Observations of Resonance Effects on Satellite Orbits Arising from the Thirteenth- and Fourteenth-Order Tesseral Gravitational Coefficients

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Abstract. Orbit parameters for several satellites, obtained on the basis of Doppler observations with gravity parameters through $(n, m) = (7, 6)$, yielded residuals of fit which showed a periodic error with an amplitude of 100 to 150 meters and a period of 2.5 to 5.5 days. Analysis of the residuals, which agree in period with the beat period between the orbit period and $1/m$ times the earth's rotational period relative to the plane of the orbit, yielded values for the gravity coefficients corresponding to $(n, m) = (15, 13)$, $(13, 13)$, and $(15, 14)$.

Periodic variation in the prediction errors for the polar satellite 1963 49*B* were noted by R. Newton (Johns Hopkins University, private communication), who attributed the errors to a resonance phenomenon existing between the nodal period of the satellite and the sidereal period corresponding to the gravitational harmonics ($C_{13,13}$, $S_{13,13}$). Calculations described below were then performed by S. J. Smith, R. W. Hill, and F. Rowell of the Naval Weapons Laboratory, Dahlgren, Virginia, which confirmed this hypothesis and yielded values for $C_{13,13}$, $S_{13,13}$, $C_{15,13}$, $S_{15,13}$, $C_{15,14}$, and $S_{15,14}$ on the basis of Doppler observations made on satellites 1963 49*B*, 1961o_1, and 1962$\beta\mu_1$, which have orbital inclinations of 90, 67, and 50 degrees, respectively.

To evaluate the magnitude of the effects we transformed the frequency observations for each pass of the satellite over each station to an 'along-track error,' which represents the distance the station would have to be moved parallel to the velocity vector of the satellite (at the time of closest approach of the satellite to

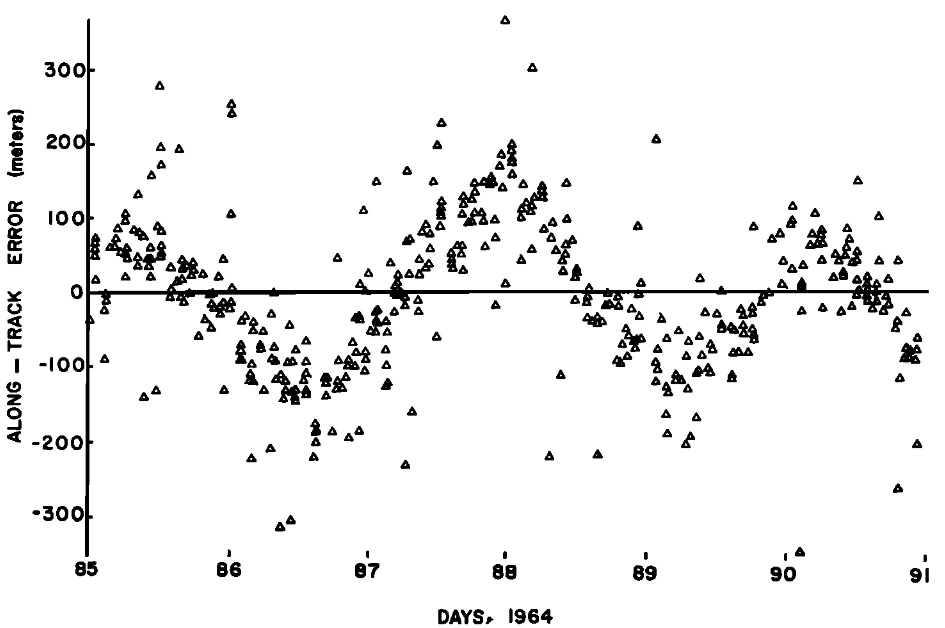

Fig. 1. Along-track errors for satellite 1963 49*B* with NWL-5A geodetic parameters.

