# Path integrals for awkward actions

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## Abstract

Time derivatives of scalar fields occur quadratically in textbook actions. A simple Legendre transformation turns the lagrangian into a hamiltonian that is quadratic in the momenta. The path integral over the momenta is gaussian. Mean values of operators are euclidian path integrals of their classical counterparts with positive weight functions. Monte Carlo simulations can estimate such mean values.

This familiar framework falls apart when the time derivatives do not occur quadratically. The Legendre transformation becomes difficult or so intractable that one can't find the hamiltonian. Even if one finds the hamiltonian, it usually is so complicated that one can't path-integrate over the momenta and get a euclidian path integral with a positive weight function. Monte Carlo simulations don't work when the weight function assumes negative or complex values.

This paper solves both problems. It shows how to make path integrals without knowing the hamiltonian. It also shows how to estimate complex path integrals by combining the Monte Carlo method with parallel numerical integration and a look-up table. This "Atlantic City method" lets one estimate the energy densities of theories that, unlike those with quadratic time derivatives, may have finite energy densities. It may lead to a theory of dark energy.

The approximation of multiple integrals over weight functions that assume negative or complex values is a long-standing problem in applied mathematics, called the sign problem. The Atlantic City method solves the sign problem for certain classes of integrals.

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## I. INTRODUCTION

Despite the success of renormalization, infinities remain a major problem in quantum field theory. This problem grows more important as cosmological observations continue to support the existence of dark energy [1], which may be the energy density of empty space. We need to be able to compute finite energy densities. This paper advances theories of scalar fields a small step closer to that goal.

The ground-state energy of a theory is the low-temperature limit of the logarithmic derivative of the partition function $Z(\beta)$ with respect to the inverse temperature $\beta$. If the action density $L$ is quadratic in the time derivatives $\dot{\phi}$ of the fields, then a linear Legendre transformation gives a hamiltonian $H$ that is quadratic in the momenta $\pi$. One then can use the hamiltonian to write the partition function as a euclidian path integral in which the momentum integrals are gaussian. The gaussian integrals give a partition function that is a path integral of a probability distribution in the fields, and one can use Monte Carlo methods to estimate it.

This simple framework falls apart when the time derivatives do not occur quadratically. This collapse is unfortunate because theories of scalar fields that are quadratic in the time derivatives of the fields have infinite energy densities.

An awkward action is one that is not quadratic in the time derivatives but that is simple enough for one to find its hamiltonian. One typically can't integrate over the momentum $\pi$, and the partition function is a double path integral with a complex weight function

$$Z(\beta) = \int \exp\left\{ \int_0^\beta \int \left[ i\dot{\phi}\pi - H(\phi, \pi) \right] dt \, d^3x \right\} D\phi D\pi. \tag{1}$$

Standard Monte Carlo methods fail when the weight function assumes negative or complex values.

A very awkward action is one in which the time derivatives of the fields are related to their momenta and to the fields and their spatial derivatives by equations that are not even quartic and that have no known solutions. Very awkward actions have no known hamiltonians. To study the ground states of this wide class of theories, we show in Sec. III how to write the partition function of such a theory as a path integral without knowledge of the hamiltonian. Our formula is a double path integral over the fields $\phi$ and over auxilliary time derivatives

$\dot\psi$

$$Z(\beta) = \int \exp\left\{ \int_0^\beta \int \Big[ (i\dot\phi_\ell - \dot\psi_\ell) \frac{\partial L(\phi,\dot\psi)}{\partial \dot\psi_\ell} + L(\phi,\dot\psi) \Big] dt d^3x \right\} \left| \det\left( \frac{\partial^2 L(\phi,\dot\psi)}{\partial\dot\psi_k \partial\dot\psi_\ell} \right) \right| D\phi D\dot\psi. \tag{2}$$

This path integral, like the one (1) for awkward actions, has a complex weight function that is not a probability distribution. Again the usual Monte Carlo methods do not work. Both path integrals are examples of the sign problem [2].

To estimate such complex path integrals, we introduce in Sec. IV a Atlantic City method that combines Monte Carlo simulation with parallel numerical integration. In theories with awkward actions, we numerically integrate over the momenta $\pi$ in the double path integral (1). In theories with very awkward actions, we numerically integrate over the auxilliary time derivatives $\dot\psi$ in the double path integral (2). In both cases, we store the values of the integrals in lookup tables. We call this the Atlantic City method. It is well suited to parallel computation and may solve some versions of the sign problem [2]. We demonstrate and test this method by applying it to a quantum-mechanical version of the scalar Born-Infeld model [3] considered as a theory with an awkward action in Sec. V and as a theory with a very awkward action in Sec. VI.

The paper ends with a summary in Sec. VII and does not discuss higher-derivative theories [4] or those in which some fields have no time derivatives [5].

## II. REVIEW OF LEGENDRE TRANSFORMATIONS AND PATH INTEGRALS

The lagrangian of a theory tells us about symmetries and equations of motion, but one needs a hamiltonian to determine the time evolution of states and their energies. To find the hamiltonian of a theory of scalar fields $\phi = \{\phi_1, \dots, \phi_n\}$, one defines the conjugate momenta $\pi = \{\pi_1, \dots, \pi_n\}$ as the derivatives of the action density

$$\pi_j = \frac{\partial L}{\partial \dot\phi_j}, \tag{3}$$

and inverts these equations so as to write the time derivatives $\dot\phi_j = \dot\phi_j(\phi, \pi)$ of the fields in terms of the fields $\phi_\ell$ and their momenta $\pi_\ell$. The hamiltonian density then is

$$H = \sum_{j=1}^n \pi_j \dot\phi_j(\phi,\pi) - L(\phi, \dot\phi(\phi,\pi)). \tag{4}$$

When the action is quadratic in the time derivatives, Legendre's equations (3) are linear.

Once one has a hamiltonian, one inserts complete sets of eigenstates of the fields $\phi_j$ and their conjugate momenta $\pi_j$ into the Boltzmann operator $\exp(-\beta H) = (\exp(-\beta H/n))^n$ and writes the partition function as the complex path integral

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H} = \int \langle \phi | e^{-\beta H} | \phi \rangle D\phi = \int \exp\left\{ \int_0^\beta \int \left[ i\dot{\phi}_j \pi_j - H(\phi, \pi) \right] dt\, d^3x \right\} D\phi D\pi. \tag{5}$$

If one can integrate over the momenta, then one gets Feynman's formula

$$Z(\beta) = \int \exp\left[ \int_0^\beta \int - L_e(\phi, \dot{\phi})\, dt\, d^3x \right] D\phi \tag{6}$$

in which $L_e$ is the euclidian action density, and $D\phi$ is suitably redefined. In textbook theories, $L_e$ is real and positive, and the exponential $\exp[-L_e(\phi, \dot{\phi})]$ is a probability distribution well-suited to Monte Carlo methods.

This procedure is straightforward when the action is quadratic in its time derivatives, and the integrals over the momenta are gaussian. But when the equations (3) that define the momenta have square roots, the hamiltonian also has a square root. When those equations are cubic or quartic, the Legendre transformation and the hamiltonian are much more complicated. When they are worse than quartic, no solution exists, and the hamiltonian is inaccessible. We solve this problem in Sec. III.

## III. PATH INTEGRALS FOR VERY AWKWARD ACTIONS

Our solution to the problem of making a path integral without a hamiltonian is to use functional integration to do Legendre's transformation by means of delta functionals that impose the relation (3) between momenta and the fields and their derivatives. Our formula for the partition function is a double path integral over the fields $\phi$ and over auxiliary time derivatives $\dot{\psi}$

$$Z(\beta) = \int \exp\left\{ \int_0^\beta \int \left[ (i\dot{\phi}_\ell - \dot{\psi}_\ell) \frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}_\ell} + L(\phi, \dot{\psi}) \right] dt d^3x \right\} \left| \det\left( \frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}_k \partial \dot{\psi}_\ell} \right) \right| D\phi D\dot{\psi} \tag{7}$$

in which the determinant converts $D\dot{\psi}$ into $D\pi$, and the energy density

$$E(\phi, \dot{\psi}) = \dot{\psi}_\ell \frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}_\ell} - L(\phi, \dot{\psi}) \tag{8}$$

is the hamiltonian density when the time derivatives $\dot\psi_\ell$ respect Legendre's relation (3). If the action is time independent, then the spatial integral of $E(\phi, \dot\psi)$ is a constant when $\dot\psi_\ell = \dot\phi_\ell(\phi,\pi)$ and the equations of motion are obeyed.

The double path integral (7) for the partition function $Z(\beta)$ is complex and ill-suited to estimation by Monte Carlo methods. We solve this problem in section IV.

