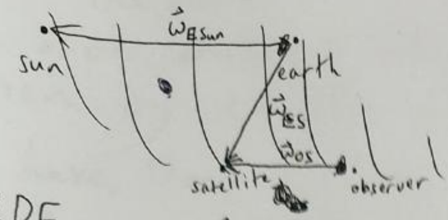
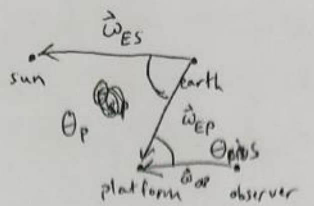
1



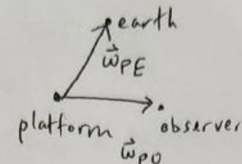


reflection relations

$BRDF_E = \frac{\rho_E}{\pi}$

$SSI: \varepsilon_s$

$BRDF_P = \frac{\rho_P}{\pi}$



emission relations

$\vec{\omega}_{AB} = -\vec{\omega}_{BA}$

θ_p: phase angle

θ_{pp}: platform phase angle

$L_{EP}(\vec{\omega}_{EP})$: radiance reflected from a point on the Earth to a point on the satellite

$$L_{EP}(\vec{\omega}_{EP}) = \int_\Omega BRDF_E(\vec{\omega}_{Ei}, \vec{\omega}_{EP})\, dE(\vec{\omega}_{Ei})$$

$$L_{Ei}(\vec{\omega}_{Ei}) = \delta(\vec{\omega}_{Ei} - \vec{\omega}_{ES})\,\varepsilon_s \qquad = \int_\Omega BRDF_E(\vec{\omega}_{Ei}, \vec{\omega}_{EP}) \langle \hat{n}_E, \hat{\omega}_{Ei} \rangle\, d\sigma(\hat{\omega}_{Ei})\, L_{Ei}(\vec{\omega}_{Ei})$$

$$= \int_\Omega \frac{\rho_E}{\pi} \langle \hat{n}_E, \hat{\omega}_{Ei} \rangle\, d\sigma(\omega_{Ei})\, L_{Ei}(\vec{\omega}_{Ei}) = \varepsilon_s \frac{\rho_E}{\pi} \langle \hat{n}_E, \hat{\omega}_{ES} \rangle = L_{EP}(\vec{\omega}_{EP})$$

$$L_{PO}(\vec{\omega}_{PO}) = \int_\Omega BRDF_P(\vec{\omega}_{PE}, \vec{\omega}_{PO})\, dE(\vec{\omega}_{PE}) = \int_\Omega BRDF_P(\vec{\omega}_{PE}, \vec{\omega}_{PO}) \langle \hat{n}_P, \hat{\omega}_{PE} \rangle\, d\sigma(\vec{\omega}_{PE})\, L_{PE}(\vec{\omega}_{PE})$$

$$= \int_\Omega \frac{\rho_P}{\pi} \langle \hat{n}_P, \hat{\omega}_{PE} \rangle\, d\sigma(\hat{\omega}_{PE})\, \frac{\varepsilon_s \rho_E}{\pi} \langle \hat{n}_E, \hat{\omega}_{ES} \rangle$$

$\vec{n}_P$: normal vector pointing out from point on surface of satellite

$\hat{n}_E$: normal vector pointing out from point on surface of Earth

Ω: solid angles formed by reflecting from a point on earth to a point on the satellite.

Now, need to parametrize the expression according to coordinate system on Earth. recall that $d\sigma$ represents a surface element on the unit sphere around the satellite. the location of that element is given by $\hat{\omega}_{PE}$. we need to convert $d\sigma(\hat{\omega}_{PE})$ to a measure that allows us to integrate over the Earth. $\langle \hat{n}_p, \hat{\omega}_{PE} \rangle$ is the cosine for the projected area on the surface of the satellite. ~~we need to account for the~~ we need to create an area $dA(\hat{\omega}_{PE})$ that allows us to write $d\sigma(\hat{\omega}_{PE}) = dA(\hat{\omega}_{PE}) / |\vec{\omega}_{PE}|^2$. ~~which~~ This area is a projection from the Earth surface area element onto a ~~surface~~ sphere of radius $|\vec{\omega}_{PE}|$.

Given the normal vector $\hat{n}_{E}$, defined for the area element dA_E, the area ~~a~~ element $dA(\hat{\omega}_{PE})$ is the projection of dA_E, which is given by $dA_E\langle \hat{n}_E, \hat{\omega}_{EP}\rangle$. Thus, $d\sigma(\hat{\omega}_{PE}) = \frac{dA_E\langle \hat{n}_E, \hat{\omega}_{EP}\rangle}{|\hat{\omega}_{PE}|^2} = \frac{dA_E\langle \hat{n}_E, -\hat{\omega}_{PE}\rangle}{|\vec{\omega}_{PE}|^2}$

We have, then,

$$= \int_\Omega \frac{\rho_P}{\pi}\langle \hat{n}_P, \hat{\omega}_{PE}\rangle \frac{E_S \rho_E}{\pi}\langle \hat{n}_E, \hat{\omega}_{ES}\rangle \frac{\langle \hat{n}_E, -\hat{\omega}_{PE}\rangle}{|\vec{\omega}_{PE}|^2}\, dA_E$$

Now, $\hat{\omega}_{ES}$ is a unit vector from a surface element on the Earth to the sun. $\hat{\omega}_{EP}$ is similar save it goes to the platform (satellite). $\hat{n}_P$ is a unit vector pointing out from the satellite surface. $\vec{\omega}_{sun}$ is the ~~dis~~ vector from Earth center to the sun (point particle approximation for sun). Thus, $\vec{\omega}_{ES} = \vec{\omega}_{sun} - \vec{n}_E$, and $\hat{\omega}_{ES} \approx \frac{1}{|\vec{\omega}_{sun}|}(\vec{\omega}_{sun} - \vec{n}_E)$.

$\hat{\omega}_{PE}$ is similarly $\vec{\omega}_{PE} = \vec{\omega}_{Platform} - \vec{n}_P$ and $\hat{\omega}_{PE} \approx \frac{1}{|\vec{\omega}_{Platform}|}(\vec{\omega}_{Platform} - \vec{n}_P)$

Later, we will integrate over the satellite-centered coordinate system, so we can define $\vec{n}_P$ in terms of (R_P, θ_P, ϕ_P), where ~~R_P, θ_P~~ these parameters define spherical coordinates around the satellite.

