AN ELEMENTARY PROOF FOR BERTRAND'S POSTULATE

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Contents

1. Abstract

In this paper we give an elementary proof for Bertrand's postulate also known as Bertrand-Chebyshev theorem.

KEYWORDS: prime numbers, prime gaps, inequalities

MSC classification : 11A41, 11N05

2. Introduction

Prime numbers have the point of discussion and fascination among mathematicians for a long time. Various attempts have been made to understand the distribution of prime numbers and the behaviour of prime gaps. In all those efforts, in 1845 Joseph Bertrand conjectured that given any integer $n > 3$, there exists a prime number between n and $2n$. Pafnuty Chebyshev proved it and the theorem came to be known as Bertrand–Chebyshev theorem. Then in 1919, S. Ramanujan gave a short proof for Bertrand's postulate [5], which then was improved by J. Meher and M. Murty in an effort to make it calculus free [4]. Then in 1932, P. Erdős gave another proof for the same [6]. In this paper we present an elementary proof for Bertrand's postulate using some logic, inequalities, properties of prime numbers and prime gaps and some basic and elementary techniques.

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Notations : p_n and g_n represents nth-prime and nth-prime gap respectively. Here n, $x_i \in \mathbf{N}$. As primes and prime gaps are natural numbers, hence $i \in \mathbf{N}$.

Now, consider statement 1: $\forall\, x > 5\,(g_x \geq p_x)$. But, this statement is false as for $x = 10$, $p_{10} > g_{10}$. Hence negation of above statement is True i.e. $\exists\, x > 5\,(p_x > g_x)$ is True.

Now, consider statement 2: $\exists\, x > 5\,(g_x \geq p_x)$. As we already showed $\forall\, x > 5\,(g_x \geq p_x)$ is false, therefore this statement says: there exist some $x > 5$ such that $g_x \geq p_x$ and for rest $x > 5$, $p_x > g_x$. Now we need to prove statement 2 is false in order to prove our claim.

3. CONSTRUCTION

Let us construct a set S such that $S = \{x_i|\, g_{x_i} \geq p_{x_i}\}$. Then $\mathbf{N} - S = \{x_i|\, p_{x_i} > g_{x_i}\}$. Also let us define following :-

1) max gap : g_x is said to be max gap if $\forall\, x_i < x$, $g_{x_i} \leq g_x$.

2) maximal gap : g_x is said to be maximal gap if $\forall\, x_i < x$, $g_{x_i} < g_x$.

So from above definitions we have not maximal gap definition as follows: g_x is said to be not max gap if $\exists\, x_i < x$ such that $g_{x_i} > g_x$.

Observation:-

$x_i, i \in \mathbf{N}$. Maximal gap is more restricted version of max gap. Also according to the definition of max gap it is clear that the complete list of prime gaps can be partitioned into two sets, one with max gaps and other with not max gaps. Also we know that prime numbers keep on increasing. Now, we move towards results and proof .

Lemma 3.1. *Given $x_k > 2$, if g_{x_k} is a max gap then $p_{x_k} > g_{x_k}$.*

Proof. As g_{x_k} is max gap, then by definition of max gap $\forall\, x_i < x_k$, $g_{x_i} \leq g_{x_k}$, where x_i runs from 1 to x_{k-1}. Given $g_{x_i} \leq g_{x_k}$ multiplying by -1 and adding p_{x_k} gives

$$p_{x_k} - g_{x_k} \leq p_{x_k} - g_{x_i}$$

Also, as $x_i < x_k$ we have $p_{x_i} < p_{x_k}$. Subtracting g_{x_k} from both sides, we get

$$p_{x_k} - g_{x_k} > p_{x_i} - g_{x_k}$$

Combining both we get

$$p_{x_i} - g_{x_k} < p_{x_k} - g_{x_k} < p_{x_k} - g_{x_i}$$

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Putting $i = k-1$, we get $p_{x_{k-1}} - g_{x_k} < p_{x_k} - g_{x_k}$