To derive our formula (7), we write the path integral (5) as

$$\begin{aligned} Z(\beta) = \int \exp\left\{ \int_0^\beta \int \left[ i\,\dot\phi_j \pi_j - (\pi_k \dot\psi_k - L(\phi,\dot\psi)) \right] dt d^3x \right\} \\ \times \exp\left[ i \int \left( \pi_\ell - \frac{\partial L}{\partial \dot\psi_\ell} \right) a_\ell \, d^4x \right] \left| \det\left( \frac{\partial^2 L}{\partial\dot\psi_k \partial\dot\psi_\ell} \right) \right| D\phi D\pi D\dot\psi D a \end{aligned} \tag{9}$$

in which the integration over the $n$ auxiliary fields $a_\ell$ makes the second exponential a delta functional $\delta(\pi - \partial L/\partial\dot\psi)$ that enforces the definition (3) of the momentum $\pi_j$ as the derivative of the action density $L$ with respect to the time derivative $\dot\phi_j$. The jacobian is an $n \times n$ determinant that converts $D\dot\psi$ to $D\pi$. The integration is over all fields that are periodic with period $\beta$. Integrating first over $a$, we get

$$\begin{aligned} Z(\beta) = \int \exp\left\{ \int_0^\beta \int \left[ i\dot\phi_j \pi_j - (\pi_k \dot\psi_k - L(\phi,\dot\psi)) \right] dt d^3x \right\} \\ \times \prod_{\ell=1}^n \left[ \delta\left( \pi_\ell - \frac{\partial L}{\partial\dot\psi_\ell} \right) \right] \left| \det\left( \frac{\partial^2 L}{\partial\dot\psi_k \partial\dot\psi_\ell} \right) \right| D\phi D\pi D\dot\psi. \end{aligned} \tag{10}$$

To integrate over the auxiliary time derivatives $\dot\psi$, we recall the delta-function rule that if a vector $g(x) = (g_1(x_1,\dots,x_n),\dots,g_n(x_1,\dots,x_n))$ is zero only at $x = x_0$, then

$$\int \delta^n(g_1(x_1,\dots,x_n),\dots,g_n(x_1,\dots,x_n)) \left| \det\left( \frac{\partial g_k(x)}{\partial x_\ell} \right) \right| d^n x = g(x_{10},\dots,x_{n0}). \tag{11}$$

Thus integrating the triple path integral (10) over $\dot\psi$, we find that the delta functional and the jacobian require the time derivatives to assume values $\dot\psi_0(\phi,\pi) = \dot\phi(\phi,\pi)$ that satisfy the definition (3) of the momenta, and we get the path integral (5) over $\phi$ and $\pi$

$$\begin{aligned} Z(\beta) &= \int \exp\left\{ \int_0^\beta \int \left[ i\dot\phi\pi - (\pi\dot\psi_0(\phi,\pi) - L(\phi,\dot\psi_0(\phi,\pi))) \right] dt d^3x \right\} D\phi D\pi \\ &= \int \exp\left\{ \int_0^\beta \int \left[ i\dot\phi_j\pi_j - H(\phi,\pi) \right] dt\, d^3x \right\} D\phi D\pi. \end{aligned} \tag{12}$$

On the other hand, if we integrate the triple path integral (10) over $\pi$, then we get our proposed formula (7)

$$Z(\beta) = \int \exp\left\{\int_0^\beta \int \left[(i\dot{\phi}_\ell - \dot{\psi}_\ell)\frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}_\ell} + L(\phi, \dot{\psi})\right] dt d^3x\right\} \left|\det\left(\frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}_k \partial \dot{\psi}_\ell}\right)\right| D\phi D\dot{\psi}. \tag{13}$$

This functional integral generalizes the path integral to theories of scalar fields in which the hamiltonian is unknown. A similar formula should work in theories of vector and tensor fields, apart from the issue of constraints.

Our first example is a free scalar field with action density

$$L = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{14}$$

The determinant in our formula (13) is unity because

$$\frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}^2} = 1. \tag{15}$$

To simplify our notation, we will use the abbreviation

$$\int_0^\beta dt \int d^3x \equiv \int^\beta d^4x. \tag{16}$$

In this notation, the proposed path integral (13) for the free field theory (14) is

$$\begin{aligned}
Z(\beta) &= \int \exp\left\{\int^\beta \left[L(\phi, \dot{\psi}) + \dot{\psi}(i\dot{\phi} - \dot{\psi})\right] d^4x\right\} D\phi D\dot{\psi} \\
&= \int \exp\left\{\int^\beta \left[\frac{1}{2}\dot{\psi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 + \dot{\psi}(i\dot{\phi} - \dot{\psi})\right] d^4x\right\} D\phi D\dot{\psi} \\
&= \int \exp\left\{\int^\beta \left[-\frac{1}{2}\left(\dot{\psi} - i\dot{\phi}\right)^2 - \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2\right] d^4x\right\} D\phi D\dot{\psi} \\
&= \int \exp\left\{\int^\beta \left[-\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2\right] d^4x\right\} D\phi \\
&= \int \exp\left[-\int^\beta L_e(\phi, \dot{\phi})\, d^4x\right] D\phi
\end{aligned} \tag{17}$$

which is the standard result.

Our second example is the scalar Born-Infeld theory [3] with action density

$$L = M^4\left(1 - \sqrt{1 - M^{-4}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)}\right) \tag{18}$$

and conjugate momentum

$$\pi = \frac{\partial L(\phi, \dot{\phi})}{\partial \dot{\phi}} = \frac{\dot{\phi}}{\sqrt{1 - M^{-4}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)}}. \tag{19}$$

The proposed path integral (13) is

$$Z(\beta) = \int \exp\left\{\int^{\beta}\left[(i\dot{\phi} - \dot{\psi})\frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}} + L(\phi, \dot{\psi})\right] d^4x\right\} \left|\frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}^2}\right| D\phi D\dot{\psi} \tag{20}$$

in which

$$\frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}} = \frac{\dot{\psi}}{\sqrt{1 - M^{-4}\left(\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)}} \tag{21}$$

and

$$\frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}^2} = \frac{1 + M^{-4}\left((\nabla\phi)^2 + m^2\phi^2\right)}{\left[1 - M^{-4}\left(\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)\right]^{3/2}}. \tag{22}$$

Substituting these formulas into (20) gives

$$\begin{aligned} Z(\beta) = \int \exp\Bigg\{\int^{\beta}\Bigg[&\frac{(i\dot{\phi} - \dot{\psi})\dot{\psi}}{\sqrt{1 - M^{-4}\left(\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)}} \\ &+ M^4\left(1 - \sqrt{1 - M^{-4}\left(\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)}\right)\Bigg] d^4x\Bigg\} \\ &\times \frac{1 + M^{-4}\left((\nabla\phi)^2 + m^2\phi^2\right)}{\left[1 - M^{-4}\left(\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)\right]^{3/2}} D\phi D\dot{\psi}. \end{aligned} \tag{23}$$

We can set

$$\pi = \frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}} = \frac{\dot{\psi}}{\sqrt{1 - M^{-4}\left(\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)}} \tag{24}$$

and so absorb the jacobian in

$$d\pi = \frac{\partial \pi}{\partial \dot{\psi}}\, d\dot{\psi} = \frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}^2}\, d\dot{\psi} = \frac{1 + M^{-4}\left((\nabla\phi)^2 + m^2\phi^2\right)}{\left[1 - M^{-4}\left(\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)\right]^{3/2}}\, d\dot{\psi}. \tag{25}$$

The partition function (23) then is

$$Z(\beta) = \int \exp\left[\int^{\beta}(i\dot{\phi} - \dot{\psi})\pi + M^4\left(1 - \sqrt{1 - \left(\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)/M^4}\right) d^4x\right] D\phi D\pi \tag{26}$$

where now $\dot{\psi}(\phi, \pi)$ is the function of $\phi$ and $\pi$ defined by (24).

