Ground-state energy densities

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Abstract

We try to find some theories that have finite ground-state energy densities.

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I. INTRODUCTION

First, we have the scalar Born-Infeld theory with action density

$$L = M^4 \left(1 - \sqrt{1 - M^{-4} \left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2 \right)} \right) \tag{1}$$

and conjugate momentum

$$\pi = \frac{\partial L(\phi, \dot{\phi})}{\partial \dot{\phi}} = \frac{\dot{\phi}}{\sqrt{1 - M^{-4} \left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2 \right)}}. \tag{2}$$

Its energy density is

$$\begin{aligned} H(\phi, \pi) &= \pi\dot{\psi} - L(\phi, \dot{\psi}) \\ &= \sqrt{\left(M^4 + \pi^2\right)\left(M^4 + (\nabla\phi)^2 + m^2\phi^2\right)} - M^4. \end{aligned} \tag{3}$$

It's not clear what the ground-state energy density of this theory is. But the hamiltonian makes sense even in the $M \to 0$ limit

$$H(\phi, \pi) = \sqrt{\pi^2 \left((\nabla\phi)^2 + m^2\phi^2\right)}. \tag{4}$$

The ground-state energy density of the free theory is $1/a^4$ where a is the lattice spacing. If ϕ and π get as big as in the free theory, that is, if $\phi \sim 1/a$ and $\pi \sim 1/a^2$, then H grows like $H \sim \sqrt{\pi^2\,(\nabla\phi)^2} \sim \sqrt{a^{-4}a^{-4}} = 1/a^4$. But the scalar Born-Infeld theory probably is less sigular. Finding out is one of the purposes of this work.

Another theory worth studying has

$$L = -M^4 \ln\left(1 - M^{-4} L_0\right) \tag{5}$$

in which L_0 is a typical action density

$$L_0 = \frac{1}{2}\dot{\phi}^2 - V \tag{6}$$

and $V = (\nabla\phi)^2/2 + m^2\phi^2/2 + \ldots$. One can solve for the time derivative of the field

$$\dot{\phi} = \frac{M^4}{\pi}\left(\sqrt{1 + 2M^{-4}\pi^2\left(1 + M^{-4}V\right)} - 1\right), \tag{7}$$

and get the hamiltonian density

$$\begin{aligned} H = {} & M^4 \left(\sqrt{1 + 2M^{-4}\pi^2\left(1 + M^{-4}V\right)} - 1\right) \\ & + M^4 \ln\left[\frac{M^4}{\pi^2}\left(\sqrt{1 + 2M^{-4}\pi^2\left(1 + M^{-4}V\right)} - 1\right)\right]. \end{aligned} \tag{8}$$

One way to get finite energy densities may be to start with an action density like

$$L = -\frac{M^4}{1 + \frac{1}{2}M^{-4}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)}. \tag{9}$$

Now the momentum is

$$\pi = \frac{\dot{\phi}}{\left[1 + \frac{1}{2}M^{-4}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)\right]^2}, \tag{10}$$

and the equation relating $\dot{\phi}$ to π, ϕ, and $\nabla\phi$ is quartic. Quartic equations have very complicated algebraic solutions. The equation (10) may say that as $\dot{\phi} \to \infty$, $\pi \sim (\dot{\phi})^{-3}$, and so $H \sim \pi\dot{\phi} - L \sim (\dot{\phi})^{-2}$, which may be finite, or as $H \sim \pi^{2/3}$, which may be infinite.