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Approximation of the Müntz–Szász Type in Weight Spaces L^p and Zeroes of Functions of Bergman Classes in a Half-Plane

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Abstract—We study the completeness of the system of exponents $\exp(-\lambda_n t)$, $\operatorname{Re}\lambda_n > 0$, in spaces L^p with the power weight on the semiaxis $\mathbb{R}_+$. We prove a sufficient condition for the completeness; one can treat it as a modification of the well-known Szász condition. With $p = 2$ it is unimprovable (in a sense). The proof is based on the results (which are also obtained in this paper) on the distribution of zeroes of functions of the Bergman classes in a half-plane.

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Dedicated to the memory of Petr Lavrent'evich Ul'yanov on the occasion of his 80th anniversary

1. INTRODUCTION

We consider the question on the *completeness of the system of exponents*

$$(e^{-\lambda_n t})_{n=1}^{\infty}, \;\; \operatorname{Re}\lambda_n > 0, \;\; \Lambda = (\lambda_n)_{n=1}^{\infty} \tag{1}$$

in spaces $L^p_\alpha = L^p_\alpha(\mathbb{R}_+)$, consisting, by definition, of *measurable functions with the norm*

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}_+} |f(t)|^p t^\alpha dt\right)^{1/p}, \quad p \ge 1, \;\; \alpha \in \mathbb{R}.$$

The completeness of system (1) in the non-weight space $L^p_0 = L^p$ is equivalent to the *completeness of the system of exponents*

$$(x^{\mu_n})_{n=1}^{\infty}, \;\; \mu_n = \lambda_n - 1/p$$

in the space $L^p(0,1)$. The question on the completeness of a system of exponents in $L^2(0,1)$ was first studied by Müntz [1] and Szász [2]. According to the Szász theorem (in its equivalent statement) the completeness of system (1) in $L^2(\mathbb{R}_+)$ is equivalent to the condition

$$\sum_{n=1}^{\infty} \frac{\operatorname{Re}\lambda_n}{1 + |\lambda_n|^2} = \infty. \tag{2}$$

It is well-known [3] that for the completeness of system (1) in L^p_α, condition (2) is sufficient for $p > 1$, $-1 < \alpha \le \min(0, p-2)$ and necessary for $p \ge 1$, $\alpha \ge \max(0, p-2)$; it is not sufficient for $p \ge 1$, $\alpha > \min(0, p-2)$.

Establishing sufficient conditions, we [4] (see also [3]) used the *Bergman classes in a circle*. We proved the following theorem with the help of the obtained assertions about zeroes of functions from these classes.

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Theorem A. *Let $1 < p \le 2$, $\alpha > p - 2$. If*

$$\delta(\Lambda) := \limsup_{x\to+\infty} \frac{N_\Lambda(x)}{x\log x} > \frac{\alpha-(p-2)}{p}, \qquad (3)$$

where $N_\Lambda(x)$ is the number of points of the sequence Λ in the circle with the radius $\sqrt{x^2-1}$ and the center at the point $x > 1$, then system (1) is complete in L^p_α. With $p = 2$ the constant in (3) is exact.

In spite of the fact that this result is relatively complete, it should be noted that the characteristic $\delta(\Lambda)$ does not properly correspond to the geometric features of the half-plane $\operatorname{Re} w > 0$, where the sequence Λ is distributed. Clearly, this characteristic is "tied" to the unit circle (the circle which enters in it is a prototype of the circle $|z| < r = r(x)$ in the mapping $z = (w-1)/(w+1)$ of the half-plane $\operatorname{Re} w > 0$ onto the circle $|z| < 1$). Therefore, it makes sense to *establish* conditions for the completeness of system (1) in L^p_α, using a *characteristic of the sequence* Λ which more adequately (with respect to the half-plane) describes its distribution. In this paper we try to do this.

To this end, let us study the distribution of zeroes of functions from the *Bergman classes in a half-plane*. In [4] we restricted ourselves with the "transfer" of the circle onto the half-plane, but now we have to treat this question as a separate one and to describe zeroes in a way which is more proper for a half-plane.

