THE LAGRANGIAN THEORY OF STÄCKEL SYSTEMS

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Abstract. A purely Lagrangian formulation and a direct proof of the separation of variables theorem is given for what is called Stäckel Systems in dynamics and celestial mechanics. The proof is essentially based on some properties of determinants and minors (given in Appendix A). In contrast with the standard literature on the subject, we avoid the use of the Hamiltonian, canonical transformations or the Hamilton–Jacobi equation, by using instead a more elementary approach based on the Lagrangian. In Appendix B we use the Kepler Problem as an illustration of the Lagrangian theory of Stäckel Systems.

1. Introduction

It is not known what is the most general *separable* system with n degrees of freedom. However it is known what is the most general *separable diagonal* (= orthogonal) system with n degrees of freedom. This was discovered by Stäckel (Ph.D. dissertation, Halle, 1891), and such systems are now called Stäckel Systems. The theory of Stäckel Systems can be found in Stäckel (1891, 1893, 1895), Charlier (1905), Eisenhart (1934), Pars (1949, 1965), Garfinkel (1966), or Schneider (1979). He will treat here the Lagrangian formulation, although the whole literature on the subject is with the Hamiltonian Canonical formulas. Generally the Hamiltonian is first given and next the Hamilton–Jacobi is derived from it and solved by separation of variables. Here we will not use the Hamilton–Jacobi equation but we will treat the Lagrange equations of motion directly. We will show that the Lagrange equations have n first integrals and these n integrals will be the key for the complete solution of a Stäckel System.

2. The Stäckel Matrix φ and the Stäckel Vector Ψ

In a Stäckel System with n degrees of freedom, we will consider an $(n \times n)$ matrix φ a vector Ψ with n components Ψ_r. In fact the $n^2 + n$ components of φ and Ψ completely determine the Stäckel System and for this reason we will call them the Stäckel matrix and the Stäckel vector. The components are all functions of the coordinates q_r, but in the following special way:

$$\varphi_{rs} = \varphi_{rs}(q_r), \qquad \Psi_r = \Psi_r(q_r). \tag{1}$$

In other words each *row* r of both φ and Ψ depend on only one coordinate q_r. We will say that the rows of φ are with '*separated variables*' or shorter, that the rows of φ are

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'separated'. It turns out that this separation property dominates the whole theory of Stäckel systems. Let us thus examine a few consequences of it.

First of all, in what follows, we will need the cofactors C_{ij} of the terms φ_{ij} of the matrix φ, and also the determinant Δ and the inverse v of the matrix φ. We will designate the elements of the inverse $v = \varphi^{-1}$ of the matrix φ by $(\varphi^{-1})_{ij}$ or by v_{ij}. We have several well-known properties of determinants and matrices. For instance

$$\sum_j \varphi_{ij} v_{jk} = \sum_j v_{ij} \varphi_{jk} = \delta_{ik}, \tag{2a}$$

$$C_{ij} = \Delta v_{ji} \quad \text{or} \quad v_{ji} = \frac{C_{ij}}{\Delta}, \tag{2b}$$

$$\sum_i \varphi_{ij} C_{ik} = \sum_i \varphi_{ji} C_{ki} = \Delta \delta_{jk}. \tag{2c}$$

A direct consequence of the separation property (1) is that the co-factors C_{ij} will depend on $(n-1)$ coordinates only: *C_{ij} does not contain the variable q_i*. This will simplify several partial derivatives: For instance

$$\frac{\partial \Delta}{\partial q_k} = \sum_i C_{ki} \frac{\partial \varphi_{ki}}{\partial q_k}. \tag{3}$$

3. The Lagrangian of a Stäckel System

In terms of the notations and preliminary developments given in the two previous sections we can now easily define a Stäckel System: It is a system with the following Lagrangian:

$$\boxed{L = \sum_{k=1}^{m} \left[\frac{1}{2} \frac{\dot{q}_k^2}{v_{1k}} + v_{1k} \Psi_k \right] = \sum_{k=1}^{n} v_{1k} \left[\frac{1}{2} \frac{\dot{q}_k^2}{v_{1k}^2} + \Psi_k \right].} \tag{4}$$