This theory is one of the few in which we can solve Legendre's equation (24) for the time derivative $\dot{\psi}$

$$\dot{\psi} = \frac{\pi}{\sqrt{1 + M^{-4}\,\pi^2}} \sqrt{1 + M^{-4}\left((\nabla\phi)^2 + m^2\phi^2\right)} \tag{27}$$

and find as the hamiltonian density

$$\begin{aligned} H(\phi, \pi) &= \pi\dot{\psi} - L(\phi, \dot{\psi}) \\ &= \frac{\pi^2\sqrt{1 + M^{-4}\left((\nabla\phi)^2 + m^2\phi^2\right)}}{\sqrt{1 + M^{-4}\,\pi^2}} - M^4 \\ &\quad + M^4\sqrt{1 - M^{-4}\left(\frac{\pi^2\left(1 + M^{-4}\left((\nabla\phi)^2 + m^2\phi^2\right)\right)}{1 + M^{-4}\,\pi^2} - (\nabla\phi)^2 - m^2\phi^2\right)} \\ &= \frac{\pi^2\sqrt{M^4 + (\nabla\phi)^2 + m^2\phi^2}}{\sqrt{M^4 + \pi^2}} + M^4\frac{\sqrt{M^4 + (\nabla\phi)^2 + m^2\phi^2}}{\sqrt{M^4 + \pi^2}} - M^4 \\ &= \sqrt{(M^4 + \pi^2)\left(M^4 + (\nabla\phi)^2 + m^2\phi^2\right)} - M^4. \end{aligned} \tag{28}$$

Thus for this theory, the double path integral (1) is

$$Z(\beta) = \int \exp\left\{\int^{\beta}\left[i\dot{\phi}\pi - \sqrt{(M^4 + \pi^2)\left(M^4 + (\nabla\phi)^2 + m^2\phi^2\right)} + M^4\right] d^4x\right\} D\phi D\pi. \tag{29}$$

Our third example is the theory defined by the action density

$$L = M^4 \exp(L_0/M^4) \tag{30}$$

in which $L_0$ is the action density (14) of the free field. The derivatives of $L$ are

$$\frac{\partial L}{\partial\dot{\psi}} = M^{-4}\dot{\psi}\,L \quad \text{and} \quad \frac{\partial^2 L}{\partial\dot{\psi}^2} = M^{-4}(1 + M^{-4}\dot{\psi}^2)\,L. \tag{31}$$

So the proposed path integral is

$$\begin{aligned} Z(\beta) &= \int \exp\left\{\int^{\beta}\left[L(\phi, \dot{\psi}) + \frac{\partial L(\phi, \dot{\psi})}{\partial\dot{\psi}}(i\dot{\phi} - \dot{\psi})\right] d^4x\right\} \left|\frac{\partial^2 L(\phi, \dot{\psi})}{\partial\dot{\psi}^2}\right| D\phi D\dot{\psi} \\ &= \int \exp\left\{\int^{\beta}\left[1 + \frac{\dot{\psi}(i\dot{\phi} - \dot{\psi})}{M^4}\right] L(\phi, \dot{\psi})\, d^4x\right\} M^{-4}(1 + M^{-4}\dot{\psi}^2)\, L\, D\phi D\dot{\psi}. \end{aligned} \tag{32}$$

Our fourth example is the Nambu-Gotō action density

$$L = -\frac{T_0}{c}\int_0^{\sigma_1} \sqrt{\left(\dot{X}\cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2} \tag{33}$$

in which the tau or time derivatives of the coordinate fields $X^\mu$ do not occur quadratically [6]. The momenta are

$$\mathcal{P}^\tau_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2}} \tag{34}$$

and the second derivatives of the Lagrange density are

$$\begin{aligned} \frac{\partial^2 L}{\partial \dot{X}^\mu \partial \dot{X}^\nu} = \frac{T_0}{c} \Bigg[ & \frac{\delta_{\mu\nu} X'^2 - X'_\mu X'_\nu}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2}} \\ & - \frac{\left((\dot{X} \cdot X')X'_\mu - (X')^2 \dot{X}_\mu\right)\left((\dot{X} \cdot X')X'_\nu - (X')^2 \dot{X}_\nu\right)}{\left[\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2\right]^{3/2}} \Bigg] . \end{aligned} \tag{35}$$

The proposed partition function (13) for the Nambu-Gotō action is then

$$Z(\beta) = \int \exp\left\{ \int^\beta \left[ (i\dot{X}^\mu - \dot{Y}^\mu) \frac{\partial L(X,\dot{Y})}{\partial \dot{Y}^\mu} + L(X,\dot{Y}) \right] d^4x \right\} \left| \det\left[ \frac{\partial^2 L(X,\dot{Y})}{\partial \dot{Y}^\mu \partial \dot{Y}^\nu} \right] \right| DX D\dot{Y} \tag{36}$$

in which the formulas (34) and (35) (with $\dot{X}^\mu \to \dot{Y}^\mu$) are to be substituted for the first and second derivatives of the action density $L$ with respect to the tau derivatives $\dot{Y}^\mu$.

## IV. THE ATLANTIC CITY METHOD

Monte Carlos let us estimate the mean values of observables weighted by probability distributions [7]. They fail when the weight function assumes negative or complex values. This failure is one aspect of the sign problem [2]. The double-ratio trick (A3–A4) outlined in the appendix is unreliable.

These problems are not hopeless however. For although the weight functions of the double path integrals (1) and (2) are complex, the integrals of these complex weight functions over the momenta $\pi$ or over the auxiliary time derivatives $\dot{\psi}$ are real and positive. They are the probability distribution that determines the partition function and the mean values of observables.

If one can't do these integrals analytically, one can do them numerically. These numerical integrations are perfectly suited to parallel computation. The Atlantic City method of computing the partition functions of theories with awkward and very awkward actions uses parallel computation to numerically integrate over the momenta $\pi$ or over the auxiliary time derivatives $\dot{\psi}$ in the double path integrals (1) and (2). One then stores the values of these integrals in a look-up table. One then uses the Monte Carlo method guided by the stored integrals to estimate the partition function.

Our main goal is to use the Atlantic City method to study the ground states of field theories, but for simplicity in this paper we will explain and test it in the context of quantum mechanics.

Suppose the action is awkward, but not very awkward. In this case, we know the hamiltonian $H(q,p)$ but can't integrate analytically over the momentum $p$. Then the partition function $Z(\beta)$ is

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H} = \int \exp\left\{ \int_0^\beta \left[ i\dot{q}p - H(q,p) \right] dt \right\} DpDq. \tag{37}$$

We use the approximation

$$\begin{aligned} \langle q_{\ell+1}|e^{-aH(q,p)}|q_\ell\rangle &\approx \langle q_{\ell+1}|e^{-aH(q_{\ell+1},p)/2} \int dp|p\rangle\langle p|e^{-aH(q_\ell,p)/2}|q_\ell\rangle \\ &= \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \exp\left[ i(q_{\ell+1}-q_\ell)p - a\,\left(H(q_{\ell+1},p)+H(q_\ell,p)\right)/2 \right] \end{aligned} \tag{38}$$

to estimate the partition function as the multiple integral

$$Z(\beta) = \prod_{j=1}^n \int \frac{dp_j dq_j}{2\pi} \exp\left[ i(q_{j+1}-q_j)p_j - aH(q_{j+1},p_j)/2 - aH(q_j,p_j)/2 \right] \tag{39}$$

in which $n = \beta/a$ and $q_n = q_0$.

We know that

$$P[q,\beta] = \int \exp\left\{ \int_0^\beta \left[ i\dot{q}p - H(q,p) \right] dt \right\} Dp \tag{40}$$

is a functional probability distribution that assigns a number $P[q,\beta]$ to every path $q(t)$. It is the limit as $n \to \infty$ and $a = \beta/n \to 0$ of the multiple integral

$$P_n[q,\beta] = \prod_{j=1}^n \int \frac{dp_j}{\sqrt{2\pi}} \exp\left[ i(q_{j+1}-q_j)p_j - aH(q_{j+1},p_j)/2 - aH(q_j,p_j)/2 \right]. \tag{41}$$

If the hamiltonian is even in the momentum, then the probability distribution is real

$$P_n[q,\beta] = \prod_{j=1}^n \int \frac{dp_j}{\sqrt{2\pi}} \cos((q_{j+1}-q_j)p_j) e^{-aH(q_{j+1},p_j)/2 - aH(q_j,p_j)/2}. \tag{42}$$

The partition function $Z(\beta)$ is

$$Z(\beta) = \int P[q,\beta]\, Dq = \int \prod_{j=1}^{n} \frac{dq_j}{\sqrt{2\pi}}\, P_n[q,\beta]. \tag{43}$$

The mean value of the energy at inverse temperature $\beta$ is

$$\langle H \rangle_\beta = \frac{\mathrm{Tr}\, H\, e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}} = -\frac{1}{Z(\beta)} \frac{dZ(\beta)}{d\beta} = -\frac{1}{Z(\beta)} \int \prod_{j=1}^{n} \frac{dq_j}{\sqrt{2\pi}} \frac{dP_n[q,\beta]}{d\beta}. \tag{44}$$

The derivative of the probability distribution with respect to $\beta = n\, a$ is

$$-\frac{dP_n[q,\beta]}{d\beta} = \frac{1}{n} \sum_{k=1}^{n} \int \prod_{j=1}^{n} \frac{dp_j}{\sqrt{2\pi}}\, H(q_k,p_k)\, e^{i(q_{j+1}-q_j)p_j - aH(q_{j+1},p_j)/2 - aH(q_j,p_j)/2}. \tag{45}$$