We can evaluate $\langle \hat{n}_P, \hat{\omega}_{PE}\rangle = \langle \hat{n}_P, \hat{\omega}_{Platform}\rangle - \langle \hat{n}_P, \frac{\vec{n}_P}{|\vec{\omega}_{platform}|}\rangle$

$= \langle \hat{n}_P, \hat{\omega}_{platform}\rangle - \langle \hat{n}_P, \hat{n}_P\left(\frac{R_P}{|\vec{\omega}_{platform}|}\right)\rangle \approx \langle \hat{n}_P, \hat{\omega}_{platform}\rangle$ for $\frac{R_P}{|\vec{\omega}_{platform}|} \to 0$

and $\langle \hat{n}_E, \hat{\omega}_{ES}\rangle = \langle \hat{n}_E, \hat{\omega}_{sun}\rangle - \langle \hat{n}_E, \frac{\vec{n}_E}{|\vec{\omega}_{sun}|}\rangle = \langle \hat{n}_E, \hat{\omega}_{sun}\rangle - \langle \hat{n}_E, \frac{R_E}{|\vec{\omega}_{sun}|}\hat{n}_E\rangle$

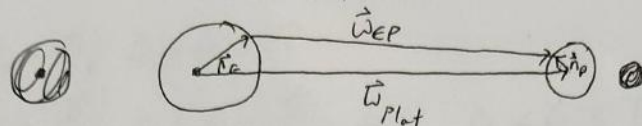
$\approx \langle \hat{n}_E, \hat{\omega}_{sun}\rangle$ for $\frac{R_E}{|\vec{\omega}_{sun}|} \to 0$.

Thus,

$$= \int_\Omega \frac{\rho_P \rho_E E_S}{\pi^2}\langle \hat{n}_P, \hat{\omega}_{Platform}\rangle\langle \hat{n}_E, \hat{\omega}_{sun}\rangle \frac{\langle \hat{n}_E, -\hat{\omega}_{PE}\rangle}{|\vec{\omega}_{PE}|^2}\, dA_E$$

$\vec{\omega}_{PE}$ is more difficult. $\vec{\omega}_{platform}$ is the vector ~~distance~~ from ~~between~~ ~~Earth~~ platform to Earth. so $-\vec{\omega}_{platform}$ is from Earth to platform in Earth coordinates.

$\vec{\omega}_{PE}$ is more difficult. let $\vec{\omega}_{plat}$ be the center-to-center vector from Earth to platform. we then make the following picture to illustrate the relation between $\vec{\omega}_{EP}$, $\vec{n}_P$, and $\vec{n}_E$.



so $\vec{\omega}_{EP} + \vec{n}_E = \vec{\omega}_{Plat} + \vec{n}_P \Rightarrow \vec{\omega}_{PE} = \vec{\omega}_{plat} + \vec{n}_p - \vec{n}_E \Rightarrow \underbrace{\vec{\omega}_{PE} = \vec{n}_E - \vec{\omega}_{plat} - \vec{n}_p}$

Thus, $\hat{\omega}_{PE} = \frac{1}{|\vec{\omega}_{PE}|}(\vec{n}_E - \vec{\omega}_{plat} - \vec{n}_P)$

Now, we can evaluate $\langle \hat{n}_E, \hat{\omega}_{ES} \rangle = \langle \hat{n}_E, \hat{\omega}_{sun} \rangle - \langle \hat{n}_E, \frac{\vec{r}_E}{|\vec{\omega}_{sun}|} \rangle$

$= \langle \hat{n}_E, \hat{\omega}_{sun} \rangle - \langle \hat{n}_E, \frac{R_E}{|\vec{\omega}_{sun}|}\hat{n}_E \rangle \approx \underbrace{\langle \hat{n}_E, \hat{\omega}_{sun} \rangle}$ for $\frac{R_E}{|\vec{\omega}_{sun}|} \to 0$

and

$\langle \hat{n}_P, \hat{\omega}_{PE} \rangle = \underbrace{\langle \hat{n}_p, \frac{R_E}{|\vec{\omega}_{PE}|}\hat{n}_E \rangle - \langle \hat{n}_p, \frac{|\vec{\omega}_{Plat}|}{|\vec{\omega}_{PE}|}\hat{\omega}_{plat} \rangle - \langle \hat{n}_P, \frac{R_P}{|\vec{\omega}_{PE}|}\hat{n}_P \rangle} = \underbrace{\frac{1}{|\vec{\omega}_{PE}|}\langle \hat{n}_P, R_E\hat{n}_E - |\vec{\omega}_{plat}|\hat{\omega}_{Plat} \rangle}$

for $\frac{R_P}{|\vec{\omega}_{PE}|} \to 0$ the last term is zero

now $\langle \hat{n}_E, -\hat{\omega}_{PE} \rangle = -\langle \hat{n}_E, \hat{\omega}_{PE} \rangle = -\langle \hat{n}_E, \frac{\vec{n}_E}{|\vec{\omega}_{PE}|} \rangle + \langle \hat{n}_E, \frac{\vec{\omega}_{Plat}}{|\vec{\omega}_{PE}|} \rangle + \underbrace{\langle \hat{n}_E, \frac{\vec{n}_P}{|\vec{\omega}_{PE}|} \rangle}_{\simeq 0}$

$= \frac{-1}{|\vec{\omega}_{PE}|}\langle \hat{n}_E, R_E\hat{n}_E - |\vec{\omega}_{plat}|\hat{\omega}_{plat} \rangle$

Thus,

$$= \int_{\Omega} \frac{\rho_P}{\pi}\left(\frac{1}{|\vec{\omega}_{PE}|}\langle \hat{n}_P, R_E\hat{n}_E - |\vec{\omega}_{plat}|\hat{\omega}_{plat} \rangle\right) \frac{\varepsilon_S \rho_E}{\pi}\langle \hat{n}_E, \hat{\omega}_{sun} \rangle \frac{1}{|\vec{\omega}_{PE}|^2}\left(\frac{-1}{|\vec{\omega}_{PE}|}\langle \hat{n}_E, R_E\hat{n}_E - |\vec{\omega}_{Plat}|\hat{\omega}_{Plat} \rangle\right) dA_E$$

we need an expression for $\vec{\omega}_{plat}$. we define $\vec{\omega}_{plat}$ as $(0, D_P, 0)$ in (x,y,z). let θ_p be defined in the θ-plane of Earth. $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$. so $\vec{\omega}_{plat}$ in (r,θ,ϕ) is $(D_P, \frac{\pi}{2}, \frac{\pi}{2})$.

for $\vec{\omega}_{sun}$, we rotate $\theta = \theta_P$ in the θ-plane, giving $(D_S\cos\theta_P, D_S\sin\theta_P, 0)$ where D_S is $|\vec{\omega}_{sun}|$ and D_p is $|\vec{\omega}_{plat}|$.

wrong see below $\hat{\omega}_{Plat} = (0,1,0)$ and $\hat{\omega}_{sun} = ~~(\cos\theta_P, \sin\theta_P, 0)~~$. We define $\hat{n}_E$ as $(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$ and $\hat{n}_p$ as $(\cos\theta_2\sin\phi_2, \sin\theta_2\sin\phi_2, \cos\phi_2)$ where (θ_2, ϕ_2) are defined as spherical coordinates centered around the satellite.

