Ground-state energy densities

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Abstract

We try to find some theories that have finite ground-state energy densities.

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I. INTRODUCTION

A. Scalar Born-Infeld Theory

First, we have the scalar Born-Infeld theory with awkward action density

$$L = M^4 \left(1 - \sqrt{1 - M^{-4}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)}\right) \tag{1}$$

and conjugate momentum

$$\pi = \frac{\partial L(\phi, \dot{\phi})}{\partial \dot{\phi}} = \dot{\phi}\left[1 - M^{-4}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)\right]^{-1/2}. \tag{2}$$

Its energy density is

$$\begin{aligned} H(\phi, \pi) &= \pi\dot{\psi} - L(\phi, \dot{\psi}) \\ &= \sqrt{\left(M^4 + \pi^2\right)\left(M^4 + (\nabla\phi)^2 + m^2\phi^2\right)} - M^4. \end{aligned} \tag{3}$$

It's not clear what the ground-state energy density of this theory is. But the hamiltonian makes sense even in the $M \to 0$ limit in which $H(\phi, \pi) = \sqrt{\pi^2\left((\nabla\phi)^2 + m^2\phi^2\right)}$. The ground-state energy density of the free theory is $1/a^4$ where a is the lattice spacing. If ϕ and π get as big as in the free theory, that is, if $\phi \sim 1/a$ and $\pi \sim 1/a^2$, then H grows like $H \sim \sqrt{\pi^2\,(\nabla\phi)^2} \sim \sqrt{a^{-4}a^{-4}} = 1/a^4$. The scalar Born-Infeld theory probably may be

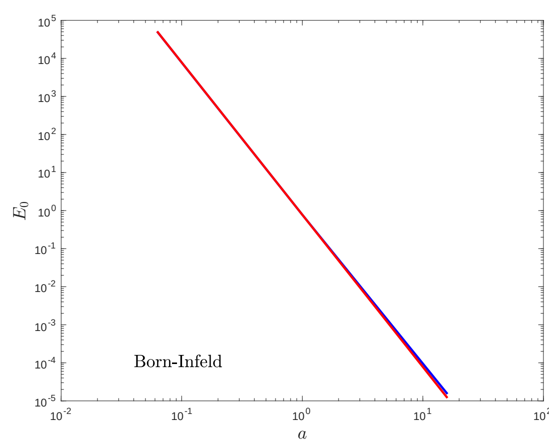

FIG. 1: Blue line is data from simulations of the scalar Born-Infeld theory (1 & 3) on a 20^4 lattice. Red line is $1/x^4$.

less sigular. Finding out is one of the purposes of this work. We ran with $m = M = 1$ at

$a_s = a_t = 1/16$, 1, 16 and got the ground-state energies shown in Fig. 1, which diverge as $1/a^4$.

B. Inverse Scalar Born-Infeld Theory

Let's set

$$L = M^4 \left[\left(1 - \frac{2L_0}{M^4} \right)^{-1/2} - 1 \right] \tag{4}$$

in which

$$L_0 = \frac{1}{2} \left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2 \right) \equiv \frac{1}{2} \left(\dot{\phi}^2 - V \right). \tag{5}$$

To find the hamiltonian, we must solve the equation

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi} \left(1 - \frac{2L_0}{M^4} \right)^{-3/2} = \dot{\phi} \left[1 - \left(\dot{\phi}^2 - V \right) / M^4 \right]^{-3/2} \tag{6}$$

for $\dot{\phi}$ in terms of π and V. This is a cubic equation in the variable $\dot{\phi}^2/M^4$

$$\frac{\pi^2}{M^4} \left(1 + \frac{V}{M^4} - \frac{\dot{\phi}^2}{M^4} \right)^3 = \frac{\dot{\phi}^2}{M^4}. \tag{7}$$

