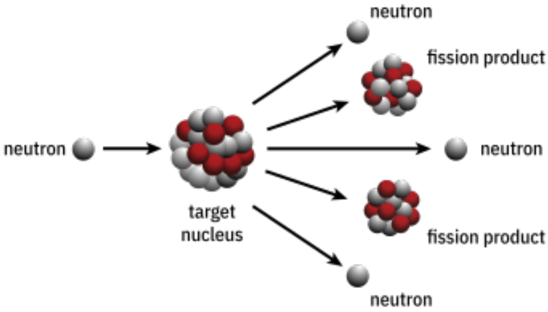
Nuclear Fission Using Algebra and Calculus

Daniel Topa
daniel.m.topa.ctr@mail.mil

Defense Threat Reduction Information Analysis Center (DTRIAC)

January 16, 2023

Compute Energy Release for Fission of ^{235}U



Overview

Strange Behavior of the Nuclear Force

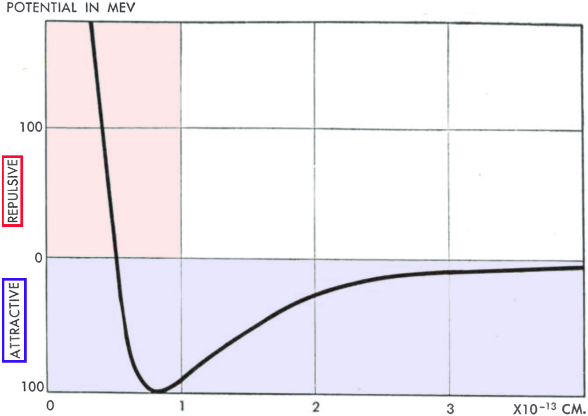
What Holds the Nucleus Together?

Electrical forces bind the electron to the atom, but they cause nuclear particles to fly apart. The $\boxed{\textit{powerful cohesion}}$ *of protons and neutrons must be explained by a* $\boxed{\textit{wholly different phenomenon}}$

by Hans A. Bethe

Scientific American, Vol. 189, No. 3 (September 1953), pp. 58-63

Strange Behavior of the Nuclear Force



NUCLEAR FORCE (measured in millions of electron volts) is plotted against the distance between particles. When the distance is less than half of 10^{-13} centimeter, the nucleons repel one another. They most strongly attract one another at just under 10^{-13} cm.

The Binding Force of Nucleons

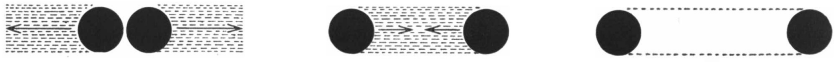
THE ATOMIC NUCLEUS

How do physicists presently visualize it? Curiously, different approaches to the nucleus suggest different pictures, notably the liquid-drop model, the shell model and the optical model

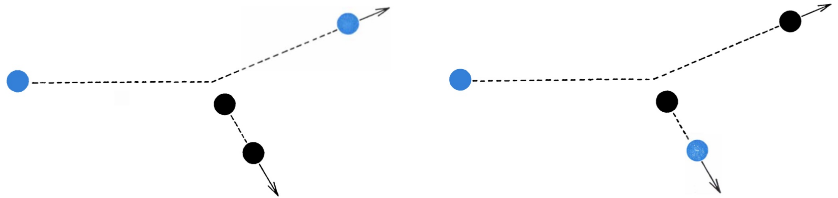
by R. E. Peierls

Scientific American, Vol. 200, No. 1 (January 1959), pp. 75-84

Vagaries of the Nuclear Force

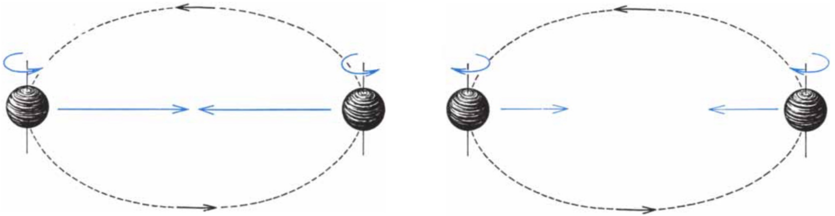


NUCLEAR FORCES are dependent on the distance between particles. If the particles are very close, they repel each other (*left*). If they are a certain distance apart, they attract each other (*center*). If they are farther apart, they have little effect on each other (*right*).



CHARGE EXCHANGE in the nucleus is schematically depicted. When protons (*black balls*) are struck by fast neutrons (*red balls*), in half the cases (*left*) the neutron continues forward. In the other half (*right*), the proton exchanges its charge with the neutron.

Spin–Orbit Interactions



SPIN-ORBIT FORCE arises from a relationship between spin and orbit. When two particles (*left*) spin in the same direction as that in which they move on an orbit, the force between them is strong. When they spin in opposite directions (*right*), force is weak.

Popular Models of the Nucleus.

1. **Liquid Drop Model: Fission**
2. **Shell Model: Energy Levels**
3. **Optical Model: Neutron capture cross section**

Popularization of the Optical Model

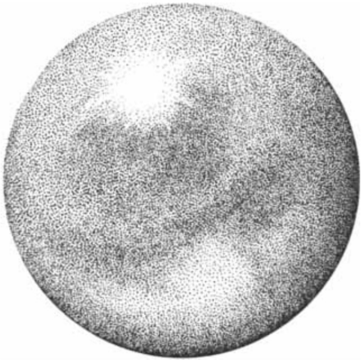
A Model of the Nucleus

As an aid to understanding the atomic nucleus, physicists visualize it in terms of simplified models. A surprisingly fruitful approach is to regard it as a cloudy crystal ball

by Victor F. Weisskopf and E. P. Rosenbaum

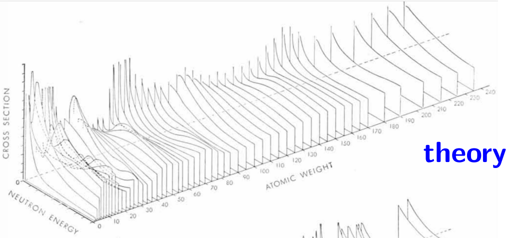
Scientific American, Vol. 193, No. 6 (December 1955), pp. 84-91

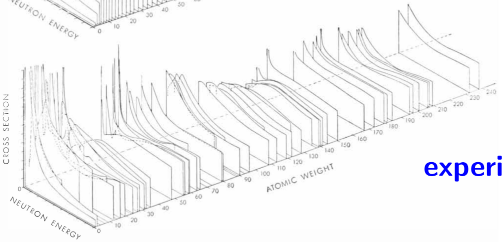
Cloudy Crystal Ball



OPTICAL MODEL pictures the nucleus as a somewhat cloudy crystal ball. The cloudiness represents the tendency of bombarding neutrons to be absorbed by the nucleus.

Neutron Cross Section

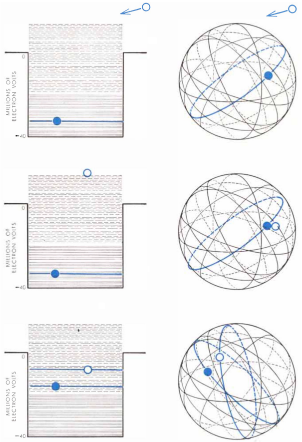




NEUTRON CROSS SECTIONS of all the elements as predicted by the cloudy-crystal-ball model are shown at the top. The actual values found by experiment appear below. The graphs are three-dimensional, with atomic weight increasing horizontally to the right, the energy of the probing neutrons increasing in the direction from the plane of the page out toward the reader, and the cross section increasing vertically upward. The range of neutron energies shown extends from zero to about three million electron volts.

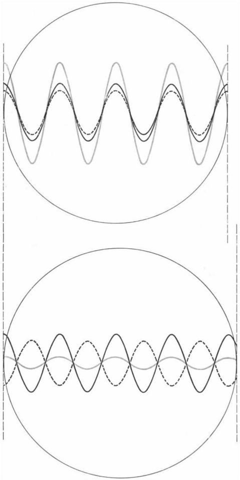
Neutron Absorption

NEUTRON ABSORPTION is illustrated in these drawings. Left-hand diagrams show energies, right-hand diagrams, locations in space. At the top, a bombarding neutron (*open dot*) approaches the nucleus. Solid dot represents the nucleon destined to be hit, colored lines, its orbit. Middle diagrams show particles just before impact, when no energy has been exchanged. In the final result (*bottom*) the neutron has lost energy, dropping into a vacant orbit, and the target nucleon has picked up the energy, jumping to a higher orbit. The energy transaction is possible only if both particles find vacant positions. Cloudiness, or tendency to absorption, increases with the energy of the bombarding neutron because high-energy impacts put both particles into the upper, sparsely populated orbit region, while low-energy impacts leave them in the lower region where they are less likely to find vacancies.



Resonance

TOTAL WAVE MOTION inside nucleus is strong (*gray curve in top diagram*) when incoming wave (*solid black curve*) and reflected wave (*broken curve*) reinforce each other. If nuclear radius is wrong length (*bottom*) individual waves interfere giving small total.



Bonus Content: Ramsauer-Townsend Effect

AMERICAN JOURNAL OF PHYSICS VOLUME 36, NUMBER 8 AUGUST 1968

Demonstration of the Ramsauer–Townsend Effect in a Xenon Thyratron

STEPHEN G. KUKOLICH
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

(Received 15 January 1968; revision received 15 April 1968)

The anomalously small scattering of electrons near 1 eV energy by noble gas atoms may be easily demonstrated using a 2D21 xenon thyratron. This experiment is suitable for a lecture demonstration or for an undergraduate physics laboratory. The probability of scattering and the scattering cross section may be obtained as a function of electron energy by measuring the grid and plate currents in the tube.

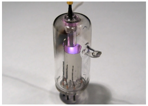
Bonus Content: Ramsauer-Townsend Effect

AMERICAN JOURNAL OF PHYSICS VOLUME 36, NUMBER 8 AUGUST 1968

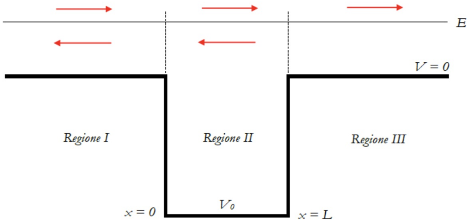
Demonstration of the Ramsauer–Townsend Effect in a Xenon Thyratron

STEPHEN G. KUKOLICH
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 15 January 1968; revision received 15 April 1968)

The anomalously small scattering of electrons near 1 eV energy by noble gas atoms may be easily demonstrated using a 2D21 xenon thyratron. This experiment is suitable for a lecture demonstration or for an undergraduate physics laboratory. The probability of scattering and the scattering cross section may be obtained as a function of electron energy by measuring the grid and plate currents in the tube.