$|\vec{\omega}_{PE}|$ can be found by noting that $\vec{\omega}_{EP} \approx \vec{\omega}_{plat} - \vec{n}_E$

$= (-R_E\cos\theta\sin\phi, D_P - R_E\sin\theta\sin\phi, -R_E\cos\phi)$

actually, $\hat{\omega}_{sun} = (\cos(\frac{\pi}{2}-\theta_p), \sin(\frac{\pi}{2}-\theta_p), 0)$

The bottom 4th-power makes the problem intractable.
instead, approximate by average of $\sin\theta\sin\phi$ for integration boundaries

for $\theta_p = 0,\ 0 \le \theta \le \pi$

for $\theta_p = \frac{\pi}{2},\ 0 \le \theta \le \frac{\pi}{2} \Rightarrow 0 \le \theta \le \pi - \theta_p$

for $\theta_p = \pi,\ 0 \le \theta \le 0$

$$\frac{\iint \sin\theta \sin^2\phi \, d\phi \, d\theta}{\iint \sin\phi \, d\phi \, d\theta} = \frac{1}{2(\pi-\theta_p)} \int_0^{\pi-\theta_p}\int_0^{\pi} \sin\theta \sin^2\phi \, d\phi \, d\theta$$

$$= \frac{1}{2(\pi-\theta_p)} \int_0^{\pi-\theta_p} \left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right)\Big|_0^{\pi} \sin\theta \, d\theta = \frac{\pi}{2\cdot 2(\pi-\theta_p)} \int_0^{\pi-\theta_p} \sin\theta \, d\theta = \frac{\pi}{4(\pi-\theta_p)} (-\cos\theta)\Big|_0^{\pi-\theta_p}$$

$$= \frac{\pi}{4(\pi-\theta_p)}(1-\cos(\pi-\theta_p)) = \cancel{\frac{\pi}{4(\pi+\theta_p)}\cos(\theta_p)}$$

~~for $\theta_p \neq 0$, $\frac{\pi}{4(\pi-\theta_p)}$ for $\theta_p = \pi$~~

$$= \frac{\pi}{4(\pi-\theta_p)}(1+\cos\theta_p)$$

for $\theta_p = 0$, $\frac{\pi}{2(\pi-\theta_p)}$

for $\theta_p = \pi$, 0 by $\lim_{x\to\pi} \frac{1+\cos x}{\pi - x}$

so $$|\vec{\omega}_{PE}|^4 \approx \left(D_p^2 - \frac{2D_pR_E\pi(1+\cos\theta_p)}{4(\pi-\theta_p)} + R_E^2\right)^2$$

we will safely extricate this constant out of the integral

This leaves us with an integral like

$$k\int_\Omega \langle \hat{n}_p, R_E\hat{n}_E - D_p\hat{\omega}_{plat}\rangle \langle \hat{n}_E, \hat{\omega}_{sun}\rangle \langle \hat{n}_E, R_E\hat{n}_E - D_p\hat{\omega}_{plat}\rangle \, dA_E$$ where k contains all constants

Expanding,

$$= k\int_\Omega (n_{p1}R_E\cos\theta\sin\phi + n_{p2}(R_E\sin\theta\sin\phi - D_p) + n_{p3}R_E\cos\phi)(\cos\theta\sin\phi\sin\theta_p + \sin\theta\sin\phi\cos\theta_p)$$
$$\cdot(R_E\cos^2\theta\sin^2\phi + R_E\sin^2\theta\sin^2\phi - D_p\sin\theta\sin\phi + R_E\cos^2\phi)R_E^2\sin\phi \, d\phi \, d\theta$$

where Ω: $0 \le \theta \le \pi - \theta_p$, $0 \le \phi \le \pi$

$\frac{1}{384}(\sin\theta_p\cos\theta + \cos\theta_p\sin\theta)$ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22,

$\cdot\big(-72 n_{p1} D_p R_E \pi\sin 2\theta + 16 n_{p1} R_E^2(\cos(\theta-3\pi)-\cos\theta) - 144 n_{p1} R_E^2(\cos(\theta-\pi)-\cos\theta)$

$-144 n_{p1} R_E^2(\cos(\theta+\pi)-\cos\theta) + 16 n_{p1} R_E^2(\cos(\theta+3\pi)-\cos\theta) + 16 n_{p2} D_p^2(\sin(\theta-3\pi)-\sin\theta)$

$-144 n_{p2} D_p^2(\sin(\theta-\pi)-\sin\theta) - 144 n_{p2} D_p^2(\sin(\theta+\pi)-\sin\theta) + 16 n_{p2} D_p^2(\sin(\theta+3\pi)-\sin\theta)$

$+72 n_{p2} D_p R\pi\cos 2\theta - 264 n_{p2} D_p R\pi + \cancel{264 n_{p2} D_p R_E} + 16 n_{p2} R_E^2(\sin(\theta-3\pi)-\sin\theta)$

$-144 n_{p2} R_E^2(\sin(\theta-\pi)-\sin\theta) - 144 n_{p2} R_E^2(\sin(\theta+\pi)-\sin\theta) + 16 n_{p2} R_E^2(\sin(\theta+3\pi)-\sin\theta)$

$= \frac{1}{384}(\sin\theta_p\cos\theta + \cos\theta_p\sin\theta)$

$\cdot\big(-72 n_{p1} D_p R_E\pi\sin 2\theta + 512 n_{p1} R_E^2\cos\theta + 512 n_{p2} D_p^2\sin\theta + 72 n_{p2} D_p R\pi\cos 2\theta$

