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REVIEW

From infinite to finite time stability in Celestial Mechanics and Astrodynamics

Alessandra Celletti[1]

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Abstract

Time scales in Celestial Mechanics and Astrodynamics vary considerably, from a few hours for the motion of Earth's artificial satellites to millions of years for planetary dynamics. Hence, the time scales on which one needs to investigate the stability of celestial objects are different. Therefore, the methods of study are themselves different and might lead to specific definitions of stability, either in the sense of bounds on the initial conditions or rather in the sense of confinement in a given region of the phase space. In this work we concentrate on three different methods: perturbation theory, Nekhoroshev's theorem, KAM theory. All theories are constructive in the sense that they provide explicit algorithms to give estimates on the parameters of the system and on the stability time. Perturbation theory gives results on finite time scales, Nekhoroshev's theorem provides stability results on exponentially long times, KAM theory ensures the confinement between invariant tori in low-dimensional systems.

We recall the basic ingredients of each theory, starting with KAM theory, then presenting Nekhoroshev's theorem and finally introducing perturbation theory. We provide examples of stability results for some objects of the Solar system. Precisely, we consider the stability of the rotational motion of the Moon (or other planetary satellites) within the spin-orbit model by means of KAM theory, we analyze the stability of asteroids, also in the triangular equilibrium Lagrangian points, using Nekhoroshev's theorem, we study the Earth's satellite dynamics through perturbation theory.

Keywords Celestial mechanics · Astrodynamics · Stability · Perturbative methods · Nekhoroshev's theorem · KAM theory · Satellite dynamics · Lagrangian points · Spin-orbit problem

Understanding the behavior over time of celestial objects is one of the main goals of Celestial Mechanics, either concerning planets, satellites, asteroids, comets or spacecraft. In particular, it is of paramount importance to understand the future of the Solar system and, especially, of planet Earth. The dynamics of satellites, asteroids and comets is interesting for several reasons, especially in view to prevent collisions. The lifetime of artificial satellites and space debris is of crucial importance in Astrodynamics. The common factor of all the above examples is the prediction of the stability time of each celestial object, either natural or artificial. A crucial remark is that the time interval over which one is interested to analyze the stability strongly depends on the object of study, varying between a few years for artificial satellites to millions of years on a planetary scale.

Celestial Mechanics offers different methods to analyze the stability of celestial objects, where the concept of *stability* can have different meanings, ranging from Lyapunov stability, ensuring that solutions stay nearby an equilibrium position over a given time, to KAM stability, ensuring that for low-dimensional systems the dynamics stays confined within a given region of the phase space.

In this work we revise three different mathematical methods to analyze the stability of models in Celestial Mechanics and Astrodynamics: perturbation theory, Nekhoroshev's theorem, KAM theory. The three methods ensure stability over different time scales, as shown in Fig. 1.

Developed during the XVIII and XIX centuries, perturbation theory applies to nearly-integrable Hamiltonian systems and it consists in the construction of a canonical transformation that allows one to find the solution of the transformed system within a better degree of approximation than

✉ A. Celletti
celletti@mat.uniroma2.it

1 Department of Mathematics, University of Rome Tor Vergata, Via della Ricerca Scientifica 1, Roma, 00133, Italy

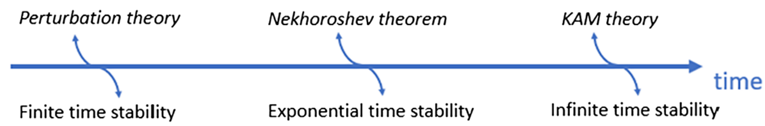

Fig. 1 Timeline of perturbative methods and stability times

the original one. It can be implemented under quite general conditions and its proof provides an explicit algorithm to construct the canonical transformation and the new approximate solution. This feature attracted astronomers that used perturbation theory to analyze the dynamics of celestial objects. Another feature of perturbation theory is that it is very efficient, but in some cases can be also very complex. These hallmarks are witnessed by results obtained already in the XIX century. In fact, using perturbation theory, J.-U. Leverrier conjectured the existence of a planet beyond Uranus (the highest planet known at that time); his computations led to the discovery of Neptune by the astronomer G. Galle, thus showing the effectiveness of the computations performed using perturbative methods. Almost at the same time, again using perturbation theory, C. Delaunay made very accurate computations of the ephemeris of the Moon, which took about 20 years of his life, since they required very long computations. Perturbative methods are still widely used nowadays in different contexts of Celestial Mechanics and Astrodynamics (see, e.g., recent works like (Danesi et al. 2023; Di Ruzza et al. 2020, 2023; Di Ruzza 2023; Mastroianni and Efthymiopoulos 2023)), from the study of satellite dynamics (see, e.g., (Lidov 1962; Steichen and Giorgilli 1997; De Blasi et al. 2021)) to the determination of low-cost interplanetary trajectories (see, e.g., (Conley 1968; Jorba and Masdemont 1999; Koon et al. 2001; Gómez and Mondelo 2001; Celletti et al. 2015; Kumar et al. 2022)).

Long-time stability of nearly-integrable Hamiltonian systems is the content of Nekhoroshev's theorem, which can be developed under a non-degeneracy condition of the integrable part of the Hamiltonian. The theorem says that the action variables stay bounded for an exponentially long time, thus limiting the time of diffusion of the trajectories. Though we will mainly consider only non-resonant motions, Nekhoroshev's theorem applies also to resonant dynamics, namely when the frequency vector (computed as the derivative of the integrable Hamiltonian) satisfies a commensurability relation. After the original proof of Nekhoroshev in 1977 (Nekhoroshev 1977), several alternative formulations and proofs have been given, often providing explicit algorithms to give an accurate estimate of the diffusion time (see, e.g., (Lochak and Neishtadt 1992; Pöschel 1993; Fassò et al. 1998; Guzzo 2007)).

Kolmogorov-Arnold-Moser theory (hereafter KAM theory, (Kolmogorov 1954; Arnol'd 1963; Moser 1962)) investigates the existence of quasi-periodic motions in non-integrable dynamical systems and, in particular, the persistence of invariant tori in nearly-integrable Hamiltonian systems. KAM theory for nearly-integrable Hamiltonian systems has been applied to several problems of Celestial Mechanics, among which the spin-orbit problem (Celletti 1990b,a, 2010), the three-body problem (Celletti and Chierchia 1997, 2006, 2007), the planetary problem (Giorgilli et al. 2009, 2017; Sansottera et al. 2013). A KAM theory for more general systems has been developed in Moser (1967) and later in Broer et al. (1996); the theory for non-Hamiltonian systems, and in particular dissipative systems, requires to consider extra-parameters to compensate the loss of energy. A KAM theory for a special class of dissipative systems, called conformally symplectic systems, has been developed in Calleja et al. (2013), which leads to very efficient algorithms for the construction of invariant tori for values of the parameters which are fully consistent with the astronomical observations. An application to the spin-orbit problem with a dissipative effect due to the tidal torque is studied in Calleja et al. (2022b) (see also (Calleja et al. 2022a, 2024), compare with (Celletti and Chierchia 2009)) with results in full agreement with the astronomical data and also in good agreement with the experimental values of the breakdown threshold of invariant tori.

This work is organized as follows: we start with KAM theory providing stability results for infinite times, then we look at exponential time stability through Nekhoroshev's theorem, we conclude with finite time stability results based on perturbation theory. More precisely, in Sect. 1 we present KAM theory (Sect. 1.1), and we give an application to a 3-body problem (Sect. 1.2), and to the conservative and dissipative spin-orbit problems (Sect. 1.3). In Sect. 2 we introduce Nekhoroshev's theorem (Sect. 2.1) and we present applications to the 3-body problem (Sect. 2.2), an Earth's satellite model (Sect. 2.3), and to the triangular Lagrangian equilibrium points (Sect. 2.4). Results based on perturbation theory are given in Sect. 3, where we recall the basic ingredients to construct a normal form (Sect. 3.1); we also present some applications to study the Earth's satellite stability (Sect. 3.2) and the space debris dynamics (Sect. 3.3). Some conclusions and perspectives are given in Sect. 4.

1 Infinite time stability: KAM theory

In this Section we give a short introduction to KAM theory (Sect. 1.1), an application to the planar, circular, restricted 3-body problem (Sect. 1.2) and an application to the conservative and dissipative spin-orbit models (Sect. 1.3).

