The Singular Value Decomposition for Pedestrians

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The Fundamental Theorem of Linear Algebra

The singular value decomposition resolves the Four Fundamental Subspaces

Example: $\mathbf{A} \in \mathbb{R}_1^{3\times 2}$: a matrix with $m = 3$ rows, $n = 2$ columns and matrix rank $\rho = 1$. The codomain is $\mathbb{R}^3$, the domain is $\mathbb{R}^2$.

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

$$\begin{aligned} \text{CODOMAIN} &= \mathbb{R}^m \\ &= \mathcal{R}(\mathbf{A}) \oplus \mathcal{N}\left(\mathbf{A}^{\mathrm{T}}\right) \\ &= \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} \oplus \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\} \end{aligned}$$

$$\begin{aligned} \text{DOMAIN} &= \mathbb{R}^n \\ &= \mathcal{R}\left(\mathbf{A}^{\mathrm{T}}\right) \oplus \mathcal{N}(\mathbf{A}) \\ &= \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \oplus \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

The basis matrix $\mathbf{Y}$ is an orthogonal decomposition of the *codomain*.
The basis matrix $\mathbf{X}$ is an orthogonal decomposition of the *domain*.

$$\begin{aligned}
\mathbf{A} &= \mathbf{Y} \quad \Sigma \quad \mathbf{X} \\
&= \left[\begin{array}{c|c} \mathcal{R}(\mathbf{A}) & \mathcal{N}\left(\mathbf{A}^{\mathrm{T}}\right) \end{array}\right] \left[\begin{array}{c|c} \mathbf{S} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right] \left[\begin{array}{c} \mathcal{R}\left(\mathbf{A}^{\mathrm{T}}\right) \\ \hline \mathcal{N}(\mathbf{A}) \end{array}\right] \\
&= \left[\begin{array}{c|c} \mathbf{Y}_{\mathcal{R}} & \mathbf{Y}_{\mathcal{N}} \end{array}\right] \left[\begin{array}{c|c} \mathbf{S} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right] \left[\begin{array}{c} \mathbf{X}_{\mathcal{R}}^{\mathrm{T}} \\ \hline \mathbf{X}_{\mathcal{N}}^{\mathrm{T}} \end{array}\right] \\
&= \left[\begin{array}{c|cc} \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} & \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} & \frac{1}{\sqrt{3}}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \end{array}\right] \left[\begin{array}{c|c} \sqrt{6} & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right] \left[\begin{array}{c} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix} \\ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix} \end{array}\right] \\
&= \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \left[\begin{array}{c|c} \sqrt{6} & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right] \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

Anatomy of the SVD: row and column rank deficiency

$$
\begin{array}{ccccc}
\mathbf{A} & = & \mathbf{Y} & \Sigma & \mathbf{X}^{\mathrm{T}} \\
\left[\begin{array}{rr} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{array}\right] & = &
\left[\begin{array}{rrr} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{array}\right] &
\left[\begin{array}{c|c} \sqrt{6} & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right] &
\frac{1}{\sqrt{2}}\left[\begin{array}{rr} 1 & -1 \\ \hline 1 & 1 \end{array}\right]
\end{array}
$$

Here the target matrix $\mathbf{A} \in \mathbb{R}_1^{3\times 2}$ has $m = 3$ rows, $n = 2$ columns and matrix rank $\rho = 1$. This matrix has rank deficiency in both the rows and the columns. All components needed for the SVD come from the product matrices $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and $\mathbf{A}\,\mathbf{A}^{\mathrm{T}}$. The square root of the nonzero eigenvalues comprise the diagonal entries of Σ.

product matrix:

$$\mathbf{W}_{\mathrm{x}} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = 3\left[\begin{array}{rr} 1 & -1 \\ -1 & 1 \end{array}\right]$$

eigenvalues:

$$\lambda\left(\mathbf{W}_{\mathrm{x}}\right) = \{6, 0\}$$

eigenvectors:

$$\left\{\left[\begin{array}{r} -1 \\ 1 \end{array}\right], \left[\begin{array}{r} 1 \\ 1 \end{array}\right]\right\}$$

domain matrix:

$$\mathbf{X} = \left[\begin{array}{c|c} \frac{1}{\sqrt{2}}\left[\begin{array}{r} -1 \\ 1 \end{array}\right] & \frac{1}{\sqrt{2}}\left[\begin{array}{r} -1 \\ 1 \end{array}\right] \end{array}\right]$$

product matrix:

$$\mathbf{W}_{\mathrm{y}} = \mathbf{A}\,\mathbf{A}^{\mathrm{T}} = 5\left[\begin{array}{rrr} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{array}\right]$$

