Ground-state energy densities

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(Dated: January 12, 2017)

Abstract

We try to find some theories that have finite ground-state energy densities.

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I. INTRODUCTION

Amplitudes are matrix elements of $\exp(-itH)$. After inserting zillions of complete sets of eigenstates of the fields and their momenta, one gets

$$\langle\phi(t)|e^{-itH}|\phi(0)\rangle = \int \exp\left[i\int(\dot\phi\pi - H)\, d^4x\right] D\phi D\pi. \tag{1}$$

The quantity inside the parentheses is the action density $L = \dot\phi\pi - H$ when π is written as a function of the field ϕ and its first derivatives $\partial_\mu\phi$. For graceful theories, one has

$$\langle\phi(t)|e^{-itH}|\phi(0)\rangle = \int \exp\left[i\int L\, d^4x\right] D\phi. \tag{2}$$

In theories that respect both of these equations (1 & 2), the exponential $\exp(i\int Ld^4x)$ is a functional Fourier transform of exponential $\exp(-i\int Hd^4x)$ evaluated at $\dot\phi$

$$\exp\left[i\int L\, d^4x\right] = \int \exp\left[i\int(\dot\phi\pi - H)\, d^4x\right] D\pi. \tag{3}$$

The partition function is

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H}) = \int \exp\left[\int(i\dot\phi\pi - H)\, d^4x\right] D\phi D\pi. \tag{4}$$

For graceful theories, one has

$$Z(\beta) = \int \exp\left[-\int L_e\, d^4x\right] D\phi \tag{5}$$

in which L_e is the euclidian action density. In theories that respect both of these equations (4 & 5), the exponential $\exp(-\int L_e d^4x)$ is a functional Fourier transform of exponential $\exp(-\int Hd^4x)$ evaluated at $\dot\phi$

$$\exp\left[-\int L_e\, d^4x\right] = \int \exp\left[\int(i\dot\phi\pi - H)\, d^4x\right] D\pi. \tag{6}$$

A. Free field with cutoff

The action is quadratic in the time derivative of the field

$$L = \frac{1}{2}\left(\dot\phi^2 - (\nabla\phi)^2\right) - m^4\left(\sqrt{1-\frac{\phi^2}{M^2}}\right)^{-1/2} + m^4 \tag{7}$$

and so is graceful. At big M, it describes a free field with particles of mass m^2/M. Its hamiltonian is

$$H = \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2\right) + m^4 \left(\sqrt{1 - \frac{\phi^2}{M^2}}\right)^{-1/2} - m^4. \tag{8}$$

Since the action is graceful, its partition function is

$$\begin{aligned} Z(\beta) = \mathrm{Tr}(e^{-\beta H}) &= \int \exp\left[\int_0^\beta dt \int d^3x \left(i\dot{\phi}\pi - H\right)\right] D\phi D\pi \\ = \mathrm{Tr}(e^{-\beta H}) &= \int \exp\left\{\int_0^\beta dt \int d^3x \left[i\dot{\phi}\pi - \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2\right) - V\right]\right\} D\phi D\pi \end{aligned} \tag{9}$$

where $V = m^4/\sqrt{1 - \phi^2/M^2} - m^4$. On a lattice of spacing a_t in the time direction and a_s in the spatial directions, the partition function is a path integral over all spacetime points $\alpha = (\boldsymbol{x}, t)$

$$Z(\beta) = \prod_\alpha \int d\phi_\alpha d\pi_\alpha \exp\left[a_t a_s^3 \sum_\alpha \left(i\dot{\phi}_\alpha \pi_\alpha - \tfrac{1}{2}\pi_\alpha^2 - \tfrac{1}{2}(\nabla\phi_\alpha)^2 - V_\alpha\right)\right] \tag{10}$$

$$= \left(\frac{2\pi}{a_t a_s^3}\right)^{N_t N_s^3/2} \prod_\alpha \int d\phi_\alpha \exp\left[-a_s^3 \sum_\alpha \left(\frac{(d\phi_\alpha)^2}{2a_t} + \frac{a_t}{2}(\nabla\phi_\alpha)^2 + a_t V_\alpha\right)\right] \tag{11}$$

and the mean value of the energy is

$$\begin{aligned} -\frac{d\log Z(\beta)}{d\beta} &= -\frac{1}{Z(\beta)}\frac{1}{N_t}\frac{dZ(\beta)}{da_t} \\ &= -\frac{1}{Z(\beta)}\frac{1}{N_t}\Bigg\{-\frac{N_t N_s^3}{2a_t} Z(\beta) \\ &\quad + \left(\frac{2\pi}{a_t a_s^3}\right)^{N_t N_s^3/2} \prod_\alpha \int d\phi_\alpha \left[a_s^3 \sum_\alpha \left(\frac{(d\phi_\alpha)^2}{2a_t^2} - \frac{1}{2}(\nabla\phi_\alpha)^2 - V_\alpha\right)\right] \\ &\quad \times \exp\left[-a_s^3 a_t \sum_\alpha \left(\frac{(d\phi_\alpha)^2}{2a_t^2} + \frac{1}{2}(\nabla\phi_\alpha)^2 + V_\alpha\right)\right]\Bigg\}. \end{aligned} \tag{12}$$

