Contents

Part I

Preliminaries

Chapter 1

The Fundamental Theorem of Linear Algebra

Out of intense complexities intense simplicities emerge.
—Winston Churchill

Three Rules of Work: Out of clutter find simplicity. From discord find harmony. In the middle of difficulty lies opportunity.
—Albert Einstein

This chapter recalls basic concepts linear algebra and specifies nomenclature.

1.1 Background

1. basis
2. rank
3. reduction

1.2 The Fundamental Theorem of Linear Algebra

Given $\mathbf{A}^* \in \mathbb{C}^{m\times n}$:

$$\begin{aligned} \mathbb{C}^n &= \mathcal{R}\left(\mathbf{A}^*\right) \oplus \mathcal{N}\left(\mathbf{A}\right), \\ \mathbb{C}^m &= \mathcal{R}\left(\mathbf{A}\right) \oplus \mathcal{N}\left(\mathbf{A}^*\right). \end{aligned} \tag{1.1}$$

Consider sample decompositions. Start with a two dimensional space. Such a space is spanned by a pair of vectors of length two. There are two cases. In the first, the range contains two linearly independent vectors:

$$\mathbb{C}^2 = \underbrace{\operatorname{sp}\left\{ \left[\begin{array}{c} \star \\ \star \end{array} \right], \left[\begin{array}{c} \star \\ \star \end{array} \right] \right\}}_{\text{range}} \oplus \underbrace{\operatorname{sp}\left\{ \left[\begin{array}{c} 0 \\ 0 \end{array} \right] \right\}}_{\text{null space}} \tag{1.2}$$

In this instance the null space is trivial. In the other case, a single vector is in the span of the range and the span of the null space:

$$\mathbb{C}^2 = \underbrace{\text{sp}\left\{ \begin{bmatrix} \star \\ \star \end{bmatrix} \right\}}_{\text{range}} \oplus \underbrace{\text{sp}\left\{ \begin{bmatrix} \star \\ \star \end{bmatrix} \right\}}_{\text{null space}} \tag{1.3}$$

By definition, any vector in the span of the range is perpendicular to any vector in the span of the null space.

The first case is when the null space is trivial:

$$\mathbb{C}^3 = \underbrace{\text{sp}\left\{ \begin{bmatrix} \star \\ \star \\ \star \end{bmatrix}, \begin{bmatrix} \star \\ \star \\ \star \end{bmatrix}, \begin{bmatrix} \star \\ \star \\ \star \end{bmatrix} \right\}}_{\text{range}} \oplus \underbrace{\text{sp}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}}_{\text{null space}} \tag{1.4}$$

The remaining case each has a nontrivial null space:

$$\mathbb{C}^3 = \underbrace{\text{sp}\left\{ \begin{bmatrix} \star \\ \star \\ \star \end{bmatrix}, \begin{bmatrix} \star \\ \star \\ \star \end{bmatrix} \right\}}_{\text{range}} \oplus \underbrace{\text{sp}\left\{ \begin{bmatrix} \star \\ \star \\ \star \end{bmatrix} \right\}}_{\text{null space}} \tag{1.5}$$

$$\mathbb{C}^3 = \underbrace{\text{sp}\left\{ \begin{bmatrix} \star \\ \star \\ \star \end{bmatrix} \right\}}_{\text{range}} \oplus \underbrace{\text{sp}\left\{ \begin{bmatrix} \star \\ \star \\ \star \end{bmatrix}, \begin{bmatrix} \star \\ \star \\ \star \end{bmatrix} \right\}}_{\text{null space}} \tag{1.6}$$

1.2.1 Sample reductions

There are two different ways to resolve $\mathbb{C}^2$:

$$\mathbb{C}^2 = \text{sp}\left\{ \begin{bmatrix} \star \\ \star \end{bmatrix}, \begin{bmatrix} \star \\ \star \end{bmatrix} \right\} \tag{1.7}$$

$$\begin{aligned} \mathbb{C}^m &= \mathcal{R}(\mathbf{A}) = \text{sp}\left\{ \begin{bmatrix} \star \\ \star \end{bmatrix}, \begin{bmatrix} \star \\ \star \end{bmatrix} \right\} \\ \mathbb{C}^n &= \mathcal{R}(\mathbf{A}^*) \oplus \mathcal{N}(\mathbf{A}) = \text{sp}\left\{ \begin{bmatrix} \star \\ \star \\ \star \end{bmatrix}, \begin{bmatrix} \star \\ \star \\ \star \end{bmatrix} \right\} \oplus \text{sp}\left\{ \begin{bmatrix} \star \\ \star \\ \star \end{bmatrix} \right\} \end{aligned} \tag{1.8}$$

1.2.2 Subspace decompositions

Row and column rank deficiency

The initial example is a matrix which has both row and column rank deficiency. The first the column space is to reduce the matrix $\mathbf{A}$ to the row echelon form $\mathbf{E}$.

The matrix is

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{1.9}$$

This matrix takes a 2−vector and returns a 3−vector. Notice that the second column is the negative of the first column. By inspection we see there is one linearly independent column and therefore this is a rank one matrix. The image of this matrix is the line through the origin and the point $(1, -1, 1)^{\mathrm{T}}$.

Similarly, we you look at the transpose matrix, we see again that there is only one linearly independent vector and the image of the transpose matrix is the line through the origin and the point $(1, -1)^{\mathrm{T}}$.

For instructional purpose, proceed with the reductions to row echelon form. The pivot elements are boxed. The matrix reduces to the following:

$$\mathbf{R}\left[\begin{array}{c|c} \mathbf{A} & \mathbf{I}_3 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{E} & \mathbf{R} \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \left[\begin{array}{rr|rrr} 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{rr|rrr} \boxed{1} & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{array}\right] \tag{1.10}$$

The range of the matrix **A** is given by the basic columns of **A**. Since there is but one pivot there is but one basic column:

$$\mathcal{R}(\mathbf{A}) = \operatorname{sp}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}. \tag{1.11}$$

The null space vectors are associated with the zero pivots in the reduced matrix **E**. These and subsequent null space vectors are shaded with a light gray. The null space vectors span the null space:

$$\mathcal{N}\left(\mathbf{A}^{\mathrm{T}}\right) = \operatorname{sp}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}. \tag{1.12}$$

Recall that these are the null space vectors for the transpose matrix. For example,

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}. \tag{1.13}$$

The codomain, or range or image space or column space, then is resolved as

$$\begin{aligned} \mathbb{C}^3 &= \mathcal{R}(\mathbf{A}) \oplus \mathcal{N}\left(\mathbf{A}^{\mathrm{T}}\right) \\ &= \operatorname{sp}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} \oplus \operatorname{sp}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}. \end{aligned} \tag{1.14}$$

To resolve the row space, resolve the column space of the transpose matrix.

$$\mathbf{R}\left[\begin{array}{c|c} \mathbf{A}^{\mathrm{T}} & \mathbf{I}_2 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{E} & \mathbf{R} \end{array}\right]$$

$$\left[\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right]\left[\begin{array}{rrr|rr} 1 & -1 & 1 & 1 & 0 \\ -1 & 1 & -1 & 0 & 1 \end{array}\right] = \left[\begin{array}{rrr|rr} \boxed{1} & -1 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 \end{array}\right] \tag{1.15}$$

