

Ground-state energy densities

David Amdahl,^{*} Kevin Cahill,[†] and Daniel Topa[‡]

Department of Physics & Astronomy, University of New Mexico,

Albuquerque, New Mexico 87131, USA and

School of Computational Sciences, Korea Institute for Advanced Study, Seoul 130-722, Korea

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Abstract

We try to find some theories that have finite ground-state energy densities.

^{*} damdahl@unm.edu

[†] cahill@unm.edu

[‡] dantopa@gmail.com

I. INTRODUCTION

Amplitudes are matrix elements of $\exp(-itH)$. After inserting zillions of complete sets of eigenstates of the fields and their momenta, one gets

$$\langle \phi(t) | e^{-itH} | \phi(0) \rangle = \int \exp \left[i \int (\dot{\phi}\pi - H) d^4x \right] D\phi D\pi. \quad (1)$$

The quantity inside the parentheses is the action density $L = \dot{\phi}\pi - H$. So one always has something like

$$\langle \phi(t) | e^{-itH} | \phi(0) \rangle = \int \exp \left[i \int L d^4x \right] D\phi. \quad (2)$$

A. Free field with cutoff

The action is quadratic in the time derivative of the field

$$L = \frac{1}{2} \left(\dot{\phi}^2 - (\nabla\phi)^2 \right) - m^4 \left(\sqrt{1 - \frac{\phi^2}{M^2}} \right)^{-1/2} + m^4 \quad (3)$$

and so is graceful. At big M , it describes a free field with particles of mass m^2/M . Its hamiltonian is

$$H = \frac{1}{2} (\pi^2 + (\nabla\phi)^2) + m^4 \left(\sqrt{1 - \frac{\phi^2}{M^2}} \right)^{-1/2} - m^4. \quad (4)$$

Since the action is graceful, its partition function is

$$\begin{aligned} Z(\beta) &= \text{Tr}(e^{-\beta H}) = \int \exp \left[\int_0^\beta dt \int d^3x \left(i\dot{\phi}\pi - H \right) \right] D\phi D\pi \\ &= \text{Tr}(e^{-\beta H}) = \int \exp \left\{ \int_0^\beta dt \int d^3x \left[i\dot{\phi}\pi - \frac{1}{2} (\pi^2 + (\nabla\phi)^2 - V) \right] \right\} D\phi D\pi \end{aligned} \quad (5)$$

where $V = m^4/\sqrt{1 - \phi^2/M^2} - m^4$. On a lattice of spacing a_t in the time direction and a_s in the spatial directions, the partition function is

$$Z(\beta) = \prod_\alpha \int d\phi_\alpha d\pi_\alpha \exp \left[a_t a_s^3 \sum_\alpha \left(i\dot{\phi}_\alpha \pi_\alpha - \frac{1}{2} \pi_\alpha^2 - \frac{1}{2} (\nabla\phi_\alpha)^2 - V_\alpha \right) \right] \quad (6)$$

$$= \left(\frac{2\pi}{a_t a_s^3} \right)^{N_t N_s^3/2} \prod_\alpha \int d\phi_\alpha \exp \left[-\frac{a_s^3}{2} \sum_\alpha \left(\frac{(d\phi_\alpha)^2}{a_t} + a_t (\nabla\phi_\alpha)^2 + a_t V_\alpha \right) \right] \quad (7)$$

