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# Stability theory and Lyapunov regularity ☆

Luis Barreira *, Claudia Valls

*Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal*

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**Abstract**

We establish the stability under perturbations of the dynamics defined by a sequence of linear maps that may exhibit both *nonuniform* exponential contraction and expansion. This means that the constants determining the exponential behavior may increase exponentially as time approaches infinity. In particular, we establish the stability under perturbations of a nonuniform exponential contraction under appropriate conditions that are much more general than uniform asymptotic stability. The conditions are expressed in terms of the so-called regularity coefficient, which is an essential element of the theory of Lyapunov regularity developed by Lyapunov himself. We also obtain sharp lower and upper bounds for the regularity coefficient, thus allowing the application of our results to many concrete dynamics. It turns out that, using the theory of Lyapunov regularity, we can show that the *nonuniform* exponential behavior is ubiquitous, contrarily to what happens with the uniform exponential behavior that although robust is much less common. We also consider the case of infinite-dimensional systems.
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* Corresponding author.
*E-mail addresses:* barreira@math.ist.utl.pt (L. Barreira), cvalls@math.ist.utl.pt (C. Valls).
*URL:* http://www.math.ist.utl.pt/~barreira/ (L. Barreira).

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## 1. Introduction

The main theme of our paper is the relation between the stability theory of dynamical systems and the so-called Lyapunov regularity theory. In particular, we are interested in establishing the stability under perturbations of *nonuniform* exponential contractions and *nonuniform* exponential dichotomies. We are mainly interested in the case of perturbations of a nonautonomous dynamics defined by a sequence of linear maps. We consider both finite-dimensional and infinite-dimensional systems.

In particular, a nonuniform exponential contraction allows a "spoiling," possibly exponential, of the uniform contraction along the solution as the initial time increases. Thus, even though we can still establish the exponential stability of the solutions, in the nonuniform case the size of the neighborhood in which we must choose the initial condition so that the corresponding solution satisfies a prescribed bound may depend on the initial time. We recall that in the uniform case this neighborhood can be chosen independently of the initial time. In a similar manner, the notion of nonuniform exponential dichotomy imitates the classical notion of (uniform) exponential dichotomy, although in the nonuniform case we allow a "spoiling," again possibly exponential, of the uniform contraction and uniform expansion along the solution as the initial time approaches infinity. The notions are recalled in the main text. We refer to the book [1] for an introduction to the theory of nonuniformly hyperbolic dynamics.

It should be emphasized that the notions of uniform exponential contraction and uniform exponential dichotomy demand considerably from the dynamics. Of course that there exist large classes of dynamical systems possessing this exponential behavior, and even more this class is robust under sufficiently small perturbations. For a detailed discussion, references, and historical comments, we strongly recommend the book [4]. See also [5–9]. On the other hand, we can show that, using the so-called Lyapunov regularity theory, the notions of nonuniform exponential contraction and nonuniform exponential dichotomy are very common. Indeed, *any* linear nonautonomous dynamics possessing only negative Lyapunov exponents admits a nonuniform exponential contraction, and essentially *any* linear nonautonomous dynamics possessing both negative and nonnegative Lyapunov exponents admits a nonuniform exponential dichotomy. This indicates that the nonuniform exponential behavior is very common, and in fact much more common than the uniform exponential behavior. Thus, it is quite reasonable to study the stability under perturbations in the nonuniform setting.

In particular, we establish the stability under perturbations of the nonautonomous dynamics defined by a sequence of linear maps, under appropriate conditions that are much more general than uniform asymptotic stability. Roughly speaking, these conditions ensure that the "nonuniformity" of the exponential behavior is sufficiently small when compared to the nonlinear perturbation. Fortunately, the theory of Lyapunov regularity already possesses an invariant that can be used to express the above conditions on the smallness of the "nonuniformity" of the exponential behavior. Namely, this is the so-called regularity coefficient. Thus, having in mind the application of the stability results to concrete systems, it is crucial to obtain sharp estimates for the regularity coefficient, that hopefully can be given somewhat explicitly in terms of the linear unperturbed dynamics. In particular, we give sharp lower and upper bounds for the regularity coefficient. In addition, we give several alternative characterizations of the situation when the dynamics is Lyapunov regular, i.e., when the regularity coefficient is zero. In this latter case, we essentially can consider any perturbation in our stability results.

## 2. Lyapunov exponents and Lyapunov regularity

Consider a sequence of invertible $k \times k$ matrices $(A_n)_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to+\infty} \frac{1}{n} \log^+ \|A_n\| = 0, \tag{1}$$

where $\log^+ x = \max\{0, \log x\}$. Set

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n, & m > n, \\ \mathrm{Id}, & m = n. \end{cases} \tag{2}$$

The Lyapunov exponent $\lambda : \mathbb{R}^k \to \mathbb{R} \cup \{-\infty\}$ associated with the sequence $(A_n)_{n\in\mathbb{N}}$ is defined by

$$\lambda(v) = \limsup_{n\to+\infty} \frac{1}{n} \log\bigl\|\mathcal{A}(n,1)v\bigr\|. \tag{3}$$

By the abstract theory of Lyapunov exponents (see [1] for details), the function $\lambda$ takes at most a number $r \leqslant k$ of distinct values on $\mathbb{R}^k \setminus \{0\}$, say $-\infty \leqslant \lambda_1 < \cdots < \lambda_r$. Moreover, for each $i = 1, \ldots, r$ the set

$$E_i = \bigl\{v \in \mathbb{R}^k \colon\ \lambda(v) \leqslant \lambda_i\bigr\} \tag{4}$$

is a linear subspace of $\mathbb{R}^k$.

In order to introduce the concept of Lyapunov regularity we also consider the sequence of matrices $B_n = (A_n^*)^{-1}$ for $n \in \mathbb{N}$, where $A^*$ denotes the transpose of $A$. In a similar manner to that in (2), set

$$\mathcal{B}(m,n) = \bigl(\mathcal{A}(m,n)^*\bigr)^{-1} = \begin{cases} (A_{m-1}^*)^{-1}\cdots(A_n^*)^{-1}, & m > n, \\ \mathrm{Id}, & m = n. \end{cases} \tag{5}$$

The Lyapunov exponent $\mu : \mathbb{R}^k \to \mathbb{R} \cup \{-\infty\}$ associated with the sequence $(B_n)_{n\in\mathbb{N}}$ is defined by

$$\mu(w) = \limsup_{n\to+\infty} \frac{1}{n} \log\bigl\|\mathcal{B}(n,1)w\bigr\|. \tag{6}$$

It follows again from the abstract theory of Lyapunov exponents that $\mu$ can take at most $s \leqslant k$ distinct values on $\mathbb{R}^k \setminus \{0\}$, say $-\infty \leqslant \mu_s < \cdots < \mu_1$.

We define the *regularity coefficient* of $\lambda$ and $\mu$ by

$$\gamma(\lambda,\mu) = \min\max\bigl\{\lambda(v_i) + \mu(w_i)\colon\ 1 \leqslant i \leqslant k\bigr\}, \tag{7}$$

where the minimum is taken over all dual bases $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ of $\mathbb{R}^k$, i.e., such that $\langle v_i, w_j\rangle = \delta_{ij}$ for every $i$ and $j$, where $\delta_{ij}$ is the Kronecker symbol.

**Proposition 1.** *If* $\langle v, w\rangle = 1$, *then* $\lambda(v) + \mu(w) \geqslant 0$.

**Proof.** Set $v_n = \mathcal{A}(n,1)v$ and $w_n = \mathcal{B}(n,1)w$. For every $n \in \mathbb{N}$,

$$\langle v_n, w_n \rangle = \big\langle \mathcal{A}(n,1)v, \big(\mathcal{A}(n,1)^*\big)^{-1} w \big\rangle = \langle v, w \rangle = 1.$$

Thus, $1 \leqslant \|\mathcal{A}(n,1)v\| \cdot \|\mathcal{B}(n,1)w\|$, and we obtain the desired inequality. □

It follows from the proposition that $\gamma(\lambda,\mu) \geqslant 0$. We say that the sequence $(A_n)_{n\in\mathbb{N}}$ is *Lyapunov regular* or simply *regular* if $\gamma(\lambda,\mu) = 0$.

## 3. Nonuniform exponential contractions

We say that the sequence $(A_n)_{n\in\mathbb{N}}$ admits a *nonuniform exponential contraction* if there exist constants $\bar{a} < 0$ and $D, c \geqslant 0$ such that for every $m \geqslant n \geqslant 1$,

$$\big\|\mathcal{A}(m,n)\big\| \leqslant De^{\bar{a}(m-n)+cn}. \tag{8}$$

We say that the sequence $(A_n)_{n\in\mathbb{N}}$ admits a *strong nonuniform exponential contraction* if there exist constants $\underline{a} \leqslant \bar{a} < 0$ and $D, c \geqslant 0$ such that for every $m \geqslant n \geqslant 1$,

$$\big\|\mathcal{A}(m,n)\big\| \leqslant De^{\bar{a}(m-n)+cn}, \qquad \big\|\mathcal{A}(m,n)^{-1}\big\| \leqslant De^{-\underline{a}(m-n)+cm}. \tag{9}$$

The following result shows that the notion of strong nonuniform exponential contraction is very natural.