In the argument of the exponential, we see the euclidian action

$$S = a_s^3 a_t \sum_\alpha \left(\frac{(d\phi_\alpha)^2}{2a_t^2} + \frac{1}{2}(\nabla\phi_\alpha)^2 + V_\alpha\right) = \int \left[\tfrac{1}{2}(\partial_\mu\phi)^2 + V\right] d^4x. \tag{13}$$

So the energy (12) is

$$-\frac{d\log Z(\beta)}{d\beta} = \frac{N_s^3}{2a_t} + \prod_\alpha \int d\phi_\alpha \left[\frac{a_s^3}{N_t} \sum_\alpha \left(-\tfrac{1}{2}\dot{\phi}_\alpha^2 + \tfrac{1}{2}(\nabla\phi_\alpha)^2 + V_\alpha\right)\right] e^{-S} \Bigg/ \prod_\alpha \int d\phi_\alpha e^{-S}. \tag{14}$$

Dividing by the spatial volume $N_s^3 a_s^3$, we get the energy density

$$\frac{\langle E(\beta)\rangle}{N_s^3 a_s^3} = \frac{1}{2a_s^3 a_t} - \langle L\rangle \tag{15}$$

in which $\langle L\rangle$ is the lattice estimate of the mean value of the action density (7) weighted by the euclidian action (13)

$$-\langle L\rangle = -\frac{\int L\, e^{-S} D\phi}{\int e^{-S} D\phi} = \frac{\int [\frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}\dot{\phi}^2 + V]\, e^{-S} D\phi}{\int e^{-S} D\phi}. \tag{16}$$

The euclidian symmetry of S implies that $\langle -\dot{\phi}^2 + (\nabla\phi)^2\rangle = 2\langle\dot{\phi}^2\rangle$. Thus the energy density of a graceful action diverges quartically, at least as fast as $1/(2a_s^3 a_t)$ unless $\langle V\rangle$ is negative and similarly divergent.

When the action is quadratic in the time derivative $\dot{\phi}$, we don't need to use the Atlantic-City Way to compute the partition function. But we can check that method by comparing its results with the ones given by the standard method (9–16). The A-C integrals are

$$A(\phi, \nabla\phi, d\phi) = \int_0^\infty \cos\left(a_s^3\, d\phi\, \pi\right) e^{-a_s^3 a_t H(\phi,\pi)}\, d\pi \tag{17}$$

and

$$C(\phi, \nabla\phi, d\phi) = \int_0^\infty \cos\left(a_s^3\, d\phi\, \pi\right) H(\phi,\pi)\, e^{-a_s^3 a_t H(\phi,\pi)}\, d\pi. \tag{18}$$

The energy density (15) should rise as $1/a^4$. Data from the A-C way are plotted in blue dashes in Fig. 1. Data from a standard Monte Carlo (red dots) overlap those from the A-C way. Both rise more slowly than $1/a^4$ (solid, black) but faster than $1/a^{3.5}$ (solid, cyan) and much faster than $1/a^2$ (solid, green).

B. Scalar Born-Infeld Theory

First, we have the scalar Born-Infeld theory with awkward action density

$$L = M^4 \left(1 - \sqrt{1 - M^{-4}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)}\right) \tag{19}$$

and conjugate momentum

$$\pi = \frac{\partial L(\phi,\dot{\phi})}{\partial\dot{\phi}} = \dot{\phi}\, \left[1 - M^{-4}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)\right]^{-1/2}. \tag{20}$$

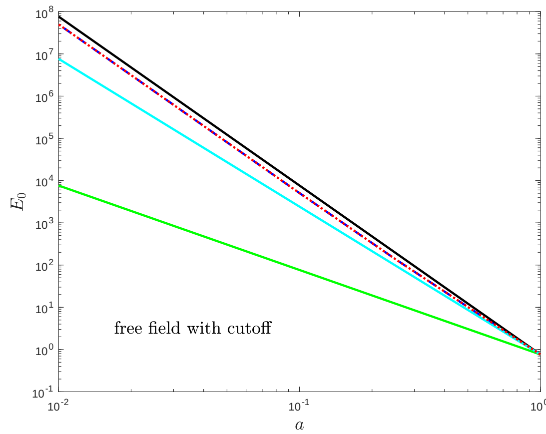

FIG. 1. Blue dashes are data from Atlantic-City Way simulations of the cutoff free scalar field theory (7 & 8) on a 20^4 lattice. The red dots are from a standard Monte Carlo simulation of the same theory; they overlap those from the Atlantic-City Way. The black line is $1/a^4$. The cyan line is $1/a^{3.5}$. The green line is $1/a^2$.

