JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 10, 368-377 (1965)

Vector Lyapunov Functions and Conditional Stability*

V. LAKSHMIKANTHAM

University of Alberta, Calgary, Alberta, Canada

Submitted by Richard Bellman

1. One of the most versatile technics in the theory of nonlinear differential equations is the second method of Lyapunov. It depends crucially on the fact that a function satisfying a scalar differential inequality can be majorized by the maximal solution of the corresponding scalar differential equation. This comparison principle has been successfully employed to study a variety of problems, in a unified way, of ordinary differential equations, functional differential equations, differential equations with unbounded operators acting in Banach spaces and parabolic differential equations [1-6]. This extension is based on the use of a single Lyapunov function. It is natural to ask whether it might be more advantageous, in some situations, to use several Lyapunov functions. The answer is postive. In fact, Bellman [7] and Matrosov [8] have shown that using a vector Lyapunov function is indeed fruitful. In this paper, we have exploited this idea further. Defining, in a natural way, the concepts of conditional stability and boundedness of solutions, we obtain sufficient conditions in terms of several Lyapunov functions such that these concepts hold. These notions, as is to be expected, include as a special case, the usual notions of stability and boundedness of solutions. Examples are worked out to demonstrate the fruitfulness of using a vector Lyapunov function.

2. Let I denote the half-line $0 \leqslant t < \infty$, and let R^n denote n-dimensional Euclidian space. We consider the differential system

$$x' = f(t, x), \qquad x(t_0) = x_0, \qquad (t_0 \geqslant 0) \tag{2.1}$$

where x and f are n-dimensional vectors and the function $f(t, x)$ is defined and continuous on the product space $I \times R^n$.

Let $x(t)$ be any solution of (2.1). Denote by $S(\alpha)$ and $S(\bar{\alpha})$ the sets $[x : |x| < \alpha]$ and $[x : |x| \leqslant \alpha]$ respectively, where $|x|$ denotes any convenient norm of x. Let $M_{(n-k)}$ denote a manifold of $(n-k)$ dimensions containing the origin. In order to unify our results on conditional stability and boundedness, it is convenient to list below the following conditions.

* This work was supported by U.S.A.F. Grant No. GAO-135.

(i) For each $\epsilon > 0$ and each $t_0 \geqslant 0$, there exists a positive function $d = d(t_0, \epsilon)$ that is continuous in t_0 for each ϵ, such that

$$x(t) \subset S(\epsilon) \qquad (t \geqslant t_0),$$

whenever

$$x(t_0) \subset S(\bar{d}) \cap M_{(n-k)} .$$

(ii) The d in (i) is independent of t_0.

(iii) For each $\epsilon > 0$, $\alpha \geqslant 0$ and $t_0 \geqslant 0$, there exists a positive number $T = T(t_0, \epsilon, \alpha)$ such that

$$x(t) \subset S(\epsilon) \qquad (t \geqslant t_0 + T),$$

provided

$$x(t_0) \subset S(\bar{\alpha}) \cap M_{(n-k)} .$$

(iv) The T in (iii) is independent of t_0.

(v) The conditions (i) and (iii) hold simultaneously.

(vi) The conditions (ii) and (iv) hold simultaneously.

(vii) For each $\alpha \geqslant 0$ and $t_0 \geqslant 0$, there exists a positive function $\beta = \beta(t_0, \alpha)$ that is continuous in t_0 for each α, such that

$$x(t) \subset S(\beta) \qquad (t \geqslant t_0),$$

whenever

$$x(t_0) \subset S(\bar{\alpha}) \cap M_{(n-k)} .$$

(viii) The β in (vii) is independent of t_0.

(ix) For each $\alpha \geqslant 0$ and $t_0 \geqslant 0$, there exist positive numbers B and $T = T(t_0, \alpha)$ such that

$$x(t) \subset S(B) \qquad (t \geqslant t_0 + T),$$

provided

$$x(t_0) \subset S(\bar{\alpha}) \cap M_{(n-k)} .$$

(x) The T in (ix) is independent of t_0.

(xi) The conditions (vii) and (ix) hold simultaneously.

(xii) The conditions (viii) and (x) hold simultaneously.

