma$ then $[\gamma]e \in E_A$

Proof: By induction on $\Gamma \vdash e : A$

Proof

Case

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}$$

$$\gamma \in V_{\Gamma}$$

By assumption

$$\gamma(x) \in V_A$$

By definition of V_{Γ}

$$\gamma(x) \in E_A$$

Since $V_A \subseteq E_A$

$$[\gamma]x \in E_A$$

By definition of $[\gamma]e$

Proof

Case

$$\Gamma \vdash () : 1$$

$$() \in V_1$$

By definition of V_1

$$() \in E_1$$

Since $V_1 \subseteq E_1$

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$$

1. $\tau \in V_\Gamma$

Assumption

2. $\Gamma \vdash e_1 : A$

Subderivation

3. $\Gamma \vdash e_2 : B$

Subderivation

Proof

Case	$\frac{\Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$
1.	$\tau \in V_\Gamma$
2.	Assumption
3.	Subderivation
4.	Subderivation
	Induction on (1), (2)

Proof

Case	$\frac{\Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$
1.	$\tau \in V_\Gamma$
2.	$\Gamma \vdash e_1 : A$
3.	$\Gamma \vdash e_2 : B$
4.	$[\delta] e_1 \in E_A$
5.	$[\delta] e_2 \in E_B$

Assumption

Subderivation

Subderivation

Induction on (1), (2)

Induction on (1), (3)

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$$

1. $\tau \in V_\Gamma$
2. $\Gamma \vdash e_1 : A$
3. $\Gamma \vdash e_2 : B$
4. $[\tau] e_1 \in E_A$
5. $[\tau] e_2 \in E_B$
6. $[\tau] e_1 \rightsquigarrow^* v_1$

Assumption

Subderivation

Subderivation

Induction on (1), (2)

Induction on (1), (3)

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$$

1. $\tau \in V_\Gamma$
2. $\Gamma \vdash e_1 : A$
3. $\Gamma \vdash e_2 : B$
4. $[\tau] e_1 \in \mathcal{E}_A$
5. $[\tau] e_2 \in \mathcal{E}_B$
6. $[\tau] e_1 \rightsquigarrow^* v_1$
7. $v_1 \in V_A$

Assumption
Subderivation
Subderivation
Induction on (1), (2)
Induction on (1), (3)
Definition of \mathcal{E}_A

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$$

1. $\tau \in V_\Gamma$
2. $\Gamma \vdash e_1 : A$
3. $\Gamma \vdash e_2 : B$
4. $[\tau]e_1 \in \mathcal{E}_A$
5. $[\tau]e_2 \in \mathcal{E}_B$
6. $[\tau]e_1 \rightsquigarrow^* v_1$
7. $v_1 \in V_A$
8. $[\tau]e_2 \rightsquigarrow^* v_2$

Assumption
Subderivation
Subderivation
Induction on (1), (2)
Induction on (1), (3)
Definition of \mathcal{E}_A

Proof

Case	$\frac{\Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$
1. $\tau \in V_\Gamma$	Assumption
2. $\Gamma \vdash e_1 : A$	Subderivation
3. $\Gamma \vdash e_2 : B$	Subderivation
4. $[\tau] e_1 \in \mathcal{E}_A$	Induction on (1), (2)
5. $[\tau] e_2 \in \mathcal{E}_B$	Induction on (1), (3)
6. $[\tau] e_1 \rightsquigarrow^* v_1$	
7. $v_1 \in V_A$	Definition of \mathcal{E}_A
8. $[\tau] e_2 \rightsquigarrow^* v_2$	
9. $v_2 \in V_B$	Definition of \mathcal{E}_B

Proof

Case	$\frac{\Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$
1. $\tau \in V_\Gamma$	Assumption
2. $\Gamma \vdash e_1 : A$	Subderivation
3. $\Gamma \vdash e_2 : B$	Subderivation
4. $[\tau]e_1 \in \mathcal{E}_A$	Induction on (1), (2)
5. $[\tau]e_2 \in \mathcal{E}_B$	Induction on (1), (3)
6. $[\tau]e_1 \rightsquigarrow^* v_1$	
7. $v_1 \in V_A$	Definition of \mathcal{E}_A
8. $[\tau]e_2 \rightsquigarrow^* v_2$	
9. $v_2 \in V_B$	Definition of \mathcal{E}_B
10. $([\tau]e_1, [\tau]e_2) \rightsquigarrow^* (v_1, [\tau]e_2)$	Reduction rules on (6)