and the mean value of the energy is

$$\begin{aligned}
-\frac{d \log Z(\beta)}{d\beta} &= -\frac{1}{Z(\beta)} \frac{1}{N_t} \frac{dZ(\beta)}{da_t} \\
&= -\frac{1}{Z(\beta)} \frac{1}{N_t} \left\{ -\frac{N_t N_s^3 Z(\beta)}{2a_t} \right. \\
&\quad \left. + \left(\frac{2\pi}{a_t a_s^3} \right)^{N_t N_s^3/2} \prod_{\alpha} \int d\phi_{\alpha} \left[\frac{a_s^3}{2} \sum_{\alpha} \left(\frac{(d\phi_{\alpha})^2}{a_t^2} - (\nabla \phi_{\alpha})^2 - V_{\alpha} \right) \right] \right. \\
&\quad \left. \times \exp \left[-\frac{a_s^3 a_t}{2} \sum_{\alpha} \left(\frac{(d\phi_{\alpha})^2}{a_t^2} + (\nabla \phi_{\alpha})^2 + V_{\alpha} \right) \right] \right\}. \tag{8}
\end{aligned}$$

In the argument of the exponential, we see the euclidian action

$$S = \frac{a_s^3 a_t}{2} \sum_{\alpha} \left(\frac{(d\phi_{\alpha})^2}{a_t^2} + (\nabla \phi_{\alpha})^2 + V_{\alpha} \right) = \int [(\partial_{\mu} \phi)^2 + V] d^4 x. \tag{9}$$

So the energy (8) is

$$-\frac{d \log Z(\beta)}{d\beta} = \frac{N_s^3}{2a_t} + \prod_{\alpha} \int d\phi_{\alpha} \left[\frac{a_s^3}{2N_t} \sum_{\alpha} \left(-\dot{\phi}_{\alpha}^2 + (\nabla \phi_{\alpha})^2 + V_{\alpha} \right) \right] e^{-S} \Big/ \prod_{\alpha} \int d\phi_{\alpha} e^{-S}. \tag{10}$$

Dividing by the spatial volume $N_s^3 a_s^3$, we get the energy density

$$\frac{\langle E(\beta) \rangle}{N_s^3 a_s^3} = \frac{1}{2a_s^3 a_t} - \langle L \rangle \tag{11}$$

in which $\langle L \rangle$ is the lattice estimate of the mean value of the action density (3) weighted by the euclidian action (9)

$$\langle L \rangle = \frac{\int L e^{-S} D\phi}{\int e^{-S} D\phi}. \tag{12}$$

Thus the energy density of a graceful action diverges quartically, as $1/(2a_s^3 a_t)$ unless $\langle L \rangle$ also diverges. The euclidian symmetry of S implies that $\langle -\dot{\phi}^2 + (\nabla \phi)^2 \rangle = 2\langle \dot{\phi}^2 \rangle$.

When the action is quadratic in the time derivative $\dot{\phi}$, we don't need to use the Atlantic-City Way to compute the partition function. But we can check that method by comparing its results with the ones given by the standard method (5–12). The A-C integrals are

$$A(\phi, \nabla \phi, d\phi) = \int_0^{\infty} \cos [a_s^3 d\phi \pi] e^{-a_s^3 a_t H(\phi, \pi)} d\pi \tag{13}$$

and

$$C(\phi, \nabla \phi, d\phi) = \int_0^{\infty} \cos [a_s^3 d\phi \pi] H(\phi, \pi) e^{-a_s^3 a_t H(\phi, \pi)} d\pi \tag{14}$$

The energy density should rise as $1/a^2$. Does it? Data from the A-C way are plotted in blue in Fig. 1. They rise more slowly than $1/a^4$ (red) but much faster than $1/a^2$ (green).

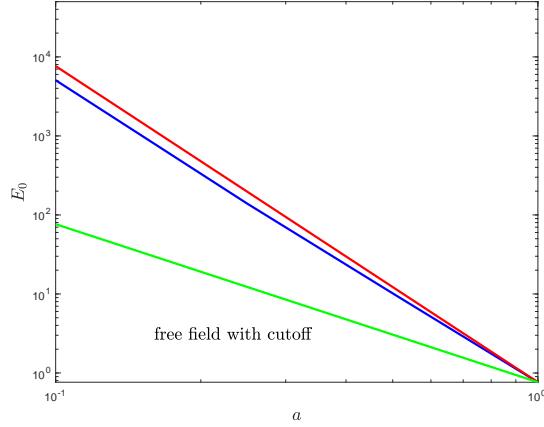


FIG. 1: Blue line is data from simulations of the cutoff free scalar field theory (3 & 4) on a 20^4 lattice.