2. ZEROES OF FUNCTIONS OF THE BERGMAN CLASS IN A HALF-PLANE

Let $p > 0$, $\alpha > -1$. Let $A^p_\alpha = A^p_\alpha(\operatorname{Re} w > 0)$ stand for the *Bergman class in the right-hand half-plane*, i.e., the class of functions which are analytic in the half-plane $\operatorname{Re} w > 0$ and satisfy the condition

$$\|F\| = \left(\iint_{\operatorname{Re} w>0} |F(u+iv)|^p u^\alpha du\, dv\right)^{1/p} < \infty, \quad w = u+iv.$$

The following theorem is true.

Theorem 1. *Let $W = (w_n)$, $\operatorname{Re} w_n > 0$ be the sequence of zeroes of a nontrivial function from the class A^p_α, $p > 0$, $\alpha > -1$. Then*

1)

$$\limsup_{x\to+0}\left(\frac{1}{\log(1/x)}\sum_{x<\operatorname{Re} w_n\le 1}\log\left|\frac{w_n-x+1}{w_n-x-1}\right|\right) \le \frac{1+\alpha}{p},$$

2) *for any $h > 1$,*

$$\Delta(W;h) := \limsup_{x\to+0}\left(\frac{1}{\log(1/x)}\sum_{x<\operatorname{Re} w_n<hx}\frac{\operatorname{Re}(w_n-x)}{1+|w_n-x|^2}\right) \le \frac{1+\alpha}{2p}. \qquad (4)$$

Lemma 1. *If $F(w) \in A^p_\alpha$, $p > 0$, $\alpha > -1$, then*

$$|F(w)| \le C_{p,\alpha}\|F\|u^{-\frac{\alpha+2}{p}}, \quad u = \operatorname{Re} w > 0.$$

Proof. Fix a point w_0, $u_0 = \operatorname{Re} w_0 > 0$. Let K_0 stand for the circle $|w-w_0| < u_0/2$. Due to the subharmonicity of the function $|F(w)|^p$ we have

$$|F(w)|^p \le \frac{4}{\pi u_0^2}\iint_{K_0}|F(w)|^p\, du\, dv.$$

Since $u_0^\alpha \le C_\alpha u^\alpha$, $w \in K_0$, we get

$$|F(w)|^p u_0^\alpha \le \frac{c_\alpha}{u_0^2}\iint_{K_0}|F(w)|^p u^\alpha du\, dv \le \frac{c_\alpha}{u_0^2}\|F\|,$$

what proves the lemma.

Lemma 2 (the Jensen formula for a half-plane). *Let a function $F(w)$ be analytic and bounded in the half-plane* $\operatorname{Re} w > 0$, *and* $F(1) \neq 0$; *let* $F(iv)$ *be the function of its boundary values and let* w_n *be its zeroes,* $\operatorname{Re} w_n > 0$. *Then*

$$\sum_{\operatorname{Re} w_n > 0} \log \left| \frac{w_n + 1}{w_n - 1} \right| = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\log |F(iv)|}{1 + v^2} dv - \log |F(1)|. \tag{5}$$

Proof. Consider the conformal mapping

$$z = \frac{w - 1}{w + 1} \tag{6}$$

of the half-plane $\operatorname{Re} w > 0$ onto the circle $|z| < 1$ and apply to the image of the function $F(w)$, i.e., to the function $\varphi(z) = F((1 + z)/(1 - z))$, the well-known Jensen formula (for a circle)

$$\sum_{|z_n| < 1} \log \frac{1}{|z_n|} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |\varphi(e^{i\theta})| d\theta - \log |\varphi(0)|, \tag{7}$$

where z_n are zeroes of the function $\varphi(z)$, and $\varphi(e^{i\theta})$ is the function of its boundary values. Since in view of (6),

$$z_n = \frac{w_n - 1}{w_n + 1}, \quad e^{i\theta} = \frac{iv - 1}{iv + 1}, \quad d\theta = -\frac{2dv}{1 + v^2},$$

formula (7) turns into that (5). □

Proof of Theorem 1. 1) Let $F(w)$ be the function, whose zeroes we seek for. Without loss of generality, assume that $F(1) = 1$. Consider the function $F_t(w) = F(w + t)$, where $t > 0$ is sufficiently small. Then $F_t(1) = 1 + o(1)$, $t \to +0$, and zeroes of the function $F_t(w)$ in the right-hand half-plane are the points $w_n - t$, $\operatorname{Re} w_n > t$. According to Lemma 1, the function $F_t(w)$ is bounded in the half-plane $\operatorname{Re} w > 0$. By Lemma 2

$$\sum_{\operatorname{Re} w_n > t} \log \left| \frac{w_n - t + 1}{w_n - t - 1} \right| = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\log |F(iv + t)|}{1 + v^2} dv + o(1), \quad t \to +0. \tag{8}$$