1.1 KAM theory

Developed at the middle of the XX century, Kolmogorov-Arnold-Moser theory (hereafter KAM theory, see (Kolmogorov 1954; Arnol'd 1963; Moser 1962)) is a powerful set of theorems, algorithms and techniques on the persistence of quasi-periodic motions in nearly-integrable systems. The original versions generated a large body of results in different directions, including the extension to non-Hamiltonian systems (Moser 1967; Broer et al. 1996), which find many applications in Celestial Mechanics and Astrodynamics. First, we precise that by quasi-periodic motion we mean a motion that can be described by a function $F = F(\omega_1, \ldots, \omega_n)$, which is multi-periodic of period 2π in each argument and with the vector $\underline{\omega} = (\omega_1, \ldots, \omega_n)$ being linearly independent.

While in classical perturbation theory (see Sect. 3) one considers the stability of the motion for given initial conditions, Kolmogorov's breakthrough (Kolmogorov 1954) consisted in studying the stability of motions with fixed frequency (instead of fixing the initial conditions). To be more precise, let us consider a nearly-integrable Hamiltonian system in action-angle variables $(\underline{I}, \underline{\varphi}) \in \mathbb{R}^n \times \mathbb{T}^n$ with n denoting the number of degrees of freedom:

$$\mathcal{H}(\underline{I}, \underline{\varphi}) = h(\underline{I}) + \varepsilon f(\underline{I}, \underline{\varphi}) , \tag{1}$$

where h is called the integrable Hamiltonian, f is the perturbing function and $0 \leq \varepsilon < 1$ is the perturbing parameter. Let us denote by $\underline{\omega} = \underline{\omega}(\underline{I})$ the frequency vector associated to (1), which can be computed as

$$\underline{\omega}(\underline{I}) = \frac{\partial h(\underline{I})}{\partial \underline{I}} . \tag{2}$$

When $\varepsilon = 0$, from Hamilton's equations we obtain that the actions are constant, say $\underline{I} = \underline{I}_0$, and the angles are linear functions of the time with frequency $\underline{\omega}_0 = \underline{\omega}(\underline{I}_0)$, say $\underline{\varphi}(t) = \underline{\omega}_0 t + \underline{\varphi}_0$. For ε sufficiently small and under suitable conditions, KAM theory ensures the persistence of an invariant torus on which the motion is quasi-periodic with frequency $\underline{\omega}_0$. The conditions for the applicability of KAM theory are essentially two:

(i) a strong non-resonance condition on the frequency, typically requiring that the frequency satisfies a Diophantine inequality;

(ii) a non-degeneracy condition on the unperturbed Hamiltonian, for example requiring that the determinant of the Hessian matrix associated to h is not zero (we will see later alternative formulations of the non-degeneracy condition).

Assumption (i) can be formulated by requiring that the vector frequency satisfies the Diophantine inequality

$$|\underline{\omega} \cdot \underline{k}|^{-1} \leq C |\underline{k}|^{\tau} , \qquad \forall \underline{k} \in \mathbb{Z}^n \backslash \{\underline{0}\} \tag{3}$$

for some positive constants C and τ. This condition ensures that the frequency vector satisfies a strong non-resonance condition, since (3) implies that $\underline{\omega}$ is rationally independent and badly approximated by rationals:

$$|\underline{\omega} \cdot \underline{k}| \geq \frac{1}{C |\underline{k}|^{\tau}} > 0 .$$

A possible formulation of assumption (ii) is obtained by requiring that the unperturbed Hamiltonian satisfies Kolmogorov's non-degeneracy condition:

$$\det \left(\frac{\partial^2 h(\underline{I})}{\partial \underline{I}^2} \right) \neq 0 , \qquad \forall \underline{I} \in \mathbb{R}^n . \tag{4}$$

In view of (2), the condition (4) can be reformulated as

$$\det \left(\frac{\partial \underline{\omega}(\underline{I})}{\partial \underline{I}} \right) \neq 0 , \qquad \forall \underline{I} \in \mathbb{R}^n .$$

It is remarkable that KAM theory provides a constructive algorithm to give a lower bound estimate on the parameter ε, such that a quasi-periodic torus with fixed frequency $\underline{\omega}_0$ exists for any value less than a given threshold, which is provided explicitly by the theorem, say $\varepsilon \leq \varepsilon_{\mathrm{KAM}}(\underline{\omega}_0)$. A crucial question is whether the estimates which can be obtained by implementing KAM theory are consistent with the astronomical prediction, in case of applications to concrete models, or rather with the prediction obtained through numerical experiments. Though applications to problems of Celestial Mechanics, carried out in the '60s, were very far from the astronomical values, more recent versions of KAM theory allow one to obtain results which are in full agreement with the physical expectations.

A great improvement came by the so-called *a-posteriori approach,* developed in de la Llave (2001), de la Llave et al. (2005), and later extended to a dissipative case in Calleja et al. (2013). This method consists in reducing the existence of invariant objects to finding solutions of a functional equation, namely the invariance equation; having an approximate solution satisfying some non-degeneracy conditions, one can prove that near the approximately invariant torus, there is a true invariant torus. Then, the method developed in de la Llave (2001), de la Llave et al. (2005) implements the so-called *automatic reducibility,* which means that in the neighborhood of the invariant torus there exists a coordinate change that transforms the linearization of the invariance equation into a constant coefficient equation, which is easier to solve. This approach does not need that the model is expressed in action-angle variables or that the Hamiltonian is nearly-integrable. The proof of KAM theory is rather technical and we refer to the literature for an exhaustive presentation. Here, we just want to mention that the proof developed in de la Llave (2001), de la Llave et al. (2005) provides a very efficient algorithm to construct invariant tori and to give explicit estimates on the parameters. However, to get realistic estimates, one needs to use a computer, which is required, for example, to compute a good approximate solution. When using the computer, to get rigorous bounds, it is necessary to control the rounding-off and propagation errors introduced by the machine; this task can be performed by implementing the so-called *interval arithmetic* technique (see, e.g., (Lanford 1987)). KAM computer-assisted estimates in models of Celestial Mechanics are presented in Sects. 1.2 and 1.3.

It is worth mentioning that an efficient version of KAM theory for some dissipative systems, precisely *conformally symplectic* systems, has been developed in Calleja et al. (2013). Such systems have the property to transform the symplectic form into a multiple of itself; the conformal factor, measuring the rate of expansion or contraction of the symplectic form, will be denoted by λ and it can be proved that it must be constant for any system with dimension greater or equal than two. An example of conformally symplectic system in Celestial Mechanics is given by the spin-orbit problem with tidal torque, in which the dissipation is modeled by a linear function of the velocity (see Sect. 1.3).

In conservative systems, when the number of degrees of freedom, say n, is small, namely less or equal to two, then the existence of invariant tori gives a very important stability property for infinite times. In fact, when $n = 2$, then the phase space is 4D; fixing the energy, we have a 3D space. Hence, the existence of two invariant tori, which are 2D, provides stability in the sense of confinement of the motion within the invariant tori. The same conclusion is valid for a system described by a 1D, non-autonomous Hamiltonian system; in this case, the phase space is 3D and any pair of 2D invariant tori traps the motion in between for infinite times. We will make use of this property to give stability results in a three-body problem (Sect. 1.2) and in the conservative spin-orbit problem (Sect. 1.3). We stress that the stability property is no more valid for systems with larger degrees of freedom. For example, if $n = 3$, then the phase space is 6D; fixing the energy, one obtains a 5D space, which is not separated into invariant regions by the 3D KAM tori.

1.2 The three-body problem

We consider the planar, circular, restricted 3-body problem (hereafter, PCR3BP) formed by three bodies, say P_1, P_2, A, with P_1 and P_2 with masses, respectively, m_1 and m_2 with $m_1 > m_2$, while we assume that the mass of A is so small that it does not influence the motion of the primaries P_1 and P_2; we assume that P_1 and P_2 move on circular orbits around their common barycenter and that all bodies move on the same plane.

This problem is described by a Hamiltonian function with 2 degrees of freedom, say

$$\mathcal{H}(L, G, \ell, g) = -\frac{1}{2L^2} - G + \varepsilon R(L, G, \ell, g) , \tag{5}$$

in action-angle variables (due to Delaunay), where ℓ is the mean-anomaly, $g = g_0 - \psi$ with g_0 the argument of perihelion, ψ the longitude of P_2 (coinciding with time if the common frequency of the primaries is 1 and if $m_1 + m_2 = 1$), while the action variables are (in normalized units) $L = \sqrt{a}$ and $G = L\sqrt{1-e^2}$, with a the semimajor axis, e the eccentricity; ε is the perturbative parameter, which is defined in terms of the masses of the primaries as $\varepsilon = m_2/(m_1 + m_2)$. The function $h(L, G) = -\frac{1}{2L^2} - G$ is the integrable part and it includes the Keplerian interaction with P_1. The function $R = R(L, G, \ell, g)$ is termed the perturbing function and it represents the gravitational interaction with P_2.

