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Bertrand's theorem and virial theorem in fractional classical mechanics

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Abstract. Fractional classical mechanics is the classical counterpart of fractional quantum mechanics. The central force problem in this theory is investigated. Bertrand's theorem is generalized, and virial theorem is revisited, both in three spatial dimensions. In order to produce stable, closed, non-circular orbits, the inverse-square law and the Hooke's law should be modified in fractional classical mechanics.

1 Introduction

Fractional quantum mechanics proposed by Laskin [1–3] is a generalization of standard quantum mechanics in terms of the Riesz fractional derivative. Just as the standard Schrödinger equation arises from the Feynman path integral over Brownian paths, a path integral over the Lévy paths leads to the fractional Schrödinger equation

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t}=H_\alpha\psi(\mathbf{r},t), \tag{1}$$

with the Hamiltonian operator [3]

$$H_\alpha(\mathbf{p},\mathbf{r})=D_\alpha|\mathbf{p}|^\alpha+V(\mathbf{r}). \tag{2}$$

Here $|\mathbf{p}|=(\mathbf{p}\cdot\mathbf{p})^{1/2}$, $1<\alpha\leq 2$, and $[D_\alpha]=\mathrm{erg}^{1-\alpha}\times\mathrm{cm}^\alpha\times\mathrm{sec}^{-\alpha}$ in CGS units. In the special case at $\alpha=2$, $D_2=1/(2m)$, it recovers the standard quantum mechanics. In the literature, fractional quantum mechanics has been studied from various perspectives. Some of them involve central potentials. For the Coulomb potential $V=-Ze^2/|\mathbf{r}|$, the energy spectrum was derived in [3], and the wave function was obtained in [4]. For the so-called fractional oscillator potential $V=q^2|\mathbf{r}|^\beta$ with $1<\beta\leq 2$, $[q]=\mathrm{erg}^{1/2}\times\mathrm{cm}^{-\beta/2}$, the energy spectrum was studied in [3].

Since the fractional calculus is more intricate than ordinary calculus, solutions to the fractional Schrödinger equation are often very challenging and sometimes controversial. To understand the fractional quantum mechanics better, it would be helpful to go to the classical limit. According to Bohr's correspondence principle, classical physics and quantum physics should give the same answer when systems become large. Based on Hamiltonian (2), the classical counterpart of fractional quantum mechanics has been developed in [5], mainly in one spatial dimension. For convenience, we will call it fractional classical mechanics though it is free of fractional derivatives.

In this article, we are interested in fractional classical mechanics in three spatial dimensions. Parallel to the one-dimensional analysis in [5], one can write down the canonical Hamilton equations from eq. (2), which give Newton's second law,

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}=-\nabla V(\mathbf{r}), \tag{3}$$

and the relations between momentum $\mathbf{p}$ and velocity

$$\dot{\mathbf{r}}=\alpha D_\alpha|\mathbf{p}|^{\alpha-2}\mathbf{p},\quad \mathbf{p}=\left(\frac{1}{\alpha D_\alpha}\right)^{\frac{1}{\alpha-1}}|\dot{\mathbf{r}}|^{\frac{2-\alpha}{\alpha-1}}\dot{\mathbf{r}}. \tag{4}$$

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In this article, we will start with the above equation of motion and focus on central potentials $V(\mathbf{r}) = V(r)$, where $r = |\mathbf{r}| = (\mathbf{r}\cdot\mathbf{r})^{1/2}$. In sect. 2, we will work out the first integrals and derive the generalized Binet equation. In sect. 3, the Bertrand theorem will be extended to fractional classical mechanics, where the inverse-square law and the Hooke's law are modified. For these two modified systems, we will confirm the maximal superintegrability and the Runge-Lenz vector in sect. 4. Before making conclusion in sect. 6, we will briefly revisit the virial theorem in sect. 5.

