# Numerical Quadrature for Keplerian Orbits: Time and Distance Approximations

Prepared for Discussion

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## 1 Introduction

In the analysis of Keplerian orbits, we often compute both distances and times along the orbital path. This report explores the role of numerical quadrature in approximating these quantities, and highlights the importance of considering errors not only in spatial position but also in timing. Special attention is given to the concepts of apogee, perigee, and orbital period.

## 2 Keplerian Formulation

Given a two-body problem under Newtonian gravitation, the radial distance $r(\theta)$ as a function of true anomaly $\theta$ is

$$r(\theta) = \frac{p}{1 + e\cos\theta}, \tag{1}$$

where $p = a(1 - e^2)$ is the semi-latus rectum, $a$ is the semi-major axis, and $e$ is the eccentricity.

The specific angular momentum $h$ is

$$h = \sqrt{\mu p}, \tag{2}$$

where $\mu$ is the standard gravitational parameter.

By Kepler's second law (conservation of areal velocity), the relationship between time and true anomaly is:

$$\frac{dt}{d\theta} = \frac{r(\theta)^2}{h}. \tag{3}$$

Thus, the time to reach a given true anomaly $\theta$ is

$$t(\theta) = \int_0^\theta \frac{r(\theta')^2}{h}\, d\theta'. \tag{4}$$

The total period $T$ of the orbit is

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}. \tag{5}$$

## 3 Numerical Quadrature Approach

The integral for $t(\theta)$ cannot always be evaluated analytically for arbitrary $e$. Instead, we approximate it using numerical quadrature techniques, such as Simpson's rule or adaptive methods like those employed by Mathematica's `NIntegrate`.

Similarly, the arc length along the orbit up to angle $\theta$ can be expressed as

$$s(\theta) = \int_0^{\theta} \sqrt{r(\theta')^2 + \left(\frac{dr}{d\theta'}\right)^2}\, d\theta'. \tag{6}$$

## 4 Sources of Error

Both time and distance are approximated when using quadrature methods. Errors arise from:

- Discretization: finite number of evaluation points.
- Round-off errors: especially when $r(\theta)$ varies sharply near perigee.
- Numerical method order: higher-order methods converge faster.

It is common to report positional errors (errors in $r$ or Cartesian coordinates), but timing errors (errors in $t(\theta)$) are equally important, particularly in mission planning and rendezvous calculations.

## 5 Apogee, Perigee, and Period

- **Perigee**: closest approach to the central body, occurring at $\theta = 0$. - **Apogee**: farthest distance, occurring at $\theta = \pi$. - **Period** $T$: full traversal time of the orbit, computable both symbolically and via numerical integration.

Accurate timing at perigee and apogee is critical for navigation and maneuvers.

## 6 Outline of Error Table

| Quantity | Ideal Value | Numerical Approximation | Absolute Error |
|---|---|---|---|
| Time to Perigee ($t = 0$) | 0 s | 0 s | 0 |
| Time to Apogee ($t(\pi)$) | $T/2$ | Approximated | Error |
| Full Period ($T$) | Calculated | Integrated | Error |
| Distance at $\theta = \pi/2$ | Analytical $r(\pi/2)$ | Approximated | Error |

## 7 Conclusion

When performing numerical quadrature on Keplerian orbits, it is important to track both distance and time errors. Convergence in position does not guarantee convergence in timing, and both are critical depending on the application. Through proper setup and analysis, we gain deeper insights into the behavior of orbital systems and the importance of numerical methods.