REMARK. We observe that if $M_{(n-k)} = R^n$, our definitions reduce to the usual definitions of stability and boundedness of solutions of (2.1) with respect to the origin. Observe also that we have not assumed that the origin is an invariant set for the system (2.1). In fact, such a possibility, whenever it exists, is implied by the definitions. For example, if the solutions of (2.1) satisfy (i), it is evident (since $d \to 0$ as $\epsilon \to 0$) that the origin is an invariant (positively) set for the system (2.1). Where as, this need not, in general, be the case with respect to the conditions (iii) (or (vii) to (xii)), since we have not assumed the uniqueness of solutions of (2.1).

3. Let w be a vector of N-dimensions and the function $w(t, r)$ be defined on the product space $I \times R^N$. Let, for each $t \in I$ and for each i, $w_i(t, r_1, \cdots, r_N)$ be nondecreasing in $r_1, \cdots, r_{i-1}, r_{i+1}, \cdots, r_N$. Then, it is known [9] that the differential system

$$r' = w(t, r), \qquad r(t_0) = r_0 \tag{3.1}$$

has the maximal solution (in the sense of component wise majorization) existing to the right of t_0.

Let V be a vector of N-dimensions and the function $V(t, x) \geqslant 0$ be defined and continuous on the product space $I \times R^N$. Suppose further that $V(t, x)$ satisfies a Lipschitz's condition in x locally. Define

$$V^*(t, x) = \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x + hf(t, x)) - V(t, x)]. \tag{3.2}$$

With respect to these functions, we state the following.

LEMMA. *Let the function $V^*(t, x)$ of (3.2) satisfy the inequality*

$$V^*(t, x) \leqslant w(t, V(t, x)), \tag{3.3}$$

where $w(t, r)$ is the same function defined above satisfying the monotonic property. Let $x(t)$ be any solution of (2.1) such that $V(t_0, x_0) \leqslant r_0$. Then

$$V(t, x(t)) \leqslant r(t; t_0, r_0) \qquad (t \geqslant t_0), \tag{3.4}$$

where $r(t; t_0, r_0)$ is the maximal solution of (3.1).

PROOF. Let $x(t)$ be any solution of (2.1) such that $V(t_0, x_0) \leqslant r_0$. Define the N-vector m by $m(t) = V(t, x(t))$. Then using the hypothesis that $V(t, x)$ satisfies Lipschitz's condition with respect to x, we obtain, for small, positive, h, the inequality

$$\begin{aligned} m(t + h) - m(t) \leqslant K \mid x(t + h) - x(t) - hf(t, x(t)) \mid \\ + V(t + h, x(t) + hf(t, x(t))) - V(t, x(t)) \end{aligned}$$

where K is the Lipschitz constant at (t, x). This together with (2.1) and (3.3) implies the inequality

$$\limsup_{h \to 0^+} \frac{1}{h} [m(t + h) - m(t)] \leqslant w(t, m(t)).$$

Now using the notion of maximal solution for the systems and the monotonic property of w, the result (3.4) can be established following the standard argument used in [9].

REMARK. Notice that this result is an extension to systems of a corresponding lemma [1, 2]. Whenever we use several Lyapunov functions instead of one, this lemma plays an important role.

4. Let $r(t; t_0, r_0)$ be the solutions of the differential system (3.1) such that

$$r_i(t_0; t_0, r_0) = \begin{cases} 0, & (i = 1, 2, \cdots, k); \\ r_{i0} & (i = k+1, \cdots, N). \end{cases} \tag{4.1}$$

Now corresponding to the conditions (i) to (xii) above, if we say that the differential system (3.1) has the property (ia), we mean the following condition is satisfied:

(ia) Given $\epsilon > 0$ and $t_0 \geqslant 0$, there exists a positive function $d = d(t_0, \epsilon)$ that is continuous in t_0 for each ϵ, such that

$$\sum_{i=1}^{N} r_i(t; t_0, r_0) < \epsilon \qquad (t \geqslant t_0),$$

whenever

$$\sum_{i=k+1}^{N} r_{i0} \leqslant d.$$

Conditions (iia) to (xiia) may be formulated similarly. It is important to note that in the conditions (ia) to (xiia), we have used the solutions $r(t; t_0, r_0)$ of (3.1) satisfying (4.1).