Setting $y = \dot{\phi}^2/M^4$, $x = \pi^2/M^4$, and $c = 1 + V/M^4$, we must solve the cubic equation

$$x(c-y)^3 = y. \tag{8}$$

In terms of the quantity

$$r = \left(\frac{27cx^2}{2} + \frac{3x}{2} \sqrt{12x + (9cx)^2} \right)^{1/3}, \tag{9}$$

its only real solution is

$$y = c + \frac{1}{r} - \frac{r}{3x} = 1 + V/M^4 + \frac{1}{r} - \frac{rM^4}{3\pi^2}. \tag{10}$$

As $x \to 0$, the cube root r tends to

$$r \to \sqrt{3x} \left(1 + \frac{c\sqrt{3x}}{2} \right) \approx \sqrt{3x} + \frac{3cx}{2}, \tag{11}$$

and so although y vanishes analytically as $x \to 0$ and $\pi \sim x^2 \to 0$, that is

$$y \to c + \frac{1}{\sqrt{3x} + 3cx/2} - \frac{\sqrt{3x} + 3cx/2}{3x} \approx c - c = 0, \tag{12}$$

it is numerically unstable as $x \to 0$. And analytically $\dot\phi \sim \sqrt{y} \to 0$ as $\pi \to 0$, as one would expect. Yet our code for H gives NaNs as π drops below 0.0009.

The hamiltonian is

$$\begin{aligned}
H &= \pi\dot\phi - L \\
&= \pi\dot\phi - M^4 \left[\left(1 - \frac{2L_0}{M^4}\right)^{-1/2} - 1 \right] \\
&= \pi\dot\phi - M^4 \left[\left(1 + \frac{(\nabla\phi)^2}{M^4} + \frac{m^2\phi^2}{M^4} - \frac{\dot\phi^2}{M^4}\right)^{-1/2} - 1 \right] \\
&= M^4 \left[1 + \frac{\pi\dot\phi}{M^4} - \left(1 + \frac{(\nabla\phi)^2}{M^4} + \frac{m^2\phi^2}{M^4} - \frac{\dot\phi^2}{M^4}\right)^{-1/2} \right] \\
&= M^4 \left(1 + \sqrt{xy} - \frac{1}{\sqrt{c-y}} \right).
\end{aligned} \tag{13}$$

Thus the energy diverges as $y \to c$ which is when $2L_0 \to M^2$, as expected. But also as $x \to \infty$, $r \sim x^{2/3} \to \infty$, and $y \to c$, and so $H \to \infty$. So H can be unstable numerically as $x \to \infty$. But we don't get NaNs as $\pi \to \infty$. All these theories seem to have energy densities that go as $1/a^4$. We ran with $m = M = 1$ at $a_s = a_t = 1/16$, 1, 16 and got the ground-state energies shown in Fig. 2, which diverge as $1/a^4$.

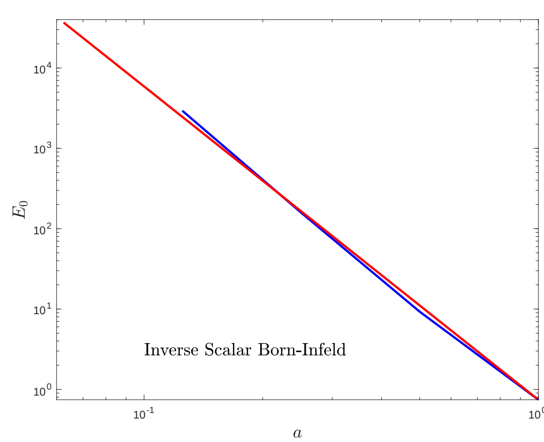

FIG. 2: Blue line is data from simulations of the inverse scalar Born-Infeld theory (4 & 16) on a 20^4 lattice. Red line is $1/x^4$.