$-264 n_{p2} D_p R\pi + 512 n_{p2} R_E^2\sin\theta\big)$

$R_E^2 k\int_0^{\pi-\theta_p}$ —— stuff above $d\theta$

$= R_E^2 k\Big[\frac{3}{32}\pi n_{p1} D_p\sin\theta_p R_E\cos\theta + \frac{1}{32}\pi n_{p1} D_p\sin\theta_p R_E\cos 3\theta - \frac{3}{32}\pi n_{p1} D_p\cos\theta_p R_E\sin\theta$

$+\frac{1}{32}\pi n_{p1} D_p\cos\theta_p R\sin 3\theta + \frac{2}{3} n_{p1}\sin\theta_p R_E^2\theta + \frac{1}{3} n_{p1}\sin\theta_p R_E^2\sin 2\theta - \frac{1}{3} n_{p1}\cos\theta_p R_E^2\cos 2\theta$

$-\frac{1}{3} n_{p2} D_p^2\sin\theta_p\cos 2\theta + \frac{2}{3} n_{p2} D_p^2\cos\theta_p\theta - \frac{1}{3} n_{p2} D_p^2\cos\theta_p\sin 2\theta - \frac{19}{32}\pi n_{p2} D_p\sin\theta_p R\sin\theta$

$+\frac{1}{32}\pi n_{p2} D_p\sin\theta_p R_E\sin 3\theta + \frac{25}{32}\pi n_{p2} D_p\cos\theta_p R\cos\theta - \frac{1}{32}\pi n_{p2} D_p\cos\theta_p R\cos 3\theta$

$-\frac{1}{3} n_{p2}\sin\theta_p R_E^2\cos 2\theta + \frac{2}{3} n_{p2}\cos\theta_p R_E^2\theta - \frac{1}{3} n_{p2}\cos\theta_p R_E^2\sin 2\theta\Big]\Big|_0^{\pi-\theta_p}$

$= R_E^2 k\Big[\frac{-3}{32}\pi n_{p1} D_p \cancel{\sin}\theta_p R_E\cos\theta_p - \cancel{\frac{3}{32}\pi n_{p1} D_p}\sin\theta_p R_E + \frac{1}{32}\pi n_{p1} D_p\sin\theta_p R_E(\cos 3\theta_p)$

$\cancel{\frac{1}{32}\pi n_{p1} D_p}\sin\theta_p R_E - \frac{3}{32}\pi n_{p1}\cancel{D_p\cos}\theta_p R_E\sin\theta_p + \frac{1}{32}\pi n_{p1} D_p\cos\theta_p R_E\sin 3\theta_p$

$+\frac{1}{3} n_{p1}\cos\theta_p R_E^2$

$+\frac{2}{3} n_{p1}\cancel{\sin\theta_p R_E^2}(\pi-\theta_p) + \frac{1}{3} n_{p1}\sin\theta_p R_E^2\sin(-2\theta_p) - \frac{1}{3} n_{p1}\cos\theta_p R_E^2\cos(-2\theta_p)$

$-\frac{1}{3} n_{p2} D_p^2\sin\theta_p\cos(-2\theta_p) + \cancel{\frac{1}{3} n_{p2} D_p^2}\sin\theta_p + \frac{2}{3} n_{p2} D_p^2\cos\theta_p(\pi-\theta_p) - \frac{19}{32}\pi n_{p2} D_p\sin\theta_p R_E\sin\theta_p$

$+\frac{1}{32}\pi n_{p2} D_p\sin\theta_p R_E\sin 3\theta_p - \frac{25}{32}\pi n_{p2} D_p\cos\theta_p R_E\cos\theta_p - \frac{25}{32}\pi n_{p2} D_p\cos\theta_p R_E$

$+\frac{1}{32}\pi n_{p2} D_p\cos\theta_p R_E\cos 3\theta_p - \cancel{\frac{1}{3} n_{p2}\sin\theta_p R_E^2\cos(-2\theta_p)} + \frac{1}{32}\pi n_{p2} D_p\cos\theta_p R_E$

$-\frac{1}{3} n_{p2}\sin\theta_p R_E^2\cos(-2\theta_p) + \frac{1}{3} n_{p2}\sin\theta_p R_E^2 + \frac{2}{3} n_{p2}\cos\theta_p R_E^2(\pi-\theta_p) - \frac{1}{3} n_{p2}\cos\theta_p R_E^2\sin(-2\theta_p)$

$-\frac{1}{3} n_{p2} D_p^2\cos\theta_p\sin(-2\theta_p)\Big]$

simplifying the above is too tedious. ignore the cross outs and gather according to n_{p1} and n_{p2}

$$\cancel{\frac{-6}{32}\pi n_{p1} D_p \sin\theta_p R_E \cos\theta_p}$$

$$\cancel{\frac{-4}{32}\pi n_{p1} D_p R_E + \frac{2}{3} n_{p1} R_E^2 (\pi-\theta_p)} + \frac{1}{3}$$

$$= R_E^2 k\left[k_{p1} n_{p1} + k_{p2} n_{p2}\right]$$

calculate k_{p1} and k_{p2}

$$k_{p1} = \frac{\pi D_p R_E}{32}\left(-3\sin\theta_p\cos\theta_p - 3\sin\theta_p - \sin\theta_p\cos 3\theta_p - \sin\theta_p - 3\sin\theta_p\cos\theta_p + \cos\theta_p \sin 3\theta_p\right) + \frac{R_E^2}{3}\left(2\sin\theta_p(\pi-\theta_p) - \sin\theta_p\sin 2\theta_p - \cos\theta_p\cos 2\theta_p + \cos\theta_p\right)$$

$$= \frac{\pi D_p R_E}{32}\left(-3\sin\theta_p(1+\cos\theta_p) - \sin\theta_p(1+\cos 3\theta_p) + \cos\theta_p(\sin 3\theta_p - \cancel{3\sin\theta_p})\right) + \frac{R_E^2}{3}\left(\sin\theta_p(2(\pi-\theta_p) - \sin 2\theta_p) + (1-\cos 2\theta_p)\cos\theta_p\right)$$

$$= \cancel{\frac{\pi D_p R_E}{32}\left(-3\sin\theta_p(1+2\cos\theta_p)\right)}$$

$$= \frac{\pi D_p R_E}{32}\left(-2\sin\theta_p(2+3\cos\theta_p) - \sin\theta_p\cos 3\theta_p + \cos\theta_p\sin 3\theta_p\right) + \frac{R_E^2}{3}\left(\sin\theta_p(2(\pi-\theta_p) - \sin 2\theta_p) + (1-\cos 2\theta_p)\cos\theta_p\right)$$