Two comments are in order here. First, although we motivated it from fractional quantum mechanics, the fractional classical mechanics may have different physical interpretations. For instance, as we will publish elsewhere, it can be obtained as the low-momentum limit of deformed special relativity with $E^\alpha = m^\alpha c^{2\alpha} + p^\alpha c^\alpha$. Second, the results in this paper are well defined at the classical level for $\alpha > 1$. Fractional quantum mechanics is restricted to the case $1 < \alpha \leq 2$, since the Lévy distribution is stable at $0 < \alpha \leq 2$, and the first moment, such as the average momentum of quantum particles becomes infinite if $0 < \alpha \leq 1$. In contrast, we see no obstacle to fractional classical mechanics with $\alpha > 2$. However, we will not touch on the situation $0 < \alpha \leq 1$ for the reason as follows. When $\alpha = 1$, eq. (4) is ill-behaved. When $0 < \alpha < 1$, the momentum and thus the energy are divergent at zero velocity $\dot{\mathbf{r}} = 0$.

2 First integrals and generalized Binet equation

Since Hamiltonian (2) is independent of time, according to Noether's theorem, the time invariance leads to the conservation theorem of energy. To see this, we calculate the time derivative of $\dot{\mathbf{r}}\cdot\mathbf{p}$ with the help of eqs. (3), (4), which turns out to be

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha D_\alpha |\mathbf{p}|^\alpha\right) = -\alpha\dot{\mathbf{r}}\cdot\nabla V. \tag{5}$$

The first integral of this equation can be worked out easily, identified as the total energy

$$D_\alpha |\mathbf{p}|^\alpha + V = E. \tag{6}$$

Indeed, this relation may also be produced from eq. (2) with the replacement $H_\alpha \to E$, and the first term on the left-hand side can be interpreted as the kinetic energy [3].

In this article, we are interested in central potentials $V(\mathbf{r}) = V(r)$. In this case, the rotational invariance gives rise to the conservation theorem of angular momentum according to Noether's theorem. This can be seen by working out the time derivative of $\mathbf{r}\times\mathbf{p}$. From eq. (3), it is not hard to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\times\mathbf{p}) = -\mathbf{r}\times\nabla V. \tag{7}$$

For central forces, the right-hand side vanishes and the equation can be integrated directly. Its first integral, $\mathbf{r}\times\mathbf{p}$, is a conserved vector, which will be identified as the angular momentum

$$\mathbf{L} = \mathbf{r}\times\mathbf{p}. \tag{8}$$

From eq. (4), we note that momentum $\mathbf{p}$ is parallel to velocity $\dot{\mathbf{r}}$ but not proportional to it unless $\alpha \neq 2$. Therefore, in fractional classical mechanics, the Kepler's second law is generalized to the form

$$\mathbf{r}\times\dot{\mathbf{r}}|\dot{\mathbf{r}}|^{\frac{2-\alpha}{\alpha-1}} = \mathrm{const.} \tag{9}$$

This quantity generalizes the specific angular momentum vector in standard classical mechanics. Along the same logic as in standard classical mechanics, we deduce that a particle in central force field is moving in a single plane perpendicular to this vector.

Since the motion is planar and the force is radial, it would be convenient to switch to polar coordinates (r,θ), in which the angular momentum has the value

$$L = |\mathbf{L}| = \left(\frac{1}{\alpha D_\alpha}\right)^{\frac{1}{\alpha-1}} r^2\dot{\theta}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)^{\frac{2-\alpha}{2(\alpha-1)}}. \tag{10}$$

Then the total energy (6) is equivalent to

$$\frac{D_\alpha L^\alpha}{(r^2\dot{\theta})^\alpha}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)^{\frac{\alpha}{2}} + V = E. \tag{11}$$

In terms of reciprocal radius $u = 1/r$, we can rewrite it as

$$D_\alpha L^\alpha\left[\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + u^2\right]^{\frac{\alpha}{2}} + V = E. \tag{12}$$

This equation can be used to derive the shape of orbit for a given potential, or reconstruct the potential from a specified orbit conversely.