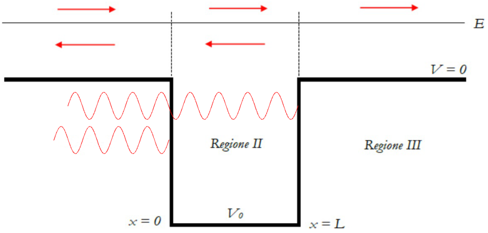
Quantum Mechanical Scattering



The red arrows indicate the phenomena of reflections that may occur in correspondence of the two "walls" of the "hole". For the potential well we can easily solve the Schrödinger equation obtaining the following equations representing the particle in the three regions :

$$\begin{cases} \Psi_I(x) = A_0 e^{ik_1x} + Ae^{-ik_1x} \\ \Psi_{II}(x) = Be^{ik_2x} + Ce^{-ik_2x} \\ \Psi_{III}(x) = De^{ik_1x} \end{cases}$$

Quantum Mechanical Scattering



The red arrows indicate the phenomena of reflections that may occur in correspondence of the two “walls” of the “hole”. For the potential well we can easily solve the Schrödinger equation obtaining the following equations representing the particle in the three regions :

$$\begin{cases} \Psi_I(x) = A_0 e^{ik_1 x} + A e^{-ik_1 x} \\ \Psi_{II}(x) = B e^{ik_2 x} + C e^{-ik_2 x} \\ \Psi_{III}(x) = D e^{ik_1 x} \end{cases}$$

Borrow From Atomic Physics

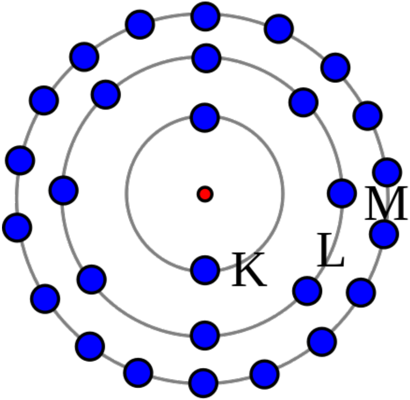
THE STRUCTURE OF THE NUCLEUS

The electrons in an atom tend to occupy distinct shells. What about particles in the nucleus? The "magic numbers" of the isotope chart suggest that they also have a shell arrangement

by Maria G. Mayer 

Scientific American, Vol. 184, No. 3 (March 1951), pp. 22-27

Atomic Physics: Shell Names



K shell full at 2

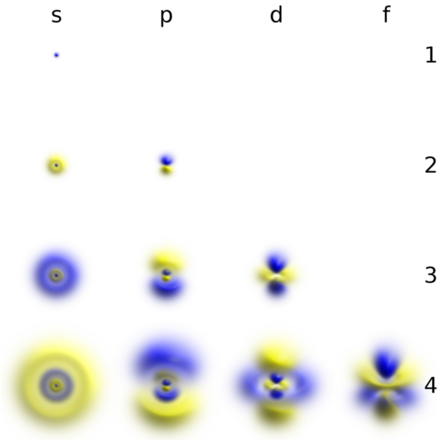
L shell full at 8

M shell full at 18

Atomic Physics: Electron Shell, Subshell Names

Shell name	Subshell name	Subshell max electrons	Shell max electrons
K	1s	2	**2**
L	2s	2	2 + 6 = **8**
	2p	6	
M	3s	2	2 + 6 + 10 = **18**
	3p	6	
	3d	10	
N	4s	2	2 + 6 + 10 + 14 = **32**
	4p	6	
	4d	10	
	4f	14	
O	5s	2	2 + 6 + 10 + 14 + 18 = **50**
	5p	6	
	5d	10	
	5f	14	
	5g	18	

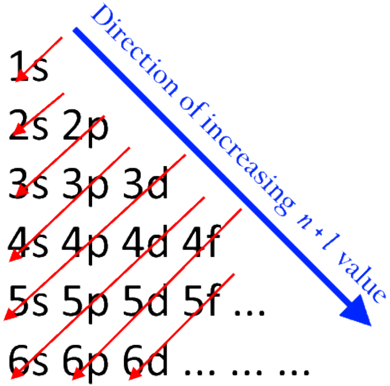
Atomic Physics: Electron Shells



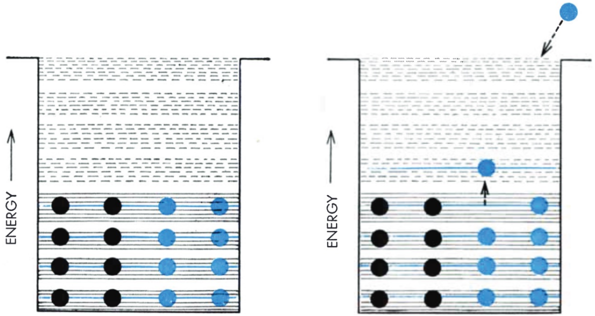
Atomic Physics: Electron Shells

Subshell label	ℓ	Max electrons	Shells containing it	Historical name
s	0	2	Every shell	**s**harp
p	1	6	2nd shell and higher	**p**rincipal
d	2	10	3rd shell and higher	**d**iffuse
f	3	14	4th shell and higher	**f**undamental
g	4	18	5th shell and higher (theoretically)	*(next in alphabet after f)*[25]

Atomic Physics: Filling the Shells

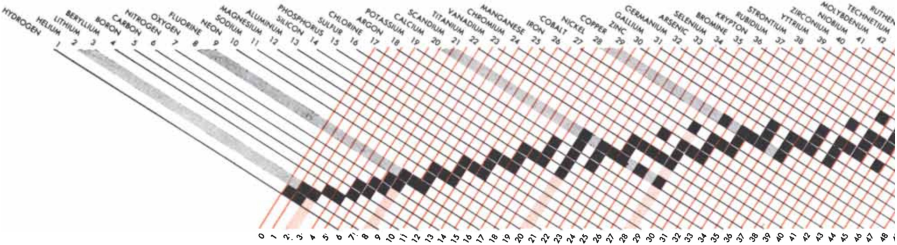


Shell Levels



LOW-ENERGY ORBITS in the shell model of the nucleus may each be occupied by only two neutrons (*colored balls*) and two protons (*black balls*). In the normal state of affairs (*left*) the low-energy orbits are filled; the particles cannot gain or lose energy, and thus cannot change their orbits. A bombarding particle (*upper right*) has energy to spare; thus it can exchange energy with a particle in nucleus and move it to orbit of higher energy.

Shell Theory Magic Numbers



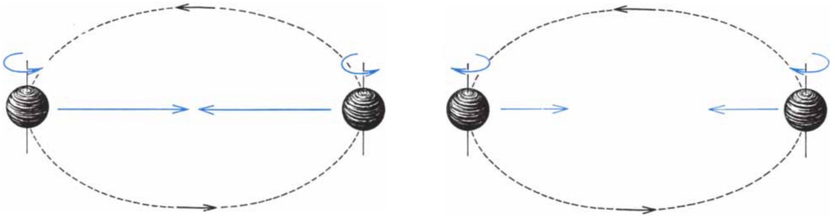
THE CHART OF THE ISOTOPES reveals seven "magic numbers" (*rows shaded red and black*). Each isotope is represented by a black square. Each row outlined by black lines includes nuclei with the same number of protons, *i.e.*, isotopes; each row enclosed in red lines shows nuclei with same number of neutrons. Since

Margenau: Naieve Shell Levels

ORBITAL ANGULAR MOMENTUM	SHELL	NUMBER OF NUCLEONS IN A LEVEL	NUMBER OF NUCLEONS UP TO A LEVEL
0	1s	2	
			2
1	1p	6	
			8
2 0	1d 2s	10 2	
			20
3 1	1f 2p	14 6	
			40
4 2 0	1g 2d 3s	18 10 2	
			70
5 3 1	1h 2f 3p	22 14 6	
			112
6 4 2 0	1i 2g 3d 4s	26 18 10 2	
			168

MARGENAU CHART of nuclear energy levels supports the shell-structure hypothesis in part. The first three numbers in column at right match the magic numbers. The next four do not; at this point the scheme breaks down.

Spin–Orbit Interactions



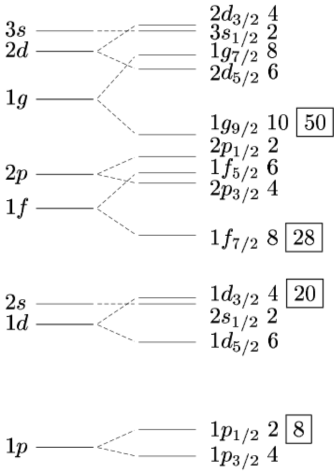
SPIN-ORBIT FORCE arises from a relationship between spin and orbit. When two particles (*left*) spin in the same direction as that in which they move on an orbit, the force between them is strong. When they spin in opposite directions (*right*), force is weak.

Mayer: Corrected Shell Levels

ORBITAL ANGULAR MOMENTUM	SHELL	TOTAL ANGULAR MOMENTUM	NUMBER OF NUCLEONS IN A LEVEL	NUMBER OF NUCLEONS UP TO A LEVEL
0	1s	1/2	2	
				2
		3/2	4	
1	1p	1/2	2	
				8
		5/2	6	
2	1d	3/2	4	
0	2s	1/2	2	
				20
		7/2	8	
				28
3	1f	5/2	6	
		3/2	4	
1	2p	1/2	2	
		9/2	10	
				50
4	1g	7/2	8	
		5/2	6	
2	2d	3/2	4	
0	3s	1/2	2	
		11/2	12	
				82
5	1h	9/2	10	
		7/2	8	
3	2f	5/2	6	
		3/2	4	
1	3p	1/2	2	
		13/2	14	
				126
6	1i	11/2	12	

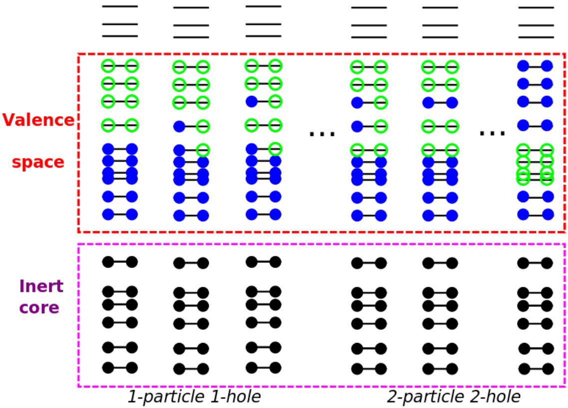
MAYER CHART accounts for the discrepancies by introducing the spins of nuclei (*fractions in the center column*). "Spin-orbit coupling" splits the close-lying levels apart and creates energy gaps where the magic numbers occur.