Although the Hamiltonian is degenerate, since the determinant of the Hessian matrix of the integrable part h is zero, it satisfies the so-called Arnold's isoenergetic non–degeneracy condition, which is an alternative to Kolmogorov's non-degeneracy condition (4) and it ensures the persistence of invariant tori on a fixed energy surface. Arnold's isoenergetic non–degeneracy condition reads as

$$\begin{aligned} &\det \begin{pmatrix} h''(L,G) & h'(L,G) \\ h'(L,G)^T & 0 \end{pmatrix} \\ &= \det \begin{pmatrix} -\frac{3}{L^4} & 0 & \frac{1}{L^3} \\ 0 & 0 & -1 \\ \frac{1}{L^3} & -1 & 0 \end{pmatrix} \\ &= \frac{3}{L^4} \neq 0 \quad \text{for all } L \neq 0 . \end{aligned} \tag{6}$$

For the PCR3BP, the dimension of the phase space is 4; fixing the energy we obtain a 3D space and, therefore, the perpetual stability of the small body can be obtained by proving the existence of bounding invariant surfaces which confine the motion of the asteroid on a preassigned energy level.

In Celletti and Chierchia (2007) (see also (Celletti and Chierchia 1997, 2006)), the following sample case is considered: Sun (P_1), Jupiter (P_2), and the asteroid 12 Victoria (A) whose orbital elements are

$$a_V \simeq 0.449 , \qquad e_V \simeq 0.220 , \tag{7}$$

while the size of the perturbing parameter measuring the mass of Jupiter w.r.t. the Sun is

$$\varepsilon_J = 0.954 \cdot 10^{-3} \tag{8}$$

(for completeness, the inclination of Jupiter on the ecliptic is 1.305^o and the inclination of Victoria is 8.363^o).

After constructing an approximated model with a finite series expansion of the perturbing function R in (5), we compute the observed values for the asteroid Victoria corresponding to (7) as $L_V = \sqrt{a_V} \simeq 0.670$, $G_V = L_V \sqrt{1-e_V^2} \simeq 0.654$. Then, we define the *osculating energy value* as

$$\overline{E}_V = E_V^{(0)} + \varepsilon_J E_V^{(1)} \simeq -1.769$$

defined in terms of the Keplerian $E_{\mathrm{V}}^{(0)}$ and secular $E_{\mathrm{V}}^{(1)}$ parts:

$$E_{\mathrm{V}}^{(0)} = -\frac{1}{2L_{\mathrm{V}}^2} - G_{\mathrm{V}} \simeq -1.768 \, ,$$

$$E_{\mathrm{V}}^{(1)} = \langle \mathcal{H}_1(L_{\mathrm{V}}, G_{\mathrm{V}}, \cdot, \cdot) \rangle_{\ell,g} \simeq -1.060 \, ,$$

where $\langle \cdot \rangle_{\ell,g}$ denotes the average with respect to ℓ, g. Finally, we consider two invariant tori on the energy level $\overline{E}_{\mathrm{V}}$, which bound from above and below the observed value L_{V}. Precisely, we define

$$\widetilde{L}_{\pm} = L_{\mathrm{V}} \pm 0.001$$

with associated frequencies

$$\underline{\widetilde{\omega}}_{\pm} = \left(\frac{\partial \mathcal{H}_0}{\partial L}, \frac{\partial \mathcal{H}_0}{\partial G}\right) = \left(\frac{1}{\widetilde{L}_{\pm}^3}, -1\right) = (\widetilde{\alpha}_{\pm}, -1) \, . \tag{9}$$

Since we need Diophantine frequencies, we compute the continued fraction representation up to the order 5 of $\widetilde{\alpha}_{\pm}$ in (9) and then we modify these numbers by adding a tail of all one's to obtain new frequencies $\underline{\omega}_{\pm}$, that satisfy the Diophantine condition. The KAM theory applied to the system formed by the Sun, Jupiter, and the asteroid 12 Victoria leads to the following result (see (Celletti and Chierchia 2007) for full details), which is valid for a value slightly bigger than the true value of the Jupiter/Sun mass ratio (namely ε_{J} as in (8)).

Proposition 1 (*Celletti and Chierchia* 2007) *Consider the Hamiltonian* (5) *satisfying the non-degeneracy condition* (6). *Let* $E = \overline{E}_{\mathrm{V}}$. *Then, for* $|\varepsilon| \leq 10^{-3}$ *the unperturbed tori with trapping frequencies* $\underline{\omega}_{\pm}$ *can be analytically continued into KAM tori for the perturbed system on the energy level* $\mathcal{H}^{-1}(\overline{E}_{\mathrm{V}})$ *keeping fixed the ratio of the frequencies.*

This result guarantees that the semimajor axis and the eccentricity of the asteroid Victoria remain forever close to the unperturbed values within an interval of size of order ε; in other words, the actual motion (in the considered mathematical model) is nearly elliptical with osculating orbital values close to the observed ones.

1.3 The conservative and dissipative spin-orbit models

The spin–orbit problem describes the motion of a triaxial satellite, say S, with principal moments of inertia $I_1 < I_2 < I_3$, whose center of mass moves on a Keplerian orbit around a planet P; the satellite rotates around a spin–axis, which is assumed to be perpendicular to the orbit plane and coinciding with the shortest physical axis. In the conservative case the satellite is assumed to be a rigid body, while in the dissipative case it is assumed to be non-rigid and therefore it feels a tidal torque. The geometry of the spin–orbit problem is described in Fig. 2: r and f are the polar coordinates of the center of the ellipsoid along the ellipse, while x is the angle between the longest axis of the satellite and the periapsis line.

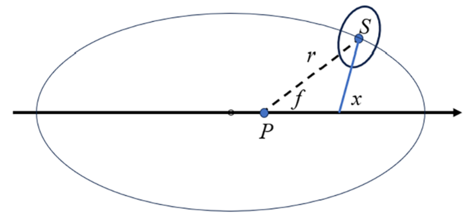

Fig. 2 The geometry of the spin–orbit problem

An important phenomenon associated to the spin–orbit problem is the *spin-orbit resonance,* which occurs whenever there is a commensurability of the period of revolution T_{rev} and the period of rotation T_{rot}, namely there exist non-zero integers p, q, such that $\frac{T_{\mathrm{rev}}}{T_{\mathrm{rot}}} = \frac{p}{q}$. The Moon, as well as many other satellites of the Solar system, moves in a 1:1 spin-orbit resonance, while Mercury is the only object found in a non-synchronous resonance, precisely a 3:2 spin-orbit resonance.

In the conservative case, the equation of motion is given by

$$\ddot{x} + \varepsilon \left(\frac{a}{r(t)}\right)^3 \sin(2x - 2f(t)) = 0 \, , \qquad \varepsilon = \frac{3}{2}\frac{I_2 - I_1}{I_3} \, ,$$

which corresponds to the following 1 degree of freedom, time–dependent Hamiltonian:

$$\mathcal{H}(y, x, t) = \frac{y^2}{2} - \frac{\varepsilon}{2}\left(\frac{a}{r(t)}\right)^3 \cos(2x - 2f(t)) \, , \tag{10}$$

where $r = r(t)$, $f = f(t)$ are known functions of the time, due to the assumption that the orbit is Keplerian.

In the dissipative case, the equation of motion includes at the r.h.s. the tidal torque:

$$\ddot{x} + \varepsilon \left(\frac{a}{r(t)}\right)^3 \sin(2x - 2f(t)) = \mathcal{T}_d(\dot{x}, t) \, , \tag{11}$$

whose expression can be taken as (see, e.g., (Peale 2005))

$$\mathcal{T}_d(\dot{x}, t) = -\eta \left(\frac{a}{r(t)}\right)^6 (\dot{x} - \dot{f}(t)) \, , \tag{12}$$

where $\eta > 0$ is named the *dissipative constant.* One can also consider (12) by averaging over one orbital period:

$$\bar{\mathcal{T}}_d(\dot{x}) = -\eta \, \bar{L}(e) \left(\dot{x} - \frac{\bar{N}(e)}{\bar{L}(e)}\right) ,$$

where

$$\bar{L}(e) \equiv \frac{1}{(1-e^2)^{9/2}}\left(1 + 3e^2 + \frac{3}{8}e^4\right),$$

$$\bar{N}(e) \equiv \frac{1}{(1-e^2)^6}\Big(1+\frac{15}{2}e^2+\frac{45}{8}e^4+\frac{5}{16}e^6\Big)\,.$$

An application of KAM theory to the conservative spin-orbit problem leads to the following result, which is proved in Celletti (1990b), see also (Celletti 1990a), by considering the Hamiltonian (10) with the potential $V(x,t)=-\frac{1}{2}\big(\frac{a}{r(t)}\big)^3\cos(2x-2f(t))$ expanded in powers of the eccentricity and in Fourier series of the angles x, t, then truncated to a finite order, thus obtaining a trigonometric function.

