

Kepler's Equation: Derivation, Implementation, Validation

Daniel Topa
daniel.topa@hii-tds.com

*Mission Technologies
Huntington Ingalls Industries
Kirtland AFB, NM*

October 25, 2024

Abstract

Kepler's law is a cornerstone of orbital mechanics.

Contents

1	Overview	1
2	Derivation	2
2.1	Orbit Equation	2
2.2	Ellipse	2
3	Geometry of Kepler's Law	2
4	Mathematics	2
4.1	Definitions	2
A	Survey	4

1 Overview

Kepler's law is a cornerstone of orbital mechanics. Kepler's equation

$$M = E - e \sin E \tag{1}$$

Kepler's Law III

$$n = \mu^{\frac{1}{2}} a^{\frac{-3}{2}} \tag{2}$$

2 Derivation

2.1 Orbit Equation

$$\frac{du^2}{d^2\lambda} + u = \frac{\mu}{h}, \quad u(0) = u_0, u'(0) = v_0 \quad (3)$$

A second order linear partial differential equation Boyce, DiPrima, and Meade 2021 First find the solution for the homogenous equation

$$\frac{du^2}{d^2\lambda} + u = 0 \quad (4)$$

which is

$$u(\theta) = A \cos \theta + B \sin \theta \quad (5)$$

Using the boundary conditions, $u(\theta) = u_0 \cos \theta + v_0 \sin \theta$.

2.2 Ellipse

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1 \quad (6)$$

$$\xi = ae + r \cos f, \eta = r \sin f, b^2 = a^2(1 - e^2) \quad (7)$$

$$(1 - e^2)\xi^2 + \eta^2 - a^2(1 - e^2) = 0 \quad (8)$$

$$f(r) = \alpha r^2 + \beta r + \gamma = 0 \quad (9)$$

$$\alpha = 1 - e^2 \cos^2 f, \quad \beta = 2ae(1 - e^2) \cos f, \quad \gamma = a^2(1 - e^2)^2 \quad (10)$$

$$r_{\pm} = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \Rightarrow \pm \frac{a(1 - e^2)}{1 \pm e \cos f} \quad (11)$$

$$r_{\pm} = \pm \frac{a(1 - e^2)}{1 \pm e \cos f} \quad (12)$$

3 Geometry of Kepler's Law

Disagreement with this YouTuber True Anomaly vs. Mean Anomaly

4 Mathematics

4.1 Definitions

Definition 1 (The ellipse). *Given $\theta \in [0, 2\pi)$, and parameters $a, b \in \mathbb{R}^+$ with $a \geq b$ the following parametric form defines an ellipse.*

$$\epsilon(\theta) = (a \cos \theta, b \sin \theta) \quad (13)$$

Definition 2 (Eccentricity of the ellipse). *The eccentricity is a scalar parameter $e \in (0, 1)$ and can be expressed in terms of fundamental parameters of the ellipse where $a > b$ as*

$$e = \frac{c}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2} \quad (14)$$

