Analysis of a Finite Difference Scheme for STC

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April 6, 2020

Abstract

A new finite difference scheme for the STC model is quickly analyzed to clarify the relationship between the discrete centered-difference derivative `nx1` and a reformulation `nx2`. Sufficient details are provided to make the report useful as an independent check.

1 Finite Difference Scheme

1.1 Inputs

A finite difference scheme was recorded in the attached spreadsheet `finite-difference`. Unfortunately, the attachment excludes the algebraic relations between the columns. However, a Mathematica notebook was composed to recreate the algebra, and the attached file `log-01.nb` explicitly details the relationship between relevant parameters. The spreadsheet features explicit entries for a, b, Δ, and, x_0 to allow for numerical experimentation.

1.2 Mesh

The mesh presented is an equipartition of the domain $x \in [0, 600]$ with an interval size of $\Delta = 15$. The domain is composed of the set of points

$$x = \{x_k\}_{k=0}^{40} = \{k\Delta\}_{k=0}^{40}, \qquad k = 0, 1, 2, \dots 40. \tag{1}$$

1.3 Functions of interest

The quantity `nx1` represents a centered difference formulation of the derivative $G'(x)$ while the quantity `nx2` is intended as a new formulation which suppresses large spikes in velocity. The foundation function is of Gaussian form:

$$G(x) = ae^{-bx^2}, \tag{2}$$

with the parameters given as $a = 10^{14}$ and $b = 1/5000$. For clarity, the Gaussian quantity `nx1` is called $G(x)$.

The goal of the analysis is to explore the relationship between the two putatively equivalent terms

$$\frac{G(x+\Delta) - G(x-\Delta)}{(x+\Delta)-(x-\Delta)} \qquad \text{and} \qquad G(x+\Delta)\left(\frac{\ln G(x+\Delta) - \ln G(x-\Delta)}{(x+\Delta)-(x-\Delta)}\right)$$

which we will see has a leading order equivalence.

Start by observing that the denominators evaluate to the same constant

$$(x + \Delta) - (x - \Delta) = 2\Delta,$$

and then define the functions $\eta(x)$, the classic discrete derivative, and $\eta(x)$, the reformulation:

$$\begin{aligned}\xi(x) &= \frac{G\left(x+\Delta\right) - G\left(x-\Delta\right)}{2\Delta}, \\ \eta(x) &= \frac{G\left(x+\Delta\right)}{2\Delta}\left(\ln G\left(x+\Delta\right) - \ln G\left(x-\Delta\right)\right).\end{aligned} \tag{3}$$

2 Analysis

The analysis entails computing series expansions for the two forms and comparing terms.

2.1 The Function $\xi(x)$

Using the identity

$$(x \pm \Delta)^2 = x^2 \mp 2\Delta x + \Delta^2, \tag{4}$$

the first function becomes

$$\begin{aligned}\xi(x) &= \frac{G\left(x+\Delta\right) - G\left(x-\Delta\right)}{2\Delta} \\ &= \frac{a}{\Delta} e^{-b\left(x^2+\Delta^2\right)} \left(e^{-2b\Delta x} - e^{2b\Delta x}\right).\end{aligned} \tag{5}$$

Simplification produces a form amenable to series expansion, a constant, an exponential function and the hyperbolic sine:

$$\xi(x) = \left(-\frac{2a}{\Delta} e^{-b\Delta^2}\right) e^{-bx^2} \sinh\left(2\Delta x\right) \tag{6}$$

2.2 The Function $\eta(x)$

The reformulation of the derivative uses the difference of the logarithmic terms:

$$\begin{aligned}\ln G\left(x+\Delta\right) - \ln G\left(x-\Delta\right) &= \left(a - b\left(x+\Delta\right)^2\right) - \left(a - b\left(x-\Delta\right)^2\right) \\ &= -b\left(\left(x+\Delta\right)^2 - \left(x-\Delta\right)^2\right) \\ &= -4b\Delta x\end{aligned} \tag{7}$$

Use this identity in (3) and reduce the function to a constant multiplying an exponential function:

$$\begin{aligned}\eta(x) &= -4b\Delta x \frac{G\left(x+\Delta\right)}{2\Delta} \\ &= -2abxe^{-b\left(x^2+\Delta^2-2\Delta x\right)} \\ &= \left(-2abxe^{-b\Delta^2}\right) e^{-b\left(x^2-2\Delta x\right)}\end{aligned} \tag{8}$$