Red line is $1/x^4$. Green line is $1/x^2$.

B. Scalar Born-Infeld Theory

First, we have the scalar Born-Infeld theory with awkward action density

$$L = M^4 \left(1 - \sqrt{1 - M^{-4} (\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2)} \right) \quad (15)$$

and conjugate momentum

$$\pi = \frac{\partial L(\phi, \dot{\phi})}{\partial \dot{\phi}} = \dot{\phi} \left[1 - M^{-4} (\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2) \right]^{-1/2}. \quad (16)$$

Its energy density is

$$\begin{aligned} H(\phi, \pi) &= \pi\dot{\phi} - L(\phi, \dot{\phi}) \\ &= \sqrt{(M^4 + \pi^2)(M^4 + (\nabla\phi)^2 + m^2\phi^2)} - M^4. \end{aligned} \quad (17)$$

It's not clear what the ground-state energy density of this theory is. But the hamiltonian makes sense even in the $M \rightarrow 0$ limit in which $H(\phi, \pi) = \sqrt{\pi^2((\nabla\phi)^2 + m^2\phi^2)}$. The ground-state energy density of the free theory is $1/a^4$ where a is the lattice spacing. If ϕ and π get as big as in the free theory, that is, if $\phi \sim 1/a$ and $\pi \sim 1/a^2$, then H grows like $H \sim \sqrt{\pi^2(\nabla\phi)^2} \sim \sqrt{a^{-4}a^{-4}} = 1/a^4$. The scalar Born-Infeld theory probably may be less singular. Finding out is one of the purposes of this work. We ran with $m = M = 1$ at $a_s = a_t = 1/16, 1, 16$ and got the ground-state energies shown in Fig. 2, which diverge as $1/a^4$.

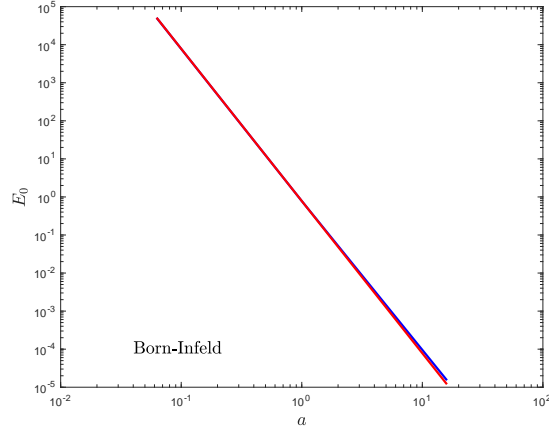


FIG. 2: Blue line is data from simulations of the scalar Born-Infeld theory (15 & 17) on a 20^4 lattice.

Red line is $1/x^4$.

C. Inverse Scalar Born-Infeld Theory

Let's set

$$L = M^4 \left[\left(1 - \frac{2L_0}{M^4} \right)^{-1/2} - 1 \right] \quad (18)$$

in which

$$L_0 = \frac{1}{2} \left(\dot{\phi}^2 - (\nabla \phi)^2 - m^2 \phi^2 \right) \equiv \frac{1}{2} \left(\dot{\phi}^2 - V \right). \quad (19)$$

To find the hamiltonian, we must solve the equation

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi} \left(1 - \frac{2L_0}{M^4} \right)^{-3/2} = \dot{\phi} \left[1 - \left(\dot{\phi}^2 - V \right) / M^4 \right]^{-3/2} \quad (20)$$

for $\dot{\phi}$ in terms of π and V . This is a cubic equation in the variable $\dot{\phi}^2/M^4$

$$\frac{\pi^2}{M^4} \left(1 + \frac{V}{M^4} - \frac{\dot{\phi}^2}{M^4} \right)^3 = \frac{\dot{\phi}^2}{M^4}. \quad (21)$$