The range of the matrix $\mathbf{A}^{\mathrm{T}}$ is given by the basic columns of $\mathbf{A}^{\mathrm{T}}$. Again there is a single pivot and one basic column:

$$\mathcal{R}\left(\mathbf{A}^{\mathrm{T}}\right) = \mathrm{sp}\left\{\left[\begin{array}{r} 1 \\ -1 \end{array}\right]\right\}. \tag{1.16}$$

The null space vector is associated with the zero pivot in the reduced matrix $\mathbf{E}$, and is the shaded vector:

$$\mathcal{N}\left(\mathbf{A}\right) = \mathrm{sp}\left\{\left[\begin{array}{c} 1 \\ 1 \end{array}\right]\right\}. \tag{1.17}$$

The domain is resolved as

$$\begin{aligned} \mathbb{C}^2 &= \mathcal{R}\left(\mathbf{A}^{\mathrm{T}}\right) \oplus \mathcal{N}\left(\mathbf{A}\right) \\ &= \mathrm{sp}\left\{\left[\begin{array}{r} 1 \\ -1 \end{array}\right]\right\} \oplus \mathrm{sp}\left\{\left[\begin{array}{c} 1 \\ 1 \end{array}\right]\right\}. \end{aligned} \tag{1.18}$$

All four fundamental spaces are resolved and we see that $\mathbf{A} \in \mathbb{R}^{3\times 2}_1$.

Column rank deficiency

The next example matrix $\mathbf{A} \in \mathbb{R}^{2\times 3}$ has full row rank, but has a column rank deficiency. We know this matrix must have a column rank deficiency by inspecting the matrix dimensions: $m < n$. There are fewer rows than columns and we cannot have a case of full column rank. The matrix is given by

$$\mathbf{A} = \left[\begin{array}{ccc} 0 & 3 & 0 \\ 1 & 2 & 2 \end{array}\right]. \tag{1.19}$$

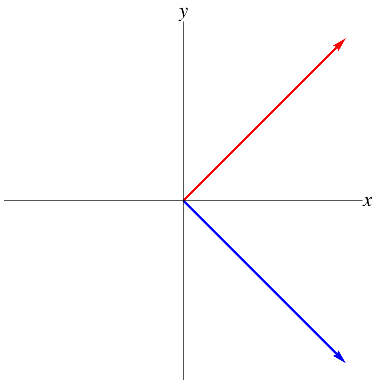
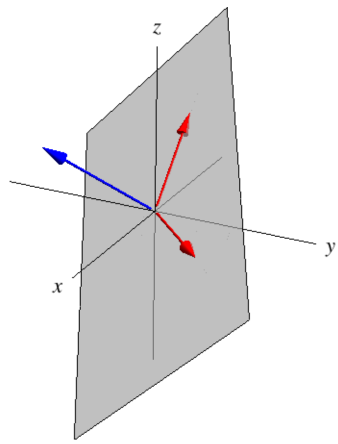
This matrix takes a 3−vector and returns a 2−vector. There is no need to reduce this matrix. By inspection we see that it is equivalent to a reduced form

$$\left[\begin{array}{ccc} 0 & 3 & 0 \\ 1 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc} \boxed{1} & 2 & 2 \\ 0 & \boxed{3} & 0 \end{array}\right]. \tag{1.20}$$

There are two nonzero pivots and therefore the first two columns of the matrix $\mathbf{A}$ are basic. These columns define the span of the range of $\mathbf{A}$:

$$\mathcal{R}\left(\mathbf{A}\right) = \mathrm{sp}\left\{\left[\begin{array}{c} 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 2 \\ 3 \end{array}\right]\right\}. \tag{1.21}$$

Table 1.1. *The Fundamental Theorem of Linear Algebra in pictures for the matrix in equation* (1.9)*. Range vectors are colored blue, null space vectors are colored red. For the codomain the null space vectors lie in the shaded plane.*

Domain Row space	Codomain Column space
	
$\mathbb{C}^2 = \mathcal{R}(\mathbf{A}^*) \oplus \mathcal{N}(\mathbf{A})$	$\mathbb{C}^3 = \mathcal{R}(\mathbf{A}) \oplus \mathcal{N}(\mathbf{A}^*)$
$\mathbb{C}^2 = \text{sp}\left\{\left[\begin{array}{r} 1 \\ -1 \end{array}\right]\right\} \oplus \text{sp}\left\{\left[\begin{array}{r} 1 \\ 1 \end{array}\right]\right\}$	$\mathbb{C}^3 = \text{sp}\left\{\left[\begin{array}{r} 1 \\ -1 \\ 1 \end{array}\right]\right\} \oplus \text{sp}\left\{\left[\begin{array}{r} 1 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right]\right\}$
$\mathbf{A} = \left[\begin{array}{rr} 1 & -1 \\ \hline -1 & 1 \\ \hline 1 & -1 \end{array}\right]$	$\mathbf{A} = \left[\begin{array}{r\|r} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{array}\right]$

Using the rank plus nullity theorem, we see that the null space must be trivial. That is

$$\mathcal{N}\left(\mathbf{A}^{\mathrm{T}}\right) = \left\{\left[\begin{array}{c} 0 \\ 0 \end{array}\right]\right\} = \{\mathbf{0}\}. \tag{1.22}$$

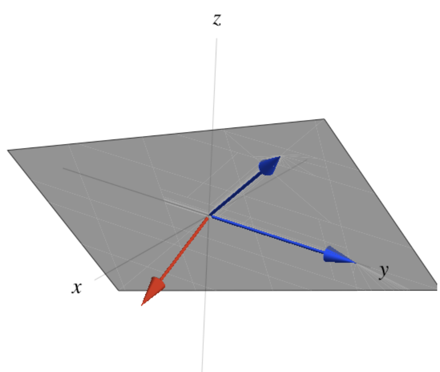
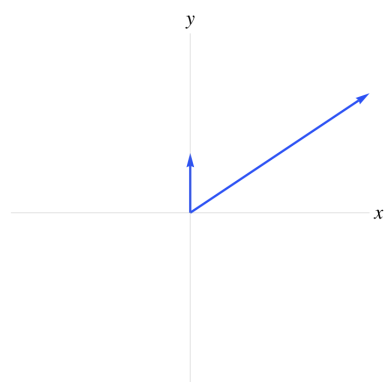
To reinforce this point, note that $\mathbb{R}^2$ will be spanned by two linearly independent 2−vectors. Two such vectors are identified in equation (1.21).

Now for the row space. Again we resolve the column space of the transpose matrix. More specifically, the equivalent form in equation (1.20):

$$
\begin{gathered}
\mathbf{R}\left[\begin{array}{c|c} \mathbf{A} & \mathbf{I}_3 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{E} & \mathbf{R} \end{array}\right] \\
\left[\begin{array}{rrr} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{array}\right]
\left[\begin{array}{rr|rrr} 1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array}\right]
=
\left[\begin{array}{rr|rrr} \boxed{1} & 0 & 1 & 0 & 0 \\ 0 & \boxed{3} & -2 & 1 & 0 \\ \hline 0 & 0 & -2 & 0 & 1 \end{array}\right]
\end{gathered}
\tag{1.23}
$$

All four fundamental spaces are resolved and we see that $\mathbf{A} \in \mathbb{R}_2^{2\times 3}$.