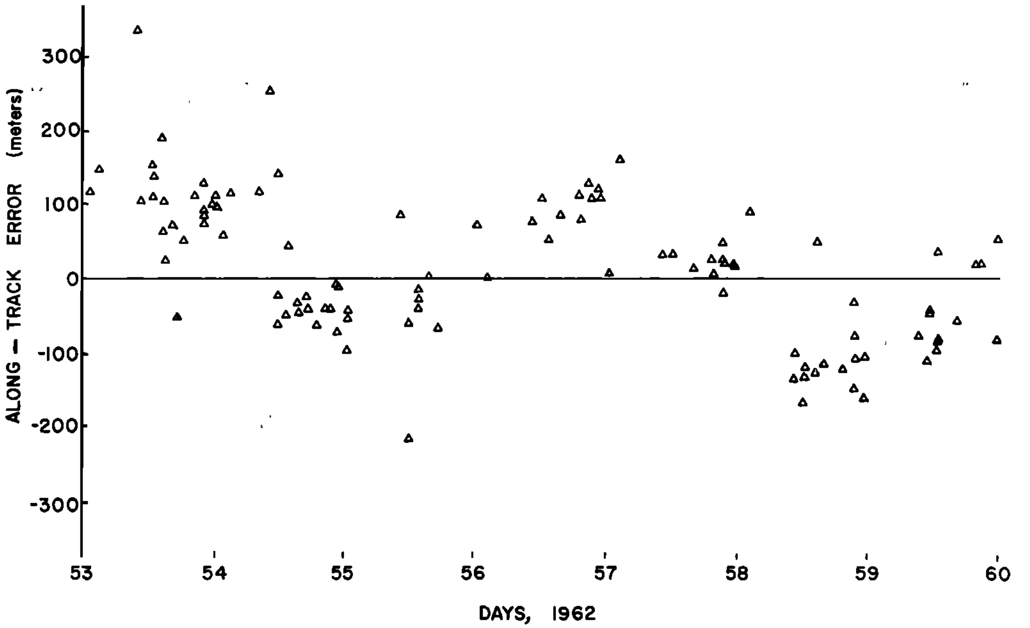

Fig. 2. Along-track errors for satellite $1961o_1$ with NWL-5A geodetic parameters.

the station) to reduce the residuals between the observed frequencies and the computed frequencies corresponding to the orbit best fitting the observations. All orbits were computed by numerical integration by means of a tenth-order Cowell process. The orbit parameters, including a drag-scaling factor, were computed on the basis of observations made in a 7-day time span at about fifteen tracking stations. The coordinates of the tracking stations and the 59 gravity coefficients used in the computations were determined in a general solution

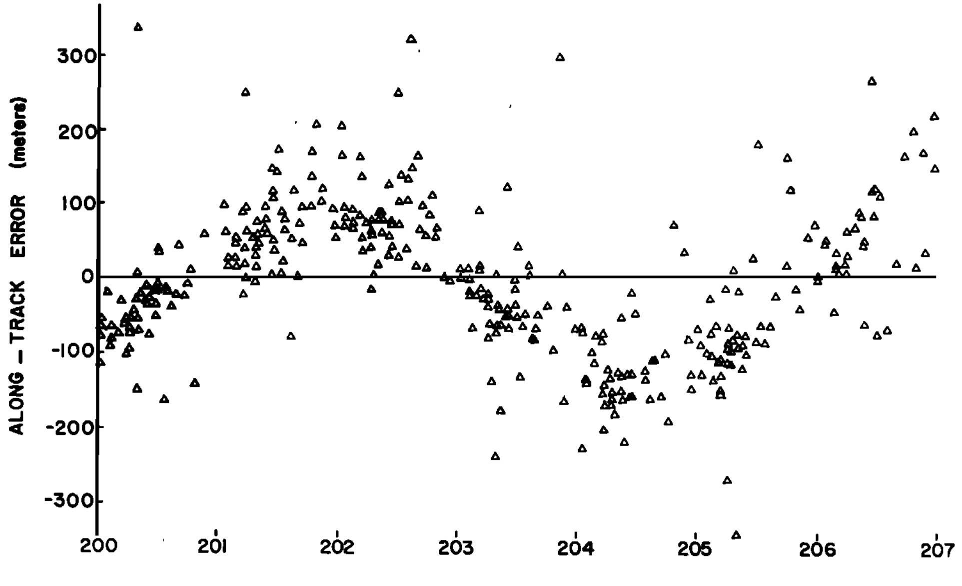

Fig. 3. Along-track errors for satellite $1962\beta\mu_1$ with NWL-5A geodetic parameters.

