Ground-state energy densities

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Abstract

We try to find some theories that have finite ground-state energy densities.

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I. THE ATLANTIC CITY WAY

Amplitudes are matrix elements of $\exp(-itH)$. After inserting zillions of complete sets of eigenstates of the fields and their momenta, one gets

$$\langle\phi(t)|e^{-itH}|\phi(0)\rangle = \int \exp\left[i\int(\dot\phi\pi - H)\,d^4x\right]\,D\phi D\pi. \tag{1}$$

The quantity inside the parentheses is the action density $L = \dot\phi\pi - H$ when π is written as a function of the field ϕ and its first derivatives $\partial_\mu\phi$. A graceful theory has an action density that is quadratic in the time derivative $\dot\phi$

$$L = \tfrac{1}{2}\dot\phi^2 - \tfrac{1}{2}(\nabla\phi)^2 - V(\phi) \tag{2}$$

and a hamiltonian

$$H = \tfrac{1}{2}\dot\phi^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi). \tag{3}$$

For such theories, one has

$$\langle\phi(t)|e^{-itH}|\phi(0)\rangle = \int \exp\left[i\int L\,d^4x\right]\,D\phi. \tag{4}$$

In theories that respect these equations (1–4), the exponential $\exp(i\int Ld^4x)$ is a functional Fourier transform of exponential $\exp(-i\int Hd^4x)$ evaluated at $\dot\phi$

$$\exp\left[i\int L\,d^4x\right] = \int \exp\left[i\int(\dot\phi\pi - H)\,d^4x\right]\,D\pi. \tag{5}$$

The partition function is

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H}) = \int \exp\left[\int(i\dot\phi\pi - H)\,d^4x\right]\,D\phi D\pi. \tag{6}$$

In graceful theories (2 & 3), $Z(\beta)$ is

$$Z(\beta) = \int \exp\left[-\int L_e\,d^4x\right]\,D\phi \tag{7}$$

in which L_e is the euclidian action density

$$L_e = \tfrac{1}{2}\dot\phi^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi). \tag{8}$$

In theories that respect these equations (6–7), the exponential $\exp(-\int L_e d^4x)$ is a functional Fourier transform of exponential $\exp(-\int H\,d^4x)$ evaluated at $\dot\phi$

$$\exp\left[-\int L_e\,d^4x\right] = \int \exp\left[\int(i\dot\phi\pi - H)\,d^4x\right]\,D\pi. \tag{9}$$

Unless the hamiltonian H is a simple function of the momentum π, one can't analytically integrate over π so as to do the Fourier transforms (5) or (9). The Atlantic City way to estimate a complex path integral like the one (6) for the partition function is first to put the theory on a lattice of spatial spacing a_s and (euclidian) temporal spacing a_t. On such a lattice, the partition function (6) is

$$\begin{aligned} Z(\beta) &= \int \exp\left[\int_0^\beta dt \int d^3x \left(i\dot{\phi}\pi - H(\phi,\pi)\right)\right] D\phi D\pi \\ &= \prod_\alpha \int \exp\left[\sum_\alpha a_s^3 \left(i\Delta\phi_\alpha \pi_\alpha - a_t H(\phi_\alpha, \pi_\alpha)\right)\right] d\phi_\alpha d\pi_\alpha \end{aligned} \tag{10}$$

in which $\alpha = (i,j,k,\ell)$ labels the sites of the lattice, and $\Delta\phi_\alpha = \phi(i,j,k,\ell+1) - \phi(i,j,k,\ell)$. Usually the hamiltonian is even in π, and so one can replace the exponential factor $\exp(a_s^3 i\Delta\phi_\alpha\pi_\alpha)$ by a cosine $\cos(a_s^3 d\phi_\alpha\pi_\alpha)$. The second step of the Atlantic City way is then to do the momentum integrals numerically, setting

$$A(\phi, \nabla\phi, \Delta\phi) = \int_0^\infty \cos\left(a_s^3\,\Delta\phi\,\pi\right) e^{-a_s^3 a_t H(\phi,\pi)}\, d\pi \tag{11}$$

and

$$C(\phi, \nabla\phi, \Delta\phi) = \int_0^\infty H(\phi,\pi)\, \cos\left(a_s^3\,\Delta\phi\,\pi\right)\, e^{-a_s^3 a_t H(\phi,\pi)}\, d\pi. \tag{12}$$

The partition function (6) then is a product of the A integrals

$$Z(\beta) = \prod_\alpha \int A(\phi_\alpha, \nabla\phi_\alpha, \Delta\phi_\alpha)\, d\phi_\alpha \tag{13}$$

in which the squared gradients are

$$(\nabla\phi)^2_{i,j,k,\ell} = \frac{(\phi_{i+1,j,k,\ell} - \phi_{i,j,k,\ell})^2}{a_s^2} + \frac{(\phi_{i,j+1,k,\ell} - \phi_{i,j,k,\ell})^2}{a_s^2} + \frac{(\phi_{i,j,k+1,\ell} - \phi_{i,j,k,\ell})^2}{a_s^2}. \tag{14}$$

The third step of the Atlantic City way is to use the Monte Carlo method to estimate the multiple integral (13).

One can use the partition function (13) to compute euclidian Green's functions like the two-point function

$$\langle \phi_\gamma \phi_\delta \rangle = \prod_\alpha \int \phi_\gamma \phi_\delta\, A(\phi_\alpha, \nabla\phi_\alpha, \Delta\phi_\alpha)\, d\phi_\alpha \Big/ Z(\beta). \tag{15}$$

