Exploring Convergence through the Computation of π

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1 Introduction

Convergence is a fundamental concept in numerical computation and analysis. It captures how a sequence of approximations approaches a target value. This report presents several methods of approximating π and discusses how convergence manifests differently in each. We also explore the philosophical and practical question: *What do we mean by "the answer"?*

2 Classical Method: Geometric Sandwiching

Archimedes (c. 250 BC) famously approximated π by inscribing and circumscribing polygons around a circle and calculating their perimeters.

Using n-sided polygons:

$$\text{Perimeter}(inscribed) < \text{Perimeter}(circle) \quad < \text{Perimeter}(circumscribed)$$

As $n \to \infty$, both perimeters converge to π.

Archimedes used 96-sided polygons to approximate:

$$3.1408 < \pi < 3.1429$$

3 Analytic Method: Taylor Series for Trigonometric Functions

The Taylor expansion[1] for $\arctan(x)$ around $x = 0$ gives:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

[1] In mathematics, the arctangent series, traditionally called Gregory's series, is the Taylor series expansion at the origin of the arctangent function.

Setting $x = 1$, we obtain:

$$\arctan(1) = \frac{\pi}{4}$$

thus:

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

This series converges very slowly but illustrates analytic convergence.

4 Modern Method: Number Theory and the Chudnovsky Algorithm

The Chudnovsky brothers developed a remarkably fast-converging series[2]:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(545140134k + 13591409)}{(3k)!(k!)^3 (640320)^{3k+3/2}}$$

This formula allows for the computation of millions of digits of π rapidly.

5 What Do We Mean by "The Answer"?

When approximating π, several questions arise:

- How many significant digits are we aiming for?
- How do we judge convergence speed (iterations, computational cost)?
- Is "the answer" a limit, an approximation within a tolerance, or a symbolic expression?

Different applications tolerate different degrees of approximation. Philosophically, "the answer" may vary between exact mathematical limits and practical engineering approximations.

6 Conclusion

By comparing methods rooted in geometry, analysis, and number theory, we appreciate different notions of convergence and the evolving idea of an "answer" in mathematics. Each method highlights trade-offs between conceptual simplicity, computational efficiency, and precision.

[2] The Chudnovsky algorithm.