5

Lyapunov Stability Theory

5.1 Introduction

The study of the stability of dynamical systems has a very rich history. Many famous mathematicians, physicists, and astronomers worked on axiomatizing the concepts of stability. A problem, which attracted a great deal of early interest was the problem of stability of the solar system, generalized under the title "the N-body stability problem." One of the first to state formally what he called the principle of "least total energy" was Torricelli (1608–1647), who said that a system of bodies was at a stable equilibrium point if it was a point of (locally) minimal total energy. In the middle of the eighteenth century, Laplace and Lagrange took the Torricelli principle one step further: They showed that if the system is conservative (that is, it conserves total energy—kinetic plus potential), then a state corresponding to zero kinetic energy and minimum potential energy is a stable equilibrium point. In turn, several others showed that Torricelli's principle also holds when the systems are dissipative, i.e., total energy decreases along trajectories of the system. However, the abstract definition of stability for a dynamical system not necessarily derived for a conservative or dissipative system and a characterization of stability were not made till 1892 by a Russian mathematician/engineer, Lyapunov, in response to certain open problems in determining stable configurations of rotating bodies of fluids posed by Poincaré. The original paper of Lyapunov of 1892, was translated into French very shortly there after, but its English translation appeared only recently in [193]. The interested reader may consult this reference for many interesting details, as well as the historical and biographical introduction in this issue of the *International Journal of Control* by A. T. Fuller. There is another interesting survey paper about the impact of Lyapunov's stability theorem on feedback control by Michel [208].

At heart, the theorems of Lyapunov are in the spirit of Torricelli's principle. They give a precise characterization of those functions that qualify as "valid energy functions" in the vicinity of equilibrium points and the notion that these "energy functions" decrease along the trajectories of the dynamical systems in question. These precise concepts were combined with careful definitions of different notions of stability to give some very powerful theorems. The exposition of these theorems is the main goal of this chapter.

5.2 Definitions

This chapter is concerned with general differential equations of the form

$$\dot{x} = f(x, t), \quad x(t_0) = x_0, \tag{5.1}$$

where $x \in \mathbb{R}^n$ and $t \geq 0$. The system defined by (5.1) is said to be *autonomous* or *time-invariant* if f does not depend explicitly on t. It is said to be *linear* if $f(x, t) = A(t)x$ for some $A(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}^{n \times n}$ and nonlinear otherwise. In this chapter, we will assume that $f(x, t)$ is *piecewise continuous* with respect to t, that is, there are only finitely many discontinuity points in any compact set. The notation B_h will be short-hand for $B(0, h)$, the ball of radius h centered at 0. Properties will be said to be true

- *locally* if they are true for all x_0 in some ball B_h.
- *globally* if they are true for all $x_0 \in \mathbb{R}^n$.
- *semi-globally* if they are true for all $x_0 \in B_h$ with h arbitrary.
- *uniformly* if they are true for all $t_0 \geq 0$.

In the development that follows, unless explicitly specified, properties are true locally. The first few definitions and estimates are repeated from Chapter 3.

5.2.1 *The Lipschitz Condition and Consequences*

Definition 5.1 Lipschitz Continuity. *The function f is said to be* locally Lipschitz continuous *in x if for some $h > 0$ there exists $l \geq 0$ such that*

$$|f(x_1, t) - f(x_2, t)| \leq l|x_1 - x_2| \tag{5.2}$$

for all $x_1, x_2 \in B_h$, $t \geq 0$. The constant l is called the Lipschitz constant. *A definition for* globally Lipschitz continuous *functions follows by requiring equation (5.2) to hold for $x_1, x_2 \in \mathbb{R}^n$. The definition of* semi-globally Lipschitz continuous *functions holds as well by requiring that equation (5.2) hold in B_h for arbitrary h but with l possibly a function of h. The Lipschitz property is by default assumed to be uniform in t.*

If f is Lipschitz continuous in x, it is continuous in x. On the other hand, if f has bounded partial derivatives in x, then it is Lipschitz. Formally, if

$$D_1 f(x,t) := \left[\frac{\partial f_i}{\partial x_j}\right]$$

denotes the partial derivative matrix of f with respect to x (the subscript 1 stands for the first argument of $f(x,t)$), then $|D_1 f(x,t)| \leq l$ implies that f is Lipschitz continuous with Lipschitz constant l (again locally, globally or semi-globally depending on the region in x that the bound on $|D_2 f(x,t)|$ is valid). We have seen in Chapter 3 that if f is locally bounded and Lipschitz continuous in x, then the differential equation (5.1) has a unique solution on some time interval (so long as $x \in B_h$).

Definition 5.2 Equilibrium Point. *x^* is said to be an* equilibrium point *of (5.1) if $f(x^*, t) \equiv 0$ for all $t \geq 0$.*

If $f(x,t)$ is Lipschitz continuous in x, then the solution $x(t) \equiv x^*$ for all t is called an *equilibrium solution.* By translating the origin to the equilibrium point x^* we can make 0 an equilibrium point. Since this is of great notational help, we will henceforth assume that 0 is an equilibrium point of (5.1). One of the most important consequences of the Lipschitz continuity hypothesis is that it gives bounds on the rate of convergence or divergence of solutions from the origin:

Proposition 5.3 Rate of Growth/Decay. *If $x = 0$ is an equilibrium point of (5.1) f is Lipschitz in x with Lipschitz constant l and piecewise constant with respect to t, then the solution $x(t)$ satisfies*

$$|x_0|e^{l(t-t_0)} \geq |x(t)| \geq |x_0|e^{-l(t-t_0)} \tag{5.3}$$

as long as $x(t)$ remains in B_h.

Proof: Since $|x|^2 = x^T x$, it follows that

$$\begin{aligned} \left|\frac{d}{dt}|x|^2\right| &= 2\,|x|\left|\frac{d}{dt}\,|x|\right| \\ &= 2\left|x^T\frac{d}{dt}x\right| \leq 2\,|x|\left|\frac{d}{dt}x\right|, \end{aligned} \tag{5.4}$$

so that

$$\left|\frac{d}{dt}\,|x|\right| \leq \left|\frac{d}{dt}x\right|.$$

Since $f(x,t)$ is Lipschitz continuous, and $f(x,0) = 0$ it follows that

$$-\,l|x| \leq \frac{d}{dt}|x| \leq l|x| \tag{5.5}$$

Using the Bellman Gronwall lemma twice (refer back to Chapter 3 and work out the details for yourself) yields equation (5.3), provided that the trajectory stays in the ball B_h where the Lipschitz condition holds. □

The preceding proposition implies that solutions starting inside B_h will stay in B_h for at least a finite time. Also, if $f(x,t)$ is globally Lipschitz, it guarantees that the solution has no finite escape time, that is, it is finite at every finite instant. The proposition also establishes that solutions $x(t)$ cannot converge to zero faster than exponentially.

We are now ready to make the stability definitions. Informally $x = 0$ is *stable* equilibrium point if trajectories $x(t)$ of (5.1) remain close to the origin if the initial condition x_0 is close to the origin. More precisely, we have the following definition.

Definition 5.4 Stability in the sense of Lyapunov. *The equilibrium point $x = 0$ is called a* stable *equilibrium point of (5.1) if for all $t_0 \geq 0$ and $\epsilon > 0$, there exists $\delta(t_0, \epsilon)$ such that*

$$|x_0| < \delta(t_0, \epsilon) \Rightarrow |x(t)| < \epsilon \quad \forall\ t \geq t_0, \tag{5.6}$$

where $x(t)$ is the solution of (5.1) starting from x_0 at t_0.

The definition is illustrated in Figure 5.1, showing the trajectories starting in a ball B_δ and not leaving the ball B_ϵ. Sometimes this definition is also called *stability in the sense of Lyapunov (i.s.L.) at time t_0.*

Definition 5.5 Uniform Stability. *The equilibrium point $x = 0$ is called a* uniformly stable *equilibrium point of (5.1) if in the preceding definition δ can be chosen independent of t_0.*

Intuitively, the definition of uniform stability captures the notion that the equilibrium point is not getting progressively less stable with time. Thus, in particular, it prevents a situation in which given an $\epsilon > 0$, the ball of initial conditions of radius $\delta(t_0, \epsilon)$ in the definition of stability required to hold trajectories in the ϵ ball tends to zero as $t_0 \to \infty$. The notion of stability is weak in that it does not require that trajectories starting close to the origin to tend to the origin asymptotically. That property is included in a definition of asymptotic stability:

Definition 5.6 Asymptotic Stability. *The equilibrium point $x = 0$ is an* asymptotically stable *equilibrium point of (5.1) if*

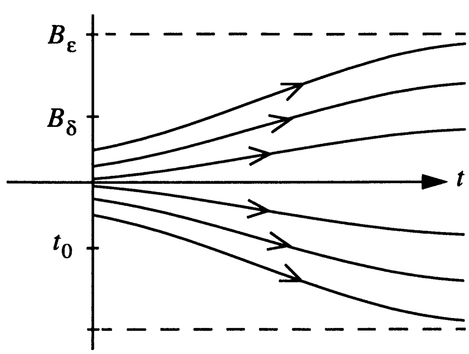

FIGURE 5.1. Illustrating the definition of stability

- $x = 0$ *is a stable equilibrium point of (5.1),*
- $x = 0$ *is attractive, that is for all* $t_0 \geq 0$ *there exists a* $\delta(t_0)$ *such that*

$$|x_0| < \delta \Rightarrow \lim_{t \to \infty} |x(t)| = 0.$$

An interesting feature of this definition is that it requires two separate conditions: one, that the equilibrium point be stable and two that trajectories tend to the equilibrium point as $t \to \infty$. Though, it may superficially appear to be the case, the requirement that trajectories converge to the origin does not imply the stability of the equilibrium point. To illustrate this, we consider the following example:

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_2^2, \\ \dot{x}_2 &= 2x_1x_2. \end{aligned} \tag{5.7}$$

The phase portrait of this system (also presented as an example of an equilibrium point with index 2 in Chapter 2) is as shown in Figure 5.2. All trajectories tend to the origin as $t \to \infty$, except for the trajectory that follows the positive x_1 axis to $+\infty$. By assuming that this "point at infinity" is the same as the point at infinity at $x_1 = -\infty$ we may assert that all trajectories tend to the origin.[1] However, the equilibrium point at the origin is not stable in the sense of Lyapunov: Given any $\epsilon > 0$, no matter how small a δ we choose for the ball of initial condition, there are always some initial conditions close to the x_1 axis which will exit the ϵ ball before converging to the origin. The trajectory starting right on the x_1 axis gives a hint to this behavior.

Definition 5.7 Uniform Asymptotic Stability. *The equilibrium point* $x = 0$ *is a* uniformly asymptotically stable *equilibrium point of (5.1) if*

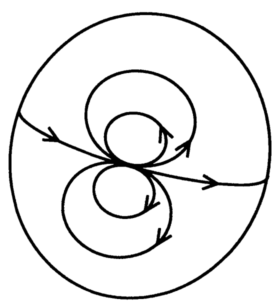

FIGURE 5.2. An equilibrium point, which is not stable, but for which all trajectories tend to the origin if one includes the point at infinity

[1]This procedure is referred to in mathematical terms as the Alexandroff or one-point compactification of $\mathbb{R}^2$. Another way of conceptualizing this procedure is to imagine that the plane is mapped onto a sphere with the origin corresponding to the north pole and the "point at infinity" to the south pole; that is, the state space of the system is really the sphere S^2.

- *$x = 0$ is a uniformly stable equilibrium point of (5.1).*
- *The trajectory $x(t)$ converges uniformly to 0, that is, there exists $\delta > 0$ and a function $\gamma(\tau, x_0) : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+$ such that $\lim_{\tau\to\infty} \gamma(\tau, x_0) = 0$ for all $x_0 \in B_\delta$ and*

$$|x_0| < \delta \Rightarrow |x(t)| \leq \gamma(t - t_0, x_0) \quad \forall\ t \geq t_0.$$

The previous definitions are *local*, since they concern neighborhoods of the equilibrium point. *Global* asymptotic stability and global uniform asymptotic stability are defined as follows:

Definition 5.8 Global Asymptotic Stability. *The equilibrium point $x = 0$ is a* globally asymptotically stable *equilibrium point of (5.1) if it is stable and* $\lim_{t\to\infty} x(t) = 0$ *for all* $x_0 \in \mathbb{R}^n$.

Definition 5.9 Global Uniform Asymptotic Stability. *The equilibrium point $x = 0$ is a* globally, uniformly, asymptotically stable *equilibrium point of (5.1) if it is globally asymptotically stable and if in addition, the convergence to the origin of trajectories is uniform in time, that is to say that there is a function $\gamma : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}$ such that*

$$|x(t)| \leq \gamma(x_0, t - t_0) \quad \forall\ t \geq 0.$$

It is instructive to note that the definitions of asymptotic stability do not quantify the speed of convergence of trajectories to the origin. For time-invariant linear systems, the speed of convergence of trajectories either to or from the origin is exponential, but for time varying and nonlinear systems, the rate of convergence can be of many different types, for example as

$$\frac{1}{t}, \qquad \frac{1}{\sqrt{t}}.$$

It is a good exercise to write down examples of this behavior. There is a strong form of stability which demands an exponential rate of convergence:

Definition 5.10 Exponential Stability, Rate of Convergence. *The equilibrium point $x = 0$ is an* exponentially stable *equilibrium point of (5.1) if there exist $m, \alpha > 0$ such that*

$$|x(t)| \leq m e^{-\alpha(t-t_0)} |x_0| \tag{5.8}$$

for all $x_0 \in B_h$, $t \geq t_0 \geq 0$. The constant α is called (an estimate of) the rate of convergence.

Global exponential stability is defined by requiring equation (5.8) to hold for all $x_0 \in \mathbb{R}^n$. Semi-global exponential stability is also defined analogously except that m, α are allowed to be functions of h. For linear (possibly time-varying) systems it will be shown that uniform asymptotic stability is equivalent to exponential stability, but in general, exponential stability is stronger than asymptotic stability.

5.3 Basic Stability Theorems of Lyapunov

The so-called second method of Lyapunov enables one to determine the stability properties of a system (5.1) without explicitly integrating the differential equation. The method is a generalization of the basic notion that some measure of "energy dissipation" in a system enables us to conclude stability. To make this precise we need to define exactly what one means by a "measure of energy," that is, energy functions. This needs the following preliminary definitions (due, in the current form to Hahn [124]):

5.3.1 Energy-Like Functions

Definition 5.11 Class K, KR Functions. *A function $\alpha(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ belongs to* class K *(denoted by $\alpha(\cdot) \in K$) if it is continuous, strictly increasing and $\alpha(0) = 0$. The function $\alpha(\cdot)$ is said to belong to class KR if α is of class K and in addition, $\alpha(p) \to \infty$ as $p \to \infty$.*

Definition 5.12 Locally Positive Definite Functions. *A continuous function $v(x, t) : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is called a* locally positive definite function (l.p.d.f) *if, for some $h > 0$ and some $\alpha(\cdot)$ of class K,*

$$v(0, t) = 0 \quad \textit{and} \quad v(x, t) \geq \alpha(|x|) \quad \forall\, x \in B_h, \quad t \geq 0. \tag{5.9}$$

An l.p.d.f. is locally like an "energy function." Functions which are globally like "energy functions" are called positive definite functions (p.d.f.s) and are defined as follows:

Definition 5.13 Positive Definite Functions. *A continuous function $v(x, t) : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is called a* positive definite function (p.d.f.) *if for some $\alpha(\cdot)$ of class KR,*

$$v(0, t) = 0 \quad \textit{and} \quad v(x, t) \geq \alpha(|x|) \quad \forall\, x \in \mathbb{R}^n, \quad t \geq 0 \tag{5.10}$$

and, in addition, $\alpha(p) \to \infty$ as $p \to \infty$.