So the mean value of the hamiltonian at inverse temperature $\beta$ is

$$\begin{aligned} \langle H \rangle_\beta = {} & \frac{1}{n} \sum_{k=1}^{n} \int \prod_{j=1}^{n} dq_j\, dp_j\; H(q_k,p_k)\, e^{i(q_{j+1}-q_j)p_j - aH(q_{j+1},p_j)/2 - aH(q_j,p_j)/2} \\ & \Big/ \int \prod_{j=1}^{n} dq_j\, dp_j\, e^{i(q_{j+1}-q_j)p_j - aH(q_{j+1},p_j)/2 - aH(q_j,p_j)/2} . \end{aligned} \tag{46}$$

In the Atlantic City method, one does the $p$ integrations numerically, setting

$$A(q_{\ell+1},q_\ell) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}}\, \exp\left[i(q_{\ell+1}-q_\ell)p\, - a\, \left(H(q_{\ell+1},p) + H(q_\ell,p)\right)/2\right] \tag{47}$$

and

$$C(q_{\ell+1},q_\ell) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}}\, H(q_\ell,p)\, \exp\left[i(q_{\ell+1}-q_\ell)p\, - a\, \left(H(q_{\ell+1},p) + H(q_\ell,p)\right)/2\right]. \tag{48}$$

In most problems of interest, the hamiltonian is an even function of the momentum, $H(q,-p) = H(q,p)$, and the integrals (47 & 48) are real

$$\begin{aligned} A(q_{\ell+1},q_\ell) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} dp\, \cos\left[(q_{\ell+1}-q_\ell)\, p\right]\, e^{-a\,(H(q_{\ell+1},p)+H(q_\ell,p))/2} \\ C(q_{\ell+1},q_\ell) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} dp\, H(q_\ell,p)\, \cos\left[(q_{\ell+1}-q_\ell)\, p\right]\, e^{-a\,(H(q_{\ell+1},p)+H(q_\ell,p))/2}. \end{aligned} \tag{49}$$

In terms of these numerical integrals, the mean value of the hamiltonian is

$$\langle H \rangle_\beta = \frac{1}{n} \sum_{k=1}^{n} \int dq_k \prod_{j=1,\, j\neq k}^{n} dq_j\, C(q_{k+1},q_k)\, A(q_{j+1},q_j) \Big/ \int \prod_{j=1}^{n} dq_j\, A(q_{j+1},q_j) \tag{50}$$

which we may write as

$$\langle H\rangle_\beta = \frac{1}{n}\sum_{k=1}^{n}\int\prod_{j=1}^{n} dq_j\,\frac{C(q_{k+1},q_k)}{A(q_{k+1},q_k)}\,A(q_{j+1},q_j)\bigg/\int\prod_{j=1}^{n} dq_j\,A(q_{j+1},q_j)\;. \tag{51}$$

We do a Monte Carlo over the probability distribution

$$P(q) = \prod_{j=1}^{n} A(q_{j+1},q_j)\bigg/\int\prod_{j=1}^{n} dq_j\,A(q_{j+1},q_j) \tag{52}$$

and measure the ratio

$$\langle H\rangle_\beta = \left\langle \frac{1}{n}\sum_{k=1}^{n}\frac{C(q_{k+1},q_k)}{A(q_{k+1},q_k)}\right\rangle = \int \frac{1}{n}\sum_{k=1}^{n}\frac{C(q_{k+1},q_k)}{A(q_{k+1},q_k)}\,P(q)\,Dq. \tag{53}$$

The probability distribution $P[q,\beta]$ is positive. For instance, the quantity

$$\langle q^2(t)\rangle = \mathrm{Tr}\, e^{-(\beta-t)H} q^2 e^{-tH} = \int q^2(t) P[q,\beta]\,Dq \tag{54}$$

is positive for every $t$ and every path $q(t)$. In fact, the value $A(q_{\ell+1},q_\ell)$ of the integral (49) is positive as shown in Appendix B.

To take a Metropolis step, we pick a new $q'_j$ and look up the value of the probability distribution

$$P(q'_j) = A(q_{j+1},q'_j)\,A(q'_j,q_{j-1}). \tag{55}$$

In general, the random points $q_{j+1}$, $q'_j$, and $q_{j-1}$ will not be among the $q_j$'s in our tables. If the value $q_{j+1}$ falls between the values $q_{j+1,+}$ and $q_{j+1,-}$, and the value $q'_j$ falls between the values $q_{j+}$ and $q_{j-}$, then we use a bilinear interpolation, setting

$$\rho(q_{j+1}) = \frac{q_{j+1}-q_{j+1,-}}{q_{j+1,+}-q_{j+1,-}} \quad \text{and} \quad \rho(q'_j) = \frac{q'_j - q_{j-}}{q_{j+}-q_{j-}} \tag{56}$$

so that

$$\begin{aligned} A(q_{j+1},q'_j) &= \rho(q'_j)A(q_{j+1},q_{j+}) + \left[1-\rho(q'_j)\right]A(q_{j+1},q_{j-}) \\ &= \rho(q'_j)\left\{\rho(q_{j+1})A(q_{j+1,+},q_{j+}) + \left[1-\rho(q_{j+1})\right]A(q_{j+1,-},q_{j+})\right\} \\ &\quad + \left[1-\rho(q'_j)\right]\left\{\rho(q_{j+1})A(q_{j+1,+},q_{j-}) + \left[1-\rho(q_{j+1})\right]A(q_{j+1,-},q_{j-})\right\} \end{aligned} \tag{57}$$

with a similar bilinear interpolation for $A(q'_j,q_{j-1})$.

If $P(q'_j) \ge P(q_j)$, then we accept the new $q'_j$. If $P(q'_j) < P(q_j)$, then we accept the new $q'_j$ with probability

$$P(q_j \to q'_j) = P(q'_j)/P(q_j). \tag{58}$$

## V. APPLICATION OF THE ATLANTIC CITY METHOD TO THE BORN-INFELD OSCILLATOR

In this section we demonstrate and test our Atlantic City model on a theory with an awkward action, the quantum-mechanical version of the scalar Born-Infeld model (18–29). The lagrangian of this model is

$$L = Mc^2 - Mc^2 \left[1 - \frac{m}{Mc^2}\left(\dot{q}^2 - \omega^2 q^2\right)\right]^{1/2}. \tag{59}$$

The momentum is

$$p = \frac{m\dot{q}}{\sqrt{1 - m\left(\dot{q}^2 - \omega^2 q^2\right)/(Mc^2)}}, \tag{60}$$

and the velocity is

$$\dot{q} = \frac{p}{m}\frac{\sqrt{1 + m\omega^2 q^2/Mc^2}}{\sqrt{1 + p^2/(mMc^2)}}. \tag{61}$$

The hamiltonian of the Born-Infeld oscillator is

$$H = \sqrt{\left(p^2/m + Mc^2\right)\ \left(Mc^2 + m\omega^2 q^2\right)} - Mc^2. \tag{62}$$

In terms of the hamiltonian $H_0 = p^2/2m + m\omega^2 q^2/2$ of the harmonic oscillator, the hamiltonian $H$ of the Born-Infeld oscillator in the limit $M/m \gg 1$ is

$$H = H_0 - \frac{1}{8Mc^2}\left(\frac{p^2}{2m} - \frac{m\omega^2 q^2}{2}\right)^2\left(1 - \frac{H_0}{Mc^2}\right) + \dots \tag{63}$$

and

$$H = \sqrt{Mc^2 m\omega^2 q^2}\left[1 + \frac{p^2}{2mMc^2} - \frac{p^4}{8(mMc^2)^2}\right]\left[1 + \frac{Mc^2}{2m\omega^2 q^2} - \frac{(Mc^2)^2}{32(m\omega^2 q^2)^2}\right] + \dots \tag{64}$$

for $M/m \ll 1$. The energy levels of the Born-Infeld oscillator, like those of the harmonic oscillator, are independent of the mass $m$ due to a symmetry that the change of variables to $q' = q\sqrt{m}$ and $p' = p/\sqrt{m}$ reveals.

