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Quaternion Regularization in Celestial Mechanics and Astrodynamics and Trajectory Motion Control. I[1]

Yu. N. Chelnokov

Saratov State University, Saratov, Russia
Institute of Precision Mechanics and Control, Russian Academy of Sciences, Saratov, Russia
e-mail: ChelnokovYuN@gmail.com

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Abstract—Regularization problems in celestial mechanics and astrodynamics are considered. The fundamental regular quaternion models of celestial mechanics and astrodynamics are presented. It is shown that the efficiency of analytical investigation and numerical solution of boundary problems of optimal trajectory motion control of spacecraft may be increased using quaternion astrodynamics models.
The regularization problem of celestial mechanics and astrodynamics that implies eliminating the feature, which arises in the equations of the two-body problem in case of impact of the second body with the central body, is considered in the first section of the paper. The quaternion method for regularizing the equations of the perturbed spatial two-body problem suggested by the author is presented; the method is compared with Kustaanheimo–Stiefel (KS) regularization. Demonstrative geometric and kinematic interpretations of regularizing transformations are provided. Regular quaternion equations for the two-body problem, which generalize the regular Kustaanheimo–Stiefel equations, as well as regular equations in quaternion osculating elements and quaternion regular equations for perturbed central motion of a material point, are considered. The papers on quaternion regularization in celestial mechanics and astrodynamics are briefly analyzed.

Keywords: two-body problem, Kustaanheimo–Stiefel regularization, quaternion regularization, perturbed central motion.

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1. THE PROBLEM OF REGULARIZING DIFFERENTIAL EQUATIONS OF THE PERTURBED SPATIAL TWO-BODY PROBLEM

Celestial mechanics and astrodynamics are based on Newton's vector differential equation for the perturbed spatial two-body problem

$$d^2\mathbf{r}/dt^2 + f(m+M)r^{-3}\mathbf{r} = \mathbf{p}(t, \mathbf{r}, d\mathbf{r}/dt), \qquad (1)$$

where $\mathbf{r}$ is the radius vector of the center of mass of the second (studied) body drawn from the center of mass of the first (central) body; $r = |\mathbf{r}|$, m and M are the masses of the second and the first bodies; f is the gravitational constant; $\mathbf{p}$ is the vector of perturbing acceleration of the center of mass of the second body (or the vector that equals the sum of perturbing and control accelerations of the center of mass of the second body), and t is the time.

This equation degenerates in the case of impact of the second body with the central one (when the distance r between the bodies is zero), which makes the use of this equation inconvenient for studying the motion of the second body within a small vicinity of the central body or its motion along highly eccentric orbits. The singularity at the origin of the coordinates causes both theoretical and practical (computational) difficulties.

The problem of eliminating the mentioned peculiarity, which is known in celestial mechanics and astrodynamics as the problem of regularizing differential equations of the perturbed two-body problem, dates back to Euler [2] and Levi-Civita [3–5], who derived solutions for one- and two-dimensional impact problems for two bodies (for linear and planar motion). The most efficient regularization of equations of the spatial two-body problem, namely spinor or KS regularization, was proposed by Kustaanheimo and Stiefel [6, 7]. This is the generalized Levi-Civita's regularization of planar equations of motion, and its most detailed description is given in the widely known monograph by Stiefel and Scheifele [8].

2. KUSTAANHEIMO–STIEFEL REGULARIZATION

Kustaanheimo–Stiefel regularization is based on the following nonlinear ambiguous transformation of

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Cartesian coordinates of the studied body, which is also called the KS transform or generalized Levi-Civita's transform:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_0 \\ u_2 & u_1 & -u_0 & -u_3 \\ u_3 & u_0 & u_1 & u_2 \\ u_0 & -u_3 & u_2 & -u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \end{pmatrix} \tag{2}$$

$$= L(\mathbf{u}_{KS}) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \end{pmatrix},$$

Here, x_k ($k = 1, 2, 3$) are the coordinates of the center of mass of the studied body in inertial system of coordinates X, whose origin is at the center of mass of the central body, and the coordinate axes are directed towards distant stars; u_j ($j = 0, 1, 2, 3$) are the new variables (KS variables), $L(\mathbf{u}_{KS})$ is the generalized Levi-Civita's matrix, which is also called the KS matrix, which includes the two-dimensional square Levi-Civita's matrix at the upper left corner.

Transform (2) is written in scalar form as follows:

$$x_1 = u_0^2 + u_1^2 - u_2^2 - u_3^2,$$
$$x_2 = 2(u_1u_2 - u_0u_3), \quad x_3 = 2(u_1u_3 + u_0u_2) \tag{3}$$

which coincides with the Hopf map up to a permutation of the indices [9].

Regular Kustaanheimo–Stiefel equations of the perturbed spatial two-body problem are written in scalar form as follows [8]:

$$d^2u_j/d\tau^2 - (h/2)u_j = (r/2)q_j \ (j = 0, 1, 2, 3), \tag{4}$$

$$dh/d\tau = 2(q_0du_0/d\tau + q_1du_1/d\tau + q_2du_2/d\tau + q_3du_3/d\tau), \tag{5}$$

$$dh/d\tau = r, r = |\mathbf{r}| = u_0^2 + u_1^2 + u_2^2 + u_3^2; \tag{6}$$

$$q_0 = u_0p_1^* - u_3p_2^* + u_2p_3^*, \quad q_1 = u_1p_1^* + u_2p_2^* + u_3p_3^*,$$
$$q_2 = -u_2p_1^* + u_1p_2^* + u_0p_3^*,$$
$$q_3 = -u_3p_1^* - u_0p_2^* + u_1p_3^*.$$

Here, τ is the new independent variable, which is called fictitious time, related to the time t by differential equation (6); h is the additional variable interpreted as the Kepler energy, p_k^* ($k = 1, 2, 3$) are the projections of the perturbing acceleration $\mathbf{p}$ of the center of mass of the second body onto axes of the inertial coordinates.

These equations form the system of ten ordinary nonlinear differential equations with respect to the Kustaanheimo–Stiefel variables u_j, the Kepler energy h, and time t. Equations (4) are equivalent to the matrix equation

$$d^2\mathbf{u}_{KS}/d\tau^2 - (h/2)\mathbf{u}_{KS} = (r/2)L(\mathbf{u}_{KS})\mathbf{P}_{KS}, \tag{7}$$

where $\mathbf{u}_{KS}$ is the four-dimensional vector of KS variables: $\mathbf{u}_{KS} = (u_1, u_2, u_3, u_0)$; $\mathbf{P}_{KS}$ is the four-dimensional vector matched with the three-dimensional vector of the acceleration $\mathbf{p}$: $\mathbf{P}_{KS} = (p_1^*, p_2^*, p_3^*, 0)$.

The following fundamental advantages of Kustaanheimo–Stiefel equations are to be mentioned [8, 10–15]:

—As opposed to Newton's equations, they are regular at the center of attraction.

—In case of nonperturbed Kepler motions, they are linear and written as follows:

$$d^2u_j/d\tau^2 - (h/2)u_j = 0,$$
$$h = \text{const}, \ (j = 0, 1, 2, 3)$$

(in the case of elliptic Kepler motion, i.e., with Kepler energy $h < 0$, these equations are equivalent to the equations of motion for the four-dimensional single-frequency harmonic oscillator, whose square frequency is equal to half the Kepler energy).

—They make it possible to develop the unified approach to studying all three types of Kepler motion.

—They are close to linear equations for perturbed Kepler motions.

—They make it possible to represent right parts of differential equations of motion of celestial bodies in polynomial form, which is convenient for solving using computers.