Its energy density is

$$\begin{aligned} H(\phi, \pi) &= \pi \dot{\psi} - L(\phi, \dot{\psi}) \\ &= \sqrt{(M^4 + \pi^2)\left(M^4 + (\nabla\phi)^2 + m^2\phi^2\right)} - M^4. \end{aligned} \tag{21}$$

The Atlantic City integrals for this awkward action are

$$A(\phi, \nabla\phi, d\phi) = \int_0^\infty \cos\left(a_s^3\, d\phi\, \pi\right) e^{-a_s^3 a_t H(\phi,\pi)}\, d\pi \tag{22}$$

and

$$C(\phi, \nabla\phi, d\phi) = \int_0^\infty H(\phi, \pi) \cos\left(a_s^3\, d\phi\, \pi\right) e^{-a_s^3 a_t H(\phi,\pi)}\, d\pi. \tag{23}$$

In the $M \to 0$ limit in which $H(\phi, \pi) = \sqrt{\pi^2 \left((\nabla\phi)^2 + m^2\phi^2\right)}$, one can do these integrals. In terms of $V = (\nabla\phi)^2 + m^2\phi^2$, they are

$$\begin{aligned} A(\phi, \nabla\phi, d\phi) &= \frac{1}{2} \int_0^\infty e^{-a_s^3 \pi (a_t \sqrt{V} + i d\phi)} + e^{-a_s^3 \pi (a_t \sqrt{V} - i d\phi)}\, d\pi \\ &= \frac{a_t}{a_s^3} \frac{\sqrt{V}}{(a_t^2 V + d\phi^2)} \end{aligned} \tag{24}$$

and

$$\begin{aligned}
C(\phi,\nabla\phi,d\phi) &= \frac{1}{2}\int_0^\infty \pi\,\sqrt{V}\left[e^{-a_s^3\pi(a_t\sqrt{V}+id\phi)}+e^{-a_s^3\pi(a_t\sqrt{V}-id\phi)}\right]\,d\pi \\
&= -\frac{1}{a_t}\frac{\partial A}{\partial a_s^3} = -\frac{1}{a_t}\frac{\partial}{\partial a_s^3}\left(\frac{a_t}{a_s^3}\frac{\sqrt{V}}{(a_t^2 V + d\phi^2)}\right) \\
&= \frac{1}{a_s^6}\frac{\sqrt{V}}{(a_t^2 V + d\phi^2)}.
\end{aligned} \tag{25}$$

The energy density therefore is

$$\frac{\langle E(\beta)\rangle}{(a_s N_s)^3} = \left\langle \frac{C(\phi,\nabla\phi,d\phi)}{A(\phi,\nabla\phi,d\phi)}\right\rangle = \frac{1}{a_t\,a_s^3}. \tag{26}$$

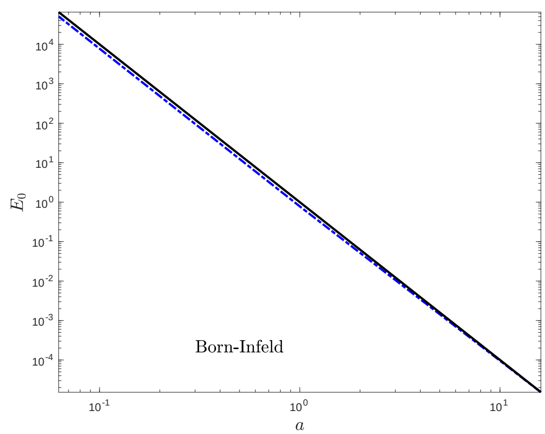

FIG. 2. Data from simulations of the scalar Born-Infeld theory (19 & 21) for $M/m = 1$ (dashdot, blue) and $M/m = 0$ (solid, black) on a 20^4 lattice.

We ran with $M/m = 1$ at $a_s = a_t = 1/16$, 1, 16 and got the ground-state energies shown in Fig. 2 (dashdot, blue). The exact energy density (26) on an infinite lattice is $1/(a_t a_s^3)$.