We see that all the ingredients are the Stäckel vector Ψ and the *first row of the inverse* of the Stäckel matrix φ. The second form of the Lagrangian shown in (4) is the product of a row-vector v_{1k} by a column-vector. The components g_{kk} of the diagonal metric tensor are thus given by

$$g_{kk} = \frac{1}{v_{1k}} = \frac{1}{(\varphi^{-1})_{1k}} = \frac{\Delta}{C_{k1}}, \qquad \left(\text{with} \quad \sum_k \frac{\varphi_{ks}}{g_{kk}} = \delta_{1s} \right). \tag{5}$$

As a consequence of the remarks of the previous section we have

$$\frac{\partial g_{kk}}{\partial q_k} = \frac{1}{C_{k1}} \frac{\partial \Delta}{\partial q_k} = \sum_i \frac{C_{ki}}{C_{k1}} \frac{\partial \varphi_{ki}}{\partial q_k}. \tag{6}$$

We easily derive the Lagrange equations of motion from the Lagrangian (4):

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\dot{q}_s}{v_{1s}} \right] = - \sum_{k=1}^{n} \left[\frac{1}{2} \frac{\dot{q}_k^2}{v_{1k}^2} - \Psi_k \right] \frac{\partial v_{1k}}{\partial q_s} + v_{1s} \frac{\partial \Psi_s}{\partial q_s}.} \tag{7}$$

The time not being present explicitly in the above Stäckel Lagrangian, we have here a conservative system with the corresponding classical energy integral:

$$\boxed{\sum_{k=1}^{n} v_{1k}\left[\frac{1}{2}\frac{\dot{q}_k^2}{v_{1k}^2} - \Psi_k\right] = \alpha_1 = \text{constant.}} \tag{8}$$

It will be useful to write this first integral also in a different form. Let us take advantage of the relation (2a) and add to (8) some terms which are zero (or one):

$$\sum_{k=1}^{n} v_{1k}\left[\frac{1}{2}\frac{\dot{q}_k^2}{v_{1k}^2} - \Psi_k\right] = \alpha_1 \sum_k v_{1k}\varphi_{k1} + \alpha_2 \sum_k v_{1k}\varphi_{k2} + \ldots + \alpha_n \sum_k v_{1k}\varphi_{kn}, \tag{9}$$

where the α's are all arbitrary constants. Grouping the terms differently gives for the energy integral the form:

$$\sum_{k=1}^{n} v_{1k}\left[\frac{1}{2}\frac{\dot{q}_k^2}{v_{1k}^2} - \Psi_k - \sum_{r=1}^{n} \varphi_{kr}\alpha_r\right] = 0. \tag{10}$$

The constants α_s are sometimes called *separation constants*. The interest of the above form of the energy integral is in the fact that the two last terms in the brackets are now with separated variables.

The most important property of Stäckel Systems resides in the following theorem: not only the expression given in (10) is zero *but also each bracket separately*:

$$\boxed{\frac{1}{2}\frac{\dot{q}_k^2}{v_{1k}^2} - \Psi_k = \sum_{r=1}^{n} \varphi_{kr}\alpha_r.} \tag{11}$$

This will be proved in the next section.

4. Proof of the Main Theorem

The equation (11) can be written in a different form: let us multiply (11) at the left-side by the matrix $v = \varphi^{-1}$. More precisely let us multiply by v_{sk} and sum in k. We find

$$\boxed{\sum_{k=1}^{n} v_{sk}\left[\frac{1}{2}\frac{\dot{q}_k^2}{v_{1k}^2} - \Psi_k\right] = \alpha_s = \text{constant;}} \quad s = 1, 2, \ldots, n. \tag{12}$$

This shows that the integral (8) holds even if the subscript *one* on the left side is replaced by any value s! This gives n first integrals of the system, and the complete solution will be derived from them.

In order to prove (12) we will show that, as a consequence of the equations of motion (7) the left-side of (12) has a zero derivative. We have to show that the following equation holds:

$$\sum_k v_{sk}\left[\frac{\dot{q}_k}{v_{1k}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{q}_k}{v_{1k}}\right)-\frac{\partial \Psi_k}{\partial q_k}\dot{q}_k\right]+\sum_k\sum_l \frac{\partial v_{sk}}{\partial q_l}\dot{q}_l\left[\frac{1}{2}\frac{\dot{q}_k^2}{v_{1k}^2}-\Psi_k\right]=0. \tag{13}$$

Eliminating the term $(\mathrm{d}/\mathrm{d}t)\ (\dot{q}_k/v_{1k})$ with the equations of motion (7) reduces the above equation to the following form:

$$\sum_k\sum_l\left\{\frac{\dot{q}_l}{v_{1l}}\left[\frac{1}{2}\frac{\dot{q}_k^2}{v_{1k}^2}-\Psi_k\right]\left[v_{1l}\frac{\partial v_{sk}}{\partial q_l}-v_{sl}\frac{\partial v_{1k}}{\partial q_l}\right]\right\}=0. \tag{14}$$

This equation can be shown to be true due to the fact that the last factor is identically zero.