2.3 Series expansions

The equivalence of the two functions is determined by a term–by–term comparison of their respective expansions. Fortunately, we only need to recall a single series expansion, and it is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}. \tag{9}$$

Two needed identities follow from this formulation. The first is the Gaussian function:

$$e^{-bx^2} = 1 - bx^2 + \frac{1}{2!}b^2x^4 - \frac{1}{3!}b^3x^6 + \cdots = 1 + \sum_{k=1}^{\infty}(-1)^k \frac{b^k x^{2k}}{k!}, \tag{10}$$

the second, the hyperbolic sine function

$$\begin{aligned} \frac{1}{2}\left(e^x - e^{-x}\right) &= \frac{1}{2}\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots\right) - \frac{1}{2}\left(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \ldots\right) \\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \\ &= \sum_{k=1}^{\infty} \frac{x^{2k-1}}{(2k-1)!}. \end{aligned} \tag{11}$$

2.4 Comparing $\xi(x)$ to $\eta(x)$

To reduce algebraic clutter for both functions, define the constant

$$\alpha = abe^{-b\Delta^2}. \tag{12}$$

2.4.1 Leading order terms for $\xi(x)$

Continue with the form of $\xi(x)$ given in (6). The two expansions to be multiplied are (10) and (11):

$$\begin{aligned} \xi(x) &= -\frac{\alpha}{b\Delta} e^{-bx^2} \sinh(2b\Delta x) \\ &= -\left(\frac{\alpha}{b\Delta}\right)\left(1 - bx^2 + \frac{1}{2!}b^2x^4 + \mathcal{O}\left(x^6\right)\right)\left(2b\Delta x + \frac{8b^3\Delta^3}{3!}x^3 + \frac{32b^5\Delta^5}{5!}x^5 + \mathcal{O}\left(x^7\right)\right) \\ &= -\left(\frac{\alpha}{b\Delta}\right)\left(2b\Delta x - 2b^2\Delta x^3 + \frac{4b^3\Delta^3}{3}x^3 + \mathcal{O}\left(x^4\right)\right) \end{aligned} \tag{13}$$

which reduces to:

$$\boxed{\xi(x) = -2\alpha x + 2\alpha b\left(1 - \tfrac{2}{3}b\Delta^2\right)x^3 + \mathcal{O}\left(x^4\right)} \tag{14}$$

2.4.2 Leading order terms for $\eta(x)$

Restate (8) using the parameter α and decompose the function into a product of basic expansions ((9) and (10)):

$$\begin{aligned}
\eta(x) &= -2\alpha x e^{-b\left(x^2-2\Delta x\right)} \\
&= -2\alpha x \left(e^{-bx^2}\right)\left(e^{2b\Delta x}\right) \\
&= -2\alpha x \left(1 - bx^2 + \frac{1}{2!}b^2x^4 + \mathcal{O}\left(x^6\right)\right)\left(1 + 2b\Delta x + 2b^2\Delta^2x^2 + \frac{4}{3}b^3\Delta^3x^3 + \mathcal{O}\left(x^4\right)\right) \\
&= -2\alpha x \left(1 + 2b\Delta x - bx^2 + 2b^2\Delta^2x^2 - 2\Delta b^2x^3 + \frac{4}{3}b^3\Delta^3x^3\right) + \mathcal{O}\left(x^4\right).
\end{aligned} \tag{15}$$

The leading order terms are

$$\boxed{\eta(x) = -2\alpha x + 4\alpha b\Delta x^2 + 2\alpha b\Delta\left(1 - 2b\Delta^2\right)x^3 + \mathcal{O}\left(x^4\right)} \tag{16}$$

3 Results

Equations (14) and (16) establish a leading order equivalence between the functions $\xi(x)$ and $\eta(x)$. The issue is when will the errors dominate. The leading terms in the error are

$$\xi(x) - \eta(x) = -4\alpha b\Delta x^2 + \tfrac{8}{3}\alpha b^2\Delta^2x^3 + \mathcal{O}(x^4). \tag{17}$$

This functions is plotted in figure 1

The constant term

$$\alpha = abe^{-b\Delta^2} \approx 1.91199 \times 10^{10}, \tag{18}$$

and the factor b controls the growth of the error term as the terms can be thought of as $\frac{\Delta x^2}{5000}$. But the error will grow quickly if the maximum value of x increases or b decreases.