C. Inverse Scalar Born-Infeld Theory

Let's set

$$L = M^4\left[\left(1-\frac{2L_0}{M^4}\right)^{-1/2} - 1\right] \tag{27}$$

in which

$$L_0 = \frac{1}{2}\left(\dot\phi^2 - (\nabla\phi)^2 - m^2\phi^2\right) \equiv \frac{1}{2}\left(\dot\phi^2 - V\right). \tag{28}$$

To find the hamiltonian, we must solve the equation

$$\pi = \frac{\partial L}{\partial \dot\phi} = \dot\phi \left(1 - \frac{2L_0}{M^4}\right)^{-3/2} = \dot\phi \left[1 - \left(\dot\phi^2 - V\right)/M^4\right]^{-3/2} \tag{29}$$

for $\dot\phi$ in terms of π and V. This is a cubic equation in the variable $\dot\phi^2/M^4$

$$\frac{\pi^2}{M^4}\left(1 + \frac{V}{M^4} - \frac{\dot\phi^2}{M^4}\right)^3 = \frac{\dot\phi^2}{M^4}. \tag{30}$$

Setting $y = \dot\phi^2/M^4$, $x = \pi^2/M^4$, and $c = 1 + V/M^4$, we must solve the cubic equation

$$x(c-y)^3 = y. \tag{31}$$

In terms of the quantity

$$r = \left(\frac{27cx^2}{2} + \frac{3x}{2}\sqrt{12x + (9cx)^2}\right)^{1/3}, \tag{32}$$

its only real solution is

$$y = c + \frac{1}{r} - \frac{r}{3x} = 1 + V/M^4 + \frac{1}{r} - \frac{rM^4}{3\pi^2}. \tag{33}$$

As $x \to 0$, the cube root r tends to

$$r \to \sqrt{3x}\left(1 + \frac{c\sqrt{3x}}{2}\right) \approx \sqrt{3x} + \frac{3cx}{2}, \tag{34}$$

and so although y vanishes analytically as $x \to 0$ and $\pi \sim x^2 \to 0$, that is

$$y \to c + \frac{1}{\sqrt{3x} + 3cx/2} - \frac{\sqrt{3x} + 3cx/2}{3x} \approx c - c = 0, \tag{35}$$

it is numerically unstable as $x \to 0$. And analytically $\dot\phi \sim \sqrt{y} \to 0$ as $\pi \to 0$, as one would expect. Yet our code for H gives NaNs as π drops below 0.0009.

The hamiltonian is

$$\begin{aligned}
H &= \pi\dot\phi - L \\
&= \pi\dot\phi - M^4\left[\left(1 - \frac{2L_0}{M^4}\right)^{-1/2} - 1\right] \\
&= \pi\dot\phi - M^4\left[\left(1 + \frac{(\nabla\phi)^2}{M^4} + \frac{m^2\phi^2}{M^4} - \frac{\dot\phi^2}{M^4}\right)^{-1/2} - 1\right] \\
&= M^4\left[1 + \frac{\pi\dot\phi}{M^4} - \left(1 + \frac{(\nabla\phi)^2}{M^4} + \frac{m^2\phi^2}{M^4} - \frac{\dot\phi^2}{M^4}\right)^{-1/2}\right] \\
&= M^4\left(1 + \sqrt{xy} - \frac{1}{\sqrt{c-y}}\right).
\end{aligned} \tag{36}$$

Thus the energy diverges as $y \to c$ which is when $2L_0 \to M^2$, as expected. But also as $x \to \infty$, $r \sim x^{2/3} \to \infty$, and $y \to c$, and so $H \to \infty$. So H can be unstable numerically as $x \to \infty$. But we don't get NaNs as $\pi \to \infty$. All these theories seem to have energy densities that go as $1/a^4$. We ran with $m = M = 1$ at $a_s = a_t = 1/16$, 1, 16 and got the ground-state energies shown in Fig. 3, which diverge as $1/a^4$.

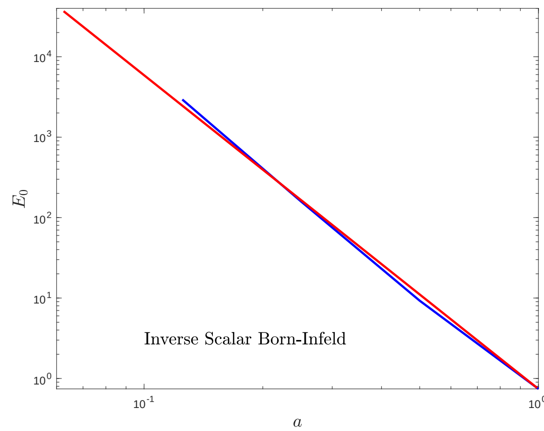

FIG. 3. Blue line is data from simulations of the inverse scalar Born-Infeld theory (27 & 39) on a 20^4 lattice. Red line is $1/x^4$.

