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ORIGINAL ARTICLE

# Fast computation of complete elliptic integrals and Jacobian elliptic functions

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**Abstract** As a preparation step to compute Jacobian elliptic functions efficiently, we created a fast method to calculate the complete elliptic integral of the first and second kinds, $K(m)$ and $E(m)$, for the standard domain of the elliptic parameter, $0 < m < 1$. For the case $0 < m < 0.9$, the method utilizes 10 pairs of approximate polynomials of the order of 9–19 obtained by truncating Taylor series expansions of the integrals. Otherwise, the associate integrals, $K(1-m)$ and $E(1-m)$, are first computed by a pair of the approximate polynomials and then transformed to $K(m)$ and $E(m)$ by means of Jacobi's nome, $q$, and Legendre's identity relation. In average, the new method runs more-than-twice faster than the existing methods including Cody's Chebyshev polynomial approximation of Hastings type and Innes' formulation based on $q$-series expansions. Next, we invented a fast procedure to compute simultaneously three Jacobian elliptic functions, $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$, by repeated usage of the double argument formulae starting from the Maclaurin series expansions with respect to the elliptic argument, $u$, after its domain is reduced to the standard range, $0 \le u < K(m)/4$, with the help of the new method to compute $K(m)$. The new procedure is 25–70% faster than the methods based on the Gauss transformation such as Bulirsch's algorithm, `sncndn`, quoted in the Numerical Recipes even if the acceleration of computation of $K(m)$ is not taken into account.

**Keywords** Numerical methods · Complete elliptic integrals · Jacobian elliptic functions · Nome expansion · Innes' method · Encke's method

## 1 Introduction

### 1.1 Jacobian elliptic functions

Three Jacobian elliptic functions, $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$, are special functions used in various fields of mathematical physics and engineering (Abramowitz and Stegun 1964;

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Byrd and Friedman 1971). In Celestial Mechanics, they appear frequently in orbital mechanics (1) to solve a high-order solution of a resonance problem (Erdi and Kovacs 1993), (2) to describe the Kepler motion in terms of the elliptic anomaly (Brumberg and Fukushima 1994), (3) to construct planetary theories by using the elliptic anomaly (Chapront and Simon 1996; Brumberg and Brumberg 2001), (4) to solve two-body orbits perturbed by the galactic torque (Brasser 2001), (5) to obtain the secular motion under a low-order non spherical gravitational field (Scheeres and Hu 2001), (6) to express an analytical solution of the problem of two fixed centers (Varvoglis et al. 2004), and (7) to write a solution of an integrable perturbed two-body problem (Poleshchikov 2004).

Also they are used in rotational dynamics as (1) the periodic solutions of a Kovalevskaya top (El-Sabaa 1992), (2) the integrable case of a rotational motion of a gyrostat with and without a rotor (Cavas and Vigueras 1994; Elipe and Lanchares 2008), (3) the descriptions of torque-free rotation of a triaxial rigid body (Barkin 1999; Fukushima 2008a), (4) the construction of symplectic integrator for rotation (Breiter and Buciora 2000; Fukushima 2009a), and (5) the application of the implicit midpoint integrator for satellite attitude dynamics (Hellstrom and Mikkola 2009).

Triggered by the recent development of formulations to study rotational motions of a general triaxial rigid body (Fukushima 2008b,c, 2009b), we started to seek faster algorithms to compute elliptic functions and integrals. We published the first result for $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, $\mathrm{dn}(u|m)$, and the incomplete elliptic integral of the second and third kinds regarded as a function of that of the first kind, $\mathrm{en}(u|m) \equiv E(\mathrm{am}(u|m)|m)$ and $\mathrm{pn}(u, n|m) \equiv \Pi(\mathrm{am}(u|m), n|m)$ (Fukushima 2009a). The developed algorithms are based on the addition theorems and run quite fast when compared with the existing routines.

However, the obtained formulation is effective only in a special case to compute quite a few function values for different values of the elliptic argument, $u$, while the elliptic parameter, $m$, and the elliptic characteristic, $n$, are fixed. This situation is similar to that Prony accelerated, as early as in 1791, the simultaneous computation of sine and cosine function tables for a number of equidistant angles by repeated multiplications of a constant complex number (Osborne and Smyth 1991). A modern interpretation of Prony's technique is described in terms of the recurrence formulae of Chebyshev polynomials (Press et al. 2007).