**Theorem 1.** *Assume that the sequence $(A_n)_{n\in\mathbb{N}}$ has only negative Lyapunov exponents, i.e., that $\lambda_r < 0$. Then for each $\varepsilon > 0$ the sequence $(A_n)_{n\in\mathbb{N}}$ admits a strong nonuniform exponential contraction with*

$$\underline{a} = \lambda_1 + \varepsilon, \qquad \bar{a} = \lambda_r + \varepsilon, \quad \textit{and} \quad c = \gamma(\lambda,\mu) + 2\varepsilon.$$

**Proof.** The argument is inspired in the proof of Theorem 15 in [2]. Given an invertible $k \times k$ matrix $C$, let $v_1(n), \dots, v_k(n)$ be the columns of $\mathcal{A}(n,1)C$, and $w_1(n), \dots, w_k(n)$ the columns of $\mathcal{B}(n,1)(C^*)^{-1}$. For $i = 1, \dots, k$, set

$$\alpha_i = \lambda\big(v_i(1)\big) \quad \text{and} \quad \beta_i = \mu\big(w_i(1)\big).$$

Given $\varepsilon > 0$, there exists $D > 0$ such that for each $i = 1, \dots, k$ and $n \in \mathbb{N}$,

$$\big\|v_i(n)\big\| \leqslant De^{(\alpha_i+\varepsilon)n} \quad \text{and} \quad \big\|w_i(n)\big\| \leqslant De^{(\beta_i+\varepsilon)n}.$$

It follows from the identity

$$\big(\mathcal{B}(n,1)\big(C^*\big)^{-1}\big)^*\big(\mathcal{A}(n,1)C\big) = \mathrm{Id}$$

that $\langle v_i(n), w_j(n) \rangle = \delta_{ij}$ for every $i, j = 1, \dots, k$ and $n \in \mathbb{N}$. Hence, since the maximum in (7) takes only finitely many values, there exists a matrix $C$ for which

$$\max\{\alpha_i + \beta_i \colon i = 1, \dots, k\} = \gamma(\lambda,\mu). \tag{10}$$

Notice that

$$\mathcal{A}(m,n) = \mathcal{A}(m,1)\mathcal{B}(n,1)^* = \big(\mathcal{A}(m,1)C\big)\big(\mathcal{B}(n,1)\big(C^*\big)^{-1}\big)^*.$$

The entries $a_{ij}(m,n)$ of this matrix are

$$a_{ij}(m,n) = \sum_{l=1}^{k} v_{il}(m) w_{jl}(n),$$

where $v_{il}(m)$ is the $i$th coordinate of $v_l(m)$, and $w_{jl}(n)$ is the $j$th coordinate of $w_l(n)$. Thus, in view of (10),

$$\begin{aligned}
\big|a_{ij}(m,n)\big| &\leqslant \sum_{l=1}^{k} \big\|v_l(m)\big\| \cdot \big\|w_l(n)\big\| \leqslant \sum_{l=1}^{k} D^2 e^{(\alpha_l+\varepsilon)m+(\beta_l+\varepsilon)n} \\
&= \sum_{l=1}^{k} D^2 e^{(\alpha_l+\varepsilon)(m-n)+(\alpha_l+\beta_l+2\varepsilon)n} \\
&\leqslant k D^2 e^{(\lambda_r+\varepsilon)(m-n)+(\gamma(\lambda,\mu)+2\varepsilon)n}.
\end{aligned}$$

Taking $v = \sum_{j=1}^{k} c_j e_j$ with $\|v\|^2 = \sum_{j=1}^{k} c_j^2 = 1$, where $e_1, \dots, e_k$ is the canonical orthonormal basis of $\mathbb{R}^k$, we obtain

$$\begin{aligned}
\big\|\mathcal{A}(m,n)v\big\|^2 &= \left\| \sum_{i=1}^{k} \sum_{j=1}^{k} c_j a_{ij}(m,n) e_i \right\|^2 = \sum_{i=1}^{k} \left( \sum_{j=1}^{k} c_j a_{ij}(m,n) \right)^2 \\
&\leqslant \sum_{i=1}^{k} \sum_{j=1}^{k} c_j^2 \sum_{j=1}^{k} a_{ij}(m,n)^2 = \sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij}(m,n)^2. \qquad (11)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\big\|\mathcal{A}(m,n)\big\| &\leqslant \left( \sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij}(m,n)^2 \right)^{1/2} \\
&\leqslant k^2 D^2 e^{(\lambda_r+\varepsilon)(m-n)+(\gamma(\lambda,\mu)+2\varepsilon)n},
\end{aligned}$$

which is the first inequality in (9). Similarly, the entries $b_{ij}(m,n) = a_{ij}(n,m)$ of the matrix

$$\mathcal{A}(m,n)^{-1} = \mathcal{A}(n,1)\mathcal{B}(m,1)^* = \big(\mathcal{A}(n,1)C\big)\big(\mathcal{B}(m,1)\big(C^*\big)^{-1}\big)^*$$

satisfy

$$
\begin{aligned}
\left|b_{ij}(m,n)\right| &\leqslant \sum_{l=1}^{k}\left\|v_l(n)\right\|\cdot\left\|w_l(m)\right\| \leqslant \sum_{l=1}^{k} D^2 e^{(\alpha_l+\varepsilon)n+(\beta_l+\varepsilon)m} \\
&= \sum_{l=1}^{k} D^2 e^{-(\alpha_l+\varepsilon)(m-n)+(\alpha_l+\beta_l+2\varepsilon)m} \\
&\leqslant kD^2 e^{-(\lambda_1+\varepsilon)(m-n)+(\gamma(\lambda,\mu)+2\varepsilon)m}.
\end{aligned}
$$

Arguing as in (11) we obtain the second inequality in (9). □

## 4. Stability for nonuniform contractions

We show in this section that the asymptotic stability of a nonuniform exponential contraction, for a given sequence $(A_n)_{n\in\mathbb{N}}$, persists under sufficiently small nonlinear perturbations, provided that the constant $c$ in (8) is also sufficiently small.

We assume that there exist continuous maps $f_n:\mathbb{R}^k\to\mathbb{R}^k$ with $f_n(0)=0$ for each $n\in\mathbb{N}$, and constants $C>0$ and $q>0$ such that

$$
\left\|f_n(u)-f_n(v)\right\| \leqslant C\|u-v\|\left(\|u\|^q+\|v\|^q\right) \tag{12}
$$

for every $n\in\mathbb{N}$ and $u,v\in\mathbb{R}^k$. For each $n\in\mathbb{N}$ we define the map $F_n:\mathbb{R}^k\to\mathbb{R}^k$ by $F_n(v)=A_nv+f_n(v)$, and given $m\geqslant n$ we also set

$$
\mathcal{F}(m,n)=\begin{cases} F_{m-1}\circ\cdots\circ F_n, & m>n,\\ \mathrm{Id}, & m=n.\end{cases} \tag{13}
$$

We denote by $B(\delta)\subset\mathbb{R}^k$ the open ball of radius $\delta>0$ centered at zero. We also consider the constant $\beta=(1+2/q)c$ with $c$ as in (8).

The following is our first stability result.

**Theorem 2.** *If the sequence $(A_n)_{n\in\mathbb{N}}$ admits a nonuniform exponential contraction and $q\bar{a}+c<0$, then there exists $\delta>0$ such that for each $v_n,\bar{v}_n\in B(\delta e^{-(\beta+c)n})$ and $m\geqslant n$,*

$$
\left\|\mathcal{F}(m,n)(v_n)-\mathcal{F}(m,n)(\bar{v}_n)\right\| \leqslant Ke^{\bar{a}(m-n)+cn}\|v_n-\bar{v}_n\|,
$$

*where $K=2\max\{1,D\}$ with $D$ as in* (8).

**Proof.** Consider a sequence of points $v_{n+1}=F_n(v_n)$ in $\mathbb{R}^k$. For each $m\geqslant n$ we have

$$
v_m=\mathcal{F}(m,n)(v_n)=\mathcal{A}(m,n)v_n+\sum_{l=n}^{m-1}\mathcal{A}(m,l+1)f_l(v_l). \tag{14}
$$

Given $\delta>0$, we consider the space

$$
\mathcal{D}=\left\{w=(w_m)_{m\geqslant n}\colon\ \|w\|'\leqslant\delta e^{-\beta n}\right\},
$$

with the norm

$$\|w\|' = \frac{1}{K} \sup\bigl\{\|w_m\| e^{-\bar{a}(m-n)-cn}\colon\ m \geqslant n\bigr\}.$$

One can easily verify that $\mathcal{D}$ is a complete metric space. Given $v_n \in B(\delta e^{-(\beta+c)n})$ we define the operator

$$(Sw)_m = \sum_{l=n}^{m-1} \mathcal{A}(m, l+1) f_l(w_l) \tag{15}$$

in the space $\mathcal{D}$. By (12), given $w, \overline{w} \in \mathcal{D}$ we have

$$\begin{aligned}\bigl\| f_l(w_l) - f_l(\overline{w}_l)\bigr\| &\leqslant C\|w_l - \overline{w}_l\|\bigl(\|w_l\|^q + \|\overline{w}_l\|^q\bigr)\\ &\leqslant 2K^{q+1} C \delta^{q+1} e^{\bar{a}(q+1)(l-n)-\beta n}\|w - \overline{w}\|'.\end{aligned}$$

Therefore, by (8),

$$\begin{aligned}\bigl\|(Sw)_m - (S\overline{w})_m\bigr\| &\leqslant \sum_{l=n}^{m-1} \bigl\|\mathcal{A}(m, l+1)\bigr\| \cdot \bigl\| f_l(w_l) - f_l(\overline{w}_l)\bigr\|\\ &\leqslant 2K^{q+1} DC\delta^{q+1}\|w - \overline{w}\|' \sum_{l=n}^{m-1} e^{\bar{a}(m-l-1)+c(l+1)} e^{\bar{a}(q+1)(l-n)-\beta n}\\ &\leqslant 2K^{q+1} DC\delta^{q+1} e^{-\bar{a}+c}\|w - \overline{w}\|' e^{\bar{a}(m-n)} \sum_{l=n}^{m-1} e^{(q\bar{a}+c)(l-n)}.\end{aligned}$$

It follows from the condition $q\bar{a} + c < 0$ that

$$\|Sw - S\overline{w}\|' \leqslant \theta\|w - \overline{w}\|', \tag{16}$$

where

$$\theta = 2K^{q+1} DC\delta^{q+1} e^{-\bar{a}+c}/\bigl(1 - e^{q\bar{a}+c}\bigr).$$

We now set

$$(Tw)_m = \mathcal{A}(m, n)v_n + (Sw)_m, \tag{17}$$

and choose $\delta > 0$ sufficiently small so that $\theta < 1/2$. It follows from (16) that $\|Tw - T\overline{w}\|' \leqslant \theta\|w - \overline{w}\|'$ and thus $T$ is a contraction. Observe that for $\overline{w} = 0$ we have $S\overline{w} = 0$ (since $f_m(0) = 0$ for every $m$). Therefore, by (16), $\|Sw\|' \leqslant \theta\|w\|'$. By (8) we have $\|(\mathcal{A}(m, n)v_n)_{m\geqslant n}\|' \leqslant \|v_n\|/2$ and hence,

$$\begin{aligned}\|Tw\|' &\leqslant \left\|\left(\mathcal{A}(m,n)v_n\right)_{m\geqslant n}\right\|' + \|Sw\|' \\ &\leqslant \frac{1}{2}\|v_n\| + \theta\|v\|' < \frac{1}{2}\delta e^{-(\beta+c)n} + \frac{1}{2}\|v\|' < \|v\|'.\end{aligned}$$