Differentiating eq. (12) with respect to θ and making use of

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+u^2\right]=2\left(\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u\right)\frac{\mathrm{d}u}{\mathrm{d}\theta}\,, \tag{13}$$

we go back to the high-order differential equation of motion

$$\alpha D_\alpha L^\alpha\left(\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u\right)\left[\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+u^2\right]^{\frac{\alpha-2}{2}}+\frac{\mathrm{d}V(1/u)}{\mathrm{d}u}=0. \tag{14}$$

This equation can be regarded as the generalized Binet equation in fractional classical mechanics, but it is not as useful as the standard Binet equation because it is highly nonlinear when $\alpha\neq 2$. Instead, it is more convenient to work with eq. (12). Given a central potential $V(r)$, the particle's orbit can be obtained by integrating eq. (12) formally,

$$\theta=\int\mathrm{d}u\left\{\frac{1}{L^2}\left[\frac{E-V(1/u)}{D_\alpha}\right]^{\frac{2}{\alpha}}-u^2\right\}^{-\frac{1}{2}}. \tag{15}$$

3 Generalized Bertrand's theorem

Armed with the results above, we can now move towards one main goal of this article: looking for the necessary and sufficient condition to produce stable, exactly closed orbits in fractional classical mechanics. In standard classical mechanics, this is captured by Bertrand's theorem, which states that the only central forces that result in closed orbits for all bound particles are the inverse-square law and the linear Hooke's law [6]. In the current section, we will see this theorem can be generalized to fractional classical mechanics with some modifications.

The key point is to observe that eq. (12) can be transformed to

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u=J(u), \tag{16}$$

if we define

$$J(u)=\frac{1}{2L^2}\frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{E-V(1/u)}{D_\alpha}\right]^{\frac{2}{\alpha}}. \tag{17}$$

This definition is equivalent to the integral form

$$V=E-D_\alpha L^\alpha\left[2\int\mathrm{d}uJ(u)\right]^{\frac{\alpha}{2}}. \tag{18}$$

Starting with eq. (16) and following the standard proof of Bertrand's theorem [6,7], we arrive at two choices:

$$J(u)=k,\ \frac{k}{u^3}\,. \tag{19}$$

In both cases, k is a positive constant. For each possibility, one can obtain the potential with the aid of eq. (18), and then the orbit utilizing eq. (15). This will be done in the rest of this section.

In the first case, $J=k$, eq. (18) can be integrated out to give

$$V=E-D_\alpha L^\alpha\left(2ku+C\right)^{\frac{\alpha}{2}}, \tag{20}$$

where C is the constant of integration. The Newtonian or Coulomb law can be reproduced from this potential by choosing $\alpha=2$ and $E=D_2L^2C$. Inserting eq. (20) into eq. (15), we get the equation of orbit

$$u=k+\left(C+k^2\right)^{\frac{1}{2}}\cos\theta. \tag{21}$$

This is exactly the polar equation of conic sections with a focus at the origin. The semi-latus rectum is $1/k$ and the eccentricity is $\sqrt{1+Ck^{-2}}$.

In the second case, $J = k/u^3$, by eqs. (15), (18), one can show that the potential and the orbit take the form

$$V = E - D_\alpha L^\alpha \left(C - \frac{k}{u^2}\right)^{\frac{\alpha}{2}}, \tag{22}$$

$$u^2 = \frac{C}{2} + \left(\frac{C^2}{4} - k\right)^{\frac{1}{2}} \cos(2\theta). \tag{23}$$

This potential reduces to the Hooke's law at $\alpha = 2$. The constant of integration E can be fixed by setting the potential to zero in the origin $r = 0$. The orbit equation is exactly the ellipse's polar equation with the origin at the center. The lengths of the semi-major and semi-minor axes are, respectively, $(\sqrt{C + 2\sqrt{k}} \pm \sqrt{C - 2\sqrt{k}})/\sqrt{4k}$.

4 Superintegrability and Runge-Lenz vector

In standard classical mechanics, the closed one-dimensional orbits of a mechanical system can be attributed to the maximal superintegrability. That is to say, the system has n degrees of freedom and $2n - 1$ constants of motion. For example, the standard Kepler problem is maximally superintegrable. With three degrees of freedom, it has five independent constants of motion: energy E, three components of angular momentum $\mathbf{L}$ and an independent component of the Runge-Lenz vector

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \frac{kL^2\mathbf{r}}{r}. \tag{24}$$

It would be interesting to investigate if this superintegrability property is also true in the fractional classical mechanics and which is the form of the new conserved quantities[1]. Indeed, the questions can be answered by extending the result of [8], as we will do in this section.