TABLE 1. Periods and Amplitudes of Along-Track Errors

Satellite	Inclination	Nodal Period, minutes	Period, days	Amplitude, meters
1963 49B	90°	107.2	2.5	150
1961o_1	67°	103.8	3.2	100
1962$\beta\mu_1$	50°	107.8	5.0	100

based on Doppler observations made on satellites having four different orbital inclinations. The along-track errors are shown in Figures 1, 2, and 3 for the three satellites. These figures show a periodic variation of the error (Table 1). Computations for satellite 1963 38C gave results similar to those for 1963 49B, since the orbital elements are similar. C. J. Cohen and F. Pearson (U. S. Naval Weapons Laboratory, private communication) developed the equations for the periodic effect of the (n, m) harmonic on the orbital phase angle of a satellite in a circular polar orbit. That part of the effect having quadratic divisors is

$$\tfrac{3}{2} C_{n,m} \left(\frac{R}{a}\right)^n \sum_{k=0}^{n} P_{nmk}^{1} \left\{ \frac{\sin\left[(n-2k+m\omega_E/n_0)n_0 t - (n-m)(\pi/2)\right]}{[n-2k+m\omega_E/n_0]^2} + \frac{\sin\left[(n-2k-m\omega_E/n_0)n_0 t - (n-m)(\pi/2)\right]}{[n-2k-m\omega_E/n_0]^2}\right\}$$

where

$$\begin{aligned} P_{nmk}^{1} = -\frac{m!}{2^{2n}} \sum_{2j_1 \le n-m} 2^{2j_1} \binom{2n-2j_1}{n}\binom{n}{j_1} & \cdot \left\{ m\binom{n-2j_1}{m} \sum_{j_2=0}^{m-1} \binom{m-1}{j_2} \right. \\ & \cdot \binom{n-m-2j_1+1}{k-j_1-j_2}(-1)^{k-j_1-j_2} \\ & + (m+1)\binom{n-2j_1}{m+1} \sum_{j_2=0}^{m+1} \binom{m+1}{j_2} \\ & \left. \cdot \binom{n-m-2j_1-1}{k-j_1-j_2}(-1)^{k-j_1-j_2} \right\} \end{aligned}$$

$C_{n,m}$ is the tesseral harmonic coefficient, a is the semimajor axis of the orbit, R is the earth's equatorial radius, n_0 is the reciprocal of the nodal period of the satellite, and ω_E is the sidereal rate of the earth. From the denominators it can be seen that amplification due to small quadratic divisors will occur when n is odd if $m\omega_E/n_0$ is near unity and when n is even if $m\omega_E/n_0$ is near 2. (Small linear divisors occur without quadratic divisors when n is even if $m\omega_E/n_0$ is near unity.) The period of the corresponding trigonometric terms is the beat period due to deviation of the orbital period from resonance with the period of rotation of the harmonic of order m. For orbits with heights between 750 and 1500 km, near resonance is encountered for $m = 13$, 14, or 15 (n odd and greater than or equal to m) and for $m = 26$, 27, 28, 29, or 30 (n even and greater than or equal to m). C. Sheffield (Computer Usage Corporation, private communication) showed that the same denominators and periods apply for the general inclination provided ω_E is taken to be the angular rate of the earth relative to the precessing orbit plane.