One also can use it to find the mean value of the energy at inverse temperature β

$$\begin{aligned}
\langle E(\beta)\rangle =& -\frac{1}{Z(\beta)N_t}\frac{dZ(\beta)}{da_t} \\
=& \frac{a_s^3}{Z(\beta)N_t}\sum_\gamma \int H(\phi_\gamma,\pi_\gamma)\cos(a_s^3\Delta\phi_\gamma\pi_\gamma)e^{-a_s^3 a_t H(\phi_\gamma,\pi_\gamma)}d\phi_\gamma d\pi_\gamma \\
& \times \prod_{\alpha\neq\gamma}\int \cos(a_s^3\Delta\phi_\alpha\pi_\alpha)e^{-a_s^3 a_t H(\phi_\alpha,\pi_\alpha)}d\phi_\alpha d\pi_\alpha \\
=& \frac{a_s^3}{N_t}\left\langle \sum_\gamma \frac{\int C(\phi_\gamma,\nabla\phi_\gamma,\Delta\phi_\gamma)\,d\phi_\gamma}{\int A(\phi_\gamma,\nabla\phi_\gamma,\Delta\phi_\gamma),d\phi_\gamma}\right\rangle
\end{aligned} \tag{16}$$

by doing a Monte Carlo guided by the unnormalized probability distribution

$$P[\phi] = Z(\beta) = \prod_\alpha \int A(\phi_\alpha,\nabla\phi_\alpha,\Delta\phi_\alpha)\,d\phi_\alpha \tag{17}$$

to estimate the mean value of this ratio. The energy density then is

$$\frac{\langle E(\beta)\rangle}{(N_s a_s)^3} = \frac{1}{N_s^3 N_t}\left\langle \sum_\gamma \frac{\int C(\phi_\gamma,\nabla\phi_\gamma,\Delta\phi_\gamma)\,d\phi_\gamma}{\int A(\phi_\gamma,\nabla\phi_\gamma,\Delta\phi_\gamma),d\phi_\gamma}\right\rangle. \tag{18}$$

We can *define* the euclidian action density as the functional Fourier transform (9)

$$\exp\left[-\int L_e\,d^4x\right] \equiv \int \exp\left[\int (i\dot\phi\pi - H)\,d^4x\right]\,D\pi. \tag{19}$$

II. DARK ENERGY

What accelerates the expansion of the universe is not the energy of the vacuum but rather the difference between its potential energy and its kinetic energy. Einstein's equations are

$$R_{ik} = -8\pi G\left(T_{ik} - \frac{T}{2}g_{ik}\right) \tag{20}$$

in which T_{ik} is the energy-momentum tensor defined by how the matter action S_m changes when the metric is varied

$$\delta S_m = -\frac{1}{2}\int T_{ik}\,\delta g^{ik}\,\sqrt{g}\,d^4x \tag{21}$$

in which $g = |\det(g_{mn})|$ and T is the trace $T = g^{ik}T_{ik}$. Suppose S_m is the action of a scalar field ϕ

$$S_m = S_\phi = \int L(L_0,\phi)\,\sqrt{g}\,d^4x, \tag{22}$$

in which all the derivatives of the field are inside the free field massless action

$$L_0 = -\frac{1}{2} g^{ik} \phi_{,i} \phi_{,k} \tag{23}$$

and the ϕ dependence of L describes the potential terms. The key identities are

$$\delta L_0 = -\frac{1}{2}\, \phi_{,i} \phi_{,k}\, \delta g^{ik} \tag{24}$$

and

$$\delta \sqrt{g} = -\frac{1}{2}\, g_{ik}\, \delta g^{ik}\, \sqrt{g}. \tag{25}$$

The change (21) in the action (22) then is

$$\delta S_m = -\frac{1}{2} \int \left(\frac{\partial L}{\partial L_0}\, \phi_{,i} \phi_{,k} + L\, g_{ik} \right) \sqrt{g}\, \delta g^{ik}\, d^4 x. \tag{26}$$

Comparing this change with the definition (21) of T_{ik}, we get for the energy-momentum tensor

$$T_{ik} = \frac{\partial L}{\partial L_0}\, \phi_{,i} \phi_{,k} + L\, g_{ik} \tag{27}$$

and its trace T

$$T = g^{ik}\, T_{ik} = \frac{\partial L}{\partial L_0}\, g^{ik}\, \phi_{,i} \phi_{,k} + L\, g^{ik}\, g_{ik} = \frac{\partial L}{\partial L_0}\, g^{ik}\, \phi_{,i} \phi_{,k} + 4L. \tag{28}$$

If we use the metric of a flat universe

$$ds^2 = -\,dt^2 + a^2(t) \left(dx^2 + dy^2 + dz^2 \right), \tag{29}$$

which is a good approximation for the era of dark energy, then L_0 is

$$L_0 = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2a^2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2, \tag{30}$$

and the trace (28) is

$$T = \frac{\partial L}{\partial L_0} \left((\nabla \phi)^2 / a^2 - \dot{\phi}^2 \right) + 4L \tag{31}$$

in which $a = a(t)$ is the scale factor —not the lattice spacing.

In a homogeneous and isotropic universe, the 00 component of Ricci's tensor is $R_{00} = 3\ddot{a}/a$, the component T_{00} is

$$T_{00} = \frac{\partial L}{\partial L_0}\, \dot{\phi}^2 - L, \tag{32}$$

and so Einstein's 00 equation is

$$
\begin{aligned}
\frac{\ddot a}{a} &= -\,\frac{8\pi G}{3}\left(T_{00}+\frac{T}{2}\right)\\
&= -\,\frac{8\pi G}{3}\left(\frac{\partial L}{\partial L_0}\,\dot\phi^2 - L + \frac{T}{2}\right)\\
&= -\,\frac{8\pi G}{3}\left[\frac{\partial L}{\partial L_0}\,\dot\phi^2 - L + \frac{1}{2}\left(\frac{\partial L}{\partial L_0}\,\left((\nabla\phi)^2/a^2-\dot\phi^2\right)+4L\right)\right]\\
&= -\,\frac{4\pi G}{3}\left[\frac{\partial L}{\partial L_0}\,\left(\dot\phi^2+(\nabla\phi)^2/a^2\right)+2L\right].
\end{aligned}
\tag{33}
$$

For a free field of mass m, the action density is $L = L_0 + m^2\phi^2/2$, the acceleration is

$$
\frac{\ddot a}{a} = -\,\frac{4\pi G}{3}\left[\dot\phi^2+(\nabla\phi)^2/a^2+\dot\phi^2-(\nabla\phi)^2/a^2-m^2\phi^2\right] = \frac{4\pi G}{3}\left(m^2\phi^2-2\dot\phi^2\right). \tag{34}
$$