$$= \left\{\frac{\pi D_p R_E}{32}\left(-2\sin\theta_p\left(2+3\cos\theta_p + \frac{1}{2}\cos 3\theta_p\right) + \cos\theta_p\sin 3\theta_p\right) + \frac{R_E^2}{3}\left(\sin\theta_p(2(\pi-\theta_p) - \sin 2\theta_p) + (1-\cos 2\theta_p)\cos\theta_p\right)\right\}$$

$$k_{p2} = \frac{D_p^2}{3}\left(-\sin\theta_p\cos 2\theta_p + \sin\theta_p + 2\cos\theta_p(\pi-\theta_p) + \cos\theta_p\sin 2\theta_p\right) + \frac{\pi D_p R_E}{32}\left(-19\sin^2\theta_p + \sin\theta_p\sin 3\theta_p - 25\cos^2\theta_p - 25\cos\theta_p + \cos\theta_p\cos 3\theta_p + \cos\theta_p\right) + \frac{R_E^2}{3}\left(-\sin\theta_p\cos 2\theta_p + \sin\theta_p + 2\cos\theta_p(\pi-\theta_p) + \cos\theta_p\sin 2\theta_p\right)$$

$$= \frac{D_p^2}{3}\left(\sin\theta_p(1-\cos 2\theta_p) + \cos\theta_p(2(\pi-\theta_p) + \sin 2\theta_p)\right) + \frac{\pi D_p R_E}{32}\left(\sin\theta_p(\sin 3\theta_p - 19\sin\theta_p) + \cos\theta_p(\cos 3\theta_p - 25\cos\theta_p - 24)\right) + \frac{R_E^2}{3}\left(\sin\theta_p(1-\cos 2\theta_p) + \cos\theta_p(2(\pi-\theta_p) + \sin 2\theta_p)\right)$$

$$\left\{ \begin{aligned} &\frac{D_p^2+R_E^2}{3}\left(\sin\theta_p(1-\cos 2\theta_p)+\cos\theta_p[2(\pi-\theta_p)+\sin 2\theta_p]\right) \\ &+\frac{\pi D_p R_E}{32}\left(\sin\theta_p[\sin 3\theta_p-19\sin\theta_p]+\cos\theta_p[\cos 3\theta_p-25\cos\theta_p-24]\right) \end{aligned} \right\}$$

Finally, we have $L_{PO}(\vec{\omega}_{PO}) = R_E^2 k[k_{P1}n_{P1} + k_{P2}n_{P2}]$

where $k = \frac{-\rho_P \varepsilon_S \rho_E}{\pi^2}\left(D_p^2 - \frac{D_p R_E \pi(1+\cos\theta_p)}{2(\pi-\theta_p)} + R_E^2\right)^{-2}$

$$k_{P1} = \frac{\pi D_p R_E}{32}\left[-2\sin\theta_p\left(2+3\cos\theta_p+\tfrac{1}{2}\cos 3\theta_p\right)+\cos\theta_p\sin 3\theta_p\right] + \frac{R_E^2}{3}\left[\sin\theta_p(2(\pi-\theta_p)-\sin 2\theta_p)+\cos\theta_p(1-\cos 2\theta_p)\right]$$

$$k_{P2} = \frac{\pi D_p R_E}{32}\left[\sin\theta_p(\sin 3\theta_p-19\sin\theta_p)+\cos\theta_p(\cos 3\theta_p-25\cos\theta_p-24)\right] + \frac{D_p^2+R_E^2}{3}\left[\sin\theta_p(1-\cos 2\theta_p)+\cos\theta_p(2(\pi-\theta_p)+\sin 2\theta_p)\right]$$

I re-did the constants to reduce their size. See the back pages

all this could be expanded further to powers of $\cos\theta_p$ and $\sin\theta_p$ if needed.

Now, we need the radiant intensity from the satellite to the observer. The observer is approximated as a point. The differential radiant intensity is then

$$dI = L_{PO}(\vec{\omega}_{PO})\langle \hat{\omega}_{PO}, \hat{n}_P\rangle \, dA$$

we define new spherical coordinates centered at the satellite. (r_2, θ_2, ϕ_2). We also define the θ_2-plane as the plane formed by the center-to-center vector between the ~~platform~~ satellite to the observer $\vec{\omega}_{obs}$ and $-\vec{\omega}_{plat}$. Thus, θ_{obs} is the angle formed by rotating $-\vec{\omega}_{plat}$ to $\vec{\omega}_{obs}$. We further define $-\vec{\omega}_{plat}$ as $\vec{\omega}_{Earth}$ and $\vec{\omega}_{Earth} = (0, D_p, 0)$ in xyz. $x = r_2\cos\theta_2\sin\phi_2$, $y = r_2\sin\theta_2\sin\phi_2$, $z = r_2\cos\phi_2$. in spherical, $\vec{\omega}_{Earth} = (D_p, \frac{\pi}{2}, \frac{\pi}{2})$. $\vec{\omega}_{obs}$ is then $(D_{obs}\cos(\frac{\pi}{2}-\theta_{obs}), D_{obs}\sin(\frac{\pi}{2}-\theta_{obs}), 0) = (D_{obs}\sin\theta_{obs}, D_{obs}\cos\theta_{obs}, 0)$. Finally, the ~~exposed~~ illuminated area on the satellite is given by

$$0 \le \theta_2 \le \pi - \theta_{obs}, \quad 0 \le \phi_2 \le \pi.$$

and n_{p2} were defined in the previous coordinate system and must be rotated and translated into the new coordinate system. In other words, given the spherical coordinates (r_2, θ_2, ϕ_2) define at the center of the satellite, express n_{p1} and n_{p2} in terms of (r_2, θ_2, ϕ_2). We must first define the relation between the planes in which lies angles θ_p and θ_{obs}. The normal vector for the θ_p plane is found by taking $\hat{\omega}_{sun} \times \hat{\omega}_{plat}$. Similarly, we take $\hat{\omega}_{obs} \times \hat{\omega}_{Earth}$ for the normal in the θ_{obs} plane. However, we note that both of these calculations require all vectors to be defined in a common reference frame. We assume that we can convert them to such a frame (ex. G(RF). Also note $\hat{\omega}_{plat} = -\hat{\omega}_{Earth}$ and define the angle of rotation between the planes as

$$\theta_{rot} = \arccos\left((\hat{\omega}_{sun} \times \hat{\omega}_{plat}) \cdot (\hat{\omega}_{obs} \times \hat{\omega}_{Earth})\right)$$

$$= \arccos\left[(\hat{\omega}_{sun} \times \hat{\omega}_{plat}) \cdot (\hat{\omega}_{obs} \times -\hat{\omega}_{plat})\right] = \arccos\left[(\hat{\omega}_{obs} \cdot \hat{\omega}_{sun})(\hat{\omega}_{plat} \cdot -\hat{\omega}_{plat})\right.$$

$$\left. - (\hat{\omega}_{plat} \cdot \hat{\omega}_{obs})(\hat{\omega}_{sun} \cdot -\hat{\omega}_{plat})\right] = \arccos\left[(\hat{\omega}_{plat} \cdot \hat{\omega}_{obs})(\hat{\omega}_{plat} \cdot \hat{\omega}_{sun}) - (\hat{\omega}_{obs} \cdot \hat{\omega}_{sun})\right]$$

This angle is then a constant for the purposes of the proof.