In the preceding definitions of l.p.d.f.s and p.d.f.s, the energy was not bounded above as t varied. This is the topic of the next definition

Definition 5.14 Decrescent Functions. *A continuous function $v(x, t) : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is called a* decrescent *function if, there exists a function $\beta(\cdot)$ of class K, such that*

$$v(x, t) \leq \beta(|x|) \quad \forall\, x \in B_h, \quad t \geq 0. \tag{5.11}$$

Example 5.15 Examples of Energy-like Functions. *Here are some examples of energy like functions and their membership in the various classes introduced above. It is an interesting exercise to check the appropriate functions of class K and KR that need to be used to verify these properties.*

1. $v(x, t) = |x|^2$: *p.d.f., decrescent.*
2. $v(x, t) = x^T P x$, *with* $P \in \mathbb{R}^{n\times n} > 0$: *p.d.f., decrescent.*
3. $v(x, t) = (t + 1)|x|^2$: *p.d.f.*
4. $v(x, t) = e^{-t}|x|^2$: *decrescent.*
5. $v(x, t) = \sin^2(|x|^2)$: *l.p.d.f., decrescent.*
6. $v(x, t) = e^t x^T P x$ *with* P *not positive definite: not in any of the above classes.*
7. $v(x, t)$ *not explicitly depending on time* t*: decrescent.*

5.3.2 Basic Theorems

Generally speaking, the basic theorem of Lyapunov states that when $v(x, t)$ is a p.d.f. or an l.p.d.f. and $dv(x, t)/dt \leq 0$ then we can conclude stability of the equilibrium point. The time derivative is taken along the trajectories of (5.1), i.e.,

$$\left.\frac{dv(x,t)}{dt}\right|_{(5.1)} = \frac{\partial v(x,t)}{\partial t} + \frac{\partial v(x,t)}{\partial x} f(x,t). \tag{5.12}$$

The rate of change of $v(x, t)$ along the trajectories of the vector field (5.1) is also called the *Lie derivative of* $v(x, t)$ *along* $f(x, t)$. In the statement of the following theorem recall that we have translated the origin to lie at the equilibrium point under consideration.

Theorem 5.16 Basic Lyapunov Theorems.

	Conditions on $v(x,t)$	Conditions on $-\dot{v}(x,t)$	Conclusions
1.	l.p.d.f.	≥ 0 locally	stable
2.	l.p.d.f., decrescent	≥ 0 locally	uniformly stable
3.	l.p.d.f., decrescent	l.p.d.f.	uniformly asymptotically stable
4.	p.d.f., decrescent	p.d.f.	globally unif. asymp. stable

Proof:

1. Since v is an l.p.d.f., we have that for some $\alpha(\cdot) \in K$,

$$v(x, t) \geq \alpha(|x|) \quad \forall \quad x \in B_s. \tag{5.13}$$

Also, the hypothesis is that

$$\dot{v}(x, t) \leq 0, \quad \forall\, t \geq t_0, \quad \forall\, x \in B_r \tag{5.14}$$

Given $\epsilon > 0$, define $\epsilon_1 = \min(\epsilon, r, s)$. Choose $\delta > 0$ such that

$$\beta(t_0, \delta) := \sup_{|x|\leq\delta} v(x, t_0) < \alpha(\epsilon_1).$$

Such a δ always exists, since $\beta(t_0, \delta)$ is a continuous function of δ and $\alpha(\epsilon_1) > 0$. We now claim that $|x(t_0)| \leq \delta$ implies that $|x(t)| < \epsilon_1$ $\forall t \geq t_0$. The proof is by contradiction. Clearly since

$$\alpha(|x(t_0)|) \le v(x(t_0), t_0) < \alpha(\epsilon_1),$$

it follows that $|x(t_0)| < \epsilon_1$. Now, if it is not true that $|x(t)| < \epsilon_1$ for all t, let $t_1 > t_0$ be the first instant such that $|x(t)| \ge \epsilon_1$. Then

$$v(x(t_1), t_1) \ge \alpha(\epsilon_1) > v(x(t_0), t_0). \tag{5.15}$$

But this is a contradiction, since $\dot{v}(x(t), t) \le 0$ for all $|x| < \epsilon_1$. Thus,

$$|x(t)| < \epsilon_1 \quad \forall\, t \ge t_0.$$

2. Since v is decrescent,

$$\beta(\delta) = \sup_{|x|\le\delta} \sup_{t\ge t_0} v(x, t) \tag{5.16}$$

is nondecreasing and satisfies for some d

$$\beta(\delta) < \infty \quad \text{for} \quad 0 \le \delta \le d.$$

Now choose δ such that $\beta(\delta) < \alpha(\epsilon_1)$.

3. If $-\dot{v}(x, t)$ is an l.p.d.f., then $\dot{v}(x, t)$ satisfies the conditions of the previous proof so that 0 is a uniformly stable equilibrium point. We need to show the existence of a $\delta_1 > 0$ such that for $\epsilon > 0$ there exists $T(\epsilon) < \infty$ such that

$$|x_0| < \delta_1 \Rightarrow |\phi(t_1 + t, x_0, t_1)| < \epsilon \quad \text{when} \quad t > T(\epsilon)$$

The hypotheses guarantee that there exist functions $\alpha(\cdot), \beta(\cdot), \gamma(\cdot) \in K$ such that $\forall t \ge t_0, \forall x \in B_r$ such that

$$\begin{aligned} \alpha(|x|) \le v(x, t) &\le \beta(|x|), \\ \dot{v}(x, t) &\le -\gamma(|x|). \end{aligned}$$

Given $\epsilon > 0$, define δ_1, δ_2 and T by

$$\begin{aligned} \beta(\delta_1) &< \alpha(r), \\ \beta(\delta_2) &< \min(\alpha(\epsilon), \beta(\delta_1)), \\ T &= \alpha(r)/\gamma(\delta_2). \end{aligned}$$

This choice is explained in Figure 5.3. We now show that there exists at least one instant $t_2 \in [t_1, t_1 + T]$ when $|x_0| < \delta_2$. The proof is by contradiction. Recall the notation that $\phi(t, x_0, t_0)$ stands for the trajectory of (5.1) starting from x_0 at time t_0. Indeed, if

$$|\phi(t, x_0, t_1)| \ge \delta_2 \quad \forall t \in [t_1, t_1 + T],$$

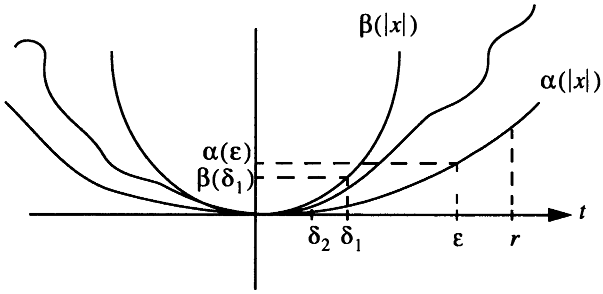

FIGURE 5.3. The choice of constants in the proof of Lyapunov's Theorem.

then it follows that

$$\begin{aligned} 0 \le \alpha(\delta_2) &\le v(s(t_1+T, x_0, t_1), t_1+T) \\ &= v(t_1, x_0) + \int_{t_1}^{t_1+T} \dot v(\tau, \phi(\tau, x_0, t_1) d\tau \\ &\le \beta(\delta_1) - T\gamma(\delta_2) \\ &\le \beta(\delta_1) - \alpha(r) \\ &< 0, \end{aligned}$$

establishing the contradiction (compare the ends of the preceding chain of inequalities to see this). Now, if $t \ge t_1 + T$, then

$$\begin{aligned} \alpha(|\phi(t, x_0, t_1)|) &\le v(t, \phi(t, x_0, t_1)) \\ &\le v(t_2, \phi(t_2, x_0, t_1)), \end{aligned}$$

since $\dot v(x,t) \le 0$. (Actually, the definition of δ_1 guarantees that the trajectory stays in B_r so that $\dot v(x,t) \le 0$.) Thus,

$$\begin{aligned} \alpha(|\phi(t, x_0, t_1)|) \le v(t_2, \phi(t_2, x_0, t_1)) &\le \beta(|\phi(t_2, x_0, t_1|) \\ &\le \beta(\delta_2) \\ &< \alpha(\epsilon) \end{aligned}$$

so that $|\phi(t_2, x_0, t_1)| < \epsilon$ for $t \ge t_1 + T$. □

Remarks:

1. The tabular version of Lyapunov's theorem is meant to highlight the following correlations between the assumptions on $v(x,t)$, $\dot v(x,t)$ and the conclusions:

   a. Decrescence of $v(x,t)$ is associated with uniform stability and the local positive definite character of $\dot v(x,t)$ being associated with asymptotic stability.

   b. $-\dot v(x,t)$ is required to be an l.p.d.f. for asymptotic stability,

   c. $v(x,t)$ being a p.d.f. is associated with global stability.

However, we emphasize that this correlation is not perfect, since $v(x,t)$ being l.p.d.f. and $-\dot{v}(x,t)$ being l.p.d.f. does not guarantee local asymptotic stability. (See Problem 5.1 for a counterexample from Massera [227]).

2. The proof of the theorem, while seemingly straightforward, is subtle in that it is an exercise in the use of contrapositives.

5.3.3 Examples of the Application of Lyapunov's Theorem

1. Consider the following model of an RLC circuit with a linear inductor, nonlinear capacitor, and inductor as shown in the Figure 5.4. This is also a model for a mechanical system with a mass coupled to a nonlinear spring and nonlinear damper as shown in Figure 5.4. Using as state variables x_1, the charge on the capacitor (respectively, the position of the block) and x_2, the current through the inductor (respectively, the velocity of the block) the equations describing the system are

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -f(x_2) - g(x_1).\end{aligned} \tag{5.17}$$

Here $f(\cdot)$ is a continuous function modeling the resistor current–voltage characteristic, and $g(\cdot)$ the capacitor charge–voltage characteristic (respectively the friction and restoring force models in the mechanical analog). We will assume that f, g both model locally passive elements, i.e., there exists a σ_0 such that

$$\begin{aligned}\sigma f(\sigma) &\geq 0 \;\; \forall \sigma \in [-\sigma_0, \sigma_0],\\ \sigma g(\sigma) &\geq 0 \;\; \forall \sigma \in [-\sigma_0, \sigma_0].\end{aligned}$$

The Lyapunov function candidate is the total energy of the system, namely,

$$v(x) = \frac{x_2^2}{2} + \int_0^{x_1} g(\sigma)d\sigma.$$

The first term is the energy stored in the inductor (kinetic energy of the body) and the second term the energy stored in the capacitor (potential energy stored in

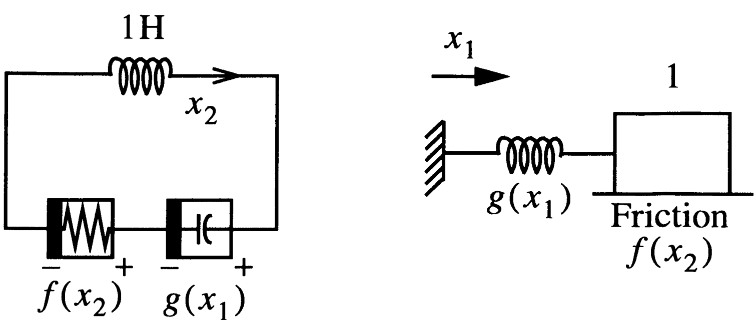

FIGURE 5.4. An RLC circuit and its mechanical analogue

the spring). The function $v(x)$ is an l.p.d.f., provided that $g(x_1)$ is not identically zero on any interval (verify that this follows from the passivity of g). Also,

$$\begin{aligned}\dot{v}(x) &= x_2[-f(x_2) - g(x_1)] + g(x_1)x_2 \\ &= -x_2 f(x_2) \leq 0\end{aligned}$$

when $|x_2|$ is less than σ_0. This establishes the stability but not asymptotic stability of the origin. In point of fact, the origin is actually asymptotically stable, but this needs the LaSalle principle, which is deferred to a later section.

2. *Swing Equation*
The dynamics of a single synchronous generator coupled to an infinite bus is given by

$$\begin{aligned}\dot{\theta} &= \omega, \\ \dot{\omega} &= -M^{-1}D\omega - M^{-1}(P - B\sin(\theta)).\end{aligned} \tag{5.18}$$

Here θ is the angle of the rotor of the generator measured relative to a synchronously spinning reference frame and its time derivative is ω. Also M is the moment of inertia of the generator and D its damping both in normalized units; P is the exogenous power input to the generator from the turbine and B the susceptance of the line connecting the generator to the rest of the network, modeled as an infinite bus (see Figure 5.5) A choice of Lyapunov function is

$$v(\theta, \omega) = \frac{1}{2}M\omega^2 + P\theta + B\cos(\theta).$$

The equilibrium point is $\theta = \sin^{-1}(\frac{P}{B})$, $\omega = 0$. By translating the origin to this equilibrium point it may be verified that $v(\theta, \omega) - v(\theta_0, 0)$ is an l.p.d.f. around it. Further it follows that

$$\dot{v}(\theta, \omega) = -D\omega^2,$$

yielding the stability of the equilibrium point. As in the previous example, one cannot conclude asymptotic stability of the equilibrium point from this analysis.

3. *Damped Mathieu Equation*
This equation models the dynamics of a pendulum with sinusoidally varying

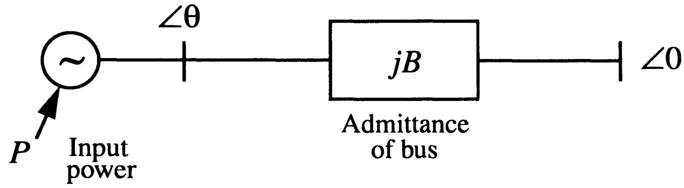

FIGURE 5.5. A generator coupled to an infinite bus

length; this is a good model for instance of a child pumping a swing.

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -x_2 - (2+\sin t)x_1.\end{aligned} \tag{5.19}$$

The total energy is again a candidate for a Lyapunov function, namely,

$$v(x,t) = x_1^2 + \frac{x_2^2}{2+\sin t}.$$

It is an l.p.d.f. since

$$x_1^2 + x_2^2 \geq x_1^2 + \frac{x_2^2}{2+\sin t} \geq x_1^2 + \frac{x_2^2}{3}.$$

Further, a small computation yields

$$\dot{v}(x,t) = -\frac{x_2^2(4+2\sin t+\cos t)}{(2+\sin t)^2} \leq 0.$$

Thus, the equilibrium point at the origin is uniformly stable. However, we cannot conclude that the origin is uniformly asymptotically stable.

4. *System with a Limit Cycle*
Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_2 + x_1(x_1^2 + x_2^2 - 1),\\ \dot{x}_2 &= x_1 + x_2(x_1^2 + x_2^2 - 1).\end{aligned}$$

A choice of $v(x) = x_1^2 + x_2^2$ yields for $\dot{v}(x) = 2(x_1^2+x_2^2)(x_1^2+x_2^2-1)$, which is the negative of an l.p.d.f. for $\{x : x_1^2 + x_2^2 < 1\}$. Thus, 0 is a locally asymptotically stable equilibrium (though it is not globally, asymptotically stable, since there is a limit cycle of radius 1, as may be verified by changing into polar coordinates, see also Chapter 2).

Comments.

1. The theorems of Lyapunov give sufficient conditions for the stability of the equilibrium point at the origin of (5.1). They do not, however, give a prescription for determining the Lyapunov function $v(x,t)$. Since the theorems are only sufficient conditions for stability the search for a Lyapunov function establishing stability of an equilibrium point could be long. However, it is a remarkable fact that the converse of theorem (5.16) also exists: for example, if an equilibrium point is stable then there exists an l.p.d.f. $v(x,t)$ with $\dot{v}(x,t) \leq 0$. However, the utility of this and other *converse theorems of Lyapunov* is limited by the fact that there is no general and *computationally non-intensive* technique for generating these Lyapunov functions (an example of such a theorem, called Zubov's theorem, is given in the Exercises (Problem 5.14). We will not give the details of the construction of converse Lyapunov functions in general, but we focus on exponentially stable equilibria in the next subsection. However, the method of construction of the Lyapunov function in Section 5.3.4 is prototypical of other converse theorems.

2. If the system has multiple equilibrium points, then by translating the equilibria in turn to the origin one may individually ascertain their stability.