Setting $y = \dot{\phi}^2/M^4$, $x = \pi^2/M^4$, and $c = 1 + V/M^4$, we must solve the cubic equation

$$x(c - y)^3 = y. \quad (22)$$

In terms of the quantity

$$r = \left(\frac{27cx^2}{2} + \frac{3x}{2} \sqrt{12x + (9cx)^2} \right)^{1/3}, \quad (23)$$

its only real solution is

$$y = c + \frac{1}{r} - \frac{r}{3x} = 1 + V/M^4 + \frac{1}{r} - \frac{rM^4}{3\pi^2}. \quad (24)$$

As $x \rightarrow 0$, the cube root r tends to

$$r \rightarrow \sqrt{3x} \left(1 + \frac{c\sqrt{3x}}{2} \right) \approx \sqrt{3x} + \frac{3cx}{2}, \quad (25)$$

and so although y vanishes analytically as $x \rightarrow 0$ and $\pi \sim x^2 \rightarrow 0$, that is

$$y \rightarrow c + \frac{1}{\sqrt{3x} + 3cx/2} - \frac{\sqrt{3x} + 3cx/2}{3x} \approx c - c = 0, \quad (26)$$

it is numerically unstable as $x \rightarrow 0$. And analytically $\dot{\phi} \sim \sqrt{y} \rightarrow 0$ as $\pi \rightarrow 0$, as one would expect. Yet our code for H gives NaNs as π drops below 0.0009.

The hamiltonian is

$$\begin{aligned} H &= \pi\dot{\phi} - L \\ &= \pi\dot{\phi} - M^4 \left[\left(1 - \frac{2L_0}{M^4} \right)^{-1/2} - 1 \right] \\ &= \pi\dot{\phi} - M^4 \left[\left(1 + \frac{(\nabla\phi)^2}{M^4} + \frac{m^2\phi^2}{M^4} - \frac{\dot{\phi}^2}{M^4} \right)^{-1/2} - 1 \right] \\ &= M^4 \left[1 + \frac{\pi\dot{\phi}}{M^4} - \left(1 + \frac{(\nabla\phi)^2}{M^4} + \frac{m^2\phi^2}{M^4} - \frac{\dot{\phi}^2}{M^4} \right)^{-1/2} \right] \\ &= M^4 \left(1 + \sqrt{xy} - \frac{1}{\sqrt{c-y}} \right). \end{aligned} \quad (27)$$

Thus the energy diverges as $y \rightarrow c$ which is when $2L_0 \rightarrow M^2$, as expected. But also as $x \rightarrow \infty$, $r \sim x^{2/3} \rightarrow \infty$, and $y \rightarrow c$, and so $H \rightarrow \infty$. So H can be unstable numerically as $x \rightarrow \infty$. But we don't get NaNs as $\pi \rightarrow \infty$. All these theories seem to have energy densities that go as $1/a^4$. We ran with $m = M = 1$ at $a_s = a_t = 1/16, 1, 16$ and got the ground-state energies shown in Fig. 3, which diverge as $1/a^4$.

D. Inverse Born-Infeld with Cutoff

So let's try

$$L = M^4 \left[\left(1 - \frac{2L_0}{M^4} \right)^{-1/2} - \left(1 - \frac{m^2\phi^2}{M^4} \right)^{-1/2} \right] \quad (28)$$