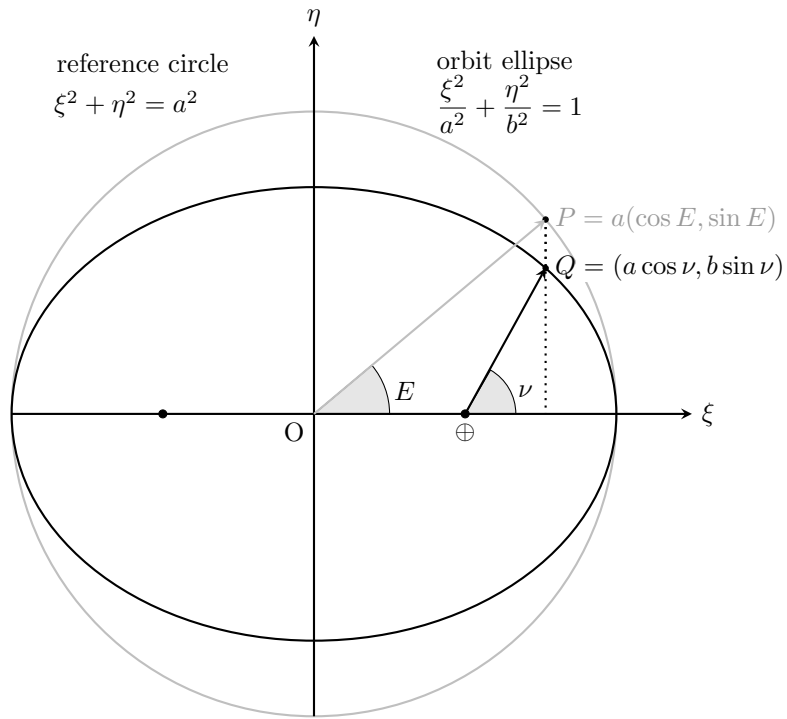


Figure 1: Orbit trajectory (black) and the auxiliary circle (gray) showing the angles for eccentric anomaly, E , and the true anomaly, ν . The true anomaly indicates position relative to the Earth, \oplus , while the eccentric anomaly points to an echo point on the auxiliary circle.

Definition 3 (The circle). *In the limit*

$$\lim_{b \rightarrow a^-} (a \cos \theta, b \sin \theta) = a (\cos \theta, \sin \theta) = C_2(\theta)$$

$$C_2(\theta) = a (\cos \theta, \sin \theta) \tag{15}$$

Definition 4 (Mean anomaly). *Kepler's Law¹ defines the mean anomaly as the angular measure $M(e, E): (0, 1) \times [0, 2\pi) \mapsto [0, 2\pi)$ as*

$$M(e, E) = E - e \sin E \tag{16}$$

Theorem 5 (Continuity of the mean anomaly). *The mean anomaly as defined in definition 16 is a continuous function.*

Proof. To prove continuity show that for any two points p and q in the domain there exists a majorization constant K such that

$$M(p) - M(q) \leq K|p - q| \tag{17}$$

Spoiler alert: the majorization constant is 2π . □

Observation: given the continuity of the mean anomaly, one may use Newton's method Gautschi 2011, §4.6 to solve the nonlinear equation.

A Survey

Disagreement with this YouTuber True Anomaly vs. Mean Anomaly

References

1. Bate, Roger R et al. (2020). *Fundamentals of astrodynamics*. Courier Dover Publications.
2. Boyce, William E, Richard C DiPrima, and Douglas B Meade (2021). *Elementary differential equations and boundary value problems*. John Wiley & Sons.
3. Gautschi, Walter (2011). *Numerical analysis*. Springer Science & Business Media.
4. Kaula, William M (2013). *Theory of satellite geodesy: applications of satellites to geodesy*. Courier Corporation.
5. Moulton, Forest Ray (1970). *An introduction to celestial mechanics*. Dover.
6. Vallado, D.A. (2022). *Fundamentals of Astrodynamics and Applications*. 5th ed. Microcosm Press;

¹Bate et al. 2020, eq 4.5 Moulton 1970, p.159 Vallado 2022, §2.2, Kaula 2013, pp. 3–19

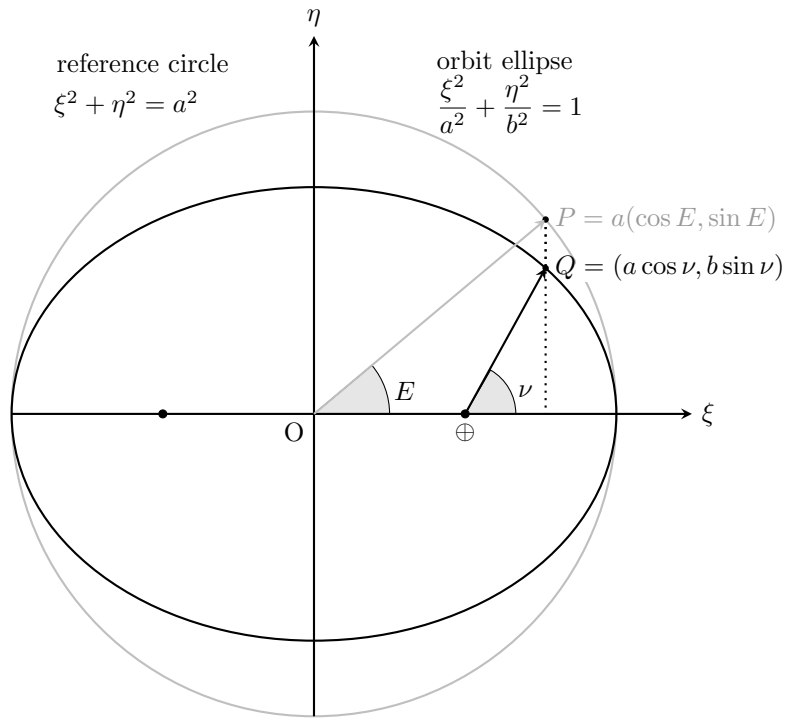


Figure 2: Orbit trajectory (black) and the auxiliary circle (gray) showing the angles for eccentric anomaly, E , and the true anomaly, ν . The true anomaly indicates position relative to the Earth, \oplus , while the eccentric anomaly points to an echo point on the auxiliary circle.