The scalar Born-Infeld action density is

$$
L = M^4\left(1-\sqrt{1-2L_0/M^4}\right), \tag{35}
$$

and so

$$
\frac{\partial L}{\partial L_0} = \frac{1}{\sqrt{1-2L_0/M^4}}. \tag{36}
$$

So our formula (33) for $\ddot a/a$ gives us

$$
\begin{aligned}
\frac{\ddot a}{a} &= -\,\frac{4\pi G}{3}\left[\frac{\partial L}{\partial L_0}\,\left(\dot\phi^2+(\nabla\phi)^2/a^2\right)+2L\right]\\
&= -\,\frac{4\pi G}{3}\left[\frac{\dot\phi^2+(\nabla\phi)^2/a^2}{\sqrt{1-2L_0/M^4}}+2M^4\left(1-\sqrt{1-2L_0/M^4}\right)\right]\\
&= \frac{4\pi G}{3}\left[\frac{(\nabla\phi)^2/a^2-3\dot\phi^2+2m^2\phi^2+2M^4\left(1-\sqrt{1-2L_0/M^4}\right)}{\sqrt{1-2L_0/M^4}}\right].
\end{aligned}
\tag{37}
$$

The right-hand side is what accelerates $\ddot a$ the expansion of the scale factor a in the Born-Infeld theory. Astronomers use coordinates in which at the present time t_0, the scale factor is unity, $a(t_0)=1$. So this dark energy is not the energy of the vacuum but something that may be much less divergent.

III. VARIOUS CALCULATIONS

A. Free field with cutoff

The action is quadratic in the time derivative of the field

$$L = \frac{1}{2}\left(\dot{\phi}^2 - (\nabla\phi)^2\right) - m^4 \left(\sqrt{1 - \frac{\phi^2}{M^2}}\right)^{-1/2} + m^4 \tag{38}$$

and so is graceful. At big M, it describes a free field with particles of mass m^2/M. Its hamiltonian is

$$H = \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2\right) + m^4 \left(\sqrt{1 - \frac{\phi^2}{M^2}}\right)^{-1/2} - m^4. \tag{39}$$

Since the action is graceful, its partition function is

$$\begin{aligned} Z(\beta) = \mathrm{Tr}(e^{-\beta H}) &= \int \exp\left[\int_0^\beta dt \int d^3x \left(i\dot{\phi}\pi - H\right)\right] D\phi D\pi \\ = \mathrm{Tr}(e^{-\beta H}) &= \int \exp\left\{\int_0^\beta dt \int d^3x \left[i\dot{\phi}\pi - \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2\right) - V\right]\right\} D\phi D\pi \end{aligned} \tag{40}$$

where $V = m^4/\sqrt{1-\phi^2/M^2} - m^4$. On a lattice of spacing a_t in the time direction and a_s in the spatial directions, the partition function is a path integral over all spacetime points $\alpha = (\boldsymbol{x}, t)$

$$Z(\beta) = \prod_\alpha \int d\phi_\alpha d\pi_\alpha \exp\left[a_t a_s^3 \sum_\alpha \left(i\dot{\phi}_\alpha \pi_\alpha - \tfrac{1}{2}\pi_\alpha^2 - \tfrac{1}{2}(\nabla\phi_\alpha)^2 - V_\alpha\right)\right] \tag{41}$$

$$= \left(\frac{2\pi}{a_t a_s^3}\right)^{N_t N_s^3/2} \prod_\alpha \int d\phi_\alpha \exp\left[-a_s^3 \sum_\alpha \left(\frac{(\Delta\phi_\alpha)^2}{2a_t} + \frac{a_t}{2}(\nabla\phi_\alpha)^2 + a_t V_\alpha\right)\right] \tag{42}$$

and the mean value of the energy is

$$\begin{aligned} -\frac{d\log Z(\beta)}{d\beta} &= -\frac{1}{Z(\beta)}\frac{1}{N_t}\frac{dZ(\beta)}{da_t} \\ &= -\frac{1}{Z(\beta)}\frac{1}{N_t}\left\{-\frac{N_t N_s^3}{2a_t} Z(\beta)\right. \\ &\quad + \left(\frac{2\pi}{a_t a_s^3}\right)^{N_t N_s^3/2} \prod_\alpha \int d\phi_\alpha \left[a_s^3 \sum_\alpha \left(\frac{(\Delta\phi_\alpha)^2}{2a_t^2} - \frac{1}{2}(\nabla\phi_\alpha)^2 - V_\alpha\right)\right] \\ &\quad \left.\times \exp\left[-a_s^3 a_t \sum_\alpha \left(\frac{(\Delta\phi_\alpha)^2}{2a_t^2} + \frac{1}{2}(\nabla\phi_\alpha)^2 + V_\alpha\right)\right]\right\}. \end{aligned} \tag{43}$$

In the argument of the exponential, we see the euclidian action

$$S = a_s^3 a_t \sum_\alpha \left(\frac{(\Delta\phi_\alpha)^2}{2a_t^2} + \frac{1}{2}(\nabla\phi_\alpha)^2 + V_\alpha \right) = \int \left[\tfrac{1}{2}(\partial_\mu\phi)^2 + V \right] d^4x. \tag{44}$$

So the energy (43) is

$$-\frac{d\log Z(\beta)}{d\beta} = \frac{N_s^3}{2a_t} + \prod_\alpha \int d\phi_\alpha \left[\frac{a_s^3}{N_t} \sum_\alpha \left(-\tfrac{1}{2}\dot\phi_\alpha^2 + \tfrac{1}{2}(\nabla\phi_\alpha)^2 + V_\alpha \right) \right] e^{-S} \Big/ \prod_\alpha \int d\phi_\alpha e^{-S}\,. \tag{45}$$

Dividing by the spatial volume $N_s^3 a_s^3$, we get the energy density

$$\frac{\langle E(\beta)\rangle}{N_s^3 a_s^3} = \frac{1}{2a_s^3 a_t} - \langle L \rangle \tag{46}$$

in which $\langle L \rangle$ is the lattice estimate of the mean value of the action density (38) weighted by the euclidian action (44)

$$-\langle L \rangle = -\frac{\int L\, e^{-S} D\phi}{\int e^{-S} D\phi} = \frac{\int [\frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}\dot\phi^2 + V]\, e^{-S} D\phi}{\int e^{-S} D\phi}. \tag{47}$$

The euclidian symmetry of S implies that $\langle -\dot\phi^2 + (\nabla\phi)^2 \rangle = 2\langle \dot\phi^2 \rangle$. Thus the energy density of a graceful action diverges quartically, at least as fast as $1/(2a_s^3 a_t)$ unless $\langle V \rangle$ is negative and similarly divergent.