Let us raise the base e to the powers which equal both parts of formula (8) multiplied by p and then apply the Jensen inequality to the integral in the right-hand side. We obtain that with sufficiently small $t > 0$,

$$\exp \left(p \sum_{\operatorname{Re} w_n > t} \log \left| \frac{w_n - t + 1}{w_n - t - 1} \right| \right) \leq \frac{2}{\pi} \int_{\mathbb{R}} \frac{|F(iv + t)|^p}{1 + v^2} dv.$$

Integrating this inequality with respect to the positive measure $t^\alpha dt$ along $(0, x)$, where x is sufficiently small, we obtain the bound

$$\int_0^x t^\alpha \exp \left(p \sum_{\operatorname{Re} w_n > t} \log \left| \frac{w_n - t + 1}{w_n - t - 1} \right| \right) dt \leq \frac{2}{\pi} \iint_{0 < \operatorname{Re} w < x} \frac{|F(u + iv)|^p}{1 + v^2} u^\alpha du\, dv \leq \frac{2}{\pi} \|F\|^p. \tag{9}$$

The left-hand side will not increase if in it we consider only points w_n such that $\operatorname{Re} w_n \leq 1$. But the sum of the series, corresponding to these points, is a function, decreasing on $(0, x)$ (we will prove it soon). Therefore, in view of (9) we have

$$x^{1+\alpha} \exp \left(p \sum_{x < \operatorname{Re} w_n \leq 1} \log \left| \frac{w_n - x + 1}{w_n - x - 1} \right| \right) \leq \frac{2}{\pi} (1 + \alpha) \|F\|^p.$$

Hence, finding the logarithm and proceeding to the limit, we obtain proposition 1).

Let us prove the proposition (which we just have used) on the decrease of the function

$$S(t) = \sum_{t < \operatorname{Re} w_n \leq 1} \log \left| \frac{w_n - t + 1}{w_n - t - 1} \right|, \quad 0 < t < 1. \tag{10}$$

If $0 < t_1 < t_2$, then the set of indices n in $S(t_2)$ is a subset of the set of indices in $S(t_1)$. Therefore, it suffices to prove the decrease of each term in (10) or, equivalently, the decrease of the function

$$\left|\frac{w-t+1}{w-t-1}\right|^2 = 1 + \frac{4(u-t)}{(u-t-1)^2+v^2}, \quad w = u+iv, \ 0 < u \le 1.$$

The sign of the derivative of this function coincides with that of the function

$$-(u-t-1)^2 - v^2 + 2(u-t)(u-t-1) = (u-t)^2 - 1 - v^2.$$

The latter expression is negative, when $0 < t < u \le 1$, and the desired intermediate proposition is proved.

2) follows from 1). Indeed, proposition 1) implies that for any $h > 1$,

$$\limsup_{x\to+0} \frac{1}{2\log(1/x)} \sum_{x<u_n<hx} \log\left(1 + \frac{4(u_n-x)}{(u_n-x-1)^2}\right) \le \frac{1+\alpha}{p}, \quad w_n = u_n + iv_n. \tag{11}$$

Further, for each $\varepsilon > 0$ one can find $\delta > 0$ such that

$$\log(1+t) > (1-\varepsilon)t, \quad 0 < t < \delta. \tag{12}$$

And since $x < u_n < hx$, for any $\delta > 0$ one can find $x_0 > 0$ such that

$$0 < \frac{4(u_n-x)}{(u_n-x-1)^2+v_n^2} < \delta, \quad 0 < x < x_0. \tag{13}$$

Relations (13), (12), and (11) yield

$$\limsup_{x\to+0} \frac{1}{\log(1/x)} \sum_{x<u_n<hx} \frac{u_n-x}{(u_n-x-1)^2+v_n^2} \le \frac{1+\alpha}{2p(1-\varepsilon)}. \tag{14}$$