C. Inverse Born-Infeld with Cutoff

So let's try

$$L = M^4 \left[\left(1 - \frac{2L_0}{M^4}\right)^{-1/2} - \left(1 - \frac{m^2\phi^2}{M^4}\right)^{-1/2} \right] \tag{14}$$

where now L_0 is just the kinetic term

$$L_0 = \frac{1}{2}\left(\dot{\phi}^2 - (\nabla\phi)^2\right). \tag{15}$$

With $c = 1 + (\nabla\phi)^2/M^4$ and the real solution (10) of the cubic, the hamiltonian is

$$\begin{aligned} H &= \pi\dot{\phi} - L \\ &= M^4 \left(\pi\sqrt{y}/M^2 - \frac{1}{\sqrt{c-y}} + \frac{1}{\sqrt{1 - m^2\phi^2/M^4}} \right). \end{aligned} \tag{16}$$

We ran with $m = M = 1$ at $a_s = a_t = 1/16,\ 1/8,\ 1/4,\ 1/2,\ 1$ and got the ground-state energies shown in Fig. 3, which diverge as $1/a^{3.9}$.

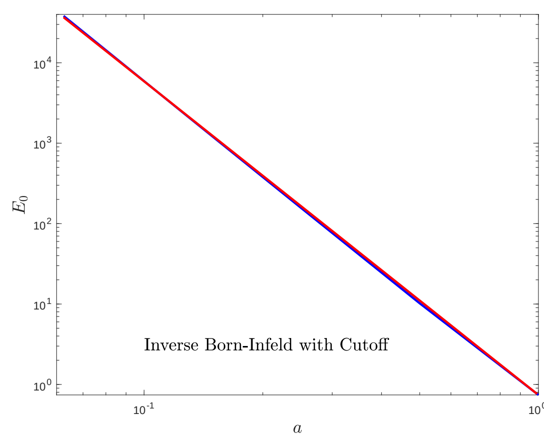

FIG. 3: Blue line is data. Red line is $1/x^{3.9}$.

D. Logarithmic Action

Another theory worth studying has

$$L = -M^4 \ln\left(1 - M^{-4}L_0\right) \tag{17}$$

in which L_0 is a typical action density

$$L_0 = \frac{1}{2}\dot{\phi}^2 - V \tag{18}$$

and $V = (\nabla\phi)^2/2 + m^2\phi^2/2 + \ldots$. One can solve for the time derivative of the field

$$\dot{\phi} = \frac{M^4}{\pi}\left(\sqrt{1 + 2M^{-4}\pi^2\left(1 + M^{-4}V\right)} - 1\right), \tag{19}$$

and get the hamiltonian density

$$\begin{aligned} H = {} & M^4\left(\sqrt{1 + 2M^{-4}\pi^2\left(1 + M^{-4}V\right)} - 1\right) \\ & + M^4 \ln\left[\frac{M^4}{\pi^2}\left(\sqrt{1 + 2M^{-4}\pi^2\left(1 + M^{-4}V\right)} - 1\right)\right]. \end{aligned} \tag{20}$$

When $\pi = 0$, the hamiltonian is $H = M^4 \ln(1 + M^{-4}V)$.

E. Exponential Action

Let's consider the exponential action

$$L = M^4\,\left[\exp\left(L_0/M^4\right) - 1\right] \tag{21}$$

in which L_0 is an free action (18) with the usual kinetic terms. The momentum π conjugate to $\dot{\phi}$ is

$$\pi = \frac{\partial L}{\partial\dot{\phi}} = \dot{\phi}\,\exp\left(L_0/M^4\right). \tag{22}$$

Its derivative with respect to $\dot{\phi}$ is

$$\frac{\partial\pi}{\partial\dot{\phi}} = \left[1 + \dot{\phi}^2\,\exp\left(L_0/M^4\right)\right]. \tag{23}$$