eigenvalues:

$$\lambda\left(\mathbf{W}_{\mathrm{y}}\right) = \{6, 0, 0\}$$

eigenvectors:

$$\left\{\left[\begin{array}{r} -1 \\ 1 \\ -1 \end{array}\right], \left[\begin{array}{r} 0 \\ -1 \\ 1 \end{array}\right], \left[\begin{array}{r} 2 \\ 1 \\ -1 \end{array}\right]\right\}$$

codomain matrix:

$$\mathbf{Y} = \left[\begin{array}{c|c} \frac{1}{\sqrt{3}}\left[\begin{array}{r} -1 \\ 1 \\ -1 \end{array}\right] & \frac{1}{\sqrt{2}}\left[\begin{array}{r} 0 \\ -1 \\ 1 \end{array}\right] \end{array}\right]$$

$$\mathbf{S} = \left[\begin{array}{c} 6 \end{array}\right], \qquad \Sigma = \left[\begin{array}{c|c} \mathbf{S} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right] = \left[\begin{array}{cc} \sqrt{6} & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right]$$

Anatomy of the SVD: full row rank

$$
\begin{array}{ccccc}
\mathbf{A} & = & \mathbf{Y} & \Sigma & \mathbf{X}^{\mathrm{T}} \\
\begin{bmatrix} 0 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix} & = & \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} & \left[\begin{array}{cc|c} \sqrt{15} & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{array}\right] & \begin{bmatrix} \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} & \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{-2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{bmatrix}
\end{array}
$$

Here the target matrix $\mathbf{A} \in \mathbb{R}_2^{2\times 3}$ has $m = 2$ rows, $n = 3$ columns and matrix rank $\rho = 2$ (full column rank).

product matrix:

$$
\mathbf{W}_{\mathrm{x}} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 13 & 4 \\ 2 & 4 & 4 \end{bmatrix}
$$

eigenvalues:

$$
\lambda\left(\mathbf{W}_{\mathrm{x}}\right) = \{15, 3, 0\}
$$

eigenvectors:

$$
\left\{ \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}
$$

domain matrix:

$$
\mathbf{X} = \left[\frac{1}{30}\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \quad \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \;\middle|\; \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right]
$$

product matrix:

$$
\mathbf{W}_{\mathrm{y}} = \mathbf{A}\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 9 & 6 \\ 6 & 9 \end{bmatrix}
$$

eigenvalues:

$$
\lambda\left(\mathbf{W}_{\mathrm{y}}\right) = \{15, 3\}
$$

eigenvectors:

$$
\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}
$$

codomain matrix:

$$
\mathbf{Y} = \frac{1}{\sqrt{2}}\left[\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right]
$$

$$
\mathbf{S} = \begin{bmatrix} \sqrt{15} & 0 \\ 0 & \sqrt{3} \end{bmatrix}, \qquad \Sigma = \left[\begin{array}{c|c} \mathbf{S} & \mathbf{0} \end{array}\right] = \left[\begin{array}{cc|c} \sqrt{15} & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{array}\right]
$$

Anatomy of the SVD: full row and column rank

$$
\begin{array}{ccccc}
\mathbf{A} & = & \mathbf{Y} & \Sigma & \mathbf{X}^{\mathrm{T}} \\
\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} & = & \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \sqrt{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\end{array}
$$

The target matrix $\mathbf{A} \in \mathbb{R}_2^{2\times 2}$ has $m = 2$ rows, $n = 2$ columns and matrix rank $\rho = 2$ (full column rank, full row rank). Notice how the eigenvalues are presented in the target matrix $\mathbf{W}_{\mathrm{x}}$. This is a diagonal matrix and it is customary (but not obligatory) to read the eigenvalues from the diagonal as $\{\lambda_1, \lambda_2\}$. When we order the singular values we must reorder the eigenvectors as well.