D. Inverse Born-Infeld with Cutoff

So let's try

$$L = M^4 \left[\left(1 - \frac{2L_0}{M^4}\right)^{-1/2} - \left(1 - \frac{m^2\phi^2}{M^4}\right)^{-1/2} \right] \tag{37}$$

where now L_0 is just the kinetic term

$$L_0 = \frac{1}{2}\left(\dot{\phi}^2 - (\nabla\phi)^2\right). \tag{38}$$

With $c = 1 + (\nabla\phi)^2/M^4$ and the real solution (33) of the cubic, the hamiltonian is

$$\begin{aligned} H &= \pi\dot{\phi} - L \\ &= M^4 \left(\pi\sqrt{y}/M^2 - \frac{1}{\sqrt{c-y}} + \frac{1}{\sqrt{1 - m^2\phi^2/M^4}} \right). \end{aligned} \tag{39}$$

We ran with $m = M = 1$ at $a_s = a_t = 1/16$, $1/8$, $1/4$, $1/2$, 1 and got the ground-state energies shown in Fig. 4, which diverge as $1/a^{3.9}$.

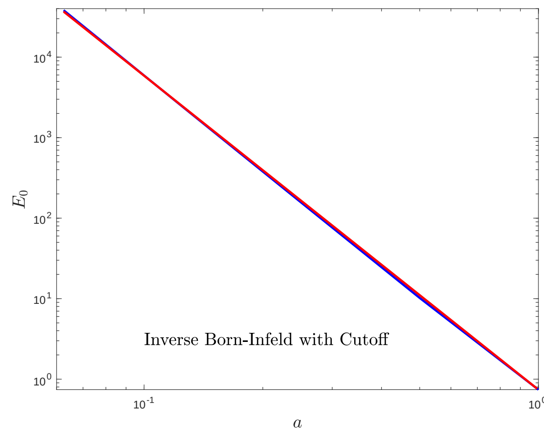

FIG. 4. Blue line is data. Red line is $1/x^{3.9}$.

E. Logarithmic Action

Another theory worth studying has

$$L = -M^4 \ln\left(1 - M^{-4} L_0\right) \tag{40}$$

in which L_0 is a typical action density

$$L_0 = \frac{1}{2}\dot{\phi}^2 - V \tag{41}$$

and $V = (\nabla\phi)^2/2 + m^2\phi^2/2 + \ldots$. One can solve for the time derivative of the field

$$\dot{\phi} = \frac{M^4}{\pi}\left(\sqrt{1 + 2M^{-4}\pi^2\left(1 + M^{-4}V\right)} - 1\right), \tag{42}$$

and get the hamiltonian density

$$\begin{aligned} H = {} & M^4\left(\sqrt{1 + 2M^{-4}\pi^2\left(1 + M^{-4}V\right)} - 1\right) \\ & + M^4 \ln\left[\frac{M^4}{\pi^2}\left(\sqrt{1 + 2M^{-4}\pi^2\left(1 + M^{-4}V\right)} - 1\right)\right]. \end{aligned} \tag{43}$$

When $\pi = 0$, the hamiltonian is $H = M^4 \ln(1 + M^{-4}V)$. These data are plotted in Fig. 5; they diverge slightly more slowly than $1/a^4$.

F. Exponential Action

Let's consider the exponential action

$$L = M^4\left[\exp\left(L_0/M^4\right) - 1\right] \tag{44}$$

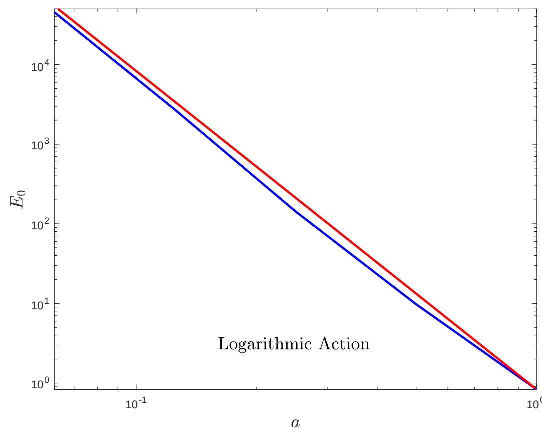

FIG. 5. Blue line is data from simulations of the theory with a logarithmic action (40 & 43) on a 20^4 lattice. Red line is $1/x^4$.

in which L_0 is a free action (41) with the usual kinetic terms. The momentum π conjugate to $\dot{\phi}$ is

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi} \, \exp\left(L_0/M^4\right). \tag{45}$$

Its derivative with respect to $\dot{\phi}$ is

$$\frac{\partial \pi}{\partial \dot{\phi}} = \left[1 + \frac{\dot{\phi}^2}{M^4} \, \exp\left(L_0/M^4\right)\right]. \tag{46}$$