4 PDF Attachments

a	1.00E+14		x	n	nx	nx1	nx1 rel err	ln n	nx2	nx2 rel err
b	0.0002		0	1.0000 E+14	0.0000 E+00			32.24		
Delta	15		15	9.5600 E+13	-5.7360 E+11	-5.4910 E+11	-4.27%	32.19	-5.7360 E+11	-1.7025 E-15
x0	0		30	8.3527 E+13	-1.0023 E+12	-9.6340 E+11	-3.88%	32.06	-1.0023 E+12	-1.7050 E-15
			45	6.6698 E+13	-1.2006 E+12	-1.1617 E+12	-3.23%	31.83	-1.2006 E+12	4.8805 E-15
			60	4.8675 E+13	-1.1682 E+12	-1.1411 E+12	-2.32%	31.52	-1.1682 E+12	-1.6719 E-15
			75	3.2465 E+13	-9.7396 E+11	-9.6285 E+11	-1.14%	31.11	-9.7396 E+11	-1.6293 E-15
			90	1.9790 E+13	-7.1244 E+11	-7.1467 E+11	0.31%	30.62	-7.1244 E+11	1.5421 E-15
			105	1.1025 E+13	-4.6305 E+11	-4.7255 E+11	2.05%	30.03	-4.6305 E+11	1.1863 E-15
			120	5.6135 E+12	-2.6945 E+11	-2.8043 E+11	4.08%	29.36	-2.6945 E+11	-1.8122 E-15
			135	2.6121 E+12	-1.4106 E+11	-1.5009 E+11	6.40%	28.59	-1.4106 E+11	-1.5145 E-15
			150	1.1109 E+12	-6.6654 E+10	-7.2679 E+10	9.04%	27.74	-6.6654 E+10	4.5785 E-16
			165	4.3178 E+11	-2.8498 E+10	-3.1917 E+10	12.00%	26.79	-2.8498 E+10	2.6772 E-16
			180	1.5338 E+11	-1.1043 E+10	-1.2733 E+10	15.30%	25.76	-1.1043 E+10	0.0000 E+00
			195	4.9796 E+10	-3.8841 E+09	-4.6202 E+09	18.95%	24.63	-3.8841 E+09	-1.2277 E-16
			210	1.4775 E+10	-1.2411 E+09	-1.5263 E+09	22.98%	23.42	-1.2411 E+09	-1.9210 E-16
			225	4.0065 E+09	-3.6059 E+08	-4.5940 E+08	27.40%	22.11	-3.6059 E+08	9.9179 E-16
			240	9.9295 E+08	-9.5323 E+07	-1.2605 E+08	32.24%	20.72	-9.5323 E+07	9.3793 E-16
			255	2.2491 E+08	-2.2940 E+07	-3.1546 E+07	37.51%	19.23	-2.2940 E+07	-4.8717 E-16
			270	4.6557 E+07	-5.0282 E+06	-7.2032 E+06	43.26%	17.66	-5.0282 E+06	0.0000 E+00
			285	8.8082 E+06	-1.0041 E+06	-1.5011 E+06	49.50%	15.99	-1.0041 E+06	-1.1594 E-16
			300	1.5230 E+06	-1.8276 E+05	-2.8558 E+05	56.26%	14.24	-1.8276 E+05	1.5925 E-16
			315	2.4067 E+05	-3.0325 E+04	-4.9608 E+04	63.59%	12.39	-3.0325 E+04	2.3994 E-16
			330	3.4759 E+04	-4.5882 E+03	-7.8695 E+03	71.52%	10.46	-4.5882 E+03	-7.9290 E-16
			345	4.5880 E+03	-6.3314 E+02	-1.1402 E+03	80.08%	8.43	-6.3314 E+02	1.7956 E-16
			360	5.5346 E+02	-7.9698 E+01	-1.5090 E+02	89.34%	6.32	-7.9698 E+01	-1.7831 E-16
			375	6.1019 E+01	-9.1529 E+00	-1.8244 E+01	99.32%	4.11	-9.1529 E+00	-1.9408 E-16
			390	6.1484 E+00	-9.5915 E-01	-2.0151 E+00	110.09%	1.82	-9.5915 E-01	-1.1575 E-16
			405	5.6620 E-01	-9.1724 E-02	-2.0336 E-01	121.71%	-0.57	-9.1724 E-02	-3.0260 E-16
			420	4.7653 E-02	-8.0057 E-03	-1.8751 E-02	134.22%	-3.04	-8.0057 E-03	-2.1669 E-16
			435	3.6654 E-03	-6.3779 E-04	-1.5798 E-03	147.71%	-5.61	-6.3779 E-04	-3.3999 E-16
			450	2.5768 E-04	-4.6382 E-05	-1.2163 E-04	162.24%	-8.26	-4.6382 E-05	1.1688 E-15
			465	1.6555 E-05	-3.0793 E-06	-8.5568 E-06	177.88%	-11.01	-3.0793 E-06	8.2523 E-16
			480	9.7210 E-07	-1.8664 E-07	-5.5010 E-07	194.73%	-13.84	-1.8664 E-07	-7.0910 E-16
			495	5.2167 E-08	-1.0329 E-08	-3.2318 E-08	212.88%	-16.77	-1.0329 E-08	-6.4066 E-16
			510	2.5586 E-09	-5.2195 E-10	-1.7351 E-09	232.42%	-19.78	-5.2195 E-10	-3.9620 E-16
			525	1.1469 E-10	-2.4084 E-11	-8.5130 E-11	253.46%	-22.89	-2.4084 E-11	-5.3664 E-16
			540	4.6984 E-12	-1.0148 E-12	-3.8171 E-12	276.12%	-26.08	-1.0148 E-12	5.9698 E-16
			555	1.7591 E-13	-3.9052 E-14	-1.5641 E-13	300.52%	-29.37	-3.9052 E-14	4.8481 E-16
			570	6.0193 E-15	-1.3724 E-15	-5.8574 E-15	326.80%	-32.74	-1.3724 E-15	2.8740 E-16
			585	1.8824 E-16	-4.4048 E-17	-2.0046 E-16	355.10%	-36.21	-4.4048 E-17	-5.5966 E-16
			600	5.3802 E-18	-1.2912 E-18			-39.76		