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$$

1. $\tau \in V_\Gamma$
2. $\Gamma \vdash e_1 : A$
3. $\Gamma \vdash e_2 : B$
4. $[\tau]e_1 \in \mathcal{E}_A$
5. $[\tau]e_2 \in \mathcal{E}_B$
6. $[\tau]e_1 \rightsquigarrow^* v_1$
7. $v_1 \in V_A$
8. $[\tau]e_2 \rightsquigarrow^* v_2$
9. $v_2 \in V_B$
10. $([\tau]e_1, [\tau]e_2) \rightsquigarrow^* (v_1, [\tau]e_2)$
11. $(v_1, [\tau]e_2) \rightsquigarrow^* (v_1, v_2)$

Assumption
Subderivation
Subderivation
Induction on (1), (2)
Induction on (1), (3)

Definition of \mathcal{E}_A

Definition of \mathcal{E}_B
Reduction rules on (6)
Reduction rules on (8)

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$$

1. $\tau \in V_\Gamma$
2. $\Gamma \vdash e_1 : A$
3. $\Gamma \vdash e_2 : B$
4. $[\tau]e_1 \in \mathcal{E}_A$
5. $[\tau]e_2 \in \mathcal{E}_B$
6. $[\tau]e_1 \rightsquigarrow^* v_1$
7. $v_1 \in V_A$
8. $[\tau]e_2 \rightsquigarrow^* v_2$
9. $v_2 \in V_B$
10. $([\tau]e_1, [\tau]e_2) \rightsquigarrow^* (v_1, [\tau]e_2)$
11. $(v_1, [\tau]e_2) \rightsquigarrow^* (v_1, v_2)$
12. $([\tau]e_1, [\tau]e_2) \rightsquigarrow^* (v_1, v_2)$

Assumption

Subderivation

Subderivation

Induction on (1), (2)

Induction on (1), (3)

Definition of \mathcal{E}_A

Definition of \mathcal{E}_B

Reduction rules on (6)

Reduction rules on (8)

Transitivity of \rightsquigarrow^* on (10), (11)

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$$

1. $\tau \in V_\Gamma$
2. $\Gamma \vdash e_1 : A$
3. $\Gamma \vdash e_2 : B$
4. $[\tau]e_1 \in \mathcal{E}_A$
5. $[\tau]e_2 \in \mathcal{E}_B$
6. $[\tau]e_1 \rightsquigarrow^* v_1$
7. $v_1 \in V_A$
8. $[\tau]e_2 \rightsquigarrow^* v_2$
9. $v_2 \in V_B$
10. $([\tau]e_1, [\tau]e_2) \rightsquigarrow^* (v_1, [\tau]e_2)$
11. $(v_1, [\tau]e_2) \rightsquigarrow^* (v_1, v_2)$
12. $([\tau]e_1, [\tau]e_2) \rightsquigarrow^* (v_1, v_2)$
13. $[\tau](e_1, e_2) \rightsquigarrow^* (v_1, v_2)$

Assumption
Subderivation
Subderivation
Induction on (1), (2)
Induction on (1), (3)

Definition of \mathcal{E}_A

Definition of \mathcal{E}_B
Reduction rules on (6)
Reduction rules on (8)
Transitivity of \rightsquigarrow^* on (10), (11)
Definition of $[\tau]e$ on (12)

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$$

1. $\tau \in V_\Gamma$
2. $\Gamma \vdash e_1 : A$
3. $\Gamma \vdash e_2 : B$
4. $[\tau]e_1, e_2 \in \mathcal{E}_A$
5. $[\tau]e_2 \in \mathcal{E}_B$
6. $[\tau]e_1 \rightsquigarrow^* v_1$

Assumption

Subderivation

Subderivation

Induction on (1), (2)

Induction on (1), (3)

7. $v_1 \in V_A$
8. $[\tau]e_2 \rightsquigarrow^* v_2$

Definition of \mathcal{E}_A

9. $v_2 \in V_B$
10. $([\tau]e_1, [\tau]e_2) \rightsquigarrow^* (v_1, [\tau]e_2)$

