

Ground-state energy densities

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Abstract

We try to find some theories that have finite ground-state energy densities.

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I. INTRODUCTION

A. Scalar Born-Infeld Theory

First, we have the scalar Born-Infeld theory with awkward action density

$$L = M^4 \left(1 - \sqrt{1 - M^{-4} (\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2)} \right) \quad (1)$$

and conjugate momentum

$$\pi = \frac{\partial L(\phi, \dot{\phi})}{\partial \dot{\phi}} = \dot{\phi} \left[1 - M^{-4} (\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2) \right]^{-1/2}. \quad (2)$$

Its energy density is

$$\begin{aligned} H(\phi, \pi) &= \pi\dot{\phi} - L(\phi, \dot{\phi}) \\ &= \sqrt{(M^4 + \pi^2)(M^4 + (\nabla\phi)^2 + m^2\phi^2)} - M^4. \end{aligned} \quad (3)$$

It's not clear what the ground-state energy density of this theory is. But the hamiltonian makes sense even in the $M \rightarrow 0$ limit in which $H(\phi, \pi) = \sqrt{\pi^2((\nabla\phi)^2 + m^2\phi^2)}$. The ground-state energy density of the free theory is $1/a^4$ where a is the lattice spacing. If ϕ and π get as big as in the free theory, that is, if $\phi \sim 1/a$ and $\pi \sim 1/a^2$, then H grows like $H \sim \sqrt{\pi^2(\nabla\phi)^2} \sim \sqrt{a^{-4}a^{-4}} = 1/a^4$. The scalar Born-Infeld theory probably may be

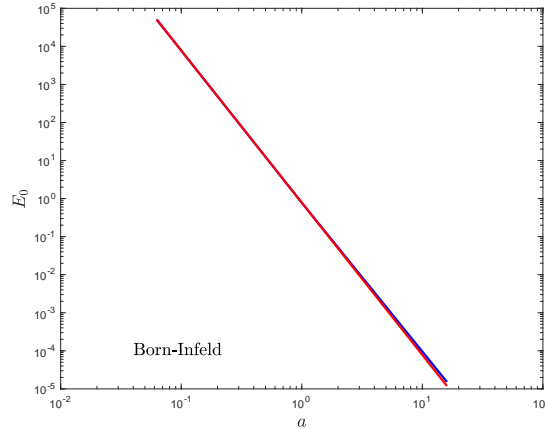


FIG. 1: Blue line is data from simulations of the scalar Born-Infeld theory (1 & 3) on a 20^4 lattice. Red line is $1/x^4$.

less singular. Finding out is one of the purposes of this work. We ran with $m = M = 1$ at

$a_s = a_t = 1/16, 1, 16$ and got the ground-state energies shown in Fig. 1, which diverge as $1/a^4$.

B. Inverse Scalar Born-Infeld Theory

Let's set

$$L = M^4 \left[\left(1 - \frac{2L_0}{M^4} \right)^{-1/2} - 1 \right] \quad (4)$$

in which

$$L_0 = \frac{1}{2} \left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2 \right) \equiv \frac{1}{2} \left(\dot{\phi}^2 - V \right). \quad (5)$$

To find the hamiltonian, we must solve the equation

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi} \left(1 - \frac{2L_0}{M^4} \right)^{-3/2} = \dot{\phi} \left[1 - \left(\dot{\phi}^2 - V \right) / M^4 \right]^{-3/2} \quad (6)$$

for $\dot{\phi}$ in terms of π and V . This is a cubic equation in the variable $\dot{\phi}^2/M^4$

$$\frac{\pi^2}{M^4} \left(1 + \frac{V}{M^4} - \frac{\dot{\phi}^2}{M^4} \right)^3 = \frac{\dot{\phi}^2}{M^4}. \quad (7)$$

Setting $y = \dot{\phi}^2/M^4$, $x = \pi^2/M^4$, and $c = 1 + V/M^4$, we must solve the cubic equation

$$x(c - y)^3 = y. \quad (8)$$

In terms of the quantity

$$r = \left(\frac{27cx^2}{2} + \frac{3x}{2} \sqrt{12x + (9cx)^2} \right)^{1/3}, \quad (9)$$

its only real solution is

$$y = c + \frac{1}{r} - \frac{r}{3x} = 1 + V/M^4 + \frac{1}{r} - \frac{rM^4}{3\pi^2}. \quad (10)$$

As $x \rightarrow 0$, the cube root r tends to

$$r \rightarrow \sqrt{3x} \left(1 + \frac{c\sqrt{3x}}{2} \right) \approx \sqrt{3x} + \frac{3cx}{2}, \quad (11)$$

and so although y vanishes analytically as $x \rightarrow 0$ and $\pi \sim x^2 \rightarrow 0$, that is

$$y \rightarrow c + \frac{1}{\sqrt{3x} + 3cx/2} - \frac{\sqrt{3x} + 3cx/2}{3x} \approx c - c = 0, \quad (12)$$

it is numerically unstable as $x \rightarrow 0$. And analytically $\dot{\phi} \sim \sqrt{y} \rightarrow 0$ as $\pi \rightarrow 0$, as one would expect. Yet our code for H gives NaNs as π drops below 0.0009.