Due to these circumstances, the efficient methods for obtaining analytical and numerical solutions of such problems as the investigation of motion in the vicinity of attracting masses or motion along orbits with high eccentricities, which are difficult to solve using classical methods, could be developed. It was shown by Stiefel, Scheifele, Bordovitsyna, and others [8, 11, 12] that the use of regular equations in Kustaanheimo–Stiefel variables makes it possible to achieve an increase in accuracy of numerical solutions of a number of celestial mechanics and astrodynamics problems (for instance, the problem of artificial satellite motion along highly eccentric orbits) of three to five orders in comparison with the solutions obtained using classical (Newton's) equations. In addition, these equations, as well as their quaternion analogues and equations in quaternion osculating elements, made it possible to construct efficient solutions for a number of problems of optimal control of the orbital motion of a spacecraft (Sapunkov, Chelnokov, Yurko [1, 13–19]).

It was mentioned earlier that Kustaanheimo–Stiefel regularization is based on nonlinear ambiguous transformation of Cartesian coordinates (3). This transformation is a transition from the three-dimensional space of Cartesian coordinates x_k to the four-dimensional space of new coordinates u_j. Therefore, according to Stiefel and Scheifele, regular equations cannot be directly derived in the three-dimensional (i.e. spatial) case. In their book [8], they postulate the regular matrix solution of the spatial two-body problem (7), which was written down similarly to the regular matrix Levi-Civita's equation for planar motion. They also use several theorems to prove that the former Newton's vector equation (1) is satisfied in this case as well. This approach to constructing regular equations of the spatial two-body problem is, to a significant extent, artificial and not very illustrative.

The use of quaternions (four-dimensional hypercomplex numbers) and four-dimensional quaternion matrices for regularizing the equations of the spatial two-body problem was considered soon after the discovery of the KS transform. However, Stiefel and Scheifele rejected this idea completely and wrote ([8], p. 29) that "any attempt to substitute the theory of the KS matrix by the more popular theory of the quaternion matrices leads to failure or at least to a very unwieldy formalism". This statement was refuted for the first time by the author of the present paper, who showed at late 70s and early 80s [20–23] that the quaternion approach does actually make it possible to directly and vividly derive regular equations in Kustaanheimo–Stiefel variables, making the fundamental principles underlying the KS regularization more natural and clear, and makes it possible to develop the theory, which generalizes the KS regularization.

3. CHELNOKOV'S QUATERNION REGULARIZATION

The key stages of construction of regular quaternion equations for a spatial two-body problem proposed by the author in [20–23] are as follows.

1) We write down the initial Newton's equations for the perturbed spatial two-body problem in rotating coordinates η, whose origin is located at the center of mass of the second body. Axis η_1 of this system of coordinates is set along radius vector $\mathbf{r}$ of the center of mass of the second body. To describe the attitude movement of coordinates η, we use Euler (Rodrigues–Hamilton) parameters.

In this case, the equations for the two-body problem are written as follows:

$$\ddot{r} - r(\omega_2^2 + \omega_3^3) + f(m+M)/r^2 = p_1, \tag{8}$$

$$\begin{aligned} 2\omega_3\dot{r} + r\dot{\omega}_3 + r\omega_1\omega_2 &= p_2, \\ 2\omega_2\dot{r} + r\dot{\omega}_2 - r\omega_1\omega_3 &= -p_3; \end{aligned} \tag{9}$$

$$\begin{aligned} 2\dot{\lambda}_0 &= -\omega_1\lambda_1 - \omega_2\lambda_2 - \omega_3\lambda_3, \\ 2\dot{\lambda}_1 &= \omega_1\lambda_0 + \omega_3\lambda_2 - \omega_2\lambda_3, \\ 2\dot{\lambda}_2 &= \omega_2\lambda_0 - \omega_3\lambda_1 + \omega_1\lambda_3, \\ 2\dot{\lambda}_3 &= \omega_1\lambda_0 + \omega_2\lambda_1 - \omega_1\lambda_2. \end{aligned} \tag{10}$$

Here, r is the modulus of the radius vector of the center of mass of the second body; λ_j ($j = 0, 1, 2, 3$) are the Euler parameters characterizing the orientation of coordinates η in the inertial system of coordinates X; ω_k ($k = 1, 2, 3$) are the projections of vector ω of the absolute rotation speed of the system of coordinates η onto its own axes; p_k is the projection of the vector of perturbing acceleration $\mathbf{p}$ onto axis η_k; the upper point indicates the derivative with respect to time t.

The equations of the two-body problem (8)–(10) form the closed system of eighth-order differential equations with respect to variables r, $\dot{r}$, ω_2, and ω_3 and parameters λ_j. In these equations, the projection of rotation speed ω_1 onto the direction of radius vector $\mathbf{r}$ (axis η_1) is an arbitrarily set parameter.

Subsystem (10), which describes the orientation of coordinates η, written down in quaternion form is as follows:

$$2\dot{\lambda} = \lambda \circ \omega_\eta, \quad \boldsymbol{\lambda} = \lambda_0 + \lambda_1\mathbf{i}_1 + \lambda_2\mathbf{i}_2 + \lambda_3\mathbf{i}_3,$$
$$\boldsymbol{\omega}_\eta = \omega_1\mathbf{i}_1 + \omega_2\mathbf{i}_2 + \omega_3\mathbf{i}_3.$$

Hereinafter, symbol $\circ$ indicates a quaternion product, $\boldsymbol{\lambda}$ is the attitude (rotation) quaternion of coordinates η in the inertial system of coordinates; quaternion $\boldsymbol{\omega}_\eta$ is the mapping of vector $\boldsymbol{\omega}$ onto the basis η; $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ are Hamilton imaginary numbers in vector form.

2) We set an arbitrarily defined projection of rotation speed ω_1 equal to zero and write it using the following expression:

$$\omega_1 = 2(-\lambda_1\dot{\lambda}_0 + \lambda_0\dot{\lambda}_1 + \lambda_3\dot{\lambda}_2 - \lambda_2\dot{\lambda}_3) = 0, \tag{11}$$

We introduce projections c_k of the vector of momentum of center-of-mass velocity of the second body relatively to the center of attraction as new variables, instead of projections of rotation speed ω_k. In this case, the new vector is related to the rotation speed vector $\boldsymbol{\omega}$ by the expression

$$\boldsymbol{\omega} = r^{-2}\mathbf{c} = r^{-2}\mathbf{r} \times d\mathbf{r}/dt.$$

Equations of the two-body problem become as follows:

$$\ddot{r} - (c_2^2 + c_3^2)r^{-3} + f(m+M)r^{-2} = p_1;$$
$$c_1 = 0, \quad \dot{c}_2 = -rp_3, \quad \dot{c}_3 = rp_2;$$

$$\begin{aligned} 2\dot{\boldsymbol{\lambda}} &= r^{-2}\boldsymbol{\lambda} \circ \mathbf{c}_\eta, \\ \boldsymbol{\lambda} = \lambda_0 + \lambda_1\mathbf{i}_1 + \lambda_2\mathbf{i}_2 + \lambda_3\mathbf{i}_3, &\quad \mathbf{c}_\eta = c_2\mathbf{i}_2 + c_3\mathbf{i}_3. \end{aligned} \tag{12}$$

It should be noted that rotating coordinates η introduced in such way are called azimuthally free or a nonholonomic accompanying trihedral of coordinates.

3) We replace quaternion variable $\boldsymbol{\lambda}$ with a new quaternion variable $\mathbf{u}$, which is defined by the tension transformation

$$\mathbf{u} = u_0 + u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3 = r^{1/2}\bar{\boldsymbol{\lambda}},$$

$$\boldsymbol{\lambda} = r^{-1/2}\bar{\mathbf{u}},\ u_0 = r^{1/2}\lambda_0,\ u_k = -r^{1/2}\lambda_k,\ (k = 1,2,3),$$

$$\|\mathbf{u}\|^2 = \mathbf{u} \circ \bar{\mathbf{u}} = \bar{\mathbf{u}} \circ \mathbf{u} = u_0^2 + u_1^2 + u_2^2 + u_3^2 = r.$$

Hereinafter, the overline indicates quaternion conjugation.