setup

overhead

tag

```
In[383]:= home = "ert/stc/algorithms/";
          Get["utility modules.m", Path → dirPack];
          stamp1;

          maximum memory: 0.210887 GB

          seed file: /Users/dantopa/Mathematica_files/nb/seed 19_12.nb

          user: dantopa, CPU: Xiuhcoatl,  MM v. 12.0.0 for Mac OS X x86

          date: Mar 30, 2020, time: 16:16:46

          nb: /Users/dantopa/Mathematica_files/nb/ert/stc/algorithms/log-01.nb
```

modules, functions, settings, ...

2 spreadsheet

```
In[417]:= (* column A *)
          x = Range[0, 600, 15]

Out[417]= {0, 15, 30, 45, 60, 75, 90, 105, 120, 135, 150, 165, 180, 195,
           210, 225, 240, 255, 270, 285, 300, 315, 330, 345, 360, 375, 390,
           405, 420, 435, 450, 465, 480, 495, 510, 525, 540, 555, 570, 585, 600}
```

In[411]:=
```
(* column B *)
```
$$n = 10^{14}\,\mathrm{Exp}\left[-\frac{x^2}{5000}\right];$$
```
% // N
```

Out[412]= $\{1.\times 10^{14}, 9.55997\times 10^{13}, 8.3527\times 10^{13}, 6.66977\times 10^{13}, 4.86752\times 10^{13}, 3.24652\times 10^{13}, 1.97899\times 10^{13}, 1.10251\times 10^{13}, 5.61348\times 10^{12}, 2.61214\times 10^{12}, 1.1109\times 10^{12}, 4.31784\times 10^{11}, 1.53381\times 10^{11}, 4.97955\times 10^{10}, 1.47748\times 10^{10}, 4.00653\times 10^{9}, 9.9295\times 10^{8}, 2.24906\times 10^{8}, 4.65572\times 10^{7}, 8.80818\times 10^{6}, 1.523\times 10^{6}, 240\,672., 34\,758.9, 4587.96, 553.461, 61.0194, 6.1484, 0.5662, 0.047653, 0.00366543, 0.000257676, 0.0000165552, 9.72099\times 10^{-7}, 5.21674\times 10^{-8}, 2.55859\times 10^{-9}, 1.14688\times 10^{-10}, 4.69835\times 10^{-12}, 1.75909\times 10^{-13}, 6.01928\times 10^{-15}, 1.88241\times 10^{-16}, 5.38019\times 10^{-18}\}$