Table 1.2. *The Fundamental Theorem of Linear Algebra in pictures for the matrix in equation* (1.19)*. Range vectors are colored blue, null space vectors are colored red. For the domain the range vectors lie in the shaded plane.*

Domain **Row space**	**Codomain** **Column space**
 	
$\mathbb{C}^2 = \mathcal{R}\left(\mathbf{A}^*\right) \oplus \mathcal{N}\left(\mathbf{A}\right)$	$\mathbb{C}^3 = \mathcal{R}\left(\mathbf{A}\right) \oplus \mathcal{N}\left(\mathbf{A}^*\right)$
$\operatorname{sp}\left\{\left[\begin{array}{r} 0 \\ 3 \\ 0 \end{array}\right], \left[\begin{array}{r} 1 \\ 2 \\ 2 \end{array}\right]\right\} \oplus \operatorname{sp}\left\{\left[\begin{array}{r} -2 \\ 0 \\ 1 \end{array}\right]\right\}$	$\operatorname{sp}\left\{\left[\begin{array}{r} 0 \\ 1 \end{array}\right], \left[\begin{array}{r} 3 \\ 2 \end{array}\right]\right\}$
$\mathbf{A} = \left[\begin{array}{rrr} 0 & 3 & 0 \\ \hline 1 & 2 & 2 \end{array}\right]$	$\mathbf{A} = \left[\begin{array}{r\|r\|r} 0 & 3 & 0 \\ 1 & 2 & 2 \end{array}\right]$

Full row and column rank

The next example matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ has full row and column rank deficiency. The matrix is given by

$$\mathbf{A} = \left[\begin{array}{rr} 1 & 2 \\ -1 & 2 \end{array} \right]. \tag{1.24}$$

This matrix takes a 2−vector and returns another 2−vector. The reduction follows:

$$\begin{array}{c} \mathbf{R} \left[\begin{array}{c|c} \mathbf{A} & \mathbf{I}_2 \end{array} \right] = \left[\begin{array}{c|c} \mathbf{E} & \mathbf{R} \end{array} \right] \\ \left[\begin{array}{rr} 1 & 0 \\ 1 & 1 \end{array} \right] \left[\begin{array}{rr|rr} 1 & 2 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array} \right] = \left[\begin{array}{rr|rr} \boxed{1} & 2 & 1 & 0 \\ 0 & \boxed{4} & 1 & 1 \end{array} \right] \end{array} \tag{1.25}$$

The two pivots reveal that both columns of the matrix $\mathbf{A}$ are basic columns of $\mathbf{A}$. These two column vectors define the span of the range:

$$\mathcal{R}(\mathbf{A}) = \text{sp} \left\{ \left[\begin{array}{r} 1 \\ -1 \end{array} \right], \left[\begin{array}{r} 2 \\ 2 \end{array} \right] \right\}. \tag{1.26}$$

The null space is trivial:

$$\mathcal{N}(\mathbf{A}^*) = \left\{ \left[\begin{array}{r} 0 \\ 0 \end{array} \right] \right\}. \tag{1.27}$$

Now we turn to the transpose matrix to resolve the row space. Since the matrix is full rank, we expect two pivots. All four fundamental spaces are resolved and we see that $\mathbf{A} \in \mathbb{R}_2^{2\times 3}$

$$\begin{array}{c} \mathbf{R} \left[\begin{array}{c|c} \mathbf{A}^{\mathrm{T}} & \mathbf{I}_2 \end{array} \right] = \left[\begin{array}{c|c} \mathbf{E} & \mathbf{R} \end{array} \right] \\ \left[\begin{array}{rr} 1 & 0 \\ -1 & 1 \end{array} \right] \left[\begin{array}{lr|rr} 1 & -1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array} \right] = \left[\begin{array}{rr|rr} \boxed{1} & -1 & 1 & 0 \\ 0 & \boxed{4} & -2 & 1 \end{array} \right] \end{array} \tag{1.28}$$

The column vectors of the transpose matrix define the span of the range of the domain:

$$\mathcal{R}(\mathbf{A}^*) = \text{sp} \left\{ \left[\begin{array}{r} 1 \\ 2 \end{array} \right], \left[\begin{array}{r} -1 \\ 2 \end{array} \right] \right\}. \tag{1.29}$$

Again, the null space is trivial:

$$\mathcal{N}(\mathbf{A}) = \left\{ \left[\begin{array}{r} 0 \\ 0 \end{array} \right] \right\} = \{\mathbf{0}\}. \tag{1.30}$$

1.2.3 FTOLA tables

Table 1.3. *The Fundamental Theorem of Linear Algebra in pictures for the matrix in equation* (1.24)*. Range vectors are colored blue. Both null spaces are trivial*

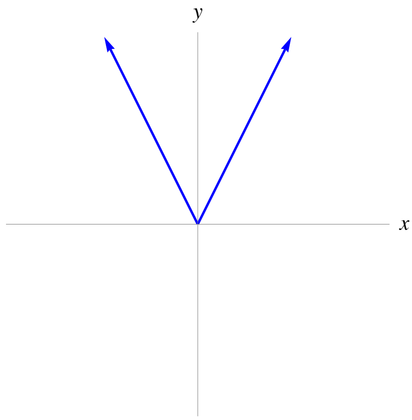
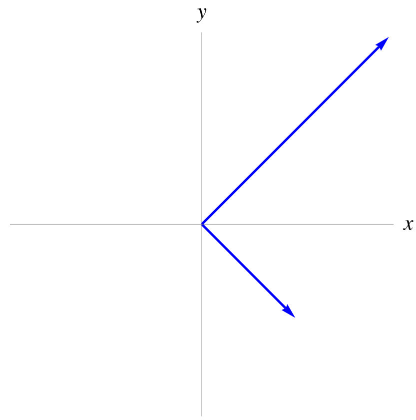
Domain **Row space**	**Codomain** **Column space**
	
$\mathbb{C}^2 = \mathcal{R}\left(\mathbf{A}^*\right) \oplus \mathcal{N}\left(\mathbf{A}\right)$	$\mathbb{C}^2 = \mathcal{R}\left(\mathbf{A}\right) \oplus \mathcal{N}\left(\mathbf{A}^*\right)$
$\mathbb{C}^2 = \text{sp}\left\{ \left[\begin{array}{r} 1 \\ 2 \end{array} \right], \left[\begin{array}{r} -1 \\ 2 \end{array} \right] \right\}$	$\mathbb{C}^2 = \text{sp}\left\{ \left[\begin{array}{r} 1 \\ -1 \end{array} \right], \left[\begin{array}{r} 2 \\ 2 \end{array} \right] \right\}$
$\mathbf{A} = \left[\begin{array}{rr} 1 & 2 \\ \hline -1 & 2 \end{array} \right]$	$\mathbf{A} = \left[\begin{array}{r\|r} 1 & 2 \\ -1 & 2 \end{array} \right]$