Wikipedia: Nuclear Shell Model



Wikipedia: Nuclear Shell Model

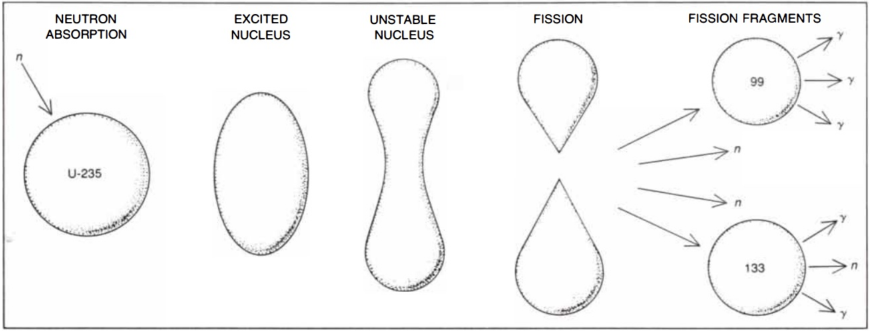
$$|\Psi\rangle = \alpha_1|\Phi\rangle_1 + \alpha_2|\Phi\rangle_2 + \alpha_3|\Phi\rangle_3 + \ldots + \alpha_k|\Phi\rangle_k + \alpha_{k+1}|\Phi\rangle_{k+1} + \ldots$$



Summary of Liquid Drop Model

1. **Nucleus is liquid drop**
2. **Fluid is incompressible**
3. **Volume is conserved**

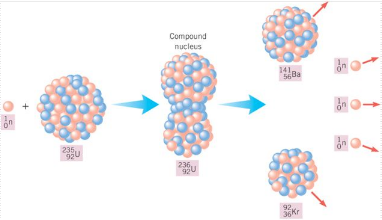
Fission of a Liquid Drop



FISSION OF A NUCLEUS of U-235 is induced by the absorption of a thermal, or slow, neutron (n), which excites the nucleus and deforms it. About 85 percent of the time the deformed nucleus becomes unstable and splits into two fragments of unequal size. The fission fragments shown have atomic mass numbers of 99 and 133; many other pairs of fragments are possible and each has a well-defined probability. The fragments are themselves unstable and are transformed by their subsequent decay, so that the total spectrum of fission products includes many isotopes of more than 30 elements. At the moment of fission high-energy photons, or gamma rays (γ), are emitted, as are a few neutrons. For a chain reaction to be sustained at least one neutron must be absorbed and must induce fission in another U-235 nucleus.

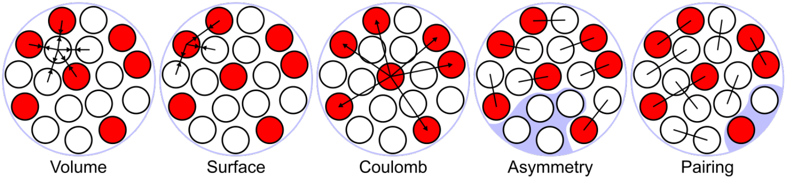
Deformation of Spherical Drop

Nuclear Fission

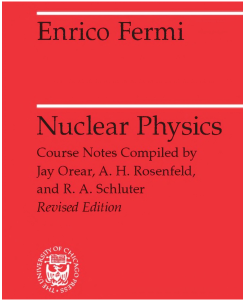


A slowly moving neutron causes the uranium nucleus to fission into barium, krypton, and three neutrons.

Interactions Within the Liquid Drop



Fission of a Liquid Drop



J. Fission **

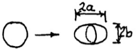
The most useful model for explaining fission phenomena is the liquid drop model (see chapter I,C, p. 6). This model permits calculation of the change in potential energy when the nuclear drop suffers an ellipsoidal deformation from spherical shape*. If the potential energy increases, spherical shape is a stable configuration.

Fission of a Liquid Drop.

The two contributions to the potential energy are 1) capillary energy, 2) electrostatic energy. We will calculate the change in these contributions to the potential energy for a constant volume prolate ellipsoidal deformation given by the equations

major semi-axis $= a = R(1+\varepsilon)$

minor semi-axis $= b = R/\sqrt{1+\varepsilon}$



VIII.58

where R = initial radius, ε = parameter giving extent of deformation. (3ε approaches the square of the eccentricity of the elliptical section as both approach 0.) Volume is invariant: $V = (4\pi/3)ab^2 = (4\pi/3)R^3$.

1) The capillary energy is proportional to the surface area.

$$\text{Ellipsoidal surface} = 4\pi\, R^2\ (1 + 2/5\,\varepsilon^2 + \ldots) \qquad \text{VIII.59}$$

The capillary energy was computed in Ch. I, p. 7 , and found to be $0.014\ A^{2/3}$ for an unexcited (spherical) nucleus, therefore

$$\underline{\text{Capillary energy}} = 0.014\ A^{2/3}(1 + 2/5\ \varepsilon^2 + \ldots) \quad \text{(mass units)} \qquad \text{VIII.60}$$

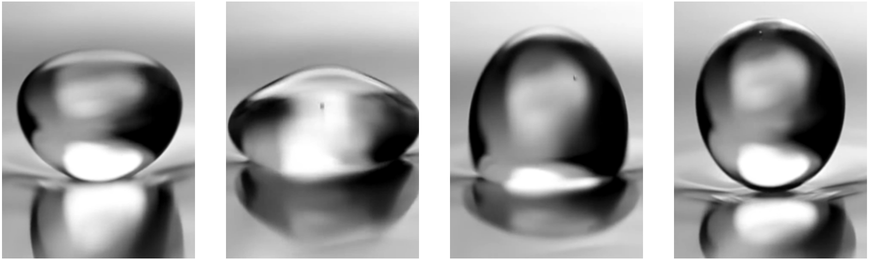
2) The electrostatic energy $= (3/5)(e^2Z^2/R)(1 - 1/5\ \varepsilon^2)$. At sphericity the energy is, from Ch. I, p. 6 , $0.000627\ Z^2/A^{1/3}$, therefore

$$\underline{\text{Electrostatic energy}} = 0.000627(Z^2/A^{1/3})(1 - 1/5\ \varepsilon^2) \qquad \text{VIII.61}$$

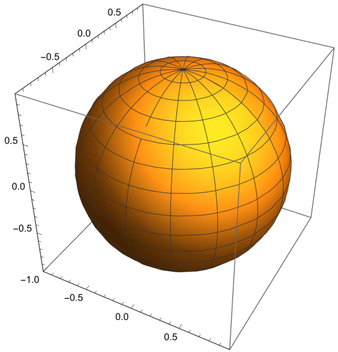
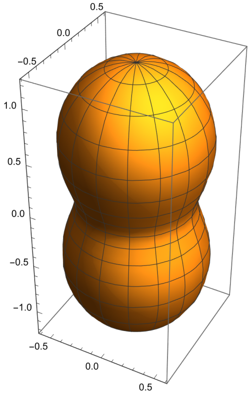
This is evidently maximum at sphericity. The total change is

$$\varepsilon^2(2/5 \times 0.014\ A^{2/3} - 1/5\ 0.000627\ Z^2/A^{1/3}) \qquad \text{VIII.62}$$

Water Drop



Comparison of Liquid Drop Models

Constant Radius	Variable Radius
	

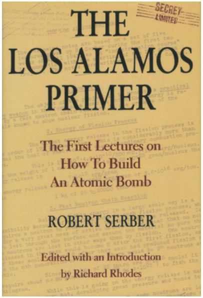
Two Schools of Thought. . .

Constant Radius	Variable Radius	
		
Serber	Bohr	Wheeler

Summary of Liquid Drop Model

1. Nucleus is liquid drop of constant radius
2. Fluid is incompressible
3. Volume is conserved Hofstadter 1956

Serber's Ingredients



...based on a set of five lectures given by R. Serber during the first two weeks of April 1943, as an "indoctrination course" in connection with the starting of the Los Alamos Project.

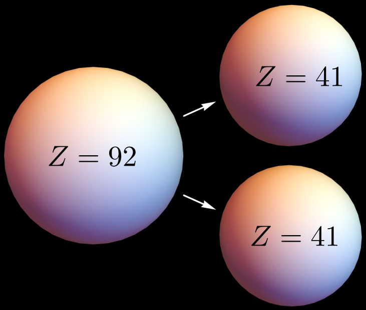
Serber's Ingredients

1. **Volume of Sphere**
2. **High School Electrostatics**

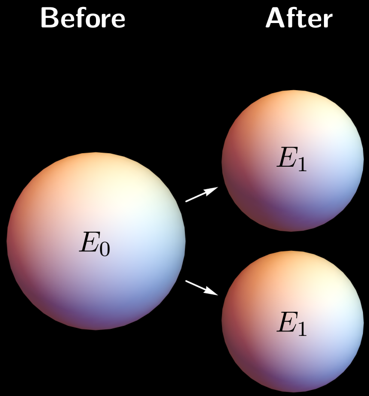
Reaction Model: Charge Bisection

$$\mathbf{U}_{92} \rightarrow \mathbf{Nb}_{41} + \mathbf{Nb}_{41}$$

Reaction Model: Charge Bisection



Quantify Change In Electrostatic Energy



Electrostatic Energy: Before and After

Compute electrostatic energy for both configurations.

Electrostatic Energy For Sphere of Charge Z

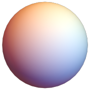
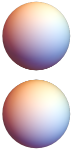
Electrostatic Energy For Sphere of Charge Z:

$$E = \frac{Z^2}{R} \tag{1}$$

Electrostatic Energy: Before and After

Before	**After**
$E_0 = \frac{92^2}{R_0}$	$E_1 = 2 \times \frac{41^2}{R_1}$

Electrostatic Energy: Before and After

Before	After
$E_0 = \frac{92^2}{R_0}$	$E_1 = 2 \times \frac{41^2}{R_1}$
	

Change in Radius

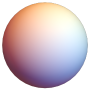
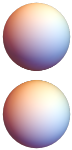
How does R_1 compare to R_0?

Change in Radius

How does R_1 compare to R_0?

Use Conservation of Nuclear Fluid.

Nuclear Fluid (Volume) is Conserved

Before	After
$E_0 = \dfrac{92^2}{R_0}$	$E_1 = 2 \times \dfrac{41^2}{R_1}$
$V_0 = \frac{4}{3}\pi R_0^3$	$V_1 = 2 \times \frac{4}{3}\pi R_1^3$
	

Solve for Radius R_1

$$V_0 = V_1 \quad \Rightarrow \quad R_0^3 = 2R_1^3$$

Solve for Radius R_1

$$V_0 = V_1 \quad \Rightarrow \quad \cancel{\tfrac{4}{3}\pi} R_0^3 = 2 \cdot \cancel{\tfrac{4}{3}\pi} R_1^3$$

$$\Downarrow$$

$$R_0^3 = 2R_1^3$$

$$\Downarrow$$

$$R_1 = 2^{-\frac{1}{3}} R_0$$

$$\Downarrow$$

$$R_1 \approx \frac{4}{5} R_0$$

Change in Electrostatic Energy

$$\Delta E = E_0 - 2E_1 = \frac{92^2}{R_0} - 2\frac{41^2}{R_1} = \frac{3}{8}E_0 \tag{2}$$

Change in Electrostatic Energy

$$\Delta E = \frac{3}{8} E_0 \qquad (2)$$

Change in Electrostatic Energy: Algebra Details

$$\begin{aligned}
\Delta E &= E_0 - 2E_1 \\
&= \frac{92^2}{R_0} - 2 \cdot \frac{41^2}{R_1} \\
&= \frac{(2 \cdot 41)^2}{R_0} - \frac{2 \times 41^2}{\frac{4}{5} R_0} \\
&= \frac{4 \cdot 41^2 - 5/2 \cdot 41^2}{R_0} \\
&= \frac{3}{2} \frac{41^2}{R_0} \\
&= \frac{3}{8} E_0
\end{aligned} \tag{3}$$

Serber's Ingredients and Result

1. Volume of Sphere
2. High School Electrostatics

Result: Fission Releases Astonishing Energy

$$\Delta E = \frac{3}{8} E_0 \tag{2}$$

Astonishing Energy Release

If TNT gives you $1 per reaction, how much will ^{235}U provide?

Astonishing Energy Release

If TNT gives you $1 per reaction, how much will ^{235}U provide?

A stack of dollar bills 7,160 ft tall.