When the action is quadratic in the time derivative $\dot\phi$, we don't need to use the Atlantic-City Way to compute the partition function. But we can check that method by comparing its results with the ones given by the standard method (40–47). The A-C integrals are

$$A(\phi, \nabla\phi, \Delta\phi) = \int_0^\infty \cos\left(a_s^3\, \Delta\phi\, \pi\right) e^{-a_s^3 a_t H(\phi,\pi)}\, d\pi \tag{48}$$

and

$$C(\phi, \nabla\phi, \Delta\phi) = \int_0^\infty \cos\left(a_s^3\, \Delta\phi\, \pi\right) H(\phi,\pi)\, e^{-a_s^3 a_t H(\phi,\pi)}\, d\pi. \tag{49}$$

The energy density (46) should rise as $1/a^4$. Data from the A-C way are plotted in blue dashes in Fig. 1. Data from a standard Monte Carlo (red dots) overlap those from the A-C way. Both rise more slowly than $1/a^4$ (solid, black) but faster than $1/a^{3.5}$ (solid, cyan) and much faster than $1/a^2$ (solid, green).

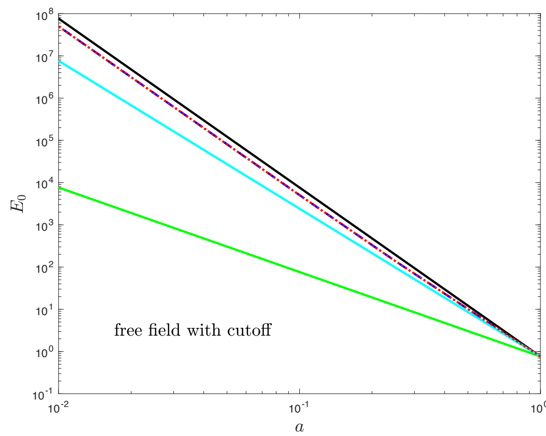

FIG. 1. Blue dashes are data from Atlantic-City Way simulations of the cutoff free scalar field theory (38 & 39) on a 20^4 lattice. The red dots are from a standard Monte Carlo simulation of the same theory; they overlap those from the Atlantic-City Way. The black line is $1/a^4$. The cyan line is $1/a^{3.5}$. The green line is $1/a^2$.

B. Scalar Born-Infeld Theory

First, we have the scalar Born-Infeld theory with awkward action density

$$L = M^4 \left(1 - \sqrt{1 - M^{-4}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)}\right) \tag{50}$$

and conjugate momentum

$$\pi = \frac{\partial L(\phi, \dot{\phi})}{\partial \dot{\phi}} = \dot{\phi}\,\left[1 - M^{-4}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)\right]^{-1/2}. \tag{51}$$

Its energy density is

$$\begin{aligned} H(\phi, \pi) &= \pi\dot{\psi} - L(\phi, \dot{\psi}) \\ &= \sqrt{\left(M^4 + \pi^2\right)\left(M^4 + (\nabla\phi)^2 + m^2\phi^2\right)} - M^4. \end{aligned} \tag{52}$$

The Atlantic City integrals for this awkward action are

$$A(\phi, \nabla\phi, \Delta\phi) = \int_0^\infty \cos\left(a_s^3\,\Delta\phi\,\pi\right) e^{-a_s^3 a_t H(\phi,\pi)}\,d\pi \tag{53}$$

and

$$C(\phi, \nabla\phi, \Delta\phi) = \int_0^\infty H(\phi, \pi)\,\cos\left(a_s^3\,\Delta\phi\,\pi\right) e^{-a_s^3 a_t H(\phi,\pi)}\,d\pi. \tag{54}$$

In the $M \to 0$ limit in which $H(\phi, \pi) = \sqrt{\pi^2 \left((\nabla\phi)^2 + m^2\phi^2 \right)}$, one can do these integrals. In terms of $V = (\nabla\phi)^2 + m^2\phi^2$, they are

$$
\begin{aligned}
A(\phi, \nabla\phi, \Delta\phi) &= \frac{1}{2} \int_0^\infty e^{-a_s^3 \pi (a_t \sqrt{V} + i\Delta\phi)} + e^{-a_s^3 \pi (a_t \sqrt{V} - i\Delta\phi)} \, d\pi \\
&= \frac{a_t}{a_s^3} \frac{\sqrt{V}}{(a_t^2 V + \Delta\phi^2)}
\end{aligned}
\tag{55}
$$

and

$$
\begin{aligned}
C(\phi, \nabla\phi, \Delta\phi) &= \frac{1}{2} \int_0^\infty \pi \sqrt{V} \left[e^{-a_s^3 \pi (a_t \sqrt{V} + i\Delta\phi)} + e^{-a_s^3 \pi (a_t \sqrt{V} - i\Delta\phi)} \right] d\pi \\
&= -\frac{1}{a_t} \frac{\partial A}{\partial a_s^3} = -\frac{1}{a_t} \frac{\partial}{\partial a_s^3} \left(\frac{a_t}{a_s^3} \frac{\sqrt{V}}{(a_t^2 V + \Delta\phi^2)} \right) \\
&= \frac{1}{a_s^6} \frac{\sqrt{V}}{(a_t^2 V + \Delta\phi^2)}.
\end{aligned}
\tag{56}
$$

The energy density therefore is

$$
\frac{\langle E(\beta) \rangle}{(a_s N_s)^3} = \left\langle \frac{C(\phi, \nabla\phi, \Delta\phi)}{A(\phi, \nabla\phi, \Delta\phi)} \right\rangle = \frac{1}{a_t \, a_s^3}. \tag{57}
$$

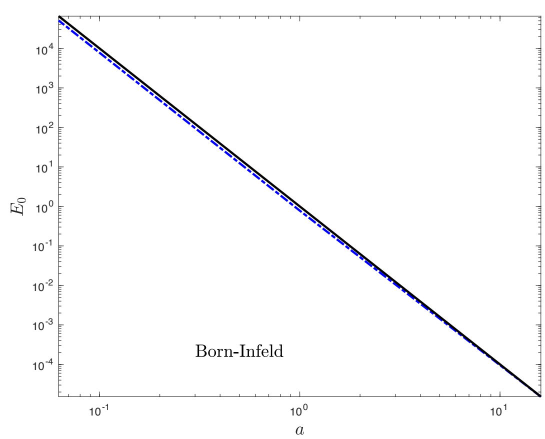

FIG. 2. The energy density E_0 of the ground-state of the scalar Born-Infeld theory (50 & 52) for $M/m = 1$ (dashdot, blue) and $M/m = 0$ (solid, black) on a 20^4 lattice of spacing $a_s = a_t = a$.