Theorem 2 (*Celletti 1990b*) *Consider the spin–orbit Hamiltonian* (10) *with a trigonometric potential defined in* $Y\times\mathbb{T}^2$ *with* $Y\subset\mathbb{R}$ *open set. Then, for the true eccentricity of the Moon* $e=0.0549$, *there exist invariant tori, bounding the motion of the Moon for any* $\varepsilon\le\varepsilon_{\text{Moon}}=3.45\cdot 10^{-4}$.

This result yields the stability of the Moon for infinite times within the conservative spin-orbit problem and for the astronomical value $\varepsilon_{\text{Moon}}$ of the equatorial oblateness of the Moon. The proof is computer-assisted: interval arithmetic is implemented to get rid of the rounding-off and propagation errors introduced by the computer.

Several results in the dissipative spin-orbit problem are presented in Calleja et al. (2022a,b, 2024). In these works, one first makes a reduction of the dissipative spin-orbit problem to a Poincaré map P_e depending on the drift parameter e (which can be identified with the eccentricity); this map is shown to be conformally symplectic according to the following definition.

Definition 1 Let $\mathcal{M}\subset\mathbb{R}\times\mathbb{T}$ be a symplectic manifold, endowed with a symplectic form Ω. The map $P_e:\mathcal{M}\to\mathcal{M}$ is conformally symplectic, if there exists a function $\lambda:\mathcal{M}\to\mathbb{R}$, such that

$$P_e^*\Omega=\lambda\Omega\,, \tag{13}$$

where P_e^* denotes the pull-back of P_e.

A numerical construction of the invariant tori is given in Calleja et al. (2022a) (see also (Bustamante et al. 2023) in a different setting, the standard map). Then, KAM estimates for the dissipative spin-orbit problem with tidal torque are developed in Calleja et al. (2022b), while a numerical computation of the break-down threshold is presented in Calleja et al. (2024). With reference to (13), the conformal factor of the spin-orbit Poincaré map P_e takes the form

$$\lambda=\exp\bigg(-\eta\,\pi\,\frac{3e^4+24e^2+8}{4\left(1-e^2\right)^{9/2}}\bigg)\,.$$

If $\eta=0$, one recovers the Hamiltonian case; if $\eta>0$ the map is contractive and if $\eta<0$, the map is expansive. We can formally introduce a KAM attractor for the map P_e as follows.

Definition 2 Let $P_e:\mathcal{M}\to\mathcal{M}$ be a family of conformally symplectic maps defined on a symplectic manifold $\mathcal{M}\subset\mathbb{R}\times\mathbb{T}$ and depending on the drift parameter e. A KAM attractor for the spin-orbit problem (11)-(12) with Diophantine frequency ω is an invariant torus described by an embedding $K:\mathbb{T}\to\mathcal{M}$ and a drift parameter e, satisfying

$$P_e\circ K(\theta)-K(\theta+\omega)=0\,. \tag{14}$$

Two values for the frequency ω have been considered in Calleja et al. (2013), the golden mean ω_1 and a frequency close to one, say ω_2:

$$\omega_1=\frac{\sqrt{5}+1}{2}\approx 1.6180339887498948\,,$$

$$\omega_2=1+\frac{1}{2+(\frac{\sqrt{5}-1}{2})}\approx 1.3819660112501051\,.$$

An efficient KAM algorithm is developed in Calleja et al. (2013), where each step needs only shifting functions, multiplying, composing and differentiating functions, solving difference equations with constant coefficients. Starting with an FFT with N Fourier modes, the Newton step needs just $O(N)$ storage and $O(N\log N)$ operations. The application of KAM theory to the dissipative spin-orbit problem is the content of the following result proved in Calleja et al. (2022b).

Theorem 3 (*Calleja et al. 2022b*) *Consider the spin-orbit dissipative Poincaré map* P_e, *let* ω *be Diophantine, and let* $\rho>0$; *let* (K_0,e_0) *be an approximate solution of* (14) *with error term* E_0, *such that* $P_{e_0}\circ K_0(\theta)-K_0(\theta+\omega)=E_0(\theta)$. *Assume smallness conditions on the norm of* E_0 *and assume that a non-degeneracy condition on coordinates and parameters is fulfilled* (*see* (*Calleja et al. 2022b*) *for the rather technical formulation of this condition*).

Then, for given parameter values, i.e. $\varepsilon=0.0116$ *for* ω_1 *and* $\varepsilon=0.0126$ *for* ω_2, *there exists an exact solution* (K_*,e_*) *of the invariance equation* (14):

$$P_{e_*}\circ K_*(\theta)-K_*(\theta+\omega)=0\,.$$

The quantities (K_*,e_*) *are close to* (K_0,e_0), *since for* $0<\delta<\rho/2$ *one has*

$$\|K_*-K_0\|_{\rho-\delta}\le C_1\,C\,\delta^{-\tau}\,\|E_0\|_\rho\,,$$

$$|e_*-e_0|\le C_2\|E_0\|_\rho\,,$$

where C_1 *and* C_2 *are suitable constants.*

The work (Calleja et al. 2022b) gives a computer-assisted validation (namely, high precision computations, obtained using many digits, rather than implementing interval arithmetic) for values of the parameters close to the numerical breakdown values, which are obtained by implementing a continuation method for fixed $\eta = 10^{-3}$, a multi-precision arithmetic with 170 bits (equivalent to ~ 50 digits or more), a Newton's method with tolerance 10^{-45}, a careful estimate of a partial tail of the Fourier expansion and a check of the KAM estimates to prove the convergence of the iteration.

It is remarkable that the results of Theorem 3 are very close to the numerical values of the breakdown threshold found in Calleja et al. (2022b) which, for $\eta = 10^{-3}$, are $\varepsilon = 0.01163296365$ for ω_1 and $\varepsilon = 0.01269763003$ for ω_2.

2 Exponentially long stability time: Nekhoroshev's theorem

Another pillar in the XX century among stability results of dynamical systems is represented by Nekhoroshev's theorem. While KAM theory gives results on the existence of tori with prescribed frequency, Nekhoroshev's theorem investigates the stability of the dynamics for given initial conditions and provides stability times which are exponentially long. More precisely, let us consider a Hamiltonian function of the form (1); under suitable assumptions (see Sect. 2.1), Nekhoroshev's theorem states that if $0 \le \varepsilon < \varepsilon_{\rm Nekh}$ with $\varepsilon_{\rm Nekh}$ given by the theorem, then for some positive constants C_0, C_1, a, b, the initial conditions in the actions can be bounded as

$$|\underline{I}(t) - \underline{I}(0)| \le C_0 \varepsilon^a \ , \qquad |t| \le C_1 \ e^{\frac{1}{\varepsilon^b}} \ ,$$

which expresses the stability of the actions for exponential times. We notice that $\varepsilon_{\rm Nekh}$ and b depend on the number of degrees of freedom, as well as the regularity of h and f. We also notice that improvements on the estimates of $\varepsilon_{\rm Nekh}$ and on the stability time can be obtained by implementing a preliminary normal form, which transforms $\mathcal{H} = h + \varepsilon f$ into a new Hamiltonian $\mathcal{H}'$ with

$$\mathcal{H}'(\underline{I}', \underline{\varphi}') = h'(\underline{\varphi}') + \varepsilon^2 f'(\underline{I}', \underline{\varphi}') \ ,$$

where $(\underline{I}', \underline{\varphi}')$ denote the transformed variables.

2.1 Nekhoroshev's theorem

Nekhoroshev's theorem, expressing that the actions remain confined over an exponentially long time, can be implemented provided that the Hamiltonian satisfies a requirement called *steepness* condition in the original formulation of the theorem (Nekhoroshev 1977). According to a later formulation (see, e.g., (Pöschel 1993)), one can require that the Hamiltonian satisfies the *convex* or *quasi–convex* hypotheses that we introduce as follows.