So, now if we show

$$p_{x_{k-1}} - g_{x_k} > 0 \tag{1}$$

then we are done. Let $\frac{p_{x_{k-1}}}{g_{x_k}} = m$. Therefore showing $m > 1$ is equivalent to showing inequality (1). We can write $\frac{p_{x_{k-1}}}{m} = g_{x_k}$. On applying componendo-dividendo rule, we get

$$\frac{p_{x_{k-1}} + m}{p_{x_{k-1}} - m} = \frac{g_{x_k} + 1}{g_{x_k} - 1} \tag{2}$$

Observe, showing $p_{x_{k-1}} - m < p_{x_{k-1}} - 1$ is equivalent to showing $m > 1$. Putting value of $p_{x_{k-1}} - m$ using equation (2) into above inequality gives

$$(p_{x_{k-1}} + m) \cdot \frac{g_{x_k} - 1}{g_{x_k} + 1} < (p_{x_{k-1}} - 1) = m g_{x_k} - 1 = m \cdot (g_{x_k} - \frac{1}{m})$$

Above inequality can be re-written as

$$\frac{p_{x_{k-1}} + m}{g_{x_k} + 1} \cdot \frac{g_{x_k} - 1}{g_{x_k} - \frac{1}{m}} < m$$

Substituting value of m and solving gives,

$$\frac{g_{x_k} - 1}{p_{x_{k-1}} - 1} < \frac{1}{m} = \frac{g_{x_k}}{p_{x_{k-1}}} \tag{3}$$

Hence showing inequality (3) is equivalent to showing $m > 1$. Now, adding 1 on both sides of inequality (3) gives,

$$\frac{(p_{x_{k-1}} + g_{x_k}) - 2}{p_{x_{k-1}} - 1} < \frac{(p_{x_{k-1}} + g_{x_k})}{p_{x_{k-1}}}$$

Putting $h = p_{x_{k-1}} + g_{x_k}$, $f = p_{x_{k-1}}$ and $g = g_{x_k}$ above inequality can be re-written as

$$\frac{h - 2}{f - 1} < \frac{h}{f} \tag{4}$$

Showing inequality (4) is equivalent to showing inequality (3).

Let $\frac{h-1}{f} = k'$, we can write $h = k'f + 1$. This gives,

$$\frac{h - 2}{f - 1} = \frac{k'f - 1}{f - 1} \text{ and } \frac{h}{f} = \frac{k'f + 1}{f} \tag{5}$$

observe as $g_{x_k} \geq 2$ and $p_{x_{k-1}} > 0$ therefore $\frac{g_{x_k} - 1}{p_{x_{k-1}}} > 0$ and hence

$$1 + \frac{g_{x_k} - 1}{p_{x_{k-1}}} = \frac{p_{x_{k-1}} + g_{x_k} - 1}{p_{x_{k-1}}} = \frac{h - 1}{f} = k' > 1$$