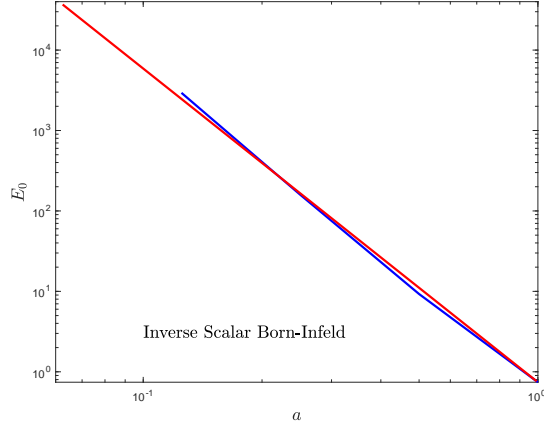


FIG. 3: Blue line is data from simulations of the inverse scalar Born-Infeld theory (18 & 30) on a 20^4 lattice. Red line is $1/x^4$.

where now L_0 is just the kinetic term

$$L_0 = \frac{1}{2} \left(\dot{\phi}^2 - (\nabla\phi)^2 \right). \quad (29)$$

With $c = 1 + (\nabla\phi)^2/M^4$ and the real solution (24) of the cubic, the hamiltonian is

$$\begin{aligned} H &= \pi\dot{\phi} - L \\ &= M^4 \left(\pi\sqrt{y}/M^2 - \frac{1}{\sqrt{c-y}} + \frac{1}{\sqrt{1-m^2\phi^2/M^4}} \right). \end{aligned} \quad (30)$$

We ran with $m = M = 1$ at $a_s = a_t = 1/16, 1/8, 1/4, 1/2, 1$ and got the ground-state energies shown in Fig. 4, which diverge as $1/a^{3.9}$.

E. Logarithmic Action

Another theory worth studying has

$$L = -M^4 \ln(1 - M^{-4}L_0) \quad (31)$$

in which L_0 is a typical action density

$$L_0 = \frac{1}{2} \dot{\phi}^2 - V \quad (32)$$

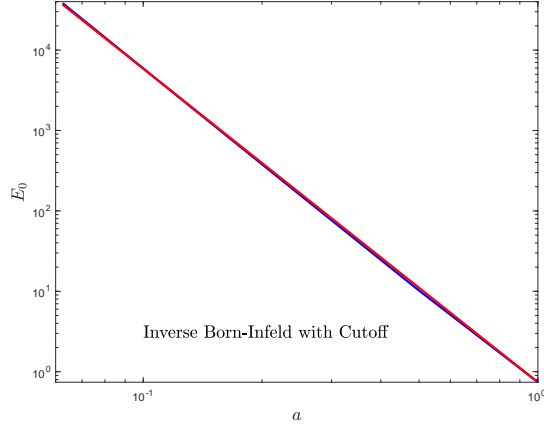


FIG. 4: Blue line is data. Red line is $1/x^{3.9}$.

and $V = (\nabla\phi)^2/2 + m^2\phi^2/2 + \dots$. One can solve for the time derivative of the field

$$\dot{\phi} = \frac{M^4}{\pi} \left(\sqrt{1 + 2M^{-4}\pi^2 (1 + M^{-4}V)} - 1 \right), \quad (33)$$

and get the hamiltonian density

$$H = M^4 \left(\sqrt{1 + 2M^{-4}\pi^2 (1 + M^{-4}V)} - 1 \right) + M^4 \ln \left[\frac{M^4}{\pi^2} \left(\sqrt{1 + 2M^{-4}\pi^2 (1 + M^{-4}V)} - 1 \right) \right]. \quad (34)$$

When $\pi = 0$, the hamiltonian is $H = M^4 \ln(1 + M^{-4}V)$. These data are plotted in Fig. 5; they diverge slightly more slowly than $1/a^4$.