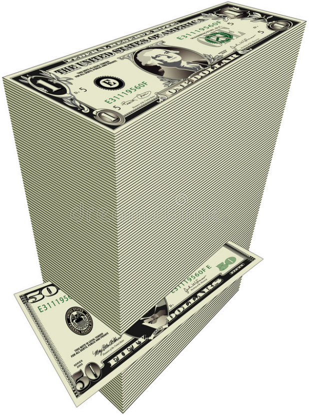
Astonishing Energy Release

If TNT gives you \$1 per reaction, how much will ^{235}U provide?

A stack of dollar bills 7,160 ft tall.

1.4 miles high.

Astonishing Energy Release



Hiroshima

Amount of ^{235}U fissioned < 1.5 cubic inches

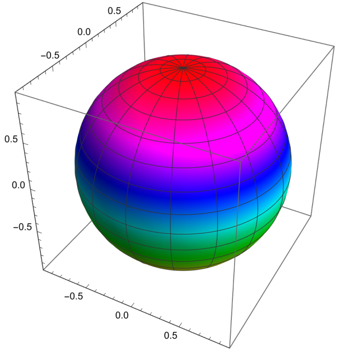
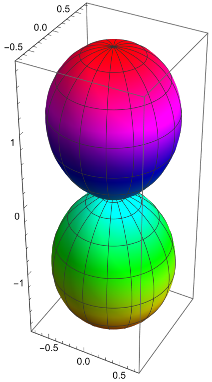
Internet References: Liquid Drop Model.

- **nuclear-power.com:** Liquid Drop Model of Nucleus
- **University of Saskatchewan:** Liquid Drop Model
- **Colorado School of Mines:** The Liquid Drop Model of the Nucleus
- **Wikipedia:** Semi-empirical mass formula
- **University of Southampton:** The Liquid Drop Model
- **Vendatu:** Liquid Drop Model in Nuclear Physics
- **hyperphysics:** Liquid Drop Model of Nucleus

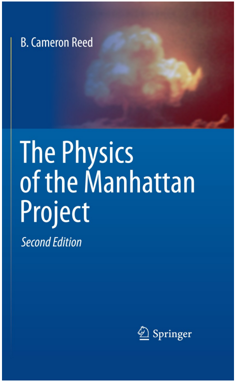
Next Step: Variable R...

Serber's argument used a constant radius: $R =$ const
Bohr and Wheeler let the radius vary $R = R(\theta)$

Comparison of Liquid Drop Models

Constant Radius	Variable Radius
	

Historical Survey



Appendix E: Formal Derivation of the Bohr–Wheeler Spontaneous Fission Limit.

Summary of Liquid Drop Model

1. Nucleus is liquid drop of variable radius
2. Fluid is incompressible
3. Volume is conserved

1939 Paper on Nuclear Fission

SEPTEMBER 1, 1939 PHYSICAL REVIEW VOLUME 56

The Mechanism of Nuclear Fission

NIELS BOHR
University of Copenhagen, Copenhagen, Denmark, and The Institute for Advanced Study, Princeton,

AND

JOHN ARCHIBALD WHEELER
Princeton University, Princeton, New Jersey

(Received June 28, 1939)

On the basis of the liquid drop model of atomic nuclei, an account is given of the mechanism of nuclear fission. In particular, conclusions are drawn regarding the variation from nucleus to nucleus of the critical energy required for fission, and regarding the dependence of fission cross section for a given nucleus on energy of the exciting agency. A detailed discussion of the observations is presented on the basis of the theoretical considerations. Theory and experiment fit together in a reasonable way to give a satisfactory picture of nuclear fission.

Deformations from Sphericity

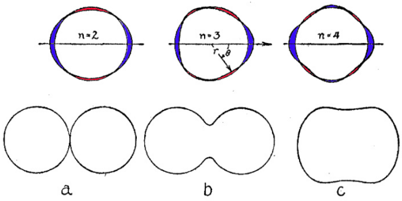


FIG. 2. Small deformations of a liquid drop of the type $\delta r(\theta)=\alpha_n P_n(\cos\theta)$ (upper portion of the figure) lead to characteristic oscillations of the fluid about the spherical form of stable equilibrium, even when the fluid has a uniform electrical charge. If the charge reaches the critical value $(10\times\text{surface tension}\times\text{volume})^{\frac{1}{2}}$, however, the spherical form becomes unstable with respect to even infinitesimal deformations of the type $n=2$. For a slightly smaller charge, on the other hand, a finite deformation (c) will be required to lead to a configuration of *unstable equilibrium*, and with smaller and smaller charge densities the critical form gradually goes over (c, b, a) into that of two uncharged spheres an infinitesimal distance from each other (a).

Bohr-Wheeler Derivation

to the value

$$\boxed{r(\theta)=R[1+\alpha_0+\alpha_2P_2(\cos\theta)+\alpha_3P_3(\cos\theta)+\cdots],} \quad (8)$$

where the α_n are small quantities. Then a straightforward calculation shows that the surface energy plus the electrostatic energy of the comparison drop has increased to the value

$$\begin{aligned}E_{S+E}&=4\pi(r_0A^{\frac{1}{3}})^2O[1+2\alpha_2^2/5+5\alpha_3^2/7+\cdots\\&\quad+(n-1)(n+2)\alpha_n^2/2(2n+1)+\cdots]\\&\quad+3(Ze)^2/5r_0A^{\frac{1}{3}}[1-\alpha_2^2/5-10\alpha_3^2/49-\cdots\\&\quad-5(n-1)\alpha_n^2/(2n+1)^2-\cdots], \end{aligned} \quad (9)$$

where we have assumed that the drop is composed of an incompressible fluid of volume $(4\pi/3)R^3=(4\pi/3)r_0^3A$, uniformly electrified to a charge Ze, and possessing a surface tension O.

Express Radius as a Function of Polar Angle

$$\xcancel{r = R_0} \tag{4}$$

$$\Downarrow$$

$$r(\theta) = R_0\left(1 + \alpha_0 + \alpha_2 P_2(\cos\theta)\right) \tag{5}$$

Express Radius as a Function of Polar Angle

Adjust α_0, α_2 so that volume is conserved:

$$r(\theta) = R_0 \left(1 + \alpha_0 + \alpha_2 P_2(\cos\theta)\right) \tag{5}$$

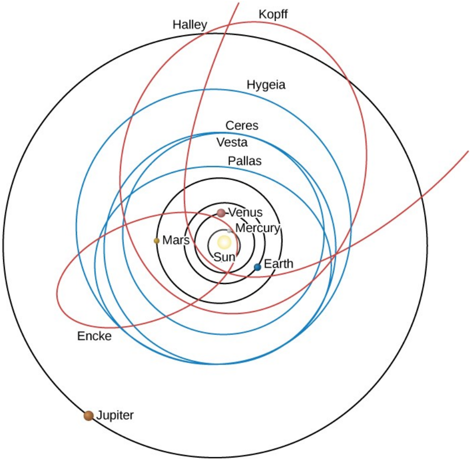
Why Legendre Polynomials?

Problem: **Planetary Orbits**

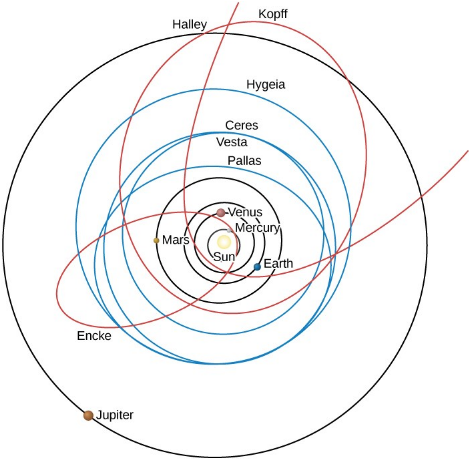
Recherches sur la figure des planètes

AM Legendre
Mémoires de l'Académie Royale des sciences, 1784

Why Legendre Polynomials?



Why Legendre Polynomials?



Orbit types:
Circular
Elliptical
Parabolic
Hyperbolic

Expansion of the Newtonian Potential

$$\frac{1}{\|d\|_2} = \frac{1}{\sqrt{R^2 + r^2 - 2Rr\cos\gamma}} = \sum_{k=0}^{\infty} \frac{r^k}{R^{k+1}} P_k\left(\cos\gamma\right) \tag{6}$$

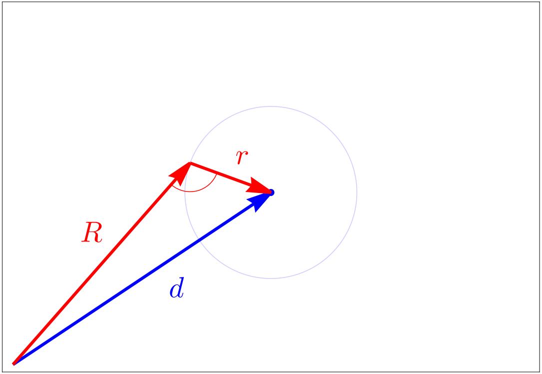
Expansion of the Newtonian Potential

$$\frac{1}{\|d\|_2} = \frac{1}{\sqrt{R^2 + r^2 - 2Rr\cos\gamma}} = \sum_{k=0}^{\infty} \frac{r^k}{R^{k+1}} P_k(\cos\gamma) \tag{7}$$

$$\Uparrow$$

I am the Law of Cosines

Law of Cosines



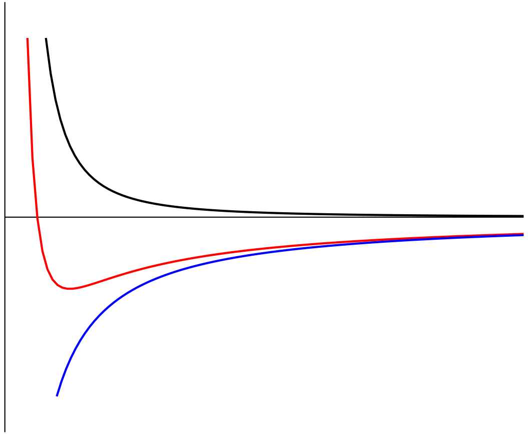
Expansion of the Newtonian Potential

$$\frac{1}{\|d\|_2} = \frac{1}{\sqrt{R^2 + r^2 - 2Rr\cos\gamma}} = \sum_{k=0}^{\infty} \frac{r^k}{R^{k+1}} P_k(\cos\gamma) \tag{8}$$