Assume that the Hamiltonian (1) is defined for $\underline{I} \in Y \subset \mathbb{R}^n$, $\underline{\varphi} \in \mathbb{T}^n$ with $Y \subset \mathbb{R}^n$ open. Define a complex neighborhood of $Y \times \mathbb{T}^n$ with radii r_0 and s_0 as

$$V_{r_0} Y \times W_{s_0} \mathbb{T}^n \ ,$$

where $V_{r_0} Y$ denotes the complex neighborhood of radius r_0 around Y with respect to the Euclidean norm $\|\cdot\|$ and $W_{s_0}\mathbb{T}^n$ is the complex strip of width s_0 around $\mathbb{T}^n$:

$$\begin{aligned} W_{s_0}\mathbb{T}^n \equiv \{ &\underline{\varphi} \in \mathbb{C}^n \ : \ Re(\varphi_j) \in \mathbb{T}, \\ &j = 1, \dots, n, \ \max_{1 \le j \le n} |\mathrm{Im}\, \varphi_j| < s_0 \} \ . \end{aligned}$$

Let $U_{r_0} Y \equiv V_{r_0} Y \cap \mathbb{R}^n$ be the real neighborhood of Y.

Definition 3 For an analytic function $g = g(\underline{I}, \underline{\varphi})$ on $V_{r_0} Y \times W_{s_0}\mathbb{T}^n$ with Fourier expansion

$$g(\underline{I}, \underline{\varphi}) = \sum_{\underline{k} \in \mathbb{Z}^n} \hat{g}_{\underline{k}}(\underline{I}) e^{i \underline{k} \cdot \underline{\varphi}} \ ,$$

we define the norm $\|g\|_{Y, r_0, s_0}$ as

$$\|g\|_{Y, r_0, s_0} \equiv \sup_{\underline{I} \in V_{r_0} Y} \sum_{\underline{k} \in \mathbb{Z}^n} |\hat{g}_{\underline{k}}(\underline{I})| \ e^{(|k_1| + \dots + |k_n|) s_0} \ .$$

Let $\underline{\omega}(\underline{I}) \equiv \frac{\partial h(\underline{I})}{\partial \underline{I}}$ be the frequency vector and let $\mathcal{Q}$ be the Hessian matrix of the unperturbed Hamiltonian h.

Definition 4 Given $m > 0$, then $h(\underline{y})$ is said m–convex if

$$\begin{aligned} &(\mathcal{Q}(\underline{I})\underline{v}, \underline{v}) \ge m \|\underline{v}\|^2 \\ &\qquad for\ all\ \underline{v} \in \mathbb{R}^n \ , \quad for\ all\ \underline{I} \in U_{r_0} Y \ . \end{aligned}$$

Given $m, \ell > 0$, then $h(\underline{I})$ is said m, ℓ–quasi–convex if, for any $\underline{I} \in U_{r_0} Y$, one of the following inequalities hold for any $\underline{v} \in \mathbb{R}^n$:

$$|(\underline{\omega}(\underline{I}), \underline{v})| > \ell \|\underline{v}\| \ , \qquad (\mathcal{Q}(\underline{I})\underline{v}, \underline{v}) \ge m \|\underline{v}\|^2 \ . \tag{15}$$

We remark that quasi–convex Hamiltonians satisfy the isoenergetic non–degeneracy condition (see (6)). Next, we state the theorem in the formulation given by J. Pöschel in Pöschel (1993).

Theorem 4 (*Pöschel* 1993) *Let us consider the Hamiltonian function* (1) *and assume that the unperturbed Hamiltonian satisfies the* m, ℓ*–quasi–convexity hypothesis* (15). *Let* M *be an upper bound on* $\mathcal{Q}$:

$$\sup_{\underline{I} \in V_{r_0} Y} \|\mathcal{Q}(\underline{I})\| \le M \ ;$$

let $r_0 \le \frac{4\ell}{m}$, $A \equiv \frac{11M}{m}$, $\varepsilon_0 \equiv \frac{m r_0^2}{2^{10} A^{2n}}$. *Assume that for* $s_0 > 0$, *one has*

$$\|f\|_{Y,r_0,s_0}\varepsilon \le \varepsilon_0\ .$$

Then, for any initial condition $(\underline{I}_0, \underline{\varphi}_0) \in Y \times \mathbb{T}^n$ *the following estimates hold*:

$$\|\underline{I}(t) - \underline{I}_0\| \le \frac{r_0}{A}\left(\frac{\varepsilon}{\varepsilon_0}\right)^{\frac{1}{2n}} \qquad \text{for } |t| \le \frac{A^2 s_0}{\Omega_0}\, e^{\frac{s_0}{6}\left(\frac{\varepsilon_0}{\varepsilon}\right)^{\frac{1}{2n}}}\ ,$$

where $\Omega_0 \equiv \sup_{\|\underline{I}-\underline{I}_0\|\le \frac{r_0}{A}} \|\underline{\omega}(\underline{I})\|$.

The proof of Theorem 4 is based on three main ingredients: the construction of a suitable normal form, the use of the convexity and quasi–convexity assumptions, a careful analysis of the geography of the resonances.

In the non-resonant case, introduced according to the definition below, the proof and the statement of the theorem are consistently simpler.

Definition 5 Let Λ be a sublattice of $\mathbb{Z}^n$. The set $D \subseteq Y$ is said α, K-non resonant modulo Λ, if for any $\underline{y} \in D$:

$$|\underline{\omega}(\underline{I}) \cdot \underline{k}| \ge \alpha\ , \qquad for\ all\ \underline{k} \in \mathbb{Z}^n_K \backslash \Lambda\ ,$$

where

$$\mathbb{Z}^n_K \equiv \{\underline{k} :\ |\underline{k}| \le K\}\ .$$

If Λ is the trivial sublattice containing only 0, then D is said completely α, K-non resonant.

In the general case, the Hamiltonian $h + \varepsilon f$ is put in Λ-resonant normal form, say $h + \varepsilon g + R$ with $g = \sum_{\underline{k}\in\Lambda} g_{\underline{k}}(\underline{I}) e^{i\underline{k}\cdot\underline{\varphi}}$ containing only terms with indexes in the set Λ and the remainder function R is made exponentially small in K. In the case of a completely α, K-non resonant Hamiltonian, then g does not depend on the angles and one can state the following result, where all the constants can be computed explicitely as in Theorem 4 (compare with (Pöschel 1993)).

Proposition 5 (*Pöschel* 1993) *Let* $D \subseteq Y$ *be completely* α, K*-non resonant. Then, there exists* $\varepsilon_{\mathrm{Nekh}}$, $\rho > 0$ (*depending on* α) *and there exists constants* C_0, C_1, $\tau > 0$, *such that if* $\varepsilon \le \varepsilon_{\mathrm{Nekh}}$, *one has*

$$\|\underline{I}(t) - \underline{I}(0)\| \le C_0\rho \quad \text{for } |t| \le C_1\, e^{K\tau}\ .$$

The stability estimates provided by Nekhoroshev's theorem are particularly relevant in Celestial Mechanics. In fact, they can be used to provide bounds on the orbital elements for an exponentially long time, possibly comparable with the age of the Solar system, namely 5 billion years.

Effective estimates have been developed for different models, among which the three–body problem and the Earth satellite problem, that we briefly summarize in Sects. 2.2 and 2.3, respectively, and the triangular Lagrangian points that we detail in Sect. 2.4.

2.2 The three-body problem

The exponential time stability of the PCR3BP is considered in Celletti and Ferrara (1996), where a preliminary normal form reduction is performed to reduce the perturbation to higher orders in the perturbing parameter. The PCR3BP is modeled by a Hamiltonian function containing 17 Fourier coefficients. A non-resonant normal form is implemented to remove the perturbation to the order 5. Then, Proposition 5 in the non-resonant case is implemented to get the confinement for an exponentially long time in phase space of the action variables, which are related to the orbital elements of the small body, namely the semimajor axis and the eccentricity. A sample case is considered, namely the stability of the dwarf planet Ceres under the attraction of Jupiter and the Sun, obtaining its stability for $4 \cdot 10^9$ years, namely a time comparable with the age of the Solar system and for a mass-ratio of the primaries equal to 10^{-6}; though not being consistent with the Jupiter-Sun mass ratio, it is a physically meaningful result, since it is consistent with a model obtained replacing Jupiter with an Earth-size planet. More precisely, taking some initial conditions close to Ceres for the Delaunay variables, the following result (which is obtained without implementing interval arithmetic) is proved in Celletti and Ferrara (1996).