Assume that

the set of points defined by $V_i(t, x) \equiv 0$ $(i = 1, 2, \cdots, k)$ $(k < N)$ constitute a manifold of $(n - k)$ dimensions, containing the origin. Let us denote it by $M_{(n-k)}$. (4.2)

the function $b(r)$ is continuous and nondecreasing in r,

$$b(r) > 0 \quad \text{for} \quad r > 0 \qquad \text{and} \qquad b(|x|) \leqslant \sum_{i=1}^{N} V_i(t, x). \tag{4.3}$$

$$\sum_{i=1}^{N} V_i(t, x) \to 0 \qquad \text{as} \qquad |x| \to 0 \qquad \text{for each} \qquad t \in I. \tag{4.4}$$

$$\sum_{i=1}^{N} V_i(t, x) \to 0 \qquad \text{as} \qquad |x| \to 0 \qquad \text{uniformly in } t. \tag{4.4*}$$

$$b(r) \to \infty \qquad \text{as} \qquad r \to \infty. \tag{4.5}$$

Then we have the following theorems on conditional stability and boundedness of solutions of (2.1).

THEOREM 1. *Let the assumptions of lemma hold together with (4.2), (4.3), and (4.4). Then, if one of the conditions* (ia), (iiia), *and* (va) *holds, the corresponding one of the conditions* (i), (iii), *and* (v) *holds. If (4.4) is strengthened to (4.4*), one of the conditions* (iia), (iva), *and* (via) *implies the corresponding one of the conditions.*

PROOF. For any $\epsilon > 0$, if $|x| = \epsilon$, we get from (4.3) that

$$b(\epsilon) \leqslant \sum_{i=1}^{N} V_i(t, x). \tag{4.6}$$

If (ia) holds, given $b(\epsilon) > 0$ and $t_0 \geqslant 0$, there exists a positive function $d = d(t_0, \epsilon)$, that is continuous in t_0 for each ϵ, such that

$$\sum_{i=1}^{N} r_i(t; t_0, r_0) < b(\epsilon) \qquad (t \geqslant t_0), \tag{4.7}$$

provided

$$\sum_{i=k+1}^{N} r_{i0} \leqslant d.$$

Suppose that $x(t)$ is any solution of (2.1). Then it follows from the lemma that

$$\sum_{i=1}^{N} V_i(t, x(t)) \leqslant \sum_{i=1}^{N} r_i(t; t_0, r_0) \qquad (t \geqslant t_0), \tag{4.8}$$

whenever

$$\sum_{i=1}^{N} V_i(t_0, x(t_0)) \leqslant \sum_{i=1}^{N} r_{i0}. \tag{4.9}$$

Now choose r_{i0} ($i = 1, 2, \cdots, N$) to satisfy the following conditions:

$$r_{i0} = 0 \qquad (i = 1, 2, \cdots, k); \tag{4.10}$$

$$\sum_{i=k+1}^{N} r_{i0} \leqslant d. \tag{4.11}$$

In view of (4.9) and the fact that $V_i(t, x) \geqslant 0$ ($i = 1, 2, \cdots, N$), (4.10) implies that $x(t_0) \subset M_{(n-k)}$ because of (4.2). Further, from the monotonic property of $b(r)$, (4.3), (4.9), and (4.11), we deduce that

$$|x(t_0)| \leqslant b^{-1}(d) \equiv d_1.$$

Also, because of (4.4), there exists a $d_2 = d_2(t_0, d)$ such that

$$\operatorname*{Sup}_{|x(t_0)| \leqslant d_2} \sum_{i=1}^{N} V_i(t_0, x(t_0)) \leqslant d.$$

Let $d_3 = \min(d_1, d_2)$. It then follows from the choice of r_{i0} and d_3 that $x(t_0) \subset S(\bar{d}_3) \cap M_{(n-k)}$ implies every solution $x(t)$ satisfies (4.8). Suppose, if possible, that a solution $x(t)$ of (2.1) satisfying $x(t_0) \subset S(\bar{d}_3) \cap M_{(n-k)}$ lies on the boundary of $S(\epsilon)$, for some $t = t_1 > t_0$. This means that $| x(t_1) | = \epsilon$. Then, using the relations (4.3), (4.6), (4.7), and (4.8), we are lead to the contradiction

$$b(\epsilon) \leqslant \sum_{i=1}^{N} V_i(t_1, x(t_1)) \leqslant \sum_{i=1}^{N} r_i(t_1, t_0, r_0) < b(\epsilon)$$

which proves the conclusion (i).