In[409]:=
```
(* column c *)
```
$$nx = \frac{-2\,x\,n}{5000};$$
```
% // N
```

Out[410]= $\{0., -5.73598\times 10^{11}, -1.00232\times 10^{12}, -1.20056\times 10^{12}, -1.16821\times 10^{12}, -9.73957\times 10^{11}, -7.12435\times 10^{11}, -4.63052\times 10^{11}, -2.69447\times 10^{11}, -1.41056\times 10^{11}, -6.6654\times 10^{10}, -2.84977\times 10^{10}, -1.10434\times 10^{10}, -3.88405\times 10^{9}, -1.24109\times 10^{9}, -3.60588\times 10^{8}, -9.53232\times 10^{7}, -2.29404\times 10^{7}, -5.02817\times 10^{6}, -1.00413\times 10^{6}, -182\,760., -30\,324.7, -4588.18, -633.139, -79.6984, -9.15291, -0.95915, -0.0917243, -0.00800571, -0.000637785, -0.0000463816, -3.07927\times 10^{-6}, -1.86643\times 10^{-7}, -1.03291\times 10^{-8}, -5.21953\times 10^{-10}, -2.40844\times 10^{-11}, -1.01484\times 10^{-12}, -3.90518\times 10^{-14}, -1.3724\times 10^{-15}, -4.40484\times 10^{-17}, -1.29124\times 10^{-18}\}$

In[407]:=
```
(* column d *)
```
$$nx1 = \mathrm{Table}\left[\frac{n[[k+1]] - n[[k-1]]}{x[[k+1]] - x[[k-1]]}, \{k, 2, \mathrm{Length}[x] - 1\}\right];$$
```
% // N
```

Out[408]= $\{-5.49099\times 10^{11}, -9.63402\times 10^{11}, -1.16173\times 10^{12}, -1.14108\times 10^{12}, -9.62845\times 10^{11}, -7.14673\times 10^{11}, -4.72546\times 10^{11}, -2.8043\times 10^{11}, -1.50086\times 10^{11}, -7.26786\times 10^{10}, -3.19173\times 10^{10}, -1.27329\times 10^{10}, -4.62021\times 10^{9}, -1.5263\times 10^{9}, -4.59396\times 10^{8}, -1.26054\times 10^{8}, -3.15464\times 10^{7}, -7.20325\times 10^{6}, -1.50114\times 10^{6}, -285\,584., -49\,608., -7869.48, -1140.18, -150.898, -18.2438, -2.01511, -0.203358, -0.0187511, -0.00157985, -0.000121629, -8.55679\times 10^{-6}, -5.50102\times 10^{-7}, -3.2318\times 10^{-8}, -1.73509\times 10^{-9}, -8.51298\times 10^{-11}, -3.81706\times 10^{-12}, -1.56411\times 10^{-13}, -5.85736\times 10^{-15}, -2.00463\times 10^{-16}\}$

```
In[437]:= (* column e *)
relerror1 = Table[
    (nx1[[k]] - nx[[k + 1]]) / nx[[k + 1]]
    , {k, Length[nx1]}];
% // N

Out[438]= {-0.0427114, -0.0388318, -0.0323447, -0.0232187, -0.0114093, 0.00314109, 0.0205036,
  0.0407632, 0.0640192, 0.090386, 0.119993, 0.152988, 0.189533, 0.22981, 0.274021,
  0.322386, 0.37515, 0.432577, 0.494961, 0.562617, 0.635893, 0.715164, 0.80084,
  0.893365, 0.99322, 1.10093, 1.21706, 1.34222, 1.47708, 1.62236, 1.77883,
  1.94735, 2.12882, 2.32423, 2.53464, 2.76122, 3.00522, 3.26798, 3.55098}

(* column f *)
ln = Log[n];
% // N

Out[421]= {32.2362, 32.1912, 32.0562, 31.8312, 31.5162, 31.1112, 30.6162, 30.0312, 29.3562,
  28.5912, 27.7362, 26.7912, 25.7562, 24.6312, 23.4162, 22.1112, 20.7162, 19.2312,
  17.6562, 15.9912, 14.2362, 12.3912, 10.4562, 8.43119, 6.31619, 4.11119, 1.81619,
  -0.568809, -3.04381, -5.60881, -8.26381, -11.0088, -13.8438, -16.7688,
  -19.7838, -22.8888, -26.0838, -29.3688, -32.7438, -36.2088, -39.7638}

In[428]:= (* column g *)
nx2 = Table[
    n[[k + 1]] (ln[[k]] - ln[[k + 2]]) / (x[[k]] - x[[k + 2]])
    , {k, Length[nx1]}];
% // N
```