With $\hbar = c = 1$ and $\beta = n\,a$, the partition function is

$$\begin{aligned}
Z(\beta) = \mathrm{Tr}\, e^{-\beta H} &= \int \exp\left\{\int_0^\beta \left[i\dot{q}p - H(q,p)\right]dt\right\} DpDq \\
&\approx \prod_{j=1}^{n}\int \frac{dp_j dq_j}{2\pi}\exp\left[i(q_{j+1} - q_j)p_j - aH(q_{j+1},p_j)/2 - aH(q_j,p_j)/2\right] \\
&\approx \prod_{j=1}^{n}\int \frac{dp_j dq_j}{2\pi}\exp\Big[i(q_{j+1} - q_j)p_j - \frac{a}{2}\left(\sqrt{p_j^2/m + M}\ \sqrt{M + m\omega^2 q_{j+1}^2} - M\right) \\
&\quad -\frac{a}{2}\left(\sqrt{p_j^2/m + M}\ \sqrt{M + m\omega^2 q_j^2} - M\right)\Big].
\end{aligned} \tag{65}$$

In terms of the variables to $q' = q\sqrt{m}$ and $p' = p/\sqrt{m}$, which satisfy the commutation relation $[q', p'] = i$, the Born-Infeld hamiltonian (62) is

$$H = \sqrt{(p'^2 + M)\ \ (M + \omega^2 q'^2)} - M, \tag{66}$$

which shows that the energy levels are independent of the mass parameter $m$. To simplify our notation and expose the actual dependence of these energies, we change variables again to $p' = \sqrt{M}p''$, $q' = \sqrt{M}q''/\omega$, and $a' = aM$. After we drop all the primes, we have

$$H = M\left[\sqrt{(p^2 + 1)\ \ (q^2 + 1)} - 1\right], \tag{67}$$

and

$$\begin{aligned} Z(\beta) \approx \prod_{j=1}^{n} \int \frac{M dp_j dq_j}{2\pi\omega} \exp\Bigg\{ \Bigg[ & i\frac{M}{\omega}(q_{j+1} - q_j)p_j - \frac{a}{2}\left(\sqrt{\left(p_j^2 + 1\right)\left(q_{j+1}^2 + 1\right)} - 1\right) \\ & -\frac{a}{2}\left(\sqrt{\left(p_j^2 + 1\right)\left(q_j^2 + 1\right)} - 1\right)\Bigg]\Bigg\}. \end{aligned} \tag{68}$$

The mean value of the hamiltonian at inverse temperature $\beta = na$ is

$$\langle H \rangle_\beta = -\frac{\operatorname{Tr} H\, e^{-\beta H}}{\operatorname{Tr} e^{-\beta H}} = -\frac{1}{Z(\beta)}\frac{dZ(\beta)}{d\beta} = -\frac{M}{n\, Z(\beta)}\frac{dZ(\beta)}{da}. \tag{69}$$

The energy $\langle H \rangle_\beta$ is a function of the ratio $M/\omega$ and is proportional to $M$

$$\begin{aligned} \langle H \rangle_\beta = \prod_{j=1}^{n} \int dp_j dq_j & \left[\frac{M}{2n}\sum_{\ell=0}^{n}\sqrt{p_\ell^2 + 1}\left(\sqrt{q_{\ell+1}^2 + 1} + \sqrt{q_\ell^2 + 1} - 2\right)\right] \\ & \times \exp\left[i\frac{M}{\omega}(q_{j+1} - q_j)p_j - \frac{a}{2}\sqrt{p_j^2 + 1}\left(\sqrt{q_{j+1}^2 + 1} + \sqrt{q_j^2 + 1}\right) + a\right] \\ & \Bigg/ \prod_{j=1}^{n} \int dp_j dq_j \exp\left[i\frac{M}{\omega}(q_{j+1} - q_j)p_j - \frac{a}{2}\sqrt{p_j^2 + 1}\left(\sqrt{q_{j+1}^2 + 1} + \sqrt{q_j^2 + 1}\right) + a\right]. \end{aligned} \tag{70}$$

The ground-state energy is the limit of the ratio as $\beta \to \infty$ and $a \to 0$.

We wrote Fortran90 codes to compute in parallel the momentum integrals

$$A(q_{\ell+1}, q_\ell) = \int_0^\infty dp \cos\left[\frac{M}{\omega}(q_{\ell+1} - q_\ell)p\right] \exp\left[-\frac{a}{2}\sqrt{p^2 + 1}\left(\sqrt{q_{\ell+1}^2 + 1} + \sqrt{q_\ell^2 + 1}\right) + a\right] \tag{71}$$

and

$$\begin{aligned} C(q_{\ell+1}, q_\ell) = M \int_0^\infty & dp \sqrt{p^2 + 1}\left(\tfrac{1}{2}\sqrt{q_{\ell+1}^2 + 1} + \tfrac{1}{2}\sqrt{q_\ell^2 + 1} - 1\right) \\ & \times \cos\left[\frac{M}{\omega}(q_{\ell+1} - q_\ell)p\right] \exp\left[-\frac{a}{2}\sqrt{p^2 + 1}\left(\sqrt{q_{\ell+1}^2 + 1} + \sqrt{q_\ell^2 + 1}\right) + a\right] \end{aligned} \tag{72}$$

for suitably large sets of values of $q_\ell$ and $q_{\ell+1}$ and stored them in look-up tables. We then used the look-up tables in a generic Monte Carlo with the Metropolis step (55–58) to estimate the mean value of the hamiltonian at inverse temperature $\beta$

$$\langle H\rangle_\beta = \left\langle \frac{1}{n}\sum_{k=1}^{n} \frac{C(q_{k+1},q_k)}{A(q_{k+1},q_k)} \right\rangle = \int \frac{1}{n}\sum_{k=1}^{n} \frac{C(q_{k+1},q_k)}{A(q_{k+1},q_k)}\, P_n[q,\beta]\, Dq \tag{73}$$

in which the unnormalized probability distribution is

$$P_n(q,\beta) = \prod_{\ell=1}^{n} A(q_{\ell+1},q_\ell) \tag{74}$$

and $q_{n+1} \equiv q_1$.

The Monte Carlo codes run fast; all the work is in the look-up tables. We made look-up tables for $0.1 \le Mc^2/(\hbar\omega) \le 10$. We plotted our Atlantic City (71–73) estimates of the ground-state energy of the Born-Infeld oscillator as blue dots in Fig. 1 and listed them in Table I. The statistical errors are smaller than the dots.

To test these results, we used Matlab to compute the exact eigenvalues of the Born-Infeld oscillator. In terms of the harmonic-oscillator variables

$$a = \sqrt{\frac{m\omega}{2}}\left(q + \frac{ip}{m\omega}\right) \quad \text{and} \quad a^\dagger = \sqrt{\frac{m\omega}{2}}\left(q - \frac{ip}{m\omega}\right), \tag{75}$$

the operators $q$ and $p$ are $q = \left(a^\dagger + a\right)/\sqrt{2m\omega}$ and $p = i\sqrt{m\omega/2}\left(a^\dagger - a\right)$, and so the hamiltonian (62) is

$$H = \sqrt{Mc^2 - \frac{\omega}{2}\left(a^\dagger - a\right)^2}\sqrt{Mc^2 + \frac{\omega}{2}\left(a^\dagger + a\right)^2} - Mc^2 \tag{76}$$

in which the mass $m$ does not appear. We made a $1000\times1000$ matrix $a$ as diag(sqrt([1:Nmax]),1) with Nmax = 1000 and $a^\dagger$ as its transpose. The Matlab command eig(sqrtm(H)) then gave the exact energy eigenvalues, which generated the red curves in the figures and the exact results in the tables.

## VI. THE ATLANTIC CITY MODEL APPLIED TO A VERY AWKWARD ACTION

In this section, we test our Atlantic City method by using it to find the ground-state energy of the Born-Infeld oscillator considered as a theory with a very awkward action.

TABLE I: Exact (Matlab) and Atlantic City results (71–73) for the ground-state energy $E_0/(\hbar\omega)$ of the Born-Infeld hamiltonian (62) for $0.1 \le Mc^2/(\hbar\omega) \le 10$.

| $Mc^2/(\hbar\omega)$ | $E_0/(\hbar\omega)$ exact | $E_0/(\hbar\omega)$ Atlantic City |
|---|---|---|
| 0.1 | 0.1881 | 0.1759 |
| 0.5 | 0.3155 | 0.3191 |
| 1.0 | 0.3702 | 0.3746 |
| 2.5 | 0.4288 | 0.4308 |
| 5.0 | 0.4587 | 0.4603 |
| 7.5 | 0.4708 | 0.4723 |
| 10.0 | 0.4774 | 0.4781 |

That is, we pretend that we don't know the Born-Infeld hamiltonian (62) and use our Atlantic City method to evaluate the complex path integral (2) for its partition function. So instead of the partition function (70), we have the partition function

$$
\begin{aligned}
Z(\beta) &= \int \exp\left\{\int^{\beta}\left[(i\dot{q}-\dot{s})\frac{\partial L(q,\dot{s})}{\partial \dot{s}} + L(q,\dot{s})\right]dt\right\}\left|\frac{\partial^2 L(q,\dot{s})}{\partial \dot{s}^2}\right| DqD\dot{s} \\
&= \int \exp\left\{\int^{\beta}\left[\frac{mi\dot{q}\dot{s} - m\omega^2 q^2 - M}{\sqrt{1-m\left(\dot{s}^2-\omega^2 q^2\right)/M}} + M\right]dt\right\} \\
&\quad \times \frac{m + m^2\omega^2 q^2/M}{\left[1-m\left(\dot{s}^2-\omega^2 q^2\right)/M\right]^{3/2}}\, DqD\dot{s}.
\end{aligned}
\tag{77}
$$