Since we use arbitrary $\varepsilon > 0$, relation (14) is also true with $\varepsilon = 0$. It remains to use the inequality

$$(u_n-x-1)^2 + v_n^2 = 1 + (u_n-x)^2 + v_n^2 - 2(u_n-x) < 1 + |w_n-x|^2.$$

Proposition 2) is proved. □

In order to prove that the constant in proposition 2) of Theorem 1 is exact, we need the result by Ch. Horowitz [5]. According to it, the power series

$$\varphi(z) = \sum_{n=0}^{\infty} c_n z^n$$

belongs to the *Bergman class* $A^p_\alpha(|z|<1)$ *in the unit circle*, i.e., to the class of functions which are analytic in the circle $|z| < 1$ and satisfy the condition

$$\iint_{|z|<1} |\varphi(z)|^p (1-|z|)^\alpha dx\,dy < \infty, \quad p > 0, \ \alpha > -1, \ z = x+iy. \tag{15}$$

Denote

$$S_n^{(q)} = \sum_{k=0}^{n} |c_k|^q, \quad q > 0.$$

Lemma 3 ([5]). *Let* $\alpha > -1$. *If*

$$S_n^{(q)} = O(n^s), \quad s < \frac{q}{p}(1+\alpha),$$

where $q = 2$ *for* $0 < p \le 2$ *and* $q = p/(p-1)$ *with* $p \ge 2$, *then* $\varphi(z) \in A^p_\alpha(|z|<1)$.

Note that the statement of this lemma in [5] contains the condition $\alpha \ge 0$, but the proof is true for all $\alpha > -1$.

Theorem 2. *Let $p > 0$, $\alpha > -1$. Then for any $\varepsilon > 0$ one can find a function $F(w) \in A^p_\alpha$ such that for all sufficiently large $h > 1$ the sequence of its zeroes $W = (w_n)$, $\operatorname{Re} w_n > 0$, satisfies the relation*

$$\Delta(W; h) > \frac{1+\alpha}{2p}(1-\varepsilon). \tag{16}$$

Recall that the characteristic $\Delta(W; h)$ is introduced in (4).

Proof consists in a proper modification of the proof proposed by E. Beller [6] for the class $A^p_0(|z| < 1)$. Fix $\varepsilon > 0$ and put

$$n_k = 2^{2^k}, \;\; k \in \mathbb{Z}_+, \;\; n_{-1} = 0, \;\; b_k = n_k^{\frac{1+\alpha}{p(1+\varepsilon)}}. \tag{17}$$

Introduce the infinite product

$$G(w) = \prod_{k=1}^{\infty} (1 + b_k \exp(-(n_k - n_{k-1})w)). \tag{18}$$

It converges uniformly in each half-plane of the form $\operatorname{Re} w \geq \delta > 0$, because, denoting $a = (1 + \alpha)/(p(1+\varepsilon))$, for such w and sufficiently large k we have

$$|b_k \exp(-(n_k - n_{k-1})w)| \leq n_k^a \exp(-\delta n_k/2) \leq \exp(-\delta n_k/4).$$

Consequently, the function $G(w)$ is analytic in the half-plane $\operatorname{Re} w > 0$. Consider zeroes of the function $G(w)$ generated by the term number k of the infinite product (18). They represent the roots of the equation

$$\exp(-(n_k - n_{k-1})w) = -1/b_k = -n_k^{-\frac{1+\alpha}{p(1+\varepsilon)}}.$$

Hence, clearly, they form the sequence

$$w_{k,l} = \frac{1}{n_k - n_{k-1}} \left(\frac{1+\alpha}{p\,(1+\varepsilon)} \log n_k + (2l+1)\pi i \right), \quad l \in \mathbb{Z}. \tag{19}$$