We ran with $M/m = 1$ at $a_s = a_t = 1/16$, 1, 16 and got the ground-state energies shown in Fig. 2 (dashdot, blue). When $M/m = 0$, the exact energy density (57) on an infinite lattice is $1/(a_t a_s^3)$.

C. Inverse Scalar Born-Infeld Theory

Let's set

$$L = M^4 \left[\left(1 - \frac{2L_0}{M^4} \right)^{-1/2} - 1 \right] \tag{58}$$

in which

$$L_0 = \frac{1}{2} \left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2 \right) \equiv \frac{1}{2} \left(\dot{\phi}^2 - V \right). \tag{59}$$

To find the hamiltonian, we must solve the equation

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi} \left(1 - \frac{2L_0}{M^4} \right)^{-3/2} = \dot{\phi} \left[1 - \left(\dot{\phi}^2 - V \right) / M^4 \right]^{-3/2} \tag{60}$$

for $\dot{\phi}$ in terms of π and V. This is a cubic equation in the variable $\dot{\phi}^2/M^4$

$$\frac{\pi^2}{M^4} \left(1 + \frac{V}{M^4} - \frac{\dot{\phi}^2}{M^4} \right)^3 = \frac{\dot{\phi}^2}{M^4}. \tag{61}$$

Setting $y = \dot{\phi}^2/M^4$, $x = \pi^2/M^4$, and $c = 1 + V/M^4$, we must solve the cubic equation

$$x(c - y)^3 = y. \tag{62}$$

In terms of the quantity

$$r = \left(\frac{27cx^2}{2} + \frac{3x}{2} \sqrt{12x + (9cx)^2} \right)^{1/3}, \tag{63}$$

its only real solution is

$$y = c + \frac{1}{r} - \frac{r}{3x} = 1 + V/M^4 + \frac{1}{r} - \frac{rM^4}{3\pi^2}. \tag{64}$$

As $x \to 0$, the cube root r tends to

$$r \to \sqrt{3x} \left(1 + \frac{c\sqrt{3x}}{2} \right) \approx \sqrt{3x} + \frac{3cx}{2}, \tag{65}$$

and so although y vanishes analytically as $x \to 0$ and $\pi \sim x^2 \to 0$, that is

$$y \to c + \frac{1}{\sqrt{3x} + 3cx/2} - \frac{\sqrt{3x} + 3cx/2}{3x} \approx c - c = 0, \tag{66}$$

it is numerically unstable as $x \to 0$. And analytically $\dot{\phi} \sim \sqrt{y} \to 0$ as $\pi \to 0$, as one would expect. Yet our code for H gives NaNs as π drops below 0.0009.

The hamiltonian is

$$
\begin{aligned}
H &= \pi\dot{\phi} - L \\
&= \pi\dot{\phi} - M^4\left[\left(1 - \frac{2L_0}{M^4}\right)^{-1/2} - 1\right] \\
&= \pi\dot{\phi} - M^4\left[\left(1 + \frac{(\nabla\phi)^2}{M^4} + \frac{m^2\phi^2}{M^4} - \frac{\dot{\phi}^2}{M^4}\right)^{-1/2} - 1\right] \\
&= M^4\left[1 + \frac{\pi\dot{\phi}}{M^4} - \left(1 + \frac{(\nabla\phi)^2}{M^4} + \frac{m^2\phi^2}{M^4} - \frac{\dot{\phi}^2}{M^4}\right)^{-1/2}\right] \\
&= M^4\left(1 + \sqrt{xy} - \frac{1}{\sqrt{c-y}}\right).
\end{aligned}
\tag{67}
$$

Thus the energy diverges as $y \to c$ which is when $2L_0 \to M^2$, as expected. But also as $x \to \infty$, $r \sim x^{2/3} \to \infty$, and $y \to c$, and so $H \to \infty$. So H can be unstable numerically as $x \to \infty$. But we don't get NaNs as $\pi \to \infty$. All these theories seem to have energy densities that go as $1/a^4$. We ran with $m = M = 1$ at $a_s = a_t = 1/16$, 1, 16 and got the ground-state energies shown in Fig. 3, which diverge as $1/a^4$.

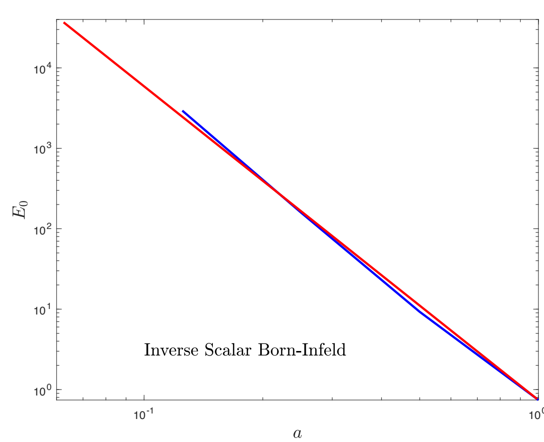

FIG. 3. Blue line is data from simulations of the inverse scalar Born-Infeld theory (58 & 70) on a 20^4 lattice. Red line is $1/x^4$.

D. Inverse Born-Infeld with Cutoff

So let's try

$$L = M^4 \left[\left(1 - \frac{2L_0}{M^4} \right)^{-1/2} - \left(1 - \frac{m^2\phi^2}{M^4} \right)^{-1/2} \right] \tag{68}$$

where now L_0 is just the kinetic term

$$L_0 = \frac{1}{2} \left(\dot{\phi}^2 - (\nabla\phi)^2 \right). \tag{69}$$

With $c = 1 + (\nabla\phi)^2/M^4$ and the real solution (64) of the cubic, the hamiltonian is

$$\begin{aligned} H &= \pi\dot{\phi} - L \\ &= M^4 \left(\pi\sqrt{y}/M^2 - \frac{1}{\sqrt{c-y}} + \frac{1}{\sqrt{1 - m^2\phi^2/M^4}} \right). \end{aligned} \tag{70}$$

We ran with $m = M = 1$ at $a_s = a_t = 1/16,\ 1/8,\ 1/4,\ 1/2,\ 1$ and got the ground-state energies shown in Fig. 4, which diverge as $1/a^{3.9}$.