New variables u_j and their derivatives $\dot{u}_j$ are related to Cartesian coordinates x_k of the center of mass of the second body and their derivatives $\dot{x}_k$ (projections v_i^+ of the vector of center-of-mass velocity $\mathbf{v}$ of the body onto axes of the inertial system of coordinates) by nonlinear relationships

$$\begin{aligned} x_1 &= r(\lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2) = u_0^2 + u_1^2 - u_2^2 - u_3^2, \\ x_2 &= 2r(\lambda_1\lambda_2 + \lambda_0\lambda_3) = 2(u_1u_2 - u_0u_3), \\ x_3 &= 2r(\lambda_1\lambda_3 - \lambda_0\lambda_2) = 2(u_1u_3 + u_0u_2); \end{aligned} \tag{13}$$

$$\begin{aligned} v_1^+ &= \dot{x}_1 = 2(u_0\dot{u}_0 + u_1\dot{u}_1 - u_2\dot{u}_2 - u_3\dot{u}_3), \\ v_2^+ &= \dot{x}_2 = 2(u_2\dot{u}_1 + u_1\dot{u}_2 - u_3\dot{u}_0 - u_0\dot{u}_3), \\ v_3^+ &= \dot{x}_3 = 2(u_3\dot{u}_1 + u_1\dot{u}_3 + u_2\dot{u}_0 + u_0\dot{u}_2). \end{aligned} \tag{14}$$

Relationships (13) and (14) in quaternion form are written as follows:

$$\mathbf{r}_x = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3 = \bar{\mathbf{u}} \circ \mathbf{i}_1 \circ \mathbf{u},$$

$$\mathbf{v}_x = v_1^+\mathbf{i}_1 + v_2^+\mathbf{i} + v_3^+\mathbf{i}_3 = d\mathbf{r}_x/dt = 2\bar{\mathbf{u}} \circ \mathbf{i}_1 \circ d\mathbf{u}/dt.$$

It follows from relationships (13) and (3) that our new variables u_j and Kustaanheimo–Stiefel variables coincide.

Relationship (11) for the rotation speed projection ω_1 in new variables is as follows:

$$\omega_1 = 2r^{-1}(u_1\dot{u}_0 - u_0\dot{u}_1 + u_3\dot{u}_2 - u_2\dot{u}_3) = 0.$$

If independent variable t is replaced with new independent variable τ, according to the formula $dt = rd\tau$, then following from that relationship is the well-known bilinear relationship [8]

$$\begin{aligned} &u_1du_0/d\tau - u_0du_1/d\tau \\ &+ u_3du_2/d\tau - u_2du_3/d\tau = 0, \end{aligned} \tag{11'}$$

which relates Kustaanheimo–Stiefel variables and their first derivatives. According to Stiefel and Scheifele, this relationship plays a major part in construction of the regular celestial mechanics [8, p. 29].

Thus, the regularizing transformation of Kustaanheimo–Stiefel coordinates indicates the transition from Cartesian coordinates of the center of mass of the second body in the inertial system of coordinates to new variables, which are the components of the conjugate quaternion of inertial orientation of coordinates η, whose axis η_1 is directed along radius vector $\mathbf{r}$ of the center of mass of the second body, normalized in a specific way. The normalizing factor equals the square root of the distance r from the center of mass of the second body to the center of attraction. Bilinear Kustaanheimo–Stiefel relationship imposes the additional (nonholonomic) constraint on the motion of trihedral η, namely, the equality of projection ω_1 of the vector of absolute rotation speed of trihedral η onto the direction of radius vector $\mathbf{r}$ (axis η_1) to zero.

Quaternion kinematic motion equation (12) of nonholonomic moving trihedral η in new KS variables is as follows:

$$\begin{aligned} 2\dot{\mathbf{u}} &= r^{-1}(\dot{r} - r^{-1}\mathbf{c}_\eta) \circ \mathbf{u}, \\ \mathbf{u} &= u_0 + u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3, \\ \mathbf{c}_\eta &= c_2\mathbf{i}_2 + c_3\mathbf{i}_3 = r^2(\omega_2\mathbf{i}_2 + \omega_3\mathbf{i}_3). \end{aligned} \tag{15}$$

4) We differentiate equation (15) with respect to time taking into account trajectory motion equations

$$\dot{v}_1 = (c_2^2 + c_3^2)r^{-3} - f(m+M)r^{-2} + p_1, \quad \dot{r} = v_1,$$

$$c_1 = 0, \quad \dot{c}_2 = -rp_3, \quad \dot{c}_3 = rp_2.$$

5) We introduce Kepler energy h determined by the expression

$$h = (1/2)v^2 - f(m+M)r^{-1}(v = |\mathbf{v}|,\ \mathbf{v} = d\mathbf{r}/dt)$$

and satisfying the differential equation $\dot{h} = \mathbf{p} \cdot \mathbf{v}$, where the central point indicates a scalar product of vectors, as a new variable.

6) We replace independent variable t (time) with a new independent variable τ using the formula

$$\begin{aligned} dt = rd\tau, \quad r = |\mathbf{r}| &= (x_1^2 + x_2^2 + x_3^2)^{1/2} \\ &= u_0^2 + u_1^2 + u_2^2 + u_3^2. \end{aligned}$$

As a result of these transformations, we obtain the following regular quaternion equations for the perturbed spatial two-body problem in Kustaanheimo–Stiefel variables [20, 21]:

$$\begin{aligned} &d^2\mathbf{u}/d\tau^2 - (h/2)\mathbf{u} = (r/2)\mathbf{q}, \\ &dh/d\tau = 2\mathrm{scal}((d\bar{\mathbf{u}}/d\tau) \circ \mathbf{q}), \quad dt/d\tau = r; \\ &r = \|\mathbf{u}\|^2 = \mathbf{u} \circ \bar{\mathbf{u}} = \bar{\mathbf{u}} \circ \mathbf{u} = u_0^2 + u_1^2 + u_2^2 + u_3^2, \\ &\mathbf{q} = -\mathbf{i}_1 \circ \mathbf{u} \circ \mathbf{p}_x, \ \mathbf{p}_x = p_1^*\mathbf{i}_1 + p_2^*\mathbf{i}_2 + p_3^*\mathbf{i}_3; \\ &\mathbf{r}_x = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3 = \bar{\mathbf{u}} \circ \mathbf{i}_1 \circ \mathbf{u}, \\ &\mathbf{v}_x = dr_x/dt = 2\bar{\mathbf{u}} \circ \mathbf{i}_1 \circ d\mathbf{u}/dt = 2r^{-1}\bar{\mathbf{u}} \circ \mathbf{i}_1 \circ d\mathbf{u}/d\tau, \end{aligned} \tag{16}$$

These equations have all the aforementioned advantages of regular equations (4)–(6) derived by Kustaanheimo and Stiefel (in equations (16) and below, scal() is the scalar part of the quaternion in parentheses).