Therefore, $T(\mathcal{D}) \subset \mathcal{D}$. Since $T$ is a contraction it has a unique fixed point $(w_m)_{m\geqslant n} \in \mathcal{D}$. This shows that the sequence $(v_m)_{m\geqslant n}$ in (14) satisfies

$$\|v_l\| \leqslant K e^{\bar{a}(l-n)+cn}\|v_n\|, \quad l \geqslant n.$$

When $v_n, \bar{v}_n \in B(\delta e^{-(\beta+c)n})$ we thus obtain

$$\begin{aligned}\left\|f_l(v_l) - f_l(\bar{v}_l)\right\| &\leqslant C\|v_l - \bar{v}_l\|\left(\|v_l\|^q + \|\bar{v}_l\|^q\right) \\ &\leqslant \eta e^{q\bar{a}(l-n)-\beta n}\|v_l - \bar{v}_l\|,\end{aligned} \tag{18}$$

where $\eta = 2K^q C\delta^q$. Set now

$$\rho_m = \|v_m - \bar{v}_m\| = \left\|\mathcal{F}(m,n)(v_n) - \mathcal{F}(m,n)(\bar{v}_n)\right\| \tag{19}$$

and $T_m = e^{-\bar{a}(m-n)}\rho_m$. Using (8) and (18), it follows from (14) that

$$\begin{aligned}\rho_m &\leqslant \left\|\mathcal{A}(m,n)\right\| \cdot \|v_n - \bar{v}_n\| + \sum_{l=n}^{m-1}\left\|\mathcal{A}(m,l+1)\right\|\eta e^{q\bar{a}(l-n)-\beta n}\rho_l \\ &\leqslant De^{\bar{a}(m-n)+cn}\|v_n - \bar{v}_n\| + D\eta e^{-\bar{a}+(1-2n/q)c}\sum_{l=n}^{m-1} e^{\bar{a}(m-l)+(q\bar{a}+c)(l-n)}\rho_l \\ &\leqslant De^{\bar{a}(m-n)}\left(e^{cn}\|v_n - \bar{v}_n\| + \eta e^{-\bar{a}+c}\sum_{l=n}^{m-1} e^{-\bar{a}(l-n)}e^{(q\bar{a}+c)(l-n)}\rho_l\right).\end{aligned}$$

Therefore

$$T_m \leqslant D\left(e^{cn}\|v_n - \bar{v}_n\| + \eta e^{-\bar{a}+c}\sum_{l=n}^{m-1} e^{(q\bar{a}+c)(l-n)}T_l\right).$$

Set $T = \sup_{m\in\mathbb{N}} T_m$. Using the condition $q\bar{a} + c < 0$, provided that $\delta$ is sufficiently small we obtain

$$T \leqslant D\left(e^{cn}\|v_n - \bar{v}_n\| + \frac{\eta e^{-\bar{a}+c}T}{1 - e^{q\bar{a}+c}}\right) \leqslant De^{cn}\|v_n - \bar{v}_n\| + \frac{T}{2}.$$

This completes the proof of the theorem. □

## 5. Nonuniform exponential dichotomies

We discuss briefly in this section the situation when the sequence of matrices $(A_n)_{n\in\mathbb{N}}$ exhibits not only contraction but also expansion. We allow both the contraction and the expansion to be nonuniform.

### 5.1. Existence of nonuniform exponential dichotomies

We say that the sequence $(A_n)_{n\in\mathbb{N}}$ admits a *nonuniform exponential dichotomy* if there exist constants $\bar{a} < 0 \leqslant \underline{b}$ and $D, c \geqslant 0$, and projections $P_n$ for $n \in \mathbb{N}$ such that for every $m \geqslant n \geqslant 1$,

$$P_m \mathcal{A}(m,n) = \mathcal{A}(m,n) P_n, \tag{20}$$

and letting $Q_n = \mathrm{Id} - P_n$ be the complementary projection,

$$\left\|\mathcal{A}(m,n)P_n\right\| \leqslant De^{\bar{a}(m-n)+cn}, \qquad \left\|\mathcal{A}(m,n)^{-1}Q_m\right\| \leqslant De^{-\underline{b}(m-n)+cm}. \tag{21}$$

We say that the sequence $(A_n)_{n\in\mathbb{N}}$ admits a *strong nonuniform exponential dichotomy* if there exist constants

$$\underline{a} \leqslant \bar{a} < 0 \leqslant \underline{b} \leqslant \bar{b} \quad \text{and} \quad D, c \geqslant 0,$$

and projections $P_n$ for $n \in \mathbb{N}$ such that for every $m \geqslant n \geqslant 1$, (20) holds and

$$\begin{aligned} \left\|\mathcal{A}(m,n)P_n\right\| &\leqslant De^{\bar{a}(m-n)+cn}, & \left\|\mathcal{A}(m,n)^{-1}P_m\right\| &\leqslant De^{-\underline{a}(m-n)+cm}, \\ \left\|\mathcal{A}(m,n)Q_n\right\| &\leqslant De^{\bar{b}(m-n)+cn}, & \left\|\mathcal{A}(m,n)^{-1}Q_m\right\| &\leqslant De^{-\underline{b}(m-n)+cm}. \end{aligned}$$

In a similar manner to that for exponential contractions (see Theorem 1) we want to show that the notion of nonuniform exponential dichotomy is also very natural. The existence of a nonuniform exponential dichotomy corresponds to the case when there exist also nonnegative Lyapunov exponents. We assume in this section that there is a decomposition $\mathbb{R}^k = E \oplus F$ with $E = E_p$ for some $p$ (see (4)), with respect to which

$$A_n = \begin{pmatrix} C_n & 0 \\ 0 & D_n \end{pmatrix} \tag{22}$$

for every $n \in \mathbb{N}$. Given $v = (x, y) \in E \times F$, one can write

$$\mathcal{A}(m,n)v = \big(\mathcal{C}(m,n)x, \mathcal{D}(m,n)y\big),$$

where

$$\mathcal{C}(m,n) = \begin{cases} C_{m-1}\cdots C_n, & m > n, \\ \mathrm{Id}, & m = n, \end{cases} \qquad \mathcal{D}(m,n) = \begin{cases} D_{m-1}\cdots D_n, & m > n, \\ \mathrm{Id}, & m = n. \end{cases}$$

The following statement provides a condition for the existence of strong nonuniform exponential dichotomies.

**Theorem 3.** *Assume that the sequence $(A_n)_{n\in\mathbb{N}}$ has the block form in* (22)*, with $\lambda(v) < 0$ for $v \in E = E_p$, and $\lambda(v) \geqslant 0$ for $v \in F \setminus \{0\}$. Then for each $\varepsilon > 0$ the sequence $(A_n)_{n\in\mathbb{N}}$ admits a strong nonuniform exponential dichotomy with $c = \gamma(\lambda, \mu) + 2\varepsilon$, and*

$$\underline{a} = \lambda_1 + \varepsilon, \qquad \bar{a} = \lambda_p + \varepsilon, \qquad \underline{b} = \lambda_{p+1} + \varepsilon, \qquad \bar{b} = \lambda_r + \varepsilon.$$

**Proof.** The statement can be obtained proceeding in a similar manner to that in the proof of Theorem 1, by considering separately the blocks $\mathcal{C}(m,n)$ and $\mathcal{D}(m,n)$. $\square$

*5.2. Stability for nonuniform dichotomies*

We now briefly discuss a version of Theorem 2 in the case of nonuniform exponential dichotomies. Namely, one can show that under sufficiently small nonlinear perturbations the nonuniform exponential contraction exhibited by the stable components $C_n$ in (22) persists along a stable invariant manifold tangent to the stable space $E$.

We assume that there exist continuous transformations $f_n : \mathbb{R}^k \to \mathbb{R}^k$ with $f_n(0) = 0$ for each $n \in \mathbb{N}$, and constants $C > 0$ and $q > 0$ such that (12) holds for every $n \in \mathbb{N}$ and $u, v \in \mathbb{R}^k$. Set again $\beta = (1 + 2/q)c$ with $c$ as in (21). We denote by $\mathcal{X}$ the space of sequences $(\varphi_n)_{n\in\mathbb{N}}$ of continuous functions $\varphi_n : B(\delta e^{-\beta n}) \to F$ such that $\varphi_n(0) = 0$ and

$$\left\|\varphi_n(x) - \varphi_n(y)\right\| \leqslant \|x - y\| \quad \text{for every } x, y \in B\left(\delta e^{-\beta n}\right).$$

Given $(\varphi_n)_{n\in\mathbb{N}} \in \mathcal{X}$, for each $n \in \mathbb{N}$ we consider the graph

$$\mathcal{V}_n = \left\{\left(\xi, \varphi_n(\xi)\right) \colon \xi \in B\left(\delta e^{-\beta n}\right)\right\}.$$

The following stability result was established in [3].

**Theorem 4.** *If the sequence $(A_n)_{n\in\mathbb{N}}$ admits a nonuniform exponential dichotomy, and the conditions $\bar{a} + \beta \leqslant 0$ and $\bar{a} + c < \underline{b}$ hold, then there exist $\delta > 0$ and a unique $\varphi \in \mathcal{X}$ such that*

$$\mathcal{F}(m,n)\left(\xi, \varphi_n(\xi)\right) \subset \mathcal{V}_m \quad \text{for every } m \geqslant n \text{ and } \xi \in B\left(\delta e^{-(\beta+c)n}\right).$$

*Furthermore, given $\xi, \bar{\xi} \in B(\delta e^{-(\beta+c)n})$ and $m \geqslant n$ we have*

$$\left\|\mathcal{F}(m,n)\left(\xi, \varphi_n(\xi)\right) - \mathcal{F}(m,n)\left(\bar{\xi}, \varphi_n(\bar{\xi})\right)\right\| \leqslant 4De^{\bar{a}(m-n)+cn}\|\xi - \bar{\xi}\|.$$

It is also shown in [3] that when the maps $f_n$ are of class $C^1$ and $q > 1$, eventually reducing $\delta$ the set

$$\mathcal{V}_n' = \left\{\left(\xi, \varphi_n(\xi)\right) \colon \xi \in B\left(\delta e^{-(\beta+c)n}\right)\right\} \subset \mathcal{V}_n$$

is a smooth manifold of class $C^1$ with $T_0\mathcal{V}_n' = E$ for each $n \in \mathbb{N}$.

## 6. Bounds for the regularity coefficient

The above Theorems 2 and 4 establish respectively the stability under perturbations of nonuniform exponential contractions and nonuniform exponential dichotomies. These theorems include conditions that involve the constant $c$ in (8) and (21), which measures the nonuniformity of the nonuniform contractions and nonuniform dichotomies. Essentially, the conditions require that $c$ is sufficiently small. Thus, having in mind the importance of the stability results in the theory of dynamical systems, it is crucial to obtain sharp estimates for the constant $c$, that hopefully can be given somewhat explicitly in terms of the matrices $(A_n)_{n\in\mathbb{N}}$. This is the main objective of this section.