Sending $q_j \to \sqrt{M/m}\, q_j/\omega$ and $\dot{s}_j \to \sqrt{M/m}\, \dot{s}_j$, we can write $Z(\beta)$ as

$$
Z(\beta) = \int \exp\left\{\int^{\beta}\left[\frac{i(M/\omega)\dot{q}\dot{s} - Mq^2 - M}{\sqrt{1+q^2-\dot{s}^2}} + M\right]dt\right\}\frac{1+q^2}{\left[1+q^2-\dot{s}^2\right]^{3/2}}\frac{M}{\omega}DqD\dot{s}. \tag{78}
$$

Sending $dt = a \to a/M$, we approximate this path integral on an $n \times n$ lattice of lattice

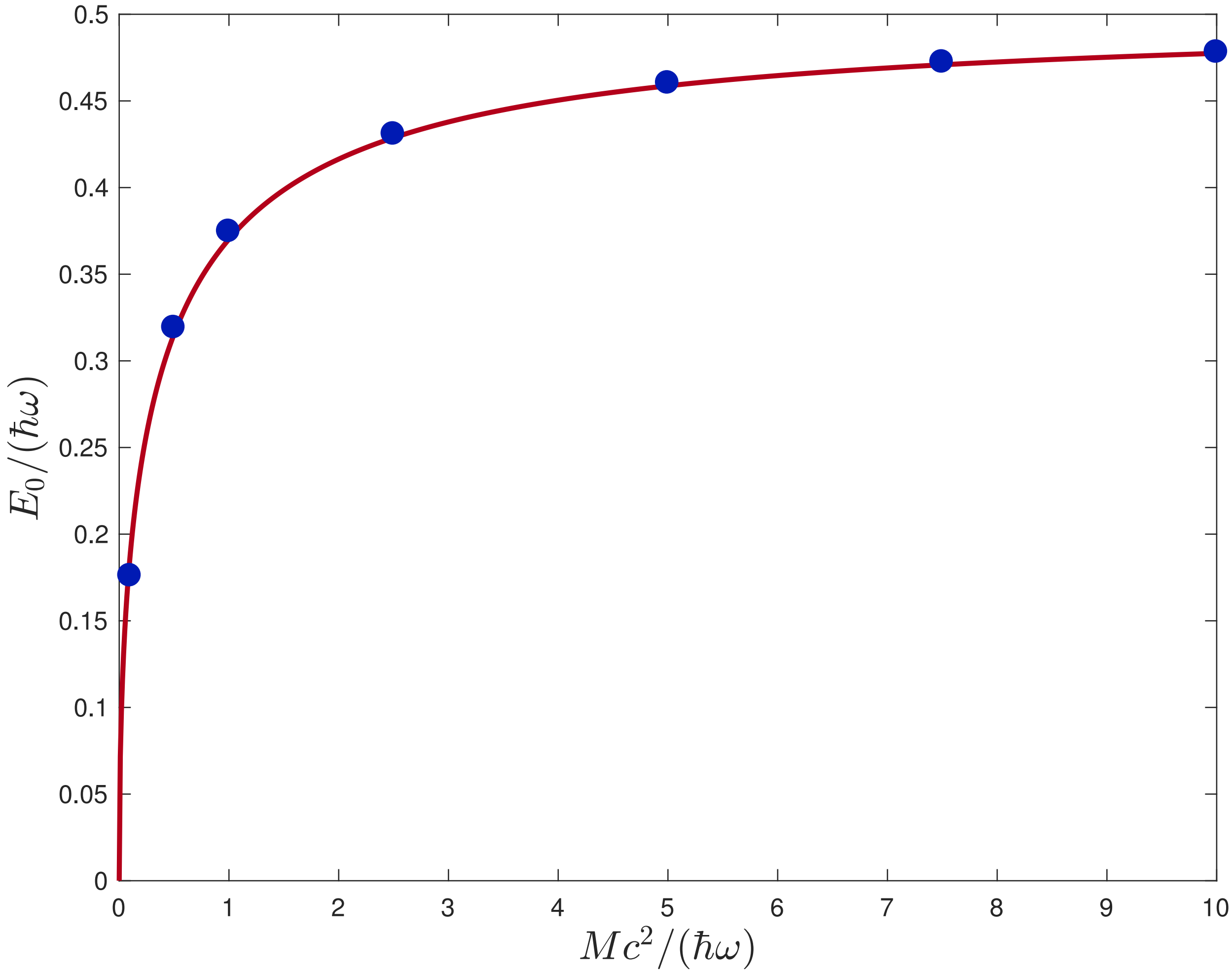


FIG. 1: Our Atlantic City estimates (71–73, blue dots) of the ground-state energies $E_0/(\hbar\omega)$ of the Born-Infeld oscillator are plotted along with the exact values (Matlab, red curve) for $0.1 \le Mc^2/(\hbar\omega) \le 10$.

spacing $a = \beta/n$ as the multiple integral

$$
\begin{aligned}
Z(\beta) = \prod_{j=1}^{n} \int \frac{M d\dot{s}_j dq_j}{2\pi\omega} \exp \Bigg[ & i\frac{M}{\omega} \left( \frac{q_{j+1}\dot{s}_j}{\sqrt{1+q_{j+1}^2-\dot{s}_j^2}} - \frac{q_j\dot{s}_j}{\sqrt{1+q_j^2-\dot{s}_j^2}} \right) \\
& -\frac{a}{2} \left( \frac{q_{j+1}^2+1}{\sqrt{1+q_{j+1}^2-\dot{s}_j^2}} + \frac{q_j^2+1}{\sqrt{1+q_j^2-\dot{s}_j^2}} \right) + a \Bigg] \\
& \times \frac{\sqrt{1+q_{j+1}^2}}{\left[1+q_{j+1}^2-\dot{s}_j^2\right]^{3/4}} \frac{\sqrt{1+q_j^2}}{\left[1+q_j^2-\dot{s}_j^2\right]^{3/4}}.
\end{aligned}
\tag{79}
$$

Apart from the phase factor, the integrand is even in $\dot{s}$. We numerically compute the integrals

$$
\begin{aligned}
A(q_{j+1}, q_j) = & \int_0^{S_j} d\dot{s} \, \cos\left[ i \frac{M}{\omega} \left( \frac{q_{j+1}\dot{s}_j}{\sqrt{1+q_{j+1}^2 - \dot{s}_j^2}} - \frac{q_j \dot{s}_j}{\sqrt{1+q_j^2 - \dot{s}_j^2}} \right) \right] \\
& \times \exp\left[ -\frac{a}{2} \left( \frac{q_{j+1}^2 + 1}{\sqrt{1+q_{j+1}^2 - \dot{s}_j^2}} + \frac{q_j^2 + 1}{\sqrt{1+q_j^2 - \dot{s}_j^2}} \right) + a \right] \\
& \times \frac{\sqrt{1+q_{j+1}^2}}{\left[1+q_{j+1}^2 - \dot{s}_j^2\right]^{3/4}} \frac{\sqrt{1+q_j^2}}{\left[1+q_j^2 - \dot{s}_j^2\right]^{3/4}}
\end{aligned} \tag{80}
$$

and

$$
\begin{aligned}
C(q_{j+1}, q_j) = M & \int_0^{S_j} d\dot{s} \, \left[ \frac{q_{j+1}^2 + 1}{2\sqrt{1+q_{j+1}^2 - \dot{s}_j^2}} + \frac{q_j^2 + 1}{2\sqrt{1+q_j^2 - \dot{s}_j^2}} - 1 \right] \\
& \times \cos\left[ i \frac{M}{\omega} \left( \frac{q_{j+1}\dot{s}_j}{\sqrt{1+q_{j+1}^2 - \dot{s}_j^2}} - \frac{q_j \dot{s}_j}{\sqrt{1+q_j^2 - \dot{s}_j^2}} \right) \right] \\
& \times \exp\left[ -\frac{a}{2} \left( \frac{q_{j+1}^2 + 1}{\sqrt{1+q_{j+1}^2 - \dot{s}_j^2}} + \frac{q_j^2 + 1}{\sqrt{1+q_j^2 - \dot{s}_j^2}} \right) + a \right] \\
& \times \frac{\sqrt{1+q_{j+1}^2}}{\left[1+q_{j+1}^2 - \dot{s}_j^2\right]^{3/4}} \frac{\sqrt{1+q_j^2}}{\left[1+q_j^2 - \dot{s}_j^2\right]^{3/4}}
\end{aligned} \tag{81}
$$

in which $S_j$ is the lesser of $\sqrt{1+q_{j+1}^2}$ and $\sqrt{1+q_j^2}$. We do these integrals in parallel and store their values in a look-up table. We then use the look-up table to estimate the mean value

$$
\langle H \rangle_\beta = \left\langle \frac{1}{n} \sum_{k=1}^{n} \frac{C(q_{k+1}, q_k)}{A(q_{k+1}, q_k)} \right\rangle . \tag{82}
$$

We plotted our results as the green dots in Fig. 2 and listed them in Table II.