The following, more general regular quaternion equations for the perturbed spatial two-body problem were also derived by the author [20, 21]:

$$\boldsymbol{\alpha}'' - \frac{1}{2}h\boldsymbol{\alpha} - \frac{3}{2}r\omega_1\boldsymbol{\alpha}' \circ \mathbf{i}_1 - \frac{1}{2}\omega_1\boldsymbol{\alpha} \circ \mathbf{i}_1 \circ \bar{\boldsymbol{\alpha}} \circ \boldsymbol{\alpha}'$$

$$-\frac{1}{2}r^2\omega_1^2\boldsymbol{\alpha} - \frac{1}{2}r^2\varepsilon_1\boldsymbol{\alpha} \circ \mathbf{i}_1 = -\frac{1}{2}r\mathbf{p}_x \circ \boldsymbol{\alpha} \circ \mathbf{i}_i, \tag{17}$$

$$h' = \mathbf{p}_x \cdot (\boldsymbol{\alpha} \circ \mathbf{i}_1 \circ \bar{\boldsymbol{\alpha}})', \quad t' = r;$$

$$\boldsymbol{\alpha} = \bar{\mathbf{u}} \circ \mathbf{i}_1 = \alpha_0 + \alpha_1\mathbf{i}_1 + \alpha_2\mathbf{i}_2 + \alpha_3\mathbf{i}_3$$

$$= u_1 + u_0\mathbf{i}_1 - u_3\mathbf{i}_2 + u_2\mathbf{i}_3,$$

$$r = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2, \quad \mathbf{p}_x = p_1^*\mathbf{i}_1 + p_2^*\mathbf{i}_2 + p_3^*\mathbf{i}_3. \tag{18}$$

$$\mathbf{r}_x = \boldsymbol{\alpha} \circ \mathbf{i}_1 \circ \bar{\boldsymbol{\alpha}}, \quad \mathbf{v}_x = r^{-1}(\boldsymbol{\alpha}' \circ \mathbf{i}_1 \circ \bar{\boldsymbol{\alpha}} + \boldsymbol{\alpha} \circ \mathbf{i}_1 \circ \bar{\boldsymbol{\alpha}}'),$$

In equations (17) and (18), the prime symbol indicates differentiation with respect to variable τ (fictitious time); $\boldsymbol{\alpha}$ is the new regular quaternion variable, ω_1 and ε_1 are the projections of the rotation speed and rotation acceleration vectors of moving trihedral η onto the direction of radius vector $\mathbf{r}$ of the center of mass of the second body (one of these projections is an arbitrarily set parameter).

We assume that $\omega_1 = 0$ and $\varepsilon_1 = 0$ and derive the quaternion form of the regular Kustaanheimo–Stiefel equations (16) from equations (17).

Thus, when the bilinear relationship (11') is rejected, we obtain more regular general equations for the spatial two-body problem than in the Kustaanheimo–Stiefel case. However, it should be noted that these equations are more complex than the Kustaanheimo–Stiefel equations.

It should be noted that regular quaternion equations for the perturbed spatial two-body problem (17) derived by the author in [21], and their matrix analogue in quaternion matrices was derived in the earlier paper [20]. It should also be mentioned that another, more general, interpretation of the regularizing Kustaanheimo–Stiefel transform based on consideration of helical motion of the introduced trihedral η and the use of quaternion differential motion equations for the two-body problem in helical motion parameters of the said trihedral was presented in [20, 21] as well. The relationship of the regularizing Kustaanheimo–Stiefel transform (2) and the Study's formula, which relates the rectangular coordinates of the origin of the introduced moving coordinates η with the biquaternion components of its finite helical displacement [24, p. 146], is revealed using this approach. This transform is recognized as a specific case of the Study's formula.

Thus, as opposed to the approach which uses the KS matrix apparatus, the quaternion approach to regularizing equations of the perturbed spatial two-body problem proposed by the author provides clear geometric and kinematic interpretations of the regularizing Kustaanheimo–Stiefel transform, reveals the geometric sense of its ambiguity, and makes it possible to directly and clearly derive regular equations for spatial two-body problems, of which regular Kustaanheimo–Stiefel equations are specific cases.

The author also proposed regular equations for the perturbed spatial two-body problem in quaternion osculating (i.e. slowly varying) elements obtained from regular quaternion equations in Kustaanheimo–Stiefel variables via alteration of arbitrary constants, which are as follows:

$$d\boldsymbol{\alpha}/d\tau^* = -\mathbf{f}\sin(\tau^*/2), \quad d\boldsymbol{\beta}/d\tau^* = \mathbf{f}\cos(\tau^*/2),$$

$$d\nu/d\tau^* = -(1/(2\nu))(d\mathbf{u}/d\tau^*, \mathbf{q}),$$

$$d\tau_e/d\tau^* = \mu/(8\nu^3) + (r/(8\nu^3))(\mathbf{u}, \mathbf{q}) \tag{19}$$

$$-(2/\nu^2)(d\nu/d\tau^*)(\mathbf{u}, d\mathbf{u}/d\tau^*);$$

$$\mathbf{f} = (2/\nu)[(r/(8\nu))\mathbf{q} - (d\nu/d\tau^*)d\mathbf{u}/d\tau^*], \qquad \mathbf{q} = -\mathbf{i}_1 \circ \mathbf{u} \circ \mathbf{p}_x; \tag{20}$$

$$\mathbf{u} = \boldsymbol{\alpha}\cos(\tau^*/2) + \boldsymbol{\beta}\sin(\tau^*/2),$$

$$d\mathbf{u}/d\tau^* = (1/2)(-\boldsymbol{\alpha}\sin(\tau^*/2) + \boldsymbol{\beta}\cos(\tau^*/2)); \tag{21}$$

$$\mathbf{r}_x = \bar{\mathbf{u}} \circ \mathbf{i}_1 \circ \mathbf{u}, \quad \mathbf{v}_x = (4\nu/r)\bar{\mathbf{u}} \circ \mathbf{i}_1 \circ d\mathbf{u}/d\tau^*,$$

$$\lambda = r^{-1/2}\bar{\mathbf{u}}, \quad r = (\mathbf{u}, \mathrm{u}), \tag{22}$$

$$t = \tau_e - (1/\nu)(\mathbf{u}, d\mathbf{u}/d\tau^*).$$

Here, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are osculating quaternion elements related to quaternion variables $\mathbf{u}$, $d\mathbf{u}/d\tau^*$ by relationships (21); ν, τ_e are the scalar osculating elements; $(\mathbf{b}, \mathbf{c})$ is the scalar product of four-dimensional vectors $\mathbf{b}$ and $\mathbf{c}$, $\mu = f(m + M)$. Element ν is interpreted as frequency and is related to energy h by the relationship $\nu = (-h/2)^{1/2}$, $h < 0$, and time element τ_e is related to time t and variables u_j and $du_j/d\tau^*$ by relationship (22).

In these equations, τ^* is the new independent variable related to variable τ used in the initial equations by the differential expression

$$d\tau^* = 2\nu d\tau = 2(-h/2)^{1/2}d\tau, \quad h < 0.$$

In case of nonperturbed Kepler motion, we have

$$\tau^* = 2\nu\tau = 2(-h/2)^{1/2}\tau, \ \nu = (-h/2)^{1/2} = \text{const}.$$

Equations (19)–(22) are the quaternion analogue of equations of the spatial two-body problem in regular elements derived by Stiefel and Scheifele [8, p. 93]. The following advantages of these equations should be mentioned: first, they are regular (i.e., they have no singular point at the origin of the coordinates); second, their right parts are slowly and uniformly changing functions in the case of perturbed elliptic motion, and in the case of nonperturbed Kepler motion the equations are integrated without methodological errors. The disadvantage of these equations is that their domain of application is confined to elliptic motions (with $h < 0$).