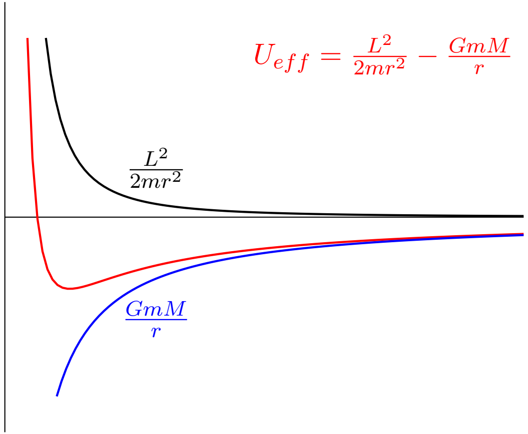
$$\Uparrow$$

I am the Law of Cosines

Effective Potential

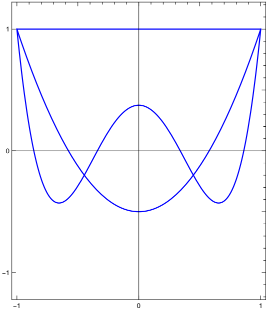


Effective Potential



Deformations from Sphericity

even parity

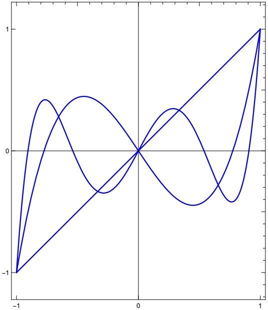


$P_0(x) = 1$

$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$

$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$

odd parity



$P_1(x) = x$

$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$

$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$

Mathematical Properties of Legendre Polynomials

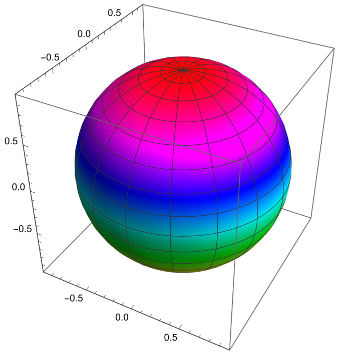
1. **Norm:** $\int_{-1}^{1} P_j(x)P_k(x)dx = \frac{1}{2k+1}\delta_j^k$
2. **Solution to** $(1-x^2)P_n(x)'' - 2xP_n(x)' + n(n+1)P_n(x) = 0$
3. **Sturm–Liouville Form:** $\frac{d}{dx}\left(\left(1-x^2\right)\frac{d}{dx}\right)P(x) = -\lambda P(x)$
4. **Rodrigues' formula:** $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$
5. **Decomposition:** $f_n(x) = \sum_{k=0}^{l} a_k P_k(x)$, $a_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)$
6. **Completeness:** $\sum_{k=0}^{\infty}\frac{2k+1}{2}P_k(x)P_k(y) = \delta(x-y)$, $x, y \in [-1, 1]$

Mathematical Properties of Legendre Polynomials

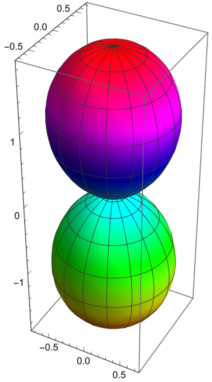
1. **First Use:** $\frac{1}{\sqrt{R^2+r^2-2Rr\cos\theta}} = \sum_{k=0}^{\infty} \frac{r^k}{R^{k+1}} P_k(\cos\theta)$
2. **Generating function:** $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{k=0}^{\infty} P_k(x) t^n$
3. **Monic Gram-Schmidt on** $(1, x, x^2, x^3, \ldots)$ **over** $x \in [-1, 1]$
4. **Bonnet's Recursion Formula:**
   $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$
5. **a.k.a. Legendre functions of the first kind**
6. **a.k.a. zonal harmonics**

How To Compute Volume of Deformed Nucleus?

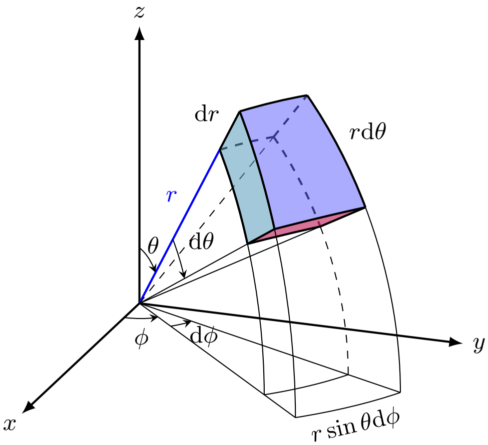
$$V = \frac{4}{3}\pi R_0^3$$



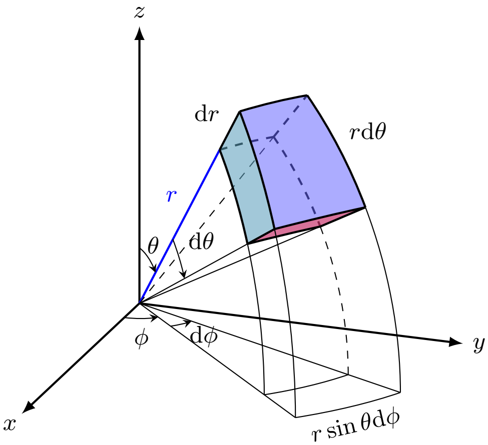
$$V = ???$$



Draw Spherical Volume Element



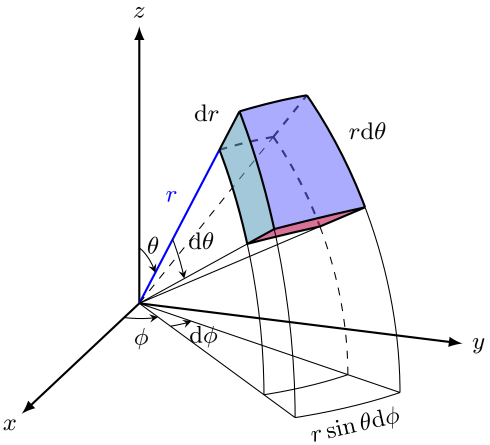
Define Angular Range



$$\theta \in [0, \pi]$$

$$\phi \in [0, 2\pi]$$

Spherical Volume Element $dV = r^2 \sin\theta \, d\theta \, dr \, d\phi$

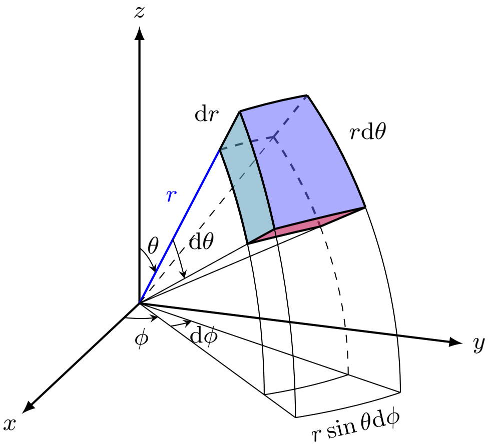


$$\theta \in [0, \pi]$$

$$\phi \in [0, 2\pi]$$

$$dV = dr(r d\phi)(r \sin\theta d\theta)$$

Volume Integral $V = \iiint dV$



$$\theta \in [0, \pi]$$

$$\phi \in [0, 2\pi]$$

$$dV = dr(rd\phi)(r\sin\theta d\theta)$$

$$V = \iiint dV$$

$$= \int_{r=0}^{R_0} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} dr(rd\phi)(r\sin\theta d\theta)$$

Deformations from Sphericity

Constant radius, spherical volume:

$$dV = dr(rd\phi)(r\sin\theta d\theta) \tag{9}$$

$$V = \iiint dV = \int_{r=0}^{R_0}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} dr(rd\phi)(r\sin\theta d\theta) = \frac{4}{3}\pi R_0^3 \tag{10}$$

Deformations from Sphericity

Constant radius, spherical volume:

$$dV = dr(rd\phi)(r\sin\theta d\theta) \tag{9}$$

$$V = \iiint dV = \int\limits_{r=0}^{R_0} \int\limits_{\phi=0}^{2\pi} \int\limits_{\theta=0}^{\pi} dr(rd\phi)(r\sin\theta d\theta) = \frac{4}{3}\pi R_0^3 \tag{10}$$

Vary the radius, keep volume fixed:

$$V(\theta) = \int_{\theta=0}^{\pi} \int_{\rho=0}^{r(\theta)} \int_{\phi=0}^{2\pi} \rho^2 \sin\theta \, d\theta \, d\rho \, d\phi \tag{11}$$

Can We Switch The Order of Integration?

$$\int_{r=0}^{R_0}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} dr(rd\phi)(r\sin\theta d\theta) \stackrel{?}{=} \int_{\theta=0}^{\pi}\int_{\rho=0}^{r(\theta)}\int_{\phi=0}^{2\pi} \rho^2 \sin\theta \, d\theta \, d\rho \, d\phi \tag{12}$$

Yes, We Can Switch The Order of Integration

The Fubini Theorem relates multiple integrals to iterated integrals:

Theorem (Fubini)

Given $f : \mathbb{R}^2 \mapsto \mathbb{R}$ a continuous function on the closed interval $R = [a, b] \times [c, d]$ then

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dxdy = \int_c^d \int_a^b f(x,y)dydx. \tag{13}$$

Proof.

See e.g. Saeed Zakeri, CUNY □

First Volume Integral: ϕ integration

$$
\begin{aligned}
V(\theta) &= \int_{\theta=0}^{\pi} \int_{\rho=0}^{r(\theta)} \int_{\phi=0}^{2\pi} \rho^2 \sin\theta \, d\theta \, d\rho \, d\phi \\
&= \int_{\theta=0}^{\pi} \int_{\rho=0}^{r(\theta)} \rho^2 \sin\theta \, d\theta \times \phi \Big|_0^{2\pi} \\
&= 2\pi \int_{\theta=0}^{\pi} \int_{\rho=0}^{r(\theta)} \rho^2 \sin\theta \, d\theta \, d\rho
\end{aligned} \tag{14}
$$

Second Volume Integral: ρ integration

$$V(\theta) = 2\pi \int_{\theta=0}^{\pi} \int_{\rho=0}^{r(\theta)} \rho^2 \sin\theta \, d\theta \, d\rho$$

Second Volume Integral: ρ integration

$$
\begin{aligned}
V(\theta) &= 2\pi \int_{\theta=0}^{\pi} \int_{\rho=0}^{r(\theta)} \rho^2 \sin\theta \, d\theta \, d\rho \\
&= \frac{2\pi}{3} \int_{\theta=0}^{\pi} \rho^3 \sin\theta \, d\theta \, \Bigg|_0^{r(\theta)}
\end{aligned}
$$

Second Volume Integral: ρ integration

$$\begin{aligned} V(\theta) &= 2\pi \int_{\theta=0}^{\pi} \int_{\rho=0}^{r(\theta)} \rho^2 \sin\theta \, d\theta \, d\rho \\ &= \frac{2\pi}{3} \int_{\theta=0}^{\pi} \rho^3 \sin\theta \, d\theta \Big|_0^{r(\theta)} \\ &= \frac{2\pi}{3} \int_{\theta=0}^{\pi} r^3(\theta) \sin\theta \, d\theta \end{aligned}$$

Second Volume Integral: ρ integration

$$
\begin{aligned}
V(\theta) &= 2\pi \int_{\theta=0}^{\pi} \int_{\rho=0}^{r(\theta)} \rho^2 \sin\theta \, d\theta \, d\rho \\
&= \frac{2\pi}{3} \int_{\theta=0}^{\pi} \rho^3 \sin\theta \, d\theta \, \Big|_0^{r(\theta)} \\
&= \frac{2\pi}{3} \int_{\theta=0}^{\pi} r^3(\theta) \sin\theta \, d\theta \\
&= \frac{2\pi}{3} \int_{\theta=0}^{\pi} \left(R_0 \left(1 + \alpha_0 + \alpha_2 P_2(\cos\theta)\right)\right)^3 \sin\theta \, d\theta
\end{aligned}
\tag{15}
$$

Radial Variations Must Conserve Volume

Using equation 5 for the Legendre expansion...

$$\begin{aligned} \frac{4\pi}{3} R_0^3 &= \frac{2\pi}{3} \int_{\theta=0}^{\pi} r^3(\theta) \sin\theta \, d\theta \\ &= \frac{2\pi}{3} R_0^3 \int_{\theta=0}^{\pi} \left(1 + \alpha_0 + \alpha_2 P_2(\cos\theta)\right)^3 \sin\theta \, d\theta \end{aligned} \tag{16}$$

Conclusion: To conserve volume,

$$1 = \int_{\theta=0}^{\pi} \left(1 + \alpha_0 + \alpha_2 P_2(\cos\theta)\right)^3 \sin\theta \, d\theta \tag{17}$$

Express Legendre Polynomials as Monomials.

$$\int_{\theta=0}^{\pi} (1 + \alpha_0 + \alpha_2 P_2(\cos\theta))^3 \sin\theta \, d\theta =$$

$$\int_{\theta=0}^{\pi} \left(1 + \alpha_0 + \frac{\alpha_2}{2}\left(3\cos^2\theta - 1\right)\right)^3 \sin\theta \, d\theta$$

Reminder: Pascal's Triangle

$n = 0$:					**1**				
$n = 1$:				**1**		**1**			
$n = 2$:			**1**		**2**		**1**		
$n = 3$:		**1**		**3**		**3**		**1**	
$n = 4$:	**1**		**4**		**6**		**4**		**1**

Resolve Cubic Expression

Recall Pascal's Triangle...