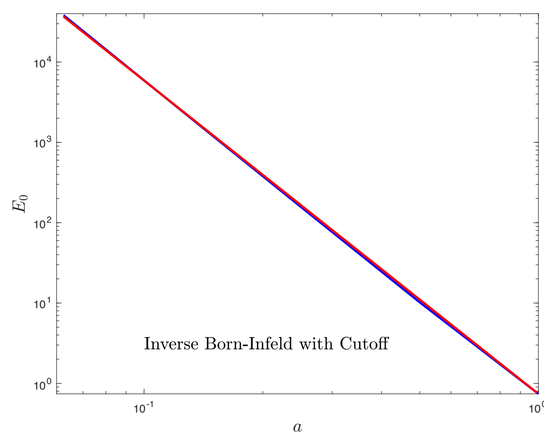

FIG. 4. Blue line is data. Red line is $1/x^{3.9}$.

E. Logarithmic Action

Another theory worth studying has

$$L = -M^4 \ln\left(1 - M^{-4}L_0\right) \tag{71}$$

in which L_0 is a typical action density

$$L_0 = \frac{1}{2}\dot{\phi}^2 - V \tag{72}$$

and $V = (\nabla\phi)^2/2 + m^2\phi^2/2 + \ldots$. One can solve for the time derivative of the field

$$\dot{\phi} = \frac{M^4}{\pi}\left(\sqrt{1 + 2M^{-4}\pi^2\left(1 + M^{-4}V\right)} - 1\right), \tag{73}$$

and get the hamiltonian density

$$\begin{aligned} H = {} & M^4\left(\sqrt{1 + 2M^{-4}\pi^2\left(1 + M^{-4}V\right)} - 1\right) \\ & + M^4 \ln\left[\frac{M^4}{\pi^2}\left(\sqrt{1 + 2M^{-4}\pi^2\left(1 + M^{-4}V\right)} - 1\right)\right]. \end{aligned} \tag{74}$$

When $\pi = 0$, the hamiltonian is $H = M^4 \ln(1 + M^{-4}V)$. These data are plotted in Fig. 5; they diverge slightly more slowly than $1/a^4$.

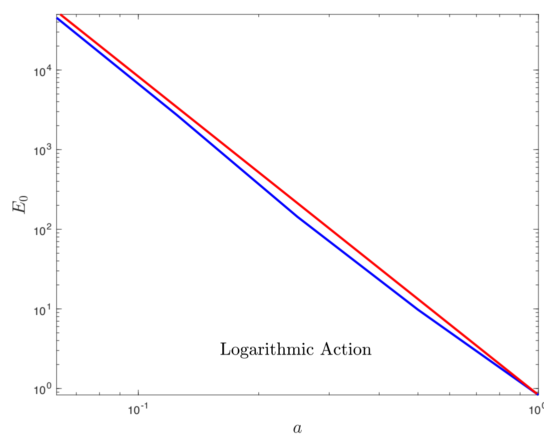

FIG. 5. Blue line is data from simulations of the theory with a logarithmic action (71 & 74) on a 20^4 lattice. Red line is $1/x^4$.

F. Exponential Action

Let's consider the exponential action

$$L = M^4 \left[\exp\left(L_0/M^4\right) - 1\right] \tag{75}$$

in which L_0 is a free action (72) with the usual kinetic terms. The momentum π conjugate to $\dot\phi$ is

$$\pi = \frac{\partial L}{\partial \dot\phi} = \dot\phi \, \exp\left(L_0/M^4\right). \tag{76}$$

Its derivative with respect to $\dot\phi$ is

$$\frac{\partial \pi}{\partial \dot\phi} = \left[1 + \frac{\dot\phi^2}{M^4} \, \exp\left(L_0/M^4\right)\right]. \tag{77}$$

The Atlantic City integrals for this very awkward action are

$$\begin{aligned} A(\phi, \nabla\phi, \Delta\phi) = & \int_0^\infty \cos\left[a_s^3 \, \Delta\phi \, \dot\psi \exp\Big(L_0(\phi, \dot\psi)/M^4\Big)\right] e^{-a_s^3 a_t H(\phi,\dot\psi)} \\ & \times \left[1 + M^{-4} \dot\psi^2 \, \exp\Big(L_0(\phi, \dot\psi)/M^4\Big)\right] d\dot\psi \end{aligned} \tag{78}$$

and

$$\begin{aligned} C(\phi, \nabla\phi, \Delta\phi) = & \int_0^\infty \cos\left[a_s^3 \, \Delta\phi \, \dot\psi \exp\Big(L_0(\phi, \dot\psi)/M^4\Big)\right] e^{-a_s^3 a_t H(\phi,\dot\psi)} \\ & \times H(\phi, \dot\psi) \, \left[1 + M^{-4} \dot\psi^2 \, \exp\Big(L_0(\phi, \dot\psi)/M^4\Big)\right] d\dot\psi \end{aligned} \tag{79}$$

in which the effective energy density is

$$H(\phi, \dot\psi) = \dot\psi \pi - L = \left(\dot\psi^2 - M^4\right) \, \exp\left(L_0(\phi, \dot\psi)/M^4\right) + M^4. \tag{80}$$

We wrote Fortran codes and found that the amplitude $A(\phi, \nabla\phi, \Delta\phi)$ increased to a constant value, largely independent of the variables $\phi, \nabla\phi$, and $\Delta\phi$. Thus the field ϕ wanders off to arbitrarily big values as the Monte Carlo runs. No table is big enough.