Note 1. The following regular equations may be derived using the considered approach to regularization without introducing Kustaanheimo–Stiefel variables [21]:

$$v_1' + h + \frac{1}{2}v_1^2 - \frac{1}{2}r^2(\omega_2^2 + \omega_3^2) = rp_1, \quad r' = rv_1,$$

$$\omega_3' + 2\omega_3 v_1 + r\omega_1\omega_2 = p_2,$$

$$\omega_2' + 2\omega_2 v_1 - r\omega_1\omega_3 = -p_3,$$

$$h' = -r(p_1 v_1 + rp_2\omega_3 - rp_3\omega_2), \quad t' = r,$$

$$2\boldsymbol{\lambda}' = r\boldsymbol{\lambda} \circ \boldsymbol{\omega}_\eta.$$

These equations form a closed system of regular tenth-order differential equations for the spatial two-body problem with respect to variables v_1, r, ω_2, ω_3, h, t, and λ_j. Cartesian coordinates x_i of the center of mass of the second body and projections of its speed $\dot{x}_i$ onto axes of the inertial system of coordinates are defined on the basis of these variables using the formulas

$$\mathbf{r}_x = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3 = r\boldsymbol{\lambda} \circ \mathbf{i}_1 \circ \bar{\boldsymbol{\lambda}},$$

$$\mathbf{v}_x = \dot{\mathbf{r}}_x = \boldsymbol{\lambda} \circ \mathbf{v}_\eta \circ \bar{\boldsymbol{\lambda}}, \quad \mathbf{v}_\eta = v_1\mathbf{i}_1 + r\omega_3\mathbf{i}_2 - r\omega_2\mathbf{i}_3.$$

The rotation speed component ω_1 in regular equations is an arbitrarily set functions of the indicated variables. The equations are simplified in the case of $\omega_1 = 0$.

When the transition to new variables $v_2 = r\omega_3$, $v_3 = -r\omega_2$, which are interpreted as projections of the center-of-mass velocity vector of the second body onto the axes of rotating coordinates η, is carried out in the presented regular equations, and assuming $\omega_1 = 0$, we derive the regular equations for the two-body problem in another, simpler and more illustrative form:

$$v_1' + h + \frac{1}{2}(v_1^2 - v_2^2 - v_3^2) = rp_1, \quad r' = rv_1,$$

$$v_2' + v_1 v_2 = rp_2, \quad v_3' + v_1 v_3 = rp_3,$$

$$h' = -r(\mathbf{p} \cdot \mathbf{v}) = -r(p_1 v_1 + p_2 v_2 + p_3 v_3), \quad t' = r,$$

$$2\boldsymbol{\lambda}' = \boldsymbol{\lambda} \circ (-v_3\mathbf{i}_2 + v_2\mathbf{i}_3).$$

The disadvantage of these regular equations for the two-body problem is their nonlinearity in case of nonperturbed non-circular Kepler motions. In case of circular Kepler motion with $p_1 = p_2 = p_3 = 0$, $r = \text{const}$, and $v_1 = 0$, it follows from the first three regular equations that velocity projections v_2 and v_3 have constant values, and Kepler energy $h = \frac{1}{2}(v_2^2 + v_3^2) = \text{const}$. At the same time, the differential equation for the rotation quaternion $\boldsymbol{\lambda}$ becomes a linear equation with constant coefficients in this case.

4. QUATERNION REGULARIZATION OF EQUATIONS FOR PERTURBED CENTRAL MOTION OF A MATERIAL POINT

The ideas of quaternion regularization of equations of the two-body problem were later used by the author [22, 23, 25–28] to develop the theory of quaternion regularization of vector differential equations for the perturbed central motion of a material point

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{1}{m}\left(\frac{d\Pi}{dr}\frac{\mathbf{r}}{r} + \frac{\partial\Pi^*}{\partial\mathbf{r}}\right) + \mathbf{p},$$

$$r = |\mathbf{r}|, \quad \Pi = \Pi(r), \quad \Pi^* = \Pi^*(t, \mathbf{r}), \tag{23}$$

$$\mathbf{p} = \mathbf{p}\left(t, \mathbf{r}, \frac{d\mathbf{r}}{dt}\right).$$

This equation describes the motion of a material point with mass m within the central force field with potential Π, which is an arbitrary differentiable function of distance r from the point to the center of the force field under the perturbing force equal to geometric sum of the force with potential Π^* and the force $m\mathbf{p}$. Here, $\mathbf{r}$ is the radius vector of the material point drawn from the center of attraction O, and $\mathbf{p}$ is the perturbing acceleration caused by the force $m\mathbf{p}$.

The equation of nonperturbed central motion of a material point is derived from (23), if we assume $\Pi^* = 0$ and $\mathbf{p} = 0$.

General quaternion equations of the perturbed central motion with regularizing functions were derived, and the necessary and sufficient conditions of their reducibility to the oscillator form (i.e., the form of equations of motion for the four-dimensional perturbed oscillator, which performs harmonic oscillations with the same frequency in the case of nonperturbed central motion), which is convenient for analytical and numerical investigation, were established in the author's papers [22, 23, 25–28]; various systems of quaternion equations (including the new regular ones) for the perturbed central motion of a material point with different structures, dimensions, and dependent and independent variables were obtained in normal and oscillator forms; comparative characterization of the systems is performed, and their properties and domains of application are indicated; the new efficient equations of motion for a satellite in the Earth's gravitational field are constructed.

For instance, the regular equations for the perturbed central motion of a material point in the oscillator form were obtained [22, 25, 28]:

$$2\frac{d^2\boldsymbol{\lambda}}{d\tau^2} + \frac{c^2}{4}\boldsymbol{\lambda} = \frac{r^3}{2}[\bar{\mathbf{Q}}^* - \mathrm{scal}(\boldsymbol{\lambda} \circ \mathbf{Q}^*)\boldsymbol{\lambda}], \tag{24}$$

$$\frac{d^2r}{d\tau^2} + c^2r + \frac{1}{m}\left[\frac{d}{dr}(r^4\Pi) - 4(h^* - \Pi^*)r^3\right] = r^4\left[\mathrm{scal}(\boldsymbol{\lambda} \circ \mathbf{q}) - \frac{1}{m}\frac{\partial \Pi^*}{\partial r}\right]; \tag{25}$$

$$\frac{dh^*}{d\tau} = r^2\frac{\partial \Pi^*}{\partial t} + m\,\mathrm{scal}(\boldsymbol{\mu}^* \circ \mathbf{q}), \quad \boldsymbol{\mu}^* = \frac{dr}{d\tau}\boldsymbol{\lambda} + 2r\frac{d\boldsymbol{\lambda}}{d\tau}, \tag{26}$$

$$\frac{dc^2}{d\tau} = 4r^3\mathrm{scal}\left(\frac{d\boldsymbol{\lambda}}{d\tau} \circ \mathbf{Q}^*\right), \tag{27}$$

$$\frac{dt}{d\tau} = r^2; \tag{28}$$

$$\begin{gathered} \mathbf{Q}^* = \mathbf{q} - \frac{r^{-1}}{2m}\frac{\partial \Pi^*}{\partial \bar{\boldsymbol{\lambda}}}, \quad \mathbf{q} = -\mathbf{i}_1 \circ \bar{\boldsymbol{\lambda}} \circ \mathbf{p}_x, \\ \frac{\partial \Pi^*}{\partial \bar{\boldsymbol{\lambda}}} = \frac{\partial \Pi^*}{\partial \lambda_0} - \sum_{i=1}^{3}\frac{\partial \Pi^*}{\partial \lambda_i}\mathbf{i}_i, \\ \bar{\mathbf{Q}}^* = \bar{\mathbf{q}} - \frac{r^{-1}}{2m}\frac{\partial \Pi^*}{\partial \boldsymbol{\lambda}}, \quad \bar{\mathbf{q}} = -\mathbf{p}_x \circ \bar{\boldsymbol{\lambda}} \circ \mathbf{i}_1, \\ \frac{\partial \Pi^*}{\partial \boldsymbol{\lambda}} = \frac{\partial \Pi^*}{\partial \lambda_0} + \sum_{i=1}^{3}\frac{\partial \Pi^*}{\partial \lambda_i}\mathbf{i}_i, \\ \Pi^* = \Pi^*(t, \mathbf{r}_x), \quad \mathbf{p}_x = \mathbf{p}_x(t, \mathbf{r}_x, \mathbf{v}_x), \\ \mathbf{r}_x = r\boldsymbol{\lambda} \circ \mathbf{i}_1 \circ \bar{\boldsymbol{\lambda}}, \quad \mathbf{v}_x = r^{-2}\boldsymbol{\lambda} \circ \mathbf{i}_1 \circ \bar{\boldsymbol{\mu}}^*. \end{gathered} \tag{29}$$

Equation (24) for attitude quaternion λ of nonholonomic moving trihedral η and equations (26) and (27) for total energy h^* and square modulus of the momentum vector of the material point speed c^2 are regular for any form of potential $\Pi(r)$ of the central force field. At the same time, equation (25) for distance r is regular for the potential

$$\Pi(r) = -a_1r^{-1} - a_2r^{-2} - a_3r^{-3} - a_4r^{-4}, \quad a_i = \mathrm{const}, \tag{30}$$

which is a polynomial having a negative fourth power of distance r. (It should be noted that Kustaanheimo–Stiefel equations are only regular for polynomials having the negative first power of distance r).