Now, we need a relation between the coordinates of the Earth system and the spherical satellite coordinates. We will describe ~~how~~ each of the steps to translate and rotate from the ~~spherical~~ satellite ~~coordinatate~~ coordinates to the Earth coordinates. First, we note the rotation between the normals. The y-axes of the two systems are collinear save for a rotation of π around the z-axis, so the rotation of θ_{rot} occurs around the y-axis.(1) Then, we rotate π radians around the z-axis.(2) This makes the x, y, z axes ~~collinear~~ parallel, and the system differ only by a translation of $-D_p$ along the y-axis.(3)

for (1), the rotation matrix is $\begin{bmatrix} \cos\theta_{rot} & 0 & \sin\theta_{rot} \\ 0 & 1 & 0 \\ -\sin\theta_{rot} & 0 & \cos\theta_{rot} \end{bmatrix} = M_{rot}$

for (2), the rotation matrix is $\begin{bmatrix} \cos\pi & -\sin\pi & 0 \\ \sin\pi & \cos\pi & 0 \\ 0 & 0 & 1 \end{bmatrix} = M_z = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

for (3), we add the following vector $\begin{bmatrix} 0 \\ -D_p \\ 0 \end{bmatrix} = \vec{V}_3$ to the resultant of (2).

In sum, $M_z M_{rot} \vec{r} + \vec{V}_3$

or the purposes of the normal vector $\vec{n}_p$, we only need to apply the rotations. Applying the rotations

$$\begin{bmatrix} \cos\theta_{rot} & 0 & \sin\theta_{rot} \\ 0 & 1 & 0 \\ -\sin\theta_{rot} & 0 & \cos\theta_{rot} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta_{rot} + z\sin\theta_{rot} \\ y \\ -x\sin\theta_{rot} + z\cos\theta_{rot} \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -x' \\ -y' \\ z' \end{bmatrix} = \begin{bmatrix} -x\cos\theta_{rot} - z\sin\theta_{rot} \\ -y \\ -x\sin\theta_{rot} + z\cos\theta_{rot} \end{bmatrix}$$

The normal vector $\vec{n}_p$ is given by $(R_p\cos\theta_z\sin\phi_z, R_p\sin\theta_z\sin\phi_z, R_p\cos\phi_z)$ in the satellite's frame. Substituting into the rotated vector, and normalizing by R_p, we have

$$\hat{n}_p = \begin{bmatrix} -\cos\theta_z\sin\phi_z\cos\theta_{rot} - \cos\phi_z\sin\theta_{rot} \\ -\sin\theta_z\sin\phi_z \\ -\cos\theta_z\sin\phi_z\sin\theta_{rot} + \cos\phi_z\cos\theta_{rot} \end{bmatrix}$$

For an expression for $\hat{\omega}_{PO}$, we do as we did for $\hat{\omega}_{ES}$. The observer is a point and the magnitude of $\vec{\omega}_{obs}$ is much greater than $\vec{n}_p$.

$$\vec{\omega}_{PO} + \vec{n}_p = \vec{\omega}_{obs} \Rightarrow \vec{\omega}_{PO} = \vec{\omega}_{obs} - \vec{n}_p \Rightarrow \hat{\omega}_{PO} = \frac{1}{|\vec{\omega}_{PO}|}(\vec{\omega}_{obs} - \vec{n}_p) \approx \frac{1}{D_{obs}}(\vec{\omega}_{obs} - \vec{n}_p)$$

$= \hat{\omega}_{obs} - \frac{R_p}{D_{obs}}\hat{n}_p = \hat{\omega}_{obs}$ for $\frac{R_p}{D_{obs}} \to 0$. We then evaluate $\langle \hat{\omega}_{PO}, \hat{n}_p \rangle$ as

$$\langle \hat{\omega}_{PO}, \hat{n}_p \rangle \approx \langle \hat{\omega}_{obs}, \hat{n}_p \rangle = \sin\theta_{obs}\cos\theta_z\sin\phi_z + \cos\theta_{obs}\sin\theta_z\sin\phi_z$$

The differential intensity is then

$$dI = R_E^2 k\left(k_{p_1}(-\cos\theta_z\sin\phi_z\cos\theta_{rot} - \cos\phi_z\sin\theta_{rot}) + k_{p_2}(-\sin\theta_z\sin\phi_z)\right)$$

$$\cdot(\sin\theta_{obs}\cos\theta_z\sin\phi_z + \cos\theta_{obs}\sin\theta_z\sin\phi_z) R_p^2 \sin\phi_z \, d\phi_z \, d\theta \qquad \text{let } -k = k'$$

$$= R_E^2 R_p^2 k' \sin\phi_z \, d\phi_z \, d\theta \big(\sin\theta_{obs}\cos\theta_{rot}\cos^2\theta_z\sin^2\phi_z k_{p_1} + \sin\theta_{obs}\sin\theta_{rot}\cos\theta_z\sin\phi_z\cos\phi_z k_{p_1}$$

$$+ k_{p_2}\sin\theta_{obs}\sin\theta_z\cos\theta_z\sin^2\phi_z + k_{p_1}\cos\theta_{obs}\sin\theta_z\cos\theta_z\sin^2\phi_z\cos\theta_{rot} + k_{p_1}\cos\theta_{obs}\sin\theta_{rot}\sin\theta_z\sin\phi_z\cos\phi_z$$

$$+ k_{p_2}\cos\theta_{obs}\sin^2\theta_z\sin^2\phi_z \big)$$

$$= R_E^2 R_p^2 k' \sin\phi_z \, d\phi_z \, d\theta \Big[\sin^2\phi_z (k_{p_1}\sin\theta_{obs}\cos\theta_{rot}\cos^2\theta_z + k_{p_2}\cos\theta_{obs}\sin^2\theta_z$$