Table 1.4. *The fundamental subspaces for the sample matrix with both row and column rank deficiency. These rank deficiencies imply nontrivial null spaces for the domain and codomain respectively. Both null spaces are nontrivial.*

Domain: row space	**Codomain: column space**
$\mathbf{A}^{\mathrm{T}} = \left[\begin{array}{r\|r\|r} 1 & -1 & 1 \\ -1 & 1 & -1 \end{array}\right]$	$\mathbf{A} = \left[\begin{array}{r\|r} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{array}\right]$
$\mathcal{R}\left(\mathbf{A}^{*}\right) = \operatorname{sp}\left\{\left[\begin{array}{r} 1 \\ -1 \end{array}\right]\right\}$	$\mathcal{R}\left(\mathbf{A}\right) = \operatorname{sp}\left\{\left[\begin{array}{r} 1 \\ -1 \\ 1 \end{array}\right]\right\}$
$\mathcal{N}\left(\mathbf{A}\right) = \operatorname{sp}\left\{\left[\begin{array}{r} 1 \\ 1 \end{array}\right]\right\}$	$\mathcal{N}\left(\mathbf{A}^{*}\right) = \operatorname{sp}\left\{\left[\begin{array}{r} 1 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right]\right\}$
$\mathbb{C}^{n} = \mathcal{R}\left(\mathbf{A}^{*}\right) \oplus \mathcal{N}\left(\mathbf{A}\right)$	$\mathbb{C}^{m} = \mathcal{R}\left(\mathbf{A}\right) \oplus \mathcal{N}\left(\mathbf{A}^{*}\right)$
$\mathbb{C}^{2} = \operatorname{sp}\left\{\left[\begin{array}{r} 1 \\ -1 \end{array}\right]\right\} \oplus \operatorname{sp}\left\{\left[\begin{array}{r} 1 \\ 1 \end{array}\right]\right\}$	$\mathbb{C}^{3} = \operatorname{sp}\left\{\left[\begin{array}{r} 1 \\ -1 \\ 1 \end{array}\right]\right\} \oplus \operatorname{sp}\left\{\left[\begin{array}{r} 1 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right]\right\}$

Table 1.5. *The fundamental subspaces for the sample matrix with column rank deficiency. This rank deficiency implies a nontrivial null space for the codomain.*

Domain: row space

$$\mathbf{A}^{\mathrm{T}} = \left[\begin{array}{c|c} 0 & 1 \\ 3 & 2 \\ 0 & 2 \end{array}\right]$$

$$\mathcal{R}\left(\mathbf{A}^{*}\right) = \operatorname{sp}\left\{ \left[\begin{array}{c} 0 \\ 3 \\ 0 \end{array}\right], \left[\begin{array}{c} 1 \\ 2 \\ 2 \end{array}\right] \right\}$$

$$\mathcal{N}\left(\mathbf{A}\right) = \operatorname{sp}\left\{ \left[\begin{array}{r} -2 \\ 0 \\ 1 \end{array}\right] \right\}$$

$$\mathbb{C}^{n} = \mathcal{R}\left(\mathbf{A}^{*}\right) \oplus \mathcal{N}\left(\mathbf{A}\right)$$

$$\mathbb{C}^{3} = \operatorname{sp}\left\{ \left[\begin{array}{c} 0 \\ 3 \\ 0 \end{array}\right], \left[\begin{array}{c} 1 \\ 2 \\ 2 \end{array}\right] \right\} \oplus \operatorname{sp}\left\{ \left[\begin{array}{r} -2 \\ 0 \\ 1 \end{array}\right] \right\}$$

Codomain: column space

$$\mathbf{A} = \left[\begin{array}{c|c|c} 0 & 3 & 0 \\ 1 & 2 & 2 \end{array}\right]$$

$$\mathcal{R}\left(\mathbf{A}\right) = \operatorname{sp}\left\{ \left[\begin{array}{c} 0 \\ 1 \end{array}\right], \left[\begin{array}{c} 3 \\ 2 \end{array}\right] \right\}$$

$$\mathcal{N}\left(\mathbf{A}^{*}\right) = \left\{ \left[\begin{array}{c} 0 \\ 0 \end{array}\right] \right\}$$

$$\mathbb{C}^{m} = \mathcal{R}\left(\mathbf{A}\right) \oplus \mathcal{N}\left(\mathbf{A}^{*}\right)$$

$$\mathbb{C}^{2} = \operatorname{sp}\left\{ \left[\begin{array}{c} 0 \\ 1 \end{array}\right], \left[\begin{array}{c} 3 \\ 2 \end{array}\right] \right\}$$

Table 1.6. *The fundamental subspaces for the sample matrix with full row and column rank. There are no nontrivial null spaces for this transformation.*

Domain: row space	**Codomain: column space**
$\mathbf{A}^{\mathrm{T}} = \left[\begin{array}{c\|c} 1 & -1 \\ 2 & 2 \end{array}\right]$	$\mathbf{A} = \left[\begin{array}{c\|c} 1 & 2 \\ -1 & 2 \end{array}\right]$
$\mathcal{R}\left(\mathbf{A}^{*}\right) = \mathrm{sp}\left\{\left[\begin{array}{r} 1 \\ 2 \end{array}\right], \left[\begin{array}{r} -1 \\ 2 \end{array}\right]\right\}$	$\mathcal{R}\left(\mathbf{A}\right) = \mathrm{sp}\left\{\left[\begin{array}{r} 1 \\ -1 \end{array}\right], \left[\begin{array}{r} 2 \\ 2 \end{array}\right]\right\}$
$\mathcal{N}\left(\mathbf{A}\right) = \mathrm{sp}\left\{\left[\begin{array}{r} 0 \\ 0 \end{array}\right]\right\}$	$\mathcal{N}\left(\mathbf{A}^{*}\right) = \left\{\left[\begin{array}{r} 0 \\ 0 \end{array}\right]\right\}$
$\mathbb{C}^{n} = \mathcal{R}\left(\mathbf{A}^{*}\right) \oplus \mathcal{N}\left(\mathbf{A}\right)$	$\mathbb{C}^{m} = \mathcal{R}\left(\mathbf{A}\right) \oplus \mathcal{N}\left(\mathbf{A}^{*}\right)$
$\mathbb{C}^{3} = \mathrm{sp}\left\{\left[\begin{array}{r} 1 \\ 2 \end{array}\right], \left[\begin{array}{r} -1 \\ 2 \end{array}\right]\right\}$	$\mathbb{C}^{2} = \mathrm{sp}\left\{\left[\begin{array}{r} 1 \\ -1 \end{array}\right], \left[\begin{array}{r} 2 \\ 2 \end{array}\right]\right\}$

1.3 Rudiments

Begin with a matrix $\mathbf{A}$ which has m rows and n columns. Entries in the matrix may be complex numbers. Then the matrix $\mathbf{A}$ is in the space of complex matrices with m rows and n columns:

$$\mathbf{A} \in \mathbb{C}^{m \times n}.$$

- The matrix $\mathbf{A}$ maps $n-$vectors into $m-$vectors.
- The matrix $\mathbf{A}^*$ maps $m-$vectors into $n-$vectors.