## VII. SUMMARY

We divide the actions of theories of scalar fields into three classes—graceful, awkward, and very awkward. An action is graceful if it is quadratic in the time derivatives of the fields, which then are linearly related to the momenta, the fields, and their spatial derivatives. The partition function is a path integral over the fields with a positive weight function. An

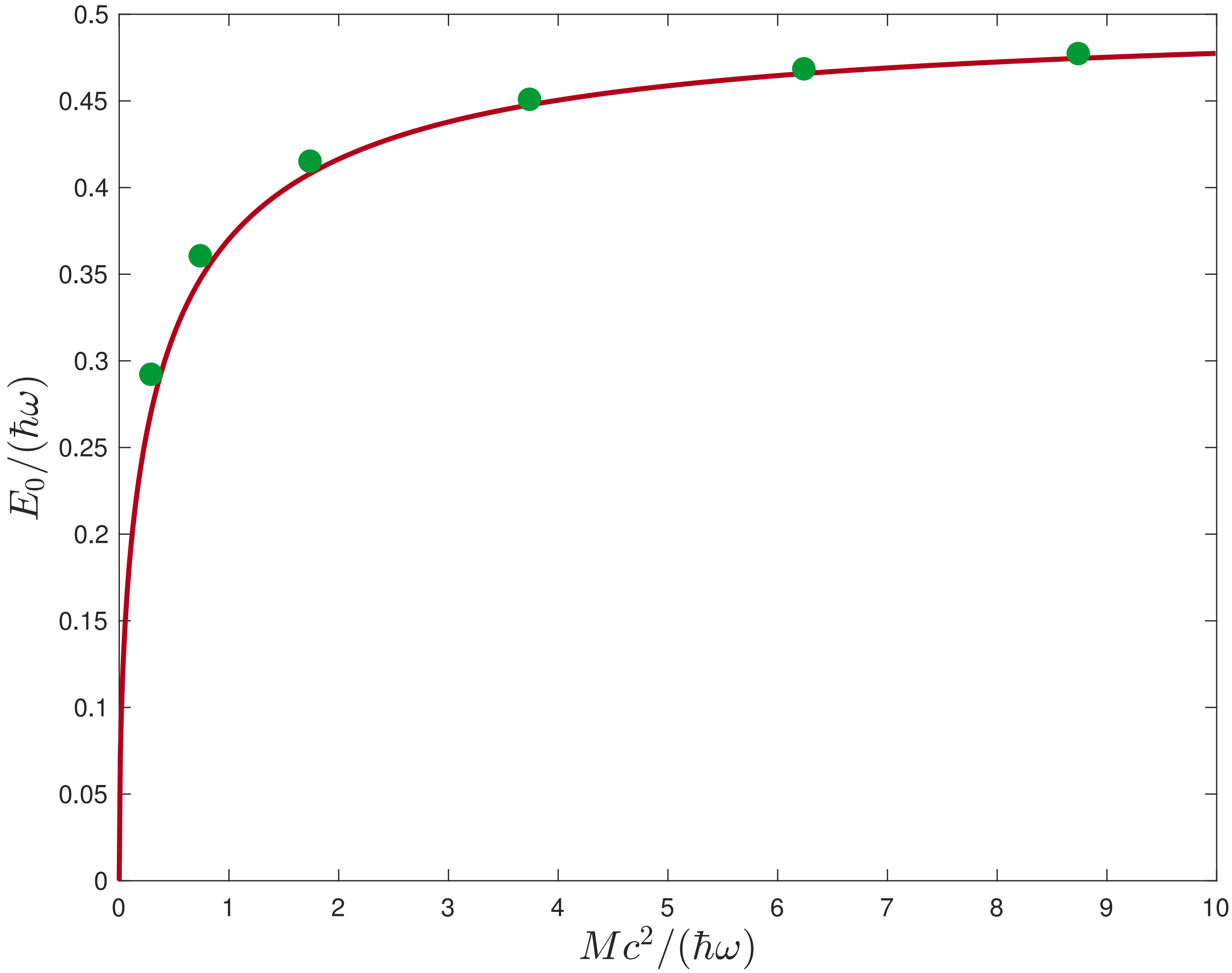


FIG. 2: Our Atlantic City estimates (80–82, green dots) of the ground-state energies of the Born-Infeld oscillator are plotted along with the exact values (red curve) for $0.1 \le Mc^2/(\hbar\omega) \le 10$.

action is awkward if it is not quadratic in the time derivatives of the fields but is simple enough for one to find its hamiltonian. One typically can't integrate over the momenta, and the partition function is a path integral over the fields and their momenta with a complex weight function. An action is very awkward action if the equations for the time derivatives are worse than quartic, and one can't find its hamiltonian.

We have shown how to write the partition function as a euclidian path integral when one doesn't know the hamiltonian. We also have shown how to estimate euclidian path integrals that have weight functions that assume negative or complex values. In this Atlantic City method, one integrates numerically over the momenta when the action is awkward or over

TABLE II: Exact (Matlab) and Atlantic City results (80–82) for the ground-state energies $E_0/(\hbar\omega)$ of the Born-Infeld oscillator (62) for $0.3 \le Mc^2/(\hbar\omega) \le 8.75$.

| $M/(\hbar\omega)$ | $E_0/(\hbar\omega)$ exact | $E_0/(\hbar\omega)$ Atlantic City |
|---|---|---|
| 0.3 | 0.2731 | 0.2915 |
| 0.75 | 0.3482 | 0.3599 |
| 1.75 | 0.4084 | 0.4145 |
| 3.75 | 0.4478 | 0.4502 |
| 6.25 | 0.4658 | 0.4678 |
| 8.75 | 0.4746 | 0.4766 |

auxiliary time derivatives when it is very awkward. One then stores the values of the integrals in a look-up table and does a standard Monte Carlo guided by the look-up table. The numerical integrations are well suited to parallel computation. We demonstrated and tested this Atlantic City method on the Born-Infeld oscillator by treating its action both as awkward and as very awkward.

Theories with graceful actions have infinite energy densities. The Atlantic City method lets us estimate the energy density of theories with awkward or very awkward actions, some of which may have finite energy densities [8]. It may lead to a theory of dark energy.

Inasmuch as the approximation of multiple integrals with integrands that assume negative or complex values is a long-standing problem in applied mathematics, the Atlantic City method may have other uses.


## ACKNOWLEDGMENTS

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**Appendix A: Ratios of complex Monte Carlos are unreliable**

The mean value of an observable $A[\phi]$ at inverse temperature $\beta$ is

$$\begin{aligned}\langle A[\phi]\rangle &= \frac{\mathrm{Tr}\, A[\phi] e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}} \\ &= \int A[\phi] \exp\left[\int \left(i\,\dot\phi_j - \dot\psi_j\right) \frac{\partial L}{\partial \dot\psi_j} + L(\phi,\dot\psi)\, d^4x\right] \left|\det\left(\frac{\partial^2 L}{\partial\dot\psi_k \partial\dot\psi_\ell}\right)\right| D\phi D\dot\psi \\ &\Bigg/ \int \exp\left[\int \left(i\,\dot\phi_j - \dot\psi_j\right) \frac{\partial L}{\partial \dot\psi_j} + L(\phi,\dot\psi)\, d^4x\right] \left|\det\left(\frac{\partial^2 L}{\partial\dot\psi_k \partial\dot\psi_\ell}\right)\right| D\phi D\dot\psi\,.\end{aligned} \tag{A1}$$

The complex action

$$S = \int \left(i\,\dot\phi_j - \dot\psi_j\right) \frac{\partial L}{\partial \dot\psi_j} + L(\phi,\dot\psi)\, d^4x \tag{A2}$$

oscillates and does not give us a probability distribution unless we can integrate over $D\dot\psi$. One can write the mean value (A1) as a ratio of mean values

$$\begin{aligned}\langle A[\phi]\rangle &= \left\langle A[\phi] \exp\left[\int i\,\dot\phi_j \frac{\partial L}{\partial \dot\psi_j}\, d^4x\right]\right\rangle \Bigg/ \left\langle \exp\left[\int i\,\dot\phi_j \frac{\partial L}{\partial \dot\psi_j}\, d^4x\right]\right\rangle \\ &= \int A[\phi] \exp\left[\int i\,\dot\phi_j \frac{\partial L}{\partial \dot\psi_j}\, d^4x\right] P(\phi,\dot\psi) D\phi D\dot\psi \\ &\Bigg/ \int \exp\left[\int i\,\dot\phi_j \frac{\partial L}{\partial \dot\psi_j}\, d^4x\right] P(\phi,\dot\psi)\, D\phi D\dot\psi\end{aligned} \tag{A3}$$

in which the functional $P(\phi,\dot\psi)$ is a normalized probability distribution

$$\begin{aligned}P(\phi,\dot\psi) =\ & \exp\left[\int \left(L(\phi,\dot\psi) - \dot\psi_j \frac{\partial L}{\partial \dot\psi_j}\right) d^4x\right] \left|\det\left(\frac{\partial^2 L}{\partial\dot\psi_k \partial\dot\psi_\ell}\right)\right| \\ &\Bigg/ \int \exp\left[\int \left(L(\phi,\dot\psi) - \dot\psi_j \frac{\partial L}{\partial \dot\psi_j}\right) d^4x\right] \left|\det\left(\frac{\partial^2 L}{\partial\dot\psi_k \partial\dot\psi_\ell}\right)\right|\ D\phi D\dot\psi\,.\end{aligned} \tag{A4}$$