The hamiltonian is

$$\begin{aligned}
H &= \pi \dot{\phi} - L \\
&= \pi \dot{\phi} - M^4 \left[\left(1 - \frac{2L_0}{M^4} \right)^{-1/2} - 1 \right] \\
&= \pi \dot{\phi} - M^4 \left[\left(1 + \frac{(\nabla\phi)^2}{M^4} + \frac{m^2\phi^2}{M^4} - \frac{\dot{\phi}^2}{M^4} \right)^{-1/2} - 1 \right] \\
&= M^4 \left[1 + \frac{\pi\dot{\phi}}{M^4} - \left(1 + \frac{(\nabla\phi)^2}{M^4} + \frac{m^2\phi^2}{M^4} - \frac{\dot{\phi}^2}{M^4} \right)^{-1/2} \right] \\
&= M^4 \left(1 + \sqrt{xy} - \frac{1}{\sqrt{c-y}} \right).
\end{aligned} \tag{13}$$

Thus the energy diverges as $y \rightarrow c$ which is when $2L_0 \rightarrow M^2$, as expected. But also as $x \rightarrow \infty$, $r \sim x^{2/3} \rightarrow \infty$, and $y \rightarrow c$, and so $H \rightarrow \infty$. So H can be unstable numerically as $x \rightarrow \infty$. But we don't get NaNs as $\pi \rightarrow \infty$. All these theories seem to have energy densities that go as $1/a^4$. We ran with $m = M = 1$ at $a_s = a_t = 1/16, 1, 16$ and got the ground-state energies shown in Fig. 2, which diverge as $1/a^4$.

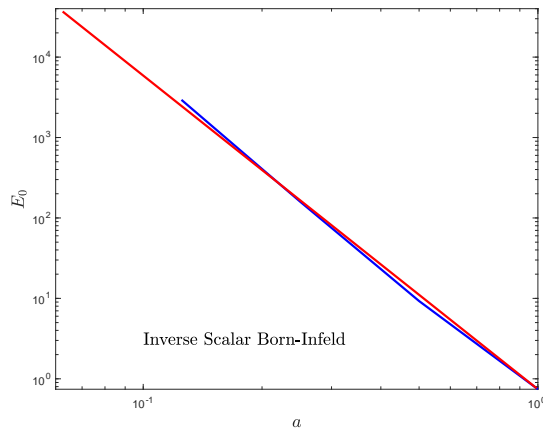


FIG. 2: Blue line is data from simulations of the inverse scalar Born-Infeld theory (4 & 16) on a 20^4 lattice. Red line is $1/x^4$.

C. Inverse Born-Infeld with Cutoff

So let's try

$$L = M^4 \left[\left(1 - \frac{2L_0}{M^4} \right)^{-1/2} - \left(1 - \frac{m^2\phi^2}{M^4} \right)^{-1/2} \right] \quad (14)$$

where now L_0 is just the kinetic term

$$L_0 = \frac{1}{2} \left(\dot{\phi}^2 - (\nabla\phi)^2 \right). \quad (15)$$

With $c = 1 + (\nabla\phi)^2/M^4$ and the real solution (10) of the cubic, the hamiltonian is

$$\begin{aligned} H &= \pi\dot{\phi} - L \\ &= M^4 \left(\pi\sqrt{y}/M^2 - \frac{1}{\sqrt{c-y}} + \frac{1}{\sqrt{1-m^2\phi^2/M^4}} \right). \end{aligned} \quad (16)$$

We ran with $m = M = 1$ at $a_s = a_t = 1/16, 1/8, 1/4, 1/2, 1$ and got the ground-state energies shown in Fig. 3, which diverge as $1/a^{3.9}$.