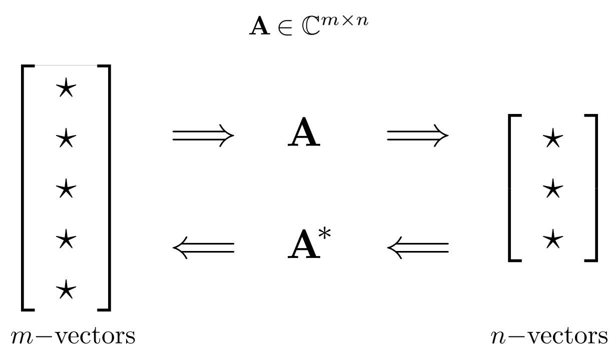


Figure 1.1. *The mapping action of the matrix* $\boldsymbol{A} \in \mathbb{C}^{m \times n}$.

1.3.1 The unit circle

1.4 Beyond the Fundamental Theorem

To some the Fundamental Theorem of Linear Algebra is a main course, to others it is an hors d'oeuvre whetting the appetite.

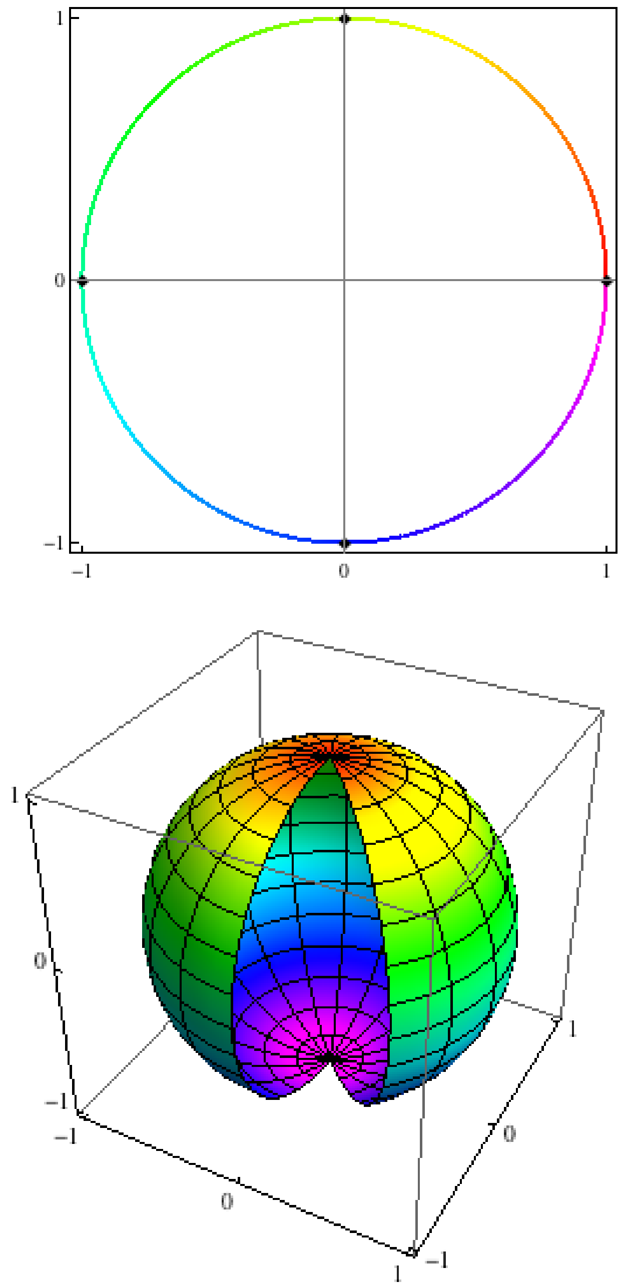


Figure 1.2. *The unit sphere in 2 and 3 dimensions.*

Table 1.7. *Mapping the unit sphere*

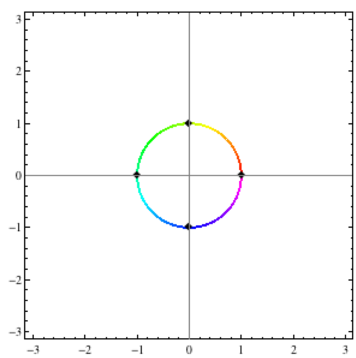
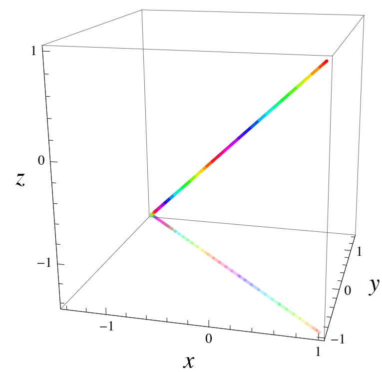
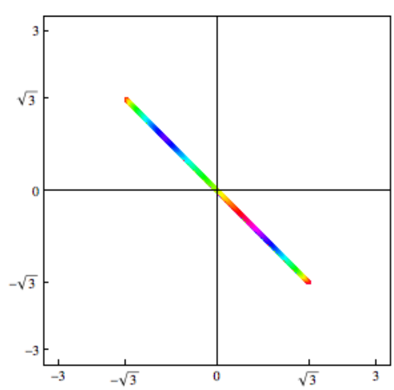
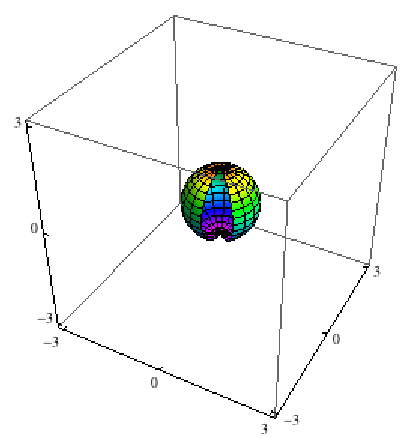
Domain: 2−**vectors**		**Codomain:** 3−**vectors**
	$\Longrightarrow$	
$x = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$		$\mathbf{A}x = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$
	$\Longleftarrow$	
$\mathbf{A}^{\mathrm{T}}y = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \cos\phi\,\sin\theta \\ \sin\phi\,\sin\theta \\ \cos\theta \end{bmatrix}$		$y = \begin{bmatrix} \cos\phi\,\sin\theta \\ \sin\phi\,\sin\theta \\ \cos\theta \end{bmatrix}$