Definition of \mathcal{E}_B

Reduction rules on (6)

11. $(v_1, [\tau]e_2) \rightsquigarrow^* (v_1, v_2)$

Reduction rules on (8)

12. $([\tau]e_1, [\tau]e_2) \rightsquigarrow^* (v_1, v_2)$

Transitivity of \rightsquigarrow^* on (10), (11)

13. $[\tau](e_1, e_2) \rightsquigarrow^* (v_1, v_2)$

Definition of $[\tau]e$ on (12)

14. $(v_1, v_2) \in V_{A \times B}$

Definition of $V_{A \times B}$

15. $[\tau](e_1, e_2) \in \mathcal{E}_{A \times B}$

Definition of $\mathcal{E}_{A \times B}$

Proof

Case

$$\frac{\Gamma, x:A \vdash e:B}{\Gamma \vdash \lambda x.e : A \rightarrow B}$$

- 0. $\Gamma, x:A \vdash e: B$
- 1. $\gamma \in V_\Gamma$

Subderivation
Assumption

Proof

Case $\frac{\Gamma, x:A \vdash e:B}{\Gamma \vdash \lambda x.e : A \rightarrow B}$

0. $\Gamma, x:A \vdash e:B$ Subderivation
1. $\gamma \in V_\Gamma$ Assumption
2. $[\gamma] \lambda x.e = \lambda x. [\gamma]e$ Definition

Proof

Case $\frac{\Gamma, x:A \vdash e:B}{\Gamma \vdash \lambda x.e : A \rightarrow B}$

0. $\Gamma, x:A \vdash e:B$ Subderivation

1. $\gamma \in V_\Gamma$ Assumption

2. $[\gamma] \lambda x.e = \lambda x. [\gamma]e$ Definition

WTS: $\lambda x.[\gamma]e \in V_{A \rightarrow B}$

Proof

Case

$$\frac{\Gamma, x:A \vdash e:B}{\Gamma \vdash \lambda x.e : A \rightarrow B}$$

0. $\Gamma, x:A \vdash e:B$

Subderivation

1. $\gamma \in V_\Gamma$

Assumption

2. $[\gamma] \lambda x.e = \lambda x. [\gamma]e$

Definition

WTS: $\lambda x. [\gamma]e \in V_{A \rightarrow B}$

3. Assume $v \in V_A$

Proof

Case

$$\frac{\Gamma, x:A \vdash e:B}{\Gamma \vdash \lambda x.e : A \rightarrow B}$$

0. $\Gamma, x:A \vdash e:B$

Subderivation

1. $\gamma \in V_\Gamma$

Assumption

2. $[\gamma] \lambda x.e = \lambda x.[\gamma]e$

Definition

WTS: $\lambda x.[\gamma]e \in V_{A \rightarrow B}$

3. Assume $v \in V_A$

WTS $(\lambda x.[\gamma]e)v \in E_B$

Proof

Case

$$\frac{\Gamma, x:A \vdash e:B}{\Gamma \vdash \lambda x.e : A \rightarrow B}$$

0. $\Gamma, x:A \vdash e:B$

Subderivation

1. $\gamma \in V_\Gamma$

Assumption

2. $[\gamma] \lambda x.e = \lambda x. [\gamma] e$

Definition

WTS: $\lambda x. [\gamma] e \in V_{A \rightarrow B}$

3. Assume $v \in V_A$

WTS $(\lambda x. [\gamma] e) v \in E_B$

4. $(\lambda x. [\gamma] e) v \rightsquigarrow [\gamma, v/x] e$ Reduction

.