### *6.1. Lower bound*

We first obtain a lower bound for the regularity coefficient. Set $a_n = |\det \mathcal{A}(n,1)|$.

**Theorem 5.** *We have*

$$\gamma(\lambda,\mu) \geqslant \frac{1}{k}\limsup_{n\to+\infty}\frac{1}{n}\log a_n - \frac{1}{k}\liminf_{n\to+\infty}\frac{1}{n}\log a_n.$$

**Proof.** Let $v_1,\dots,v_k$ be a basis of $\mathbb{R}^k$. Then

$$\bigl|\det\bigl(\mathcal{A}(n,1)C\bigr)\bigr| \leqslant \prod_{i=1}^{k}\bigl\|\mathcal{A}(n,1)v_i\bigr\|,$$

where $C$ is the $k\times k$ matrix whose columns are the vectors $v_1,\dots,v_k$. Thus,

$$\limsup_{n\to+\infty}\frac{1}{n}\log a_n = \limsup_{n\to+\infty}\frac{1}{n}\log\bigl|\det\bigl(\mathcal{A}(n,1)C\bigr)\bigr| \leqslant \sum_{i=1}^{k}\lambda(v_i). \tag{23}$$

Similarly, if $w_1,\dots,w_k$ is another basis of $\mathbb{R}^k$, then

$$\liminf_{n\to+\infty}\frac{1}{n}\log a_n = -\limsup_{n\to+\infty}\frac{1}{n}\log\bigl|\det\mathcal{B}(n,1)\bigr| \geqslant -\sum_{i=1}^{k}\mu(w_i). \tag{24}$$

Therefore,

$$\limsup_{n\to+\infty}\frac{1}{n}\log a_n - \liminf_{n\to+\infty}\frac{1}{n}\log a_n \leqslant \sum_{i=1}^{k}\bigl(\lambda(v_i)+\mu(w_i)\bigr).$$

We now assume that, in addition, $v_1,\dots,v_k$ and $w_1,\dots,w_k$ are dual bases at which the minimum in (7) is attained, i.e., that

$$\gamma(\lambda,\mu) = \max\bigl\{\lambda(v_i)+\mu(w_i)\colon\ 1\leqslant i\leqslant k\bigr\}.$$

Then $\sum_{i=1}^{k}(\lambda(v_i)+\mu(w_i)) \leqslant k\,\gamma(\lambda,\mu)$. This completes the proof. □

### 6.2. Upper bound

To obtain an upper bound for the regularity coefficient we first assume that the matrices $A_n$ are upper triangular. Analogously, we could also consider the case when all the matrices $A_n$ are lower triangular.

Let $a_{ij}(n)$ be the entries of $A_n$. For $i = 1, \dots, k$, set

$$\underline{\alpha}_i = \liminf_{n\to+\infty} \frac{1}{n} \log \prod_{l=1}^{n} a_{ii}(l) \quad \text{and} \quad \bar{\alpha}_i = \limsup_{n\to+\infty} \frac{1}{n} \log \prod_{l=1}^{n} a_{ii}(l). \tag{25}$$

We obtain an upper bound for the regularity coefficient in terms of these numbers.

**Theorem 6.** *If $A_n$ is upper triangular for every $n \in \mathbb{N}$, then*

$$\gamma(\lambda, \mu) \leqslant \sum_{i=1}^{k} (\bar{\alpha}_i - \underline{\alpha}_i). \tag{26}$$

The proof of Theorem 6 is given in Appendix A.

### 6.3. Upper triangular reduction

We describe here how to reduce the study of an arbitrary sequence $(A_n)_{n\in\mathbb{N}}$ to the study of upper triangular matrices.

**Theorem 7.** *For each sequence of matrices $(A_n)_{n\in\mathbb{N}}$ there exists a sequence $(U_n)_{n\in\mathbb{N}}$ of orthogonal matrices such that $C_n = U_{n+1}^* A_n U_n$ is upper triangular for each $n \in \mathbb{N}$.*

**Proof.** We apply the Gram–Schmidt orthogonalization procedure to the vectors $v_i(n) = \mathcal{A}(n,1)e_i$, where $e_1, \dots, e_k$ is the canonical basis of $\mathbb{R}^k$. For each $n \geqslant 1$ we obtain vectors $u_1(n), \dots, u_k(n)$ such that:

1. $\langle u_i(n), u_j(n)\rangle = \delta_{ij}$ for each $i$ and $j$;
2. each $u_i(n)$ is a linear combination of the vectors $v_1(n), \dots, v_k(n)$.

Clearly, each vector $v_k(n)$ is also a linear combination of $u_1(n), \dots, u_k(n)$, and thus

$$\big\langle v_i(n), u_j(n)\big\rangle = 0 \quad \text{for } i < j. \tag{27}$$

Let now $U_n$ be the matrix with columns $u_1(n), \dots, u_k(n)$. Clearly $U_1 = \mathrm{Id}$. By the orthogonalization procedure each $U_n$ is orthogonal. Furthermore, setting $C_n = U_{n+1}^* A_n U_n$, since $U_n$ is orthogonal for each $n$ we obtain

$$\mathcal{C}(n,1) = C_{n-1} \cdots C_1 = U_n^* \mathcal{A}(n,1), \tag{28}$$

where $\mathcal{C}(n,1)$ is defined in a similar manner to that in (2). Therefore,

$$\big\langle \mathcal{C}(n,1)e_i, e_j\big\rangle = \big\langle U_n^* \mathcal{A}(n,1)e_i, e_j\big\rangle = \big\langle \mathcal{A}(n,1)e_i, U_n e_j\big\rangle = \big\langle v_i(n), u_j(n)\big\rangle = 0$$

whenever $i < j$, in view of (27). This shows that $\mathcal{C}(n,1)$ is upper triangular for each $n$. The desired statement follows now readily from the identity $C_n = \mathcal{C}(n+1,1)\mathcal{C}(n,1)^{-1}$. □

Since $U_n$ is orthogonal for each $n$, it follows from (28) that

$$\limsup_{n\to+\infty} \frac{1}{n} \log\big\|\mathcal{A}(n,1)v\big\| = \limsup_{n\to+\infty} \frac{1}{n} \log\big\|\mathcal{C}(n,1)v\big\| \tag{29}$$

for every $v \in \mathbb{R}^k$. That is, the Lyapunov exponents defined by any two sequences $(A_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$ as in Theorem 7 coincide. Furthermore, from the point of view of regularity theory we can always replace an arbitrary sequence of matrices $(A_n)_{n\in\mathbb{N}}$ by the upper triangular sequence $(C_n)_{n\in\mathbb{N}}$ constructed in Theorem 7. This is the content of the following result.

**Theorem 8.** *For any sequences of matrices $(A_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$ as in Theorem 7, the regularity coefficient defined by the matrices $A_n$ and $(A_n^*)^{-1}$ is equal to the regularity coefficient defined by the matrices $C_n$ and $(C_n^*)^{-1}$.*

**Proof.** Notice that for the matrices $B_n = (A_n^*)^{-1}$ and $D_n = (C_n^*)^{-1}$, we have

$$\mathcal{B}(n,1) = \big(\mathcal{A}(n,1)^*\big)^{-1} \quad \text{and} \quad \mathcal{D}(n,1) = \big(\mathcal{C}(n,1)^*\big)^{-1}.$$

Since $\mathcal{C}(n,1) = U_n^*\mathcal{A}(n,1)$ (see (28)) we obtain $\mathcal{D}(n,1) = U_n^*\mathcal{B}(n,1)$. Therefore, in addition to (29), we also have the identity

$$\limsup_{n\to+\infty} \frac{1}{n} \log\big\|\mathcal{B}(n,1)v\big\| = \limsup_{n\to+\infty} \frac{1}{n} \log\big\|\mathcal{D}(n,1)v\big\| \tag{30}$$

for every $v \in \mathbb{R}^k$. Since the regularity coefficient is defined solely in terms of the Lyapunov exponents and these coincide for the sequences $A_n$ and $C_n$ (see (29)), as well as for the sequences $B_n$ and $D_n$ (see (30)), we obtain the statement in the theorem. □

## 7. Characterizations of regularity

The purpose of this section is to give several alternative characterizations of regularity. Some of them are expressed solely in terms of the matrices $\mathcal{A}(n,1)$ (and do not use the matrices $\mathcal{B}(n,1)$).

We first introduce another coefficient that also measures the regularity of a given sequence of matrices. Consider the values

$$\lambda_1' \leqslant \cdots \leqslant \lambda_k' \quad \text{and} \quad \mu_1' \geqslant \cdots \geqslant \mu_k'$$

of the Lyapunov exponents $\lambda$ and $\mu$ on $\mathbb{R}^k \setminus \{0\}$ counted with their multiplicities (for example, the values $\lambda_i'$ are obtained by repeating each value $\lambda_i$ a number of times equal to the difference $\dim E_i - \dim E_{i-1}$; see (4)) where $E_0 = \{0\}$. We define the *Perron coefficient* of the pair of Lyapunov exponents $\lambda$ and $\mu$ by

$$\pi(\lambda,\mu) = \max\big\{\lambda_i' + \mu_i' \colon 1 \leqslant i \leqslant k\big\}. \tag{31}$$

**Proposition 2.** [1, Theorem 1.2.6] *We have* $0 \leqslant \pi(\lambda, \mu) \leqslant \gamma(\lambda, \mu) \leqslant k\pi(\lambda, \mu)$.

We say that a basis $v_1, \dots, v_k$ of $\mathbb{R}^k$ is *normal* for the family of spaces $E_i$ in (4) if for each $i = 1, \dots, r$ there exists a basis of $E_i$ composed of vectors in $\{v_1, \dots, v_k\}$. We also consider the subspaces

$$F_i = \left\{ w \in \mathbb{R}^k\colon\ \mu(w) \leqslant \mu_i \right\},$$

and we consider a normal basis $w_1, \dots, w_k$ of $\mathbb{R}^k$ for the family of spaces $F_i$. One can easily verify that there always exist normal bases $v_1, \dots, v_k$ and $w_1, \dots, w_k$ (respectively of the families of spaces $E_i$ and $F_i$) which are dual.

Given vectors $v_1, \dots, v_m \in \mathbb{R}^k$ we denote the $m$-volume defined by these vectors by $\Gamma(v_1, \dots, v_m)$. It is equal to $|\det K|^{1/2}$, where $K$ is the $m \times m$ matrix with entries $k_{ij} = \langle v_i, v_j \rangle$ for each $i$ and $j$.