Let $\delta(x) > 0$, $\delta(x) = o(x)$, $x \to +0$. Then if k is sufficiently large, then one can find $x > 0$ such that

$$\operatorname{Re} w_{k,l} = hx - \delta(x) = hx(1 + o(1)) > x. \tag{20}$$

Let $\Sigma(x)$ stand for the sum of the series in (4). This sum will not be larger if it contains only the chain of zeroes of (19) which belong (due to (20)) to the strip $x < \operatorname{Re} w < hx$. Therefore,

$$\Sigma(x) \geq (h-1)(1+o(1))x \sum_{l \in \mathbb{Z}} \frac{1}{1 + (h-1)^2(1+o(1))x^2 + v_{k,l}^2}, \tag{21}$$

where

$$v_{k,l} = \operatorname{Im} w_{k,l} = \frac{(2l+1)\pi}{n_k - n_{k-1}}.$$

In order to estimate the sum in the right-hand side of (21), let us use the inequality

$$\sum_{l \in \mathbb{Z}} \frac{1}{A^2 + (2l+1)^2 B^2} > 2 \int_0^{\infty} \frac{dl}{A^2 + ((2l+1)B)^2} = \frac{1}{AB} \left(\frac{\pi}{2} - \arctan \frac{B}{A} \right),$$

putting

$$A = \sqrt{1 + (h-1)^2(1+o(1))x^2} = 1 + o(1), \quad B = \frac{\pi}{n_k - n_{k-1}}.$$

We obtain

$$\Sigma(x) \geq \frac{(h-1)(1+o(1))}{\pi} x(n_k - n_{k-1}) \left(\frac{\pi}{2} - \arctan \frac{\pi(1+o(1))}{n_k - n_{k-1}} \right). \tag{22}$$

Relations (20) and (19) yield

$$x = \frac{1+o(1)}{h}\operatorname{Re} w_{k,l} = \frac{(1+o(1))(1+\alpha)}{hp(1+\varepsilon)}\frac{\log n_k}{n_k - n_{k-1}}. \tag{23}$$

Hence

$$\log\frac{1}{x} = (1+o(1))\log n_k, \tag{24}$$

and substituting (23) in (22), and then doing the obtained expression and (24) under the sign of the upper limit in (4), we conclude that this upper limit is not less than

$$\frac{1+\alpha}{2p}\frac{h-1}{h(1+\varepsilon)}.$$

This value, in turn, exceeds $\frac{1+\alpha}{2p}(1-\varepsilon)$ if $h > \varepsilon^{-2}$. Inequality (16) is proved.

It remains to choose a function $H(w)$ analytic in the half-plane $\operatorname{Re} w > 0$ so that

$$F(w) := G(w)H(w) \in A^p_\alpha = A^p_\alpha(\operatorname{Re} w > 0).$$

To this end, consider the infinite product obtained from (18) with the help of the substitution $e^{-w} = z$, i.e., the function

$$\varphi(z) = \prod_{k=0}^{\infty}(1 + b_k z^{n_k - n_{k-1}}). \quad |z| < 1, \tag{25}$$

Evidently, this function is analytic in the unit circle. Let c_k be the Taylor coefficients of the function $\varphi(z)$. The following formula is true:

$$S^{(q)}_{n_m} = \prod_{k=0}^{m}(1 + b_k^q), \quad q > 0.$$

It is proved in [6] for $\alpha = 0$. However, the proof is based only on the structure of the sequence n_k, and thus it is also true for $\alpha > -1$. Taking into account the explicit form of b_k and denoting $s = q(1+\alpha)/(p(1+\varepsilon))$, where q is the value defined in Lemma 3, with the help of this formula we obtain

$$S^{(q)}_{n_m} = \prod_{k=0}^{m}(1 + b_k^{-q})\prod_{k=0}^{m} b_k^q \le M\prod_{k=0}^{m} b_k^q = M\prod_{k=0}^{m} 2^{2^k s} = M2^{s(2^{m+1}-1)} \le M2^{2^m s} = Mn_m^s.$$

Therefore, for the subsequence $S^{(q)}_{n_m}$ the assumption of Lemma 3 is fulfilled.