Proof

Case

$$\frac{\Gamma, x:A \vdash e:B}{\Gamma \vdash \lambda x.e : A \rightarrow B}$$

0. $\Gamma, x:A \vdash e:B$

Subderivation

1. $\gamma \in V_\Gamma$

Assumption

2. $[\gamma] \lambda x.e = \lambda x. [\gamma] e$

Definition

WTS: $\lambda x. [\gamma] e \in V_{A \rightarrow B}$

3. Assume $v \in V_A$

WTS $(\lambda x. [\gamma] e) v \in E_B$

4. $(\lambda x. [\gamma] e) v \rightsquigarrow [\gamma, v/x] e$

Reduction

5. $[\gamma, v/x] \in V_{\Gamma, x:A}$

Definition of V_Γ

Proof

Case

$$\frac{\Gamma, x:A \vdash e:B}{\Gamma \vdash \lambda x.e : A \rightarrow B}$$

0. $\Gamma, x:A \vdash e:B$

Subderivation

1. $\gamma \in V_\Gamma$

Assumption

2. $[\gamma] \lambda x.e = \lambda x. [\gamma] e$

Definition

WTS: $\lambda x. [\gamma] e \in V_{A \rightarrow B}$

3. Assume $v \in V_A$

WTS $(\lambda \bar{x}. [\gamma] e) v \in E_B$

4. $(\lambda x. [\gamma] e) v \rightsquigarrow [\gamma, v/x] e$

Reduction

5. $[\gamma, v/x] \in V_{\Gamma, x:A}$

Definition of V_Γ

6. $[\gamma, v/x] e \in E_B$

Induction on (0), (5)

Proof

Case

$$\frac{\Gamma, x:A \vdash e:B}{\Gamma \vdash \lambda x.e : A \rightarrow B}$$

0. $\Gamma, x:A \vdash e:B$

Subderivation

1. $\gamma \in V_\Gamma$

Assumption

2. $[\gamma] \lambda x.e = \lambda x. [\gamma] e$

Definition

WTS: $\lambda x. [\gamma] e \in V_{A \rightarrow B}$

3. Assume $v \in V_A$

WTS $(\lambda x. [\gamma] e) v \in E_B$

4. $(\lambda x. [\gamma] e) v \rightsquigarrow [\gamma, v/x] e$

Reduction

5. $[\gamma, v/x] \in V_{\Gamma, x:A}$

Definition of V_Γ

6. $[\gamma, v/x] e \in E_B$

Induction on (0), (5)

7. $(\lambda x. [\gamma] e) v \in E_B$

Closure on (4), (6)

Proof

Case	$\frac{\Gamma, x:A \vdash e:B}{\Gamma \vdash \lambda x.e : A \rightarrow B}$
0.	$\Gamma, x:A \vdash e:B$ Subderivation
1.	$\gamma \in V_\Gamma$ Assumption
2.	$[\gamma] \lambda x.e = \lambda x. [\gamma] e$ Definition
	WTS: $\lambda x. [\gamma] e \in V_{A \rightarrow B}$
3.	Assume $v \in V_A$
	WTS $(\lambda \bar{x}. [\gamma] e) v \in E_B$
4.	$(\lambda x. [\gamma] e) v \rightsquigarrow [\gamma, v/x] e$ Reduction
5.	$[\gamma, v/x] \in V_{\Gamma, x:A}$ Definition of V_Γ
6.	$[\gamma, v/x] e \in E_B$ Induction on (0), (5)
7.	$(\lambda x. [\gamma] e) v \in E_B$ Closure on (4), (6)
8.	$\lambda x. [\gamma] e \in V_{A \rightarrow B}$ Definition of $V_{A \rightarrow B}$

Proof

Case

$$\frac{\Gamma, x:A \vdash e:B}{\Gamma \vdash \lambda x.e : A \rightarrow B}$$

0. $\Gamma, x:A \vdash e:B$

Subderivation

1. $\gamma \in V_\Gamma$

Assumption

2. $[\gamma] \lambda x.e = \lambda x.[\gamma]e$

Definition

WTS: $\lambda x.[\gamma]e \in V_{A \rightarrow B}$

3. Assume $v \in V_A$

WTS $(\lambda \bar{x}.[\gamma]e)v \in E_B$

4. $(\lambda x.[\gamma]e)v \rightsquigarrow [\gamma, v/x]e$

Reduction

5. $[\gamma, v/x] \in V_{\Gamma, x:A}$

Definition of V_Γ

6. $[\gamma, v/x]e \in E_B$

Induction on (0), (5)

7. $(\lambda x.[\gamma]e)v \in E_B$

Closure on (4), (6)

8. $\lambda x.[\gamma]e \in V_{A \rightarrow B}$

Definition of $V_{A \rightarrow B}$

9. $\lambda x.[\gamma]e \in E_{A \rightarrow B}$

$V_{A \rightarrow B} \subseteq E_{A \rightarrow B}$

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \rightarrow B \quad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1 e_2 : B}$$