product matrix:	product matrix:
$\mathbf{W}_{\mathrm{x}} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$	$\mathbf{W}_{\mathrm{y}} = \mathbf{A}\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$
eigenvalues:	eigenvalues:
$\lambda(\mathbf{W}_{\mathrm{x}}) = \{2, 8\}$	$\lambda(\mathbf{W}_{\mathrm{y}}) = \{2, 8\}$
eigenvectors:	eigenvectors:
$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$	$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$
domain matrix:	codomain matrix:
$\mathbf{X} = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} \; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$	$\mathbf{Y} = \left[\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \; \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right]$

$$
\mathbf{S} = \sqrt{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \mathbf{S} \end{bmatrix} = \sqrt{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}
$$

form		decomposition products			dimension
$\mathbf{A}$	$=$	$\mathbf{Y}\,\Sigma\,\mathbf{X}^*$			$m \times n$
$\mathbf{A}^*$	$=$	$\mathbf{X}\,\Sigma^{\mathrm{T}}\,\mathbf{Y}^*$			$n \times m$
$\mathbf{A}^+$	$=$	$\mathbf{X}\,\Sigma^{(+)}\,\mathbf{Y}^*$			$n \times m$
$\mathbf{A}^*\mathbf{A}$	$=$	$\left(\mathbf{X}\,\Sigma^{\mathrm{T}}\,\mathbf{Y}^*\right)\left(\mathbf{Y}\,\Sigma\,\mathbf{X}^*\right)$	$=$	$\mathbf{Y}\,\Sigma\,\Sigma^{\mathrm{T}}\,\mathbf{Y}^*$	$n \times n$
$\mathbf{A}\,\mathbf{A}^*$	$=$	$\left(\mathbf{Y}\,\Sigma\,\mathbf{X}^*\right)\left(\mathbf{X}\,\Sigma^{\mathrm{T}}\,\mathbf{Y}^*\right)$	$=$	$\mathbf{X}\,\Sigma^{\mathrm{T}}\,\Sigma\,\mathbf{X}^*$	$m \times m$
$\mathbf{A}^+\mathbf{A}$	$=$	$\left(\mathbf{X}\,\Sigma^{(+)}\,\mathbf{Y}^*\right)\left(\mathbf{Y}\,\Sigma\,\mathbf{X}^*\right)$	$=$	$\mathbf{X}\,\Sigma^{(+)}\,\Sigma\,\mathbf{X}^* = \mathbf{X}_{\mathcal{R}}\mathbf{X}_{\mathcal{R}}^*$	$n \times n$
$\mathbf{A}\mathbf{A}^+$	$=$	$\left(\mathbf{Y}\,\Sigma\,\mathbf{X}^*\right)\left(\mathbf{X}\,\Sigma^{(+)}\,\mathbf{Y}^*\right)$	$=$	$\mathbf{Y}\,\Sigma\,\Sigma^{(+)}\,\mathbf{Y}^* = \mathbf{Y}_{\mathcal{R}}\mathbf{Y}_{\mathcal{R}}^*$	$m \times m$

pseudoinverse products		basis products		target projector	space	dimension
$\mathbf{A}^+\mathbf{A}$	$=$	$\mathbf{Y}_{\mathcal{R}}\mathbf{Y}_{\mathcal{R}}^*$	$=$	$P_{\mathcal{R}(\mathbf{A})}$	$\mathcal{R}\left(\mathbf{A}\right)$	$n \times n$
$\mathbf{A}\mathbf{A}^+$	$=$	$\mathbf{X}_{\mathcal{R}}\mathbf{X}_{\mathcal{R}}^*$	$=$	$P_{\mathcal{R}(\mathbf{A}^*)}$	$\mathcal{R}\left(\mathbf{A}^*\right)$	$m \times m$
$\mathbf{I}_m - \mathbf{A}^+\mathbf{A}$	$=$	$\mathbf{Y}_{\mathcal{N}}\mathbf{Y}_{\mathcal{N}}^*$	$=$	$P_{\mathcal{N}(\mathbf{A}^*)}$	$\mathcal{N}\left(\mathbf{A}^*\right)$	$m \times m$
$\mathbf{I}_n - \mathbf{A}\mathbf{A}^+$	$=$	$\mathbf{X}_{\mathcal{N}}\mathbf{X}_{\mathcal{N}}^*$	$=$	$P_{\mathcal{N}(\mathbf{A})}$	$\mathcal{N}\left(\mathbf{A}^*\right)$	$n \times n$

Linear least squares and the SVD

Given the linear system

$$\mathbf{A}x = b, \qquad \mathbf{A} \in \mathbb{R}^{m\times n}_{\rho}, \quad x \in \mathbb{R}^n, \quad b \in \mathbb{R}^m$$

the least squares x_{LS} solution is given by this expression:

$$\begin{aligned} x_{LS} &= \mathbf{A}^+b + \left(\mathbf{I}_n - \mathbf{A}^+\mathbf{A}\right)z, \qquad z \in \mathbb{R}^n \\ &= \mathbf{A}^+b + \mathbf{X}_{\mathcal{N}}\mathbf{X}_{\mathcal{N}}^* z \end{aligned}$$

with the vector z being arbitrary. An equivalent formulation is this:

$$x_{LS} = x_p + \left(\alpha_1\mathbf{X}_{*,\rho+1} + \alpha_2\mathbf{X}_{*,\rho+2} + \cdots + \alpha_{n-\rho}\mathbf{X}_{*,n}\right)$$

where the scalars α_k are arbitrary. This solution directly encodes the $n-\rho$ null space vectors in the domain matrix $\mathbf{X}$.

