

# Kepler's Law

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## Abstract

Kepler's law is a cornerstone of orbital mechanics.

## Contents

### 1 Overview

Kepler's law is a cornerstone of orbital mechanics.

1. env
2. ldd

### 2 Derivation

$$\frac{du^2}{d^2\lambda} + u = \frac{\mu}{h}$$

### 3 Geometry of Kepler's Law

Disagreement with this YouTuber True Anomaly vs. Mean Anomaly

### 4 Mathematics

#### 4.1 Definitions

**Definition 1** (The ellipse). *Given  $\theta \in [0, 2\pi)$ , and parameters  $a, b \in \mathbb{R}^+$  with  $a > b$  the following parametric form defines an ellipse.*

$$\epsilon(\theta) = (a \cos \theta, b \sin \theta) \tag{1}$$

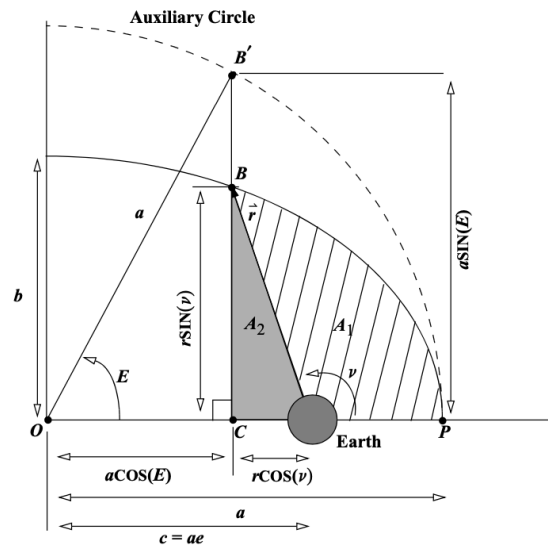


Figure 2-2. **Geometry of Kepler's Equation.** The eccentric anomaly uses an auxiliary circle as shown. The ultimate goal is to determine the area,  $A_1$ , which allows us to calculate the time.

Figure 1: Vallado's figure 2-2 showing  $E$  and  $\nu$ .

#### Sec 4.2 TIME-OF-FLIGHT – ECCENTRIC ANOMALY 183

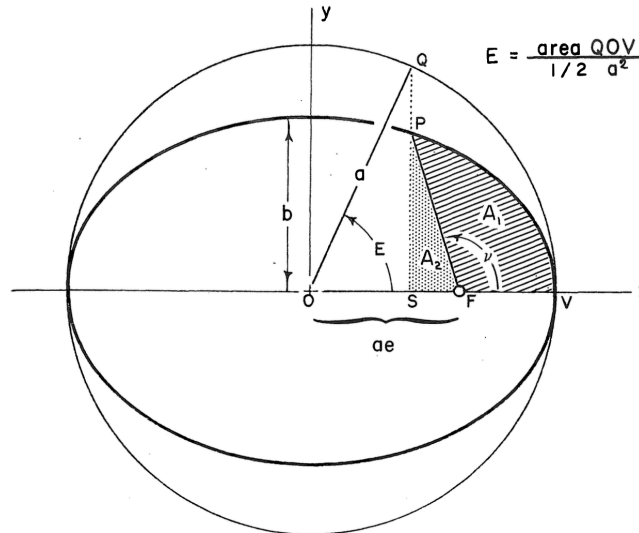


Figure 4.2-2 Eccentric anomaly,  $E$

Figure 2: Figure 4-4 in Bate *et al.* showing  $E$  and  $\nu$ .

$ACQ$  will be defined as the eccentric anomaly,  $E$ , and it will be shown that the relation between  $M$  and  $E$  is given by Kepler's equation.

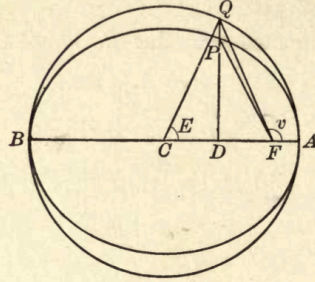


Fig. 28.

From the law of areas and the properties of the auxiliary circle, it follows that

$$\frac{M}{2\pi} = \frac{\text{area } AFP}{\text{area ellipse}} = \frac{\text{area } AFQ}{\text{area circle}}.$$

$$\text{Area } AFQ = \text{area } ACQ - \text{area } FCQ = \frac{a^2 E}{2} - \frac{a}{2} ae \sin E.$$

Therefore

$$\frac{M}{2\pi} = \frac{a^2}{2} \frac{(E - e \sin E)}{\pi a^2};$$

or,

$$\begin{cases} M = E - e \sin E, \\ FP = r = \frac{a(1 - e^2)}{1 + e \cos v} = \sqrt{PD^2 + FD^2} = a(1 - e \cos E), \end{cases}$$

Figure 3: Moulton's figure 28 showing  $E$  and  $\nu$ .

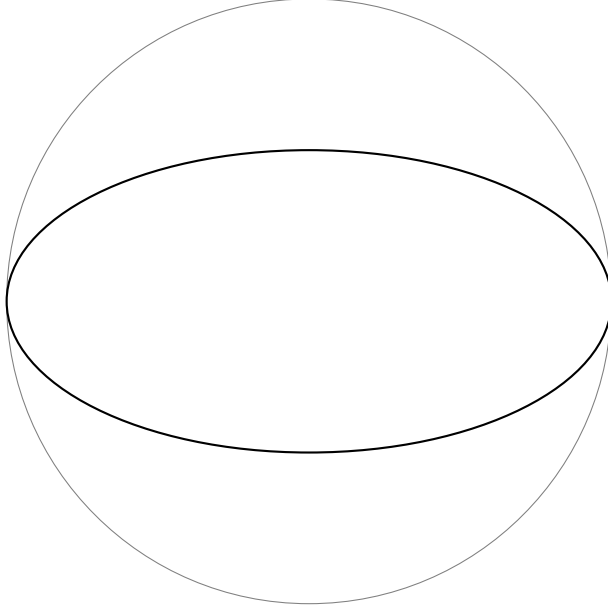


Figure 4: Geometry

**Definition 2** (Eccentricity of the ellipse). *The eccentricity is a scalar parameter  $e \in (0, 1)$  and can be expressed in terms of fundamental parameters of the ellipse where  $a > b$  as*

$$e = \frac{c}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2} \quad (2)$$

**Definition 3** (Mean anomaly). *Kepler's Law<sup>1</sup> defines the mean anomaly as the angular measure  $M(e, E) : (0, 1) \times [0, 2\pi) \mapsto [0, 2\pi)$  as*

$$M(e, E) = E - e \sin E \quad (3)$$

**Theorem 4** (Continuity of the mean anomaly). *The mean anomaly as defined in definition ?? is a continuous function.*

*Proof.* To prove continuity show that for any two points  $p$  and  $q$  in the domain there exists a majorization constant  $K$  such that

$$M(p) - M(q) \leq K|p - q| \quad (4)$$

Spoiler alert: the majorization constant is  $2\pi$ . □

Observation: given the continuity of the mean anomaly, one may use Newton's method Gautschi 2011, §4.6 to solve the nonlinear equation.

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<sup>1</sup>Bate et al. 2020, eq 4.5 Moulton 1970, p.159 Vallado 2022, §2.2, Kaula 2013, pp. 3–19

## References

1. Bate, Roger R et al. (2020). *Fundamentals of astrodynamics*. Courier Dover Publications.
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4. Moulton, Forest Ray (1970). *An introduction to celestial mechanics*. Dover.
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