Let us thus show that the elements v_{rl} of the inverse of the matrix φ satisfy the relation

$$v_{rl}\frac{\partial v_{sk}}{\partial q_l}-v_{sl}\frac{\partial v_{rk}}{\partial q_l}=0. \tag{15}$$

This is trivial when $r=s$, but we will prove that it is also true when $r\neq s$. To do it, we will express all the terms with the cofactors $C_{lr}=v_{rl}\,\Delta$. The relation (15) that has to be proven becomes

$$\frac{C_{lr}}{\Delta}\frac{\partial}{\partial q_l}\left(\frac{C_{ks}}{\Delta}\right)-\frac{C_{ls}}{\Delta}\frac{\partial}{\partial q_l}\left(\frac{C_{kr}}{\Delta}\right)=0. \tag{16}$$

Now because of the fact that C_{lr} and C_{ls} do not contain the coordinate q_l (see Section 2), the previous relation may be written as

$$\frac{\partial}{\partial q_l}\left[\frac{C_{lr}C_{ks}-C_{ls}C_{kr}}{\Delta}\right]=0. \tag{17}$$

This relation is again trivial when $l=k$ or when $r=s$ but it is also true in all other cases. This is simply a consequence of the properties of cofactors of a determinant. The quantity in the brackets of Equation (17) is known to be

$$\frac{C_{lr}C_{ks}-C_{ls}C_{kr}}{\Delta}=\frac{\partial^2\Delta}{\partial\varphi_{lr}\partial\varphi_{ks}}. \tag{18}$$

Thus this expression contains no elements of row l and row k (also: no elements of column r and column s) of the original matrix φ, because taking the partial derivative of a determinant with respect to one of it's elements removes all the terms of the corresponding row and column.

As a consequence, the expression (18) does not contain the coordinates q_l and q_k; and the partial derivative with respect to q is identically zero.

For the sake of completeness we will give an elementary proof of Equation (18) in the Appendix A.

7. Completion of the Solution of the Stäckel System

The first integrals (11) may be written in the form

$$\frac{\dot{q}_k^2}{v_{1k}^2} = 2\left[\Psi_k + \sum_r \varphi_{kr}\alpha_r\right] = f_k(q_k). \tag{19}$$

We have thus also

$$\frac{\dot{q}_k}{\sqrt{f_k(q_k)}} = v_{1k}. \tag{20}$$

Multiplying by φ_{kr} and summing in k gives

$$\sum_{k=1}^{n} \frac{\dot{q}_k \varphi_{kr}}{\sqrt{f_k(q_k)}} = \sum_k v_{1k}\varphi_{kr} = \delta_{1r}. \tag{21}$$

We see that each term in the sum on the left-side is a function of one variable q_k only: we may integrate in q_k:

$$\sum_{k=1}^{n} \int \frac{\varphi_{kr}\,\mathrm{d}q_k}{\sqrt{f_k(q_k)}} = \beta_r = \text{constant}; \qquad r = 2, 3, \ldots, n. \tag{22}$$

$$\sum_{k=1}^{n} \int \frac{\varphi_{k1}\,\mathrm{d}q_k}{\sqrt{f_k(q_k)}} = (t - t_0). \tag{23}$$

This introduces n new constants of integration; altogether $2n$ constants of integration have been introduced. The last n equations can be solved and give the n coordinates q_k as a function of the time t and these constants. The velocities are then given by the Equation (11). We have to use the Equations (12), (22), and (23) to determine the values of the constants of integration with the initial conditions.