Out[429]= $\{-5.73598\times10^{11}, -1.00232\times10^{12}, -1.20056\times10^{12}, -1.16821\times10^{12}, -9.73957\times10^{11}, -7.12435\times10^{11}, -4.63052\times10^{11}, -2.69447\times10^{11}, -1.41056\times10^{11}, -6.6654\times10^{10}, -2.84977\times10^{10}, -1.10434\times10^{10}, -3.88405\times10^{9}, -1.24109\times10^{9}, -3.60588\times10^{8}, -9.53232\times10^{7}, -2.29404\times10^{7}, -5.02817\times10^{6}, -1.00413\times10^{6}, -182760., -30324.7, -4588.18, -633.139, -79.6984, -9.15291, -0.95915, -0.0917243, -0.00800571, -0.000637785, -0.0000463816, -3.07927\times10^{-6}, -1.86643\times10^{-7}, -1.03291\times10^{-8}, -5.21953\times10^{-10}, -2.40844\times10^{-11}, -1.01484\times10^{-12}, -3.90518\times10^{-14}, -1.3724\times10^{-15}, -4.40484\times10^{-17}\}$

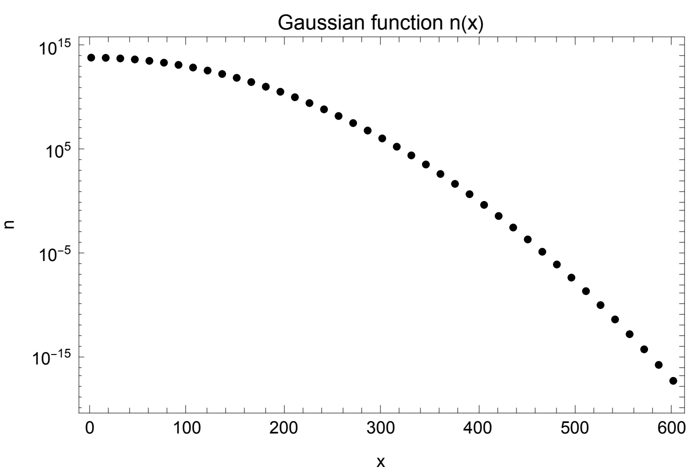
In[442]:=
```
(* column h *)
relerror2 = Table[
    (nx2[[k]] - nx[[k + 1]]) / nx[[k + 1]]
    , {k, Length[nx1]}];
% // N
```

Out[443]= $\{-1.4897\times10^{-15}, -1.58323\times10^{-15}, 4.88054\times10^{-15}, -1.6719\times10^{-15}, -1.50401\times10^{-15}, 1.88477\times10^{-15}, 1.31811\times10^{-15}, -1.58564\times10^{-15}, -1.51446\times10^{-15}, 3.43388\times10^{-16}, 2.67719\times10^{-16}, 1.72713\times10^{-16}, 0., -1.92105\times10^{-16}, -1.65299\times10^{-16}, 9.37935\times10^{-16}, 6.49561\times10^{-16}, 0., 0., -4.77739\times10^{-16}, 3.59903\times10^{-16}, 1.98226\times10^{-16}, 1.79561\times10^{-16}, 0., -1.94076\times10^{-16}, 0., 0., -2.16686\times10^{-16}, -1.69995\times10^{-16}, 0., -2.75076\times10^{-16}, 7.09102\times10^{-16}, 4.80493\times10^{-16}, -3.96195\times10^{-16}, -4.02481\times10^{-16}, -3.97989\times10^{-16}, 4.84809\times10^{-16}, -4.31104\times10^{-16}, -5.59655\times10^{-16}\}$