Now let
$$\frac{h-2}{f-1} = k \cdot \frac{h}{f}$$
putting in (4) tells showing $k < 1$ is equivalent to showing inequality (4).
Here using equation (5) we get,
$$k = \frac{f}{h} \cdot \frac{h-2}{f-1} = \frac{f}{k'f+1} \cdot \frac{k'f-1}{f-1}$$
Now we need to show $k < 1$ or $\frac{f}{k'f+1} \cdot \frac{k'f-1}{f-1} < 1$
So, let
$$\frac{k'f-1}{k'f+1} = a \cdot \frac{f}{f-1}$$
This leads to $k = \frac{af^2}{(f-1)^2}$ and now showing
$$\frac{af^2}{(f-1)^2} < 1 \tag{6}$$
is same as showing $k < 1$ which is equivalent to showing inequality (4). We
can obtain
$$a = \frac{(k'f-1) \cdot (f-1)}{(k'f+1) \cdot f}$$
This shows $a < 1$. Also as $k' > 1$, $k'f > f > 1$ and hence $a > 0$.
Now, let us consider
$$A1 = \frac{\frac{af^2}{(f-1)^2} + 1}{2} = \frac{(a+1)f^2 - 2f + 1}{2f^2 - 4f + 2}$$
As $A1$ is avg of $\frac{af^2}{(f-1)^2}$ and 1, hence showing $A1 < 1$ will ensure inequality
(6) [$a > 0$, $f > 0$, $1 > 0$ therefore $A1 > 0$]. So we need to show
$$A1 = \frac{(a+1)f^2 - 2f + 1}{2f^2 - 4f + 2} < 1$$
which is same as
$$(a+1)f^2 - 2f + 1 < 2f^2 - 4f + 2$$
It can also be re-written as
$$0 < (2f^2 - 4f + 2) - [(1+a)f^2 - 2f + 1] = (1-a)f^2 - 2f + 1$$
$$0 < (1-a)f^2 - 2f + 1 = (f - \frac{1}{1+\sqrt{a}})(f - \frac{1}{1-\sqrt{a}}) \tag{7}$$
Note that $\frac{1}{1-\sqrt{a}} > \frac{1}{1+\sqrt{a}}$
Now, two cases arise:-
case 1: $f > \frac{1}{1-\sqrt{a}} > \frac{1}{1+\sqrt{a}}$

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case 2: $\frac{1}{1-\sqrt{a}} > \frac{1}{1+\sqrt{a}} > f$

Observe case 2 is not possible as if $\frac{1}{1+\sqrt{a}} > f$, then $\frac{1}{f} > 1+\sqrt{a}$ but $1 > \frac{1}{f}$ and given $a > 0$ implies $\sqrt{a} > 0$ and therefore $1+\sqrt{a} > 1$, which leads to a contradiction. Hence, only case possible is case 1 and now for inequality (7) to hold, we need to show $f > \frac{1}{1-\sqrt{a}}$.

The above inequality is same as $1-\sqrt{a} > \frac{1}{f}$, to which multiplying -1 and adding 1 on both sides give

$$\sqrt{a} < 1 - \frac{1}{f}$$

We know the expression for a, hence we need to show

$$\sqrt{\frac{(k'f-1)(f-1)}{(k'f+f)f}} = \sqrt{a} < 1 - \frac{1}{f}$$

Let

$$\sqrt{(k'f-1)(f-1)} = f + x \ \text{ and } \ \sqrt{(k'f+1)f} = f + y \qquad (8)$$

Observe, $y > 0$, x can be both positive and negative and $y > x$ as $0 < a < 1$ Therefore the above inequality becomes $\frac{f+x}{f+y} < \frac{f-1}{f}$. On cross-multiplying we get, $f(f+x) < (f-1)(f+y)$. On further simplifying $xf < (y-1)f - y$. As $f > 0$, we can divide by f thus giving $x + 1 + \frac{y}{f} < y$.

Now, let

$$y = x + 1 + m' + \frac{y}{f} \qquad (9)$$

To show above inequality we will have to show $m' > 0$.

Observe, from equation (9) we get $y = \frac{x+1+m'}{1-\frac{1}{f}}$ but $y > 0$ and $f > 1$, therefore $x + 1 + m' > 0$

From equation (8), squaring both sides and taking LHS to RHS gives,

$$(1-k')f^2 + f(2x+1+k') + (x^2-1) = 0 \qquad (10)$$

Similarly for other equation, squaring and substituting value of y gives,

$$(1-k')f^2 + f[2(\frac{x+1+m'}{1-\frac{1}{f}}) - 1] + (\frac{x+1+m'}{1-\frac{1}{f}})^2 = 0 \qquad (11)$$