The Atlantic City integrals for this very awkward action are

$$\begin{aligned} A(\phi, \nabla\phi, d\phi) = & \int_0^\infty \cos\left[a_s^3\,d\phi\,\dot{\psi}\exp\left(L_0(\phi,\dot{\psi})/M^4\right)\right] e^{-a_s^3 a_t H(\phi,\dot{\psi})} \\ & \times \left[1 + \dot{\phi}^2\,\exp\left(L_0(\phi,\dot{\psi})/M^4\right)\right] d\dot{\psi} \end{aligned} \tag{24}$$

and

$$\begin{aligned} C(\phi, \nabla\phi, d\phi) = & \int_0^\infty \cos\left[a_s^3\,d\phi\,\dot{\psi}\exp\left(L_0(\phi,\dot{\psi})/M^4\right)\right] e^{-a_s^3 a_t H(\phi,\dot{\psi})} \\ & \times H(\phi,\dot{\psi})\,\left[1 + \dot{\phi}^2\,\exp\left(L_0(\phi,\dot{\psi})/M^4\right)\right] d\dot{\psi} \end{aligned} \tag{25}$$

in which the effective energy density is

$$H(\phi,\dot{\psi}) = \left(\dot{\psi}^2 - M^4\right)\,\exp\left(L_0(\phi,\dot{\psi})/M^4\right) + M^4. \tag{26}$$

F. A Quartic Hamiltonian

One way to get finite energy densities may be to start with an action density like

$$L = -\frac{M^4}{1 + \frac{1}{2}M^{-4}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)}. \tag{27}$$

Now the momentum is

$$\pi = \frac{\dot{\phi}}{\left[1 + \frac{1}{2}M^{-4}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)\right]^2}, \tag{28}$$

and the equation relating $\dot{\phi}$ to π, ϕ, and $\nabla\phi$ is quartic. Quartic equations have very complicated algebraic solutions.

G. Inverse Born-Infeld as Very Awkward Action

The equation (28) may say that as $\dot{\phi} \to \infty$, $\pi \sim (\dot{\phi})^{-3}$, and so $H \sim \pi\dot{\phi} - L \sim (\dot{\phi})^{-2}$, which may be finite, or as $H \sim \pi^{2/3}$, which may be infinite.

Now

$$\frac{\partial L}{\partial \dot{\psi}} = \dot{\psi}\left(1 - \frac{2L_0}{M^4}\right)^{-3/2}, \tag{29}$$

and the second derivative is

$$\frac{\partial^2 L}{\partial \dot{\psi}^2} = \left[1 + M^{-4}\left(2\dot{\psi}^2 + V\right)\right]\left(1 - \frac{2L_0}{M^4}\right)^{-5/2} \tag{30}$$

where $V = (\nabla\phi)^2 + m^2\phi^2$.

The time derivative $\dot{\phi}$ is the solution of a cubic equation in $\dot{\phi}^2$, and the hamiltonian is very complicated. It is simpler to use the formula

$$Z(\beta) = \int \exp\left\{\int_0^\beta \int \left[(i\dot{\phi} - \dot{\psi})\frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}} + L(\phi, \dot{\psi})\right] dt d^3x\right\} \left|\frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}^2}\right| D\phi D\psi. \tag{31}$$

for the partition function of a very awkward action. For the action density (4), this formula is

$$
\begin{aligned}
Z(\beta) = \int \exp \Bigg\{ \int_0^\beta \int \Big[(i\dot{\phi} - \dot{\psi})\dot{\psi}\left(1 - 2M^{-4}L_0\right)^{-3/2} + M^4 \Big[\left(1 - 2M^{-4}L_0\right)^{-1/2} - 1 \Big] \Big] dt d^3x \Bigg\} \\
\times \left[1 + M^{-4}\left(2\dot{\psi}^2 + V\right) \right] \left(1 - \frac{2L_0}{M^4}\right)^{-5/2} D\phi D\dot{\psi} \\
= \int \exp \Bigg\{ \int_0^\beta \int \Big[\left(1 - 2M^{-4}L_0\right)^{-3/2} \left(i\dot{\phi}\dot{\psi} - 2\dot{\psi}^2 + (\nabla\phi)^2 + m^2\phi^2 + M^4 \right) - M^4 \Big] dt d^3x \Bigg\} \\
\times \left[1 + M^{-4}\left(2\dot{\psi}^2 + V\right) \right] \left(1 - \frac{2L_0}{M^4}\right)^{-5/2} D\phi D\dot{\psi}
\end{aligned}
\tag{32}
$$