5.3.4 Exponential Stability Theorems

The basic theorem of Lyapunov, Theorem 5.16, stops short of giving explicit rates of convergence of solutions to the equilibria. It may however be modified to do so in the instance of exponentially stable equilibria. We will pay special attention to exponentially stable equilibria, since they are robust to perturbation and are consequently desirable from the viewpoint of applications. We will now state necessary and sufficient conditions for the existence of an exponentially stable equilibrium point, that is, we state a converse theorem for exponentially stable systems:

Theorem 5.17 Exponential Stability Theorem and Its Converse. *Assume that $f(x,t) : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ has continuous first partial derivatives in x and is piecewise continuous in t. Then the two statements below are equivalent:*

1. $x = 0$ *is a locally* exponentially stable *equilibrium point of*

$$\dot{x} = f(x,t),$$

i.e., if $x \in B_h$ for h small enough, there exist $m, \alpha > 0$ such that

$$|\phi(\tau, x, t)| \leq m e^{-\alpha(\tau - t)}.$$

2. *There exists a function $v(x,t)$ and some constants $h, \alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$ such that for all $x \in B_h$, $t \geq 0$*

$$\begin{aligned} \alpha_1|x|^2 \leq v(x,t) &\leq \alpha_2|x|^2, \\ \left.\frac{dv(x,t)}{dt}\right|_{(5.1)} &\leq -\alpha_3|x|^2, \\ \left|\frac{\partial v(x,t)}{\partial x}\right| &\leq \alpha_4|x|. \end{aligned} \tag{5.20}$$

Proof:
(1) $\Rightarrow$ (2)
We prove the three inequalities of (5.20) in turn, starting from the definition of $v(t,x)$:

- Denote by $\phi(\tau, x, t)$ the solution of (5.1) at time τ starting from x at time t, and define

$$v(x,t) := \int_t^{t+T} |\phi(\tau, x, t)|^2 d\tau, \tag{5.21}$$

where T will be defined later. From the exponential stability of the system at rate α and the lower bound on the rate of growth given by Proposition 5.3, we have

$$m|x|e^{-\alpha(\tau-t)} \geq |\phi(\tau, x, t)| \geq |x|e^{-l(\tau-t)} \tag{5.22}$$

for $x \in B_h$ for some h. Also l the Lipschitz constant of $f(x, t)$ exists because of the assumption that $f(x, t)$ has continuous first partial derivatives with respect to x. This, when used in (5.21), yields the first inequality of (5.20) for $x \in B_{h'}$ (where h' is chosen to be h/m) with

$$\alpha_1 := \frac{(1 - e^{-2lT})}{2l}, \quad \alpha_2 := m^2 \frac{(1 - e^{-2\alpha T})}{2\alpha}. \tag{5.23}$$

- Differentiating (5.21) with respect to t yields

$$\begin{aligned} \frac{dv(x,t)}{dt} &= |\phi(t+T, x, t)|^2 - |\phi(t, x, t)|^2 \\ &\quad + \int_t^{t+T} \frac{d}{dt}(|\phi(\tau, x(t), t)|^2) d\tau. \end{aligned} \tag{5.24}$$

Note that d/dt is the derivative with respect to the initial time t along the trajectories of (5.1). However, since for all Δt the solution satisfies

$$\phi(\tau, x(t + \Delta t), t + \Delta t) = \phi(\tau, x(t), t),$$

we have that that $\frac{d}{dt}(|\phi(\tau, x(t), t)|^2) \equiv 0$. Using the fact that $\phi(t, x, t) = x$ and the exponential bound on the solution, we have that

$$\frac{dv(x,t)}{dt} \leq -(1 - m^2 e^{-2\alpha T})|x|^2.$$

The second inequality of (5.20) now follows, provided that $T > (1/\alpha) \ln m$ and

$$\alpha_3 := 1 - m^2 e^{-2\alpha T}.$$

- Differentiating (5.21) with respect to x_j, we have

$$\frac{\partial v(x,t)}{\partial x_i} = 2 \int_t^{t+T} \sum_{j=1}^{n} \phi_j(\tau, x, t) \frac{\partial \phi_j(\tau, x, t)}{\partial x_i} d\tau. \tag{5.25}$$

By way of notation define

$$Q_{ij}(\tau, x, t) := \frac{\partial \phi_j(\tau, x, t)}{\partial x_i}$$

and

$$A_{ij}(x, t) := \frac{\partial f_i(t, x)}{\partial x_j}.$$

Interchanging the order of differentiation by τ, with differentiation by x_j yields that

$$\frac{d}{d\tau}Q(\tau,x,t) = A(\phi(\tau,x,t),t).Q(\tau,x,t). \tag{5.26}$$

Thus $Q(\tau,x,t)$ is the state transition matrix associated with the matrix $A(\phi(\tau,x,t),t)$. By the assumption on boundedness of the partials of f with respect to x, it follows that $|A(\cdot,\cdot)| \leq k$ for some k, so that

$$|Q(\tau,x,t)| \leq e^{k(\tau-t)}.$$

using this and the bound for exponential convergence in (5.25) yields

$$\left|\frac{\partial v(t,x)}{\partial x}\right| \leq 2\int_t^{t+T} m|x|e^{(k-\alpha)(\tau-t)}d\tau,$$

which is the last equation of (5.20) if we define

$$\alpha_4 := \frac{2m(e^{(k-\alpha)T}-1)}{(k-\alpha)}.$$

This completes the proof, but note that $v(x,t)$ is only defined for $x \in B_{h'}$ with $h' = h/m$, to guarantee that $\phi(\tau,x,t) \in B_h$ for all $\tau \geq t$ (convince yourself of this point).

(2) $\Rightarrow$ (1)
This direction is straightforward, as may be verified by noting that equation (5.20) implies that

$$\dot{v}(x,t) \leq -\frac{\alpha_3}{\alpha_2}v(x,t). \tag{5.27}$$

This in turn implies that

$$v(t,x(t)) \leq v(t_0,x(t_0))e^{-\frac{\alpha_3}{\alpha_2}(t-t_0)}. \tag{5.28}$$

Using the lower bound for $v(t,x(t))$ and the upper bound for $v(t_0,x(t_0))$ we get

$$\alpha_1|x(t)|^2 \leq \alpha_2|x(t_0)|^2e^{-\frac{\alpha_3}{\alpha_2}(t-t_0)}. \tag{5.29}$$

Using the estimate of (5.29) it follows that

$$|x(t)| \leq m|x(t_0)|e^{-\alpha(t-t_0)}.$$

with

$$m := \left(\frac{\alpha_2}{\alpha_1}\right)^{1/2}, \quad \alpha := \frac{\alpha_3}{2\alpha_2}.$$ □

5.4 LaSalle's Invariance Principle

LaSalle's invariance principle has two main applications:

1. It enables one to conclude asymptotic stability even when $-\dot{v}(x,t)$ is not an l.p.d.f.
2. It enables one to prove that trajectories of the differential equation starting in a given region converge to one of many equilibrium points in that region.

However, the principle applies primarily to autonomous or periodic systems, which are discussed in this section. Some generalizations to the non–autonomous case are also discussed in the next section. We will deal with the autonomous case first. We recall some definitions from Chapter 2:

Definition 5.18 ω limit set. *A set $S \subset \mathbb{R}^n$ is the ω limit set of a trajectory $\phi(\cdot, x_0, t_0)$ if for every $y \in S$, there exists a sequence of times $t_n \to \infty$ such that $\phi(t_n, x_0, t_0) \to y$.*

Definition 5.19 Invariant set. *A set $M \subset \mathbb{R}^n$ is said to be an* invariant set *if whenever $y \in M$ and $t_0 \geq 0$, we have*

$$\phi(t, y, t_0) \in M \quad \forall t \geq t_0.$$

The following propositions establish some properties of ω limit sets and invariant sets.

Proposition 5.20. *If $\phi(\cdot, x_0, t_0)$ is a bounded trajectory, its ω limit set is compact. Further, $\phi(t, x_0, t_0)$ approaches its ω limit set as $t \to \infty$.*

Proof: As in Chapter 2, see pages 46–50 of [329]. The proof is no different in $\mathbb{R}^n$.

Proposition 5.21. *Assume that the system (5.1) is autonomous and let S be the ω-limit set of any trajectory. Then S is invariant.*

Proof: Let $y \in S$ and $t_1 \geq 0$ be arbitrary. We need to show that $\phi(t, y, t_1) \in S$ for all $t \geq t_1$. Now $y \in S \Rightarrow \exists t_n \to \infty$ such that $\phi(t_n, x_0, t_0) \to y$ as $n \to \infty$. Since trajectories are continuous in initial conditions, it follows that

$$\begin{aligned} \phi(t, y, t_1) &= \lim_{n\to\infty} \phi(t, \phi(t_n, x_0, t_0), t_1) \\ &= \lim_{n\to\infty} \phi(t + t_n - t_1, x_0, t_0), \end{aligned}$$

since the system is autonomous. Now, $t_n \to \infty$ as $n \to \infty$ so that by Proposition 5.20 the right hand side converges to an element of S. □

Proposition 5.22 LaSalle's Principle. *Let $v : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and suppose that*

$$\Omega_c = \{x \in \mathbb{R}^n : v(x) \leq c\}$$

is bounded and that $\dot{v} \leq 0$ for all $x \in \Omega_c$. Define $S \subset \Omega_c$ by

$$S = \{x \in \Omega_c : \dot{v}(x) = 0\}$$

and let M be the largest invariant set in S. Then, whenever $x_0 \in \Omega_c$, $\phi(t, x_0, 0)$ approaches M as $t \to \infty$.

Proof: Let $x_0 \in \Omega_c$. Now, since $v(\phi(t, x_0, 0))$ is a nonincreasing function of time we see that $\phi(t, x_0, 0) \in \Omega_c \ \forall t$. Further, since Ω_c is bounded $v(\phi(t, x_0, 0))$ is also bounded below. Let

$$c_0 = \lim_{t \to \infty} v(\phi(t, x_0, 0))$$

and let L be the ω limit set of the trajectory. Then, $v(y) = c_0$ for $y \in L$. Since L is invariant we have that $\dot{v}(y) = 0 \forall y \in L$ so that $L \subset S$. Since, M is the largest invariant set inside S, we have that $L \subset M$. Since $s(t, x_0, 0)$ approaches L as $t \to \infty$, we have that $s(t, x_0, 0)$ approaches M as $t \to \infty$. $\square$.

Theorem 5.23 LaSalle's Principle to Establish Asymptotic Stability. *Let $v : \mathbb{R}^n \mapsto \mathbb{R}$ be such that on $\Omega_c = \{x \in \mathbb{R}^n : v(x) \leq c\}$, a compact set we have $\dot{v}(x) \leq 0$. As in the previous proposition define*

$$S = \{x \in \Omega_c : \dot{v}(x) = 0\}.$$

Then, if S contains no trajectories other than $x = 0$ then 0 is asymptotically stable.

Proof: follows directly from the preceding lemma.

An application of LaSalle's principle is to prove global asymptotic stability is as follows:

Theorem 5.24 Application of LaSalle's Principle to prove Global Asymptotic Stability. *Let $v(x) : \mathbb{R}^n \mapsto \mathbb{R}$ be a p.d.f. and $\dot{v}(x) \leq 0$ for all $x \in \mathbb{R}^n$. Also, let the set*

$$S = \{x \in \mathbb{R}^n : \dot{v}(x) = 0\}$$

contain no nontrivial trajectories. Then 0 is globally, asymptotically stable.

Examples:

1. *Spring–mass system with damper*
   This system is described by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -f(x_2) - g(x_1). \end{aligned} \tag{5.30}$$

   If f, g are locally passive, i.e.,

$$\sigma f(\sigma) \geq 0 \ \ \forall \sigma \in [-\sigma_0, \sigma_0],$$

   then it may be verified that a suitable Lyapunov function (l.p.d.f.) is

$$v(x_1, x_2) = \frac{x_2^2}{2} + \int_0^{x_1} g(\sigma) d\sigma.$$

   Further, it is easy to see that

$$\dot{v}(x_1, x_2) = -x_2 f(x_2) \leq 0 \ \text{ for } \ x_2 \in [-\sigma_0, \sigma_0].$$

Now choose

$$c = \min(v(-\sigma_0, 0), v(\sigma_0, 0)).$$

Then $\dot{v} \leq 0$ for $x \in \Omega_c = \{(x_1, x_2) : v(x_1, x_2) \leq c\}$. As a consequence of LaSalle's principle, the trajectory enters the largest invariant set in $\Omega_c \cap \{(x_1, x_2) : \dot{v} = 0\} = \Omega_c \cap \{x_1, 0\}$. To obtain the largest invariant set in this region note that $x_2(t) \equiv 0 \Rightarrow x_1(t) \equiv x_{10} \Rightarrow \dot{x}_1(t) = 0 = -f(0) - g(x_{10})$. Consequently, we have that $g(x_{10}) = 0 \Rightarrow x_{10} = 0$. Thus, the largest invariant set inside $\Omega_c \cap \{(x_1, x_2) : \dot{v} = 0\}$ is the origin. Thus, the origin is locally asymptotically stable.

The application of LaSalle's principle shows that one can give interesting conditions for the convergence of trajectories of the system of (5.30) even when $g(\cdot)$ is not passive. It is easy to see that the arguments given above can be easily modified to obtain convergence results for the system (5.30), provided that $\int_0^{x_1} g(\sigma)d\sigma$ is merely *bounded below*. The next examples are in this spirit.

2. *Single generator coupled to an infinite bus*

If θ is the angle of the rotor of a generator with respect to a synchronously rotating frame of reference and $\omega = \dot{\theta}$, then the equations of the dynamics of the synchronous generator are given by

$$M\ddot{\theta} + D\dot{\theta} + B\sin\theta = P_m - P_e. \tag{5.31}$$

Here, M, D stand for the moment of inertia and damping of the generator rotor; P_m stands for the exogenous power input, and P_e stands for the local power demand. This equation can be written in state space form with $x_1 = \theta, x_2 = \omega$. The equilibrium points of the system are at $\theta = \sin^{-1}(\frac{P_m - P_e}{B})$, $\omega = 0$. Note that the equilibria repeat every 2π radians. Consider a choice of

$$v(\theta, \omega) = \frac{M\omega^2}{2} + (P_e - P_m)\theta - B\cos(\theta) \tag{5.32}$$

with

$$\dot{v} = -D\omega^2.$$

Since $v(\theta, \omega)$ is not bounded below, $\Omega_c = \{(\theta, \omega) : v(\theta, \omega) \leq c\}$ is not compact. However, if we know *a priori* that a trajectory is bounded, then we know from LaSalle's principle that it converges to the largest invariant set in a compact region with $\omega = 0$. Reasoning exactly as in the previous example guarantees that the trajectory converges to one of the equilibrium points of the system. (These are the only invariant sets in the region where $\dot{v} = 0$.) This behavior is referred to by power systems engineers as "trajectories that skip only finitely many cycles (2π multiples in the θ variable) converge to an equilibrium" owing to the periodic dependence of the dynamics on θ. However, if we view the dynamics as resident on the state space $S^1 \times \mathbb{R}^1$, we can no longer use the function $v(\theta, \omega)$ to conclude convergence of the trajectories. (Why? See Problem 5.11 for details on this point.)

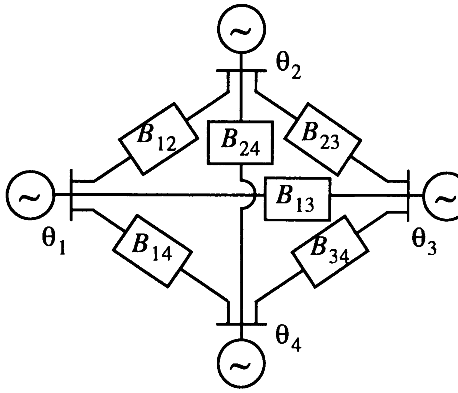

FIGURE 5.6. An interconnected power system

3. *Power system swing equations*
The dynamics of a power system modeled as shown in Figure 5.6 as a collection of generators interconnected by transmission lines of susceptance B_{ij} between generators i and j is a generalization of the swing equation for a single generator coupled to an infinite bus:

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i + \sum_{j=1, j\neq i}^{n} B_{ij}\sin(\theta_i - \theta_j) = P_{mi} - P_{ei} \tag{5.33}$$

for $i = 1, \ldots, n$. Here M_i, D_i stand for the moment of inertia and damping of the ith generator. and P_{mi}, P_{ei} stand for the mechanical power input and electrical power output from the i-th generator. The total energy function for this system is given by

$$v(\theta, \omega) = \frac{1}{2}\dot{\theta}^T M\dot{\theta} - \sum_{i<j, j=1}^{n} B_{ij}\cos(\theta_i - \theta_j) + \theta^T(P_e - P_m), \tag{5.34}$$

with its derivative given by

$$\dot{v} = -\dot{\theta}^T D\dot{\theta}.$$

As before the function $v(\theta, \omega)$ is not bounded below so that we cannot guarantee that Ω_c is compact for any c. However, we may establish as before that all *a priori bounded* trajectories converge. The same comments about convergence of finite–cycle–skipping trajectories 2π variations in one or more of the θ_i hold here as well. Also $v(\theta, \omega)$ is no longer a valid function to consider when the dynamics of the power system are considered on the state space $T^n \times \mathbb{R}^n$ (see Problem 5.11 for details).