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Resolve Cubic Expression

Recall Pascal's Triangle...

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$\begin{aligned} a &\to 1 + \alpha_0, \\ b &\to \frac{\alpha_2}{2}\left(3\cos^2\theta - 1\right) \end{aligned} \tag{18}$$

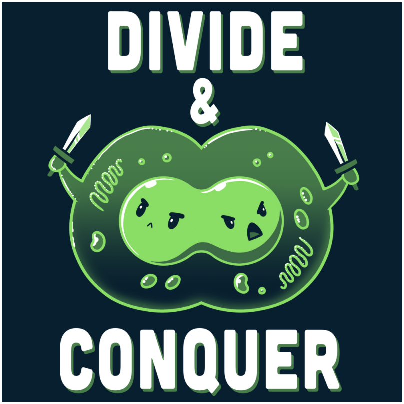
$$r^3(\theta) = R_0^3 \left(\underbrace{1 + \alpha_0}_{a} + \underbrace{\frac{\alpha_2}{2}\left(3\cos^2\theta - 1\right)}_{b} \right)^3 \tag{5}$$

Mathematica Moment

In[35]:= Expand$\left[r[\theta]^3\,\text{Sin}[\theta]\right]$

Out[35]= $R^3\,\text{Sin}[\theta] + 3\,R^3\,\alpha 0\,\text{Sin}[\theta] + 3\,R^3\,\alpha 0^2\,\text{Sin}[\theta] + R^3\,\alpha 0^3\,\text{Sin}[\theta] - \frac{3}{2}\,R^3\,\alpha 2\,\text{Sin}[\theta] - 3\,R^3\,\alpha 0\,\alpha 2\,\text{Sin}[\theta] - \frac{3}{2}\,R^3\,\alpha 0^2\,\alpha 2\,\text{Sin}[\theta] + \frac{3}{4}\,R^3\,\alpha 2^2\,\text{Sin}[\theta] + \frac{3}{4}\,R^3\,\alpha 0\,\alpha 2^2\,\text{Sin}[\theta] - \frac{1}{8}\,R^3\,\alpha 2^3\,\text{Sin}[\theta] + \frac{9}{2}\,R^3\,\alpha 2\,\text{Cos}[\theta]^2\,\text{Sin}[\theta] + 9\,R^3\,\alpha 0\,\alpha 2\,\text{Cos}[\theta]^2\,\text{Sin}[\theta] + \frac{9}{2}\,R^3\,\alpha 0^2\,\alpha 2\,\text{Cos}[\theta]^2\,\text{Sin}[\theta] - \frac{9}{2}\,R^3\,\alpha 2^2\,\text{Cos}[\theta]^2\,\text{Sin}[\theta] - \frac{9}{2}\,R^3\,\alpha 0\,\alpha 2^2\,\text{Cos}[\theta]^2\,\text{Sin}[\theta] + \frac{9}{8}\,R^3\,\alpha 2^3\,\text{Cos}[\theta]^2\,\text{Sin}[\theta] + \frac{27}{4}\,R^3\,\alpha 2^2\,\text{Cos}[\theta]^4\,\text{Sin}[\theta] + \frac{27}{4}\,R^3\,\alpha 0\,\alpha 2^2\,\text{Cos}[\theta]^4\,\text{Sin}[\theta] - \frac{27}{8}\,R^3\,\alpha 2^3\,\text{Cos}[\theta]^4\,\text{Sin}[\theta] + \frac{27}{8}\,R^3\,\alpha 2^3\,\text{Cos}[\theta]^6\,\text{Sin}[\theta]$

Strategy: Divide and Conquer



Resolve Cubic Expression

$$\begin{aligned} r^3(\theta) &= R_0^3 \left(1 + \alpha_0 + \alpha_2 P_2(\cos\theta)\right)^3 \\ &= R_0^3 \left(1 + \alpha_0 + \frac{\alpha_2}{2}\left(3\cos^2\theta - 1\right)\right)^3 \end{aligned} \tag{5}$$

Resolve Cubic Expression

$$\begin{aligned} r^3(\theta) &= R_0^3 \left(1 + \alpha_0 + \alpha_2 P_2(\cos\theta)\right)^3 \\ &= R_0^3 \left(1 + \alpha_0 + \frac{\alpha_2}{2}\left(3\cos^2\theta - 1\right)\right)^3 \end{aligned} \tag{5}$$

Collect terms in powers of $\cos\theta$

$$\begin{aligned} r^3(\theta) = R_0^3 \big(\textbf{constant} &+ \textbf{quadratic} \quad \cos^2\theta \\ &+ \textbf{quartic} \quad \cos^4\theta \\ &+ \textbf{sextic} \quad \cos^6\theta\big) \end{aligned} \tag{19}$$

Resolve θ Components

$$
\begin{aligned}
\textbf{constant} &= \frac{3}{24}\left(-\alpha_2{}^3 + 6\alpha_2{}^2\left(1+\alpha_0\right) + 12\alpha_2\left(-\alpha_0{}^2 - 2\alpha_0 + 1\right)\right) \\
\textbf{quadratic} &= \frac{9}{8}\alpha_2(\alpha_2 - 2\alpha_0 - 2)^2 \\
\textbf{quartic} &= \frac{27}{8}\alpha_2{}^2(2 + 2\alpha_0 - \alpha_2) \\
\textbf{sextic} &= \frac{27}{8}\alpha_2{}^2
\end{aligned}
\tag{20}
$$

Resolve θ Components

Let's look at the θ terms, for example,

$$\frac{27}{8}{\alpha_2}^3 \int \sin\theta \cos^6\theta \, d\theta$$

Simple Integrals

Let $w = \cos\theta$**, then** $dw = -\sin\theta d\theta$ **and for** $k \in \mathbb{N}$

$$\int \sin\theta \cos^k\theta \, d\theta \Rightarrow \int w^k(-dw) \Rightarrow -\frac{1}{k+1}w^{k+1} \Rightarrow -\frac{1}{k+1}\cos^{k+1}\theta \quad (21)$$

For example...

$$\begin{aligned}
\int \sin\theta \cos^2\theta \, d\theta &= -\tfrac{1}{3}\cos^3\theta \\
\int \sin\theta \cos^4\theta \, d\theta &= -\tfrac{1}{5}\cos^5\theta \\
\int \sin\theta \cos^6\theta \, d\theta &= -\tfrac{1}{7}\cos^7\theta
\end{aligned}$$

Really Simple Integrals

For k even,

$$\int_0^{\pi} \sin\theta \cos^{k}\theta \, d\theta = -\tfrac{1}{k+1}\cos^{k+1}\theta \Big|_0^{\pi} = \frac{2}{k+1} \tag{22}$$

Basic Integrations

$$\begin{aligned} \int_0^{\pi} r^3(\theta) \sin\theta \, d\theta \quad = & \\ 2R_0^3 \,(\quad & \textbf{constant} \times \frac{1}{1} \\ + \quad & \textbf{quadratic} \times \frac{1}{3} \\ + \quad & \textbf{quartic} \times \frac{1}{5} \\ + \quad & \textbf{sextic} \times \frac{1}{7} \quad) \end{aligned} \tag{23}$$

Final Volume Integral: θ

$$
\begin{aligned}
&\int_0^\pi r^3(\theta)\sin\theta\, d\theta = \\
&2R_0^3\left(\tfrac{3}{24}\left(-{\alpha_2}^3 + 6{\alpha_2}^2(1+\alpha_0) + 12\alpha_2\left(-{\alpha_0}^2 - 2\alpha_0 + 1\right)\right)\right. \\
&+\tfrac{9}{8}\alpha_2(\alpha_2 - 2\alpha_0 - 2)^2 \times \tfrac{1}{3} \\
&+\tfrac{27}{8}{\alpha_2}^2(2 + 2\alpha_0 - \alpha_2) \times \tfrac{1}{5} \\
&\left.+\tfrac{27}{8}{\alpha_2}^2 \times \tfrac{1}{7}\right)
\end{aligned}
\tag{23}
$$

Integration over θ

$$\int_0^\pi r^3(\theta)\sin\theta\, d\theta = 2R_0^3\left((1+\alpha_0)^3 + \tfrac{3}{5}{\alpha_2}^2(1+\alpha_0) + \tfrac{2}{35}{\alpha_2}^3\right) \qquad (23)$$

Volume Integration

$$\int_0^{\pi} r^3(\theta)\sin\theta\, d\theta = 2R_0^3\left((1+\alpha_0)^3 + \tfrac{3}{5}{\alpha_2}^2(1+\alpha_0) + \tfrac{2}{35}{\alpha_2}^3\right) \qquad (23)$$

Volume is Conserved

$$\begin{aligned}
\textbf{sphere} &= \textbf{intermediates} \\
\frac{4\pi}{3} R_0^3 &= \frac{2\pi}{3} \int_0^\pi r^3(\theta) \sin\theta \, d\theta \\
&= \frac{4\pi}{3} R_0^3 \left((1+\alpha_0)^3 + \frac{3}{5}\alpha_2{}^2 (1+\alpha_0) + \frac{2}{35}\alpha_2{}^3 \right)
\end{aligned} \tag{24}$$

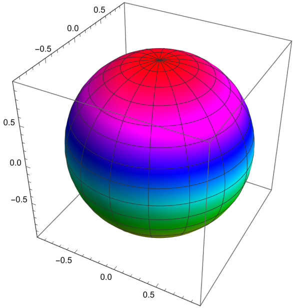
Volume is Conserved

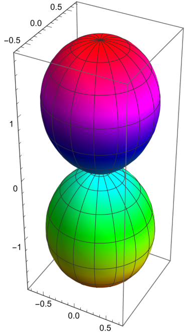
$$\begin{aligned}
\textbf{sphere} &= \textbf{intermediates} \\
\cancel{\frac{4\pi}{3} R_0^3} &= \frac{2\pi}{3}\int_0^{\pi} r^2(\theta)\sin\theta\, d\theta \\
&= \cancel{\frac{4\pi}{3} R_0^3}\left((1+\alpha_0)^3 + \frac{3}{5}{\alpha_2}^2(1+\alpha_0) + \frac{2}{35}{\alpha_2}^3 \right) \\
1 &= (1+\alpha_0)^3 + \frac{3}{5}{\alpha_2}^2(1+\alpha_0) + \frac{2}{35}{\alpha_2}^3
\end{aligned} \tag{24}$$