$$+ k_{p_2}\sin\theta_{obs}\sin\theta_z\cos\theta_z + k_{p_1}\cos\theta_{obs}\sin\theta_z\cos\theta_z\cos\theta_{rot}) + \sin\phi_z\cos\phi_z (k_{p_1}\sin\theta_{obs}\sin\theta_{rot}\cos\theta_z$$

$$+ k_{p_1}\cos\theta_{obs}\sin\theta_{rot}\sin\theta_z) \Big]$$

we must evaluate

$$\int_0^\pi \sin^3\phi_2\, d\phi_2 = \int_0^\pi (1-\cos^2\phi_2)\sin\phi_2\, d\phi_2 = \int_0^0 -(1-u^2)\, du = -u + \frac{u^3}{3} = -\cos\phi_2 + \frac{\cos^3\phi_2}{3}\Big|_0^\pi = \frac{4}{3}$$

$$\int_0^\pi \sin^2\phi_2 \cos\phi_2\, d\phi_2 = \int_0^\pi u^2\, du = \frac{u^3}{3} = \frac{\sin^3\phi_2}{3}\Big|_0^\pi = \emptyset$$

Thus we have

$$\sin^2\theta_2 = \frac{1-\cos 2\theta_2}{2} \qquad \cos^2\theta_2 = 1 - \frac{1-\cos 2\theta_2}{2} = \frac{1+\cos 2\theta_2}{2}$$

$$\frac{4}{3} R_E^2 R_p^2 k' \int_0^{\pi-\theta_{obs}} d\theta \left(k_{p1}\sin\theta_{obs}\cos\theta_{rot}\cos^2\theta_2 + k_{p2}\cos\theta_{obs}\sin^2\theta_2 + (k_{p2}\sin\theta_{obs} + k_{p1}\cos\theta_{obs}\cos\theta_{rot})\sin\theta_2\cos\theta_2\right)$$

$$= \frac{4}{3} R_E^2 R_p^2 k' \left[k_{p1}\sin\theta_{obs}\cos\theta_{rot}\left(\frac{\theta_2}{2} + \frac{\sin 2\theta_2}{4}\right) + k_{p2}\cos\theta_{obs}\left(\frac{\theta_2}{2} - \frac{\sin 2\theta_2}{4}\right) + (k_{p2}\sin\theta_{obs} + k_{p1}\cos\theta_{obs}\cos\theta_{rot})\left(\frac{\sin^2\theta_2}{2}\right)\right]\Bigg|_0^{\pi-\theta_{obs}}$$

$$= \frac{2}{3} R_E^2 R_p^2 k' \left[k_{p1}\sin\theta_{obs}\cos\theta_{rot}\left(\pi - \theta_{obs} - \frac{\sin 2\theta_{obs}}{2}\right) + k_{p2}\cos\theta_{obs}\left(\pi - \theta_{obs} + \frac{\sin 2\theta_{obs}}{2}\right) + (k_{p2}\sin\theta_{obs} + k_{p1}\cos\theta_{obs}\cos\theta_{rot})\sin^2\theta_{obs}\right]$$

$$= \frac{2}{3} R_E^2 R_p^2 k' \Big[(k_{p1}\sin\theta_{obs}\cos\theta_{rot} + k_{p2}\cos\theta_{obs})(\pi - \theta_{obs}) - \cancel{k_{p1}\sin^2\theta_{obs}\cos\theta_{rot}}\cos\theta_{obs} + k_{p2}\cos\theta_{obs}\sin\theta_{obs}\cos\theta_{obs} + k_{p2}\sin^3\theta_{obs} + \cancel{k_{p1}\cos\theta_{obs}\cos\theta_{rot}}\sin^2\theta_{obs}\Big]$$

$$= \frac{2}{3} R_E^2 R_p^2 k' \left[(\pi - \theta_{obs})(k_{p1}\sin\theta_{obs}\cos\theta_{rot} + k_{p2}\cos\theta_{obs}) + k_{p2}\sin\theta_{obs}(\cos^2\theta_{obs} + \sin^2\theta_{obs})\right]$$

$$= \boxed{\frac{2}{3} R_E^2 R_p^2 k' \left[(\pi - \theta_{obs})(k_{p1}\sin\theta_{obs}\cos\theta_{rot} + k_{p2}\cos\theta_{obs}) + k_{p2}\sin\theta_{obs}\right]}$$

$$k_{p1} = \frac{2R_E^2}{3}(\pi - \theta_p)\sin\theta_p - \frac{\pi D_p R_E}{8}\sin\theta_p(1 + \cos\theta_p)$$

$$k_{p2} = \frac{2(D_p^2 + R_E^2)}{3}(\sin\theta_p + (\pi - \theta_p)\cos\theta_p) - \frac{\pi D_p R_E}{8}(1 + \cos\theta_p)(5 + \cos\theta_p)$$

$$k' = \frac{\rho_p \varepsilon_s \rho_E}{\pi^2}\left(D_p^2 - \frac{D_p R_E \pi(1 + \cos\theta_p)}{2(\pi - \theta_p)} + R_E^2\right)^{-2}$$

$$k_{p1} = \frac{\pi D_p R_E}{32}\left[-2\sin\theta_p\left(2+3\cos\theta_p + \tfrac{1}{2}(4\cos^3\theta_p - 3\cos\theta_p)\right) + \cos\theta_p(-4\sin^3\theta_p + 3\sin\theta_p)\right]$$

$$+\frac{R_E^2}{3}\left[\sin\theta_p\,(2(\pi-\theta_p) - 2\sin\theta_p\cos\theta_p) + \cos\theta_p(1-(1-2\sin^2\theta_p))\right]$$

$$= \frac{\pi D_p R_E}{32}\left[-2\sin\theta_p\left(2+3\cos\theta_p - \tfrac{3}{2}\cos\theta_p + 2\cos^3\theta_p\right) + \cos\theta_p(-4\sin^3\theta_p + 3\sin\theta_p)\right]$$

$$+\frac{R_E^2}{3}\left[\sin\theta_p(2(\pi-\theta_p) - 2\sin\theta_p\cos\theta_p) + 2\cos\theta_p\sin^2\theta_p\right]$$

$$= \frac{\pi D_p R_E}{32}\sin\theta_p\left[-4 - 6\cos\theta_p + \overset{3\cos\theta_p}{\cancel{3\cos\theta_p}} - 4\cos^3\theta_p - 4\cos\theta_p\sin^2\theta_p + \overset{3\cos\theta_p}{\cancel{3\cos\theta_p}}\right]$$