4. *Hopfield Neural Networks [143]*
Figure 5.7 shows a circuit diagram of a so-called Hopfield network consisting of N operational amplifiers interconnected by an RC network. The operational amplifier input voltages are labeled u_i and the outputs x_i. The relationship between

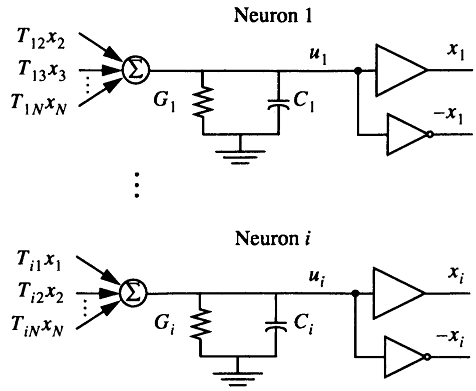

FIGURE 5.7. A Hopfield network

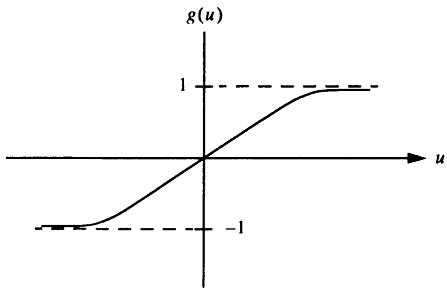

FIGURE 5.8. A sigmoidal nonlinearity

u_i and x_i for each i is given by a single, so-called *sigmoidal nonlinearity*: i.e., a monotone increasing odd-symmetric nonlinearity of the form shown in Figure 5.8 asymptotic to ± 1 as its argument tends to $\pm\infty$, respectively. Examples of sigmoidal functions $g(u)$ are

$$\begin{aligned} g(u) &= \frac{2}{\pi}\tan^{-1}\frac{\lambda\pi u}{2}, \\ g(u) &= \frac{e^{\lambda u} - e^{-\lambda u}}{e^{\lambda u} + e^{-\lambda u}}. \end{aligned} \tag{5.35}$$

The equations of the Hopfield network are given by

$$\begin{aligned} C_i\frac{du_i}{dt} &= \sum_{j=1}^{N} T_{ij}x_j - G_iu_i + I_i, \\ x_i &= g(u_i). \end{aligned} \tag{5.36}$$

Here $C_i > 0$ is the capacitance of the i th capacitor, and $G_i > 0$ the conductance of the i-th amplifier, T_{ij} stands for the strength of the interconnection between the x_j and the i-th capacitor. I_i stands for the current injection at the i-th node. Using the relationship between u_i and x_i, we get the equations of the Hopfield network:

$$\dot{x}_i = h_i(x_i)\left[\sum_{j=1}^{N} T_{ij}x_j - G_i g^{-1}(x_i) + I_i\right], \tag{5.37}$$

where

$$h_i(x) = \frac{1}{C_i}\frac{dg}{du}\bigg|_{u=g^{-1}(x)} > 0 \quad \forall x.$$

Consider the function

$$v(x) = -\frac{1}{2}x^T T x + \sum_{i=1}^{N} G_i \int_0^{x_i} g^{-1}(y)dy - \sum_{i=1}^{N} I_i x_i. \tag{5.38}$$

It is now easy to verify that equation (5.37) may be rewritten as

$$\dot{x}_i = -h_i(x_i)\frac{\partial v}{\partial x_i}. \tag{5.39}$$

Thus, it follows that the Hopfield network is a gradient flow of v (along the "metric" given by the $h_i(\cdot)$) and

$$\dot{v}|_{(5.37)} = -\sum_{i=1}^{N} h_i(x_i)\left(\frac{\partial v}{\partial x_i}\right)^2 \leq 0.$$

Further, it follows that

$$\dot{v} = 0 \quad \Rightarrow \quad \frac{\partial v}{\partial x_i} = 0 \;\; \forall i \quad \Rightarrow \quad \dot{x} = 0.$$

Thus, to apply LaSalle's principle to conclude that trajectories converge to one of the equilibrium points, it is enough to find a region which is either invariant or on which v is bounded. Consider the set

$$\Omega_\epsilon = \{x \in \mathbb{R}^n : |x_i| \leq 1 - \epsilon\}.$$

We will prove that the region Ω_ϵ is invariant for ϵ small enough. Note that

$$\frac{dx_i^2}{dt} = 2x_i h_i(x_i)\left(\sum_{j=1}^{N} T_{ij}x_j - g^{-1}(x_i) + I_i\right).$$

Since $g^{-1}(x_i) \to \pm\infty$ as $x_i \to \pm 1$, it follows that for $\epsilon > 0$ small enough,

$$\frac{dx_i^2}{dt} < 0 \quad \text{for} \quad 1 - \epsilon \leq |x_i| < 1,$$

thus establishing that Ω_ϵ is invariant. In turn, this implies that $v(x)$ is bounded below on Ω_ϵ, so that initial conditions beginning inside Ω_ϵ tend to the equilibrium points inside Ω_ϵ. Thus, for symmetric interconnections T_{ij} the Hopfield network does not oscillate, and all trajectories converge to one of the equilibria. It is of great interest to estimate the number of equilibria of a Hopfield network (see [143] and a very large literature on neural networks for more details).

There is a globalization of LaSalle's principle, which is as follows:

Theorem 5.25 Global LaSalle's Principle. *Consider the system of (5.1). Let $v(x)$ be a p.d.f. with $\dot{v} \leq 0 \forall x \in \mathbb{R}^n$. If the set*

$$s = \{x \in \mathbb{R}^n, \quad \dot{v}(x) = 0\}$$

contains no invariant sets other than the origin, the origin is globally asymptotically stable.

There is also a version of LaSalle's theorem that holds for periodic systems as well.

Theorem 5.26 LaSalle's Principle for Periodic Systems. *Assume that the system of (5.1) is periodic, i.e.,*

$$f(x,t) = f(x, t+T), \quad \forall t\ \forall x \in \mathbb{R}^n.$$

Further, let $v(x,t)$ be a p.d.f. which is periodic in t also with period T. Define

$$S = \{x \in \mathbb{R}^n : \dot{v}(x,t) = 0, \forall t \geq 0\}$$

Then if $\dot{v}(x,t) \leq 0 \quad \forall t \geq 0, \quad \forall x \in \mathbb{R}^n$ and the largest invariant set in S is the origin, then the origin is globally (uniformly) asymptotically stable.

5.5 Generalizations of LaSalle's Principle

LaSalle's invariance principle is restricted in applications because it holds only for time-invariant and periodic systems. For extending the result to arbitrary time-varying systems, two difficulties arise:

1. $\{x : \dot{v}(x,t) = 0\}$ may be a time-varying set.
2. The ω limit set of a trajectory is itself not invariant.

However, if we have the hypothesis that

$$\dot{v}(x,t) \leq -w(x) \leq 0,$$

then the set S may be defined to be

$$\{x : w(x) = 0\},$$

and we may state the following generalization of LaSalle's theorem:

Theorem 5.27 Generalization of LaSalle's Theorem. *Assume that the vector field $f(x,t)$ of (5.1) is locally Lipschitz continuous in x, uniformly in t, in a ball of radius r. Let $v(x,t)$ satisfy for functions α_1, α_2 of class K*

$$\alpha_1(|x|) \leq v(x,t) \leq \alpha_2(|x|). \tag{5.40}$$

Further, for some non-negative function $w(x)$, assume that

$$\dot{v}(x,t) = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} f(x,t) \leq -w(x) \leq 0. \tag{5.41}$$

Then for all $|x(t_0)| \leq \alpha_2^{-1}(\alpha_1(r))$, *the trajectories* $x(\cdot)$ *are bounded and*

$$\lim_{t\to\infty} w(x(t)) = 0. \tag{5.42}$$

Proof: The proof of this theorem needs a fact from analysis called Barbalat's lemma (which we have left as an exercise: Problem 5.9; see also [259]), which states that if $\phi(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ is a *uniformly continuous* integrable function with

$$\int_0^\infty \phi(t) < \infty,$$

then

$$\lim_{t\to\infty} \phi(t) = 0.$$

The requirement of uniform continuity of $\phi(\cdot)$ is necessary for this lemma, as easy counterexamples will show. We will use this lemma in what follows.

First, note that a simple contradiction argument shows that for any $\rho < r$,

$$|x(t_0)| \leq \alpha_2^{-1}(\alpha_1(\rho)) \Rightarrow |x(t)| \leq \rho \quad \forall\ t \geq t_0.$$

Thus $|x(t)| < r$ for all $t \geq t_0$, so that $v(x(t), t)$ is monotone decreasing. This yields that

$$\begin{aligned} \int_{t_0}^{t} w(x(\tau))d\tau &\leq -\int_{t}^{t_0} \dot{v}(x(\tau), \tau)d\tau \\ &= v(x(t_0), t_0) - v(x(t), t). \end{aligned} \tag{5.43}$$

Since $v(x, t)$ is bounded below by 0, it follows that

$$\int_{t_0}^{\infty} w(x(\tau))d\tau < \infty.$$

By the continuity of $f(x, t)$ (Lipschitz in x, uniformly in t) and the boundedness of $x(t)$, it follows that $x(t)$ is uniformly continuous, and so is $w(x(t))$. Using Barbalat's lemma it follows that

$$\lim_{t\to\infty} w(x(t)) = 0. \qquad \square$$

Remarks: The preceding theorem implies that $x(t)$ approaches a set E defined by

$$E := \{x \in B_r : w(x) = 0\}.$$

However, the set E is not guaranteed to be an invariant set; so that one cannot, for example, assert that $x(t)$ tends to the largest invariant set inside E. It is difficult, in general, to show that the set E is invariant. However, this can be shown to be the case when $f(x, t)$ is autonomous, T-periodic, or asymptotically autonomous. Another generalization of LaSalle's theorem is given in the exercises (Problem 5.10).

5.6 Instability Theorems

Lyapunov's theorem presented in the previous section gives a sufficient condition for establishing the stability of an equilibrium point. In this section we will give some sufficient conditions for establishing the instability of an equilibrium point.

Definition 5.28 Unstable Equilibrium. *The equilibrium point* 0 *is* unstable at t_0, *if it is not stable.*

This definition is more subtle than it seems: we may parse the definition, by systematically negating the definition of stability: *There exists* an $\epsilon > 0$, such that *for all* δ balls of initial conditions (no matter how small the ball), there exists at least one initial condition, such that the trajectory is not confined to the ϵ ball; that is to say,

$$\forall \delta \ \ \exists x_0 \in B_\delta$$

such that $\exists t_\delta$ with

$$|x_{t_\delta}| \geq \epsilon.$$

Instability is of necessity a local concept. One seldom has a definition for uniform instability. Note that the definition of instability does not require every initial condition starting arbitrarily near the origin to be expelled from a neighborhood of the origin, it just requires one from each arbitrarily small neighborhood of the origin to be expelled away. As we will see in the context of linear time-invariant systems in the next section, the linear time-invariant system

$$\dot{x} = Ax$$

is unstable, if just one eigenvalue of A lies in $\mathbb{C}_+^\circ$! The instability theorems have the same flavor: They insist on $\dot{v}$ being an l.p.d.f. so as to have a mechanism for the increase of v. However, since we do not need to guarantee that every initial condition close to the origin is repelled from the origin, we do not need to assume that v is an l.p.d.f.

We state and prove two examples of instability theorems:

Theorem 5.29 Instability Theorem. *The equilibrium point* 0 *is unstable at time* t_0 *if there exists a decrescent function* $v : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}$ *such that*

1. $\dot{v}(x, t)$ *is an l.p.d.f.*
2. $v(0, t) = 0$ *and there exist points x arbitrarily close to* 0 *such that* $v(x, t_0) > 0$.

Proof: We are given that there exists a function $v(x, t)$ such that

$$v(x,t) \leq \beta(|x|) \quad x \in B_r,$$
$$\dot{v}(x,t) \geq \alpha(|x|) \quad x \in B_s.$$

We need to show that for some $\epsilon > 0$, there is no δ such that

$$|x_0| < \delta \Rightarrow |x(t)| < \epsilon \ \ \forall t \geq t_0.$$

Now choose $\epsilon = \min(r, s)$. Given $\delta > 0$ choose x_0 with $|x_0| < \delta$ and $v(x_0, t_0) > 0$. Such a choice is possible by the hypothesis on $v(x, t_0)$. So long as $\phi(t, x_0, t_0)$ lies in B_ϵ we have $\dot{v}(x(t), t) \geq 0$, which shows that

$$v(x(t), t) \geq v(x_0, t_0) > 0$$

This implies that $|x(t)|$ is bounded away from 0. Thus $\dot{v}(x(t), t)$ is bounded away from zero. Thus, $v(x(t), t)$ will exceed $\beta(\epsilon)$ in finite time. In turn this will guarantee that $|x(t)|$ will exceed ϵ in finite time. □

Theorem 5.30 Chetaev's theorem. *The equilibrium point* 0 *is unstable at time* t_0 *if there is a decrescent function* $v : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}$ *such that*

1. $\dot{v}(x, t) = \lambda v(x, t) + v_1(x, t)$, *where* $\lambda > 0$ *and* $v_1(x, t) \geq 0 \;\; \forall t \geq 0 \;\; \forall x \in B_r$.
2. $v(0, t) = 0$ *and there exist points* x *arbitrarily close to 0 such that* $v(x, t_0) > 0$.

Proof: Choose $\epsilon = r$ and given $\delta > 0$ pick x_0 such that $|x_0| < \delta$ and $v(x_0, t_0) > 0$. When $|x(t)| \leq r$, we have

$$\dot{v}(x, t) = \lambda v(x, t) + v_1(x, t) \geq \lambda v(x, t).$$

If we multiply the inequality above by integrating factor $e^{-\lambda t}$, it follows that

$$\frac{dv(x, t)e^{-\lambda t}}{dt} \geq 0.$$

Integrating this inequality from t_0 to t yields

$$v(x(t), t) \geq e^{\lambda(t-t_0)} v(x_0, t_0).$$

Thus, $v(x(t), t)$ grows without bound. Since $v(x, t)$ is decrescent,

$$v(x, t) \geq \beta(|x|)$$

for some function of class K, so that for some t_δ, $v(x(t), t) > \beta(\epsilon)$, establishing that $|x(t_\delta)| > \epsilon$. □

5.7 Stability of Linear Time-Varying Systems

Stability theory for linear time varying systems is very easy and does not really need the application of the theorems of Lyapunov. However, Lyapunov theory provides valuable help in understanding the construction of Lyapunov functions for nonlinear systems. Thus, in this section we will consider linear time varying systems of the form

$$\dot{x} = A(t)x, \quad x(t_0) = x_0. \tag{5.44}$$

where $A(t) \in \mathbb{R}^{n \times n}$ is a piecewise continuous bounded function. As a consequence the system of (5.44) satisfies the conditions for the existence and uniqueness of solutions.

Definition 5.31 State Transition Matrix. *The state transition matrix $\Phi(t, t_0) \in \mathbb{R}^{n\times n}$ associated with $A(t)$ is by definition the unique solution of the matrix differential equation*

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0), \quad \Phi(t_0, t_0) = I. \tag{5.45}$$

The flow of (5.44) can be expressed in terms of the solutions to (5.45) as

$$x(t) = \Phi(t, t_0)x(t_0). \tag{5.46}$$

In particular, this expression shows that the trajectories are proportional to the size of the initial conditions, so that local and global properties are identical. Two important properties of the state transition matrix are as follows:

1. *Group Property*

$$\Phi(t, t_0) = \Phi(t, \tau)\Phi(\tau, t_0) \ \ \forall \ \ t, \tau, t_0 \tag{5.47}$$

2. *Inverse*

$$\Phi(t, t_0)^{-1} = \Phi(t_0, t) \ \ \forall t, t_0. \tag{5.48}$$

Recalling the formula for differentiating the inverse of a matrix:

$$\frac{d}{dt}[M(t)]^{-1} = M^{-1}(t)\frac{d}{dt}M(t)M^{-1}(t),$$

we have

$$\begin{aligned}\frac{d}{dt_0}\Phi(t, t_0) &= \frac{d}{dt_0}\left[(\Phi(t_0, t))^{-1}\right] \\ &= -\Phi(t_0, t)^{-1}A(t_0)\Phi(t_0, t)\Phi(t_0, t)^{-1} \\ &= -\Phi(t, t_0)A(t_0).\end{aligned}$$

The following theorems are elementary characterizations of stability, uniform stability, and asymptotic stability of linear systems. Uniform asymptotic stability is characterized in the next theorem. Recall that for linear systems local and global stability notions are identical.