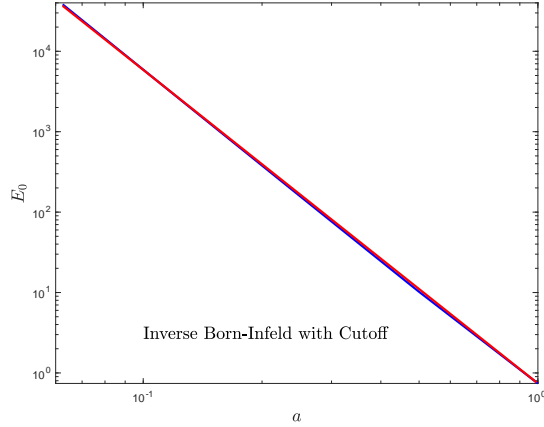


FIG. 3: Blue line is data. Red line is $1/x^{3.9}$.

D. Logarithmic Action

Another theory worth studying has

$$L = -M^4 \ln \left(1 - M^{-4} L_0 \right) \quad (17)$$

in which L_0 is a typical action density

$$L_0 = \frac{1}{2}\dot{\phi}^2 - V \quad (18)$$

and $V = (\nabla\phi)^2/2 + m^2\phi^2/2 + \dots$. One can solve for the time derivative of the field

$$\dot{\phi} = \frac{M^4}{\pi} \left(\sqrt{1 + 2M^{-4}\pi^2 (1 + M^{-4}V)} - 1 \right), \quad (19)$$

and get the hamiltonian density

$$H = M^4 \left(\sqrt{1 + 2M^{-4}\pi^2 (1 + M^{-4}V)} - 1 \right) + M^4 \ln \left[\frac{M^4}{\pi^2} \left(\sqrt{1 + 2M^{-4}\pi^2 (1 + M^{-4}V)} - 1 \right) \right]. \quad (20)$$

When $\pi = 0$, the hamiltonian is $H = M^4 \ln(1 + M^{-4}V)$. These data are plotted in Fig. 4; they diverge slightly more slowly than $1/a^4$.

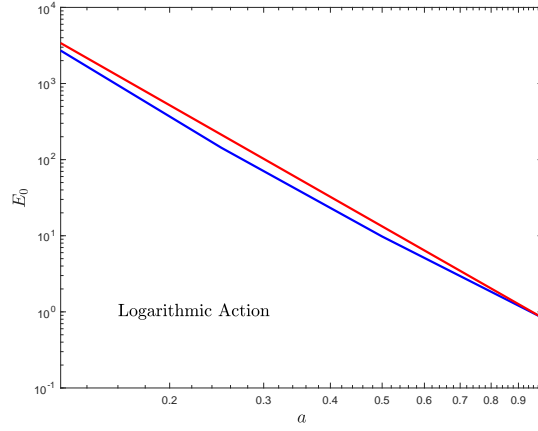


FIG. 4: Blue line is data from simulations of the theory with a logarithmic action (17 & 20) on a 20^4 lattice. Red line is $1/x^4$.

E. Exponential Action

Let's consider the exponential action

$$L = M^4 \left[\exp(L_0/M^4) - 1 \right] \quad (21)$$

in which L_0 is a free action (18) with the usual kinetic terms. The momentum π conjugate to $\dot{\phi}$ is

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi} \exp(L_0/M^4). \quad (22)$$

Its derivative with respect to $\dot{\phi}$ is

$$\frac{\partial \pi}{\partial \dot{\phi}} = \left[1 + \frac{\dot{\phi}^2}{M^4} \exp(L_0/M^4) \right]. \quad (23)$$

The Atlantic City integrals for this very awkward action are

$$\begin{aligned} A(\phi, \nabla \phi, d\phi) = & \int_0^\infty \cos \left[a_s^3 d\phi \dot{\psi} \exp \left(L_0(\phi, \dot{\psi})/M^4 \right) \right] e^{-a_s^3 a_t H(\phi, \dot{\psi})} \\ & \times \left[1 + M^{-4} \dot{\psi}^2 \exp \left(L_0(\phi, \dot{\psi})/M^4 \right) \right] d\dot{\psi} \end{aligned} \quad (24)$$

and

$$\begin{aligned} C(\phi, \nabla \phi, d\phi) = & \int_0^\infty \cos \left[a_s^3 d\phi \dot{\psi} \exp \left(L_0(\phi, \dot{\psi})/M^4 \right) \right] e^{-a_s^3 a_t H(\phi, \dot{\psi})} \\ & \times H(\phi, \dot{\psi}) \left[1 + M^{-4} \dot{\psi}^2 \exp \left(L_0(\phi, \dot{\psi})/M^4 \right) \right] d\dot{\psi} \end{aligned} \quad (25)$$

in which the effective energy density is

$$H(\phi, \dot{\psi}) = \dot{\psi} \pi - L = \left(\dot{\psi}^2 - M^4 \right) \exp \left(L_0(\phi, \dot{\psi})/M^4 \right) + M^4. \quad (26)$$

We wrote Fortran codes and found that the amplitude $A(\phi, \nabla \phi, d\phi)$ increased to a constant value, largely independent of the variables $\phi, \nabla \phi$, and $d\phi$. Thus the field ϕ wanders off to arbitrarily big values as the Monte Carlo runs. No table is big enough.