For preparation, let us recall some useful equalities:

$$\mathbf{r} \cdot \dot{\mathbf{r}} = r\dot{r}, \qquad \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \dot{r}^2 + r^2\dot{\theta}^2, \qquad \nabla V(r) = \frac{\mathbf{r}}{r}\frac{\mathrm{d}V}{\mathrm{d}r}. \tag{25}$$

From eqs. (4), (10), it is easy to derive $\mathrm{d}u/\mathrm{d}\theta = -\mathbf{r} \cdot \mathbf{p}/(Lr)$. Then one can follow ref. [8] to get a unit Runge-Lenz vector,

$$\frac{\mathbf{A}}{|\mathbf{A}|} = \frac{1}{L^2}\frac{\mathrm{d}\cos\theta}{\mathrm{d}u}\mathbf{p} \times \mathbf{L} + \left(\cos\theta - u\frac{\mathrm{d}\cos\theta}{\mathrm{d}u}\right)\frac{\mathbf{r}}{r}, \tag{26}$$

normal to $\mathbf{L}$ apparently, where $\cos\theta$ is taken as a function of u according to the orbit equation like eqs. (21), (23). In the following, we will apply this formula to eqs. (21), (23) and multiply it by a function of L and E. As we will see, the result is a conserved vector.

For the first case in sect. 3, substituting eq. (21) into (26), we find that eq. (24) remains a candidate for the Runge-Lenz vector. Calculating its time derivative with the help of eqs. (3), (4), (8), one obtains

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \left(r\frac{\mathrm{d}V}{\mathrm{d}r} - \frac{\alpha D_\alpha |\mathbf{p}|^{\alpha-2}kL^2}{r}\right)\left(\mathbf{p} - \frac{\dot{r}}{\alpha D_\alpha |\mathbf{p}|^{\alpha-2}}\frac{\mathbf{r}}{r}\right). \tag{27}$$

At the same time, the magnitude of $\mathbf{A}$ is

$$|\mathbf{A}| = L\left(|\mathbf{p}|^2 - \frac{2kL^2}{r} + k^2L^2\right)^{\frac{1}{2}}. \tag{28}$$

To proceed, we compare eq. (6) with (20) and find a relation $|\mathbf{p}| = L(2ku + C)^{1/2}$. As a result, whith the potential V given by eq. (20), the right-hand side of eq. (27) vanishes, and eq. (28) becomes $|\mathbf{A}| = L^2(C + k^2)^{1/2}$. Therefore, eq. (24) is a qualified Runge-Lenz vector in this case.

For the second case in sect. 3, substituting eq. (23) into (26) leads to

$$\mathbf{A} = \left\{1 + \left[\left(\frac{C^2}{4} - k\right)^{\frac{1}{2}} - \frac{C}{2}\right]r^2\right\}^{-\frac{1}{2}}\left\{\mathbf{p} \times \mathbf{L} + \left[\left(\frac{C^2}{4} - k\right)^{\frac{1}{2}} - \frac{C}{2}\right]L^2\mathbf{r}\right\}. \tag{29}$$

[1] We are grateful to the referee for pointing it out.

In terms of a brief notation,

$$u_-^2 = \frac{C}{2} - \left(\frac{C^2}{4} - k\right)^{\frac{1}{2}}, \tag{30}$$

we can make use of eqs. (3), (4), (8) and $|\mathbf{r} \times \mathbf{p}|^2 + (\mathbf{r} \cdot \mathbf{p})^2 = |\mathbf{r}|^2|\mathbf{p}|^2$ to verify

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \left(1 - u_-^2 r^2\right)^{-\frac{3}{2}} \left[\left(1 - u_-^2 r^2\right) r \frac{\mathrm{d}V}{\mathrm{d}r} - \alpha D_\alpha |\mathbf{p}|^{\alpha-2} u_-^2 r^2 \left(|\mathbf{p}|^2 - u_-^2 L^2\right)\right] \left(\mathbf{p} - \frac{\dot{r}}{\alpha D_\alpha |\mathbf{p}|^{\alpha-2}} \frac{\mathbf{r}}{r}\right), \tag{31}$$

$$|\mathbf{A}| = \left(1 - u_-^2 r^2\right)^{-\frac{1}{2}} L \left(|\mathbf{p}|^2 - 2u_-^2 L^2 + u_-^4 L^2 r^2\right)^{\frac{1}{2}}. \tag{32}$$

Note that $u_-^2(C - u_-^2) = k$. It follows, from eqs. (6), (22), that $|\mathbf{p}| = L(C - kr^2)^{1/2}$ and thus $\mathrm{d}\mathbf{A}/\mathrm{d}t = 0$, $|\mathbf{A}| = L^2(C^2 - 4k)^{1/2}$. It confirms that eq. (29) is an appropriate generalization of the Runge-Lenz vector in this case.