**Theorem 9.** *The following properties are equivalent*:

1. $\gamma(\lambda, \mu) = 0$;
2. $\pi(\lambda, \mu) = 0$;
3. *for any dual normal bases* $v_1, \dots, v_k$ *and* $w_1, \dots, w_k$ *of* $\mathbb{R}^k$ *we have*

$$\lambda(v_i) + \mu(w_i) = 0 \quad \textit{for } i = 1, \dots, k; \tag{32}$$

4. $\lambda_i' + \mu_i' = 0$ *for* $i = 1, \dots, k$;
5. 

$$\lim_{n \to +\infty} \frac{1}{n} \log\bigl|\det \mathcal{A}(n, 1)\bigr| = \sum_{i=1}^{r} (\dim E_i - \dim E_{i-1})\lambda_i; \tag{33}$$

6. *for any vectors* $v_1, \dots, v_k \in \mathbb{R}^k$ *and* $m \leqslant k$ *there exists the limit*

$$\lim_{n \to +\infty} \frac{1}{n} \log \Gamma\bigl(\mathcal{A}(n, 1)v_1, \dots, \mathcal{A}(n, 1)v_m\bigr). \tag{34}$$

**Proof.** The equivalence of the first two properties is immediate from Proposition 2. Let now $v_1, \dots, v_k$ and $w_1, \dots, w_k$ be dual normal bases. By Proposition 1 we have $\lambda(v_i) + \mu(w_i) \geqslant 0$ for each $i$. Therefore,

$$0 \leqslant \sum_{i=1}^{k} \bigl(\lambda(v_i) + \mu(w_i)\bigr) = \sum_{i=1}^{k} \bigl(\lambda_i' + \mu_i'\bigr) \leqslant k\pi(\lambda, \mu) \tag{35}$$

(the identity follows form the fact that the bases are normal). Furthermore, by Proposition 2 and (31) we have $\lambda_i' + \mu_i' \leqslant 0$ for $i = 1, \dots, k$. It follows from (35) that if property 2 holds, then

$$\sum_{i=1}^{k} \bigl(\lambda(v_i) + \mu(w_i)\bigr) = 0,$$

and thus (32) holds. Furthermore, if property 3 holds, then $\gamma(\lambda,\mu)=0$ and thus, by Proposition 2, we also have $\pi(\lambda,\mu)=0$. Hence, by (35), $\sum_{i=1}^{k}(\lambda_i'+\mu_i')=0$, and thus $\lambda_i'+\mu_i'=0$ for every $i$, i.e., property 4 holds. On the other hand, property 4 clearly implies property 2.

We now establish the equivalence of the first four properties to properties 5 and 6. Assume that $\gamma(\lambda,\mu)=0$. By Theorem 5 we have

$$\liminf_{n\to+\infty}\frac{1}{n}\log\left|\det\mathcal{A}(n,1)\right|=\limsup_{n\to+\infty}\frac{1}{n}\log\left|\det\mathcal{A}(n,1)\right|.$$

Choose now dual normal bases, that thus satisfy (32). In particular,

$$\sum_{i=1}^{k}\lambda(v_i)=-\sum_{i=1}^{k}\mu(w_i).$$

It follows readily from (23) and (24) that (33) holds.

We now assume that property 5 holds. Note that it is sufficient to prove the existence of the limit in (34) for a single basis $v_1,\dots,v_k$. Consider the upper triangular matrices $C_n$ constructed in Theorem 7. Since each $\mathcal{C}(n,1)$ is upper triangular,

$$\Gamma\bigl(\mathcal{A}(n,1)e_1,\dots,\mathcal{A}(n,1)e_k\bigr)=\Gamma\bigl(\mathcal{C}(n,1)e_1,\dots,\mathcal{C}(n,1)e_k\bigr)=\prod_{i=1}^{k}\left|c_i(n)\right|,\tag{36}$$

where

$$c_i(n)=\bigl\langle\mathcal{C}(n,1)e_i,e_i\bigr\rangle.\tag{37}$$

Let now $v_1,\dots,v_k$ be a normal basis. By (28), since $\mathcal{C}(n,1)$ is upper triangular we obtain

$$\sum_{i=1}^{k}\lambda(v_i)=\sum_{i=1}^{k}\limsup_{n\to+\infty}\frac{1}{n}\log\bigl\|\mathcal{C}(n,1)v_i\bigr\|\geqslant\sum_{i=1}^{k}\limsup_{n\to+\infty}\frac{1}{n}\log\left|c_i(n)\right|.$$

By property 5, since $v_1,\dots,v_k$ is a normal basis,

$$\begin{aligned}\lim_{n\to+\infty}\frac{1}{n}\log\Gamma\bigl(\mathcal{A}(n,1)v_1,\dots,\mathcal{A}(n,1)v_k\bigr)&=\sum_{i=1}^{k}\lambda(v_i)\\&\geqslant\sum_{i=1}^{k}\limsup_{n\to+\infty}\frac{1}{n}\log\left|c_i(n)\right|.\end{aligned}\tag{38}$$

On the other hand, by (36),

$$\begin{aligned}&\limsup_{n\to+\infty}\frac{1}{n}\log\Gamma\bigl(\mathcal{A}(n,1)v_1,\dots,\mathcal{A}(n,1)v_k\bigr)\\&\quad=\limsup_{n\to+\infty}\frac{1}{n}\log\Gamma\bigl(\mathcal{A}(n,1)e_1,\dots,\mathcal{A}(n,1)e_k\bigr)\\&\quad=\lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{k}\log\left|c_i(n)\right|\leqslant\sum_{i=1}^{k}\limsup_{n\to+\infty}\frac{1}{n}\log\left|c_i(n)\right|.\end{aligned}\tag{39}$$

Comparing (38) and (39) we find that

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{k}\log\left|c_i(n)\right|=\sum_{i=1}^{k}\limsup_{n\to+\infty}\frac{1}{n}\log\left|c_i(n)\right|. \tag{40}$$

It follows easily from this identity that each lim sup in (40) is indeed a limit. Indeed, if

$$\underline{c}_i:=\liminf_{n\to+\infty}\frac{1}{n}\log\left|c_i(n)\right|<\limsup_{n\to+\infty}\frac{1}{n}\log\left|c_i(n)\right|=:\bar{c}_i$$

for some $i=j$, then choosing a subsequence $k_n$ such that $\frac{1}{k_n}\log|c_j(k_n)|\to\underline{c}_j$ as $n\to+\infty$, we obtain

$$\begin{aligned}
\lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{k}\log\left|c_i(n)\right| &= \lim_{n\to+\infty}\frac{1}{k_n}\sum_{i=1}^{k}\log\left|c_i(k_n)\right| \\
&= \underline{c}_j+\lim_{n\to+\infty}\frac{1}{k_n}\sum_{i\neq j}\log\left|c_i(k_n)\right| \\
&< \bar{c}_j+\sum_{i\neq j}\bar{c}_i=\sum_{i=1}^{k}\bar{c}_i,
\end{aligned}$$

which contradicts to (40). By (39), we obtain

$$\lim_{n\to+\infty}\frac{1}{n}\log\Gamma\bigl(\mathcal{A}(n,1)v_1,\dots,\mathcal{A}(n,1)v_m\bigr)=\sum_{i=1}^{m}\lim_{n\to+\infty}\frac{1}{n}\log\left|c_i(n)\right|.$$

This establishes property 6.

We now assume that property 6 holds. We take $v_i=e_i$ for $i=1,\dots,k$. Consider again the upper triangular matrices $C_n$ constructed in Theorem 7. Since $\mathcal{C}(n,1)=U_n^*\mathcal{A}(n,1)$ (see (28)) with $U_n$ orthogonal, it follows from property 6 that

$$\begin{aligned}
&\lim_{n\to+\infty}\frac{1}{n}\log\frac{\Gamma(\mathcal{A}(n,1)e_1,\dots,\mathcal{A}(n,1)e_m)}{\Gamma(\mathcal{A}(n,1)e_1,\dots,\mathcal{A}(n,1)e_{m-1})} \\
&\quad=\lim_{n\to+\infty}\frac{1}{n}\log\frac{\Gamma(\mathcal{C}(n,1)e_1,\dots,\mathcal{C}(n,1)e_m)}{\Gamma(\mathcal{C}(n,1)e_1,\dots,\mathcal{C}(n,1)e_{m-1})}=\lim_{n\to+\infty}\frac{1}{n}\log\left|c_m(n)\right|,
\end{aligned}$$

with $c_m(n)$ as in (37). It follows from Theorem 6 that the regularity coefficient defined by the Lyapunov exponents associated with the matrices $\mathcal{C}(n,1)$ and $(\mathcal{C}(n,1)^*)^{-1}$ is zero. On the other hand, by (28), the Lyapunov exponents defined by the sequences $\mathcal{A}(n,1)$ and $\mathcal{C}(n,1)$ take exactly the same values. The same happens, again by (28), with the Lyapunov exponents defined by the sequences $(\mathcal{A}(n,1)^*)^{-1}$ and $(\mathcal{C}(n,1)^*)^{-1}$. Therefore, the regularity coefficient $\gamma(\lambda,\mu)$ of the pair of sequences $\mathcal{A}(n,1)$ and $(\mathcal{A}(n,1)^*)^{-1}$ is equal to the one of the pair of sequences $\mathcal{C}(n,1)$ and $(\mathcal{C}(n,1)^*)^{-1}$, and thus $\gamma(\lambda,\mu)=0$. This completes the proof of the theorem. $\square$

## 8. Infinite-dimensional regularity theory

We also would like to introduce an appropriate concept of regularity in infinite-dimensional spaces. We shall imitate as much as possible the finite-dimensional theory, although our approach requires several nontrivial modifications to treat this new case. We consider only separable Hilbert spaces. Due to some technical problems it is in particular unclear how to effect a corresponding theory for arbitrary Banach spaces.

### *8.1. The notion of regularity*

Let $H$ be a separable Hilbert space. We denote by $B(H)$ the space of bounded linear operators on $H$. Consider a sequence of linear operators $A_n \in B(H)$, $n \in \mathbb{N}$, with bounded inverse. We always assume that

$$\limsup_{n\to+\infty} \frac{1}{n} \log^+ \|A_n\| = 0. \tag{41}$$

We define the Lyapunov exponent $\lambda : H \to \mathbb{R} \cup \{-\infty\}$ associated with the sequence $(A_n)_{n\in\mathbb{N}}$ as in (2)–(3).