$$+\frac{R_E^2}{3}\sin\theta_p\left[2(\pi-\theta_p) - 2\sin\theta_p\cos\theta + 2\cos\theta_p\sin\theta_p\right]$$

$$= \frac{\pi D_p R_E \sin\theta_p}{32}\left[-4 - 4\cos^3\theta_p - 4\cos\theta_p\sin^2\theta_p\right]$$

$$+\frac{R_E^2}{3}\sin\theta_p\left[2(\pi-\theta_p)\cancel{-2\sin\theta_p\cos\theta_p}\right]$$

$$= -\frac{\pi D_p R_e \sin\theta_p}{8}\left[1+\cos\theta_p\right]$$

$$+\frac{R_E^2}{3}\sin\theta_p\left(2(\pi-\theta_p)\cancel{-2\sin\theta_p\cos\theta_p}\right)$$

$$k_{p1} = \frac{\pi D_p R_E}{32}\left[\sin\theta_p\left(-4\sin^3\theta_p + \cancel{3\sin\theta_p} - 16\sin\theta_p\right) + \cos\theta_p\left(4\cos^3\theta_p \cancel{-3\cos\theta_p} - 28\cos\theta_p - 24\right)\right]$$

$$\frac{D_p^2+R_E^2}{3}\left[\sin\theta_p\left(2\sin^2\theta_p\right) + \cos\theta_p\left(2(\pi-\theta_p) + 2\sin\theta_p\cos\theta_p\right)\right]$$

$$= \frac{4\pi D_p R_E}{32}\left[-\sin^2\theta_p(\sin^2\theta_p + 4) + \cos\theta_p\left(\cos^3\theta_p - 7\cos\theta_p - 6\right)\right]$$

$$\frac{D_p^2+R_E^2}{3}\left[2\sin\theta_p(1-\cancel{\cos^2\theta_p}) + 2\cos\theta_p(\pi-\theta_p) + \cancel{2\sin\theta_p\cos^2\theta_p}\right]$$

$$= \frac{\pi D_p R_E}{8}\left[-\sin^2\theta_p(\sin^2\theta_p + 4) + (1-\sin^2\theta_p)(1-\sin^2\theta_p) - 7(1-\sin^2\theta_p) - 6\cos\theta_p\right]$$

$$\frac{2(D_p^2+R_E^2)}{3}\left[\sin\theta_p + \cos\theta_p(\pi-\theta_p)\right]$$

$$= \frac{\pi D_p R_E}{8}\left[\cancel{-\sin^4\theta_p} - 4\sin^2\theta_p + 1 - 2\sin^2\theta_p \cancel{+\sin^4\theta_p} - 7 + 7\sin^2\theta_p - 6\cos\theta_p\right]$$

$$\frac{2}{3}(D_p^2+R_E^2)\left[\sin\theta_p + \cos\theta_p(\pi-\theta_p)\right]$$

$$= \frac{\pi D_p R_E}{8}\left[\sin^2\theta_p - 6\cos\theta_p - 6\right] + \frac{2}{3}(D_p^2 + R_E^2)\left[\sin\theta_p + \cos\theta_p(\pi - \theta_p)\right]$$

$$= \frac{-\pi D_p R_E}{8}(\cos^2\theta_p + 6\cos\theta_p + 5) + \cdots$$

verify the constant values

$$k_{p_1} = \frac{\pi D_p R_E}{32}\left[\cancel{-3\sin\theta_p\cos\theta_p} \cancel{-3\sin\theta_p} + \cancel{5\sin\theta_p}(-\cos3\theta_p) - \sin\theta_p - 3\cancel{\cos\theta_p\sin\theta_p} + \cos\theta_p\sin3\theta_p\right]$$

$$+ \frac{R_E^2}{3}\left[2\sin\theta_p(\pi - \theta_p) + \sin\theta_p(-\sin(2\theta_p)) - \cos\theta_p\cos2\theta_p + \cos\theta_p\right]$$

$$= \frac{\pi D_p R_E}{32}\left[-6\sin\theta_p\cos\theta_p - 4\sin\theta_p - \sin\theta_p\cos3\theta_p + \cos\theta_p\sin3\theta_p\right]$$

$$+ \frac{R_E^2}{3}\left[2\sin\theta_p(\pi - \theta_p) - \sin\theta_p\sin2\theta_p - \cos\theta_p\cos2\theta_p + \cos\theta_p\right]$$

$$= \pi D_p R_E\left[-2\sin\theta_p\left(2 + 3\cos\theta_p + \tfrac{1}{2}\cos3\theta_p\right) + \cos\theta_p\sin3\theta_p\right]$$

$$+ \frac{R_E^2}{3}\left[\sin\theta_p(2\pi - \theta_p) - \sin2\theta_p) + (1 - \cos2\theta_p)\cos\theta_p\right]$$

$$k_{p_2} = \frac{D_p^2}{3}\left[-\sin\theta_p\cos2\theta_p + \sin\theta_p + 2\cos\theta_p(\pi - \theta_p) + \cos\theta_p\sin(2\theta_p)\right]$$

$$+ \frac{\pi D_p R_E}{32}\left[-19\sin^2\theta_p + \sin\theta_p\sin3\theta_p - 25\cos^2\theta_p - 25\cos\theta_p + \cos\theta_p\cos3\theta_p + \cos\theta_p\right]$$

$$+ \frac{R_E^2}{3}\left[-\sin\theta_p\cos2\theta_p + \sin\theta_p + 2\cos\theta_p(\pi - \theta_p) + \cos\theta_p\sin2\theta_p\right]$$

---

after checking through wolfram alpha, the simplified constants are

$$k_{p1} = \frac{-\pi D_p R_E \sin\theta_p}{8}\left[1 + \cos\theta_p\right] + 2\frac{R_E^2}{3}\sin\theta_p(\pi - \theta_p)$$

$$k_{p2} = \frac{-\pi D_p R_E}{8}\left[\cos^2\theta_p + 6\cos\theta_p + 5\right] + \frac{2}{3}(D_p^2 + R_E^2)\left[\sin\theta_p + \cos\theta_p(\pi - \theta_p)\right]$$

$$= \frac{-\pi D_p R_E}{8}(\cos\theta_p + 1)(\cos\theta_p + 5) + \frac{2}{3}(D_p^2 + R_E^2)\left[\sin\theta_p + \cos\theta_p(\pi - \theta_p)\right]$$