Theorem 5.32 Stability of Linear Systems. *The right-hand-side of the following table gives the stability conclusions of the equilibrium point* 0 *of the linear time-varying system (5.44).*

Conditions on $\Phi(t, t_0)$	**Conclusions**
$\sup_{t\geq t_0} \lvert\Phi(t, t_0)\rvert := m(t_0) < \infty$	Stable at t_0
$\sup_{t_0\geq 0}\sup_{t\geq t_0} \lvert\Phi(t, t_0)\rvert < \infty$	Uniformly Stable
$\lim_{t\to\infty} \lvert\Phi(t, t_0)\rvert = 0$	Asymptotically stable

Theorem 5.33 Exponential and Uniform Asymptotic Stability. *The equilibrium point* 0 *of (5.44) is* uniformly asymptotically stable *if it is uniformly stable and* $|\Phi(t, t_0)| \to 0$ *as* $t \to \infty$, *uniformly in* t_0. *The point* $x = 0$ *is a uniform asymptotically stable equilibrium point of (5.44) iff* $x = 0$ *is an exponentially stable equilibrium point of (5.44).*

Proof: The first part of the theorem is a statement of the definition of uniform asymptotic stability for linear time varying systems. For the second half, the fact that exponential stability implies uniform asymptotic stability follows from the definition. For the converse implication let us assume that the equilibrium is uniformly asymptotically stable, From the first part of the theorem, we have that uniform stability implies that $\forall t_1$, there exist m_0, T such that

$$|\Phi(t, t_1)| \le m_0, \quad t \ge t_1.$$

Uniform convergence of $\Phi(t, t_0$ to 0 implies that

$$|\Phi(t, t_1)| \le \frac{1}{2}, \quad \forall t \ge t_1 + T.$$

Now given any t, t_0 pick k such that

$$t_0 + kT \le t \le t_0 + (k+1)T.$$

Since

$$\Phi(t, t_0) = \Phi(t, t_0 + kT) \prod_{j=1}^{k} \Phi(t_0 + jT, t_0 + jT - T),$$

it follows that

$$\begin{aligned} |\Phi(t, t_0)| &\le |\Phi(t, t_0 + kT) \prod_{j=1}^{k} |\Phi(t_0 + jT, t_0 + jT - T)| \\ &\le m_0 2^{-k} \\ &\le 2m_0 2^{-(t-t_0)/T} \\ &\le m e^{-\lambda(t-t_0)} \end{aligned}$$

where $m = 2m_0$ and $\lambda = \log 2/T$. □

5.7.1 Autonomous Linear Systems

The results stated above can be specialized to linear time-invariant systems of the form

$$\dot{x} = Ax \tag{5.49}$$

with $x \in \mathbb{R}^n$.

Theorem 5.34 Stability of Linear Time-Invariant Systems.

1. *The equilibrium at the origin of (5.49) is* stable *iff all the eigenvalues of A are in $\mathbb{C}_-$ and those on the $j\omega$ axis are simple zeros of the minimal polynomial of A.*
2. *The equilibrium at the origin of (5.49) is* asymptotically stable *iff all the eigenvalues lie in $\mathbb{C}_-^\circ$.*

An alternative way of studying the stability of (5.49) is by using the Lyapunov function with symmetric, positive definite $P \in \mathbb{R}^{n\times n}$,

$$v(x) = x^T P x.$$

An easy calculation yields that

$$\dot{v}(x) = x^T(A^T P + PA)x. \tag{5.50}$$

If there exists a symmetric, positive definite $Q \in \mathbb{R}^{n\times n}$ such that

$$A^T P + PA = -Q, \tag{5.51}$$

then we see that $-\dot{v}(x)$ is a p.d.f. Equation (5.51), which is a linear equation of the form

$$\mathcal{L}(P) = Q$$

where $\mathcal{L}$ is a map from $\mathbb{R}^{n\times n} \mapsto \mathbb{R}^{n\times n}$ referred to as a *Lyapunov equation.* Given a symmetric $Q \in \mathbb{R}^{n\times n}$, it may be shown that (5.51) has a unique symmetric solution $P \in \mathbb{R}^{n\times n}$ when $\mathcal{L}$ is invertible, that is, iff all the n^2 eigenvalues of the linear map $\mathcal{L}$ are different from 0:

$$\lambda_i + \lambda_j^* \neq 0 \quad \forall\ \lambda_i, \lambda_j \in \sigma(A).$$

Here $\sigma(A)$ refers to the spectrum, or the set of eigenvalues of A.

Claim 5.35. *If A has all its eigenvalues in $\mathbb{C}_-^\circ$, then the solution of (5.51) is given by*

$$P = \int_0^\infty e^{A^T t} Q e^{At}\, dt.$$

Proof: Define

$$S(t) = \int_0^t e^{A^T \tau} Q e^{A\tau}\, d\tau,$$

Using a change of variables, we may rewrite this as

$$S(t) = \int_0^t e^{A^T(t-\tau)} Q e^{A(t-\tau)}\, d\tau.$$

Differentiating this expression using the Leibnitz rule yields

$$\dot{S}(t) = A^T S + SA + Q.$$

Also, since A has eigenvalues in $\mathbb{C}_-^\circ$ the eigenvalues of the linear operator

$$S \mapsto A^T S + SA$$

lie in $\mathbb{C}_-^\circ$ as well, so that $\dot{S} \to 0$ as $t \to \infty$, whence $P = S(\infty)$ exists, and satisfies

$$A^T S(\infty) + S(\infty)A + Q = 0.$$

Hence the P of the claim satisfies equation (5.51). □

Theorem 5.36 Lyapunov Lemma. *For $A \in \mathbb{R}^{n\times n}$ the following three statements are equivalent:*

1. *The eigenvalues of A are in $\mathbb{C}_-^\circ$.*
2. *There exists a symmetric $Q > 0$ such that the unique symmetric solution to (5.51)satisfies $P > 0$.*
3. *For all symmetric $Q > 0$ (5.51) the unique symmetric solution satisfies $P > 0$.*
4. *There exists $C \in \mathbb{R}^{m\times n}$ (with m arbitrary) with the pair A, C* observable,[2] *there exists a unique symmetric solution $P > 0$ of the Lyapunov equation with $Q = C^T C \geq 0$,*

$$A^T P + PA + C^T C = 0. \tag{5.52}$$

5. *For all $C \in \mathbb{R}^{m\times n}$ (with m arbitrary), such that the pair A, C is observable, there exists a unique solution $P > 0$ of (5.52).*

Proof: (iii) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (iv) are obvious.
(ii) $\Rightarrow$ (i) Choose $v(x) = x^T Px$ a p.d.f. and note that $-\dot{v}(x) = x^T Qx$ is also a p.d.f.
(iv) $\Rightarrow$ (i) Choose $v(x) = x^T Px$ a p.d.f. and note that $-\dot{v}(x) = x^T C^T Cx$. By LaSalle's principle, the trajectories starting from arbitrary initial conditions converge to the largest invariant set in the null space of C, where $\dot{v}(x) = 0$. By the assumption of observability of the pair A, C it follows that the largest invariant set inside the null space of C is the origin.
(i) $\Rightarrow$ (iii) By the preceding claim it follows that the unique solution to (5.51) is

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt.$$

That $P \geq 0$ is obvious. It only needs to be shown that $P > 0$. Indeed, if $x^T Px = 0$ for some $x \in \mathbb{R}^n$ then it follows with $Q = M^T M$ that

$$0 = x^T Px = \int_0^\infty x^T e^{A^T t} M^T M e^{At} x dt = \int_0^\infty |Me^{At}x|^2 dt$$

Thus we have that $Me^{At}x \equiv 0$. In particular, $Mx = 0$. But M is nonsingular since Q is. Hence $x = 0$, establishing the positive definiteness of P.

[2]Recall from linear systems theory that a pair A, C is said to be *observable* iff the largest A-invariant space in the null space of C is the origin.

(i) $\Rightarrow$ (v) By the preceding claim it follows that the unique solution to (5.51) is

$$P = \int_0^\infty e^{A^T t} C^T C e^{At} dt.$$

That $P \geq 0$ is obvious. It only needs to be shown that $P > 0$. Indeed, if $x^T P x = 0$ for some $x \in \mathbb{R}^n$ then it follows that $Ce^{At}x \equiv 0$. In particular, by differentiating $Ce^{At}x$ n-times at the origin, we get $Cx = CAx = \cdots = CA^{n-1}x = 0$. Since the pair A, C are observable, it follows that $x = 0$ establishing the positive definiteness of P. □

The following generalization of the Lyapunov lemma, called the Taussky lemma, is useful when $\sigma(A) \not\subset \mathbb{C}_-^\circ$. We drop the proof since it is not central to the developments of this chapter (see [296]).

Lemma 5.37 Taussky Lemma. *For $A \in \mathbb{R}^{n\times n}$ and given $Q \in \mathbb{R}^{n\times n}$ positive definite, if $\sigma(A) \cap \{j\omega : \omega \in]-\infty, \infty[\} = \emptyset$ the unique symmetric solution P to the Lyapunov equation (5.51)*

$$A^T P + PA = -Q$$

has as many positive eigenvalues as the number of eigenvalues of A in $\mathbb{C}_-^\circ$ and as many negative eigenvalues as the number of eigenvalues of A in $\mathbb{C}_+^\circ$.

5.7.2 *Quadratic Lyapunov Functions for Linear Time Varying Systems*

The time varying counterpart to the claim of (5.35) may be derived after a few preliminary results.

Claim 5.38. *Consider the system of (5.44) with uniformly asymptotically stable equilibrium 0:*

$$\dot{x} = A(t)x, \quad x(t_0) = x_0.$$

Further, assume that $Q(.)$ is a continuous, bounded function. Then for $t \geq 0$ the matrix

$$P(t) = \int_t^\infty \Phi^T(\tau, t) Q(\tau) \Phi(\tau, t) d\tau \tag{5.53}$$

is well-defined. Further, $P(t)$ is bounded.

Proof: Uniform asymptotic stability of (5.44) guarantees exponential stability so that

$$|\Phi(\tau, t)| \leq m e^{-\lambda(\tau - t)} \quad \forall \tau \geq t.$$

This estimate may be used to show that $P(t)$ as defined in (5.53) is well defined and bounded. □

Claim 5.39. *If $Q(t) > 0 \ \forall \ t \geq 0$ and further,*

$$\alpha x^T x \leq x^T Q(t) x, \quad \forall x \in \mathbb{R}^n, \ \ t \geq 0,$$

and $A(t)$ is bounded, say by m, then $P(t)$ of (5.53) is uniformly positive definite for $t \geq 0$, i.e., there exists $\beta > 0$ such that

$$\beta x^T x \leq x^T P(t) x.$$

Proof:

$$\begin{aligned} x^T P(t) x &= \int_t^\infty x^T \Phi^T(\tau, t) Q(\tau) \Phi(\tau, t) x \, d\tau \\ &\geq \alpha \int_t^\infty |\Phi(\tau, t) x|^2 d\tau \\ &\geq \alpha \int_t^\infty x^T x e^{-2k(t-\tau)} d\tau. \end{aligned}$$

Here the last inequality follows from the Bellman Gronwall lemma which gives a lower bound on $|\Phi(\tau, t)x|$ from the norm bound $|A(t)| \leq k$, as

$$|\Phi(\tau, t) x| \geq e^{-k(\tau - t)} |x|.$$

From the last inequality, it also follows that

$$x^T P(t) x \geq \frac{\alpha}{2k} x^T x,$$

establishing the claim. □

Theorem 5.40 Time-Varying Lyapunov Lemma. *Assume that $A(\cdot)$ is bounded. If for some $Q(t) \geq \alpha I$, $P(t)$ as defined by (5.53) is bounded, then the origin is the uniformly asymptotically stable equilibrium point of (5.44).*

Proof: By the preceding claim, $P(t)$ satisfying (5.53) is bounded below, so that

$$\gamma |x|^2 \geq x^T P(t) x \geq \beta |x|^2$$

whence $x^T P(t) x$ is a decrescent p.d.f. Further, it is easy to see that

$$\begin{aligned} \dot{v}(x, t) &= x^T (\dot{P}(t) + A^T(t) P(t) + P(t) A(t)) x \\ &= -x^T Q(t) x \\ &\leq -\alpha |x|^2. \end{aligned}$$

This establishes exponential stability of the origin and as a consequence uniform asymptotic stability. □

Uniform Complete Observability and Stability

A version of the time-varying Lyapunov lemma for systems that are uniformly completely observable may also be stated and proved (see Exercise 5.12 for the definition of uniform complete observability and the statement of the lemma). The key philosophical content of this result and the corresponding time invariant one stated extremely loosely is that if for some Lyapunov function candidate $v(x, t)$, the nonlinear system

$$\dot{x} = f(x, t)$$

with output map

$$y(t) = h(x,t) := \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} f(x,t)$$

is "locally uniformly observable" near the origin, then the origin is uniformly asymptotically stable. The reason this statement is loose is that the concept of "local uniform observability" needs to be made precise. This is straightforward for the linear time varying case that we are presently discussing, as discussed in Problem 5.12.

5.8 The Indirect Method of Lyapunov

Consider an autonomous version of the system of (5.1), namely,

$$\dot{x} = f(x)$$

with $f(0) = 0$, so that 0 is an equilibrium point of the system. Define $A = \left.\frac{\partial f}{\partial x}\right|_{x=0} \in \mathbb{R}^{n \times n}$ to be the Jacobian matrix of $f(x)$. The system

$$\dot{z} = Az$$

is referred to as the linearization of the system (5.1) *around the equilibrium point* 0. For non-autonomous systems, the development is similar: Consider

$$\dot{x} = f(x,t)$$

with $f(0,t) \equiv 0$ for all $t \geq 0$. With

$$A(t) = \left.\frac{\partial f(x,t)}{\partial x}\right|_{x=0}$$

it follows that the remainder

$$f_1(x,t) = f(x,t) - A(t)x$$

is $o(|x|)$ for each fixed $t \geq 0$. It may not, however, be true that

$$\lim_{|x|\to 0} \sup_{t \geq 0} \frac{|f_1(x,t)|}{|x|} = 0. \tag{5.54}$$

that is $f(x,t)$ is a function for which the remainder is *not uniformly* of order $o(|x|)$. For a scalar example of this consider the function

$$f(x,t) = -x + tx^2.$$

In the instance that the higher order terms are uniformly, $o(|x|)$, i.e., if (5.54) holds, then the system of (5.55)

$$\dot{z} = A(t)z \tag{5.55}$$

is referred to as the *linearization of (5.1)* about the origin. It is important to note that analysis of the linearization of (5.55) yields conclusions about the nonlinear system (5.1) *when the uniform higher-order condition of (5.54) holds.*

Theorem 5.41 Indirect Theorem of Lyapunov: Stability from Linearization. *Consider the system of (5.1) and let*

$$\lim_{|x|\to 0} \sup_{t\geq 0} \frac{|f_1(x,t)|}{|x|} = 0.$$

Further, assume that $A(\cdot)$ is bounded. If 0 *is uniformly asymptotically stable equilibrium point of (5.55), then it is a* locally *uniformly asymptotically stable equilibrium point of (5.1).*

Proof: Since $A(\cdot)$ is bounded and 0 is a uniformly asymptotically stable equilibrium point of (5.55), it follows from Claims 5.38, 5.39 that $P(t)$ defined by

$$P(t) = \int_t^\infty \Phi^T(\tau, t)\Phi(\tau, t)d\tau$$

is bounded above and below, that is, it satisfies

$$\beta x^T x \geq x^T P(t) x \geq \alpha x^T x$$

for some $\alpha, \beta > 0$. Thus $v(x,t) = x^T P(t) x$ is a decrescent p.d.f. Also,

$$\begin{aligned}\dot{v}(x,t) &= x^T[\dot{P}(t) + A^T(t)P(t) + P(t)A(t)]x + 2x^T P(t) f_1(x,t),\\ &= -x^T x + 2x^T P(t) f_1(x,t).\end{aligned}$$

Since (5.54) holds, there exists $r > 0$ such that

$$|f_1(x,t)| \leq \frac{1}{3\beta}|x| \quad \forall x \in B_r \quad t \geq 0,$$

and

$$|2x^T P(t) f_1(x,t)| \leq \frac{2|x|^2}{3} \quad \forall x \in B_r,$$

so that

$$\dot{v}(x,t) \leq -\frac{x^T x}{3} \quad \forall x \in B_r.$$

Thus $-\dot{v}(x,t)$ is an l.p.d.f. so that 0 is a locally (since $-\dot{v}$ is an l.p.d.f. only in B_r) uniformly asymptotically stable equilibrium point of (5.1). □

Remarks:

1. The preceding theorem requires ***uniform*** asymptotic stability of the linearized system to prove uniform asymptotic stability of the nonlinear system. Counterexamples to the theorem exist if the linearized system is not uniformly asymptotically stable.
2. The converse of this theorem is also true (see Problem 5.18).
3. If the linearization is time-invariant, then $A(t) \equiv A$ and $\sigma(A) \subset \mathbb{C}^\circ_-$, then the nonlinear system is uniformly asymptotically stable.