From equation (10), substituting value of $(1-k')f^2$ into equation (11) gives,

$$-f(2x+1+k') + f[2(\frac{x+1+m'}{1-\frac{1}{f}}) - 1] + (\frac{x+1+m'}{1-\frac{1}{f}})^2 - (x^2-1) = 0$$

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It can be re-written as

$$f[\frac{2x+2+2m'f}{f-1}-k'+\frac{1}{f}]+(\frac{x+1+m'}{1-\frac{1}{f}}-x)(\frac{x+1+m'}{1-\frac{1}{f}}+x)=0 \quad (12)$$

Let $f[\frac{2x+2+2m'f}{f-1}-k'+\frac{1}{f}]$ be term1 and $(\frac{x+1+m'}{1-\frac{1}{f}}-x)(\frac{x+1+m'}{1-\frac{1}{f}}+x)$ be term2.

Term2 can also be written as $(\frac{x+1+m'+\frac{(f-1)}{f}x}{1-\frac{1}{f}})(\frac{x+1+m'-\frac{(f-1)}{f}x}{1-\frac{1}{f}})$. Now as term1 + term2 =0 we have the following cases:-

case 1: term1 > 0 and term2 < 0 with |term1|=|term2|

case 2: term1 < 0 and term2 > 0 with |term1|=|term2|

case 3: term1 = term2 = 0

Now, consider $y > 0$ and $x > 0$

For case 1 :

$f > 0$, therefore $\frac{2x+2+2m'f}{f-1}-k'+\frac{1}{f} > 0$ and as $x > 0$, then $x > \frac{f-1}{f}x$.

So, term2 can be negative only when $m' < 0$ such that

$$x+1+\frac{(f-1)}{f}x > |m'| > x+1-\frac{(f-1)}{f}x$$

(so as to ensure one factor is negative and other is positive)

As $m' < 0$, $|m'| = -m'$ and from equation (9) we get $-m' = x+1+\frac{y}{f}-y$.

On putting this in above inequality we get,

$$\frac{(f-1)}{f}x > -y(\frac{f-1}{f}) > -\frac{(f-1)}{f}x$$

multiplying by $\frac{f}{(1-f)}$ gives

$$-x < y < x$$

But this implies $y < x$, which is a contradiction.

Hence case 1 is not possible.

For case 3:

We get $\frac{2x+2+2m'f}{f-1}-k'+\frac{1}{f} = 0$ and for term 2 to be zero, m' must be negative with (1). $|m'| = x+1-\frac{(f-1)}{f}x$ or (2). $|m'| = x+1+\frac{(f-1)}{f}x$. For (1), putting value of $-m'$ gives $y = x$ but then $y < x$ which is a contradiction. So (1) is not possible. For (2), putting value of $-m'$ gives $y = -x$ but then as $x > 0$ means $-x < 0$ and that leads to $y < 0$ which is again a contradiction. So (2) is not possible.

Hence case 3 is also not possible.

For case 2:

$f > 0$, therefore $\frac{2x+2+2m'f}{f-1}-k'+\frac{1}{f} < 0$ and for term 2 to be positive,

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either $m' < 0$ such that $|m'| > x + 1 + \frac{(f-1)}{f}x$. Putting value of $-m'$ gives $\frac{y}{f} - y > \frac{(f-1)}{f}x$ but then as $f > 1$ implies $y > \frac{y}{f}$ and this leads to $\frac{(f-1)}{f}x < 0$ which is a contradiction as $f > 1$ and x is positive. So $m' < 0$ is not possible. This leaves $m' \geq 0$ only possibility. But observe $m' = 0$ is not possible or else $y = 0$.

Hence the only possible option when $y > 0$ and $x > 0$ is $m' > 0$.

Now consider $y > 0$ and $x < 0$

Then from our observation earlier as $y > 0$ then $x + 1 + m' > 0$. This implies $m' + 1 > -x = |x|$. Also, as $|x| \geq 0$, we have $m' + 1 > 0$ or $m' > -1$. This leads to two cases:

case A: $0 \geq m' > -1$ and case B: $m' > 0$

Now again we consider the previous 3 cases.