Table 1.8. *Mapping the unit sphere*

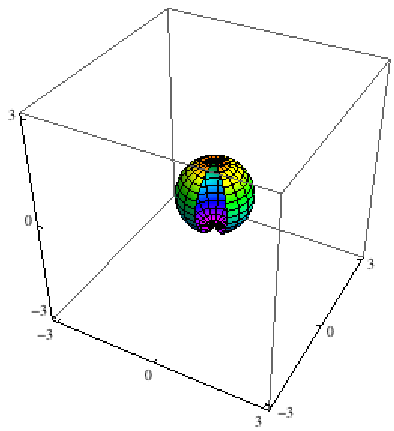
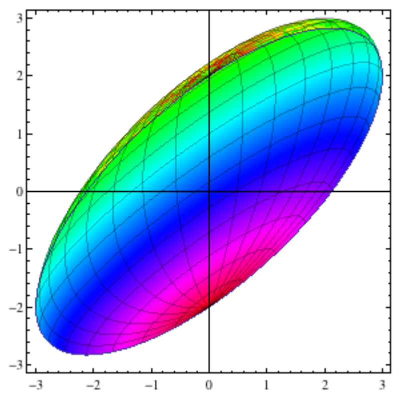
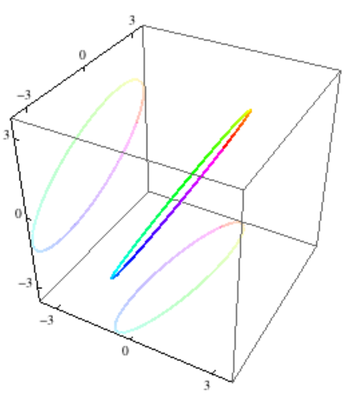
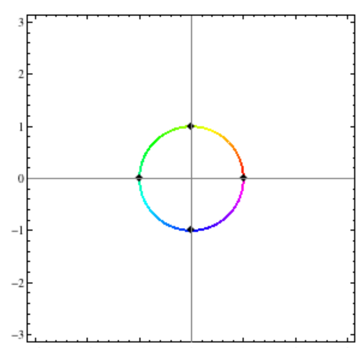
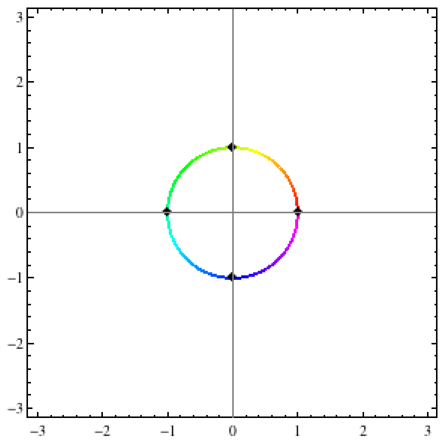
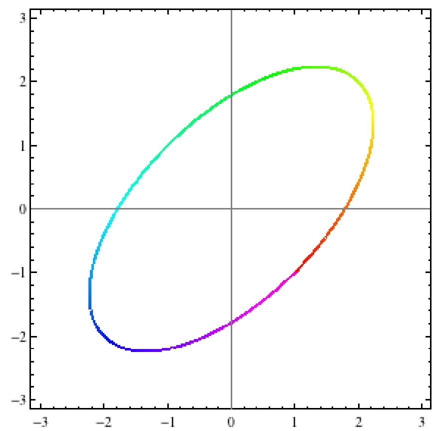
Domain: $3-$**vectors**		**Codomain:** $2-$**vectors**
	$\Longrightarrow$	
$x = \begin{bmatrix} \cos\phi\,\sin\theta \\ \sin\phi\,\sin\theta \\ \cos\theta \end{bmatrix}$		$\mathbf{A}x = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} \cos\phi\,\sin\theta \\ \sin\phi\,\sin\theta \\ \cos\theta \end{bmatrix}$
	$\Longleftarrow$	
$\mathbf{A}^{\mathrm{T}}y = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$		$y = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

Table 1.9. *Mapping the unit sphere*

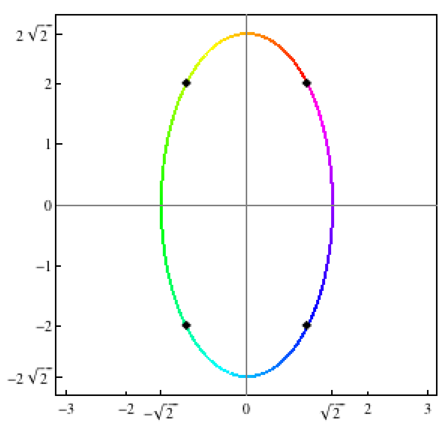
Domain: 2−**vectors**		**Codomain:** 2−**vectors**



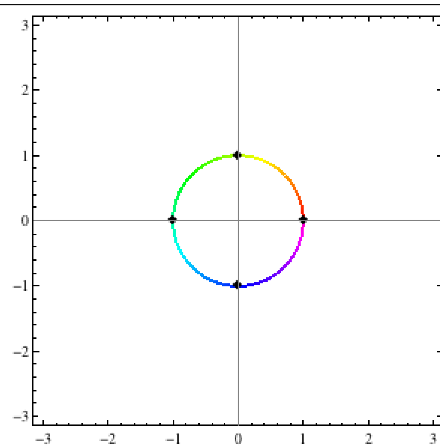
$\Longrightarrow$



$$x = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad \mathbf{A}x = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$



$\Longleftarrow$



$$\mathbf{A}^{\mathrm{T}}y = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad y = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Chapter 2

The Singular Value Decomposition

Out of intense complexities intense simplicities emerge.
—Winston Churchill

Three Rules of Work: Out of clutter find simplicity. From discord find harmony. In the middle of difficulty lies opportunity.
—Albert Einstein

The singular value decomposition represents a vantage point in linear algebra. Like other summits, there are well-worn paths to the peak. A few shall be explored in this work.

2.1 Refining the Fundamental Theorem

To some the Fundamental Theorem of Linear Algebra is a main course, to others it is an hors d'oeuvre whetting the appetite.

2.1.1 Dimension and rank

We can say a great deal about the subspaces without even seeing the values in the matrix.

- Given $\mathbf{A} \in \mathbb{C}_{\rho}^{m\times n}$,
- the codomain is $\mathbb{C}^{m}$;
- the domain is $\mathbb{C}^{n}$;
- the matrix rank is ρ.

This can be confusing. For example it means that if the column space has a null space the matrix has a row rank deficiency.

$$
\begin{aligned}
\text{row rank deficiency} \quad &\Longleftrightarrow \quad \mathcal{N}\left(\mathbf{A}^*\right) \text{ is nontrivial}\\
\text{column rank deficiency} \quad &\Longleftrightarrow \quad \mathcal{N}\left(\mathbf{A}\right) \text{ is nontrivial}
\end{aligned}
$$

Row and column rank deficiency

Rank deficiency implies the existence of a null space. In this case both the domain and the codomain will have a nontrivial null space.

- Given $\mathbf{A} \in \mathbb{C}_1^{3\times 2}$,
- the codomain is $\mathbb{C}^3$;
- the domain is $\mathbb{C}^2$;
- the matrix rank is $\rho = 1$.