F. Another Exponential Action

Let's consider the exponential action

$$L = -M^4 \left[\exp(-L_0/M^4) - 1 \right] \quad (27)$$

in which L_0 is a free action (18) with the usual kinetic terms. The momentum π conjugate to $\dot{\phi}$ is

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi} \exp(-L_0/M^4). \quad (28)$$

Its derivative with respect to $\dot{\phi}$ is

$$\frac{\partial \pi}{\partial \dot{\phi}} = \left[1 - M^{-4} \dot{\phi}^2 \exp(-L_0/M^4) \right]. \quad (29)$$

The Atlantic City integrals for this very awkward action are

$$A(\phi, \nabla\phi, d\phi) = \int_0^\infty \cos \left[a_s^3 d\phi \dot{\psi} \exp \left(- L_0(\phi, \dot{\psi})/M^4 \right) \right] e^{-a_s^3 a_t H(\phi, \dot{\psi})} \times \left[1 - M^{-4} \dot{\psi}^2 \exp \left(- L_0(\phi, \dot{\psi})/M^4 \right) \right] d\dot{\psi} \quad (30)$$

and

$$C(\phi, \nabla\phi, d\phi) = \int_0^\infty \cos \left[a_s^3 d\phi \dot{\psi} \exp \left(- L_0(\phi, \dot{\psi})/M^4 \right) \right] e^{-a_s^3 a_t H(\phi, \dot{\psi})} \times H(\phi, \dot{\psi}) \left[1 - M^{-4} \dot{\psi}^2 \exp \left(- L_0(\phi, \dot{\psi})/M^4 \right) \right] d\dot{\psi} \quad (31)$$

in which the effective energy density is

$$H(\phi, \dot{\psi}) = \dot{\psi}\pi - L = \left(\dot{\psi}^2 + M^4 \right) \exp \left(- L_0(\phi, \dot{\psi})/M^4 \right) - M^4. \quad (32)$$

G. A Quartic Hamiltonian

One way to get finite energy densities may be to start with an action density like

$$L = - \frac{M^4}{1 + \frac{1}{2}M^{-4} \left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2 \right)}. \quad (33)$$

Now the momentum is

$$\pi = \frac{\dot{\phi}}{\left[1 + \frac{1}{2}M^{-4} \left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2 \right) \right]^2}, \quad (34)$$

and the equation relating $\dot{\phi}$ to π , ϕ , and $\nabla\phi$ is quartic. Quartic equations have very complicated algebraic solutions.

H. Inverse Born-Infeld as Very Awkward Action

The equation (35) may say that as $\dot{\phi} \rightarrow \infty$, $\pi \sim (\dot{\phi})^{-3}$, and so $H \sim \pi\dot{\phi} - L \sim (\dot{\phi})^{-2}$, which may be finite, or as $H \sim \pi^{2/3}$, which may be infinite.

Now

$$\frac{\partial L}{\partial \dot{\psi}} = \dot{\psi} \left(1 - \frac{2L_0}{M^4} \right)^{-3/2}, \quad (35)$$

and the second derivative is

$$\frac{\partial^2 L}{\partial \dot{\psi}^2} = \left[1 + M^{-4} \left(2\dot{\psi}^2 + V \right) \right] \left(1 - \frac{2L_0}{M^4} \right)^{-5/2} \quad (36)$$

where $V = (\nabla\phi)^2 + m^2\phi^2$.