Proposition 6 (*Celletti and Ferrara* 1996) *Within the PCR3BP, consider the initial conditions*

$$(L_0, G_0, \ell_0, g_0) = (0.72925656, 0.72710370, 0, 0)\ ;$$

for a mass-ratio $\varepsilon \le 10^{-6}$, *the action variables satisfy the inequalities*

$$|L(t) - L_0| \le 1.6051 \cdot 10^{-7}\ ,$$

$$|G(t) - G_0| \le 1.5492 \cdot 10^{-7}$$

for all times $|t| \le 4.9277 \cdot 10^9$ *years.*

2.3 Satellite stability

Exponentially long time estimates are developed in Celletti et al. (2023a) (see also (De Blasi 2023)) for a model describing the dynamics of a satellite around the Earth at altitudes above the atmosphere and up to the geosynchronous orbit. We consider a model composed by the geopotential truncated to J_2 in spherical harmonics to which we add the attractions of Sun and Moon, which are assumed to move on the ecliptic (this is a simplifying hypothesis that makes the model simpler). We refer to this model as the geolunisolar model, which satisfies the quasi-convexity condition (remarkably, the addition of Sun and Moon make the problem quasi-convex, otherwise it would have failed to satisfy such condition). The geolunisolar model is averaged over the fast angles (mean anomaly of the satellite and the fast angles of the Moon and Sun), expanded to order 12 around reference values a_*, e_*, i_*, and normalized to order 6. In this way, one obtains a Hamiltonian with 2 degrees of freedom of the form

$$\mathcal{H}(I_1, I_2, \varphi_1, \varphi_2) = h(I_1, I_2) + \varepsilon f(I_1, I_2, \varphi_1, \varphi_2) .$$

Implementing Pöschel version of Nekhoroshev's theorem (Pöschel 1993) for quasi-convex Hamiltonians in non-resonant domains (see Proposition 5), one obtains stability estimates for different values of a_*, e_*, i_*. At the altitude of 11,000 km, for values of the inclination between 0 and 90^o and of the eccentricity between 0 and 0.5, one obtains long stability times up to 32,000 years; the stability time decreases to 8600 years at 14,000 km, down to 250 years at 18,000 km and decreasing to 100 years at 19,000 km. This means that by increasing the altitude, the domain of applicability of the theorem shrinks, due to the closer proximity to Moon and Sun, and also due to the effect of nearby resonances.

2.4 The Lagrangian points

Within the PCR3BP, we consider a small particle S moving in the gravitational field of two primaries with masses, respectively, μ and $1-\mu$, located on the negative and positive horizontal axis. In a synodic reference frame, let (x, y, z) be the coordinates of S and (p_x, p_y, p_z) the associated momenta; then, the Hamiltonian can be written as

$$\begin{aligned}\mathcal{H}(p_x, p_y, p_z, x, y, z) &= \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + y p_x - x p_y \\ &\quad - \frac{1-\mu}{\sqrt{(x-\mu)^2 + y^2 + z^2}} \\ &\quad - \frac{\mu}{\sqrt{(x+1-\mu)^2 + y^2 + z^2}}. \end{aligned} \tag{16}$$

In a reference frame centered at the Lagrangian point L_4 with coordinates $(\mu - \frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$, the Hamiltonian (keeping the same notation as in (16)) reads as

$$\begin{aligned}\mathcal{H}(p_x, p_y, p_z, x, y, z) &= \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + y p_x - x p_y - x(\mu - \frac{1}{2}) \\ &\quad - y\frac{\sqrt{3}}{2} - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} , \end{aligned} \tag{17}$$

where $r_1 = (1 - x + \sqrt{3}y + x^2 + y^2 + z^2)^{\frac{1}{2}}$, $r_2 = (1 + x + \sqrt{3}y + x^2 + y^2 + z^2)^{\frac{1}{2}}$.

Exponential estimates for a particle located at L_4 can be obtained as follows. Expanding the Hamiltonian around the equilibrium point, one can write the Hamiltonian (17) as

$$\mathcal{H}(p_x, p_y, p_z, x, y, z) = \sum_{k \geq 2} \mathcal{H}_k(p_x, p_y, p_z, x, y, z) ,$$

where $\mathcal{H}_k$ is a polynomial of order k. Setting $a \equiv -\frac{3\sqrt{3}}{4}(1-2\mu)$, the quadratic part is given by

$$\begin{aligned}\mathcal{H}_2(p_x, p_y, p_z, x, y, z) &= \frac{1}{2}(p_x^2 + p_y^2) + y p_x - x p_y + \frac{x^2}{8} \\ &\quad - \frac{5}{8}y^2 - axy + \frac{1}{2}(p_z^2 + z^2) , \end{aligned}$$

while the higher order terms are given by

$$\begin{aligned}\mathcal{H}_k(p_x, p_y, p_z, x, y, z) &= (1-\mu) r^k P_k(\frac{x - \sqrt{3}y}{2r}) + \mu r^k P_k(\frac{-x - \sqrt{3}y}{2r}) , \end{aligned}$$

being $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, with P_k denoting the Legendre polynomial of order k.

Then, one proceeds to diagonalize the quadratic part through a suitable change of variables from (x, y, z, p_x, p_y, p_z) to $(x_1, x_2, x_3, y_1, y_2, y_3)$, that leads to the transformed Hamiltonian

$$\begin{aligned}\mathcal{H}_{\text{new}}(x_1, x_2, x_3, y_1, y_2, y_3) &= \frac{1}{2}\sum_{k=1}^{3} \omega_k (y_k^2 + x_k^2) \\ &\quad + \sum_{k \geq 3} \mathcal{H}_{k,\text{new}}(x_1, x_2, x_3, y_1, y_2, y_3) , \end{aligned}$$

with $\mathcal{H}_{k,\text{new}}$ denoting higher order terms.

We notice that the quadratic part admits the first integrals

$$J_k = \frac{1}{2}(y_k^2 + x_k^2) , \qquad k = 1, 2, 3 ;$$

therefore, we look for first integrals of the perturbed system in the form

$$\Phi^{(k)} \equiv J_k + \Phi_3^{(k)} + \Phi_4^{(k)} + \cdots ,$$

where $\Phi_j^{(k)}$ is a homogeneous polynomial of degree j in x_k, y_k, that can be explicitly constructed or recursively estimated.

Let $\Phi^{(k,r)}$, $r \geq 3$, be the truncation of the integral, defined as

$$\Phi^{(k,r)} = J_k + \Phi_3^{(k)} + \cdots + \Phi_r^{(k)} .$$

For $\rho > 0$ and $\underline{R} = (R_1, R_2, R_3) \in \mathbb{R}_+^3$, to make quantitative estimates of the escape time, we define the domain

$$\Delta_{\rho \underline{R}} = \{(\underline{x}, \underline{y}) \in \mathbb{R}^6 \ : \ y_k^2 + x_k^2 \leq \rho^2 R_k^2 \ , \ 1 \leq k \leq 3\} .$$

For some $\rho_0 > 0$, assume that the initial condition at $t = 0$ lies in $\Delta_{\rho_0 \underline{R}}$. Then, look for the time $t_{\max}$ such that the solution is confined within the domain $\Delta_{\rho \underline{R}}$ with $\rho > \rho_0$ up to a maximal time $t \leq t_{\max}$. To this end, by the definition of the domains, it is enough to require that for $t \leq t_{\max}$:

$$|J_k(t) - J_k(0)| \leq \frac{1}{2} R_k^2 (\rho^2 - \rho_0^2).$$

We start by noticing that, adding and subtracting the same terms, one can replace the above inequality by

$$\begin{aligned} |J_k(t) - J_k(0)| \leq & \, |J_k(t) - \Phi^{(k,r)}(t)| \\ & + |\Phi^{(k,r)}(t) - \Phi^{(k,r)}(0)| \\ & + |\Phi^{(k,r)}(0) - J_k(0)| \\ \leq & \, \frac{1}{2} R_k^2 (\rho^2 - \rho_0^2) . \end{aligned} \tag{18}$$