Because $\rho < m$, the matrix does not have full row rank. Because $\rho < n$, the matrix does not have full column rank.

$$
\begin{aligned}
\mathbb{C}^m &= \mathcal{R}\left(\mathbf{A}\right) \oplus \mathcal{N}\left(\mathbf{A}^*\right) = \text{sp}\left\{\left[\begin{array}{c}\star\\\star\\\star\end{array}\right]\right\} \oplus \text{sp}\left\{\left[\begin{array}{c}\star\\\star\\\star\end{array}\right], \left[\begin{array}{c}\star\\\star\\\star\end{array}\right]\right\}\\
\mathbb{C}^n &= \mathcal{R}\left(\mathbf{A}^*\right) \oplus \mathcal{N}\left(\mathbf{A}\right) = \text{sp}\left\{\left[\begin{array}{c}\star\\\star\end{array}\right]\right\} \oplus \text{sp}\left\{\left[\begin{array}{c}\star\\\star\end{array}\right]\right\}
\end{aligned}
\tag{2.1}
$$

Full row rank

- Given $\mathbf{A} \in \mathbb{C}_2^{2\times 3}$,
- the codomain is $\mathbb{C}^2$;
- the domain is $\mathbb{C}^3$;
- the matrix rank is $\rho = 2$.

Because $\rho = m$, the matrix has full row rank. Because $\rho < n$, the matrix does not have full column rank.

$$
\begin{aligned}
\mathbb{C}^m &= \mathcal{R}\left(\mathbf{A}\right) = \text{sp}\left\{\left[\begin{array}{c}\star\\\star\end{array}\right], \left[\begin{array}{c}\star\\\star\end{array}\right]\right\}\\
\mathbb{C}^n &= \mathcal{R}\left(\mathbf{A}^*\right) \oplus \mathcal{N}\left(\mathbf{A}\right) = \text{sp}\left\{\left[\begin{array}{c}\star\\\star\\\star\end{array}\right], \left[\begin{array}{c}\star\\\star\\\star\end{array}\right]\right\} \oplus \text{sp}\left\{\left[\begin{array}{c}\star\\\star\\\star\end{array}\right]\right\}
\end{aligned}
\tag{2.2}
$$

Full row and column rank

- Given $\mathbf{A} \in \mathbb{C}_2^{2\times 2}$,
- the codomain is $\mathbb{C}^2$;
- the domain is $\mathbb{C}^2$;
- the matrix rank is $\rho = 2$.

Because $\rho = m = n$, the matrix has full rank.

$$\begin{aligned}
\mathbb{C}^m &= \mathcal{R}\left(\mathbf{A}\right) = \text{sp}\left\{ \left[\begin{array}{c} \star \\ \star \end{array} \right], \left[\begin{array}{c} \star \\ \star \end{array} \right] \right\} \\
\mathbb{C}^n &= \mathcal{R}\left(\mathbf{A}^*\right) = \text{sp}\left\{ \left[\begin{array}{c} \star \\ \star \end{array} \right], \left[\begin{array}{c} \star \\ \star \end{array} \right] \right\}
\end{aligned} \tag{2.3}$$

2.1.2 Comparisons

Repetition the root of learning.

2.2 Collection

$$\mathbf{X} = \left[\begin{array}{ccc|ccc} x_1 & \dots & x_\rho & x_{\rho+1} & \dots & x_m \end{array} \right] \tag{2.4}$$

Table 2.1. *The singular value decomposition for the sample matrix with both row and column rank deficiency.*

Linear transformation $\mathbf{A} \in \mathbb{C}_1^{3\times 2}$

$$\mathbf{A} = \mathbf{Y}\,\Sigma\,\mathbf{X}^{\mathrm{T}} = \left[\begin{array}{cc} \mathcal{Y}_{\mathcal{R}} & \mathcal{Y}_{\mathcal{N}} \end{array}\right] \left[\begin{array}{c|c} \mathbf{S} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right] \left[\begin{array}{c} \mathcal{X}_{\mathcal{R}}^{\mathrm{T}} \\ \mathcal{X}_{\mathcal{N}}^{\mathrm{T}} \end{array}\right]$$

$$\left[\begin{array}{rr} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{array}\right] = \left[\begin{array}{ccc} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{array}\right] \left[\begin{array}{c|cc} \sqrt{6} & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \frac{1}{\sqrt{2}} \left[\begin{array}{rr} 1 & -1 \\ 1 & 1 \end{array}\right]$$

$$\begin{aligned} \mathbb{C}^m &= \mathcal{R}(\mathbf{A}) \oplus \mathcal{N}(\mathbf{A}^*) = \mathrm{sp}\{\mathcal{Y}_{\mathcal{R}}\} \oplus \mathrm{sp}\{\mathcal{Y}_{\mathcal{N}}\} \\ \mathbb{C}^n &= \mathcal{R}(\mathbf{A}^*) \oplus \mathcal{N}(\mathbf{A}) = \mathrm{sp}\{\mathcal{X}_{\mathcal{R}}\} \oplus \mathrm{sp}\{\mathcal{X}_{\mathcal{N}}\} \end{aligned}$$

Codomain

$$\mathcal{Y}_{\mathcal{R}} = \left\{ \frac{1}{\sqrt{3}} \left[\begin{array}{r} 1 \\ -1 \\ 1 \end{array}\right] \right\}, \quad \mathcal{Y}_{\mathcal{N}} = \left\{ \frac{1}{\sqrt{2}} \left[\begin{array}{r} 0 \\ 1 \\ 1 \end{array}\right], \frac{1}{\sqrt{6}} \left[\begin{array}{r} 2 \\ 1 \\ -1 \end{array}\right] \right\}$$

$$\mathbb{C}^m = \mathrm{span} \left\{ \frac{1}{\sqrt{3}} \left[\begin{array}{r} 1 \\ -1 \\ 1 \end{array}\right] \right\} \oplus \mathrm{span} \left\{ \frac{1}{\sqrt{2}} \left[\begin{array}{r} 0 \\ 1 \\ 1 \end{array}\right], \frac{1}{\sqrt{6}} \left[\begin{array}{r} 2 \\ 1 \\ -1 \end{array}\right] \right\}$$

Domain

$$\mathcal{X}_{\mathcal{R}} = \left\{ \frac{1}{\sqrt{2}} \left[\begin{array}{r} 1 \\ -1 \end{array}\right] \right\}, \quad \mathcal{X}_{\mathcal{N}} = \left\{ \frac{1}{\sqrt{2}} \left[\begin{array}{r} 1 \\ 1 \end{array}\right] \right\}$$

$$\mathbb{C}^n = \mathrm{span} \left\{ \frac{1}{\sqrt{2}} \left[\begin{array}{r} 1 \\ -1 \end{array}\right] \right\} \oplus \mathrm{span} \left\{ \frac{1}{\sqrt{2}} \left[\begin{array}{r} 1 \\ 1 \end{array}\right] \right\}$$

Scale factors

$$\mathbf{S} = \left[\begin{array}{c} \sqrt{6} \end{array}\right]$$

Table 2.2. *The singular value decomposition for the sample matrix with full row rank.*