In both cases, there are three degrees of freedom. Five independent constants of motion are obtained from the energy E, the angular momentum $\mathbf{L}$ and the Runge-Lenz vector $\mathbf{A}$ subject to $\mathbf{A} \cdot \mathbf{L} = 0$ and a constraint on $|\mathbf{A}|$. According to discussions in the beginning of this section, the two systems are maximally superintegrable.

5 Generalized virial theorem

The generalization of virial theorem to fractional classical mechanics and fractional quantum mechanics is straightforward, see its application to the fractional Bohr atom in [3]. Let us revisit it briefly here for completeness.

Consider a collection of N particles, whose total kinetic energy is given by

$$T = \sum_{j=1}^{N} D_\alpha |\mathbf{p}_j|^\alpha, \tag{33}$$

where $\mathbf{p}_j$ is the momentum of the j-th particle. Then similar to standard classical mechanics, the virial theorem can be derived by studying the time derivative of

$$G = \sum_{j=1}^{N} \mathbf{r}_j \cdot \mathbf{p}_j. \tag{34}$$

Applying eqs. (3), (4) to each particle, it is straightforward to show

$$\begin{aligned}\frac{\mathrm{d}G}{\mathrm{d}t} &= \sum_{j=1}^{N} \dot{\mathbf{r}}_j \cdot \mathbf{p}_j + \sum_{j=1}^{N} \mathbf{r}_j \cdot \frac{\mathrm{d}\mathbf{p}_j}{\mathrm{d}t} \\ &= \alpha T - \sum_{j=1}^{N} \mathbf{r}_j \cdot \nabla_{\mathbf{r}_j} V. \end{aligned}\tag{35}$$

For periodic motion, or provided that the coordinates and velocities of all particles remain finite after a sufficiently long time τ, the time average of eq. (35) leads to

$$\langle T \rangle_\tau = \frac{1}{\alpha} \sum_{j=1}^{N} \langle \mathbf{r}_j \cdot \nabla_{\mathbf{r}_j} V \rangle_\tau, \tag{36}$$

which generalizes the virial theorem to fractional classical mechanics. Especially, if V is a power-law function of r, $V = kr^{n+1}$, then this equation becomes

$$\langle T \rangle_\tau = \frac{n+1}{\alpha} \langle V \rangle_\tau. \tag{37}$$

The same result holds also in fractional quantum mechanics and has been applied to the fractional Bohr atom in [3].

It is also interesting to consider an ideal gas composed of N particles with known volume and temperature. As demonstrated in [9], the equipartition theorem is enhanced by a factor $2/\alpha$ in fractional classical mechanics. Then one can use the virial theorem to derive the equation of state [6], in which the parameter α disappears, thanks to the cancellation of $2/\alpha$ with the same factor on the right-hand side of eq. (36).

6 Conclusion

In this article, based on central force problem in fractional classical mechanics, we have revised two well-known theorems in standard classical mechanics: Bertrand's theorem and virial theorem. Although the Binet equation becomes nonlinear in fractional classical mechanics, luckily there is a linear equation (16), from which we succeeded in generalizing the Bertrand's theorem. As indicated by eqs. (20), (21), (22), (23), there remain non-circular orbits bounded and closed for arbitrary initial conditions, but the required potentials are more complicated. The revised virial theorem, eq. (36), is in agreement with that in [3] for a power-law potential, and is useful in statistical mechanics.

Bertrand's theorem and virial theorem are playing crucial roles in celestial mechanics, thus their generalized version presented here should have subtle and profound implications to astrophysics. Their revised form, as well as other results, such as eqs. (9), (14), (20) in this article, can be utilized to constrain the value of α with observational data in turn.

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