For the satellites investigated, the beat periods of the terms with small divisors are shown in Table 2, together with the periods of residuals which were shown in Table 1. The discrepancies between the theoretical beat periods and the period of the residuals probably arise primarily from a partial absorption of the periodic error in displacements of the mean motion and drag coefficient of the satellite during the least-squares adjustment of orbit parameters.

For the thirteenth-order harmonic, numerical tests revealed that the polar satellite was three times more sensitive to the coefficients for $(n, m) = (13, 13)$ than the satellite at 50 degrees in-

TABLE 2. Beat Periods of Terms with Small Divisors

Satellite	m	Theoretical Beat Period, days	Period of Residuals, days
1963 49B	13	2.5	2.5
1962$\beta\mu_1$	13	5.4	5.0
1961o_1	14	3.7	3.2

clination; but the situation was reversed for the coefficients corresponding to $(n, m) = (15, 13)$. Therefore, the coefficients for $(n, m) = (13, 13)$ were determined primarily on the basis of the observations on the polar satellite while the coefficients for $(n, m) = (15, 13)$ were found primarily on the basis of the observations on the 50 degree satellite. Since only the satellite $1961o_1$ was sensitive to the fourteenth-order coefficient, the principal error for this satellite was assumed to be due to the coefficients corresponding to $(n, m) = (15, 14)$ which were obtained by using the predetermined values for coefficients corresponding to $(n, m) = (13, 13)$ and $(15, 13)$.

The effect of the gravity coefficients on the along-track errors was computed indirectly in order to facilitate manual optimization of the coefficients. For each time of observation, a synthetic observation was computed which corresponded to the orbit that best fit the actual observations with zero values for the coefficients corresponding to $(n, m) = (13, 13)$, $(15, 13)$, and $(15, 14)$. The residuals obtained for these synthetic observations with respect to the best fit of an orbit computed with non-zero values for the coefficients are the corrections to the observed data for the assumed coefficients. Comparison of these corrections with the observed effects for various assumed coefficients eventually led to the following results:

$$C_{13,13} = -0.10 \times 10^{-19}$$

$$S_{13,13} = 0.39 \times 10^{-19}$$

$$C_{15,14} = 0.25 \times 10^{-22}$$

$$S_{15,14} = -0.65 \times 10^{-22}$$

$$C_{15,13} = -0.11 \times 10^{-20}$$

$$S_{15,13} = -0.10 \times 10^{-20}$$

where the coefficients are defined by the potential

$$V = \mu \sum \left[R^n C_{nm} \frac{P_n^{\ m}(z/r)}{r^{n+1}} \cos m\lambda + R^n S_{nm} \frac{P_n^{\ m}(z/r)}{r^{n+1}} \sin m\lambda \right]$$

where $p_n^{\ m}$ is the associated Legendre polynomial, R is the earth's radius, μ is the earth's gravity constant, λ is longitude with respect to Greenwich, and z and r are distances above the equatorial plane and from the center of the earth,