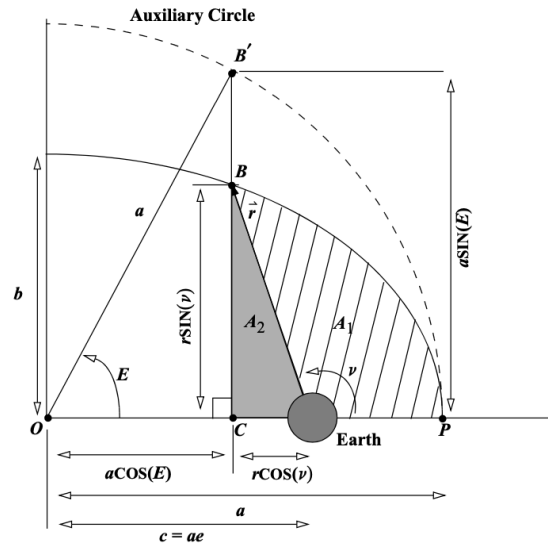


Figure 2-2. **Geometry of Kepler's Equation.** The eccentric anomaly uses an auxiliary circle as shown. The ultimate goal is to determine the area, A_1 , which allows us to calculate the time.

Figure 3: Vallado's figure 2-2 showing E and ν .

Sec 4.2 TIME-OF-FLIGHT – ECCENTRIC ANOMALY 183

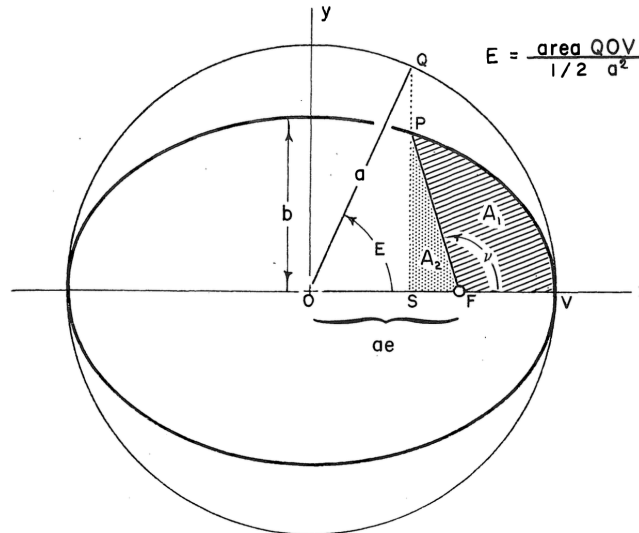


Figure 4.2-2 Eccentric anomaly, E

Figure 4: Figure 4-4 in Bate *et al.* showing E and ν .

ACQ will be defined as the eccentric anomaly, E , and it will be shown that the relation between M and E is given by Kepler's equation.

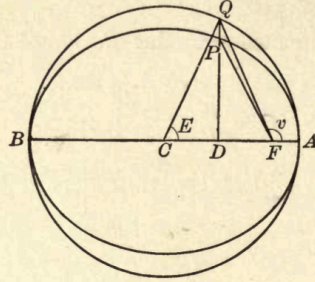


Fig. 28.

From the law of areas and the properties of the auxiliary circle, it follows that

$$\frac{M}{2\pi} = \frac{\text{area } AFP}{\text{area ellipse}} = \frac{\text{area } AFQ}{\text{area circle}}.$$

$$\text{Area } AFQ = \text{area } ACQ - \text{area } FCQ = \frac{a^2 E}{2} - \frac{a}{2} ae \sin E.$$

Therefore

$$\frac{M}{2\pi} = \frac{a^2}{2} \frac{(E - e \sin E)}{\pi a^2};$$

or,

$$\begin{cases} M = E - e \sin E, \\ FP = r = \frac{a(1 - e^2)}{1 + e \cos v} = \sqrt{PD^2 + FD^2} = a(1 - e \cos E), \end{cases}$$

Figure 5: Moulton's figure 28 showing E and ν .