The Atlantic City integrals for this very awkward action are

$$\begin{aligned} A(\phi, \nabla\phi, d\phi) = & \int_0^\infty \cos\left[a_s^3 \, d\phi \, \dot{\psi} \exp\left(L_0(\phi, \dot{\psi})/M^4\right)\right] e^{-a_s^3 a_t H(\phi, \dot{\psi})} \\ & \times \left[1 + M^{-4} \dot{\psi}^2 \, \exp\left(L_0(\phi, \dot{\psi})/M^4\right)\right] d\dot{\psi} \end{aligned} \tag{47}$$

and

$$\begin{aligned} C(\phi, \nabla\phi, d\phi) = & \int_0^\infty \cos\left[a_s^3 \, d\phi \, \dot{\psi} \exp\left(L_0(\phi, \dot{\psi})/M^4\right)\right] e^{-a_s^3 a_t H(\phi, \dot{\psi})} \\ & \times H(\phi, \dot{\psi}) \, \left[1 + M^{-4} \dot{\psi}^2 \, \exp\left(L_0(\phi, \dot{\psi})/M^4\right)\right] d\dot{\psi} \end{aligned} \tag{48}$$

in which the effective energy density is

$$H(\phi, \dot{\psi}) = \dot{\psi}\pi - L = \left(\dot{\psi}^2 - M^4\right) \, \exp\left(L_0(\phi, \dot{\psi})/M^4\right) + M^4. \tag{49}$$

We wrote Fortran codes and found that the amplitude $A(\phi, \nabla\phi, d\phi)$ increased to a constant value, largely independent of the variables $\phi, \nabla\phi$, and $d\phi$. Thus the field ϕ wanders off to arbitrarily big values as the Monte Carlo runs. No table is big enough.

G. Another Exponential Action

Let's consider the exponential action

$$L = \, - \, M^4 \, \left[\exp \left(\, - \, L_0 / M^4 \right) - 1 \right] \tag{50}$$

in which L_0 is a free action (41) with the usual kinetic terms. The momentum π conjugate to $\dot{\phi}$ is

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi} \, \exp \left(\, - \, L_0 / M^4 \right) . \tag{51}$$

Its derivative with respect to $\dot{\phi}$ is

$$\frac{\partial \pi}{\partial \dot{\phi}} = \left[1 - M^{-4} \dot{\phi}^2 \, \exp \left(\, - \, L_0 / M^4 \right) \right] . \tag{52}$$

The Atlantic City integrals for this very awkward action are

$$\begin{aligned} A(\phi, \nabla\phi, d\phi) = & \int_0^\infty \cos \left[a_s^3 \, d\phi \, \dot{\psi} \exp \Big(\, - \, L_0(\phi, \dot{\psi}) / M^4 \Big) \right] e^{-a_s^3 a_t H(\phi, \dot{\psi})} \\ & \times \left[1 - M^{-4} \dot{\psi}^2 \, \exp \Big(\, - \, L_0(\phi, \dot{\psi}) / M^4 \Big) \right] d\dot{\psi} \end{aligned} \tag{53}$$

and

$$\begin{aligned} C(\phi, \nabla\phi, d\phi) = & \int_0^\infty \cos \left[a_s^3 \, d\phi \, \dot{\psi} \exp \Big(\, - \, L_0(\phi, \dot{\psi}) / M^4 \Big) \right] e^{-a_s^3 a_t H(\phi, \dot{\psi})} \\ & \times H(\phi, \dot{\psi}) \, \left[1 - M^{-4} \dot{\psi}^2 \, \exp \Big(\, - \, L_0(\phi, \dot{\psi}) / M^4 \Big) \right] d\dot{\psi} \end{aligned} \tag{54}$$

in which the effective energy density is

$$H(\phi, \dot{\psi}) = \dot{\psi} \pi - L = \Big(\dot{\psi}^2 + M^4 \Big) \, \exp \Big(\, - \, L_0(\phi, \dot{\psi}) / M^4 \Big) - M^4 . \tag{55}$$

The factor $\exp(\, - \, a_s^3 a_t H)$ goes to unity as $\dot{\psi} \to \infty$, so these integrals do not converge.

H. A Quartic Hamiltonian

One way to get finite energy densities may be to start with an action density like

$$L = \, - \, \frac{M^4}{1 + \frac{1}{2} M^{-4} \Big(\dot{\phi}^2 - (\nabla\phi)^2 - m^2 \phi^2 \Big)} . \tag{56}$$

Now the momentum is

$$\pi = \frac{\dot{\phi}}{\left[1 + \frac{1}{2} M^{-4} \Big(\dot{\phi}^2 - (\nabla\phi)^2 - m^2 \phi^2 \Big) \right]^2} , \tag{57}$$

and the equation relating $\dot{\phi}$ to π, ϕ, and $\nabla\phi$ is quartic. Quartic equations have very complicated algebraic solutions.

I. Inverse Born-Infeld as Very Awkward Action

The equation (58) may say that as $\dot{\phi} \to \infty$, $\pi \sim (\dot{\phi})^{-3}$, and so $H \sim \pi\dot{\phi} - L \sim (\dot{\phi})^{-2}$, which may be finite, or as $H \sim \pi^{2/3}$, which may be infinite.