G. Another Exponential Action

Let's consider the exponential action

$$L = \, - M^4 \, \left[\exp\left(- L_0/M^4\right) - 1\right] \tag{81}$$

in which L_0 is a free action (72) with the usual kinetic terms. The momentum π conjugate to $\dot\phi$ is

$$\pi = \frac{\partial L}{\partial \dot\phi} = \dot\phi \, \exp\left(- L_0/M^4\right). \tag{82}$$

Its derivative with respect to $\dot\phi$ is

$$\frac{\partial \pi}{\partial \dot\phi} = \left[1 - M^{-4} \dot\phi^2 \, \exp\left(- L_0/M^4\right)\right]. \tag{83}$$

The Atlantic City integrals for this very awkward action are

$$A(\phi, \nabla\phi, \Delta\phi) = \int_0^\infty \cos\left[a_s^3\, \Delta\phi\, \dot\psi \exp\Big(- L_0(\phi, \dot\psi)/M^4\Big)\right] e^{-a_s^3 a_t H(\phi,\dot\psi)} \times \left[1 - M^{-4}\dot\psi^2 \exp\Big(- L_0(\phi, \dot\psi)/M^4\Big)\right] d\dot\psi \tag{84}$$

and

$$C(\phi, \nabla\phi, \Delta\phi) = \int_0^\infty \cos\left[a_s^3\, \Delta\phi\, \dot\psi \exp\Big(- L_0(\phi, \dot\psi)/M^4\Big)\right] e^{-a_s^3 a_t H(\phi,\dot\psi)} \times H(\phi, \dot\psi)\, \left[1 - M^{-4}\dot\psi^2 \exp\Big(- L_0(\phi, \dot\psi)/M^4\Big)\right] d\dot\psi \tag{85}$$

in which the effective energy density is

$$H(\phi, \dot\psi) = \dot\psi\pi - L = \Big(\dot\psi^2 + M^4\Big) \exp\Big(- L_0(\phi, \dot\psi)/M^4\Big) - M^4. \tag{86}$$

The factor $\exp(- a_s^3 a_t H)$ goes to unity as $\dot\psi \to \infty$, so these integrals do not converge.

H. A Quartic Hamiltonian

One way to get finite energy densities may be to start with an action density like

$$L = - \frac{M^4}{1 + \frac{1}{2}M^{-4}\Big(\dot\phi^2 - (\nabla\phi)^2 - m^2\phi^2\Big)}. \tag{87}$$

Now the momentum is

$$\pi = \frac{\dot\phi}{\left[1 + \frac{1}{2}M^{-4}\Big(\dot\phi^2 - (\nabla\phi)^2 - m^2\phi^2\Big)\right]^2}, \tag{88}$$

and the equation relating $\dot\phi$ to π, ϕ, and $\nabla\phi$ is quartic. Quartic equations have very complicated algebraic solutions.

I. Inverse Born-Infeld as Very Awkward Action

The equation (89) may say that as $\dot\phi \to \infty$, $\pi \sim (\dot\phi)^{-3}$, and so $H \sim \pi\dot\phi - L \sim (\dot\phi)^{-2}$, which may be finite, or as $H \sim \pi^{2/3}$, which may be infinite.

Now

$$\frac{\partial L}{\partial \dot\psi} = \dot\psi \left(1 - \frac{2L_0}{M^4}\right)^{-3/2}, \tag{89}$$

and the second derivative is

$$\frac{\partial^2 L}{\partial \dot{\psi}^2} = \left[1 + M^{-4}\left(2\dot{\psi}^2 + V\right)\right]\left(1 - \frac{2L_0}{M^4}\right)^{-5/2} \tag{90}$$

where $V = (\nabla\phi)^2 + m^2\phi^2$.

The time derivative $\dot{\phi}$ is the solution of a cubic equation in $\dot{\phi}^2$, and the hamiltonian is very complicated. It is simpler to use the formula

$$Z(\beta) = \int \exp\left\{\int_0^\beta \int \left[(i\dot{\phi} - \dot{\psi})\frac{\partial L(\phi,\dot{\psi})}{\partial \dot{\psi}} + L(\phi,\dot{\psi})\right] dt d^3x\right\} \left|\frac{\partial^2 L(\phi,\dot{\psi})}{\partial \dot{\psi}^2}\right| D\phi D\dot{\psi}. \tag{91}$$

for the partition function of a very awkward action. For the action density (58), this formula is

$$\begin{aligned} Z(\beta) = &\int \exp\left\{\int_0^\beta \int \left[(i\dot{\phi} - \dot{\psi})\dot{\psi}\left(1 - 2M^{-4}L_0\right)^{-3/2} + M^4\left[\left(1 - 2M^{-4}L_0\right)^{-1/2} - 1\right]\right] dt d^3x\right\} \\ &\times \left[1 + M^{-4}\left(2\dot{\psi}^2 + V\right)\right]\left(1 - \frac{2L_0}{M^4}\right)^{-5/2} D\phi D\dot{\psi} \\ = &\int \exp\left\{\int_0^\beta \int \left[\left(1 - 2M^{-4}L_0\right)^{-3/2}\left(i\dot{\phi}\dot{\psi} - 2\dot{\psi}^2 + (\nabla\phi)^2 + m^2\phi^2 + M^4\right) - M^4\right] dt d^3x\right\} \\ &\times \left[1 + M^{-4}\left(2\dot{\psi}^2 + V\right)\right]\left(1 - \frac{2L_0}{M^4}\right)^{-5/2} D\phi D\dot{\psi} \end{aligned} \tag{92}$$

in which $L_0 = L_0(\phi,\dot{\psi})$. The upper limit on the $\dot{\psi}$ integral is $2L_0 = M^4$ or $\dot{\psi} = \sqrt{M^4 + (\nabla\phi)^2 + m^2\phi^2}$. The effective hamiltonian is

$$H = \frac{2\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2 - M^4}{\left[1 - M^{-4}\left(\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)\right]^{3/2}} + M^4. \tag{93}$$