Let $n_{m-1} < n < n_m - n_{m-1}$ (the explicit form of n_m makes evident that the left-hand side is less than the right-hand one). Then, multiplying the terms in the infinite product (25), we make sure that $c_n = 0$. Therefore, for such n,

$$S^{(q)}_n = S^{(q)}_{n_{m-1}} \le Mn^s_{m-1} < Mn^s.$$

But if $n_m - n_{m-1} \le n < n_m$, then $n \ge n_m/2$ in view of the explicit form (17) and, consequently,

$$S^{(q)}_n \le S^{(q)}_{n_m} \le Mn^s_m \le M(2n)^s.$$

As a result, the assumption of Lemma 3 is fulfilled. According to this lemma, $\varphi(z) \in A^p_\alpha(|z| < 1)$. Thus, the integral in (15) is finite. Let us change the variables in it with the help of the univalent mapping $z = e^{-w}$ of the half-strip $P = (w \mid \operatorname{Re} w > 0,\ -\pi < \operatorname{Im} w \le \pi)$ onto the circle $|z| < 1$. We have $|z| = e^{-u}$, $x = e^{-u}\cos v$, $y = -e^{-u}\sin v$, the Jacobian of the mapping $J(u,v) = e^{-2u}$, and since $\varphi(e^{-w}) = G(w)$, formula (15) implies the finiteness of the integral

$$\iint_P |\,G(w)|^p(1-e^{-u})^\alpha e^{-2u}du\,dv.$$

Hence

$$\iint_{\mathrm{Re}\, w>0} \frac{|G(w)|^p (1-e^{-u})^\alpha e^{-2u}}{|1+w|^2} du\, dv < \infty. \tag{26}$$

Since $1-e^{-u} = u(1+o(1))$, $u \to +0$; $1-e^{-u} \to 1$, $u \to +\infty$, relation (26) means that in order to insure the inclusion $F(w) \in A^p_\alpha(\mathrm{Re}\, w > 0)$, it suffices to put

$$F(w) = G(w)e^{-\frac{2}{p}w}(1+w)^{-\frac{2}{p}}, F(w) = G(w)e^{-\frac{2}{p}w}(1+w)^{-\frac{2+\alpha}{p}}$$

correspondingly, for $-1 < \alpha \le 0$ and $\alpha > 0$. □

3. APPROXIMATION OF THE MÜNTZ–SZÁSZ TYPE

Along with Theorems 1, 2, we will use the following assertions (see, for example, [3]) on the Laplace transform

$$G(w) = \int_{\mathbb{R}_+} e^{-wt} g(t)dt, \quad \mathrm{Re}\, w > 0. \tag{27}$$

Lemma 4. *Let $q > 2$, $\beta < q-2$. If $g(t) \in L^q_\beta$, then function* (27) *belongs to the class $A^q_{q-3-\beta}$* *(*$\mathrm{Re}\, w > 0$*).*

Lemma 5. *Let $1 < q \le 2$, $\beta > -1$. If $G(w) \in A^q_\beta(\mathrm{Re}\, w > 0)$, then representation* (27) *is true with $g(t) \in L^q_{q-3-\beta}$.*

Theorem 3. 1) *Let $1 < p \le 2$, $\alpha > p-2$. If for certain $h > 1$,*

$$\Delta(\Lambda; h) > \frac{\alpha + 2 - p}{2p}, \tag{28}$$

then system (1) *is complete in L^p_α.*

2) *Let $p \ge 2$, $\alpha > p-2$. Then for any $\varepsilon > 0$ one can find a sequence $\Lambda = (\lambda_n)$,* $\mathrm{Re}\, \lambda_n > 0$*, such that*

a) *for all sufficiently large h,*

$$\Delta(\Lambda; h) > \frac{\alpha + 2 - p}{2p} - \varepsilon,$$

b) *system* (1) *is incomplete in L^p_α.*

Therefore, for $p = 2$ the constant in (28) is exact.

Proof. Let $p' = p/(p-1)$. Since $(L^p_\alpha)' = L^{p'}_{\alpha'}$, $\alpha' = -\alpha p'/p$, we conclude that the incompleteness of system (1) in L^p_α is equivalent to the existence of a nontrivial function in the form (27), where $g(t) \in L^{p'}_{\alpha'}$, vanishing at points of Λ.