4. This theorem proves that *global* uniform asymptotic stability of the linearization implies *local* uniform asymptotic stability of the original nonlinear system. The estimates of the theorem can be used to give a (conservative) bound on the domain of attraction of the origin. For example, the largest level set of $v(x,t)$ that is uniformly contained in B_r is such a bound and can be further estimated using the bounds on $P(t)$ as $\{x \in \mathbb{R}^n : |x| \leq \frac{r}{\beta}\}$ Also, the estimates on r in the preceding proof could be refined. Systematic techniques for estimating the bounds on the regions of attraction of equilibrium points of nonlinear systems is an important area of research and involves searching for the 'best' Lyapunov functions.

To prove the instability counterpart to the above theorem one needs to assume that the linearized system is autonomous even though the original system is non-autonomous.

Theorem 5.42 Instability from Linearization. *Consider the nonlinear system (5.1) and let the linearization satisfy the conditions of equation (5.54). Further, let the linearization be time invariant, i.e.,*

$$A(t) = \frac{\partial f(x,t)}{\partial x}|_{x=0} \equiv A_0.$$

If A_0 has at least one eigenvalue in $\mathbb{C}_+^\circ$, then the equilibrium 0 of the nonlinear system is unstable.

Proof: Consider the Lyapunov equation

$$A_0^T P + P A_0 = I$$

Assume, temporarily, that A_0 has no eigenvalues on the $j\omega$ axis. Then by the Taussky lemma the Lyapunov equation has a unique solution, and further more if A_0 has at least one eigenvalue in $\mathbb{C}_+^\circ$, then P has at least one positive eigenvalue. Then $v(x,t) = x^T P x$ takes on positive values arbitrarily close to the origin and is decrescent. Further,

$$\begin{aligned} \dot{v}(x,t) &= x^T (A_0^T P + P A_0) x + 2x^T P f_1(x,t) \\ &= x^T x + 2x^T P f_1(x,t). \end{aligned}$$

Using the assumption of (5.54) it is easy to see that there exists r such that

$$\dot{v}(x,t) \geq \frac{x^T x}{3} \quad \forall x \in B_r,$$

so that it is an l.p.d.f. Thus, the basic instability theorem yields that 0 is unstable.

In the instance that A_0 has some eigenvalues on the $j\omega$ axis in addition to at least one in the open right half plane, the proof follows by continuity. □

Remarks:

1. We have shown using Lyapunov's basic theorems that if the the linearization of a nonlinear system is time invariant then:

- Having all eigenvalues in the open left half plane guarantees local uniform asymptotic stability of the origin for the nonlinear system.
- If at least one of the eigenvalues of the linearization lies in the open right half plane, then the origin is unstable.

The only situation not accounted for is the question of stability or instability when the eigenvalues of the linearization lie in the closed left half plane and include at least one on the $j\omega$ axis. The study of stability in these cases is delicate and relies on higher order terms than those of the linearization. This is discussed in Chapter 7 as stability theorems on the center manifold.

2. The technique provides only *local* stability theorems. To show global stability for equilibria of nonlinear systems there are no short cuts to the basic theorems of Lyapunov.

5.9 Domains of Attraction

Once the local asymptotic stability of an equilibrium point has been established, it is of interest to determine the set of initial conditions that converges to the equilibrium point.

Definition 5.43 Domain of Attraction at t_0. *Consider the differential equation (5.1) with equilibrium point x_0 at t_0. The* domain of attraction *of x_0 at t_0 is the set of all initial conditions x at time t_0, denote $\phi(t, t_0, x)$ satisfies*

$$\lim_{t\to\infty} \phi(t, t_0, x) = x_0.$$

For autonomous systems,

$$\dot{x} = f(x) \tag{5.56}$$

the domain of attraction of the equilibrium point x_0 (assumed to be 0, without loss of generality) is a set that is independent of the initial time, since the flow only depends on the time difference $t - t_0$. For this case, we use the notation of Chapter 2, namely $\phi_t(x)$, to mean the state at time t starting from x at time 0. We may characterize this set as follows:

$$\Omega := \left\{x : \lim_{t\to\infty} \phi_t(x) = 0\right\}.$$

Proposition 5.44 Topological Properties of the Domain of Attraction. *Let 0 be an asymptotically stable equilibrium point of (5.56). Then, the domain of attraction, Ω of 0 is an open, invariant set. Moreover, the boundary of Ω is invariant as well.*

Proof: Let $x \in \Omega$. For the invariance of Ω we need to show that $\phi_s(x) \in \Omega$ for all s. To this end, note that

$$\phi_t(\phi_s(x)) = \phi_{t+s}(x),$$

so that for all s, we have

$$\lim_{t\to\infty} \phi_t(\phi_s(x)) = 0,$$

establishing that $\phi_s(x) \in \Omega$ for all s.

To show that Ω is open, we first note that a neighborhood of the origin is contained in Ω: Indeed, this follows from the definition of asymptotic stability. Let us denote this by B_a, a ball of radius a. Now let $x \in \Omega$ since $\phi_t(x) \to 0$ as $t \to \infty$, it follows that there exists T such that $|\phi_t(x)| < a/2$ for $t \geq T$. From the fact that solutions of the differential equation (5.56) depend continuously on initial conditions on compact time intervals, it follows that there exists $\delta > 0$ such that for $|y - x| < \delta$, we have that $|\phi_T(y) - \phi_T(x)| < a/2$. In turn, this implies that $\phi_T(y) \in B_a$, so that $\lim_{t\to\infty} \phi_t(y) = 0$. Hence a ball of radius δ around x is contained in Ω, and this proves that Ω is open.

To show that the boundary of Ω, labeled $\partial\Omega$ is invariant, let $x \in \partial\Omega$. Then $x_n \to x$ with $x_n \in \Omega$. Given any time t, then there exists a sequence x_n such that $\phi_t(x_n) \to \phi_t(x)$. Now, $\phi_t(x_n) \in \Omega$, since Ω is invariant. Further, $\phi_t(x) \notin \Omega$ since $x \in \partial\Omega$. Hence, $\phi_t(x) \in \partial\Omega$ for all t. Hence, $\partial\Omega$ is invariant. □

The proof of the preceding proposition hints at a way of constructing domains of attraction of an asymptotically stable equilibrium point. One starts with a small ball of initial conditions close to the equilibrium point and integrates them backwards in time. More specifically, let B_a be the initial ball of attraction around the equilibrium point at the origin. Then $\phi_{-T}(B_a) := \{\phi_{-T}(x) : x \in B_a\}$ corresponds to integrating equation (5.56) backwards for T seconds and is contained in Ω. By choosing T large, we get a good estimate for the domain of attraction.

Example 5.45 Power Systems: Critical Clearing Times, Potential Energy Surfaces, Alert States. *The dynamics of an interconnected power system with n buses may be modeled by the so-called swing equations, which we have seen somewhat earlier in this chapter:*

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i + \sum_{i=1}^{n} B_{ij}\sin(\theta_i - \theta_j) = P_{mi} - P_{ei}.$$

Here θ_i stands for the angle of the i-th generator bus, M_i the moment of inertia of the generator, D_i the generator damping, P_{mi}, P_{ei} the mechanical input-power and the electrical output-power at the i-th bus, respectively. During the normal operation of the power system, the system dynamics are located at an equilibrium point of the system, $\theta^e \in \mathbb{R}^n$. In the event of a system disturbance, usually in the form of a lightning strike on a bus, say, B_{ij}, protective equipment on the bus opens, interrupting power transfer from generator i to generator j. In this instance, the dynamics of the system move from θ^e, since θ^e is no longer an equilibrium point of the modified system dynamics. Of interest to power utility companies is the so-called reclosure time *for the bus hit by lightning. This is the maximum time before the trajectory of the system under the disturbance drifts out of the domain* *of attraction of the equilibrium point θ^e. It is of interest to find the maximum such time under a variety of different fault conditions so as to give the fault as long as possible to clear.*

In the past, utilities relied on lengthy off-line simulations to determine these critical clearing times. More recently, analytic techniques for estimating the boundaries of the domains of attraction of equilibria have been developed. In particular, the potential energy boundary surface (PEBS) method relies on the "potential energy" function [239].

$$W(\theta) = \sum_{i=1}^{n} - \sum_{j \geq i} B_{ij} \cos(\theta_i - \theta_j) + \sum_{i=1}^{n} (P_{ei} - P_{mi})\theta_i \qquad (5.57)$$

which is a modification of the function of equation (5.34) in Section 5.4 to exclude the "kinetic energy" terms. It may now be verified that local minima of $W(\theta)$ correspond to stable equilibria of the swing dynamics and that level sets of the function $W(\theta)$ about local minima, which do not intersect any saddles of the function give conservative estimates of the domain of attraction (see Problem 5.30). In particular, a great deal of effort is spent on computing the "nearest unstable" equilibrium of the swing dynamics (i.e., the saddle extremum or maximum of the function $W(\theta)$) closest to a given operating point θ_0. An operating point or equilibrium of the form $(\theta_0, 0)$ is called an alert *state when the nearest unstable equilibrium point is close (in Euclidean norm) to θ_0.*

We will now illustrate some qualitative considerations about the computation of domains of attraction of equilibria of power systems. This is an extremely important part of the operational considerations of most electrical utilities which spend a great deal of resources on stability assessment of operating points [314]. We begin with some general considerations and remarks. Our treatment follows Chiang, Wu, and Varaiya [62]. Let x_s be a locally asymptotically stable equilibrium point of the time-invariant differential equation $\dot{x} = f(x)$ and define the *domain of attraction* of x_s to be

$$A(x_s) := \left\{ x : \lim_{t \to \infty} \phi_t(x) = x_s \right\}.$$

As we have shown earlier in this section $A(x_s)$ is an open set containing x_s. We will be interested in characterizing the boundary of the set $A(x_s)$. It is intuitive that the boundary of $A(x_s)$ contains other unstable equilibrium points, labeled x_i. We define their stable and unstable manifolds (in analogy to Chapter 2) as

$$\begin{aligned} W_s(x_i) &= \left\{ x : \lim_{t \to \infty} \phi_t(x) = x_i \right\}, \\ W_u(x_i) &= \left\{ x : \lim_{t \to -\infty} \phi_t(x) = x_i \right\}. \end{aligned} \qquad (5.58)$$

In Chapter 7 we will state the generalizations of the Hartman–Grobman and stable unstable manifold theorems which guarantee that the sets W_s, W_u defined above are manifolds. We now make three assumptions about the system $\dot{x} = f(x)$:

A1. All equilibrium points of the system are *hyperbolic*. Actually, we only need the equilibrium points on the boundary of $A(x_s)$ to be hyperbolic.

A2. If the stable manifold $W_s(x_i)$ of one equilibrium point intersects the unstable manifold of another equilibrium point $W_u(x_j)$, they intersect transversally. The reader may wish to review the definition of transversal intersection from Chapter 3, namely, if $x \in W_s(x_i) \cap W_u(x_j)$, then

$$T_x(W_s(x_i)) + T_x(W_u(x_j)) = \mathbb{R}^n.$$

A3. There exists an energy like function $v(x)$ satisfying:

a. $\dot{v}(x) < 0 \; x \notin \{x : f(x) = 0\}$.
b. $\{x : \dot{v}(x) = 0\}$ is the set of equilibrium points.
c. $v(x)$ is a proper function, that is, the inverse image of compact sets is compact.

The first two hypotheses are "generically valid hypotheses." The last hypothesis is a strong "global" version of a converse Lyapunov hypothesis, which insists that all trajectories converge to one or the other of the equilibrium points and that there are no closed orbits or other complex ω limit sets. The following proposition can now be proved.

Proposition 5.46 Unstable Equilibrium Points on the Boundary of $A(x_s)$. *Consider the time-invariant nonlinear system satisfying assumptions A1–A3 above, with x_s an asymptotically stable equilibrium point. Then, an unstable equilibrium point x_i belongs to the boundary of $A(x_s)$ if and only if*

$$W_u(x_i) - \{x_i\} \cap \bar{A}(x_s) \neq \varnothing. \tag{5.59}$$

Proof: The proof relies on the following lemma, which we leave as exercise to the reader (taking the help of [62] if necessary):

Lemma 5.47.

1. *An equilibrium point x_i belongs to $\partial A(x_i)$ if and only if*

$$W_u(x_i) - \{x_i\} \cap \bar{A}(x_s) \neq \varnothing.$$

2. *Let x_i, x_j be two equilibrium points. Then if the unstable manifold of x_i intersects the stable manifold of x_j, that is*

$$W_u(x_i) - \{x_i\} \cap W_s(x_j) - \{x_j\} \neq \varnothing,$$

then the dimension of $W_u(x_i)$ is greater than the dimension of $W_u(x_j)$.
3. *Assumption A3 implies that every trajectory starting on the boundary of $A(x_s)$ converges to one of the equilibrium points on the boundary of $A(x_s)$.*
4. *If for equilibria x_1, x_2, x_3, $W_s(x_1)$ intersects $W_u(x_2)$ transversely and $W_u(x_2)$ intersects $W_s(x_3)$ transversely, then $W_u(x_1)$ intersects $W_s(x_3)$ transversely.*

The necessity of condition (5.59) follows in straightforward fashion from the first statement of Lemma 5.47.

For the sufficiency, we first start with the case that x_i has an unstable manifold of dimension 1 on the boundary of $A(x_s)$. We would like to make sure that $W_u(x_i)$ is not entirely contained in $\partial A(x_s)$. Indeed, if this were true, since all points on the boundary of $A(x_s)$ converge to one of the equilibrium points on the boundary $A(x_s)$, say $\bar{x}$, then we have that

$$W_u(x_i) - \{x_i\} \cap W_s(\bar{x}) - \{\bar{x}\} \neq \emptyset,$$

and from the preceding lemma, we would have that

$$1 = \dim W_u(x_i) > \dim W_u(\bar{x}),$$

implying that the dimension of $W_u(\bar{x}) = 0$, or that $\bar{x}$ is a stable equilibrium point on the boundary of $A(x_s)$. This establishes the contradiction.

