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David A. Ross

1. INTRODUCTION. The early proofs for Lyapunov's theorem on the convexity of vector measures are long and elaborate (see [**3**] or [**2**]). More recent proofs use powerful tools such as the theorems of Banach-Alaoglu and/or Krein-Milman (see [**4**], [**1**], [**5**], [**6**], or [**7**]). In this note we obtain the Lyapunov theorem as a consequence of the intermediate value theorem.

Suppose that $\mu_1, \mu_2, \ldots, \mu_n$ are finite atomless measures on $(X, \mathcal{A})$. (Recall that a measure μ is *atomless* provided for any E with $\mu E > 0$ there is a subset A of E with $0 < \mu A < \mu E$.) Denote by $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ the corresponding $\mathbb{R}^n$-valued measure.

Theorem 1 (Lyapunov). *The set* $\{\boldsymbol{\mu} E : E \in \mathcal{A}\}$ *is convex.*

This result is an application of the following, more elaborate theorem:

Theorem 2.

(LT1) *For each E in $\mathcal{A}$ and r in* $[0, 1]$ *there is a subset A of E with A in $\mathcal{A}$ and* $\boldsymbol{\mu} A = r\boldsymbol{\mu} E$.

(LT2) *For each E in $\mathcal{A}$ there is an r in* $(0, 1)$ *and a subset A of E with A in $\mathcal{A}$ and* $\boldsymbol{\mu} A = r\boldsymbol{\mu} E$.

(LT3) *For each E in $\mathcal{A}$ there is a subfamily* $\{A_r\}_{r\in[0,1]}$ *of $\mathcal{A}$ such that* $A_r \subseteq A_s \subseteq E$ *whenever* $0 \le r \le s \le 1$ *and* $\boldsymbol{\mu} A_r = r\boldsymbol{\mu} E$ *for each r in* $[0, 1]$.

To see that Lyapunov's theorem follows from Theorem 2, let E and F belong to $\mathcal{A}$, and let $0 \le r \le 1$. By LT1 there are subsets E_1 of $E \setminus F$ and F_1 of $F \setminus E$ with $\boldsymbol{\mu} E_1 = r\boldsymbol{\mu}(E \setminus F)$ and $\boldsymbol{\mu} F_1 = (1-r)\boldsymbol{\mu}(F \setminus E)$. Then $\boldsymbol{\mu}(E_1 \cup [E \cap F] \cup F_1) = r\boldsymbol{\mu} E + (1-r)\boldsymbol{\mu} F$.

The proof of Theorem 2 is carried out by an induction on the number n of atomless measures. It is convenient to assume LT3, the strongest of the three statements, as the induction hypothesis for n measures, then prove the weakest of the statements, LT2, for $n + 1$ measures. We therefore need to show that LT1, LT2, and LT3 are equivalent, in the sense that a vector measure $\boldsymbol{\mu}$ satisfying any one of the statements must satisfy all three. The proof of this equivalence is postponed until the last section.

Given the measures $\mu_1, \dots, \mu_n$, put

$$\boldsymbol{\nu} = (\mu_1, \mu_1 + \mu_2, \mu_1 + \mu_2 + \mu_3, \dots, \mu_1 + \cdots + \mu_n).$$

If A belongs to $\mathcal{A}$ and $\boldsymbol{\nu}A = r\boldsymbol{\nu}E$, then one readily verifies that $\boldsymbol{\mu}A = r\boldsymbol{\mu}E$. It follows that we may always assume that $\mu_1 \ll \mu_2 \ll \cdots \ll \mu_n$, where $\ll$ signifies absolute continuity of measures.

2. PROOF OF THEOREM 2. When $n = 1$, LT2 is simply the definition of atomlessness for μ_1, so Theorem 2 is immediate. We proceed by induction on n. Suppose that Theorem 2 holds for up to n measures, and that E in $\mathcal{A}$ and measures $\mu_1, \mu_2, \dots, \mu_{n+1}$ are given. Put $\boldsymbol{\nu} = (\mu_2, \dots, \mu_{n+1})$. By two applications of LT1 there are disjoint sets E^1, E^2, and E^3 in $\mathcal{A}$ with $E = E^1 \cup E^2 \cup E^3$ and $\boldsymbol{\nu}E^i = (1/3)\boldsymbol{\nu}E$ for each i. By LT3, there is for each i a subfamily $\{A^i_r\}_{r\in[0,1]}$ of $\mathcal{A}$ such that $A^i_r \subseteq A^i_s \subseteq E^i$ whenever $0 \le r \le s \le 1$ and $\boldsymbol{\nu}A^i_r = r\boldsymbol{\nu}E^i = (r/3)\boldsymbol{\nu}E$. We may assume that $A^i_1 = E^i$.

There are now three cases to consider.

Case 1: $\mu_1 E^i = (1/3)\mu_1 E$ for some i. Since also $\boldsymbol{\nu}E^i = (1/3)\boldsymbol{\nu}E$, LT2 holds with $r = 1/3$.