Then, we fix the norm as follows: given a complex polynomial, say

$$f(\underline{x}, \underline{y}) = \sum_{\underline{j}, \underline{k}} f_{\underline{j}\underline{k}} x_1^{j_1} y_1^{k_1} x_2^{j_2} y_2^{k_2} x_3^{j_3} y_3^{k_3}$$

with $f_{\underline{j}\underline{k}} \in \mathbb{C}$, setting $\underline{R} = (R_1, R_2, R_3) \in \mathbb{R}_+^3$, let

$$\|f\|_{\underline{R}} \equiv \sum_{\underline{j}, \underline{k} \in \mathbb{Z}^3} |f_{\underline{j}\underline{k}}| R_1^{j_1+k_1} R_2^{j_2+k_2} R_3^{j_3+k_3} ;$$

for any $(\underline{x}, \underline{y}) \in \Delta_{\rho \underline{R}}$ and for any homogeneous function $f = f(\underline{x}, \underline{y})$ of order s, we have

$$|f(\underline{x}, \underline{y})| \leq \|f\|_{\underline{R}} \, \rho^s .$$

With these settings, one has the following estimate, where we introduce the definition of the function $\eta_r^{(k)}(\rho_0)$ as given at the right hand side:

$$\begin{aligned} & |\Phi^{(k,r)}(0) - J_k(0)| \\ & \quad = |\Phi_3^{(k)}(0) + \cdots + \Phi_r^{(k)}(0)| \\ & \quad \leq \sum_{j=3}^{r} \|\Phi_j^{(k)}\|_{\underline{R}} \, \rho_0^j \equiv \eta_r^{(k)}(\rho_0) . \end{aligned} \tag{19}$$

Similarly one gets

$$|\Phi^{(k,r)}(t) - J_k(t)| \leq \eta_r^{(k)}(\rho) . \tag{20}$$

Combining (18), (19), (20), the maximum variation of $\Phi^{(k,r)}$ during the evolution is given by

$$\begin{aligned} & |\Phi^{(k,r)}(t) - \Phi^{(k,r)}(0)| \\ & \quad \leq R^{(k,r)}(\rho_0, \rho) \\ & \quad \equiv \frac{1}{2} R_k^2 (\rho^2 - \rho_0^2) - \eta_r^{(k)}(\rho) - \eta_r^{(k)}(\rho_0) . \end{aligned}$$

Using the estimate

$$|\Phi^{(k,r)}(t) - \Phi^{(k,r)}(0)| \leq |\dot{\Phi}^{(k,r)}| \, |t| \leq F^{(k,r)}(\rho) \, |t| ,$$

where $F^{(k,r)}$ is an upper bound on $|\dot{\Phi}^{(k,r)}|$, then the stability time can be computed as

$$|t| \leq \min_r \frac{R^{(k,r)}(\rho_0, \rho)}{F^{(k,r)}(\rho)} .$$

Using this procedure, one can get estimates of the stability times for objects around L_4, as it is done in Celletti and Giorgilli (1990) with particular reference to an asteroid under the gravitational field of Jupiter and the Sun. Although not including real asteroids, stability regions have been found in Celletti and Giorgilli (1990) for a time interval of the order of the age of the universe; refined estimates including a few asteroids are developed later in Giorgilli and Skokos (1997) (we refer the readers also to the remarkable work on the same subject developed in Giorgilli et al. (1989)).

3 Finite time stability: perturbation theory

Perturbation theory can be used to provide stability results over finite times; among the many examples existing in the literature, we mention again that of Earth's satellites at altitudes above 2000 km, so that the effect of the atmosphere is negligible, and up to the geosynchronous orbit. Implementing normal forms (see Sect. 3.1), one gets that the Delaunay actions (related to the orbital elements) are quasi-integrals of motion and the remainder can be used to give bounds on the stability of the orbital elements as a function of the distance from the Earth (see Sect. 3.2). The quasi-integrals obtained implementing the normal forms provide the so-called *proper elements* (see Sect. 3.3), that have been used to characterize families of asteroids (since the work (Hirayama 1918)) and, more recently, families of space debris (see, e.g., (Celletti et al. 2021, 2022; Celletti and Vartolomei 2023; Celletti et al. 2023b; Wu and Rosengren 2023)).

3.1 Perturbation theory

We briefly recall the main result of perturbation theory, that allows one to find an approximate solution by pushing the perturbation to higher orders in ε (see, e.g., (Celletti 2010)).

Theorem 7 *Let $\mathcal{H}(\underline{I},\underline{\varphi}) = h(\underline{I}) + \varepsilon f(\underline{I},\underline{\varphi})$ with $(\underline{I},\underline{\varphi}) \in Y \times \mathbb{T}^n$ for $Y \subset \mathbb{R}^n$ open, where h is the integrable part, f is the remainder function and ε is the perturbing parameter; assume that f is analytic and trigonometric with N Fourier coefficients on $Y \times \mathbb{T}^n$. Assume that the frequency $\underline{\omega}(\underline{I}) = \frac{\partial h(\underline{I})}{\partial \underline{I}}$ satisfies the following non-resonance condition for any $\underline{I}_0 \in Y$:*

$$|\underline{\omega}(\underline{I}_0)\cdot \underline{k}| > 0 \quad \textit{for all } \ 0 < |\underline{k}| \le N\ .$$

Let $S_\rho(\underline{I}_0) = \{\underline{I} \in \mathbb{R}^n :\ |\underline{I} - \underline{I}_0| \le \rho\}$. Then, there exists $\rho_0 > 0$, $\varepsilon_0 > 0$ and for $|\varepsilon| < \varepsilon_0$ there exists a canonical transformation $(\underline{I},\underline{\varphi}) \to (\underline{I}',\underline{\varphi}')$ defined in $S_{\frac{\rho_0}{2}}(\underline{I}_0) \times \mathbb{T}^n \subset Y \times \mathbb{T}^n$ and with values in $S_{\rho_0}(\underline{I}_0) \times \mathbb{T}^n$, which transforms $\mathcal{H}$ into

$$\mathcal{H}'(\underline{I}',\underline{\varphi}') \ = \ h'(\underline{I}') + \varepsilon^2 f'(\underline{I}',\underline{\varphi}')\ . \tag{21}$$

The proof is constructive and allows one to obtain the new unperturbed Hamiltonian as well as the new remainder function.

3.2 The satellite orbital lifetime

Theorem 7 can be used to obtain stability results for Earth's satellites as in De Blasi et al. (2021) (see also (De Blasi 2023)), where the geolunisolar model is used; precisely, the geopotential truncated to the spherical harmonic coefficient J_2 is considered, then expanded to order 15 in Taylor series of the actions around some reference values, and normalized to order 12 to eliminate the mean anomaly. Using Delaunay variables with L, G, ℓ, g_0 defined in Sect. 1.2 and adding, in the spatial case, the conjugated variables (H, h_0) with $H = G\cos i$, i being the orbital inclination, and h_0 the longitude of the ascending node, this procedure leads to a Hamiltonian of the form

$$\mathcal{H}^{(12)}(\delta L, G, H, \ell, g_0, h_0) = \mathcal{H}^{(12)}_{sec}(\delta L, G, H, g_0, h_0) + R^{(12)}_{J_2}(\delta L, G, H, \ell, g_0, h_0)$$

with $\mathcal{H}^{(12)}_{sec}$ the normal form at order 12, $\delta L = L - L_*$ with L_* the reference value of the Taylor expansion, and $R^{(12)}_{J_2}$ denoting the remainder function at order 12. For the normalized Hamiltonian, one obtains

$$\frac{dL}{dt} = \frac{d(\delta L + L_*)}{dt} = \frac{d(\delta L)}{dt} = -\frac{\partial \mathcal{H}^{(12)}}{\partial \ell} = -\frac{\partial R^{(12)}_{J_2}}{\partial \ell}\ .$$

Let $Y \subset [0, 1) \times [0, \frac{\pi}{2}]$ be a bounded domain in e, i and let us define the norm

$$\|f\|_{\infty,Y} \equiv \sup_{(e,i)\in Y,\ (\ell,g_0,h_0)\in\mathbb{T}^3} |f(e,i,\ell,g_0,h_0)|\ .$$

Taking $L(0) = L_*$ and denoting by $L(t)$ the evolution at time t, one finds the estimate

$$|L(t) - L_*| \le \|\frac{dL}{dt}\|_{\infty,Y}\, t = \|\frac{dR^{(12)}_{J_2}}{d\ell}\|_{\infty,Y}\, t \equiv \Delta L\ ,$$

which gives the stability time

$$t = \frac{\Delta L}{\|\frac{dR^{(12)}_{J_2}}{d\ell}\|_{\infty,Y}}\ .$$

As shown in De Blasi et al. (2021), perturbative methods provide also stability times for the eccentricity and the inclination, whose secular timescale is much longer than the timescale associated to the fast angle. However, one has to keep in mind that, within this model, eccentricity and inclination are not preserved independently, but they are linked by the Lidov-Kozai integral (Lidov 1962), say $\mathcal{I} \simeq e^2 + i^2$ (this is a major difference with what obtained in Sect. 2.3). Stability times for $\mathcal{I}$ at 3000 km of altitude reach $4.61 \cdot 10^{13}$ years, while at 100,000 km of altitude reach $3.37 \cdot 10^4$ years.