If x_i has an unstable manifold of dimension 2 on the boundary of $A(x_s)$, as before assuming for the sake of contradiction that $W_u(x_i)$ lies entirely in $\partial A(x_s)$, then there exists $\bar{x}$ such that

$$W_u(x_i) - \{x_i\} \cap W_s(\bar{x}) - \{\bar{x}\} \neq \emptyset.$$

In going through the steps of the previous argument, we see that $\bar{x}$ is either stable or has an unstable manifold of dimension 1. Since $\bar{x}$ cannot be stable, it should have a stable manifold of dimension 1. But then, by the last claim of Lemma 5.47, it follows that

$$W_u(x_i) - \{x_i\} \cap W_s(\bar{x}) - \{\bar{x}\} \neq \emptyset \quad \text{and} \quad W_u(\bar{x}) \cap A(x_s) \neq \emptyset$$

implying that

$$W_u(x_i) \cap A(x_s) \neq \emptyset.$$

By induction on the dimension of the unstable manifold of x_i, we finish the proof. □

Using this proposition, we may state the following theorem

Theorem 5.48 Characterization of the Stability Boundary. *For the time invariant nonlinear system $\dot{x} = f(x)$ satisfying assumptions A1–A3 above, let x_i, $i = 1, \ldots, k$ be the unstable equilibrium points on the boundary of the domain of attraction $\partial A(x_s)$ of a stable equilibrium point. Then,*

$$\partial A(x_s) = \bigcup_{i=1}^{k} W_s(x_i). \tag{5.60}$$

Proof: By the third statement of Lemma 5.47 it follows that

$$\partial A(x_s) \subset \bigcup_{i=1}^{k} W_s(x_i).$$

To show the reverse inclusion, note from the previous proposition that

$$W_u(x_i) \cap A(x_s) \neq \emptyset.$$

Choose $p \in W_u(x_i) \cap A(x_s)$ and let $p \in D \subset A(x_s)$ be a disk of dimension $n - \dim W_u(x_i)$ transversal to $W_u(x_i)$ at p. It is intuitive that as $t \to -\infty$ the flow $\phi_t(D)$ tends to the stable manifold of x_i. The precise proof of this fact is referred to as the λ lemma of Smale ([277], [140]) and is beyond the scope of this book. This lemma along with the fact that the invariance of $A(x_s)$ guarantees that

$$\phi_t(D) \subset A(x_s),$$

and yields that

$$W_s(x_i) = \lim_{t \to -\infty} \phi_t(D) \subset \bar{A}(x_s).$$

Since $W_s(x_i) \cap A(x_s) = \varnothing$, it follows that

$$W_s(x_i) \subset \partial A(x_s).$$

Consequently, we have that

$$\bigcup_{i=1}^{k} W_s(x_i) \subset \partial A(x_s),$$

completing the proof. □

This theorem characterizing the boundary of the stability region can be combined with the Lyapunov function to establish the following theorem, whose proof we leave to the exercises (Problem 5.29).

Theorem 5.49 Lyapunov Functions and the Boundary of the Domain of Attraction. *Consider the time invariant nonlinear system $\dot{x} = f(x)$ satisfying assumptions A1–A3, with x_s a locally asymptotically stable equilibrium point and $x_i, i = 1, \dots, k$ the equilibrium points on the boundary of its domain of attraction $A(x_s)$. Then we have*

1. *On the stable manifold $W_s(x_i)$ of an equilibrium point, the Lyapunov function $v(x)$ attains its minimum at the equilibrium point x_i.*
2. *On $\partial A(x_s)$ the minimum of $v(\cdot)$ is achieved at a type 1 equilibrium point, that is, one whose stable manifold has dimension $n - 1$.*
3. *If the domain of attraction $A(x_s)$ is bounded, the maximum of $v(\cdot)$ on the boundary is achieved at a source (that is, an equilibrium with a zero dimensional stable manifold).*

Remark: The preceding theorem is extremely geometrical. It gives a picture of a dynamical system satisfying assumptions A1–A3 as "gradient like" [277]. The stable equilibria lie at the base of valleys. The region of attraction of these equilibria are separated by "ridges" which are the stable manifolds of saddles or peaks. This is very much the landscape of the earth as we fly over it in an aircraft.

Numerical Methods of Computing Domains of Attraction

There are many interesting methods for computing domains of attraction. A detailed discussion of these is deferred to the literature: An interesting set of methods was proposed by Brayton and Tong in [41]. While we have shown that quadratic Lyapunov functions are the most natural for linear systems, there has been a recent resurgence in methods for obtaining piecewise quadratic Laypunov functions for nonlinear systems [156]. In this paper, the search for piecewise quadratic Lyapunov functions is formulated as a convex optimization problem using linear matrix inequalities. In turn, algorithms for optimization with linear matrix inequality constraints have made a great deal of progress in recent years (see, for example, the papers of Packard [238], [19] and the monograph by Boyd, Balakrishnan, El Ghaoui, and Feron [36]). We refer the reader to this literature for what is a very interesting new set of research directions.

5.10 Summary

This chapter was an introduction to the methods of Lyapunov in determining the stability and instability of equilibria of nonlinear systems. The reader will enjoy reading the original paper of Lyapunov to get a sense of how much was spelled out over a hundred years ago in this remarkable paper. There are several textbooks which give a more detailed version of the basic theory with theorems stated in greater generality. An old classic is the book of Hahn [124]. Other more modern textbooks with a nice treatment include Michel and Miller [209], Vidyasagar [317], and Khalil [162]. The recent book by Liu and Michel [210] contains some further details on the use of Lyapunov theory to study Hopfield neural networks and generalizations to classes of nonlinear systems with saturation nonlinearities.

5.11 Exercises

Problem 5.1 A p.d.f. function v with $-\dot{v}$ not a p.d.f.. Consider a function $g^2(t)$ with the form shown in Figure 5.9 and the scalar differential equation

$$\dot{x} = \frac{\dot{g}(t)}{g(t)} x.$$

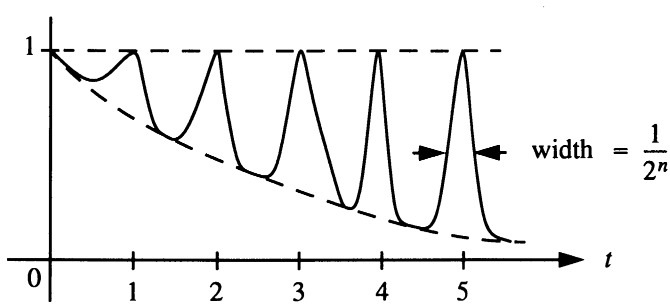

FIGURE 5.9. A bounded function $g^2(t)$ converging to zero but not uniformly.

Consider the Lyapunov function candidate

$$v(x,t) = \frac{x^2}{g^2(t)}\left[3 - \int_0^t g^2(\tau)d\tau\right].$$

Verify that $v(x, t)$ is a p.d.f. and so is $\dot{v}(t, x)$. Also, verify that the origin is stable but **not asymptotically stable**. What does this say about the missing statement in the table of Lyapunov's theorem in Theorem 5.16.

Problem 5.2 Krasovskii's theorem generalized. (a) Consider the differential equation $\dot{x} = f(x)$ in $\mathbb{R}^n$. If there exist $P, Q \in \mathbb{R}^{n\times n}$, two constant, symmetric, positive definite matrices such that

$$P\frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^T P = -Q,$$

then prove that $x = 0$ is globally asymptotically stable. *Hint: Try both the Lyapunov function candidates $f(x)^T Pf(x)$ and $x^T Px$. Be sure to justify why the former is a p.d.f. if you decide to use it!*

(b) Consider the differential equation $\dot{x} = A(x)x$ in $\mathbb{R}^n$. If there exist $P, Q \in \mathbb{R}^{n\times n}$, two constant, symmetric, positive definite matrices such that

$$PA(x) + A(x)^T P = -Q,$$

then prove that $x = 0$ is globally asymptotically stable.

Problem 5.3. Consider the second-order nonlinear system:

$$\begin{aligned}\dot{x}_1 &= -x_2 + \epsilon x_1(x_1^2 + x_2^2)\sin(x_1^2 + x_2^2),\\ \dot{x}_2 &= x_1 + \epsilon x_2(x_1^2 + x_2^2)\sin(x_1^2 + x_2^2).\end{aligned}$$

Show that the linearization is inconclusive in determining the stability of the origin. Use the direct method of Lyapunov and your own creative instincts to pick a V to study the stability of the origin for $-1 \le \epsilon \le 1$.

Problem 5.4. Consider the second order nonlinear system:

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -x_1 + (1 - x_1^2 - x_2^2)x_2.\end{aligned}$$

- Discuss the stability of the origin.
- Find the (only) limit cycle of the system.
- Prove using a suitable Lyapunov function and LaSalle's principle that all trajectories not starting from the origin converge to the limit cycle.

Problem 5.5 Rayleigh's equation. Consider the second-order Rayleigh system:

$$\begin{aligned}\dot{x}_1 &= -\epsilon(x_1^3/3 - x_1 + x_2),\\ \dot{x}_2 &= -x_1.\end{aligned}$$

This system is known to have one equilibrium and one limit cycle for $\epsilon \neq 0$. Show that the limit cycle does not lie inside the strip $-1 \leq x_1 \leq 1$. Show, using a suitable Lyapunov function, that it lies outside a circle of radius $\sqrt{3}$.

Problem 5.6 Discrete-time Lyapunov theorems. Consider the discrete time nonlinear system

$$x(k+1) = f(x(k), k). \tag{5.61}$$

Check the definitions for stability, uniform stability, etc. and derive discrete time Lyapunov theorems for this system with suitable conditions on a Lyapunov function candidate $v(x, k)$ and on

$$\begin{aligned} \Delta v(x(k), k) &:= v(x(k+1), k+1) - v(x(k), k) \\ &= v(f(x(k), k), k+1) - v(x(k), k). \end{aligned}$$

Also, discuss theorems for exponential stability and instability. Is there a counterpart to LaSalle's principle for discrete time systems?

Problem 5.7 Discrete-time Lyapunov equation. Prove that the following statements are equivalent for the discrete-time linear time invariant system

$$x(k+1) = Ax(k):$$

1. $x = 0$ is exponentially stable.
2. $|\lambda_i| < 1$ for all the eigenvalues of A.
3. Given any positive definite symmetric matrix $Q \in \mathbb{R}^{n\times n}$, there exists a unique, positive definite symmetric matrix $P \in \mathbb{R}^{n\times n}$ satisfying
$$A^T PA - P = -Q.$$
4. Given a matrix $C \in \mathbb{R}^{n_o\times n}$ such that A, C is observable, there exists a unique positive definite matrix P such that
$$A^T PA - P = -C^T C.$$

Problem 5.8 Discrete-time stability from the linearization. Give a proof of the uniform local stability of the equilibrium at the origin of the discrete-time nonlinear system (5.61) when the linearization is uniformly stable.

Problem 5.9 Barbalat's lemma [259]. Assume that $\phi(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ is **uniformly continuous** and integrable, that is,

$$\int_0^\infty \phi(t)dt < \infty.$$

Then show that

$$\lim_{t\to\infty} \phi(t) = 0.$$

Give a counterexample to show the necessity of uniform continuity.

Problem 5.10 Another generalization of LaSalle's principle. Consider a nonlinear time varying system Lipschitz continuous in x uniformly in t. Let $v; \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$ be such that for $x \in B_r$, $\delta > 0$, and functions α_1, α_2 of class K,

$$\begin{aligned} \alpha_1(|x|) \leq v(x,t) \leq \alpha_2(|x|), \\ \dot{v}(t,x) \leq 0, \\ \int_t^{t+\delta} \dot{v}(s(\tau,t,x),\tau)d\tau \leq -\lambda v(x,t) \qquad 0 < \lambda < 1. \end{aligned} \tag{5.62}$$

Now show that for $|x(t_0)| < \alpha_2^{-1}(\alpha_1(r))$ show that the trajectories converge uniformly, asymptotically to 0.

Problem 5.11 Power system swing dynamics on $T^n \times \mathbb{R}^n$ [9]. Consider the swing equations of the power system (5.33). Since the right-hand side is periodic in each of the θ_i with period 2π, we can assume that the state space of the system is $S^1 \times \cdots \times S^1 \times \mathbb{R}^n$ or $T^n \times \mathbb{R}^n$. Note that the function $v(\theta, \omega)$ of (5.34) is not a well defined function from $T^n \times \mathbb{R}^n \mapsto \mathbb{R}$, unless $P_{mi} - P_{ei} = 0$. What are the implications of the assertion in the text that bounded trajectories (on $\mathbb{R}^n \times \mathbb{R}^n$) converge to equilibrium points of the swing dynamics on the manifold $T^n \times \mathbb{R}^n$?

Prove that trajectories of the swing equations are bounded on the manifold $T^n \times \mathbb{R}^n$. Can you give counterexamples to the assertion that bounded trajectories on $T^n \times \mathbb{R}^n$ converge to equilibria (you may wish to review the Josephson junction circuit of Chapter 2 for hints in the case that $n = 1$)? Prove that all trajectories of the swing equation converge to equilibria for $P_{mi} - P_{ei}$ sufficiently small for $i = 1, \ldots, n$.

Problem 5.12 Generalization of the time-varying Lyapunov lemma. Consider the linear time varying system

$$\dot{x} = A(t)x.$$

Define it to be *uniformly completely observable* from the output $y(t) = C(t)x(t)$, if there exists a $\delta > 0$ such that the observability Gramian defined in (5.63) is uniformly positive definite.

$$W_0(t, t+\delta) := \int_t^{t+\delta} \Phi^T(\tau,t)C^T(\tau)C(\tau)\Phi(\tau,t)d\tau) \geq kI \quad \forall t. \tag{5.63}$$

Now use Problem 5.10 to show that if there exists a uniformly positive definite bounded matrix $P(\cdot) : \mathbb{R} \mapsto \mathbb{R}^{n \times n}$ satisfying

$$\begin{aligned} k_1 I \leq P(t) \leq k_2 I, \\ -\dot{P}(t) = A^T(t)P(t) + P(t)A(t) + C^T(t)C(t). \end{aligned} \tag{5.64}$$

with $A(t), C(t)$ a uniformly completely observable pair, then the origin is uniformly asymptotically stable and hence exponentially stable. Prove the converse as well.

Problem 5.13 Time-varying Lyapunov lemma generalized with output injection [259]. Prove that if a pair $(A(t), C(t))$ is uniformly completely observable, that is the observability Gramian of (5.63) is uniformly positive definite, then for any $K(t) \in \mathbb{R}^{n \times n_{\circ}}$ that $(A(t) + K(t)C(t), C(t))$ is also uniformly completely observable. You will need to do a perturbation analysis to compare trajectories of the two systems

$$
\begin{aligned}
\dot{x} &= A(t)x(t), \\
\dot{w} &= (A(t) + K(t)C(t))w(t),
\end{aligned}
$$

with the same initial condition at time t_0 and give conditions on the terms of the second observability Gramian using a perturbation analysis. You may wish to refer to the cited reference for additional help.

Problem 5.14 Zubov's converse theorem.

1. For time-invariant nonlinear systems assume that there exists a function $v : \Omega \mapsto [0, 1[$ with $v(x) = 0$ if and only if $x = 0$. Further, assume that if Ω is bounded, then as $x \to \partial\Omega$, $v(x) \to 1$. On the other hand, if Ω is unbounded assume that $|x| \to \infty \Rightarrow v(x) \to 1$. Assume also that there exists a p.d.f. $h(x)$, such that for $x \in \Omega$
$$
\dot{v}(x) = -h(x)(1 - v(x)). \tag{5.65}
$$
Show that Ω is the domain of attraction of the asymptotically stable equilibrium point 0.
2. Let v, h be as in the previous part, but replace equation (5.65) by
$$
\dot{v}(x) = -h(x)\left(1 - v(x))(1 + |f(x)|^2\right)^{\frac{1}{2}}. \tag{5.66}
$$
Prove the same conclusion as in the previous part.

Problem 5.15 Modified exponential stability theorem. Prove the following modification of the exponential stability theorem (5.17) of the text: If there exists a function $v(x, t)$ and some constants $h, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that for all $x \in B_h$, $t \geq 0$,

$$
\begin{aligned}
\alpha_1 |x|^2 \leq v(x, t) &\leq \alpha_2 |x|^2, \\
\left.\frac{dv(x, t)}{dt}\right|_{(5.1)} &\leq 0, \\
\int_t^{t+\delta} \left.\frac{dv(x, t)}{dt}\right|_{(5.1)} &\leq -\alpha_3 |x(t)|^2,
\end{aligned} \tag{5.67}
$$

then $x(t)$ converges exponentially to zero. This problem is very interesting in that it allows for a decrease in $v(x, t)$ proportional to the norm squared of $x(t)$ over a window of length δ rather than instantaneously.

Problem 5.16 Perturbations of exponentially stable systems. Consider the following system on $\mathbb{R}^n$

$$\dot{x} = f(x,t) + g(x,t). \tag{5.68}$$

Assume that $f(0,t) = g(0,t) \equiv 0$. Further, assume that:

1. 0 is an exponentially stable equilibrium point of

$$\dot{x} = f(x,t).$$

2.

$$|g(x,t)| \leq \mu|x| \quad \forall x \in \mathbb{R}^n.$$

Show that 0 is an exponentially stable equilibrium point of 5.68 for μ small enough. The moral of this exercise is that *exponential stability is robust*!