Now

$$\frac{\partial L}{\partial \dot{\psi}} = \dot{\psi} \left(1 - \frac{2L_0}{M^4}\right)^{-3/2}, \tag{58}$$

and the second derivative is

$$\frac{\partial^2 L}{\partial \dot{\psi}^2} = \left[1 + M^{-4}\left(2\dot{\psi}^2 + V\right)\right]\left(1 - \frac{2L_0}{M^4}\right)^{-5/2} \tag{59}$$

where $V = (\nabla\phi)^2 + m^2\phi^2$.

The time derivative $\dot{\phi}$ is the solution of a cubic equation in $\dot{\phi}^2$, and the hamiltonian is very complicated. It is simpler to use the formula

$$Z(\beta) = \int \exp\left\{\int_0^\beta \int \left[(i\dot{\phi} - \dot{\psi})\frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}} + L(\phi, \dot{\psi})\right] dt d^3x\right\} \left|\frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}^2}\right| D\phi D\dot{\psi}. \tag{60}$$

for the partition function of a very awkward action. For the action density (27), this formula is

$$\begin{aligned}
Z(\beta) = \int \exp\Bigg\{\int_0^\beta \int \Big[(i\dot{\phi} - \dot{\psi})\dot{\psi}\left(1 - 2M^{-4}L_0\right)^{-3/2} + M^4\left[\left(1 - 2M^{-4}L_0\right)^{-1/2} - 1\right]\Big] dt d^3x\Bigg\} \\
\times \left[1 + M^{-4}\left(2\dot{\psi}^2 + V\right)\right]\left(1 - \frac{2L_0}{M^4}\right)^{-5/2} D\phi D\dot{\psi} \\
= \int \exp\Bigg\{\int_0^\beta \int \Big[\left(1 - 2M^{-4}L_0\right)^{-3/2}\left(i\dot{\phi}\dot{\psi} - 2\dot{\psi}^2 + (\nabla\phi)^2 + m^2\phi^2 + M^4\right) - M^4\Big] dt d^3x\Bigg\} \\
\times \left[1 + M^{-4}\left(2\dot{\psi}^2 + V\right)\right]\left(1 - \frac{2L_0}{M^4}\right)^{-5/2} D\phi D\dot{\psi}
\end{aligned} \tag{61}$$

in which $L_0 = L_0(\phi, \dot{\psi})$. The upper limit on the $\dot{\psi}$ integral is $2L_0 = M^4$ or $\dot{\psi} = \sqrt{M^4 + (\nabla\phi)^2 + m^2\phi^2}$. The effective hamiltonian is

$$H = \frac{2\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2 - M^4}{\left[1 - M^{-4}\left(\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)\right]^{3/2}} + M^4. \tag{62}$$

The Atlantic City integrals are

$$\begin{aligned}
A(\phi, \nabla\phi, d\phi) = \int_0^{\dot{\Psi}} \cos\left(a_s^3\, d\phi\, \dot{\psi}\left(1 - 2M^{-4}L_0\right)^{-3/2}\right) e^{-a_s^3 a_t H} \\
\times \left[1 + M^{-4}\left(2\dot{\psi}^2 + V\right)\right]\left(1 - \frac{2L_0}{M^4}\right)^{-5/2} d\dot{\psi}
\end{aligned} \tag{63}$$

where $d\phi$ is

$$d\phi(i,j,k,\ell) = \phi(i,j,k,\ell+1) - \phi(i,j,k,\ell), \tag{64}$$

L_0 is a lattice version

$$L_0 = \frac{1}{2}\left(\dot{\psi}^2 - (\delta\phi)^2/a_s^2 - m^2\phi^2\right) \tag{65}$$

of the action density of the free field, H is a lattice version

$$H = \frac{2\dot{\psi}^2 - (\delta\phi)^2/a_s^2 - m^2\phi^2 - M^4}{\left[1 - M^{-4}\left(\dot{\psi}^2 - (\delta\phi)^2/a_s^2 - m^2\phi^2\right)\right]^{3/2}} + M^4 \tag{66}$$

of the hamiltonian density (62), $(\delta\phi)^2$ is

$$\begin{aligned}(\delta\phi(i,j,k,\ell))^2 = {} & (\phi(i+1,j,k,\ell) - \phi(i,j,k,\ell))^2 + (\phi(i,j+1,k,\ell) - \phi(i,j,k,\ell))^2 \\ & + (\phi(i,j,k+1,\ell) - \phi(i,j,k,\ell))^2,\end{aligned} \tag{67}$$

and the upper limit on the integral is

$$\dot{\Psi} = \sqrt{M^4 + (\delta\phi)^2/a_s^2 + m^2\phi^2}. \tag{68}$$

J. An Action that Bounds the Absolute Value of L_0

We could try

$$L = M^4\left[\frac{1}{2}\left(1 - \frac{2L_0}{M^4}\right)^{-1/2} - \frac{1}{2}\left(1 + \frac{2L_0}{M^4}\right)^{-1/2} + \left(1 - \frac{m^2\phi^2}{M^4}\right)^{-1/2} - 1\right] \tag{69}$$