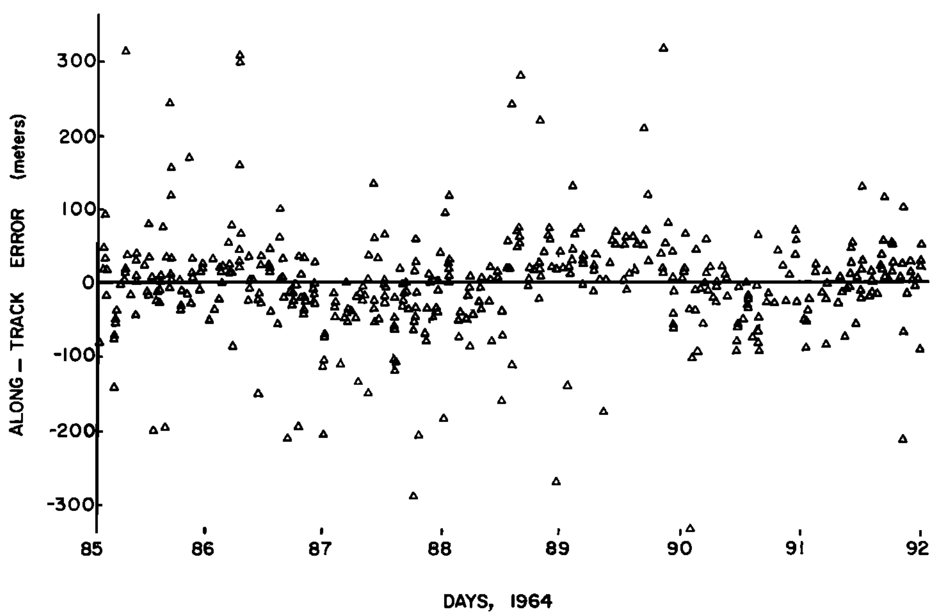

Fig. 4. Along-track errors for satellite 1963 49*B* with NWL-5E-6 geodetic parameters.

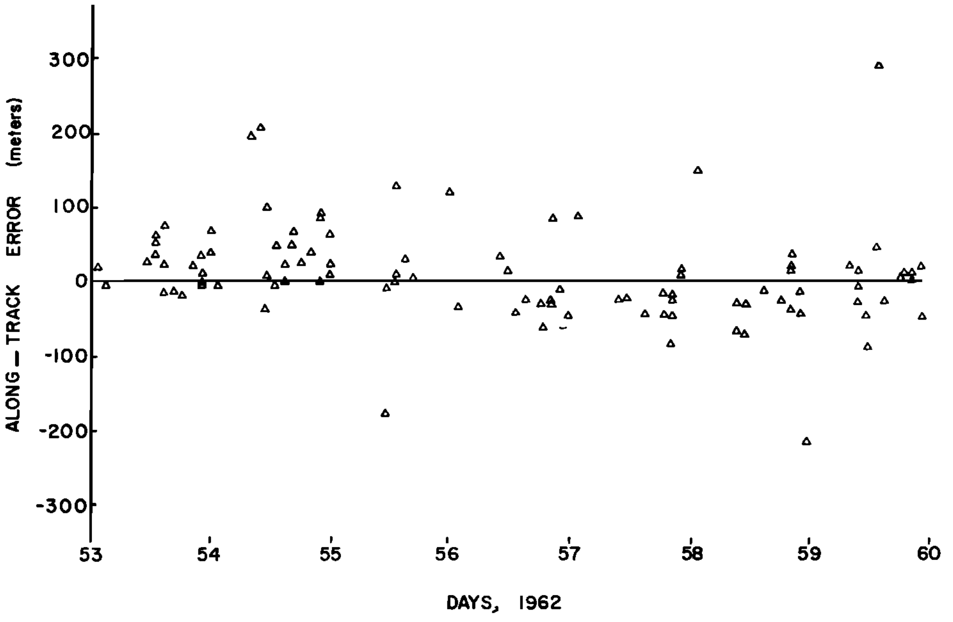

Fig. 5. Along-track errors for satellite 1961o_1 with NWL-5E-6 geodetic parameters.