The proof corresponding to (iia) is essentially the same since d and d_3 are independent of t_0 in this case. The proofs of other statements are also similar with necessary modifications. We shall only indicate the proof of the conclusion (iii).

Let $\epsilon > 0$, $\alpha > 0$, and $t_0 \geqslant 0$ be given and let $| x(t_0) | \leqslant \alpha$. Then because of (4.4), we can choose a number $\alpha_1 = \alpha_1(t_0, \alpha)$ such that

$$\operatorname*{Sup}_{|x(t_0)| \leqslant \alpha} \sum_{i=1}^{N} V_i(t_0, x(t_0)) \leqslant \alpha_1 .$$

Since (4.9) implies (4.8), we choose r_{i0} $(i = 1, 2, \cdots, N)$ such that (4.10) and $\sum_{i=k+1}^{N} r_{i0} \leqslant \alpha_1$ hold. As before, one concludes from (4.2), (4.10), and $| x(t_0) | \leqslant \alpha$, that wherever $x(t_0) \subset S(\bar{\alpha}) \cap M_{(n-k)}$; every solution $x(t)$ of (2.1) satisfies (4.8).

Now since (iiia) holds, given $b(\epsilon) > 0$, $\alpha_1 > 0$, there exists a positive number $T = T(t_0, \alpha_1, \epsilon)$ such that

$$\sum_{i=1}^{N} r_i(t; t_0, r_0) < b(\epsilon) \qquad (t \geqslant t_0 + T), \tag{4.13}$$

if $\sum_{i=k+1}^{N} r_{i0} \leqslant \alpha_1$. Let $\{t_n\}$ be a sequence such that $t_n \to \infty$ as $n \to \infty$ and $t_n \geqslant t_0 + T$. Then, the assumption that a solution $x(t)$ of (2.1) such that $x(t_0) \subset S(\bar{\alpha}) \cap M_{(n-k)}$ has the property that $x(t_n) \not\subset S(\epsilon)$ leads to the inequality

$$b(\epsilon) \leqslant \sum_{i=1}^{N} V_i(t_n; x(t_n)) \leqslant \sum_{i=1}^{N} r_i(t_n; t_0, r_0) < b(\epsilon)$$

because of the monotonic property of $b(r)$, (4.3), (4.8), and (4.12). This contradiction proves (iii) and the proof of the theorem is complete.

THEOREM 2. *Let the assumptions of the lemma hold together with (4.2), (4.3), (4.4), and (4.5). Then if one of the conditions* (viia), (ixa), *and* (xia) *holds, the corresponding one of the conditions* (vii), (ix), *and* (xia) *holds. If (4.4) is strengthened to (4.4*), one of the conditions* (viiia), (xa), *and* (xiia) *implies the corresponding one of the conditions* (viii), (x), *and* (xii).

PROOF. Let $\alpha > 0$ and $t_0 \geqslant 0$ be given. Then, proceeding as in the proof of Theorem 1, we conclude that whenever $x(t_0) \subset S(\bar{\alpha}) \cap M_{(n-k)}$, every solution $x(t)$ of (2.1) satisfies (4.8).

Since (viia) is satisfied, given $\alpha_1 > 0$ and $t_0 \geqslant 0$, there exists a $\beta = \beta(t_0, \alpha_1)$ such that

$$\sum_{i=1}^{N} r_i(t; t_0, r_0) < \beta \qquad (t \geqslant t_0) \tag{4.14}$$

if $\sum_{i=k+1}^{N} r_{i0} \leqslant \alpha_1$, where α_1 is defined by (4.12). Since $b(r) \to \infty$ as $r \to \infty$, there exists a $L = L(t_0, \alpha_1)$ such that

$$\beta < b(L). \tag{4.15}$$

Now assume that there is a solution $x(t)$ of (2.1) such that

$$x(t_0) \subset S(\bar{\alpha}) \cap M_{(n-k)}$$

lies on the boundary of $S(L)$ for some $t = t_1 > t_0$. This, because of the relations (4.3), (4.8), (4.14) and (4.15), leads to the contradiction

$$b(L) \leqslant \sum_{i=1}^{N} V_i(t_1; x(t_1)) \leqslant \sum_{i=1}^{N} r_i(t_1; t_0, r_0) < \beta < b(L)$$

which proves the conclusion (vii).