We now fix an increasing sequence of subspaces $H_1 \subset H_2 \subset \cdots$ of dimension $\dim H_n = n$ for each $n \in \mathbb{N}$, such that the closure of its union is equal to $H$. It follows from the abstract theory of Lyapunov exponents (see [1] for details) that for each $n \in \mathbb{N}$ the function $\lambda$ restricted to $H_n \setminus \{0\}$ can take at most $n$ values, say

$$-\infty < \lambda_{1,n} < \cdots < \lambda_{p_n,n}$$

for some integer $p_n \leqslant n$. Furthermore, the set

$$E_{i,n} = \big\{ v \in H_n \colon \lambda(v) \leqslant \lambda_{i,n} \big\}$$

is a linear subspace of $H_n$. We also consider the sequence of operators $(A_n^*)^{-1} \in B(H)$ and we define its associated Lyapunov exponent $\mu : H \to \mathbb{R} \cup \{-\infty\}$ as in (5)–(6). We define the *regularity coefficient* of $\lambda$ and $\mu$ by

$$\gamma(\lambda, \mu) = \sup\big\{ \gamma_n(\lambda, \mu) \colon n \in \mathbb{N} \big\},$$

where

$$\gamma_n(\lambda, \mu) = \min \max\big\{ \lambda(v_i) + \mu(w_i) \colon 1 \leqslant i \leqslant n \big\},$$

with the minimum taken over all dual bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ of the space $H_n$. It follows from Proposition 1 with $\mathbb{R}^k$ replaced by $H_n$ that $\gamma_n(\lambda, \mu) \geqslant 0$ for each $n \in \mathbb{N}$, and thus $\gamma(\lambda, \mu) \geqslant 0$. We say that the sequence $(A_n)_{n\in\mathbb{N}}$ is *Lyapunov regular* or simply *regular* if $\gamma(\lambda, \mu) = 0$.

### 8.2. Upper triangular reduction

In what follows we shall always reduce the operators in $(A_n)_{n\in\mathbb{N}}$ to a sequence of "upper triangular" operators, with respect to some fixed basis. This reduction is convenient due to the consideration of infinite-dimensional spaces. Essentially it allows us to reduce the study of an infinite-dimensional system to an infinite family of finite-dimensional systems.

More precisely, we fix an orthonormal basis $u_1, u_2, \dots$ of $H$ (recall that $H$ is a separable Hilbert space), such that $H_n = \operatorname{span}\{u_1, \dots, u_n\}$ for each $n$. We shall say that $A \in B(H)$ is *upper triangular* with respect to the basis $u_1, u_2, \dots$ if $\langle Au_i, u_j\rangle = 0$ whenever $i < j$.

The following is a version of Theorem 7 in the present setting.

**Theorem 10.** *For each sequence of operators $(A_n)_{n\in\mathbb{N}} \subset B(H)$ there exists a sequence $(U_n)_{n\in\mathbb{N}}$ of orthogonal operators such that $C_n = U_{n+1}^* A_n U_n$ is upper triangular for each $n \in \mathbb{N}$.*

**Proof.** The statement is a simple modification of the proof of Theorem 7, replacing the basis $e_1, \dots, e_k$ of $\mathbb{R}^k$ by the basis $u_1, u_2, \dots$ of $H$. $\square$

It follows from Theorem 10 and (41) that

$$\limsup_{n\to+\infty} \frac{1}{n} \log^+ \|C_n\| = \limsup_{n\to+\infty} \frac{1}{n} \log^+ \|A_n\| = 0.$$

### 8.3. Nonuniform exponential contractions

As in Section 3, for a sequence of linear operators $A_n \in B(H)$ with bounded inverse, we can introduce the notions of *nonuniform exponential contraction* (see (8)) and of *strong nonuniform exponential contraction* (see (9)). Applying Theorem 10 we obtain a sequence of upper triangular operators $(C_n)_{n\in\mathbb{N}} \subset B(H)$ associated with each given sequence of operators $(A_n)_{n\in\mathbb{N}} \subset B(H)$.

**Theorem 11.** *Assume that the sequence $(A_n)_{n\in\mathbb{N}} \subset B(H)$ has only negative Lyapunov exponents. Then for each $m \in \mathbb{N}$ and $\varepsilon > 0$, the restriction $(C_n|H_m)_{n\in\mathbb{N}}$ admits a strong nonuniform exponential contraction with*

$$\underline{a} = \lambda_{1,m} + \varepsilon, \qquad \bar{a} = \lambda_{p_m,m} + \varepsilon, \quad \text{and} \quad c = \gamma_m(\lambda, \mu) + 2\varepsilon.$$

**Proof.** Since $C_n$ is upper triangular, we have $C_n(H_m) = H_m$. Thus, we can repeat the arguments in the proof of Theorem 1, replacing $\mathbb{R}^k$ by $H_m$, to obtain the desired statement. $\square$

### 8.4. Stability for nonuniform contractions

We also obtain a version of Theorem 2 in the infinite-dimensional setting. We assume that:

H1. $(A_n)_{n\in\mathbb{N}} \subset B(H)$ is a sequence of upper triangular operators with bounded inverse;
H2. $f_n : H \to H$ are continuous functions for each $n \in \mathbb{N}$, and there exist $C > 0$ and $q > 0$ such that (12) holds for every $n \in \mathbb{N}$ and $u, v \in H$;

H3. For some sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ we have

$$\left|\left\langle f_m(u) - f_m(v), u_n\right\rangle\right| \leqslant \frac{1}{a_n}\|u-v\|\left(\|u\|^q + \|v\|^q\right)$$

for every $m, n \in \mathbb{N}$ and $u, v \in H$, with $q$ as above.

Set $F_n = A_n + f_n$ and consider the transformations $\mathcal{F}(m,n)$ in (13). Given $\delta > 0$, we also consider the set

$$D(\delta) = \left\{v \in H\colon \left|\langle v, u_m\rangle\right| < \delta/a_m \text{ for every } m \in \mathbb{N}\right\}.$$

The following is our first stability result in the infinite-dimensional setting.

**Theorem 12.** *Assume that the conditions* H1–H3 *hold, and that there exist constants* $\alpha < 0$ *and* $\kappa > 0$ *such that* $q\alpha + \kappa < 0$, *and a sequence of positive numbers* $(c_n)_{n\in\mathbb{N}}$ *with* $\sum_{n=1}^{\infty} c_n/a_n < \infty$ *such that for every* $k \in \mathbb{N}$,

$$\left\|\mathcal{A}(m,n)|H_k\right\| \leqslant c_k e^{\alpha(m-n)+\kappa n}, \quad m \geqslant n. \tag{42}$$

*Then there exist* $\delta, K > 0$ *such that for each* $v_n, \bar{v}_n \in D(\delta e^{-(1+1/q)\kappa n})$,

$$\left\|\mathcal{F}(m,n)(v_n) - \mathcal{F}(m,n)(\bar{v}_n)\right\| \leqslant K e^{\alpha(m-n)+\kappa n}\|v_n - \bar{v}_n\|, \quad m \geqslant n.$$

**Proof.** The proof is an elaboration of the proof of Theorem 2 by considering separately each finite-dimensional space $H_n$. Consider a sequence of points $v_{n+1} = F_n(v_n)$ in $H$. As in (14), we have

$$v_m = \mathcal{A}(m,n)v_n + \sum_{l=n}^{m-1} \mathcal{A}(m, l+1) f_l(w_l). \tag{43}$$

Set $\beta = (1 + 1/q)\kappa$. Given $v_n \in X$ we consider the operator

$$(Tw)_m = \mathcal{A}(m,n)v_n + \sum_{l=n}^{m-1} \mathcal{A}(m, l+1) f_l(w_l)$$

in the space

$$\mathcal{D} = \left\{w = (w_m)_{m\geqslant n}\colon \|w\|' \leqslant \delta e^{-\beta n}\right\},$$

with the norm

$$\|w\|' = \sup\left\{\|w_m\| e^{-\alpha(m-n)-\kappa n}\colon m \geqslant n\right\}.$$

One can easily verify that $\mathcal{D}$ is a complete metric space.

Let $w, \overline{w} \in \mathcal{D}$. Since $\mathcal{A}(m,n)$ is upper triangular, using (42) and condition H3, we obtain that

$$
\begin{aligned}
&\bigl\|\mathcal{A}(m,l+1)\bigl(f_l(w_l)-f_l(\overline{w}_l)\bigr)\bigr\| \\
&\quad=\left\|\mathcal{A}(m,l+1)\sum_{n=1}^{\infty}\bigl\langle f_l(w_l)-f_l(\overline{w}_l),u_n\bigr\rangle u_n\right\| \\
&\quad\leqslant\sum_{n=1}^{\infty}\bigl|\bigl\langle f_l(w_l)-f_l(\overline{w}_l),u_n\bigr\rangle\bigr|\cdot\bigl\|\mathcal{A}(m,l+1)|H_n\bigr\| \\
&\quad\leqslant\sum_{n=1}^{\infty}\frac{1}{a_n}\|w_l-\overline{w}_l\|\bigl(\|w_l\|^q+\|\overline{w}_l\|^q\bigr)c_n e^{\alpha(m-l-1)+\kappa(l+1)} \\
&\quad\leqslant 2\delta^q d\|w_l-\overline{w}_l\|e^{q\alpha(l-n)}e^{\alpha(m-l-1)+\kappa(l-n+1)},
\end{aligned}
\tag{44}
$$

where $d=\sum_{k=1}^{\infty}c_k/a_k$. We now define the operator $S$ in $\mathcal{D}$ by (15). Then,

$$
\begin{aligned}
\bigl\|(Sw)_m-(S\overline{w})_m\bigr\| &\leqslant 2\delta^q d\sum_{l=n}^{m-1}\|w_l-\overline{w}_l\|e^{(q-1)\alpha(l-n)}e^{\alpha(m-n-1)+\kappa(l-n+1)} \\
&\leqslant 2\delta^q d\|w-\overline{w}\|'\sum_{l=n}^{m-1}e^{q\alpha(l-n)}e^{\alpha(m-n-1)+\kappa(l+1)} \\
&\leqslant 2\delta^q d\|w-\overline{w}\|'e^{-\alpha+\kappa}e^{\alpha(m-n)+\kappa n}\sum_{l=n}^{\infty}e^{(q\alpha+\kappa)(l-n)} \\
&\leqslant \frac{2\delta^q d}{1-e^{q\alpha+\kappa}}\|w-\overline{w}\|'e^{-\alpha+\kappa}e^{\alpha(m-n)+\kappa n},
\end{aligned}
$$

using the condition $q\alpha+\kappa<0$. Therefore,

$$
\|Sw-S\overline{w}\|'\leqslant\frac{2\delta^q de^{-\alpha+\kappa}}{1-e^{q\alpha+\kappa}}\|w-\overline{w}\|'.
$$