System of equations (24)–(29) uses the value τ, which is related to time t by the differential relationship $d\tau = r^{-2}dt$, as an independent variable. When the transition to a new independent variable φ occurs in these equations in accordance with the differential relationship $d\varphi = cr^{-2}dt$, we obtain the system of equations (31)–(36). This system includes quaternion equation (31) in oscillator form, which corresponds to the equation of motion of a four-dimensional single-frequency harmonic oscillator with constant frequency of 1/2 in case of non-perturbed central motion:

$$\frac{d^2\boldsymbol{\lambda}}{d\varphi^2} + \frac{1}{4}\boldsymbol{\lambda} = -\frac{1}{2c^2}\left[\frac{dc^2}{d\varphi}\frac{d\boldsymbol{\lambda}}{d\varphi} - r^3(\bar{\mathbf{Q}}^* - \mathrm{scal}(\boldsymbol{\lambda} \circ \mathbf{Q}^*)\boldsymbol{\lambda})\right], \tag{31}$$

$$\frac{d^2r}{d\varphi^2} + r + \frac{1}{mc^2}\left[\frac{d}{dr}(r^4\Pi) - 4(h^* - \Pi^*)r^3\right] = \frac{1}{c^2}\left[-\frac{1}{2}\frac{dc^2}{d\varphi}\frac{dr}{d\varphi} + r^4\left(\mathrm{scal}(\boldsymbol{\lambda} \circ \mathbf{q}) - \frac{1}{m}\frac{\partial \Pi^*}{\partial r}\right)\right], \tag{32}$$

$$\frac{dh^*}{d\varphi} = \frac{r^2}{c}\frac{\partial \Pi^*}{\partial t} + m\,\mathrm{scal}(\boldsymbol{\mu}^{**} \circ \mathbf{q}), \quad \boldsymbol{\mu}^{**} = \frac{dr}{d\varphi}\boldsymbol{\lambda} + 2r\frac{d\boldsymbol{\lambda}}{d\varphi}, \tag{33}$$

$$\frac{dc^2}{d\varphi} = 4r^3\mathrm{scal}\left[\frac{d\boldsymbol{\lambda}}{d\varphi} \circ \mathbf{Q}^*\right], \tag{34}$$

$$\frac{dt}{d\varphi} = \frac{r^2}{c}, \quad c \neq 0, \tag{35}$$

$$\mathbf{r}_x = r\boldsymbol{\lambda} \circ \mathbf{i}_1 \circ \bar{\boldsymbol{\lambda}}, \quad \mathbf{v}_x = cr^{-2}\boldsymbol{\lambda} \circ \mathbf{i}_1 \circ \bar{\boldsymbol{\mu}}^{**}. \tag{36}$$

Here, quaternions $\mathbf{Q}^*$ and $\mathbf{q}$ are defined by relationships (29) and (36).

The author also derived regular systems of equations for perturbed central motion, which include quaternion regular equations in normal form [22, 28] (in these systems, vect() is the vector part of the quaternion in parentheses):

1) The system of regular equations with independent variable τ:

$$\frac{d\mathbf{c}_\eta}{d\tau} = -r^3\mathrm{vect}(\bar{\mathbf{Q}}^* \circ \boldsymbol{\lambda}), \quad \mathbf{c}_\eta = c_2\mathbf{i}_2 + c_3\mathbf{i}_3,$$

$$2\frac{d\boldsymbol{\lambda}}{d\tau} = \boldsymbol{\lambda} \circ \mathbf{c}_\eta, \quad \frac{dt}{d\tau} = r^2.$$

The system is supplemented with differential equations (25) and (26) for distance r and total energy h^*, as well as relationships (29) for perturbing forces.

2) The system of regular equations with independent variable φ:

$$\frac{d\mathbf{c}_\eta}{d\varphi} = -r^3c^{-1}\mathrm{vect}(\mathbf{Q}^*\circ\boldsymbol{\lambda}), \quad c = (c_2^2+c_3^3)^{1/2},$$

$$2\frac{d\boldsymbol{\lambda}}{d\varphi} = c^{-1}\boldsymbol{\lambda}\circ\mathbf{c}_\eta, \quad \frac{dt}{d\varphi} = \frac{r^2}{c}, \quad c\neq 0.$$

The system is supplemented with equations (32) and (33) and relationships (29) and (36).

The presented quaternion equations for the perturbed central motion for variables $\mathbf{c}_\eta$ and $\boldsymbol{\lambda}$, as well as the equation for total energy h^* are regular for any form of the potential of the central force field, since they do not include it. However, the equation for distance r is only regular for the potential of the central force field (30). From this perspective, regular quaternion equations in normal form are in no way superior to regular quaternion equations in oscillator form. However, the order of systems of regular equations in normal form is 10 (i.e., the same as the order of the system of equations in regular Kustaanheimo–Stiefel variables), which is three orders lower than the order of systems of equations (24)–(29) and (31)–(36) in oscillator form.

It should be noted that if quaternion variable $\mathbf{c}_\eta$ (quaternion of the momentum of the material point speed) and total energy h^* are constant, and the differential equation for quaternion variable $\boldsymbol{\lambda}$ becomes a linear equation with constant coefficients, then the differential equation for distance r remains significantly nonlinear.

5. SATELLITE MOTION EQUATIONS IN THE EARTH'S GRAVITATIONAL FIELD

The author derived [22, 26] new equations of motion for a satellite in the Earth's gravitational field, which, in addition to all the advantages of the known equations in Kustaanheimo–Stiefel variables [8], have a simpler and more symmetric structure and tolerate a reduction of two in their order of magnitude.

The equations of motion for an artificial satellite in the Earth's gravitational field are written in vector form as follows:

$$\frac{d^2\mathrm{r}}{dt^2} = -\frac{1}{m}\frac{\partial\Pi_1}{\partial\mathbf{r}}, \quad m = \frac{m_0}{m_0+m_1},$$

Here, $\mathbf{r}$ is the geocentric radius vector of the satellite, m_0 is the Earth's mass, m_1 is the satellite's mass, Π_1 is the potential of the Earth's gravitational field.

When we discard tesseral and sectorial harmonics, we obtain [29]

$$\Pi_1 = -\frac{fm_0}{r}+\frac{fm_0}{r}\sum_{n=2}^{\infty}J_n\left(\frac{R}{r}\right)^nP_n(\sin\varphi_1),$$
$$r = |\mathbf{r}|,$$

where f is the gravitational constant, R is the Earth's average equatorial radius, J_n are the dimensionless constants which characterize the shape of the Earth, P_n is the nth-order Legendre polynomial, and φ_1 is the geocentric latitude.

The coordinates X are introduced as follows: the origin of coordinates O is located at the center of the Earth, OX_3 axis is directed towards the Earth's North Pole, and OX_1 axis is set towards the vernal equinox point.