By following the proof of Theorem 1 and that given above, one can easily construct the proofs of the remaining statements. We leave the details to the reader.

5. In this section we indicate the modifications necessary in order to obtain the usual stability and boundedness of solution of (2.1) from the previous results.

Let $1 \leqslant k \leqslant N$ and suppose that $V_i(t, x) \geqslant 0$ $(i = 1, 2, \cdots, k)$. Let us replace the assumptions (4.3), (4.4), and (4.4*) accordingly; i.e., the summation is from 1 to k. The restriction (4.1) is no longer necessary. Conditions corresponding to (ia) to (xiia) are to be changed as follows:

(ia*) Given $\epsilon > 0$ and $t_0 \geqslant 0$, there exists a positive function $d = d(t_0, \epsilon)$ which is continuous in t_0 for each ϵ, such that

$$\sum_{i=1}^{k} r_i(t; t_0, r_0) < \epsilon \qquad (t \geqslant t_0)$$

if $\sum_{i=1}^{k} r_{i0} \leqslant d$. The conditions (iia) to (xiia) are to be understood similarly. Suppose that $M_{(n-k)} = R^n$. Then, as observed already, the conditions (i) to (xii) reduce to the usual definitions. Let us denote these reduced definitions by (i*) to (xii*). Then, the following theorems result from Theorems 1 and 2. We merely state

THEOREM 3. *Let the assumptions of lemma hold together with (4.2), (4.3), and (4.4). Then, if one of the conditions* (ia*), (iiia*), *and* (va*) *holds, the corresponding one of the conditions* (i*), (iii*), *and* (v*) *holds. If (4.4) is strengthenend to (4.4*), one of the conditions* (iia*), (iva*), *and* (via*) *implies the corresponding one of the conditions* (ii*), (iv*), *and* (vi*).

THEOREM 4. *Let the assumptions of lemma hold together with (4.2), (4.3), (4.4), and (4.5). Then, if one of the conditions* (viia*), (ixa*), *and* (xia*) *holds, the corresponding one of the conditions* (vii*), (ix*), *and* (xi*) *holds. If (4.4) is strengthened to (4.4*), one of the conditions* (viiia*), (xa*), *and* (xiia*) *implies the corresponding one of the conditions* (viii*), (x*), *and* (xii*).

REMARKS. The conclusions (i*) and (v*) include the stability results of Matrosov [8].

6. Lastly we shall give two examples to illustrate the results.

EXAMPLE 1. Consider the differential system

$$\begin{aligned} x' &= (1 + \cos t)\, x + (1 - \cos t)\, y + (\cos t - 1)\, z, \\ y' &= (1 - e^{-t})\, x + (1 + e^{-t})\, y + (e^{-t} - 1)\, z, \\ z' &= (\cos t - e^{-t})\, x + (e^{-t} \cos t)\, y + (e^{-t} + \cos t)\, z. \end{aligned} \tag{6.1}$$

Take $N = 3$ and

$$\begin{aligned} V_1 &= V_1(t, x, y, z) = (x + y - z)^2, \\ V_2 &= V_2(t, x, y, z) = (x - y + z)^2, \\ V_3 &= V_3(t, x, y, z) = (-x + y + z)^2. \end{aligned}$$

Since

$$\sum_{i=1}^{3} V_i = x^2 + y^2 + z^2 + (x-y)^2 + (y-z)^2 + (z+x)^2,$$

the assumption (4.3) reduces to (taking the Euclidian norm)

$$(\sqrt{x^2+y^2+z^2})^2 \leqslant \sum_{i=1}^{3} V_i(t, x, y, z).$$

Further, we deduce from (3.3) that

$$\begin{aligned} V_1{}^*(t, x, y, z) &\leqslant w_1(t, V_1, V_2, V_3) \equiv 4\, V_1(t, x, y, z), \\ V_2{}^*(t, x, y, z) &\leqslant w_2(t, V_1, V_2, V_3) \equiv 4 \cos t\, V_2(t, x, y, z), \\ V_3{}^*(t, x, y, z) &\leqslant w_3(t, V_1, V_2, V_3) \equiv 4e^{-t}\, V_3(t, x, y, z). \end{aligned}$$