For case 1:

As $f > 0$, therefore $\frac{2x+2+2m'f}{f-1} - k' + \frac{1}{f} > 0$ and for term 2 to be negative, we have following two cases:

(1). $1 + x + m' + x\frac{(f-1)}{f} > 0$ and $\frac{x}{f} + 1 + m' < 0$. Here the second inequality can be re-written as $(1 + m' + x) + (-x + \frac{x}{f}) < 0$, but observe as $x < 0$ and $1 > \frac{1}{f}$ implies $x(\frac{1}{f} - 1) = -x + \frac{x}{f} > 0$ and we know $1 + x + m' > 0$, hence it leads to contradiction as sum of two positive quantities cannot be negative. So, (1) is not possible.

(2). $1 + x + m' + x\frac{(f-1)}{f} < 0$ and $\frac{x}{f} + 1 + m' > 0$. The second inequality can be re-written as $1 + \frac{x}{f} > -m'$. Now if $m' < 0$ then $-m' > 0$ and putting $-m' = x + 1 + \frac{y}{f} - y$ gives $\frac{x}{f} - x > \frac{y}{f} - y$ which implies $x > y$ which is a contradiction. Suppose if $m' > 0$ then $1 + \frac{x}{f} > 0 > -m'$, which then will be true. Hence only $m' > 0$ will be possible.

For case 2:

As $f > 0$, therefore $\frac{2x+2+2m'f}{f-1} - k' + \frac{1}{f} < 0$ and for term 2 to be positive, we have following two cases:

(1). $1 + x + m' + x\frac{(f-1)}{f} < 0$ and $\frac{x}{f} + 1 + m' < 0$. But as showed above [in $y > 0$ and $x < 0$, case 1, (1)], this case is not possible.

(1). $1 + x + m' + x\frac{(f-1)}{f} > 0$ and $\frac{x}{f} + 1 + m' > 0$. But as showed above [in $y > 0$ and $x < 0$, case 1, (2)] $m' < 0$ will create contradiction. Hence only $m' > 0$ will be possible.

For case 3:

As $f > 0$, $\frac{2x+2+2m'f}{f-1} - k' + \frac{1}{f} = 0$ and for term 2, $x + 1 + m' + x - \frac{x}{f} = 0$ and $\frac{x}{f} + 1 + m' = 0$. Again, second equation can be re-written as $x + 1 +$

$m' - x + \frac{x}{f} = 0$ which implies $x + 1 + m' = x - \frac{x}{f}$. Now as $x < 0$ and $1 > \frac{1}{f}$ implies $x + 1 + m' < 0$ which leads to a contradiction. Hence, case 3 is not possible.

Hence the only possible option when $y > 0$ and $x < 0$ is $m' > 0$.

So, now we showed as $m' > 0$ then $y > x + 1 + \frac{y}{f}$. This implies $\frac{f+x}{f+y} < \frac{f-1}{f}$ holds, which is equivalent to $\sqrt{a} < 1 - \frac{1}{f}$. This shows $f > \frac{1}{1-\sqrt{a}}$ and therefore inequality (7) holds. This implies inequality (6) holds i.e. $k < 1$, which implies inequality (4) holds. As

$$\frac{(p_{x_{k-1}} + g_{x_k}) - 2}{p_{x_{k-1}} - 1} < \frac{(p_{x_{k-1}} + g_{x_k})}{p_{x_{k-1}}}$$

holds, this means inequality (3) holds which implies $p_{x_{k-1}} - m < p_{x_{k-1}} - 1$ holds. This means $m > 1$ or $p_{x_{k-1}} > g_{x_k}$.

