A NOTE ON THE RELATIONS BETWEEN TRUE AND ECCENTRIC ANOMALIES IN THE TWO-BODY PROBLEM

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Abstract. Two simple formulas are given to relate the eccentric and true anomalies in the two-body problem. The problem of the maximum difference between these two angles is alo considered.

The relation between the tangents of half the eccentric anomaly E and half the true anomaly v is well known. In fact, this is one of the most classical equations in Kepler's problem (see Brouwer and Clemence, 1961, Equation (12), p. 62). However, there is a possibility of numerical trouble when this equation is used with angles that are near to $\pm 90°$, as the two tangents become infinite. Of course, there are many ways of avoiding this difficulty, for instance by using formulas with the sine and cosine of $E/2$ and $v/2$ (see Stumpff, 1959). But the object of the present note is to propose two remarkably elegant formulas that are free of numerical trouble, no matter what the values of the angles are:

$$\tan\frac{v-E}{2} = \frac{\sin v}{\beta' + \cos v} = \frac{\sin E}{\beta' - \cos E}, \tag{1}$$

where the constant β' is a function of the eccentricity alone (see Brouwer and Clemence, p. 62):

$$\beta' = (1 + \sqrt{1-e^2})/e. \tag{2}$$

Equation (1) provides a convenient relation between the true and eccentric anomalies. This equation can be used to obtain simple formulas for the quantity $v - M$, which is usually expressed as a Poisson series in either M or v. These results are:

$$\begin{aligned} v - M &= 2\tan^{-1}\left(\frac{\sin E}{\beta' - \cos E}\right) + e\sin E = \\ &= 2\tan^{-1}\left(\frac{\sin v}{\beta' + \cos v}\right) + \frac{e\sqrt{1-e^2}\sin v}{1 + e\cos v}. \end{aligned} \tag{3}$$

The above Equations (1) and (3) can be easily derived from the classical equation by some elementary trigonometric manipulations. Equation (1) is especially easy to use because the angle $(v-E)/2$ is always less than $90°$ for all elliptic orbits.

The last statement brings up some other questions relating to the Kepler problem: What is the maximum value of $v-E$ and when is this maximum reached? A more general question can even be posed: What is the maximum difference between any two of the three anomalies M, E, and v of the two-body problem (M being the mean anomaly)?* These questions can be answered by considering the derivative of one anomaly with respect to the other. For instance, $v-E$ is maximum when

$$\frac{\mathrm{d}(v-E)}{\mathrm{d}E}=0 \quad \text{or} \quad \frac{\mathrm{d}v}{\mathrm{d}E}=1\,.$$

The classical two-body relations show that this happens when the radius-vector r is equal to the semi-minor axis b of the ellipse. In the same way it can be shown that the difference $E-M$ is maximum when the radius vector is equal to the semi-major axis a. Finally it can be shown that $v-M$ is maximum when the radius vector is the geometric mean $\sqrt{ab}$ of both semi axes.

The reader could also derive the values of M, E or v when one of the differences is maximum. For instance, when $v-E$ is maximum, we have $E+v=\pi$ with $E<\pi/2$ and $v>\pi/2$. Also when $E-M$ is maximum, we have $E=\pi/2$ and $M=\pi/2-e$.

References

Brouwer, D. and Clemence, G.: 1961, *Methods of Celestial Mechanics*, Academic Press, New York, p. 62.

Danby, J. M. A.: 1962, *Fundamentals of Celestial Mechanics*, MacMillan, New York, p. 128.

Stumpff, K.: 1959, *Himmelsmechanik*, Vol. I, Deutscher Verlag der Wissenschaften, Berlin, p. 110.

* This problem occurs in the analysis of the libration in the longitude of the Moon (see Danby, pp. 128, 129).