Problem 5.17 Bounded-input bounded-output or finite gain stability of nonlinear systems. Another application of Problem 5.16 is to prove L_∞ bounded input bounded output stability of nonlinear systems (you may wish to review the definition from Chapter 4). Consider the nonlinear control system

$$\begin{aligned} \dot{x} &= f(x,t) + g(x,t)u, \\ y &= h(x). \end{aligned} \tag{5.69}$$

Prove that if 0 is an exponentially stable equilibrium point of $f(x,t)$, $|g(x,t)| \leq \mu|x|$ and finally

$$h(0) = 0, \quad |h(x)| \leq \beta|x| \quad \forall x,$$

then the control system (5.69) is L_∞ finite gain stable. Give the bound on the L_∞ stability.

Problem 5.18 Local exponential stability implies exponential stability of the linearization. Prove the converse of Theorem 5.41, that is, if 0 is a locally exponentially stable equilibrium point of the nonlinear system (5.1) then it is a (globally) exponentially stable equilibrium point of the linearized system, provided that the linearization is uniform in the sense of Theorem 5.41. Use the exponential converse theorem of Lyapunov.

Problem 5.19 Interconnected systems. The methods of Problem 5.16 may be adapted to prove stability results for interconnected systems. Consider an interconnection of N systems on $\mathbb{R}^{n_i}$, $i = 1, \dots N$, given by

$$\dot{x}_i = f_i(x_i, t) + \sum_{j=1}^{N} g_{ij}(x_1, \dots, x_N, t) \tag{5.70}$$

with $f_i(x_i,t)$, $g_{ij}(x_1, \dots, x_N, t)$ vector fields on $\mathbb{R}^{n_i}$ for $i = 1, \dots, N$. Now assume that in each of the decoupled systems

$$\dot{x}_i = f_i(x_i, t) \tag{5.71}$$

the origin is an exponentially stable equilibrium point, and that for $i,\ j = 1, \ldots, N$,

$$|g_{ij}(x_1, \ldots, x_N, t)| \le \sum_{j=1}^{N} \gamma_{ij}|x_j|.$$

Using the converse Lyapunov theorem for exponentially stable systems to generate Lyapunov functions $v_i(x_i, t)$ satisfying

$$\begin{aligned} \alpha_1^i|x_i|^2 \le v_i(x_i, t) &\le \alpha_2^i|x_i|^2, \\ \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_i} f_i(x_i, t) &\le -\alpha_3^i|x_i|^2, \\ \left|\frac{\partial v_i}{\partial x_i}\right| &\le \alpha_4^i|x_i|, \end{aligned} \tag{5.72}$$

give conditions on the $\alpha_1^i, \alpha_2^i, \alpha_3^i, \alpha_4^i, \gamma_{ij}$ to prove the exponential stability of the origin. *Hint: Consider the Lyapunov function*

$$\sum_{i=1}^{N} d_i v_i(x_i),$$

where $d_i > 0$ *and* $v_i(x_i)$ *is the converse Lyapunov function for each of the systems (5.71).*

See [162] for a generalization of this problem to the case when the stability conditions of equations (5.72) are modified to replace $|x_i|$ by $\phi_i(x_i)$ where $\phi_i(\cdot)$ is a positive definite function, and

$$|g_{ij}(x_1, \ldots, x_N)| \le \sum_{j=1}^{N} \gamma_{ij}\phi_j(x_j).$$

Problem 5.20 Hopfield networks with asymmetric interconnections. Give a counterexample to the lack of oscillations of the Hopfield network of (5.37) for asymmetric interconnections T_{ij}. Modify the method of Problem 5.19 to give conditions under which the Hopfield network with asymmetric interconnections does not have limit cycles.

Problem 5.21 Satellite stabilization. Consider the satellite rigid body B with an orthonormal body frame attached to its center of mass with axes aligned with the principal axes of inertia as shown in Figure 5.10. Let $b_0 \in \mathbb{R}^3$ be a unit vector representing the direction of an antenna on the satellite and $d_0 \in \mathbb{R}^3$ a fixed unit vector representing the direction of the antenna of the receiving station on the ground. Euler's equations of motion say that if $\omega \in \mathbb{R}^3$ is the angular velocity vector of the satellite about the body axes shown in Figure 5.10, then

$$I\dot{\omega} + \omega \times I\omega = \tau \tag{5.73}$$

Here $I = \mathrm{diag}\,(I_1, I_2, I_3)$. The aim of this problem is to find a control law to align b_0 with $-d_0$. In the body frame, d_0 is not fixed. It rotates with angular velocity $-\omega$

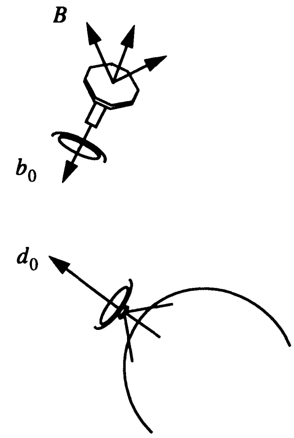

FIGURE 5.10. Satellite and earth station

so that

$$\dot{d}_0 = -\omega \times d_0 \tag{5.74}$$

Now consider the control law

$$\tau = -\alpha\omega + d_0 \times b_0$$

applied to (5.73), (5.74). What are the equilibrium points of the resulting system in $\mathbb{R}^{3+3}$? Linearize the system about each equilibrium point and discuss the stability of each. Is it possible for a system in $\mathbb{R}^6$ to have this number and type of equilibria. Do the system dynamics actually live on a submanifold of $\mathbb{R}^6$? If so, find the state space of the system. Prove that the control law causes all trajectories to converge to one of the equilibrium points. It may help to use the Lyapunov function candidate

$$v(\omega, d_0) = \frac{1}{2}\omega^T I \omega + \frac{1}{2}|d_0 + b_0|^2.$$

Another approach to satellite stabilization is in a recent paper by D'Andrea Novel and Coron [77].

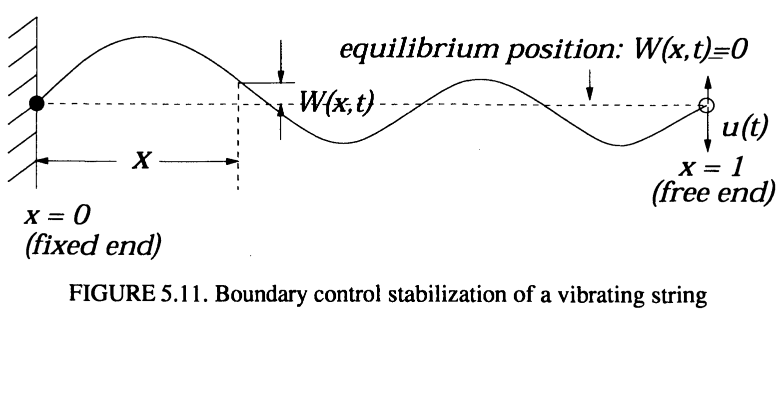

FIGURE 5.11. Boundary control stabilization of a vibrating string

Problem 5.22 Boundary control of a vibrating string. The equation of motion of a linear, frictionless, vibrating string of length 1 is given by the following partial differential equation:

$$m\frac{\partial^2 W(x,t)}{\partial t^2} - T\frac{\partial^2 W(x,t)}{\partial x^2} = 0, \tag{5.75}$$

where $W(x,t)$ denotes the displacement of the string at location x at time t, with $x \in]0,1[$, $t \geq 0$. The quantities m and T are the mass per unit length of the string and the tension in the string, respectively. Let the string be fixed at $x = 0$ as shown in Figure 5.11. Thus, $W(0,t) = 0$ for all $t \geq 0$. Let a vertical control force $u(\cdot)$ be applied to the free end of the string at $x = 1$. The balance of the forces in the vertical direction (at $x = 1$) yields:

$$u(t) = T\left.\frac{\partial W(x,t)}{\partial x}\right|_{x=1} \tag{5.76}$$

for all $t \geq 0$. The energy of the spring at time t is

$$E(t) = \frac{1}{2}\int_0^1 m\left(\frac{\partial W(x,t)}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^1 T\left(\frac{\partial W(x,t)}{\partial x}\right)^2 dx \tag{5.77}$$

which consists of the kinetic energy (first integral in the expression above) and the strain energy (second integral above). Show that if a *boundary control*

$$u(t) = -k\left.\frac{\partial W(x,t)}{\partial t}\right|_{x=1} \tag{5.78}$$

is applied, where $k > 0$, then $\dot{E}(t) \leq 0$ for all $t \geq 0$. See also [274].

Problem 5.23 Brockett's H dot equation [26], [132].

1. Consider the matrix differential equation ($H(t) \in \mathbb{R}^{n\times n}$

$$\dot{H} = [H, G(t)] \tag{5.79}$$

Here $G(t) \in \mathbb{R}^{n\times n}$ is a skew-symmetric matrix. The notation $[A,B] := AB - BA$ refers to the Lie bracket, or commutator, between A and B. Show that if $H(0)$ is symmetric, then $H(t)$ is symmetric for all t. Further, show in this case that the eigenvalues of $H(t)$ are the same as the eigenvalues of $H(0)$. Equation (5.79) is said to generate an *iso-spectral* flow on the space of symmetric matrices $SS(n)$.

2. Consider the so-called double bracket equation

$$\dot{H} = [H, [H, N]] \tag{5.80}$$

with $N \in \mathbb{R}^{n\times n}$ a constant matrix. The Lie bracket is defined as above. Show that if $H(0)$ and N are symmetric, then (5.80) generates an iso-spectral flow as well on $SS(n)$.

3. Now using the Lyapunov function candidate

$$v(H) = -\operatorname{trace}(H^T N)$$

and a norm on $\mathbb{R}^{n\times n}$ given by $\|A\| = (\text{trace } A^T A)^{\frac{1}{2}}$, prove that

$$\dot{v}(H) = -\|[H, N]\|^2$$

Now use the fact that

$$|\text{trace } A^T B| \leq \|A\| \, \|B\|$$

and LaSalle's principle to find the limits as $t \to \infty$ of $H(t)$.

4. Consider the double bracket equation with

$$N = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

with $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$. Use the iso-spectrality of (5.80) and the previous calculations to characterize the equilibrium points of (5.80). How many are there?

5. Linearize (5.80) about each equilibrium point and try to determine the stability from the linearization. What can you say if some of the eigenvalues of N, $H(0)$ are repeated?

Discuss how you might use the H dot equation to sort a given set of n numbers $\lambda_1, \cdots, \lambda_n$ in descending order. How about sorting them in ascending order?

Problem 5.24 Controlled Lagrangians for the satellite stabilization problem [28], [30]. Consider a satellite system with an internal rotor as shown in Figure 5.12 with equations given by

$$\begin{aligned}
\lambda_1 \dot{\omega}_1 &= \lambda_2 \omega_2 \omega_3 - (\lambda_3 \omega_3 + J_3 \dot{\alpha}) \omega_2, \\
\lambda_2 \dot{\omega}_2 &= \lambda_1 \omega_1 \omega_3 - (\lambda_3 \omega_3 + J_3 \dot{\alpha}) \omega_1, \\
\lambda_3 \dot{\omega}_3 + J_3 \ddot{\alpha} &= (\lambda_1 - \lambda_2) \omega_1 \omega_2, \\
J_3 \dot{\omega}_3 + J_3 \ddot{\alpha} &= u,
\end{aligned} \tag{5.81}$$

where the body angular velocity is given by $(\omega_1, \omega_2, \omega_3)^T \in \mathbb{R}^3$, and I_1, I_2, I_3 are the satellite moments of inertia, $J_1 = J_2, J_3$ the rotor moments of intertia, and α the relative angle of the rotor. Furthermore, $\lambda_i = I_i + J_i$. It is easy to verify that regardless of the choice of u we have an equilibrium manifold $\omega_1 = \omega_3 = 0$ and $\omega_2 = m$ arbitrary. This corresponds to the satellite spinning freely about the intermediate axis. Prove that the control law

$$u = k(\lambda_1 - \lambda_2)\omega_1 \omega_2$$

stabilizes the satellite about each of these equilibria. Note that you are not required to prove anything about the variables $\alpha, \dot{\alpha}$.

Problem 5.25 Rates of convergence estimates for the Lyapunov equation. Consider the Lyapunov equation

$$A^T P + PA = -Q.$$

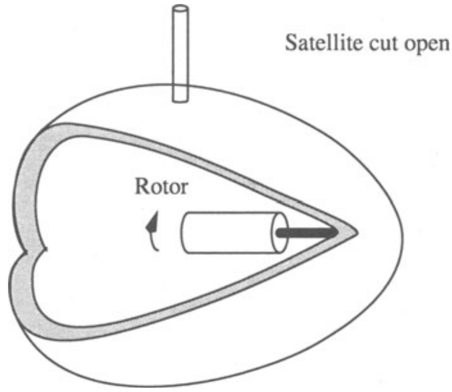

FIGURE 5.12. Satellite with a spinning rotor aligned with its central principal axis as controller

Note that an estimate of the rate of convergence is given by

$$\mu_A(Q) := \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}.$$

Show the following three properties of $\mu_A(Q)$:

1. Show that $\mu_A(kQ) = \mu_A(Q)$ for all $k \in \mathbb{R}_+$.
2. Show that if $\lambda_{\min}(Q) = 1$, then $\mu_A(I) \geq \mu_A(Q)$.
3. Using the two previous facts, show that $\mu_A(I) \geq \mu_A(Q)$ for arbitrary Q.

Thus, if one were to use the Lyapunov equation to estimate the rate of convergence of a linear equation, it is best to use $Q = I$! Now, use this to study the stability of the perturbed linear system

$$\dot{x} = Ax + f(x, t). \tag{5.82}$$

Here, $A \in \mathbb{R}^{n \times n}$ has its eigenvalues in $\mathbb{C}^\circ_-$, and $|f(x, t)| \leq \gamma |x|$. Now prove that the perturbed linear system is stable if $\gamma < \mu_A(I)$.

Problem 5.26 Stability conditions based on the measure of a matrix. Recall the definition of the *measure of a matrix* $\mu(A)$ from Problem 3.2. Use this definition to show that if for some matrix measure $\mu(\cdot)$, there exists an $m > 0$, such that

$$\int_{t_0}^{t_0+T} \mu(A(\tau))d\tau < -m \quad \forall t \geq T \quad t_0 \geq 0,$$

then the origin of the linear time-varying system of (5.44) is exponentially stable. Apply this test to various kinds of matrix measures (induced by different norms).

Problem 5.27 Smooth stabilizability [279]. Assume that the origin $x = 0$ of the system

$$\dot{x} = f(x, u)$$

is stabilizable by smooth feedback, that is, there exists $u = k(x)$ such that 0 is a locally asymptotically stable equilibrium point of $\dot{x} = f(x, k(x))$. Now prove that the extended system

$$\begin{aligned} \dot{x} &= f(x, z), \\ \dot{z} &= h(x, z) + u \end{aligned} \tag{5.83}$$

is also stabilizable by smooth feedback. *Hint:* Use the converse Lyapunov function associated with the x system and try to get z to converge to $k(x)$. Repeat the problem when the original problem is respectively globally or exponentially stabilizable. By setting $h(x, z) \equiv 0$, we see that smooth stabilizability is a property which is not lost when integrators are added to the input!

Problem 5.28. Prove Lemma 5.47.

Problem 5.29. Prove Theorem 5.49. Now apply this theorem to the following generalization of the power system swing dynamics

$$\begin{aligned} \dot{x}_1 &= x_2. \\ \dot{x}_2 &= -Dx_2 - \nabla w(x_1). \end{aligned} \tag{5.84}$$

Here $x_1, x_2 \in \mathbb{R}^n$, $w(x_1)$ is a proper function with isolated non-critical stationary points and $D \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix. Is this "second order" system gradient like? Characterize the domains of attraction of its stable equilibria. Show that all the equilibria of the system are either stable nodes, unstable nodes, or saddles (i.e., that the eigenvalues of their linearizations are real).

Problem 5.30 Potential energy boundary surfaces. Prove rigorously that the method sketched in the Example 5.45, a level set of $W(\theta)$ around a local minimum θ_0 just touching a saddle or local maximum of W, is a (conservative) estimate of the domain of attraction of the equilibrium θ_0 of the swing dynamics. You may wish to use the "Lyapunov" function of (5.34) for helping you along with the proof.