How To Compute Volume of Deformed Nucleus?

$$V = \frac{4}{3}\pi R_0^3$$

$$V = \frac{4}{3}\pi R_0^3 \times$$
$$\left(1+\alpha_0\right)^3 + \frac{3}{5}{\alpha_2}^2\left(1+\alpha_0\right) + \frac{2}{35}{\alpha_2}^3$$





Recap

In order to conserve volume, we require

$$(1+\alpha_0)^3 + \frac{3}{5}{\alpha_2}^2\,(1+\alpha_0) + \frac{2}{35}{\alpha_2}^3 = 1 \tag{24}$$

To Understand Variation of α_0, α_2

To characterize the connection between α_0 and α_2, drop the term with ${\alpha_2}^3$:

$$(1+\alpha_0)^3 + \frac{3}{5}{\alpha_2}^2(1+\alpha_0) + \xcancel{\frac{2}{35}{\alpha_2}^3} = 1 \tag{25}$$

Characterize Variation of α_0, α_2

Solve for α_2 with basic algebra:

$$(1+\alpha_0)^3 + \tfrac{3}{5}(1+\alpha_0)\,{\alpha_2}^2 = 1 \tag{26}$$

$$\Downarrow$$

$$\tfrac{3}{5}(1+\alpha_0)\,{\alpha_2}^2 = 1 - (1+\alpha_0)^3$$

$$\Downarrow$$

$${\alpha_2}^2 = \tfrac{5}{3}\,\frac{1-(1+\alpha_0)^3}{1+\alpha_0}$$

To Understand Variation of α_0, α_2

Solve for α_2 with basic algebra:

$$\alpha_2 = \pm\sqrt{\frac{5}{3}\frac{-3\alpha_0(1+\alpha_0+{\alpha_0}^2)}{1+\alpha_0}}$$

$$\alpha_2 = \pm\sqrt{\frac{5}{3}\frac{-3\alpha_0(\cancel{1+\alpha_0}+\cancel{\alpha_0^2})}{\cancel{1+\alpha_0}}}$$

$$\alpha_2 \approx \pm\sqrt{-5\alpha_0} + \mathcal{O}\left({\alpha_0}^{5/2}\right) \tag{27}$$

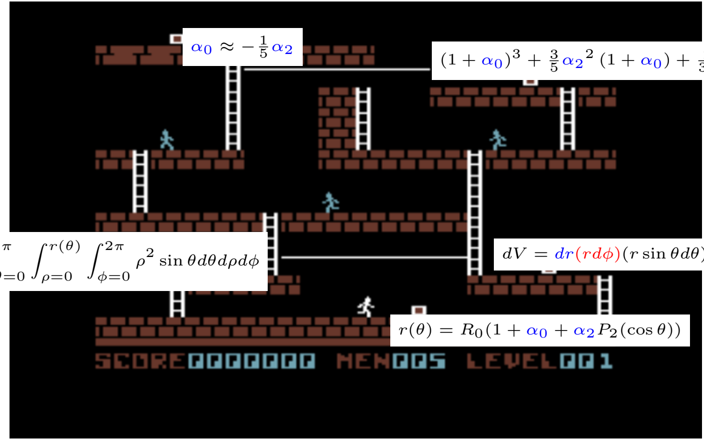
Variations of α_0, α_2 Which Conserve Volume

Let the drop radius vary so that volume is conserved.

$$\alpha_2 \approx \sqrt{-5\alpha_0} \tag{27}$$

$$\alpha_0 \approx -\tfrac{1}{5}\alpha_2 \tag{28}$$

Looking Back



$$\alpha_0 \approx -\frac{1}{5}\alpha_2$$

$$(1+\alpha_0)^3 + \frac{3}{5}{\alpha_2}^2(1+\alpha_0) + \frac{2}{35}{\alpha_2}^3 = 1$$

$$\int_{\theta=0}^{\pi}\int_{\rho=0}^{r(\theta)}\int_{\phi=0}^{2\pi}\rho^2 \sin\theta \, d\theta \, d\rho \, d\phi$$

$$dV = dr(r\,d\phi)(r\sin\theta\, d\theta)$$

$$r(\theta) = R_0(1+\alpha_0+\alpha_2 P_2(\cos\theta))$$

Bohr–Wheeler Craftsmanship

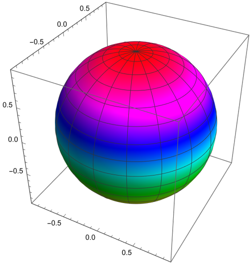
quadratic

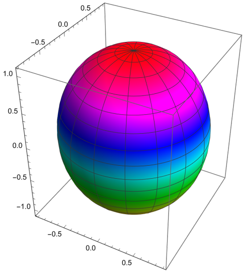
$$r(\theta) = R[1 + \alpha_0 + \alpha_2 P_2(\cos\,\theta) + \alpha_3 P_3(\cos\,\theta) + \cdots], \quad (8)$$

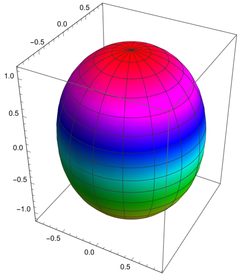
linear

cubic

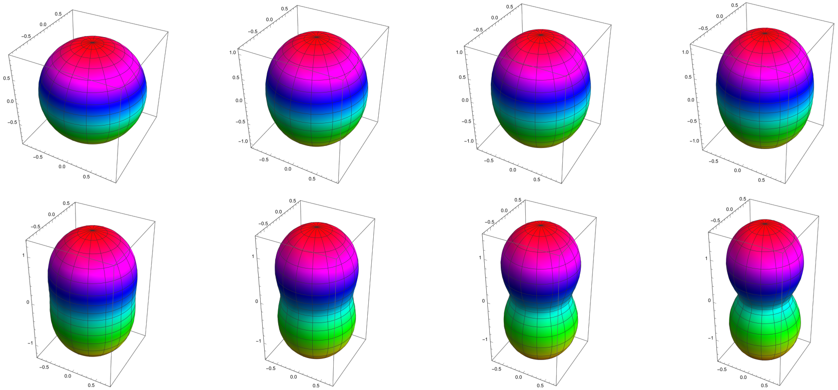
Scission Sequence



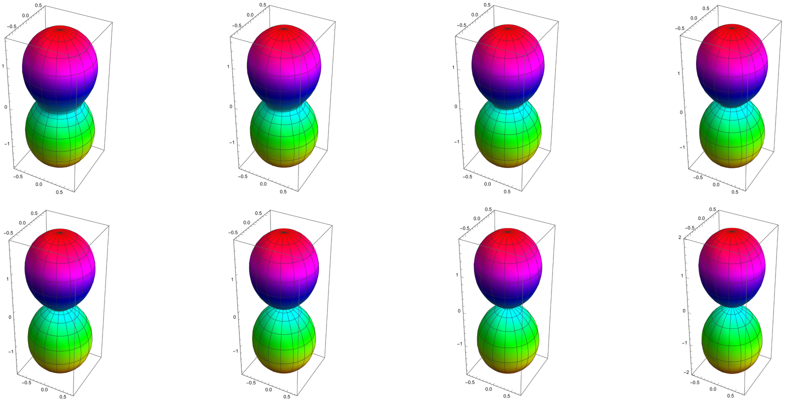




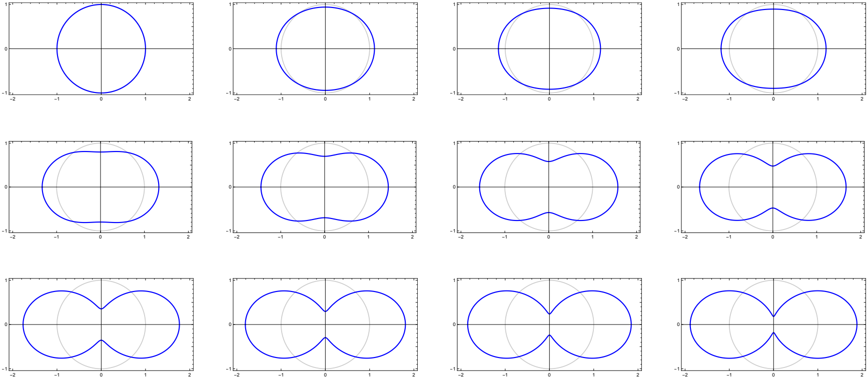
Early Scission Sequence



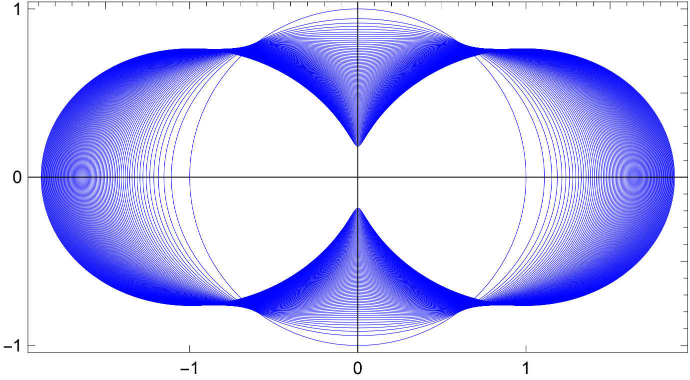
Late Scission Sequence



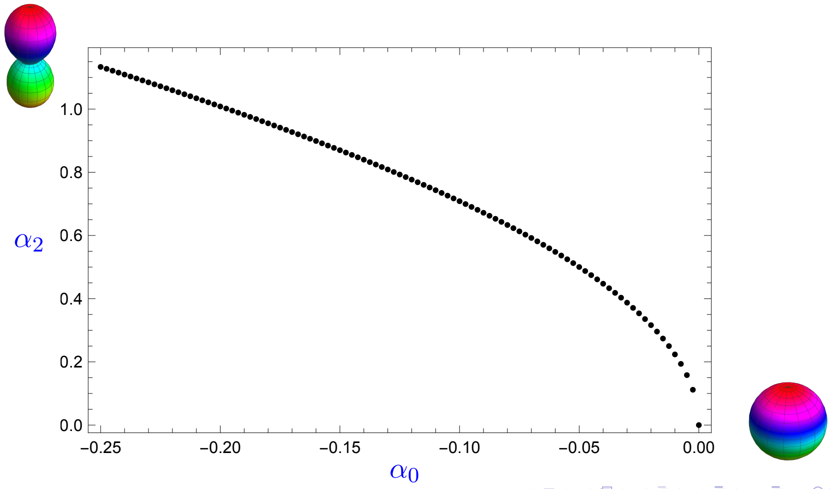
Scission Sequence Outlines



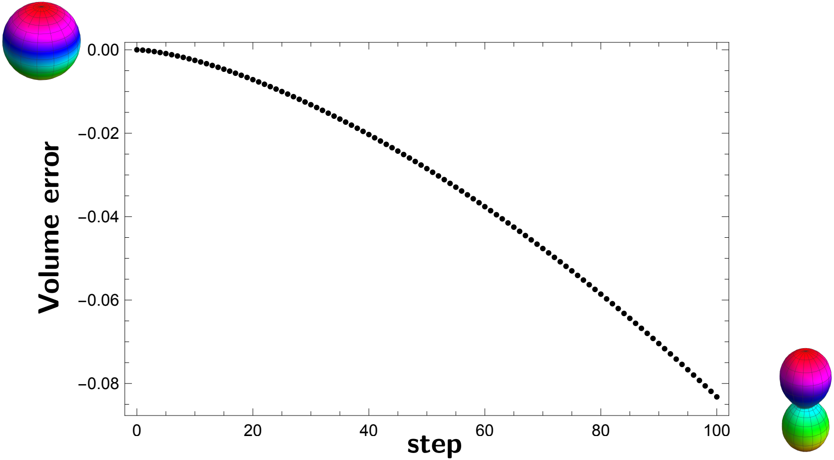
Separation Sequence



Bohr–Wheeler Deformation Sequence



Bohr–Wheeler Error in Volume



Birth of the Legendre Polynomials

"In the study of the attraction of spheroids and planetary motions"(Sansone 1959, ch. 3, p. 169)

$$\frac{1}{r} = \left(1 - 2\rho\cos\gamma + \rho^2\right)^{-\frac{1}{2}} \tag{29}$$

