Inverse Scalar Born-Infeld Theory

Ground-State Energy Densities

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1 Solve for the momentum

Start with equation (29):

$$\pi = \dot{\phi}\left[1 - \left(\dot{\phi}^2 - V\right)/M^4\right]^{-3/2} \tag{1}$$

where $M \in \mathbb{R}^+$, $V \in \mathbb{R}$. Solve for $\dot{\phi}\left(\pi, M, V\right)$.

Introducing intermediate variables

$$\begin{aligned} \xi &= 27\pi\left(M^4 + V\right), \\ \eta &= \left(M^{12}\pi^3\left(\sqrt{27\left(4M^{12}+\xi\right)} - \xi\right)\right)^{1/3}, \end{aligned} \tag{2}$$

the solution is taken as

$$\dot{\phi}\left(\pi, M, V\right) = \pm\sqrt{\left(3\times 2^{1/3}\right)^{-1}\frac{\eta}{\pi^{-2}} - M^{12}2^{1/3}\eta^{-1} + M^4 + V} \tag{3}$$

2 Behavior

The solution in (3) is plotted with $M = V = 1$ in figure 1.

2.1 Asymptotics

For unbounded momenta,

$$\lim_{\pi\to\infty}\dot{\phi}\left(\pi, M, V\right) = \sqrt{M^4 + V}. \tag{4}$$

The numerical behavior of the small momentum sequence requires attention. Analytically,

$$\lim_{\pi\to 0}\dot{\phi}\left(\pi, M, V\right) = 0. \tag{5}$$

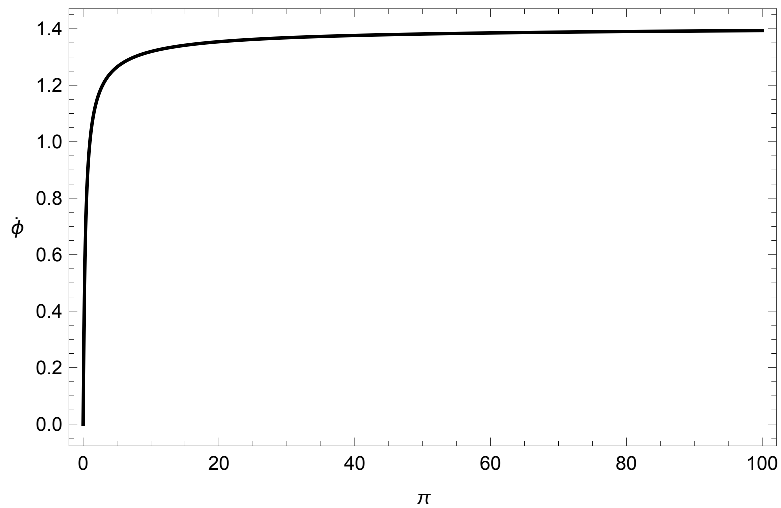

Figure 1: $\dot{\phi}(\pi, M, V)$ with $M = V = 1$

Table 1: Function evaluations, naive double precision.

π	$\dot{\phi}(\pi, M = 1, V = 1)$
10^{-12}	0.0413399
10^{-11}	0.0123526
10^{-10}	0.00378221
10^{-9}	0.00109183
10^{-8}	0.000345267
10^{-7}	0.0000965051
10^{-6}	0.0000285466
0.00001	0.0000295485
0.0001	0.000282851
0.001	0.00282841
0.01	0.0282673
0.1	0.26777
1.	1.
10.	1.31941
100.	1.39359

A table of values was built in Mathematica: The gray shaded values signal a computation problem due to the fact that subtraction is an ill-conditioned process in finite precision arithmetic. As $\pi \to 0$, the quantity

$$\left(3 \times 2^{1/3}\right)^{-1} \frac{\eta}{\pi^{-2}} - M^{12} 2^{1/3} \eta^{-1}$$

becomes the difference of two large numbers. Catastrophic cancellation destroys the precision of the result.

One solution is to use a Maclaurin expansion near the origin. The foundation series that describes η in (2):

$$\eta \approx 2^{1/3}\sqrt{3}M^6 p - \frac{3(M^4+V)}{2^{2/3}}\pi^2 + \frac{3\sqrt{3}\left(M^4+V\right)^2}{4 \times 2^{2/3}M^6}\pi^3 + \frac{3\left(M^4+V\right)^3}{2^{2/3}M^{12}}\pi^4 + \mathcal{O}\left(\pi^4\right).$$

Manipulation of these two terms yields the Laurent expansions

$$\begin{aligned}
\frac{\eta}{3 \times 2^{1/3}\pi^{-2}} &\approx \frac{M^6}{\sqrt{3}p} - \frac{1}{2}\left(M^4+V\right) + \frac{\sqrt{3}}{8M^6}\left(M^4+V\right)^2 p + \frac{\left(M^4+V\right)^3}{2M^{12}}\pi^2 + \mathcal{O}(\pi^3) \\
M^{12}2^{1/3}\eta^{-1} &\approx \frac{M^6}{\sqrt{3}p} + \frac{1}{2}\left(M^4+V\right) + \frac{\sqrt{3}}{8M^6}\left(M^4+V\right)^2 p - \frac{\left(M^4+V\right)^3}{2M^{12}}\pi^2 + \mathcal{O}(\pi^3)
\end{aligned}$$

The nature of the convergence is linear, as given by

$$\lim_{\pi \to 0} \dot{\phi}\left(\pi, M, V\right) \approx \sqrt{\frac{\left(M^4+V\right)^3}{M^{12}}\pi^2 + \mathcal{O}(\pi^3)}. \tag{6}$$

2.2 Invariance

Equation (3) has at least one invariant. When $M^4 + V = 0$, $\dot{\phi}\left(\pi, M, V\right) = 0$.