We also let the operator $T$ be as in (17). We obtain

$$
\|Tw-T\overline{w}\|'\leqslant\theta\|w-\overline{w}\|'.
$$

Choosing $\delta\in(0,1)$ such that $\theta:=2\delta^q de^{-\alpha+\kappa}/(1-e^{q\alpha+\kappa})<1/2$ the operator $T$ becomes a contraction. On the other hand, for $v_n\in D(\delta e^{-\beta n}/(2d))$,

$$
\begin{aligned}
\bigl\|\mathcal{A}(m,n)v_n\bigr\| &\leqslant\lim_{j\to\infty}\sum_{k=1}^{j}\bigl|\bigl\langle v_n,u_k\bigr\rangle\bigr|\cdot\bigl\|\mathcal{A}(m,n)|H_k\bigr\| \\
&\leqslant\sum_{k=1}^{\infty}\frac{c_k}{a_k}e^{\alpha(m-n)+\kappa n}\frac{\delta}{2d}e^{-\beta n}=\frac{\delta}{2}e^{\alpha(m-n)-\kappa n/q}.
\end{aligned}
\tag{45}
$$

Since $\theta < 1/2$, setting $\overline{w} = 0$ in (33) yields

$$\|Tw\|' \leqslant \frac{\delta}{2} e^{-\beta n} + \theta \|w\|' < \delta e^{-\beta n}.$$

Therefore $T(\mathcal{D}) \subset \mathcal{D}$ and the operator $T$ is a contraction in the complete metric space $\mathcal{D}$. Hence, the sequence $(v_m)_{m \geqslant n}$ satisfying (43) is in $\mathcal{D}$.

Let $v_n, \bar{v}_n \in D(\delta e^{-\beta n}/(2d))$. Then $v_n - \bar{v}_n \in D(\delta e^{-\beta n}/d)$. For the vector

$$w = \frac{\delta e^{-\beta n}}{d} \cdot \frac{v_n - \bar{v}_n}{\|v_n - \bar{v}_n\|},$$

proceeding in a similar manner to that in (45) we obtain

$$\left\|\mathcal{A}(m,n)w\right\| \leqslant d e^{\alpha(m-n)+\kappa n} \delta e^{-\beta n}/d = d e^{\alpha(m-n)+\kappa n} \|w\|.$$

Therefore, by linearity,

$$\left\|\mathcal{A}(m,n)v_n - \mathcal{A}(m,n)\bar{v}_n\right\| \leqslant d e^{\alpha(m-n)+\kappa n} \|v_n - \bar{v}_n\|. \tag{46}$$

Set now $\rho_m$ as in (19) and $T_m = e^{-\alpha(m-n)} \rho_m$. Using (43), it follows from (46) and (44) that

$$\begin{aligned}
\rho_m &\leqslant \left\|\mathcal{A}(m,n)(v_n - \bar{v}_n)\right\| + \sum_{l=n}^{m-1} \left\|\mathcal{A}(m,l+1)\big(f_l(v_l) - f_l(\bar{v}_l)\big)\right\| \\
&\leqslant d e^{\alpha(m-n)+\kappa n} \|v_n - \bar{v}_n\| + 2\delta^q \sum_{l=n}^{m-1} e^{q\alpha(l-n)} e^{\alpha(m-l-1)+\kappa(l-n+1)} \rho_l \\
&\leqslant d e^{\alpha(m-n)} \left( e^{\kappa n} \|v_n - \bar{v}_n\| + 2\delta^q e^{-\alpha+\kappa} \sum_{l=n}^{m-1} e^{(q\alpha+\kappa)(l-n)} e^{-\alpha(l-n)} \rho_l \right),
\end{aligned}$$

and hence,

$$T_m \leqslant d \left( e^{\kappa n} \|v_n - \bar{v}_n\| + 2\delta^q e^{-\alpha+\kappa} \sum_{l=n}^{m-1} e^{(q\alpha+\kappa)(l-n)} T_l \right).$$

Set $T = \sup_{m \in \mathbb{N}} T_m$. Using the condition $q\alpha + \kappa < 0$, provided that $\delta$ is sufficiently small we obtain

$$T \leqslant d \left( e^{\kappa n} \|v_n - \bar{v}_n\| + \frac{2\delta^q e^{-\alpha+\kappa} T}{1 - e^{q\alpha+\kappa}} \right) \leqslant d e^{\kappa n} \|v_n - \bar{v}_n\| + \frac{T}{2}.$$

This completes the proof of the theorem. □

We note that Theorem 12 is a discrete time version of results in [2].

The following is another stability result.

**Theorem 13.** *Assume that the conditions* H1–H2 *hold, and that* $(A_n)_{n\in\mathbb{N}}$ *admits a nonuniform exponential contraction such that* $q\bar{a}+c<0$. *Then there exist* $\delta, K>0$ *such that for each* $v_n, \bar{v}_n \in B(\delta e^{-2(1+q)cn})$,

$$\left\|\mathcal{F}(m,n)(v_n)-\mathcal{F}(m,n)(\bar{v}_n)\right\| \leqslant K e^{\bar{a}(m-n)+cn}\|v_n-\bar{v}_n\|, \quad m \geqslant n.$$

The proof of Theorem 13 can be obtained by repeating arguments in the proof of Theorem 2.

One can also consider the notions of *nonuniform exponential dichotomy* and *strong nonuniform exponential dichotomy*, and obtain related results. In particular we can obtain an appropriate version of the stable manifold theorem (see Theorem 4), although the work is more involved since we need to deal simultaneously with the nonuniformity and the infinite-dimensionality.

## Appendix A. Upper bound for the regularity coefficient

Recall the numbers $\underline{\alpha}_i$ and $\bar{\alpha}_i$ in (25).

**Proof of Theorem 6.** Let $z_{ij}(n)$ be the entries of $\mathcal{A}(n+1,1)$, and write

$$\lambda_{ij} = \limsup_{n\to+\infty} \frac{1}{n}\log\left|z_{ij}(n)\right|.$$

**Lemma A.1.** *For every* $i, j = 1, \ldots, k$, *we have* $\lambda_{ii} = \bar{\alpha}_i$ *and*

$$\lambda_{ij} \leqslant \bar{\alpha}_j + \sum_{m=i}^{j-1}(\bar{\alpha}_m - \underline{\alpha}_m), \quad i<j.$$

**Proof.** Clearly $\lambda_{ii} = \bar{\alpha}_i$ for $i = 1, \ldots, k$. We now consider the entries with $i<j$. We proceed by backwards induction on $i$. Assume that for a given $i<k$,

$$\lambda_{lj} \leqslant \bar{\alpha}_j + \sum_{m=l}^{j-1}(\bar{\alpha}_m - \underline{\alpha}_m) \quad \text{whenever } i+1 \leqslant l \leqslant j. \tag{A.1}$$

We want to prove that for $j \geqslant i+1$,

$$\lambda_{ij} \leqslant \bar{\alpha}_j + \sum_{m=i}^{j-1}(\bar{\alpha}_m - \underline{\alpha}_m). \tag{A.2}$$

The entries $z_{ij}(n)$ are given inductively by

$$z_{ij}(n) = \begin{cases} 0 & \text{if } i>j, \\ \prod_{l=1}^{n} a_{ii}(l) & \text{if } i=j, \\ \sum_{l=i}^{j} a_{il}(n) z_{lj}(n-1) & \text{if } i<j. \end{cases} \tag{A.3}$$

We shall obtain another formula for $z_{ij}(n)$, depending on the sign of the number

$$\beta_{ij} = \bar{\alpha}_j - \underline{\alpha}_i + \sum_{m=i+1}^{j-i} (\bar{\alpha}_m - \underline{\alpha}_m). \tag{A.4}$$

Assume first that $\beta_{ij} \geqslant 0$. Using (A.3) we obtain

$$\begin{aligned}
z_{ij}(n) &= \sum_{l=i+1}^{j} a_{il}(n) z_{lj}(n-1) + a_{ii}(n) z_{ij}(n-1) \\
&= \sum_{l=i+1}^{j} a_{il}(n) z_{lj}(n-1) + a_{ii}(n) \sum_{l=i+1}^{j} a_{il}(n-1) z_{lj}(n-2) \\
&\quad + a_{ii}(n) a_{ii}(n-1) z_{ij}(n-2).
\end{aligned}$$

Proceeding by induction yields

$$z_{ij}(n) = \sum_{p=0}^{n-1} \sum_{l=i+1}^{j} a_{il}(n-p) z_{lj}(n-p-1) \prod_{m=n-p+1}^{n} a_{ii}(m). \tag{A.5}$$

Assume now that $\beta_{ij} < 0$. Again by (A.3),

$$\begin{aligned}
z_{ij}(n+2) &= \sum_{l=i+1}^{j} a_{il}(n+2) z_{lj}(n+1) + a_{ii}(n+2) \sum_{l=i+1}^{j} a_{il}(n+1) z_{lj}(n) \\
&\quad + a_{ii}(n+2) a_{ii}(n+1) z_{ij}(n).
\end{aligned}$$

By induction, for each $q \in \mathbb{N}$,

$$\begin{aligned}
&z_{ij}(n+q) \prod_{m=n+1}^{n+q} a_{ii}(m)^{-1} \\
&\quad = \sum_{p=1}^{q} \sum_{l=i+1}^{j} a_{il}(n+p) z_{lj}(n+p-1) \prod_{m=n+1}^{n+p} a_{ii}(m)^{-1} + z_{ij}(n).
\end{aligned} \tag{A.6}$$

We want to show that there exists the limit in (A.6) when $q \to +\infty$.