Linear transformation $\mathbf{A} \in \mathbb{C}_2^{2\times 3}$

$$\mathbf{A} = \mathbf{Y}\,\Sigma\,\mathbf{X}^{\mathrm{T}} = \left[\begin{array}{c} \mathcal{Y}_{\mathcal{R}} \end{array}\right] \left[\begin{array}{c|c} \mathbf{S} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right] \left[\begin{array}{c} \mathcal{X}_{\mathcal{R}}^{\mathrm{T}} \\ \mathcal{X}_{\mathcal{N}}^{\mathrm{T}} \end{array}\right]$$

$$\left[\begin{array}{ccc} 0 & 3 & 0 \\ 1 & 2 & 2 \end{array}\right] = \frac{1}{\sqrt{2}} \left[\begin{array}{rr} 1 & -1 \\ 1 & 1 \end{array}\right] \left[\begin{array}{cc|c} \sqrt{15} & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{array}\right] \left[\begin{array}{ccc} \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} & \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{array}\right]$$

$$\mathbb{C}^m = \mathcal{R}(\mathbf{A}) = \mathrm{sp}\left\{\mathcal{Y}_{\mathcal{R}}\right\}$$
$$\mathbb{C}^n = \mathcal{R}(\mathbf{A}^*) \oplus \mathcal{N}(\mathbf{A}) = \mathrm{sp}\left\{\mathcal{X}_{\mathcal{R}}\right\} \oplus \mathrm{sp}\left\{\mathcal{X}_{\mathcal{N}}\right\}$$

Domain

$$\mathcal{X}_{\mathcal{R}} = \left\{ \frac{1}{\sqrt{30}} \left[\begin{array}{c} 1 \\ 5 \\ 2 \end{array}\right], \frac{1}{\sqrt{5}} \left[\begin{array}{c} 1 \\ 5 \\ 2 \end{array}\right] \right\}, \quad \mathcal{X}_{\mathcal{N}} = \left\{ \frac{1}{\sqrt{5}} \left[\begin{array}{r} -2 \\ 0 \\ 1 \end{array}\right] \right\}$$

$$\mathbb{C}^n = \mathrm{span}\left\{ \frac{1}{\sqrt{30}} \left[\begin{array}{c} 1 \\ 5 \\ 2 \end{array}\right], \frac{1}{\sqrt{5}} \left[\begin{array}{c} 1 \\ 5 \\ 2 \end{array}\right] \right\} \oplus \mathrm{span}\left\{ \frac{1}{\sqrt{5}} \left[\begin{array}{r} -2 \\ 0 \\ 1 \end{array}\right] \right\}$$

Codomain

$$\mathcal{Y}_{\mathcal{R}} = \left\{ \frac{1}{\sqrt{2}} \left[\begin{array}{c} 1 \\ 1 \end{array}\right], \frac{1}{\sqrt{2}} \left[\begin{array}{r} 1 \\ -1 \end{array}\right] \right\}$$

$$\mathbb{C}^m = \mathrm{span}\left\{ \frac{1}{\sqrt{2}} \left[\begin{array}{c} 1 \\ 1 \end{array}\right], \frac{1}{\sqrt{2}} \left[\begin{array}{r} 1 \\ -1 \end{array}\right] \right\}$$

Scale factors

$$\mathbf{S} = \left[\begin{array}{cc} \sqrt{15} & 0 \\ 0 & \sqrt{5} \end{array}\right]$$

Part II

Explorations

Chapter 3

Jordan Canonical Form

The Jordan Canonical Form

3.1 Canonical form $\mathbf{J}_2$

3.1.1 Basic forms

Canonical form $\mathbf{J}_2$

$$\mathbf{J}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{3.1}$$

$$\begin{aligned} \mathbf{J}_2 &= \mathbf{Y}\,\Sigma\,\mathbf{X}^{\mathrm{T}} \\ &= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \cos\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} a_+ & 0 \\ 0 & a_- \end{bmatrix} \begin{bmatrix} \sin\theta_1 & \cos\theta_1 \\ -\cos\theta_1 & \cos\theta_1 \end{bmatrix} \end{aligned} \tag{3.2}$$

where the intermediate variables are

$$\begin{aligned} \cos\theta_1 &= \sqrt{\frac{1}{10}\left(5+\sqrt{5}\right)}, \\ a_\pm &= \pm 1 + \sqrt{5}. \end{aligned} \tag{3.3}$$

The domain matrices are related in this fashion

$$\mathbf{Y} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{X} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.4}$$

Canonical form $\mathbf{J}_3$

$$\mathbf{J}_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.5}$$

$$
\begin{aligned}
\mathbf{J}_3 &= \mathbf{Y}\Sigma\,\mathbf{X}^{\mathrm{T}} \\
&= \begin{bmatrix} \alpha_+ & -\alpha_- & -\alpha_0 \\ \alpha_- & -\alpha_0 & -\alpha_+ \\ -\alpha_0 & \alpha_+ & \alpha_- \end{bmatrix} \begin{bmatrix} b_0 & 0 & 0 \\ 0 & b_+ & 0 \\ 0 & 0 & b_- \end{bmatrix} \begin{bmatrix} -\alpha_0 & \alpha_- & \alpha_+ \\ -\alpha_+ & \alpha_0 & \alpha_- \\ \alpha_- & -\alpha_+ & -\alpha_0 \end{bmatrix}
\end{aligned} \tag{3.6}
$$

where the intermediate variables for the domain matrices are

$$
\begin{aligned}
\tan(3\theta_3) &= \frac{3}{13}\sqrt{3}, \\
\alpha_0 &= \frac{1}{3}\left(1 - 2\cos\theta_3\right), \\
\alpha_\pm &= \frac{1}{3}\left(1 + \cos\theta_3 \pm \sqrt{3}\sin\theta_3\right).
\end{aligned} \tag{3.7}
$$

and the intermediate variables for the singular values are

$$
\begin{aligned}
b_0 &= \frac{2}{3}\left(1 + \sqrt{7}\cos\theta_3\right), \\
b_\pm &= \sqrt{21}\arcsin\theta_3 \pm \frac{1}{3}\left(2 - \sqrt{7}\cos\theta_3\right)
\end{aligned} \tag{3.8}
$$

The domain matrices are related in this fashion

$$
\mathbf{Y} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{X} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.9}
$$

Canonical form $\mathbf{J}_4$

Canonical form $\mathbf{J}_4$

$$
\mathbf{J}_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.10}
$$

$$
\mathbf{Y} = \begin{bmatrix} \frac{1}{\sqrt{3}}\cos 2\theta - \frac{1}{3}\sin 2\theta & \frac{2}{3}\cos\theta & \frac{1}{\sqrt{3}} & -\frac{2}{3}\sin 2\theta \\ \frac{2}{3}\cos\theta & -\frac{2}{3}\sin 2\theta & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\cos 2\theta - \frac{1}{3}\sin 2\theta \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{2}{3}\sin 2\theta & \frac{1}{\sqrt{3}}\cos 2\theta - \frac{1}{3}\sin 2\theta & -\frac{2}{3}\sin 2\theta & \frac{2}{3}\cos\theta \end{bmatrix} \tag{3.11}
$$

where

$$
\theta_4 = \frac{pi}{18} \tag{3.12}
$$

Part III

Backmatter