Appendix A: Lemma On Cofactors of a Determinant

We know that the partial derivative of a determinant with respect to the element φ_{lr} is the cofactor of this term:

$$\frac{\partial \Delta}{\partial \varphi_{lr}} = C_{lr}.$$

Taking now the partial derivative of both sides of this equation with respect to another element, say φ_{ks}, and then multiplying by Δ gives:

$$\Delta \frac{\partial^2 \Delta}{\partial \varphi_{lr}\,\partial \varphi_{ks}} = \Delta \frac{\partial C_{lr}}{\partial \varphi_{ks}}.$$

The expression on the right-side of this equation may be written in a different form. For this purpose let us start from the identity

$$\Delta\delta_{ij} = \sum_k \varphi_{ik} C_{jk},$$

which we differentiate first with respect to φ_{mn}:

$$\begin{aligned}\delta_{ij}\frac{\partial\Delta}{\partial\varphi_{mn}} = \delta_{ij}C_{mn} &= \sum_k\left[C_{jk}\frac{\partial\varphi_{ik}}{\partial\varphi_{mn}} + \varphi_{ik}\frac{\partial C_{jk}}{\partial\varphi_{mn}}\right]\\ &= \sum_k\left[C_{jk}\delta_{im}\delta_{kn} + \varphi_{ik}\frac{\partial C_{jk}}{\partial\varphi_{mn}}\right]\\ &= C_{jn}\delta_{im} + \sum_k \varphi_{ik}\frac{\partial C_{jk}}{\partial\varphi_{mn}}.\end{aligned}$$

We now multiply by C_{il} and take the sum in i

$$\sum_i\left[\delta_{ij}C_{il}C_{mn} = C_{il}C_{jn}\delta_{im} + \sum_k \varphi_{ik}C_{il}\frac{\partial C_{jk}}{\partial\varphi_{mn}}\right]$$

$$C_{jl}C_{mn} = C_{ml}C_{jn} + \sum_k \Delta\delta_{kl}\frac{\partial C_{jk}}{\partial\varphi_{mn}},$$

or

$$C_{jl}C_{mn} = C_{ml}C_{jn} + \Delta\frac{\partial C_{jl}}{\partial\varphi_{mn}}.$$

We have thus finished proving the Equation (18) relating the second partial derivative of Δ to the cofactors of Δ. Similar relations could in fact be proved for all higher-order partial derivatives of Δ. We find the following determinants:

$$\frac{\partial\Delta}{\partial\varphi_{mn}} = [C_{mn}],$$

$$\Delta\frac{\partial^2\Delta}{\partial\varphi_{mn}\partial\varphi_{jl}} = \begin{bmatrix} C_{mn} & C_{ml}\\ C_{jn} & C_{jl}\end{bmatrix},$$

$$\Delta^2\frac{\partial^3\Delta}{\partial\varphi_{mn}\partial\varphi_{jl}\partial\varphi_{rs}} = \begin{bmatrix} C_{mn} & C_{ml} & C_{ms}\\ C_{jn} & C_{jl} & C_{js}\\ C_{rn} & C_{rl} & C_{rs}\end{bmatrix}.$$

A consequence of these relations is the following property of the adjoint matrix (the matrix with cofactors): If the determinant Δ of a matrix is zero, then the adjoint matrix is of rank zero or one.

Appendix B: The Two-Body Problem as a Stäckel System

The most important Stäckel System that is known deals with the three-dimensional motion of a particle, expressed in spherical coordinates (r, θ, φ) and with a rather general potential function U. The Lagrangian of such a system may be written as

$$L = T + U = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\cdot\dot{\varphi}^2) + \left(\Psi_1(r) + \frac{\Psi_2(\theta)}{r^2} + \frac{\Psi_3(\varphi)}{r^2\sin^2\theta}\right).$$

The three functions Ψ_1, Ψ_2, and Ψ_3 are arbitrary. The above Lagrangian is separable in three systems with only one degree of freedom.

The classical two-body problem is obtained by setting $\Psi_2 = \Psi_3 = 0$ and $\Psi_1 = 1/r$. This problem has thus the following simple Lagrangian in spherical coordinates:

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\cdot\dot{\varphi}^2) + \frac{1}{r}.$$

The mass factor $\mu = G_m$ has been taken equal to unity. The coordinate $q_1 = r$ is the radius-vector, $q_2 = \theta$ the colatitude and $q_3 = \varphi$ the longitude.