3.3 Proper elements for Earth space debris

As a byproduct of perturbation theory, one can construct the so-called *proper elements,* which are quasi-integrals of motion, obtained as follows. Implementing perturbation theory given as in Theorem 7, one obtains a Hamiltonian of the form (21). If we consider only the normal form h' and neglect $O(\varepsilon^2)$, we obtain the Hamiltonian

$$\mathcal{H}'(\underline{I}',\underline{\varphi}') \ = \ h'(\underline{I}')\ ,$$

so that we obtain

$$\dot{\underline{I}}' = \frac{\partial h'(\underline{I}')}{\partial \underline{\varphi}'} = \underline{0}\ .$$

This implies that the actions are quasi-integrals of motion for the full Hamiltonian, namely

$$\dot{\underline{I}}' = \underline{0} + O(\varepsilon^2)\ ;$$

then, we can back-transform to the original set of variables through the generating function introduced in Theorem 7 to determine the *proper elements.* More specifically, we provide the following definitions of osculating, mean and proper elements.

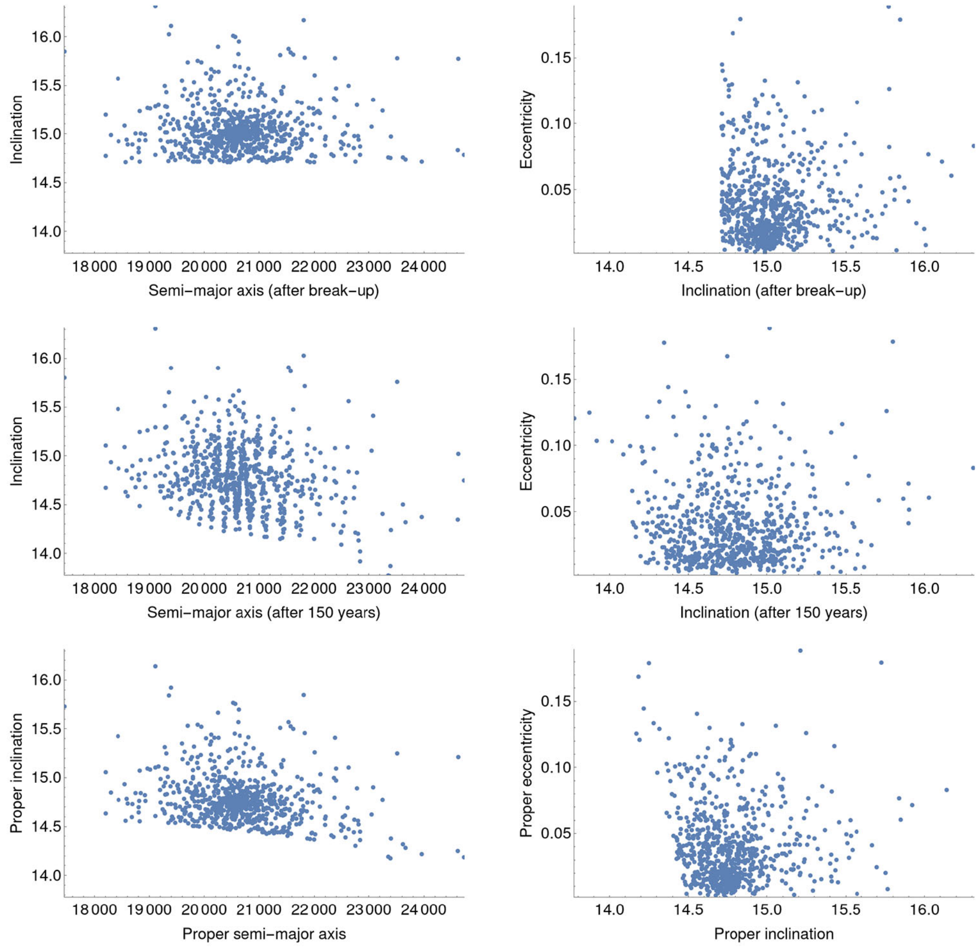

Fig. 3 Distribution of a-i (left), i-e (right) with the parent body at the initial position $a = 20{,}600$ km, $e = 0.01$, $i = 15^o$, $\omega = 10^o$, $\Omega = 20^o$ for the osculating elements after break-up (first row), the mean elements after 150 years (second row), and the proper elements computed after 150 years (third row) (reproduced with permission from Celletti et al. (2022), copyright by the authors)

Definition 6 Osculating orbital elements are obtained integrating the full Hamiltonian function (equivalently, the Cartesian equations of motion).

Definition 7 Mean orbital elements are obtained integrating the Hamiltonian averaged with respect to short-period variables, namely the mean anomaly and the sidereal time.

Definition 8 Proper elements are obtained averaging the Hamiltonian function with respect to all (fast, semi-fast and long-period) variables.

In the case of the Earth's satellite problem, proper elements are computed implementing a hierarchical normal form procedure, which is shortly described as follows (see (Celletti et al. 2022)):

1. consider the Hamiltonian with Earth, Moon and Sun, average with respect to fast and semi-fast angles (namely, the mean anomalies of the debris, Sun and Moon, and the sidereal time), so that the semimajor axis is constant and becomes the first proper element;
2. fix reference values e_0 and i_0, and expand the averaged Hamiltonian in power series around e_0, i_0 up to the order 3;
3. split the resulting Hamiltonian into a linear part and a remainder; implement the normal form to remove the remainder to higher orders;
4. disregarding the remainder, the two actions corresponding to e and i become constants of motion;
5. back-transform the canonical transformation to get the original variables.

Several applications have been considered in Celletti et al. (2021, 2022), Celletti and Vartolomei (2023), Celletti et al. (2023b). Using the simulator and propagator of space debris break-up events SIMPRO developed in (Apetrii et al. 2023), one can compute the orbital elements for each fragment, then propagate each fragment for a given period of time to compute the osculating elements or the mean elements, and finally implement the normal form to compute the proper elements of each fragment at the initial time and after the propagation time. A comparison among the proper and mean (or osculating) elements at different times allows one to cluster the fragments at any epoch (also using machine learning methods), and even to reconnect the fragments to their parent body.

Figure 3 provides an example of a break-up event generated by a parent body with $a = 20{,}600$ km, $e = 0.01$, $i = 15^o$, $\omega = 10^o$, $\Omega = 20^o$. The osculating elements at the break-up are shown in the first row, while the second row gives the mean elements after 150 ys and the third row refers to the proper elements after 150 ys. The similarity between the proper elements at the break-up and after 150 ys is evident, even just by graphical inspection; as well, it is clear that the osculating and mean elements lose the original configuration and spread. This is a crucial observation, which has been substantiated through statistical methods and clusterization tools in (Celletti et al. 2021, 2022; Celletti and Vartolomei 2023; Celletti et al. 2023b).

4 Conclusions and perspectives

Perturbation theory was developed about three centuries ago and the number of applications in Celestial Mechanics and Astrodynamics has been really impressing. Besides, it gave rise to new mathematical theories among which KAM theory on the persistence of invariant tori and Nekhoroshev's theorem on the long-time stability of dynamical systems. The constructive character of perturbation theory and its developments turns out to be extremely useful in practical problems of astronomical interest. The implementation of machine learning methods seems also very promising and deserves further study in the future. However, this is not the only direction of future research, because there are several open questions which need to be investigated.

Concerning KAM theory, we mention the extension to more general dissipative systems (beside the conformally symplectic systems), the implementation of the theory in higher dimensional models (for example, the spatial case or the N-body problem with 4 or more objects), the construction of lower dimensional and whiskered tori, the implementation of converse KAM theory in problems of Celestial Mechanics (see, e.g., (Celletti and Mackay 2007; Kallinikos et al. 2022)) to get an upper bound estimate on the non-existence of invariant tori.

Within long-term stability estimates, we mention the application of Nekhoroshev's theorem to the resonant case which could find applications in many models of Celestial Mechanics, the link with Arnold diffusion (Arnold 1964), the determination of graveyard orbits using the information on stable and unstable orbits over long times.

Perturbative methods can be exploited in more realistic situations that include uncertainty and noise in the parameters and/or in the initial conditions. This is a crucial point, if one aims to get results that can have realistic applications in astronomical problems.

Obviously, the above list is only partial and many other directions of research can be envisaged for the future.

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Author contributions I am the only author.

Declarations

Competing interests The authors declare no competing interests.

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