F. Exponential Action

Let's consider the exponential action

$$L = M^4 \left[\exp(L_0/M^4) - 1 \right] \quad (35)$$

in which L_0 is a free action (32) with the usual kinetic terms. The momentum π conjugate to $\dot{\phi}$ is

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi} \exp(L_0/M^4). \quad (36)$$

Its derivative with respect to $\dot{\phi}$ is

$$\frac{\partial \pi}{\partial \dot{\phi}} = \left[1 + \frac{\dot{\phi}^2}{M^4} \exp(L_0/M^4) \right]. \quad (37)$$

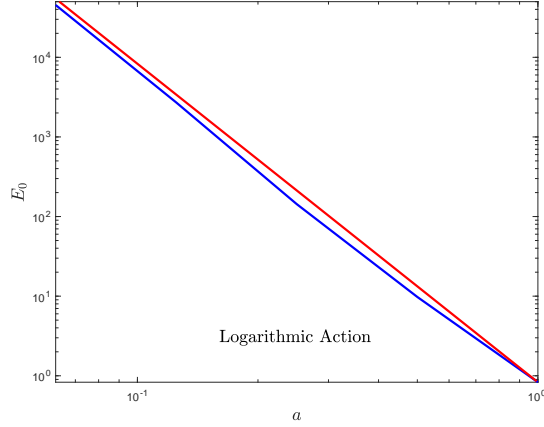


FIG. 5: Blue line is data from simulations of the theory with a logarithmic action (31 & 34) on a 20^4 lattice. Red line is $1/x^4$.

The Atlantic City integrals for this very awkward action are

$$A(\phi, \nabla\phi, d\phi) = \int_0^\infty \cos \left[a_s^3 d\phi \dot{\psi} \exp \left(L_0(\phi, \dot{\psi})/M^4 \right) \right] e^{-a_s^3 a_t H(\phi, \dot{\psi})} \times \left[1 + M^{-4} \dot{\psi}^2 \exp \left(L_0(\phi, \dot{\psi})/M^4 \right) \right] d\dot{\psi} \quad (38)$$

and

$$C(\phi, \nabla\phi, d\phi) = \int_0^\infty \cos \left[a_s^3 d\phi \dot{\psi} \exp \left(L_0(\phi, \dot{\psi})/M^4 \right) \right] e^{-a_s^3 a_t H(\phi, \dot{\psi})} \times H(\phi, \dot{\psi}) \left[1 + M^{-4} \dot{\psi}^2 \exp \left(L_0(\phi, \dot{\psi})/M^4 \right) \right] d\dot{\psi} \quad (39)$$

in which the effective energy density is

$$H(\phi, \dot{\psi}) = \dot{\psi}\pi - L = \left(\dot{\psi}^2 - M^4 \right) \exp \left(L_0(\phi, \dot{\psi})/M^4 \right) + M^4. \quad (40)$$

We wrote Fortran codes and found that the amplitude $A(\phi, \nabla\phi, d\phi)$ increased to a constant value, largely independent of the variables $\phi, \nabla\phi$, and $d\phi$. Thus the field ϕ wanders off to arbitrarily big values as the Monte Carlo runs. No table is big enough.

G. Another Exponential Action

Let's consider the exponential action

$$L = -M^4 \left[\exp \left(-L_0/M^4 \right) - 1 \right] \quad (41)$$

in which L_0 is a free action (32) with the usual kinetic terms. The momentum π conjugate to $\dot{\phi}$ is

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi} \exp(-L_0/M^4). \quad (42)$$

Its derivative with respect to $\dot{\phi}$ is

$$\frac{\partial \pi}{\partial \dot{\phi}} = \left[1 - M^{-4} \dot{\phi}^2 \exp(-L_0/M^4)\right]. \quad (43)$$