The Stäckel matrix is made up with the two functions $\varphi_1 = 1/r^2$ and $\varphi_2 = 1/\sin^2\theta$:

$$\varphi = \begin{bmatrix} +1 & \dfrac{-1}{r^2} & 0 \\ 0 & +1 & \dfrac{-1}{\sin^2\theta} \\ 0 & 0 & +1 \end{bmatrix}$$

The inverse of the Stäckel matrix is easily found to be

$$V = \varphi^{-1} = \begin{bmatrix} 1 & \dfrac{1}{r^2} & \dfrac{1}{r^2\sin\theta} \\ 0 & 1 & \dfrac{1}{\sin^2\theta} \\ 0 & 0 & 1 \end{bmatrix}.$$

The three components of the Stäckel vector are $\Psi_1 = 1/r$, $\Psi_2 = \Psi_3 = 0$. The Lagrange equations of motion take here the form

$$\begin{cases} \dfrac{d}{dt}[\dot{r}] = \ddot{r} = r\dot{\theta}^2 + r\sin^2\theta\cdot\dot{\varphi}^2 - \dfrac{1}{r^2}, \\ \dfrac{d}{dt}[r^2\dot{\theta}] = r^2\sin\theta\cos\theta\cdot\dot{\varphi}^2, \\ \dfrac{d}{dt}[r^2\sin^2\theta\dot{\varphi}] = 0\ \cdot \end{cases}$$

The three known first integrals have here the following very simple form:

$$\begin{cases} \dfrac{\dot r^2}{2} - \dfrac{1}{r} = \alpha_1 - \dfrac{\alpha_2}{r^2}, \\[2ex] \dfrac{r^4 \dot\theta^2}{2} = \alpha_2 - \dfrac{\alpha_3}{\sin^2\theta}, \\[2ex] \dfrac{r^4 \sin^4\theta \cdot \dot\varphi^2}{2} = \alpha_3. \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{\dot r^2}{2\left[\dfrac{1}{r} + \alpha_1 - \dfrac{\alpha_2}{r^2}\right]} = 1, \\[2ex] \dfrac{\dot\theta^2}{2\left[\alpha_2 - \dfrac{\alpha_3}{\sin^2\theta}\right]} = \dfrac{1}{r^4}, \\[2ex] \dfrac{\dot\varphi^2}{2[\alpha_3]} = \dfrac{1}{r^4 \sin^4\theta}. \end{cases}$$

They may also be written as:

$$\begin{cases} \dfrac{1}{2}[\dot r^2 + r^2\dot\theta^2 + r^2 \sin^2\theta \cdot \dot\varphi^2] - \dfrac{1}{r} = \alpha_1, \\[1ex] \dfrac{r^4}{2}[\dot\theta^2 + \dot\varphi^2 \sin^2\theta] = \alpha_2, \\[1ex] \dfrac{r^4 \sin^4\theta \cdot \dot\varphi^2}{2} = \alpha_3. \end{cases}$$

Let us note that the use of the second first integral (in α_2) allows a remarkable simplification of the equation in r. This equation transforms into:

$$\ddot r = -\frac{1}{r^2} + \frac{2\alpha_2}{r^3}.$$

It has it's own Lagrangian and energy equation:

$$L = \frac{\dot r^2}{2} + \left(\frac{1}{r} - \frac{\alpha_2}{r^2}\right); \qquad \frac{\dot r^2}{2} - \left(\frac{1}{r} - \frac{\alpha_2}{r^2}\right) = \alpha_1.$$

The integrals (22) and (23) take the form

$$\begin{cases} \displaystyle\int_{r_0}^{r} \frac{\mathrm{d}r}{\sqrt{2\left[\dfrac{1}{r} + \alpha_1 - \dfrac{\alpha_2}{r^2}\right]}} = t - t_0, \\[3ex] \displaystyle\int_{r_0}^{r} \frac{-\dfrac{\mathrm{d}r}{r^2}}{\sqrt{2\left[\dfrac{1}{r} + \alpha_1 - \dfrac{\alpha_2}{r^2}\right]}} + \int_{\theta_0}^{\theta} \frac{\mathrm{d}\theta}{\sqrt{2\left[\alpha_2 - \dfrac{\alpha_3}{\sin^2\theta}\right]}} = \beta_2, \\[3ex] \displaystyle\int_{\theta_0}^{\theta} \frac{-\dfrac{\mathrm{d}\theta}{\sin^2\theta}}{\sqrt{2\left[\alpha_2 - \dfrac{\alpha_3}{\sin^2\theta}\right]}} + \int_{\varphi_0}^{\varphi} \frac{\mathrm{d}\varphi}{\sqrt{2\alpha_3}} = \beta_3. \end{cases}$$

These equations are all well known in the theory of the two-body problem. They are usually obtained by solving the corresponding Hamilton–Jacobi equation of the problem. It has been our main purpose here to derive the Stäckel theory directly, without using the Hamilton–Jacobi equation.

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