Potential Π_1 is represented as follows:

$$\Pi_1 = \Pi(r)+\Pi^*(r,\gamma), \quad \gamma = \sin\varphi_1 = \cos\upsilon = x_3/r,$$

$$\Pi(r) = -\frac{fm_0}{r}, \quad \Pi^*(r,\gamma) = \frac{fm_0}{r}\sum_{n=2}^{\infty}J_n\left(\frac{R}{r}\right)^nP_n(\gamma),$$

where ϑ is the angle between the OX_3 axis and vector $\mathbf{r}$.

In Kustaanheimo–Stiefel case, axis η_1 of coordinates η is directed along the radius vector $\mathbf{r}$. Satellite coordinates x_i in the system of coordinates X are related to Kustaanheimo–Stiefel variables u_j by relationships

$$\begin{aligned} x_1 &= r(\lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2) = u_0^2+u_1^2-u_2^2-u_3^2,\\ x_2 &= 2r(\lambda_1\lambda_2+\lambda_0\lambda_3) = 2(u_1u_2-u_0u_3),\\ x_3 &= 2r(\lambda_1(\lambda_3-\lambda_0\lambda_2)) = 2(u_1u_3+u_0u_2);\end{aligned} \tag{37}$$

$$\begin{gathered} u_0 = r^{1/2}\lambda_0 \quad u_i = -r^{1/2}\lambda_i, \quad i = 1, 2, 3,\\ r = u_0^2+u_1^2+u_2^2+u_3^2,\end{gathered} \tag{38}$$

where λ_j are the Rodrigues–Hamilton parameters, which characterize the orientation of coordinates η relative to X.

Instead of axis η of coordinates η_1, we set axis η_3 along the radius vector $\mathbf{r}$. In this case, new variables u_j determined based on the Rodrigues–Hamilton parameters using formulas (38), similarly to the Kustaanheimo–Stiefel case, will be related to coordinates x_i by the relationships

$$\begin{aligned} x_1 &= 2r(\lambda_1\lambda_3+\lambda_0\lambda_2) = 2(u_1u_3-u_0u_2),\\ x_2 &= 2r(\lambda_2(\lambda_3-\lambda_0\lambda_1)) = 2(u_2u_3+u_0u_1),\\ x_3 &= r(\lambda_0^2-\lambda_1^2-\lambda_2^2+\lambda_3^2) = u_0^2-u_1^2-u_2^2+u_3^2.\end{aligned} \tag{39}$$

In Kustaanheimo–Stiefel case, we derive the following from (37):

$$\begin{aligned} \gamma &= \cos\vartheta = x_3/r = 2(\lambda_1\lambda_3-\lambda_0\lambda_2)\\ &= 2r^{-1}(u_1u_3+u_0u_2).\end{aligned} \tag{40}$$

In our case, we derive the following from (39):

$$\gamma = \lambda_0^2-\lambda_1^2-\lambda_2^2+\lambda_3^2 = r^{-1}(u_0^2-u_1^2-u_2^2+u_3^2). \tag{41}$$

Hence, taking into account (38) and the equality $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$, we derive:

$$\begin{aligned}\gamma &= 1 - 2(\lambda_1^2 + \lambda_2^2) = 2(\lambda_0^2 + \lambda_3^2) - 1 \\ &= 1 - 2r^{-1}(u_1^2 + u_2^2) = 2r^{-1}(u_0^2 + u_3^2) - 1.\end{aligned} \tag{42}$$

It can be seen from the comparison of (40) with (41) and (42) that the expressions of variable γ, on which the perturbing potential Π^* depends, based on variables u_j have a simpler and more symmetric structure in our case. Therefore, it is possible to obtain simpler and more symmetric equations of motion for the satellite, than in the Kustaanheimo–Stiefel case.

We introduce

$$\Pi^+(r, \gamma) = r\Pi^*(r, \gamma) = fm_0 \sum_{n=2}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\gamma). \tag{43}$$

Then, equations of motion for a satellite in the Earth's gravitational field may be written as follows [22, 26]:

$$\frac{d^2 u_k}{d\tau^2} - \frac{h^*}{2m} u_k = \frac{1}{2m}\left(\frac{\gamma - 1}{r}\frac{\partial \Pi^+}{\partial \gamma} - \frac{\partial \Pi^+}{\partial r}\right) u_k, \quad k = 0, 3, \tag{44}$$

$$\frac{d^2 u_s}{d\tau^2} - \frac{h^*}{2m} u_s = \frac{1}{2m}\left(\frac{\gamma + 1}{r}\frac{\partial \Pi^+}{\partial \gamma} - \frac{\partial \Pi^+}{\partial r}\right) u_s, \quad s = 1, 2, \tag{45}$$

$$\frac{dt}{d\tau} = r, \quad h^* = \text{const}, \tag{46}$$

$$\frac{d^2 r}{d\tau^2} - \frac{2h^*}{m} r = \frac{1}{m}\left[fm_0 - \frac{\partial}{\partial r}(r\Pi^+)\right], \tag{47}$$

Here, $h^* = h + \Pi^*$ is the satellite's total energy.

Equations (44)–(46) supplemented with relationships (42) and (43) and the relationship

$$r = u_0^2 + u_1^2 + u_2^2 + u_3^2,$$

form a closed system of differential equations with respect to variables u_j ($j = 0, 1, 2, 3$) and t (equations (44) and (45) may be considered independently from equation (46)). The order of this system of equations, which has all the advantages of the equations of motion in Kustaanheimo–Stiefel variables, is 9, i.e., it is the same as that of the system of equations in Kustaanheimo–Stiefel variables [8]. However, in comparison with the Kustaanheimo–Stiefel equations, the right-hand side of equations (44) and (45) have a simpler and more symmetric structure. It can be easily validated by comparison of the right-hand sides of f_j the equations of motion of the satellite in Kustaanheimo–Stiefel variables

$$f_0 = \frac{1}{2m}(l_1 u_0 - l_2 u_2), \quad f_1 = \frac{1}{2m}(l_1 u_1 - l_2 u_3),$$

$$f_2 = \frac{1}{2m}(l_1 u_2 - l_2 u_0), \quad f_3 = \frac{1}{2m}(l_1 u_3 - l_2 u_1),$$

$$l_1 = \frac{\gamma}{r}\frac{\partial \Pi^+}{\partial \gamma} - \frac{\partial \Pi^+}{\partial r}, \quad l_2 = \frac{1}{r}\frac{\partial \Pi^+}{\partial \gamma}$$

with the right-hand sides of equations (44) and (45).

To obtain coordinates x_i and projections of satellite velocity $\dot{x}_i$ based on variables u_j and their first variables, in our case, it is necessary to use formulas (39) and the relationship

$$\mathbf{v}_x = 2\bar{\mathbf{u}} \circ \mathbf{i}_3 \circ \dot{\mathbf{u}} = 2r^{-1}\bar{\mathbf{u}} \circ \mathbf{i}_3 \circ \frac{d\mathbf{u}}{d\tau}.$$

It should be noted that, in contrast with the equations in Kustaanheimo–Stiefel variables, equations (44) and (45) decompose into two independent symmetric subsystems, if distance r is defined from equation (47), which can be seen from (42). Therefore, the closed system of sixth-order differential equations is singled out from equations (44)–(47). It is either formed by equations (44) and (47) and relationship (43), where we assume

$$\gamma = 2r^{-1}(u_0^2 + u_3^2) - 1,$$

or by equations (45) and (47) and relationship (43), where we assume

$$\gamma = 1 - 2r^{-1}(u_1^2 + u_2^2),$$

If we assume that solutions $u_0(\tau)$, $u_3(\tau)$, $r(\tau)$ ($u_1(\tau)$, $u_2(\tau)$, $r(\tau)$) of the system of equations (44), (47) ((45), (47)) are obtained, then equations (45) ((44)) will be independent linear second-order differential equations with variable coefficients.