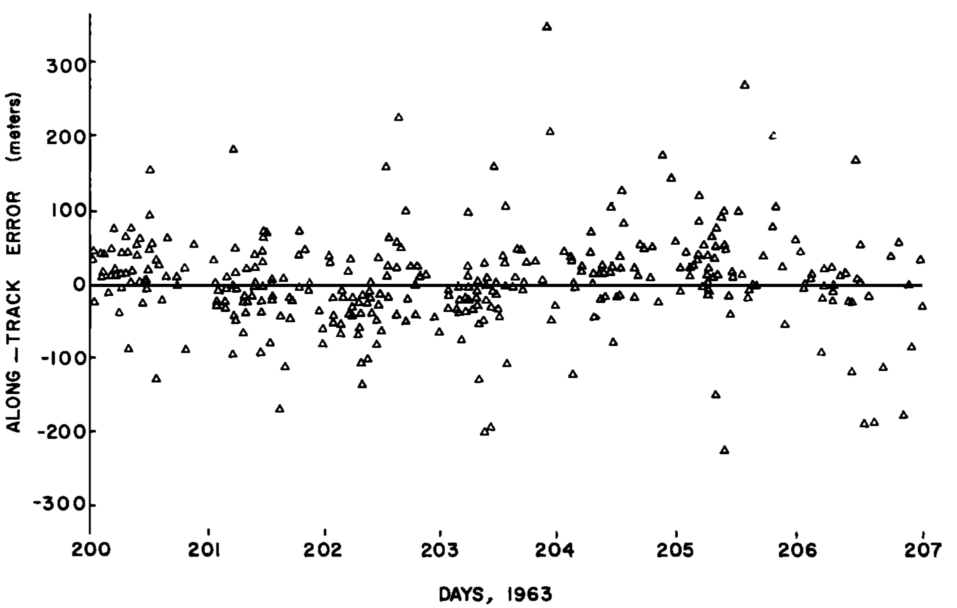

Fig. 6. Along-track errors for satellite 1962$\beta\mu_1$ with NWL-5E-6 geodetic parameters.

TABLE 3. NWL-5E-6 Normalized Gravity Coefficients*

n	m	$\bar{C}_{nm}$	$\bar{S}_{nm}$
2	0	−484.19	
3	0	0.98	
4	0	0.51	
5	0	0.05	
6	0	−0.22	
7	0	0.11	
2	1	0.02	0.06
2	2	2.45	−1.52
3	1	2.15	0.28
3	2	0.97	−0.91
3	3	0.57	1.65
4	1	−0.50	−0.58
4	2	0.27	0.67
4	3	1.00	−0.17
4	4	−0.47	0.47
5	1	0.03	−0.12
5	2	0.61	−0.31
5	3	−0.30	−0.12
5	4	−0.51	0.13
5	5	0.20	−0.41
6	1	−0.09	0.19
6	2	0.16	−0.48
6	3	−0.02	−0.14
6	4	−0.26	−0.26
6	5	−0.12	−0.74
6	6	−0.43	−0.43
7	1	0.03	0.01
7	2	0.30	−0.20
7	3	0.35	0.07
7	4	−0.47	−0.24
7	5	0.06	0.03
7	6	−0.44	−0.29
13	13	−0.03	0.11
15	13	−0.06	−0.06
15	14	0.01	−0.03

* Multiply all coefficients by 10^{-6}. $\mu = 398605.42$ km^3/sec^2.

respectively. The final residuals obtained for each satellite by using the coefficients given above are shown in Figures 4 through 6. The estimated accuracy of the determination is about 10%.

The complete set of gravity coefficients used is shown in normalized form in Table 3. The normalized coefficient $\bar{C}_{n,m} = [(n\text{-}m)!\ (2n+1)K/(n+m)!]^{-1/2}\ C_{n,m}$, where $K = 1$ when $m = 0$ and $K = 2$ when $m \neq 0$. It will be noted that the coefficients listed in Table 3 do not decrease very rapidly as the degree of the harmonics increases. However, the effects of the higher-degree hormonics on the orbit are small, except for the resonant effects. For example, the nonresonant effects of the coefficients reported above are less than 10 m. Furthermore, the orders for which simple resonance is a problem do not extend beyond about $m = 15$ because the orbit period for higher orders is so small that air drag would be more serious than the resonance problem.

For beat periods beyond simple resonance—that is, when $m\omega_B/n_0 \approx 2, 3 \ldots$—the effects of the harmonics on the orbit are small. For example, the effects of the 27th-order harmonic on the orbit of satellite 1963 49B have been observed. The beat period is 5 days, which shows that the orbit period is very close to resonance; yet the effect was found to be about 30 m. If the orbit period were closer to resonance, the effect would be greater but the beat period would be long enough so that we could treat the effect as linear instead of periodic for many applications.