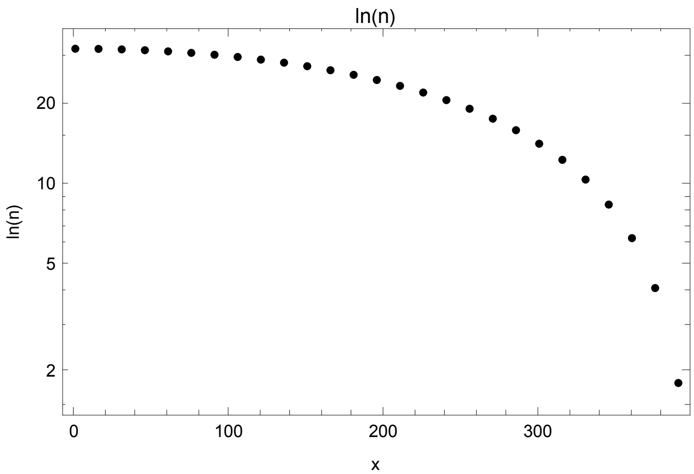
3 plots

In[436]:=
```
ListLogPlot[{x, n}ᵀ,
 PlotStyle → Black,
 PlotLabel → "Gaussian function n(x)",
 FrameLabel → {"x", "n"},
 Frame → True]
```

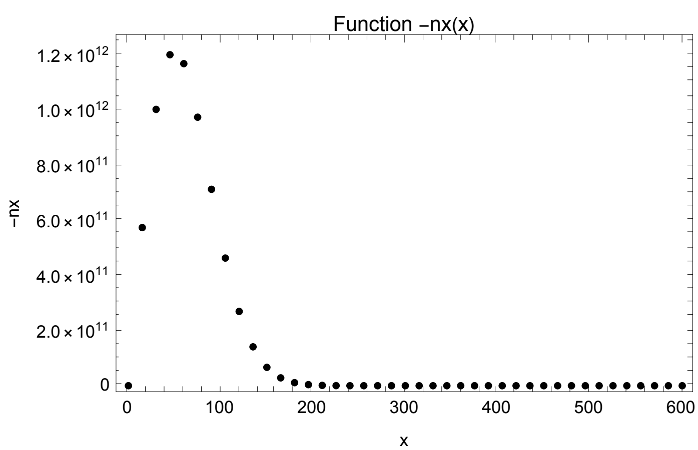
Out[436]=



In[435]:=
```
ListLogPlot[{x, ln}ᵀ,
 PlotStyle → Black,
 PlotLabel → "ln(n)",
 FrameLabel → {"x", "ln(n)"},
 Frame → True]
```

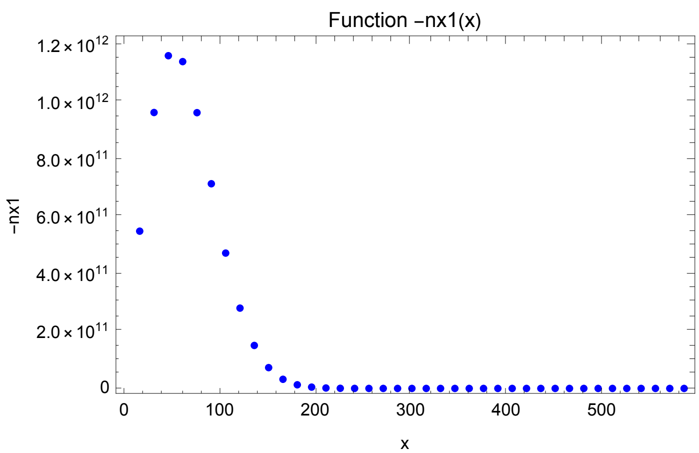
Out[435]=


In[434]:=
```
ListPlot[{x, Abs[nx]}ᵀ,
 PlotStyle → Black,
 PlotLabel → "Function -nx(x)",
 FrameLabel → {"x", "-nx"},
 PlotRange → All,
 Frame → True]
```