The Atlantic City integrals are

$$\begin{aligned} A(\phi,\nabla\phi,\Delta\phi) = &\int_0^{\dot{\Psi}} \cos\left(a_s^3\,\Delta\phi\,\dot{\psi}\left(1 - 2M^{-4}L_0\right)^{-3/2}\right) e^{-a_s^3 a_t H} \\ &\times \left[1 + M^{-4}\left(2\dot{\psi}^2 + V\right)\right]\left(1 - \frac{2L_0}{M^4}\right)^{-5/2} d\dot{\psi} \end{aligned} \tag{94}$$

where $d\phi$ is

$$d\phi(i,j,k,\ell) = \phi(i,j,k,\ell+1) - \phi(i,j,k,\ell), \tag{95}$$

L_0 is a lattice version

$$L_0 = \frac{1}{2}\left(\dot{\psi}^2 - (\delta\phi)^2/a_s^2 - m^2\phi^2\right) \tag{96}$$

of the action density of the free field, H is a lattice version

$$H = \frac{2\dot{\psi}^2 - (\delta\phi)^2/a_s^2 - m^2\phi^2 - M^4}{\left[1 - M^{-4}\left(\dot{\psi}^2 - (\delta\phi)^2/a_s^2 - m^2\phi^2\right)\right]^{3/2}} + M^4 \tag{97}$$

of the hamiltonian density (93), $(\delta\phi)^2$ is

$$\begin{aligned}(\delta\phi(i,j,k,\ell))^2 = (\phi(i+1,j,k,\ell) - \phi(i,j,k,\ell))^2 + (\phi(i,j+1,k,\ell) - \phi(i,j,k,\ell))^2 \\ + (\phi(i,j,k+1,\ell) - \phi(i,j,k,\ell))^2,\end{aligned} \tag{98}$$

and the upper limit on the integral is

$$\dot{\Psi} = \sqrt{M^4 + (\delta\phi)^2/a_s^2 + m^2\phi^2}. \tag{99}$$

J. An Action that Bounds the Absolute Value of L_0

We could try

$$L = M^4\left[\frac{1}{2}\left(1 - \frac{2L_0}{M^4}\right)^{-1/2} - \frac{1}{2}\left(1 + \frac{2L_0}{M^4}\right)^{-1/2} + \left(1 - \frac{m^2\phi^2}{M^4}\right)^{-1/2} - 1\right] \tag{100}$$

where

$$L_0 = \frac{1}{2}\left(\dot{\phi}^2 - (\nabla\phi)^2\right). \tag{101}$$

But this leads to the equation

$$\pi = \frac{\dot{\phi}}{2}\left[\left(1 + \frac{(\nabla\phi)^2 - \dot{\phi}^2}{M^4}\right)^{-3/2} + \left(1 + \frac{\dot{\phi}^2 - (\nabla\phi)^2}{M^4}\right)^{-3/2}\right]. \tag{102}$$

So this action is very awkward.

In other words: In the limit $M \to 0$, the hamiltonian is

$$H = |\pi|\sqrt{V} \tag{103}$$

in which $V = (\nabla\phi)^2 + m^2\phi^2$. So the partition function (6) is

$$
\begin{aligned}
Z(\beta) = & \int \exp\left[\int_0^\beta dt \int d^3x \left(i\dot{\phi}\pi - H\right)\right] D\phi D\pi \\
= & \int \exp\left[\int_0^\beta dt \int d^3x \left(i\dot{\phi}\pi - |\pi|\sqrt{V}\right)\right] D\phi D\pi \\
= & \prod_\alpha \int \exp\left[\sum_\alpha a_t a_s^3 \left(i\dot{\phi}_\alpha \pi_\alpha - |\pi_\alpha|\sqrt{V_\alpha}\right)\right] d\phi_\alpha d\pi_\alpha \qquad (104) \\
= & \prod_\alpha \int d\phi_\alpha \left[\frac{1}{a_t a_s^3(\sqrt{V_\alpha} + i\dot{\phi})} + \frac{1}{a_t a_s^3(\sqrt{V_\alpha} - i\dot{\phi}_\alpha)}\right] \\
= & \prod_\alpha \int d\phi_\alpha \left[\frac{2\sqrt{V_\alpha}}{a_t a_s^3(V_\alpha + \dot{\phi}_\alpha^2)}\right] = a_t^{-N^3 s N_t} \prod_\alpha \int d\phi_\alpha \left[\frac{2\sqrt{V_\alpha}}{a_s^3(V_\alpha + \dot{\phi}_\alpha^2)}\right]
\end{aligned}
$$

The mean value of the energy (43) then is

$$
\begin{aligned}
-\frac{d\log Z(\beta)}{d\beta} = & -\frac{1}{Z(\beta)}\frac{1}{N_t}\frac{dZ(\beta)}{da_t} \qquad (105) \\
= & -\frac{1}{Z(\beta)}\frac{1}{N_t}\left\{-\frac{N_t N_s^3}{a_t} Z(\beta) + a_t^{-N^3 s N_t}\frac{d}{da_t}\prod_\alpha \int d\phi_\alpha \left[\frac{2\sqrt{V_\alpha}}{a_s^3(V_\alpha + \dot{\phi}_\alpha^2)}\right]\right\}.
\end{aligned}
$$

Differentiating, we get

$$
-\frac{d\log Z(\beta)}{d\beta} = \frac{N_s^3}{a_t} - \frac{a_t^{-N_s^3 N_t}}{Z(\beta)N_t}\frac{d}{da_t}\prod_\alpha \int d\phi_\alpha \frac{2\sqrt{V_\alpha}}{a_s^3(V_\alpha + \dot{\phi}_\alpha^2)} \qquad (106)
$$

or

$$
-\frac{d\log Z(\beta)}{d\beta} = \frac{N_s^3}{a_t} - \frac{1}{a_t N_t}\sum_\alpha \int d\phi_\alpha \frac{2\dot{\phi}_\alpha^2 \sqrt{V_\alpha}}{(V_\alpha + \dot{\phi}_\alpha^2)^2} \Bigg/ \int d\phi_\alpha \frac{\sqrt{V_\alpha}}{(V_\alpha + \dot{\phi}_\alpha^2)} \,. \qquad (107)
$$

So dividing by the volume $(a_s N_s)^3$, we get

$$
\frac{\langle E(\beta)\rangle}{N_s^3 a_s^3} = \frac{1}{a_s^3 a_t}\left[1 - \frac{1}{N_t N_s^3}\sum_\alpha \int d\phi_\alpha \frac{2\dot{\phi}_\alpha^2 \sqrt{V_\alpha}}{(V_\alpha + \dot{\phi}_\alpha^2)^2} \Bigg/ \int d\phi_\alpha \frac{\sqrt{V_\alpha}}{(V_\alpha + \dot{\phi}_\alpha^2)}\right]. \qquad (108)
$$