Birth of the Legendre Polynomials

"In the study of the attraction of spheroids and planetary motions"(Sansone 1959, ch. 3, p. 169)

$$\frac{1}{r} = \left(1 - 2\rho\cos\gamma + \rho^2\right)^{-\frac{1}{2}} \tag{29}$$

Think: Law of Cosines Again

Binomial Series

For $|\rho| < 1$

$$\begin{aligned} \frac{1}{r} &= \left(1 - \rho e^{i\gamma}\right)^{-\frac{1}{2}} \left(1 - \rho e^{-i\gamma}\right)^{-\frac{1}{2}} \\ &= \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \rho^n e^{iny} \end{aligned} \tag{30}$$

Using the Binomial Series

For $|\rho| < 1$

$$\begin{aligned} \frac{1}{r} &= \left(1 - \rho e^{i\gamma}\right)^{-\frac{1}{2}} \qquad \left(1 - \rho e^{-i\gamma}\right)^{-\frac{1}{2}} \\ &= \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \rho^n e^{in\gamma} \quad \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \rho^n e^{-in\gamma} \end{aligned} \tag{31}$$

Using the Cauchy Product

Multiplication of two absolutely convergent series when $n > 0$

$$(f \circ g)(n) = \sum_{k=0}^{n} f(n)g(n-k)$$

Using the Cauchy Product

$$\frac{1}{r} = \sum_{n=0}^{\infty} \rho^n \sum_{j=0}^{[n/2]} \binom{-\frac{1}{2}}{j} \binom{-\frac{1}{2}}{n-j} \times$$
$$\left((-1)^j (-1)^{n-j} e^{ij\gamma} e^{-i(n-j)\gamma} + (-1)^j (-1)^{n-j} e^{-ij\gamma} e^{i(n-j)\gamma} \right)$$

Introducing the Legendre Polynomials!

$$\frac{1}{r} = \left(1 - 2\rho\cos\gamma + \rho^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \rho^n P_n\left(\cos\gamma\right) \tag{29}$$

Introducing the Legendre Polynomials!

$$\frac{1}{r} = \left(1 - 2\rho\cos\gamma + \rho^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \rho^n P_n\left(\cos\gamma\right) \tag{29}$$

$$P_n\left(\cos\gamma\right)$$
$$= (-1)^n \left(\binom{-\frac{1}{2}}{n} 2\cos n\gamma + \binom{-\frac{1}{2}}{n-1} \binom{-\frac{1}{2}}{1} 2\cos(n-2)\gamma + \dots \right)$$

Deriving Bonnet's Recursion Formula

Start with the generating function

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{k=0}^{\infty} P_k(x) t^k \tag{32}$$

Differentiate w.r.t.:

$$\begin{aligned} \partial_t \left(\frac{1}{\sqrt{1 - 2xt + t^2}} \right) &= \partial_t \left(\sum_{k=0}^{\infty} P_k(x) t^k \right) \\ \frac{x - t}{(1 - 2xt + t^2)^{3/2}} &= \sum_{k=1}^{\infty} k P_k(x) t^{k-1} \end{aligned} \tag{33}$$

Continuing...

Multiply both sides by the polynomial term...

$$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{k=1}^{\infty} kP_k(x)t^{k-1}$$

Continuing...

Multiply both sides by the polynomial term...

$$\begin{aligned} \frac{x-t}{(1-2xt+t^2)^{3/2}} &= \sum_{k=1}^{\infty} kP_k(x)t^{k-1} \\ \frac{x-t}{\sqrt{1-2xt+t^2}} &= (1-2xt+t^2)\sum_{k=1}^{\infty} kP_k(x)t^{k-1} \end{aligned} \quad (34)$$

Continuing...

Multiply both sides by the polynomial term...

$$\begin{aligned} \frac{x-t}{\left(1-2xt+t^2\right)^{3/2}} &= \sum_{k=1}^{\infty} kP_k(x)t^{k-1} \\ \frac{x-t}{\sqrt{1-2xt+t^2}} &= \left(1-2xt+t^2\right)\sum_{k=1}^{\infty} kP_k(x)t^{k-1} \end{aligned} \tag{34}$$

Substitute using equation (32)

$$(x-t)\sum_{k=0}^{\infty} P_k(x)t^k = \left(1-2xt+t^2\right)\sum_{k=1}^{\infty} kP_k(x)t^{k-1} \tag{35}$$

Distribute Powers of t

$$(x-t)\sum_{k=0}^{\infty} P_k(x)t^k \quad = \quad \left(1-2xt+t^2\right)\sum_{k=1}^{\infty} kP_k(x)t^{k-1} \tag{35}$$

Distribute powers of t

Distribute Powers of t

$$(x-t)\sum_{k=0}^{\infty} P_k(x)t^k = \left(1-2xt+t^2\right)\sum_{k=1}^{\infty} kP_k(x)t^{k-1} \tag{35}$$

Distribute powers of t

$$\begin{aligned}\sum_{k=0}^{\infty} xP_k(x)t^{k} &- \sum_{k=0}^{\infty} P_k(x)t^{k+1} = \\ \sum_{k=0}^{\infty} kP_k(x)t^{k-1} &- \sum_{k=0}^{\infty} 2kxP_k(x)t^{k} + \sum_{k=0}^{\infty} kP_k(x)t^{k+1}\end{aligned} \tag{36}$$

Collect Powers of t

$$\sum_{k=0}^{\infty} xP_k(x)t^{k} - \sum_{k=0}^{\infty} P_k(x)t^{k+1} = \sum_{k=0}^{\infty} kP_k(x)t^{k-1} - \sum_{k=0}^{\infty} 2xP_k(x)t^{k} + \sum_{k=0}^{\infty} kP_k(x)t^{k+1} \tag{36}$$

Rearrangement..

$$\sum_{k=0}^{\infty} kP_k(x)t^{k+1} + \sum_{k=0}^{\infty} P_k(x)t^{k+1} = \sum_{k=0}^{\infty} 2kxP_k(x)t^{k} + \sum_{k=0}^{\infty} kxP_k(x)t^{k} - \sum_{k=0}^{\infty} kP_k(x)t^{k-1} \tag{36}$$

Collect Powers of t

$$\begin{aligned}\sum_{k=0}^{\infty} kP_k(x)t^{k+1} \quad + \quad \sum_{k=0}^{\infty} P_k(x)t^{k+1} \quad &= \\ \sum_{k=0}^{\infty} 2kxP_k(x)t^{k} \quad + \quad \sum_{k=0}^{\infty} kxP_k(x)t^{k} \quad &- \quad \sum_{k=0}^{\infty} kP_k(x)t^{k-1}\end{aligned} \tag{36}$$

Group powers of t

$$\sum_{k=0}^{\infty}(k+1)P_k(x)t^{k+1} = \sum_{k=0}^{\infty}(2k+1)xP_k(x)t^{k} - \sum_{k=0}^{\infty}kP_k(x)t^{k-1} \tag{37}$$

When Are Series Equal?

Definition

Two summable infinite series

$$\sum_{k=0}^{\infty} a_k < \infty, \sum_{k=0}^{\infty} b_k < \infty$$

are equal iff they exhibit equality of the partial sums:

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{n} b_k \quad \forall n \geq 0$$

The series are equal term by term:

$$a_k = b_k \quad k = 0 : \infty$$

Equate Coefficients Term by Term

"Eliminate" the summations

$$\sum_{k=0}^{\infty}(k+1)P_k(x)t^{k+1} = \sum_{k=0}^{\infty}(2k+1)xP_k(x)t^{k} - \sum_{k=0}^{\infty}kP_k(x)t^{k-1} \quad (37)$$

Revealing Bonnet's Recursion Formula

$$(k+1)P_{k+1}(x) = (2k+1)xP_k(x) - nP_{k-1}(x) \quad (38)$$

Bibliography I

- Sam M Austin and George F Bertsch. "Halo nuclei". In: Scientific American 272.6 (1995), pp. 90–95.
- George F Bertsch. "Vibrations of the atomic nucleus". In: Scientific American 248.5 (1983), pp. 62–73.
- Niels Bohr and John Archibald Wheeler. "The mechanism of nuclear fission". In: Physical Review 56.5 (1939), p. 426.
- Enrico Fermi. Notes on quantum mechanics. University of Chicago Press, 1995.
- Enrico Fermi. Nuclear physics: a course given by Enrico Fermi at the University of Chicago. University of Chicago Press, 1950.

Bibliography II

S Frankel and N Metropolis. "Calculations in the liquid-drop model of fission". In: Physical Review 72.10 (1947), p. 914.

JH Hamilton and JA Maruhn. "Exotic atomic nuclei". In: Scientific American 255.1 (1986), pp. 80–89.

Robert Hofstadter. "The atomic nucleus". In: Scientific American 195.1 (1956), pp. 55–70.

Maria G Mayer. "The structure of the nucleus". In: Scientific American 184.3 (1951), pp. 22–27.

RE Peierls. "The atomic nucleus". In: Scientific American 200.1 (1959), pp. 75–84.

Bibliography III

B Cameron Reed. “A physicists guide to The Los Alamos Primer”. In: Physica Scripta 91.11 (2016), p. 113002.

B Cameron Reed. “The Bohr–Wheeler spontaneous fission limit: an undergraduate-level derivation”. In: European journal of physics 30.4 (2009), p. 763.

Bruce Cameron Reed and Dimitris Mihailidis. the Physics of the Manhattan Project. Springer, 2011.

G Sansone. Orthogonal Functions. Interscience, 1959.

Robert Serber. Los Alamos Primer. Tech. rep. 1943.

Robert Serber. “The Los Alamos Primer”. In: The Los Alamos Primer. University of California Press, 2020.

Bibliography IV

Victor F Weisskopf and EP Rosenbaum. "A Model of the Nucleus". In: Scientific American 193.6 (1955), pp. 84–91.

Nuclear Fission Using Algebra and Calculus

Daniel Topa
daniel.m.topa.ctr@mail.mil

Defense Threat Reduction Information Analysis Center (DTRIAC)

January 16, 2023