Out[434]=


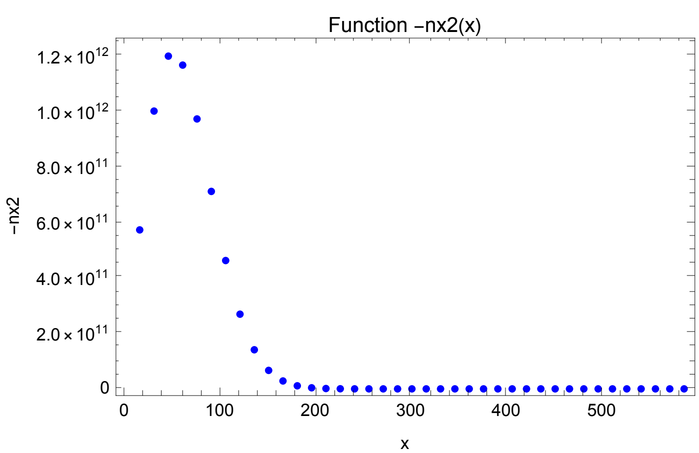
In[432]:=
```
x = Drop[Drop[x, -1], 1];
ListPlot[{x, Abs[nx1]}ᵀ,
 PlotStyle → Blue, PlotLabel → "Function -nx1(x)",
 FrameLabel → {"x", "-nx1"},
 PlotRange → All,
 Frame → True]
```

Out[433]=



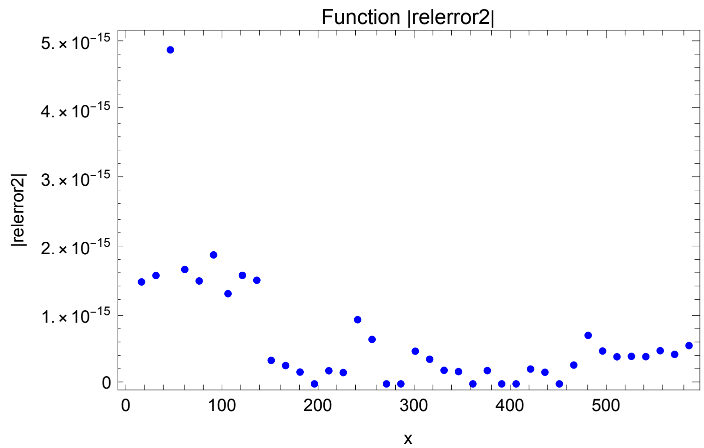
In[431]:=
```
ListPlot[{x, Abs[nx2]}ᵀ,
 PlotStyle → Blue,
 PlotLabel → "Function -nx2(x)",
 FrameLabel → {"x", "-nx2"},
 PlotRange → All,
 Frame → True]
```

Out[431]=



In[447]:=
```
ListPlot[{x, Abs[relerror2 // N]}ᵀ,
  PlotStyle → Blue,
  PlotLabel → "Function  |relerror2|",
  FrameLabel → {"x", "|relerror2|"},
  PlotRange → All,
  Frame → True]
```

Out[447]=




```
relerror2
```

end

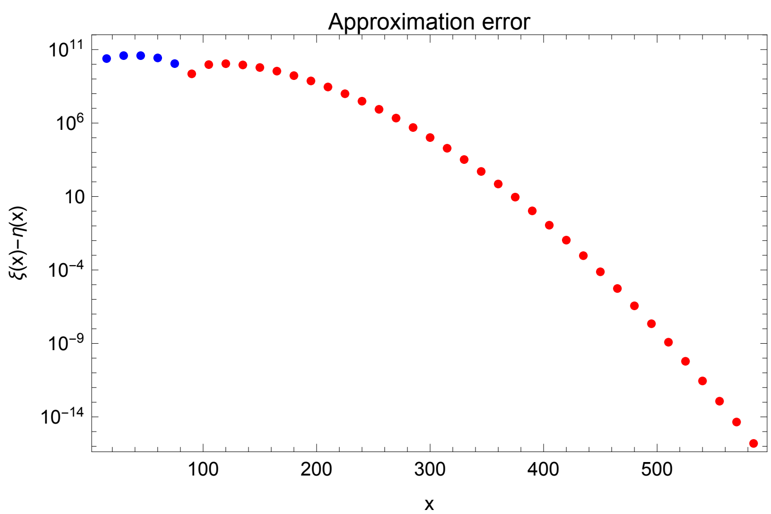


Figure 1: The error term in (17). Positive values are plotted in blue; the absolute value of the negative terms is plotted in red.