Although in principle one can use the Monte Carlo method [7] to estimate the numerator $N$ and the denominator $D$ of the ratio (A3), both $N$ and $D$ are the mean values of complex oscillating functionals. In many if not most cases of interest, both $N$ and $D$ are smaller than the measurement errors $\delta N$ and $\delta D$ in computations of reasonable lengths. The error in the observable $A[\phi]$ is

$$\delta\langle A[\phi]\rangle = \delta\frac{N}{D} = \frac{\delta N}{D} - \frac{N}{D^2}\delta D = \frac{\delta N}{D} - \langle A[\phi]\rangle \frac{\delta D}{D}, \tag{A5}$$

and in many cases both $N$ and $D$ are zero in the limit in which $\beta \to \infty$.

For instance, suppose we apply the technique (A3–A4) to the computation of the ground-state energy $\langle H\rangle = N/D$ of the harmonic oscillator in which the numerator is

$$N = \int \left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2\right) \exp\left[\int \left(i\dot{q}p - \frac{p^2}{2m} - \frac{1}{2}m\omega^2 q^2\right) dt\right] DpDq \\ \Big/ \int \exp\left[\int \left(-\frac{p^2}{2m} - \frac{1}{2}m\omega^2 q^2\right) dt\right] DpDq \,, \tag{A6}$$

the denominator $D$ is

$$D = \int \exp\left[\int \left(i\dot{q}p - \frac{p^2}{2m} - \frac{1}{2}m\omega^2 q^2\right) dt\right] DpDq \\ \Big/ \int \exp\left[\int \left(-\frac{p^2}{2m} - \frac{1}{2}m\omega^2 q^2\right) dt\right] DpDq \,, \tag{A7}$$

and the measure $DpDq$ is

$$DpDq = \prod_{j=1}^{n} \frac{1}{2\pi} dp_j dq_j. \tag{A8}$$

In the continuum limit ($n \to \infty$, $dt \to 0$, with $\beta = n\,dt$ fixed), the numerator $\mathcal{N}$ of the denominator $D \equiv \mathcal{N}/\mathcal{D}$ of the ratio $N/D$ is the partition function

$$\begin{aligned} \mathcal{N} &= Z(\beta) = \mathrm{Tr}\, e^{-\beta H} \\ &= \int \exp\left[\int \left(i\dot{q}p - \frac{p^2}{2m} - \frac{1}{2}m\omega^2 q^2\right) dt\right] DpDq \\ &= \frac{1}{2\sinh(\beta\omega/2)}, \end{aligned} \tag{A9}$$

and the denominator is

$$\begin{aligned} \mathcal{D} &= \int \exp\left[\int \left(-\frac{p^2}{2m} - \frac{1}{2}m\omega^2 q^2\right) dt\right] DpDq \\ &= \prod_{j=1}^{n} \int \frac{dp_j dq_j}{2\pi} \exp\left[\sum_{j=1}^{n} \left(-\frac{p_j^2}{2m} - \frac{1}{2}m\omega^2 q_j^2\right) \frac{\beta}{n}\right] \\ &= \left(\frac{1}{2\pi}\right)^n \left(\frac{2\pi m n}{\beta}\right)^{n/2} \prod_{j=1}^{n} \int dq_j \exp\left[\sum_{j=1}^{n} \left(-\frac{1}{2}m\omega^2 q_j^2\right) \frac{\beta}{n}\right] \\ &= \left(\frac{1}{2\pi}\right)^n \left(\frac{2\pi m n}{\beta}\right)^{n/2} \left(\frac{2\pi n}{m\omega^2\beta}\right)^{n/2} = \left(\frac{n}{\beta\omega}\right)^n. \end{aligned} \tag{A10}$$

This denominator goes to infinity as $n \to \infty$ and $\beta/n \to 0$ for any $\beta \neq 0$. So the denominator $D$ vanishes

$$D = \frac{1}{2\sinh(\beta\omega/2)\,\mathcal{D}} = 0. \tag{A11}$$

The numerator also vanishes, so the ratio $\langle H \rangle = N/D$ is hard to estimate, being $0/0$.

In general, the double-ratio method (A3–A4) is not a reliable way to estimate the mean value of an observable (A1).

## Appendix B: Positivity

In most problems of interest, the hamiltonian increases monotonically with the momentum when the momentum is positive

$$\frac{\partial H(q,p)}{\partial p} > 0 \quad \text{when} \quad p > 0. \tag{B1}$$

Thus the exponentials $\exp[-a\, H(q_{\ell+1},p)/2]$ and $\exp[-a\, H(q_\ell,p)/2]$ in the integral $A(q_{\ell+1},q_\ell)$ (49) decrease monotonically with $p$ for $p > 0$. Let us divide the domain $[0,\infty]$ of the integration (49) into the successive intervals

$$I_n = \left[ \frac{n\pi}{2|q_{\ell+1} - q_\ell|}, \frac{(n+1)\pi}{2|q_{\ell+1} - q_\ell|} \right] \tag{B2}$$

for $n = 0, 1, \ldots$. Because the exponentials decrease monotonically with $p$, the value $A(q_{\ell+1},q_\ell)$ of the integral (49) over any pair of intervals $I_{4n} \cup I_{4n+1}$ for $n \geq 0$ is positive. Invoking monotonicity (B1) once again, we see that the integral over any quartet of intervals $I_{4n} \cup I_{4n+1} \cup I_{4n+2} \cup I_{4n+3}$ for $n \geq 0$ also is positive. It follows that the value $A(q_{\ell+1},q_\ell)$ of the integral (47) is positive as long as the hamiltonian is a monotonically increasing, even function of the momentum.

For instance, when $H$ represents a harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}, \tag{B3}$$

the integral (47) is obviously positive

$$A(q_{\ell+1},q_\ell) = \sqrt{\frac{m}{a}}\, \exp\left\{ -\, a \left[ \frac{m(q_{\ell+1} - q_\ell)^2}{2a^2} + \frac{m\omega^2(q_{\ell+1}^2 + q_\ell^2)}{4} \right] \right\}. \tag{B4}$$

It is a simple matter to have one's Monte Carlo code report the minimum value of the integral $A(q_{\ell+1},q_\ell)$ (49) and to check that it is positive.

[1] P. Ade and others (Planck Collaboration), Astron. Astrophys. **566**, A54 (2014) (2013).

[2] A. Behtash, G. V. Dunne, T. Schaefer, T. Sulejmanpasic, and M. Unsal, (2015), arXiv:1510.03435 [hep-th]; L. Scorzato, *Proceedings, 33rd International Symposium on Lattice Field Theory (Lattice 2015): Kobe, Japan, July 14-18, 2015*, PoS **LATTICE2015**, 016 (2016), arXiv:1512.08039 [hep-lat]; C. Gattringer and K. Langfeld, Int. J. Mod. Phys. **A31**, 1643007 (2016), arXiv:1603.09517 [hep-lat].

[3] M. Born and L. Infeld, Proc. Roy. Soc. Lond. **A144**, 425 (1934); **A147**, 522 (1934); **A150**, 141 (1935).

[4] M. Ostrogradsky, Mem. Acad. St. Petersbourg VI **4**, 385 (1850); C. M. Bender and P. D. Mannheim, Phys. Rev. Letters **100**, 110402 (2008); J. Phys. A **41**, 304018 (2008).

[5] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950); Proc. Roy. Soc. London, ser. A **246**, 326 (1958); *Lectures on Quantum Mechanics* (Yeshiva University and Dover Publications, 1964).

[6] K. Cahill, *Physical Mathematics* (Cambridge University Press, 2013) Chap. 19.

[7] K. Cahill, *Physical Mathematics* (Cambridge University Press, 2013) Chap. 14.

[8] S. Boettcher and C. Bender, J. Math. Phys. **31(11)**, 2579 (1990); K. Cahill, Phys. Rev. D **87**, 065024 (2013); **88**, 125014 (2013).