Measuring the gradient

Examples

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1 $n = 2$

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \mathbf{A}^\dagger = \frac{1}{2}\begin{pmatrix} -2 & -1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix} \tag{1.1}$$

$$\mathbf{A} = \mathbf{Y}\,\Sigma\,\mathbf{X}^\mathrm{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{6}}(1 \ \ -2 \ \ 1) \\ \frac{1}{\sqrt{2}}(-1 \ \ 0 \ \ 1) \\ \frac{1}{\sqrt{3}}(1 \ \ 1 \ \ 1) \end{pmatrix} \tag{1.2}$$

$$\Phi = \mathbf{A}^\dagger\mathbf{Y} = \begin{pmatrix} -2y_1 - y_2 \\ y_1 - y_2 \\ y_1 + 2y_2 \end{pmatrix} \tag{1.3}$$

2 $n = 3$

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \qquad \mathbf{A}^\dagger = \frac{1}{12}\begin{pmatrix} -9 & -6 & -3 \\ 3 & -6 & -3 \\ 3 & 6 & -3 \\ 3 & 6 & 9 \end{pmatrix} \tag{2.1}$$

$$\mathbf{A} = \mathbf{Y}\,\Sigma\,\mathbf{X}^{\mathrm{T}} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \sqrt{2+\sqrt{2}} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2-\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{2\sqrt{2+\sqrt{2}}} & \frac{1}{2}\sqrt{1+\frac{1}{\sqrt{2}}} & -\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{2}}} & \frac{1}{2\sqrt{2+\sqrt{2}}} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{2}}} & -\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{2}}} & \frac{1}{2}\sqrt{1-\frac{1}{\sqrt{2}}} & \frac{1}{2}\sqrt{1+\frac{1}{\sqrt{2}}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(2.2)

$$\Phi = \mathbf{A}^{\dagger}\mathbf{Y} = \frac{1}{12}\begin{pmatrix} -9y_1 - 6y_2 - 3y_3 \\ 3y_1 - 6y_2 - 3y_3 \\ 3y_1 + 6y_2 - 3y_3 \\ 3y_1 + 6y_2 + 9y_3 \end{pmatrix}$$

(2.3)

3 $n = 4$

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \qquad \mathbf{A}^{\dagger} = \frac{1}{15}\begin{pmatrix} -12 & -9 & -6 & -3 \\ 3 & -9 & -6 & -3 \\ 3 & 6 & -6 & -3 \\ 3 & 6 & 9 & -3 \\ 3 & 6 & 9 & 12 \end{pmatrix}$$

(3.1)

$$\mathbf{A} = \mathbf{Y}\,\Sigma\,\mathbf{X}^{\mathrm{T}}$$

$$\mathbf{Y} = \begin{pmatrix}
-\frac{1}{\sqrt{5+\sqrt{5}}} & \frac{1}{\sqrt{5-\sqrt{5}}} & -\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}} & \frac{1}{\sqrt{5+\sqrt{5}}} \\
\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}} & -\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}} & -\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}} & \frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}} \\
-\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}} & -\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}} & \frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}} & \frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}} \\
\frac{1}{\sqrt{5+\sqrt{5}}} & \frac{1}{\sqrt{5-\sqrt{5}}} & \frac{1}{\sqrt{5-\sqrt{5}}} & \frac{1}{\sqrt{5+\sqrt{5}}}
\end{pmatrix}$$

$$\Sigma = \begin{pmatrix}
\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2}\left(1+\sqrt{5}\right) & 0 & 0 & 0 \\
0 & 0 & \sqrt{\frac{1}{2}\left(5-\sqrt{5}\right)} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}\left(-1+\sqrt{5}\right) & 0
\end{pmatrix} \tag{3.2}$$

$$\mathbf{X}^{\mathrm{T}} = \begin{pmatrix}
\frac{1}{\sqrt{5\left(3+\sqrt{5}\right)}} & -\frac{1}{2}\sqrt{\frac{3}{5}+\frac{1}{\sqrt{5}}} & \sqrt{\frac{2}{5}} & -\frac{1}{2}\sqrt{\frac{3}{5}+\frac{1}{\sqrt{5}}} & \frac{1}{\sqrt{5\left(3+\sqrt{5}\right)}} \\
-\frac{1}{\sqrt{5+\sqrt{5}}} & \frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}} & 0 & -\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}} & \frac{1}{\sqrt{5+\sqrt{5}}} \\
\frac{1}{\sqrt{15-5\sqrt{5}}} & -\frac{1}{2}\sqrt{\frac{1}{5}\left(3-\sqrt{5}\right)} & -\sqrt{\frac{2}{5}} & -\frac{1}{2}\sqrt{\frac{1}{5}\left(3-\sqrt{5}\right)} & \frac{1}{\sqrt{15-5\sqrt{5}}} \\
-\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}} & -\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}} & 0 & \frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}} & \frac{1}{\sqrt{5-\sqrt{5}}} \\
\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}}
\end{pmatrix}$$

$$\Phi = \mathbf{A}^{\dagger}\mathbf{Y} = \frac{1}{60}\begin{pmatrix}
-48y_1 - 36y_2 - 24y_3 - 12y_4 \\
12y_1 - 36y_2 - 24y_3 - 12y_4 \\
12y_1 + 24y_2 - 24y_3 - 12y_4 \\
12y_1 + 24y_2 + 36y_3 - 12y_4 \\
12y_1 + 24y_2 + 36y_3 + 48y_4
\end{pmatrix} \tag{3.3}$$

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