Any other disturbances affecting the satellite may be easily taken into account in the equations of motion of the satellite, if necessary.

Note 2. It was mentioned earlier that, in accordance with Levi-Civita [4], the procedure for eliminating the features of differential equations of motion is called regularization. The instability of solutions of the equations of motion of celestial bodies in the Lyapunov sense is another peculiarity, which significantly reduces the accuracy of numerical solutions of these equations. According to Baumgart and Stiefel, transformations of the equations of motion of celestial bodies, which lead to a weaker effect of the Lyapunov instability on the numerical solution of equations, are called stabilization [8, 11, 12, 30]. The transformation regularizing equations of motion and their solutions in the case of binary impacts of bodies with simultaneous stabilization of nonperturbed parts of solutions was introduced for the first time by Levi-Civita [3] for planar motion.

However, Baumgart [30] and Nacozy [31] are considered the pioneers of stabilization in celestial mechanics. Their stabilizing methods were based [12] on the use of the known first integrals, which include additional data on the solution and are considered the necessary conditions for the solution. The authors of stabilizing methods prioritize energy-based relationships above the other integral relationships, since practical experience shows that it is the stabilization in terms of energy that is the best support for overcoming the Lyapunov instability (see, for example, [11, 12]).

Similarly to regular Kustaanheimo–Stiefel equations, the regular quaternion equations for the perturbed two-body problem presented above include the differential equation for Kepler and total energy. The other forms of these equations presented in the paper include the equation for the square modulus of the momentum vector or the equation for the momentum vector, in addition to the equation for energy. Thus, the effect of stabilization of solutions of quaternion equations is achieved, to an extent, for the perturbed two-body problem, along with the regularization effect.

Note 3. A few more regularization and stabilization methods for equations of the perturbed two-body problem may be indicated [11]. For example, the method based on the introduction of the energy integral and Laplace integrals into the equations of motion [32–34]. The regularizing method for pair impacts in the problem for n bodies based on introduction of additional dependent variables may be found in papers by Duboshin [35] and Myachin [36, 37].

6. PAPERS ON KS AND QUATERNION REGULARIZATION AND STABILIZATION OF EQUATIONS OF CELESTIAL MECHANICS AND ASTRODYNAMICS BY OTHER AUTHORS

An investigation of the perturbed Kepler motion was carried out by Stiefel and Scheifele in their book [8], not only based on regular equations in oscillator form and oscillation theory methods, but also using regular equations in canonical form, which required developing the theory of canonical KS transform. This canonical approach to the regularization problem, which uses the KS transform, was developed in papers by Lidov [38–40]. Application of the generalized KS matrix and the related transformations in regularization theory for canonical equations of the two-body problem was considered later in [41].

The paper [42], where differential equations of motion of the Earth's artificial satellite written in orbital coordinates are obtained, should be mentioned as well. To describe the motion of coordinates, the rotation quaternion normalized by a factor equal to the square root of the modulus of the momentum vector of the satellite is used. These equations are linear in the case of nonperturbed Kepler motion of the satellite.

The papers by Vivareli [43] and Vrbik [44, 45] should be mentioned among the papers by foreign authors, since they demonstrated the applicability of quaternions for regularizing the equations of celestial mechanics later than the author of the present paper, but apparently independently. In recent years, the papers by the recognized western scientist Jorg Waldvogel, who also collaborated with Stiefel [48], appeared in foreign journals [46, 47]. For instance, he published the paper "Quaternions for regularizing celestial mechanics: the right way" [47] in the journal "Celestial Mechanics and Astrodynamics" in 2008, where he stated that "quaternions have been found to be the ideal tool for describing and developing the theory of spatial regularization in celestial mechanics".

Yes, indeed. However, it was shown by the author of the present paper more than twenty years ago, and the quaternion method for regularizing the equations of the spatial two-body problem described by Jorg Waldvogel in his paper has no advantages over the quaternion regularization method proposed by the author of the present paper much earlier. Moreover, in our opinion, it lacks the flexibility, geometric and kinematic clarity, and possibilities for further generalization of the latter method.

It should also be noted that Waldvogel admits the priority of the author of the present paper in the field of quaternion regularization in [47] stating the following. "This statement[2] was first refuted by Chelnokov (1981) who presented a regularization theory of the spatial Kepler problem using geometrical considerations in a rotating coordinate system and quaternion matrices. In a series of papers, including Chelnokov (1992, 1999), the same author extended the theory of quaternion regularization and also presented practical applications".

The specific key features of the quaternion regularization method by Waldvogel [47] are as follows. He suggests using the star conjugate quaternion

$$\mathbf{u}^* = -k\bar{\mathbf{u}}k = u_0 + iu_1 + ju_2 - ku_3 \tag{48}$$

(the quaternion is star conjugate with the quaternion $\mathbf{u} = u_0 + iu_1 + ju_2 + ku_3$) and the mapping

$$\mathbf{u} \in U \to \boldsymbol{x} = \mathbf{u}\mathbf{u}^*. \tag{49}$$

for regularization.

Unconventional representation of the three-dimensional vector $\mathbf{x}$ by the quaternion $\mathbf{x} = x_0 + ix_1 + jx_2$ with zero k-component is used in this mapping. This quaternion $\mathbf{x}$ is a formal generalization (increment) of the complex variable $\mathbf{x} = x_0 + ix_1$, which was

[2] Implied here is the statement by Stiefel and Scheifele on the lack of prospects for the use of the regularization theory of quaternion matrices cited above.

used by Levi-Civita in the regularization theory for planar motion.

Taking into account (48), mapping (49) becomes as follows:

$$\mathbf{x} = \mathbf{u}\mathbf{u}^* = -\mathbf{u}k\bar{\mathbf{u}}k. \tag{50}$$

On the basis of (50) written in scalar form, we have

$$\begin{gathered} x_0 = u_0^2 - u_1^2 - u_2^2 + u_3^2, \quad x_1 = 2(u_0u_1 - u_2u_3), \\ x_2 = 2(u_0u_2 + u_1u_3), \end{gathered} \tag{51}$$

"which is exactly the KS transformation in its classical form or, up to a permutation of the indices, the Hopf map" (cited from the author of [47]).

In classical quaternion theory, three-dimensional vector $\mathbf{x}$ is matched with the quaternion $\mathbf{x} = ix_1 + jx_2 + kx_3$ with a zero scalar part. The author of the present paper uses the quaternion variable $\mathbf{u} = u_0 + iu_1 + ju_2 + ku_3$, which does not (essentially) coincide with the Waldvogel's quaternion variable, and the quaternion $\mathbf{x} = ix_1 + jx_2 + kx_3$ with zero scalar parts for regularization. In the author's papers (the present one included), the mapping $\mathbf{x} = \bar{\mathbf{u}}\, i\mathbf{u}$ and the mapping $\mathbf{x} = \bar{\mathbf{u}}\, k\mathbf{u}$ are used.

When written down in scalar form, the first mapping is exactly the Kustaanheimo–Stiefel transformation (3), which differs from (51) in form.

It should be mentioned that no essentially (rather than formally) new regular equations for the two-body problem were obtained in [47]. Nevertheless, due to the special significance of the regularization problem in celestial mechanics and astrodynamics, the quaternion regularization method by Waldvogel is of undoubted interest. The elegant quaternion representation of the spatial Birkhoff transformation used in the regularization theory for the equations of the restricted three-body problem obtained by Waldvogel in his paper is of undoubted interest as well. Waldvogel provided this representation as an addition to his earlier papers on regularization theory [48–50].

To conclude, the book by the author of the paper [15] published in 2011 should be mentioned. In particular, the quaternion regularization method for differential equations of the spatial two-body problem and the perturbed central motion of a material point is presented in this book, regular quaternion models of celestial mechanics and astrodynamics are provided, and their applications to solving problems of the optimal control of the trajectory of a spacecraft are described.

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