1. $\tau \in V_r$
2. $\Gamma \vdash e_1 : A \rightarrow B$
3. $\Gamma \vdash e_2 : A$

Assumption
Subderivation
Subderivation

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \rightarrow B \quad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1 e_2 : B}$$

1. $\tau \in V_\Gamma$
2. $\Gamma \vdash e_1 : A \rightarrow B$
3. $\Gamma \vdash e_2 : A$
4. $[\tau] e_1 \in \ell_{A \rightarrow B}$

Assumption

Subderivation

Subderivation

Induction on 1,2

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \rightarrow B \quad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1 e_2 : B}$$

1. $\tau \in V_\Gamma$
2. $\Gamma \vdash e_1 : A \rightarrow B$
3. $\Gamma \vdash e_2 : A$
4. $[\delta] e_1 \in \mathcal{E}_{A \rightarrow B}$
5. $[\delta] e_1 \rightsquigarrow^* v_1, v_1 \in V_{A \rightarrow B}$

Assumption

Subderivation

Subderivation

Induction on 1,2

Def. of $\mathcal{E}_{A \rightarrow B}$

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \rightarrow B \quad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1 e_2 : B}$$

1. $\tau \in V_\Gamma$
2. $\Gamma \vdash e_1 : A \rightarrow B$
3. $\Gamma \vdash e_2 : A$
4. $[\sigma] e_1 \in \mathcal{E}_{A \rightarrow B}$
5. $[\sigma] e_1 \rightsquigarrow^* v_1, v_1 \in V_{A \rightarrow B}$
6. $[\sigma] e_2 \in \mathcal{E}_A$

Assumption

Subderivation

Subderivation

Induction on 1,2

Def. of $\mathcal{E}_{A \rightarrow B}$

Induction on 1,3

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \rightarrow B \quad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1 e_2 : B}$$

1. $\tau \in V_\Gamma$
2. $\Gamma \vdash e_1 : A \rightarrow B$
3. $\Gamma \vdash e_2 : A$
4. $[\delta] e_1 \in \mathcal{E}_{A \rightarrow B}$
5. $[\delta] e_1 \rightsquigarrow^* v_1, v_1 \in V_{A \rightarrow B}$
6. $[\delta] e_2 \in \mathcal{E}_A$
7. $[\delta] e_2 \rightsquigarrow^* v_2, v_2 \in V_A$

Assumption

Subderivation

Subderivation

Induction on 1,2

Def. of $\mathcal{E}_{A \rightarrow B}$

Induction on 1,3

Def. of \mathcal{E}_A

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \rightarrow B \quad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1 e_2 : B}$$

1. $\tau \in V_\Gamma$
2. $\Gamma \vdash e_1 : A \rightarrow B$
3. $\Gamma \vdash e_2 : A$
4. $[\delta] e_1 \in \mathcal{E}_{A \rightarrow B}$
5. $[\delta] e_1 \rightsquigarrow^* v_1, v_1 \in V_{A \rightarrow B}$
6. $[\delta] e_2 \in \mathcal{E}_A$
7. $[\delta] e_2 \rightsquigarrow^* v_2, v_2 \in V_A$
8. $[\delta] e_1 [\delta] e_2 \rightsquigarrow^* v_1 [\delta] e_2$

Assumption

Subderivation

Subderivation

Induction on 1,2

Def. of $\mathcal{E}_{A \rightarrow B}$

Induction on 1,3

Def. of \mathcal{E}_A

Reduction rules on 5

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \rightarrow B \quad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1 e_2 : B}$$

1. $\tau \in V_\Gamma$
2. $\Gamma \vdash e_1 : A \rightarrow B$
3. $\Gamma \vdash e_2 : A$
4. $[\tau]e_1 \in \mathcal{E}_{A \rightarrow B}$
5. $[\tau]e_1 \rightsquigarrow^* v_1, v_1 \in V_{A \rightarrow B}$
6. $[\tau]e_2 \in \mathcal{E}_A$
7. $[\tau]e_2 \rightsquigarrow^* v_2, v_2 \in V_A$
8. $[\tau]e_1, [\tau]e_2 \rightsquigarrow^* v_1, [\tau]e_2$
9. $v_1, [\tau]e_2 \rightsquigarrow^* v_1, v_2$