The time derivative $\dot{\phi}$ is the solution of a cubic equation in $\dot{\phi}^2$, and the hamiltonian is very complicated. It is simpler to use the formula

$$Z(\beta) = \int \exp \left\{ \int_0^\beta \int \left[(i\dot{\phi} - \dot{\psi}) \frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}} + L(\phi, \dot{\psi}) \right] dt d^3x \right\} \left| \frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}^2} \right| D\phi D\dot{\psi}. \quad (37)$$

for the partition function of a very awkward action. For the action density (4), this formula is

$$\begin{aligned} Z(\beta) &= \int \exp \left\{ \int_0^\beta \int \left[(i\dot{\phi} - \dot{\psi}) \dot{\psi} (1 - 2M^{-4}L_0)^{-3/2} + M^4 \left[(1 - 2M^{-4}L_0)^{-1/2} - 1 \right] \right] dt d^3x \right\} \\ &\quad \times \left[1 + M^{-4} (2\dot{\psi}^2 + V) \right] \left(1 - \frac{2L_0}{M^4} \right)^{-5/2} D\phi D\dot{\psi} \\ &= \int \exp \left\{ \int_0^\beta \int \left[(1 - 2M^{-4}L_0)^{-3/2} \left(i\dot{\phi}\dot{\psi} - 2\dot{\psi}^2 + (\nabla\phi)^2 + m^2\phi^2 + M^4 \right) - M^4 \right] dt d^3x \right\} \\ &\quad \times \left[1 + M^{-4} (2\dot{\psi}^2 + V) \right] \left(1 - \frac{2L_0}{M^4} \right)^{-5/2} D\phi D\dot{\psi} \end{aligned} \quad (38)$$

in which $L_0 = L_0(\phi, \dot{\psi})$. The upper limit on the $\dot{\psi}$ integral is $2L_0 = M^4$ or $\dot{\psi} = \sqrt{M^4 + (\nabla\phi)^2 + m^2\phi^2}$. The effective hamiltonian is

$$H = \frac{2\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2 - M^4}{\left[1 - M^{-4} \left(\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2 \right) \right]^{3/2}} + M^4. \quad (39)$$

The Atlantic City integrals are

$$\begin{aligned} A(\phi, \nabla\phi, d\phi) &= \int_0^{\dot{\psi}} \cos \left(a_s^3 d\phi \dot{\psi} (1 - 2M^{-4}L_0)^{-3/2} \right) e^{-a_s^3 a_t H} \\ &\quad \times \left[1 + M^{-4} (2\dot{\psi}^2 + V) \right] \left(1 - \frac{2L_0}{M^4} \right)^{-5/2} d\dot{\psi} \end{aligned} \quad (40)$$

where $d\phi$ is

$$d\phi(i, j, k, \ell) = \phi(i, j, k, \ell + 1) - \phi(i, j, k, \ell), \quad (41)$$

L_0 is a lattice version

$$L_0 = \frac{1}{2} \left(\dot{\psi}^2 - (\delta\phi)^2/a_s^2 - m^2\phi^2 \right) \quad (42)$$

of the action density of the free field, H is a lattice version

$$H = \frac{2\dot{\psi}^2 - (\delta\phi)^2/a_s^2 - m^2\phi^2 - M^4}{\left[1 - M^{-4} \left(\dot{\psi}^2 - (\delta\phi)^2/a_s^2 - m^2\phi^2 \right) \right]^{3/2}} + M^4 \quad (43)$$

of the hamiltonian density (39), $(\delta\phi)^2$ is

$$\begin{aligned} (\delta\phi(i, j, k, \ell))^2 &= (\phi(i+1, j, k, \ell) - \phi(i, j, k, \ell))^2 + (\phi(i, j+1, k, \ell) - \phi(i, j, k, \ell))^2 \\ &\quad + (\phi(i, j, k+1, \ell) - \phi(i, j, k, \ell))^2, \end{aligned} \quad (44)$$

and the upper limit on the integral is

$$\dot{\Psi} = \sqrt{M^4 + (\delta\phi)^2/a_s^2 + m^2\phi^2}. \quad (45)$$

I. An Action that Bounds the Absolute Value of L_0

We could try

$$L = M^4 \left[\frac{1}{2} \left(1 - \frac{2L_0}{M^4} \right)^{-1/2} - \frac{1}{2} \left(1 + \frac{2L_0}{M^4} \right)^{-1/2} + \left(1 - \frac{m^2\phi^2}{M^4} \right)^{-1/2} - 1 \right] \quad (46)$$

where

$$L_0 = \frac{1}{2} \left(\dot{\phi}^2 - (\nabla\phi)^2 \right). \quad (47)$$

But this leads to the equation

$$\pi = \frac{\dot{\phi}}{2} \left[\left(1 + \frac{(\nabla\phi)^2 - \dot{\phi}^2}{M^4} \right)^{-3/2} + \left(1 + \frac{\dot{\phi}^2 - (\nabla\phi)^2}{M^4} \right)^{-3/2} \right]. \quad (48)$$

So this action is very awkward.