1) Assume the contrary: Let system (1) be incomplete in L^p_α. Then a certain nontrivial function $G(w)$ in the form (27), where $g(t) \in L^{p'}_{\alpha'}$, vanishes at points of Λ. Therefore, if W is the sequence of all zeroes of function (27), then $W \supset \Lambda$. Conditions imposed on p, α imply that $p' \ge 2$, $\alpha' < p' - 2$. According to Lemma 4, $G(w) \in A^{p'}_{p'-3-\alpha'}$. Using condition (28) and Theorem 1, we obtain

$$\frac{\alpha+2-p}{2p} < \Delta(\Lambda; h) \le \Delta(W; h) \le \frac{1 + (p' - 3 - \alpha')}{2p'} = \frac{1}{2} - \frac{1}{p'} + \frac{\alpha}{2p},$$

i.e., $1/p + 1/p' < 1$. But this is false. Proposition 1) is proved.

2) Now $1 < p' \le 2$. Let us define β, using the condition

$$p' - 3 - \beta = \alpha' = -\alpha p'/p. \tag{29}$$

Then

$$\beta = p'(1 + \alpha/p) - 3. \tag{30}$$

Since $\alpha > p - 2$, we have $\beta > -1$, i.e., for $q = p'$, β assumptions of Lemma 5 are fulfilled. According to Theorem 2, a function $G(w) \in A^{p'}_{\beta}(\mathrm{Re}\, w > 0)$ exists such that if Λ is the sequence of all its zeroes, then with sufficiently large h the inequality

$$\Delta(\Lambda; h) > \frac{1 + \beta}{2p'} - \varepsilon$$

is true. Inserting the value β defined in formula (30) in this inequality, we obtain proposition a).

In view of Lemma 5 and formula (29), the function $G(w)$ is representable in the form (27) with $g(t) \in L^{p'}_{\alpha'}$. Consequently, proposition b) is also true. □

The following lemma (see [3]) allows us to widen the set of parameters p, α in Theorem 3.

Lemma 6. 1) *If system* (1) *is complete in* $L^{p_1}_{\alpha_1}$ *with certain* $p_1 \geq 1$, $\alpha_1 > -1$, *then it is complete in* L^p_α *with all* $p \geq p_1$, $-1 < \alpha \leq p(\alpha_1/p_1)$.

2) *If system* (1) *is incomplete in* $L^{p_1}_{\alpha_1}$ *with certain* $p_1 \geq 1$, $\alpha > -1$, *then it is incomplete in* L^p_α *with all* $1 \leq p \leq p_1$, $\alpha \geq p(\alpha_1/p_1)$.

Corollary. 1) Let $p > 2$, $\alpha > 0$. If with certain $h > 1$,

$$\Delta(\Lambda; h) > \frac{\alpha}{2p}, \tag{31}$$

then system (1) is complete in L^p_α.

2) Let $1 \leq p < 2$, $\alpha > 0$. Then with any $\varepsilon > 0$ one can find a sequence $\Lambda = (\lambda_n)$, $\mathrm{Re}\,\lambda_n > 0$, such that system (1) is incomplete in L^p_α and with all sufficiently large h,

$$\Delta(\Lambda; h) > \frac{\alpha}{2p} - \varepsilon.$$

Proof. 1) Put $p_1 = 2$, $\alpha_1 > 0$, $\alpha = p(\alpha_1/2)$. Then formula (31) turns into the condition $\Delta(\Lambda; h) > \alpha_1/4$, and by Theorem 1 (proposition 1)) system (1) is complete in $L^2_{\alpha_1}$. In accordance with proposition 1) of Lemma 6 system (1) is complete in L^p_α.

Analogously, with the help of the second propositions of Theorem 3 and Lemma 6 one can prove proposition 2). □

In conclusion, note that the cases, when points of Λ amass at the imaginary axis (see [3]), represent the most difficulty in the approximation of the Müntz–Szász type (i.e., in the theory of approximative properties of systems (1) in functional spaces on the semiaxis $\mathbb{R}_+$). A positive sense of the characteristic $\Delta(\Lambda; h)$ proposed in this paper consists in the fact that it indicates the growth of "segments" of the series mentioned in the Szász condition (2) near the imaginary axis.

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