Case 2: for some choice of i_1, i_2, and i_3,

$$\mu_1 E^{i_1} \ge \mu_1 E^{i_3} > (1/3)\mu_1 E > \mu_1 E^{i_2}.$$

For definiteness, take $i_1 = 1$, $i_2 = 2$, and $i_3 = 3$. Define a subfamily $\{A_r\}_{r\in[0,1]}$ of $\mathcal{A}$ by

$$A_r = \begin{cases} A^1_{3r} & \text{if } 0 \le r \le 1/3; \\ A^1_1 \cup A^2_{3r-1} & \text{if } 1/3 < r \le 2/3; \\ A^1_1 \cup A^2_1 \cup A^3_{3r-2} & \text{if } 2/3 < r \le 1. \end{cases}$$

One readily verifies that $\boldsymbol{\nu}A_r = r\boldsymbol{\nu}E$ for r in $[0, 1]$. If $\mu_1 E = 0$, then case 1 applies. Otherwise the function ϕ given on $[0, 1]$ by $\phi(r) = \mu_1 A_r/\mu_1 E$ is well defined. Note that ϕ is increasing. Also, the assumption that $\mu_1 \ll \mu_2$ ensures that ϕ is continuous. Moreover,

$$\phi(1/3) = \frac{\mu_1 A^1_1}{\mu_1 E} = \frac{\mu_1 E^1}{\mu_1 E} > 1/3,$$

and

$$\phi(2/3) = \frac{\mu_1(A^1_1 \cup A^2_1)}{\mu_1 E} = \frac{\mu_1(E^1 \cup E^2)}{\mu_1 E} = \frac{\mu_1 E - \mu_1 E^3}{\mu_1 E} < 1 - 1/3 = 2/3.$$

By the intermediate value theorem, $\phi(r) = r$ for some r in $(1/3, 2/3)$. In other words, $\mu_1 A_r = r\mu_1 E$. Since already $\boldsymbol{\nu}A_r = r\boldsymbol{\nu}E$, $A = A_r$ ensures that LT2 holds.

Case 3: for some choice of i_1, i_2, and i_3,

$$\mu_1 E^{i_1} \le \mu_1 E^{i_3} < (1/3)\mu_1 E < \mu_1 E^{i_2}.$$

In this case the argument from case 2 applies without change, except now $\phi(1/3) < 1/3$ and $\phi(2/3) > 2/3$.

This exhausts the cases (since $\mu_1 E = \mu_1 E^1 + \mu_1 E^2 + \mu_1 E^3$), and proves the theorem.

3. EQUIVALENCE OF LT1, LT2, AND LT3. The implications

$$\text{LT3} \Rightarrow \text{LT1} \Rightarrow \text{LT2}$$

are clear. Assume then that LT2 holds, and fix a set E in $\mathcal{A}$. If A is a subset of E that belongs to $\mathcal{A}$, r is in $[0, 1]$, and $\boldsymbol{\mu} A = r\boldsymbol{\mu} E$, then write $r_A = r$. Let $\mathcal{E}$ be the set of all such A. Order $\mathcal{E}$ by $A \lhd B$ if $A \subseteq B$ and $r_A < r_B$. Let $\mathcal{C}$ be a maximal chain in $\mathcal{E}$ with $\emptyset$ and E belonging to $\mathcal{C}$. Put $I = \{r_A : A \in \mathcal{C}\}$. It suffices to show that $I = [0, 1]$ (since we can then take $\{A_r\}_{r\in[0,1]} = \mathcal{C}$).

Suppose (in search of a contradiction) that a is a point of $(0, 1)$ that is not in I. Put

$$a_\infty = \sup\big(I \cap [0, a)\big) \leq a$$

and

$$b_\infty = \inf\big(I \cap (a, 1]\big) \geq a.$$

There exist A_n and B_n in $\mathcal{E}$ for $n = 1, 2, 3, \ldots$ such that r_{A_n} increases to a_∞ and r_{B_n} decreases to b_∞. Put $A_\infty = \bigcup_n A_n$ and $B_\infty = \bigcap_n B_n$. It is easy to see that A_∞ and B_∞ are members of $\mathcal{E}$ with $a_\infty = r_{A_\infty}$ and $b_\infty = r_{B_\infty}$. Since $\mathcal{C}$ is simply ordered, $A_\infty = \bigcup\{A \in \mathcal{C} : r_A < a\}$ and $B_\infty = \bigcap\{A \in \mathcal{C} : r_A > a\}$. Because $\mathcal{C}$ is maximal, A_∞ and B_∞ belong to $\mathcal{C}$, and $(a_\infty, b_\infty) \cap I = \emptyset$.

By LT2 there is a subset C of $B_\infty \setminus A_\infty$ with $\boldsymbol{\mu} C = r\boldsymbol{\mu}(B_\infty \setminus A_\infty)$. Then $A_\infty \cup C$ is in $\mathcal{E}$ and $a_\infty < r_{(A_\infty \cup C)} < b_\infty$, so $\mathcal{C} \cup \{A_\infty \cup C\}$ is a chain in $\mathcal{E}$ that properly extends $\mathcal{C}$, a contradiction.

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