Assumption

Subderivation

Subderivation

Induction on 1,2

Def. of $\mathcal{E}_{A \rightarrow B}$

Induction on 1,3

Def. of \mathcal{E}_A

Reduction rules on 5

Reduction rules on 7

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \rightarrow B \quad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1 e_2 : B}$$

1. $\tau \in V_\Gamma$
2. $\Gamma \vdash e_1 : A \rightarrow B$
3. $\Gamma \vdash e_2 : A$
4. $[\tau]e_1 \in E_{A \rightarrow B}$
5. $[\tau]e_1 \rightsquigarrow^* v_1, v_1 \in V_{A \rightarrow B}$
6. $[\tau]e_2 \in E_A$
7. $[\tau]e_2 \rightsquigarrow^* v_2, v_2 \in V_A$
8. $[\tau]e_1, [\tau]e_2 \rightsquigarrow^* v_1, [\tau]e_2$
9. $v_1, [\tau]e_2 \rightsquigarrow^* v_1, v_2$
10. $v_1, v_2 \in E_B$

Assumption

Subderivation

Subderivation

Induction on 1,2

Def. of $E_{A \rightarrow B}$

Induction on 1,3

Def. of E_A

Reduction rules on 5

Reduction rules on 7

Def of $V_{A \rightarrow B}$

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \rightarrow B \quad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1 e_2 : B}$$

1. $\tau \in V_\Gamma$
2. $\Gamma \vdash e_1 : A \rightarrow B$
3. $\Gamma \vdash e_2 : A$
4. $[\tau]e_1 \in E_{A \rightarrow B}$
5. $[\tau]e_1 \rightsquigarrow^* v_1, v_1 \in V_{A \rightarrow B}$
6. $[\tau]e_2 \in E_A$
7. $[\tau]e_2 \rightsquigarrow^* v_2, v_2 \in V_A$
8. $[\tau]e_1, [\tau]e_2 \rightsquigarrow^* v_1, [\tau]e_2$
9. $v_1, [\tau]e_2 \rightsquigarrow^* v_1, v_2$
10. $v_1, v_2 \in E_B$
11. $[\tau]e_1, [\tau]e_2 \in E_B$

Assumption

Subderivation

Subderivation

Induction on 1,2

Def. of $E_{A \rightarrow B}$

Induction on 1,3

Def. of E_A

Reduction rules on 5

Reduction rules on 7

Def of $V_{A \rightarrow B}$

Closure x2

Proof

Case

$$\frac{\Gamma \vdash e_1 : A \rightarrow B \quad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1 e_2 : B}$$

1. $\tau \in V_\Gamma$
2. $\Gamma \vdash e_1 : A \rightarrow B$
3. $\Gamma \vdash e_2 : A$
4. $[\tau]e_1 \in \mathcal{E}_{A \rightarrow B}$
5. $[\tau]e_1 \rightsquigarrow^* v_1, v_1 \in V_{A \rightarrow B}$
6. $[\tau]e_2 \in \mathcal{E}_A$
7. $[\tau]e_2 \rightsquigarrow^* v_2, v_2 \in V_A$
8. $[\tau]e_1, [\tau]e_2 \rightsquigarrow^* v_1, [\tau]e_2$
9. $v_1, [\tau]e_2 \rightsquigarrow^* v_1, v_2$
10. $v_1, v_2 \in V_B$
11. $[\tau]e_1, [\tau]e_2 \in \mathcal{E}_B$
12. $[\tau](e_1, e_2) \in \mathcal{E}_B$

Assumption

Subderivation

Subderivation

Induction on 1,2

Def. of $\mathcal{E}_{A \rightarrow B}$

Induction on 1,3

Def. of \mathcal{E}_A

Reduction rules on 5

Reduction rules on 7

Def of $V_{A \rightarrow B}$

Closure x2

Def. of $[\tau]$

Termination

If $\cdot \vdash e : A$ then $e \rightsquigarrow^* \checkmark$

Proof:

1. $[] \in V$. Definition

2. $[e] \in \mathcal{E}_A$ Fundamental Lemma

3. $e \in \mathcal{E}_A$ Definition of $[e]$

4. $e \rightsquigarrow^* \checkmark$ Definition of \mathcal{E}_A