As $p_{x_{k-1}} - g_{x_k} > 0$ and $p_{x_k} - g_{x_k} > p_{x_{k-1}} - g_{x_k}$, this implies $p_{x_k} - g_{x_k} > 0$ or can be re-written as

$$p_{x_k} > g_{x_k}$$

$\square$

Theorem 3.2. *For all $x \geq 1$, we have $p_{x+1} < 2p_x$.*

Proof. The lemma (3.1) ensures that the constructed set S does not have any $x_n > 2$ such that g_{x_n} is a max gap. Now consider any $x_i \in S$, as its not a max gap then by definition

$$\exists x_k < x_i \text{ such that } g_{x_k} > g_{x_i}$$

where g_{x_k} is a max gap with $x_k > 2$. As g_{x_k} is max gap, by lemma (3.1) $p_{x_k} > g_{x_k}$ and as $x_i \in S$, $g_{x_i} \geq p_{x_i}$. So, combining all gives $p_{x_k} > g_{x_k} > g_{x_i} \geq p_{x_i}$, which implies $p_{x_k} > p_{x_i}$. But as $x_i > x_k$, by property of prime numbers $p_{x_i} > p_{x_k}$. This leads to a contradiction. Now, this contradiction arises due wrong assumption that such x exists for which $p_x \leq g_x$. Hence there doesn't exists any such $x > 5$ such that $p_x \leq g_x$ or

for all $x > 5$ we have $p_x > g_x$, to which adding p_x on both sides give $2p_x > p_{x+1}$.

Manually checking for $x = 1, 2, 3, 4, 5$ also tells $2p_x > p_{x+1}$ holds true. $\square$

Theorem 3.3. *Suppose for all $n \geq 2$ we have $2p_n > p_{n+1}$, then there is atleast one prime p such that $n < p < 2n$.*

Proof. We have to show given $2p_n > p_{n+1}$ for all $n \geq 2$, there exists atleast one prime between n and $2n$. Let n be a prime, more specifically let it be the kth-prime i.e. $n = p_k$, then we know $p_k < p_{k+1}$ and given $p_{k+1} < 2p_k$, hence $p_k < p_{k+1} < 2p_k$ or we can write

$$n < p_{k+1} < 2n$$

Now, let n be a composite. As we know any natural number is either prime or composite, therefore given any natural number, it will be either prime or will lie between primes. Hence there exist a $k > 1$ such that $p_k < n < p_{k+1}$. Now $p_k < n$, hence $2p_k < 2n$ and given $p_{k+1} < 2p_k$ therefore we can write

$$p_k < n < p_{k+1} < 2p_k < 2n$$

This shows

$$n < p_{k+1} < 2n$$

Hence, both the cases show that there always exists at least one prime between n and 2n, given $n > 1$ thereby proving Bertrand's Postulate. □

4. Notes

This proof uses elementary methods and inequalities to arrive at the result. We first partition the list of prime gaps into two, one that contains max gaps and other that contain not max gaps. As observed earlier maximal gaps list is a subset of max gap list. It then starts with need to show for max gaps $p_{x_{k-1}} > g_{x_k}$ or $m > 1$, then we arrive at an equivalent inequality. We then use properties of arithmetical average, polynomial and factors. We also use properties of fractions. We then move on to show $y > x + 1 + \frac{y}{f}$ using m'. To show this, we contradict possibilities of m' being negative by using properties of y and x. Also, we remain in Domain of **R**. This shows $m' > 0$ and we trace back proving our claim. We then extend result to all prime gaps. We also show $p_x > g_x$ implies Bertrand's postulate. On numerical verification, we find out indeed $m' > 0$ for all y, x. Also all inequalities does hold true thereby verifying the proof.
The unique thing about this proof is that it does not rely on advanced analytic techniques and tools or combinatorial methods, it uses elementary methods and basic knowledge of primes to show the theorem. Also, its important because it opens up new dimensions in study of prime number

and distribution of gaps using elementary methods. The method used in this proof can also be generalized further.

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