in which $L_0 = L_0(\phi, \dot{\psi})$. The upper limit on the $\dot{\psi}$ integral is $2L_0 = M^4$ or $\dot{\psi} = \sqrt{M^4 + (\nabla\phi)^2 + m^2\phi^2}$. The effective hamiltonian is

$$
H = \frac{2\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2 - M^4}{\left[1 - M^{-4}\left(\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2 \right) \right]^{3/2}} + M^4. \tag{33}
$$

The Atlantic City integrals are

$$
\begin{aligned}
A(\phi, \nabla\phi, d\phi) = \int_0^{\dot{\Psi}} \cos\left(a_s^3 \, d\phi \, \dot{\psi} \left(1 - 2M^{-4}L_0\right)^{-3/2} \right) e^{-a_s^3 a_t H} \\
\times \left[1 + M^{-4}\left(2\dot{\psi}^2 + V\right) \right] \left(1 - \frac{2L_0}{M^4}\right)^{-5/2} d\dot{\psi}
\end{aligned}
\tag{34}
$$

where $d\phi$ is

$$
d\phi(i,j,k,\ell) = \phi(i,j,k,\ell+1) - \phi(i,j,k,\ell), \tag{35}
$$

L_0 is a lattice version

$$
L_0 = \frac{1}{2}\left(\dot{\psi}^2 - (\delta\phi)^2/a_s^2 - m^2\phi^2 \right) \tag{36}
$$

of the action density of the free field, H is a lattice version

$$
H = \frac{2\dot{\psi}^2 - (\delta\phi)^2/a_s^2 - m^2\phi^2 - M^4}{\left[1 - M^{-4}\left(\dot{\psi}^2 - (\delta\phi)^2/a_s^2 - m^2\phi^2 \right) \right]^{3/2}} + M^4 \tag{37}
$$

of the hamiltonian density (32), $(\delta\phi)^2$ is

$$
\begin{aligned}
(\delta\phi(i,j,k,\ell))^2 = (\phi(i+1,j,k,\ell) - \phi(i,j,k,\ell))^2 + (\phi(i,j+1,k,\ell) - \phi(i,j,k,\ell))^2 \\
+ (\phi(i,j,k+1,\ell) - \phi(i,j,k,\ell))^2,
\end{aligned}
\tag{38}
$$

and the upper limit on the integral is

$$
\dot{\Psi} = \sqrt{M^4 + (\delta\phi)^2/a_s^2 + m^2\phi^2}. \tag{39}
$$

H. An Action that Bounds the Absolute Value of L_0

We could try

$$L = M^4 \left[\frac{1}{2} \left(1 - \frac{2L_0}{M^4} \right)^{-1/2} - \frac{1}{2} \left(1 + \frac{2L_0}{M^4} \right)^{-1/2} + \left(1 - \frac{m^2 \phi^2}{M^4} \right)^{-1/2} - 1 \right] \tag{40}$$

where

$$L_0 = \frac{1}{2} \left(\dot{\phi}^2 - (\nabla \phi)^2 \right) . \tag{41}$$

But this leads to the equation

$$\pi = \frac{\dot{\phi}}{2} \left[\left(1 + \frac{(\nabla \phi)^2 - \dot{\phi}^2}{M^4} \right)^{-3/2} + \left(1 + \frac{\dot{\phi}^2 - (\nabla \phi)^2}{M^4} \right)^{-3/2} \right] . \tag{42}$$

So this action is very awkward.