where

$$L_0 = \frac{1}{2}\left(\dot{\phi}^2 - (\nabla\phi)^2\right). \tag{70}$$

But this leads to the equation

$$\pi = \frac{\dot{\phi}}{2}\left[\left(1 + \frac{(\nabla\phi)^2 - \dot{\phi}^2}{M^4}\right)^{-3/2} + \left(1 + \frac{\dot{\phi}^2 - (\nabla\phi)^2}{M^4}\right)^{-3/2}\right]. \tag{71}$$

So this action is very awkward.

In other words: In the limit $M \to 0$, the hamiltonian is

$$H = |\pi|\sqrt{V} \tag{72}$$

in which $V = (\nabla\phi)^2 + m^2\phi^2$. So the partition function (4) is

$$
\begin{aligned}
Z(\beta) = & \int \exp\left[\int_0^\beta dt \int d^3x \left(i\dot{\phi}\pi - H\right)\right] D\phi D\pi \\
= & \int \exp\left[\int_0^\beta dt \int d^3x \left(i\dot{\phi}\pi - |\pi|\sqrt{V}\right)\right] D\phi D\pi \\
= & \prod_\alpha \int \exp\left[\sum_\alpha a_t a_s^3 \left(i\dot{\phi}_\alpha \pi_\alpha - |\pi_\alpha|\sqrt{V_\alpha}\right)\right] d\phi_\alpha d\pi_\alpha \\
= & \prod_\alpha \int d\phi_\alpha \left[\frac{1}{a_t a_s^3(\sqrt{V_\alpha} + i\dot{\phi}_)} + \frac{1}{a_t a_s^3(\sqrt{V_\alpha} - i\dot{\phi}_\alpha)}\right] \\
= & \prod_\alpha \int d\phi_\alpha \left[\frac{2\sqrt{V_\alpha}}{a_t a_s^3(V_\alpha + \dot{\phi}_\alpha^2)}\right] = a_t^{-N^3 s N_t} \prod_\alpha \int d\phi_\alpha \left[\frac{2\sqrt{V_\alpha}}{a_s^3(V_\alpha + \dot{\phi}_\alpha^2)}\right]
\end{aligned}
\tag{73}
$$

The mean value of the energy (12) then is

$$
\begin{aligned}
-\frac{d\log Z(\beta)}{d\beta} = & -\frac{1}{Z(\beta)}\frac{1}{N_t}\frac{dZ(\beta)}{da_t} \\
= & -\frac{1}{Z(\beta)}\frac{1}{N_t}\left\{-\frac{N_t N_s^3}{a_t} Z(\beta) + a_t^{-N^3 s N_t}\frac{d}{da_t}\prod_\alpha \int d\phi_\alpha \left[\frac{2\sqrt{V_\alpha}}{a_s^3(V_\alpha + \dot{\phi}_\alpha^2)}\right]\right\}.
\end{aligned}
\tag{74}
$$

Differentiating, we get

$$
-\frac{d\log Z(\beta)}{d\beta} = \frac{N_s^3}{a_t} - \frac{a_t^{-N_s^3 N_t}}{Z(\beta) N_t}\frac{d}{da_t}\prod_\alpha \int d\phi_\alpha \frac{2\sqrt{V_\alpha}}{a_s^3(V_\alpha + \dot{\phi}_\alpha^2)}
\tag{75}
$$

or

$$
-\frac{d\log Z(\beta)}{d\beta} = \frac{N_s^3}{a_t} - \frac{1}{a_t N_t}\sum_\alpha \int d\phi_\alpha \frac{2\dot{\phi}_\alpha^2\sqrt{V_\alpha}}{(V_\alpha + \dot{\phi}_\alpha^2)^2} \Bigg/ \int d\phi_\alpha \frac{\sqrt{V_\alpha}}{(V_\alpha + \dot{\phi}_\alpha^2)}\ .
\tag{76}
$$

So dividing by the volume $(a_s N_s)^3$, we get

$$
\frac{\langle E(\beta)\rangle}{N_s^3 a_s^3} = \frac{1}{a_s^3 a_t}\left[1 - \frac{1}{N_t N_s^3}\sum_\alpha \int d\phi_\alpha \frac{2\dot{\phi}_\alpha^2\sqrt{V_\alpha}}{(V_\alpha + \dot{\phi}_\alpha^2)^2} \Bigg/ \int d\phi_\alpha \frac{\sqrt{V_\alpha}}{(V_\alpha + \dot{\phi}_\alpha^2)}\right].
\tag{77}
$$