We see that w fulfils the monotonic requirement. Choose $k = 1$. Then the solution $r(t; t_0, r_0)$ of (3.1) satisfying (4.1) is

$$\begin{aligned} r_1(t; t_0, 0) &\equiv 0, \\ r_2(t; t_0, r_{20}) &= r_{20} \exp[\sin t - \sin t_0], \\ r_3(t; t_0, r_{30}) &= r_{30} \exp[4(e^{-t_0} - e^{-t})]. \end{aligned}$$

In this case, $M_{(n-k)} = M_{(2)}$, the set of points (x, y, z) satisfying $x + y = z$. It is also clear that the condition (ia) is fulfilled and consequently, it follows from Theorem 1 that the system (6.1) satisfies (i).

EXAMPLE 2. Consider the differential system

$$\begin{aligned} x' &= \frac{(t^2+t-1)}{t+1}\, x + \frac{(t^2+t+1)}{t+1}(y-z), \\ y' &= (t+1)(x-z) + (t-1)\,y + (t-t_0)\, e^{-2(t-t_0)}, \\ z' &= \frac{t}{t+1}(x-y) - \frac{(t+2)}{(t+1)}\, z + (t-t_0)\, e^{-2(t-t_0)}. \end{aligned} \tag{6.2}$$

Taking the same Lyapunov functions as in the previous example, the functions w take the form

$$\begin{aligned} w_1(t, V_1, V_2, V_3) &= 4tV_1, \\ w_2(t, V_1, V_2, V_3) &= \frac{-4}{t+1}\, V_2, \\ w_3(t, V_1, V_2, V_3) &= -4V_3 + 4(t-t_0)\, e^{-2(t-t_0)}\, V_3^{1/2}. \end{aligned}$$

Clearly, w_i satisfy the monotonic requirements and the solution $r(t; t_0, r_0)$ of (3.1) satisfying (4.1) is

$$r_1(t; t_0, 0) \equiv 0,$$

$$r_2(t; t_0, r_{20}) = r_{20} \frac{(t_0 + 1)}{t + 1},$$

$$r_3(t; t_0, r_{30}) = e^{-4(t-t_0)}[r_{30} + 2r_{30}^{1/2}(t - t_0)^2 + (t - t_0)^4].$$

Evidently the condition (iiia) is satisfied and an application of Theorem 1 yields that (6.2) satisfies the condition (iii) with the same $M_{(2)}$ as in Example 1.

We note that the origin is not an invariant set in this example.

References

1. V. Lakshmikantham. Differential systems and extension of Lyapunov method. *Mich. Math. J.* **9**, 311-320 (1962).
2. V. Lakshmikantham. Differential equations in Banach spaces and extension of Lyapunov method. *Proc. Cambridge Phil. Soc.* **59**, 373-381 (1963).
3. V. Lakshmikantham. Upper and lower bounds of the norm of solutions of differential systems. *Proc. Am. Math. Soc.* **14**, 509-513 (1963).
4. V. Lakshmikantham. Lyapunov function and a basic inequality in delay-differential equations. *Arch. Rat. Mech. Anal.* **10**, 305-310 (1962).
5. V. Lakshmikantham. Functional differential systems and extension of Lyapunov's method. *J. Math. Anal. Appl.*, **8**, 392-405 (1964).
6. V. Lakshmikantham. Parabolic differential equations and Lyapunov like functions. *J. Math. Anal. Appl.*, **9**, 234-251 (1964).
7. R. Bellman. Vector Lyapunov functions. *J. Soc. Ind. Appl. Math. Ser.* **A1**, 32-34 (1962).
8. V. M. Matrosov. On the theory of stability of motion (Russian). *Prikl. Mat. Mekh.* **26**, 992-1002 (1962).
9. T. Wazewski. Systèmes des équations et des inégalités différentielles ordinaires aux deuxièmes membres monotones et leurs applications. *Ann. Soc. Pol. Math.* **23**, 112-166 (1950).