The Atlantic City integrals for this very awkward action are

$$\begin{aligned} A(\phi, \nabla\phi, d\phi) &= \int_0^\infty \cos \left[a_s^3 d\phi \dot{\psi} \exp(-L_0(\phi, \dot{\psi})/M^4) \right] e^{-a_s^3 a_t H(\phi, \dot{\psi})} \\ &\times \left[1 - M^{-4} \dot{\psi}^2 \exp(-L_0(\phi, \dot{\psi})/M^4) \right] d\dot{\psi} \end{aligned} \quad (44)$$

and

$$\begin{aligned} C(\phi, \nabla\phi, d\phi) &= \int_0^\infty \cos \left[a_s^3 d\phi \dot{\psi} \exp(-L_0(\phi, \dot{\psi})/M^4) \right] e^{-a_s^3 a_t H(\phi, \dot{\psi})} \\ &\times H(\phi, \dot{\psi}) \left[1 - M^{-4} \dot{\psi}^2 \exp(-L_0(\phi, \dot{\psi})/M^4) \right] d\dot{\psi} \end{aligned} \quad (45)$$

in which the effective energy density is

$$H(\phi, \dot{\psi}) = \dot{\psi} \pi - L = \left(\dot{\psi}^2 + M^4 \right) \exp(-L_0(\phi, \dot{\psi})/M^4) - M^4. \quad (46)$$

The factor $\exp(-a_s^3 a_t H)$ goes to unity as $\dot{\psi} \rightarrow \infty$, so these integrals do not converge.

H. A Quartic Hamiltonian

One way to get finite energy densities may be to start with an action density like

$$L = - \frac{M^4}{1 + \frac{1}{2} M^{-4} \left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2 \phi^2 \right)}. \quad (47)$$

Now the momentum is

$$\pi = \frac{\dot{\phi}}{\left[1 + \frac{1}{2} M^{-4} \left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2 \phi^2 \right) \right]^2}, \quad (48)$$

and the equation relating $\dot{\phi}$ to π , ϕ , and $\nabla\phi$ is quartic. Quartic equations have very complicated algebraic solutions.

I. Inverse Born-Infeld as Very Awkward Action

The equation (49) may say that as $\dot{\phi} \rightarrow \infty$, $\pi \sim (\dot{\phi})^{-3}$, and so $H \sim \pi\dot{\phi} - L \sim (\dot{\phi})^{-2}$, which may be finite, or as $H \sim \pi^{2/3}$, which may be infinite.

Now

$$\frac{\partial L}{\partial \dot{\psi}} = \dot{\psi} \left(1 - \frac{2L_0}{M^4}\right)^{-3/2}, \quad (49)$$

and the second derivative is

$$\frac{\partial^2 L}{\partial \dot{\psi}^2} = \left[1 + M^{-4} (2\dot{\psi}^2 + V)\right] \left(1 - \frac{2L_0}{M^4}\right)^{-5/2} \quad (50)$$

where $V = (\nabla\phi)^2 + m^2\phi^2$.

The time derivative $\dot{\phi}$ is the solution of a cubic equation in $\dot{\phi}^2$, and the hamiltonian is very complicated. It is simpler to use the formula

$$Z(\beta) = \int \exp \left\{ \int_0^\beta \int \left[(i\dot{\phi} - \dot{\psi}) \frac{\partial L(\phi, \dot{\psi})}{\partial \dot{\psi}} + L(\phi, \dot{\psi}) \right] dt d^3x \right\} \left| \frac{\partial^2 L(\phi, \dot{\psi})}{\partial \dot{\psi}^2} \right| D\phi D\dot{\psi}. \quad (51)$$

for the partition function of a very awkward action. For the action density (18), this formula is

$$\begin{aligned} Z(\beta) &= \int \exp \left\{ \int_0^\beta \int \left[(i\dot{\phi} - \dot{\psi}) \dot{\psi} (1 - 2M^{-4}L_0)^{-3/2} + M^4 \left[(1 - 2M^{-4}L_0)^{-1/2} - 1 \right] \right] dt d^3x \right\} \\ &\quad \times \left[1 + M^{-4} (2\dot{\psi}^2 + V) \right] \left(1 - \frac{2L_0}{M^4}\right)^{-5/2} D\phi D\dot{\psi} \\ &= \int \exp \left\{ \int_0^\beta \int \left[(1 - 2M^{-4}L_0)^{-3/2} \left(i\dot{\phi}\dot{\psi} - 2\dot{\psi}^2 + (\nabla\phi)^2 + m^2\phi^2 + M^4 \right) - M^4 \right] dt d^3x \right\} \\ &\quad \times \left[1 + M^{-4} (2\dot{\psi}^2 + V) \right] \left(1 - \frac{2L_0}{M^4}\right)^{-5/2} D\phi D\dot{\psi} \end{aligned} \quad (52)$$