condition	pseudoinverse	chirality
$\rho = n$	$\mathbf{A}^{+} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}$	$= \mathbf{A}^{-\mathrm{L}}$
$\rho = m$	$\mathbf{A}^{+} = \mathbf{A}\left(\mathbf{A}\,\mathbf{A}^{\mathrm{T}}\right)^{-1}$	$= \mathbf{A}^{-\mathrm{R}}$
$m = n = \rho$	$\mathbf{A}^{+} = \mathbf{A}^{-1}$	$= \mathbf{A}^{-\mathrm{L}} = \mathbf{A}^{-\mathrm{R}}$

Σ gymnastics

$$\mathbf{S} = \mathbf{S}^{\mathrm{T}} = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_\rho \end{bmatrix}, \qquad \mathbf{S}^{-1} = \begin{bmatrix} \sigma_1^{-1} & & & \\ & \sigma_2^{-1} & & \\ & & \ddots & \\ & & & \sigma_\rho^{-1} \end{bmatrix}$$

$$\mathbf{S}\,\mathbf{S}^{\mathrm{T}} = \mathbf{S}^{\mathrm{T}}\mathbf{S} = \mathbf{S}^2$$

form		block matrix			dimension
Σ	$=$	$\begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$			$m \times n$
Σ^{T}	$=$	$\begin{bmatrix} \mathbf{S}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$			$n \times m$
$\Sigma^{(+)}$	$=$	$\begin{bmatrix} \mathbf{S}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$			$n \times m$
$\Sigma\Sigma^{\mathrm{T}}$	$=$	$\begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{S}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$	$=$	$\begin{bmatrix} \mathbf{S}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$	$m \times m$
$\Sigma^{\mathrm{T}}\Sigma$	$=$	$\begin{bmatrix} \mathbf{S}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$	$=$	$\begin{bmatrix} \mathbf{S}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$	$n \times n$
$\Sigma\Sigma^{(+)}$	$=$	$\begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{S}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$	$=$	$\begin{bmatrix} \mathbf{I}_\rho & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$	$m \times m$
$\Sigma^{(+)}\Sigma$	$=$	$\begin{bmatrix} \mathbf{S}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$	$=$	$\begin{bmatrix} \mathbf{I}_\rho & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$	$n \times n$
$\Sigma^{(+)}\Sigma\Sigma^{\mathrm{T}}$	$=$	$\begin{bmatrix} \mathbf{I}_\rho & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$	$=$	Σ	$m \times n$
$\Sigma^{\mathrm{T}}\Sigma\Sigma^{(+)}$	$=$	$\begin{bmatrix} \mathbf{S}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I}_\rho & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$	$=$	Σ^{T}	$n \times m$

Table 0.1. *Note that although the matrices Σ and Σ^{T} have the same block structure (because $\boldsymbol{S} = \boldsymbol{S}^{\mathrm{T}}$), the matrix and its transpose have different dimensions.*

$$\mathbf{S} = \mathrm{diag}\left(2, \frac{1}{3}\right) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$\begin{aligned}
\Sigma &= \begin{bmatrix} \mathbf{S} \\ \mathbf{0} \end{bmatrix} &= \left[\begin{array}{cc} 2 & 0 \\ 0 & \frac{1}{3} \\ \hline 0 & 0 \end{array}\right] \\
\Sigma^{\mathrm{T}} &= \begin{bmatrix} \mathbf{S}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} &= \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{array}\right] \\
\Sigma^{(+)} &= \begin{bmatrix} \mathbf{S}^{-1} & \mathbf{0} \end{bmatrix} &= \left[\begin{array}{cc|c} \frac{1}{2} & 0 & 0 \\ 0 & 3 & 0 \end{array}\right]
\end{aligned}$$

$$\Sigma = \begin{array}{cl} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} & \begin{array}{l} \rho \\ m-\rho \end{array} \end{array} \tag{0.1}$$

$$\Sigma = \begin{array}{cl} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} & \begin{array}{l} \rho \\ m-\rho \end{array} \\ \begin{array}{cc} \rho & n-\rho \end{array} & \end{array} \tag{0.2}$$