By (1) and (A.1), for each $\varepsilon > 0$ there exists $D > 0$ such that

$$\left|a_{il}(n)\right| \leqslant De^{\varepsilon n}, \qquad D^{-1} e^{(\underline{\alpha}_i - \varepsilon)n} \leqslant \prod_{l=1}^{n} \left|a_{ii}(l)\right| \leqslant De^{(\bar{\alpha}_i + \varepsilon)n}, \tag{A.7}$$

and

$$\left|z_{lj}(n)\right| \leqslant De^{[\bar{\alpha}_j + \sum_{m=l}^{j-1} (\bar{\alpha}_m - \underline{\alpha}_m) + \varepsilon](n+1)}$$

for every $n \geqslant 1$ and $i+1 \leqslant l \leqslant j$. Using these inequalities we obtain

$$\begin{aligned}
A &:= \sum_{p=1}^{q} \sum_{l=i+1}^{j} \left|a_{il}(n+p)z_{lj}(n+p-1)\right| \prod_{m=n+1}^{n+p} \left|a_{ii}(m)\right|^{-1} \\
&\leqslant \sum_{p=1}^{q} \sum_{l=i+1}^{j} De^{\varepsilon(n+p)} De^{[\bar{\alpha}_j+\sum_{m=l}^{j-1}(\bar{\alpha}_m-\underline{\alpha}_m)+\varepsilon](n+p)} \prod_{m=1}^{n+p} \left|a_{ii}(m)\right|^{-1} \prod_{m=1}^{n} \left|a_{ii}(m)\right| \\
&\leqslant D^3 e^{\varepsilon} \sum_{p=1}^{q} \sum_{l=i+1}^{j} e^{[\bar{\alpha}_j-\underline{\alpha}_i+\sum_{m=l}^{j-1}(\bar{\alpha}_m-\underline{\alpha}_m)+3\varepsilon](n+p)} \prod_{m=1}^{n} \left|a_{ii}(m)\right|.
\end{aligned}$$

Choose now $\varepsilon > 0$ sufficiently small so that $\beta_{ij} + 3\varepsilon < 0$. Then

$$\begin{aligned}
A &\leqslant D^3 e^{\varepsilon} k \sum_{p=1}^{q} e^{(\beta_{ij}+3\varepsilon)(n+p)} \prod_{m=1}^{n} \left|a_{ii}(m)\right| \\
&< D^3 e^{\varepsilon} k \frac{e^{(\beta_{ij}+3\varepsilon)(n+1)}}{1-e^{\beta_{ij}+3\varepsilon}} \prod_{m=1}^{n} \left|a_{ii}(m)\right| < +\infty.
\end{aligned} \tag{A.8}$$

This shows that we can let $q \to +\infty$ in (A.6) and thus,

$$z_{ij}(n) = -\sum_{p=1}^{+\infty} \sum_{l=i+1}^{j} a_{il}(n+p)z_{lj}(n+p-1) \prod_{m=n+1}^{n+p} a_{ii}(m)^{-1}.$$

We estimate $\lambda_{ij}$ starting with the case when $\beta_{ij} \geqslant 0$. Using (A.5), we obtain

$$\begin{aligned}
\lambda_{ij} &\leqslant \limsup_{n\to+\infty} \frac{1}{n} \log\left| \sum_{p=0}^{n-1} \sum_{l=i+1}^{j} a_{il}(n-p)z_{lj}(n-p-1) \prod_{m=n-p+1}^{n} a_{ii}(m)\right| \\
&\leqslant \limsup_{n\to+\infty} \frac{1}{n} \log \sum_{p=0}^{n-1} \sum_{l=i+1}^{j} D^2 e^{\varepsilon(n-p)+[\bar{\alpha}_j+\sum_{m=l}^{j-1}(\bar{\alpha}_m-\underline{\alpha}_m)+\varepsilon](n-p)} \prod_{m=1}^{n} \left|a_{ii}(m)\right| De^{(-\underline{\alpha}_i+\varepsilon)(n-p)} \\
&\leqslant \bar{\alpha}_i + \limsup_{n\to+\infty} \frac{1}{n} \log\left( kD^3 e^{3\varepsilon n} \sum_{p=0}^{n-1} e^{\beta_{ij}(n-p)} \right) \\
&= \bar{\alpha}_i + \limsup_{n\to+\infty} \frac{1}{n} \log\left( e^{\beta_{ij}} \frac{1-e^{\beta_{ij}n}}{1-e^{\beta_{ij}}} \right) + 3\varepsilon \\
&= \bar{\alpha}_j + \sum_{m=i}^{j-1} (\bar{\alpha}_m - \underline{\alpha}_m) + 3\varepsilon.
\end{aligned} \tag{A.9}$$

Assume now that $\beta_{ij} < 0$. Proceeding in a similar manner to that in (A.8) and using (A.7), we obtain

$$\begin{aligned}
\lambda_{ij} &\leqslant \limsup_{n\to+\infty} \frac{1}{n} \log\left( D^3 e^{\varepsilon} k \frac{e^{(\beta_{ij}+3\varepsilon)(n+1)}}{1-e^{\beta_{ij}+3\varepsilon}} \prod_{m=1}^{n} \left|a_{ii}(m)\right| \right) \\
&= \beta_{ij} + 3\varepsilon + \limsup_{n\to+\infty} \frac{1}{n} \log\bigl(De^{(\overline{\alpha}_i+\varepsilon)n}\bigr) \\
&= \overline{\alpha}_j + \sum_{m=i}^{j-1} (\overline{\alpha}_m - \underline{\alpha}_m) + 3\varepsilon. \qquad \text{(A.10)}
\end{aligned}$$

Since $\varepsilon$ is arbitrary, the inequality in (A.2) follows readily from the ones in (A.9) and (A.10). $\square$

We now consider the matrices $B_n$. Let $w_{ij}(n)$ be the entries of $\mathcal{B}(n+1, 1)$, and write

$$\mu_{ij} = \limsup_{n\to+\infty} \frac{1}{n} \log\left|w_{ij}(n)\right|.$$

**Lemma A.2.** *For every $i, j = 1, \dots, k$, we have $\mu_{jj} = -\underline{\alpha}_j$ and*

$$\mu_{ij} \leqslant -\underline{\alpha}_j + \sum_{m=j+1}^{i} (\overline{\alpha}_m - \underline{\alpha}_m), \quad i > j.$$

**Proof.** Clearly $\mu_{jj} = -\underline{\alpha}_j$. For the entries with $i > j$, in a similar manner to that in the proof of Lemma A.1 we assume that

$$\mu_{lj} \leqslant -\underline{\alpha}_j + \sum_{m=j+1}^{l} (\overline{\alpha}_m - \underline{\alpha}_m) \quad \text{whenever } j \leqslant l \leqslant i-1. \qquad \text{(A.11)}$$

We want to prove that

$$\mu_{ij} \leqslant -\underline{\alpha}_j + \sum_{m=j+1}^{i} (\overline{\alpha}_m - \underline{\alpha}_m).$$

When $\beta_{ji} \geqslant 0$ (see (A.4)) the entries $w_{ij}(n)$ are given inductively by

$$w_{ij}(n) = \begin{cases} 0 & \text{if } i < j, \\ \prod_{l=1}^{n} a_{ii}(l)^{-1} & \text{if } i = j, \\ -\sum_{p=0}^{n-1} \sum_{l=j}^{i-1} a_{il}(n-p) w_{lj}(n-p) \prod_{m=n-p}^{n} a_{ii}(m)^{-1} & \text{if } i > j. \end{cases} \qquad \text{(A.12)}$$

By (1) and (A.11), for each $\varepsilon > 0$ there exists $D > 0$ such that (A.7) holds as well as

$$\left|w_{lj}(n)\right| \leqslant De^{[-\underline{\alpha}_j + \sum_{m=j+1}^{l} (\overline{\alpha}_m - \underline{\alpha}_m) + \varepsilon]n}$$

for every $n \geqslant 1$ and $j \leqslant l \leqslant i-1$. Therefore,

$$\begin{aligned}
\mu_{ij} &\leqslant \limsup_{n\to+\infty}\frac{1}{n}\log\left|\sum_{p=0}^{n-1}\sum_{l=j}^{i-1}a_{il}(n-p)w_{lj}(n-p)\prod_{m=n-p}^{n}a_{ii}(m)^{-1}\right| \\
&\leqslant \limsup_{n\to+\infty}\frac{1}{n}\log\sum_{p=0}^{n-1}\sum_{l=j}^{i-1}D^2e^{\varepsilon(n-p)+[-\underline{\alpha}_j+\sum_{m=j+1}^{l}(\overline{\alpha}_m-\underline{\alpha}_m)+\varepsilon](n-p)} \\
&\quad\times\prod_{m=1}^{n}\left|a_{ii}(m)\right|^{-1}De^{(\overline{\alpha}_i+\varepsilon)(n-p)} \\
&\leqslant -\underline{\alpha}_i+\limsup_{n\to+\infty}\frac{1}{n}\log\bigl(kD^3e^{3\varepsilon n+\sum_{p=0}^{n-1}e^{\beta_{ji}(n-p)}}\bigr) \\
&= -\underline{\alpha}_j+\sum_{m=j+1}^{i}(\overline{\alpha}_m-\underline{\alpha}_m)+3\varepsilon.
\end{aligned}$$

When $\beta_{ij}<0$, proceeding in a similar manner to that in the proof of Lemma A.1 and using the third formula in (A.12), we can show that

$$w_{ij}(n)=\sum_{p=1}^{+\infty}\sum_{l=j}^{i-1}a_{il}(n+p)w_{lj}(n+p)\prod_{m=n+1}^{n+p-1}a_{ii}(m).$$

Proceeding in a similar manner to that in (A.9) we readily obtain that

$$\lambda_{ij}\leqslant -\underline{\alpha}_j+\sum_{m=j+1}^{i}(\overline{\alpha}_m-\underline{\alpha}_m)+3\varepsilon.$$

Since $\varepsilon$ is arbitrary we obtain the desired inequality. □

We proceed with the proof of the theorem. Consider the columns

$$z_j(n)=\bigl(z_{1j}(n),\dots,z_{kj}(n)\bigr)\quad\text{and}\quad w_j(n)=\bigl(w_{1j}(n),\dots,w_{kj}(n)\bigr),$$

respectively of the matrices $\mathcal{A}(n+1,1)$ and $\mathcal{B}(n+1,1)$. Let $e_1,\dots,e_k$ be the canonical basis of $\mathbb{R}^k$. We have $z_j(n)=\mathcal{A}(n+1,1)e_j$ and $w_j(n)=\mathcal{B}(n+1,1)e_j$ for each $j$. Therefore,

$$\lambda(e_j)=\max\{\lambda_{ij}\colon 1\leqslant i\leqslant k\}\leqslant\overline{\alpha}_j+\sum_{m=1}^{j-1}(\overline{\alpha}_m-\underline{\alpha}_m),$$

and

$$\mu(e_j)=\max\{w_{ij}\colon 1\leqslant i\leqslant k\}\leqslant -\underline{\alpha}_j+\sum_{m=j+1}^{n}(\overline{\alpha}_m-\underline{\alpha}_m).$$

Hence, for $j = 1, \dots, k$,

$$\lambda(e_j) + \mu(e_j) \leqslant \sum_{m=1}^{n} (\overline{\alpha}_m - \underline{\alpha}_m).$$

Since $e_1, \dots, e_k$ is dual to itself, the inequality in (26) follows readily from the definition of the regularity coefficient. □

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