in which $L_0 = L_0(\phi, \dot{\psi})$. The upper limit on the $\dot{\psi}$ integral is $2L_0 = M^4$ or $\dot{\psi} = \sqrt{M^4 + (\nabla\phi)^2 + m^2\phi^2}$. The effective hamiltonian is

$$H = \frac{2\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2 - M^4}{\left[1 - M^{-4} \left(\dot{\psi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)\right]^{3/2}} + M^4. \quad (53)$$

The Atlantic City integrals are

$$\begin{aligned} A(\phi, \nabla\phi, d\phi) &= \int_0^{\dot{\Psi}} \cos \left(a_s^3 d\phi \dot{\psi} (1 - 2M^{-4}L_0)^{-3/2} \right) e^{-a_s^3 a_t H} \\ &\quad \times \left[1 + M^{-4} (2\dot{\psi}^2 + V) \right] \left(1 - \frac{2L_0}{M^4}\right)^{-5/2} d\dot{\psi} \end{aligned} \quad (54)$$

where $d\phi$ is

$$d\phi(i, j, k, \ell) = \phi(i, j, k, \ell + 1) - \phi(i, j, k, \ell), \quad (55)$$

L_0 is a lattice version

$$L_0 = \frac{1}{2} \left(\dot{\psi}^2 - (\delta\phi)^2/a_s^2 - m^2\phi^2 \right) \quad (56)$$

of the action density of the free field, H is a lattice version

$$H = \frac{2\dot{\psi}^2 - (\delta\phi)^2/a_s^2 - m^2\phi^2 - M^4}{\left[1 - M^{-4} \left(\dot{\psi}^2 - (\delta\phi)^2/a_s^2 - m^2\phi^2 \right) \right]^{3/2}} + M^4 \quad (57)$$

of the hamiltonian density (53), $(\delta\phi)^2$ is

$$\begin{aligned} (\delta\phi(i, j, k, \ell))^2 &= (\phi(i+1, j, k, \ell) - \phi(i, j, k, \ell))^2 + (\phi(i, j+1, k, \ell) - \phi(i, j, k, \ell))^2 \\ &\quad + (\phi(i, j, k+1, \ell) - \phi(i, j, k, \ell))^2, \end{aligned} \quad (58)$$

and the upper limit on the integral is

$$\dot{\Psi} = \sqrt{M^4 + (\delta\phi)^2/a_s^2 + m^2\phi^2}. \quad (59)$$

J. An Action that Bounds the Absolute Value of L_0

We could try

$$L = M^4 \left[\frac{1}{2} \left(1 - \frac{2L_0}{M^4} \right)^{-1/2} - \frac{1}{2} \left(1 + \frac{2L_0}{M^4} \right)^{-1/2} + \left(1 - \frac{m^2\phi^2}{M^4} \right)^{-1/2} - 1 \right] \quad (60)$$

where

$$L_0 = \frac{1}{2} \left(\dot{\phi}^2 - (\nabla\phi)^2 \right). \quad (61)$$

But this leads to the equation

$$\pi = \frac{\dot{\phi}}{2} \left[\left(1 + \frac{(\nabla\phi)^2 - \dot{\phi}^2}{M^4} \right)^{-3/2} + \left(1 + \frac{\dot{\phi}^2 - (\nabla\phi)^2}{M^4} \right)^{-3/2} \right]. \quad (62)$$

So this action is very awkward.