At any rate, the representative of existing algorithms to compute the Jacobian elliptic functions for a randomly chosen pair of the elliptic argument and parameter, $u$ and $m$, is that due to the Gauss transformation and implemented by Bulirsch (1965a). It is so popular as to be quoted in the Numerical Recipes (Press et al. 2007). Nevertheless, it is true that these procedures are relatively time-consuming when compared with those of trigonometric functions provided in the standard mathematical function libraries. In fact, its CPU time is 11.4–16.7 times that of the sine function call as shown in Fig. 1. This is a significant computational load if we consider the frequent appearance of the functions in practical applications.

In order to improve this situation, we developed a fast procedure to compute the elliptic functions by means of the double argument formulae with respect to $u$. As illustrated in Fig. 1, the new procedure runs 25–70% faster than Bulirsch's method.

### 1.2 Complete elliptic integrals of first and second kinds

During the process to seek for a faster algorithm to compute Jacobian elliptic functions, we realized that essential is to establish a fast computation of the complete elliptic integral of the first kind, $K(m)$. This is because $4K(m)$ is the real period of the elliptic functions, and therefore, its numerical value is always required in reducing the given value of elliptic argument into the standard range, $0 \le u < K(m)/4$.

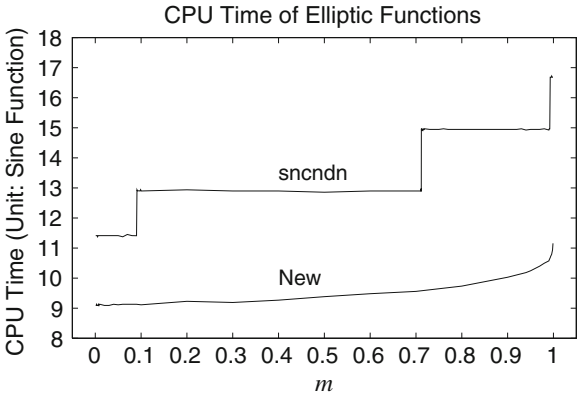


**Fig. 1** Shown are the averaged CPU times of two procedures to compute $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$ simultaneously; Bulirsch's `sncndn` (Press et al. 2007) and the new procedure explained in Sect. 4 of the main text. The CPU times are uniformly averaged with respect to $u$ in its standard domain, $0 < u < K(m)/4$, and shown as functions of the elliptic parameter, $m$, in its standard domain, $0 < m < 1$. Here $K(m)$ is the complete elliptic integral of the first kind (Byrd and Friedman 1971). The unit of CPU times is that to compute sine function. The result shown here does not include the effect of accelerated computation of $K(m)$, which is explained Sect. 3 in the main text

The complete elliptic integrals of the first and second kinds, $K(m)$ and $E(m)$, themselves frequently appear in various fields of mathematical physics and engineering (Abramowitz and Stegun 1964; Byrd and Friedman 1971). In astrophysics, for example, these are used in studying (1) the image generation formulae in a Kerr spacetime (Viergutz 1993), (2) the exact solutions of Einstein's field equation (Maharaj et al. 1996; Nolan 1998; Sussman and Triginer 1999; Conway 2000; Halburd 2001; Gair 2001), (3) the solution of an axisymmetric Poisson equation (Hure 2005; Pierens and Hure 2005), (4) the gravitational potential of power-law disks (Hure et al. 2008), (5) the self-gravitation of a mass element in a disk (Hure et al. 2009), and (6) the magnetic field of interplanetary coronal mass ejections (Nieves-Chinchilla et al. 2009).

In Celestial Mechanics and Dynamical Astronomy, the integrals are used in (1) developing the secular perturbation theories of orbital motions (Musen 1970), (2) describing the disk or ring potential (Krogh et al. 1982; Abad and Belizon 1997; Alberti and Vidal 2007), (3) discussing the torque-free rotation of a triaxial rigid body (Kinoshita 1992), (4) obtaining the orbit around a rotating black hole (Vokrouhlicky and Karas 1993), (5) solving the Jeans equation for triaxial galaxies (Ven et al. 2003), and (6) expressing the rotation curves of galactic disks (Pierens and Hure 2004).

The integrals are defined as

$$K(m) \equiv \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}}, \tag{1}$$

$$E(m) \equiv \int_0^{\pi/2} \left(\sqrt{1 - m\sin^2\theta}\right) \mathrm{d}\theta, \tag{2}$$

where $m$ is the elliptic parameter. The integrals are real-valued when $m \leq 1$.

The integrals for negative parameters are computed from those for the parameter in the standard domain, (0, 1), by using the transformation,

$$K(m) = \left(\frac{1}{\sqrt{1-m}}\right) K\left(\frac{-m}{1-m}\right), \quad E(m) = \left(\sqrt{1-m}\right) E\left(\frac{-m}{1-m}\right). \tag{3}$$

If $m = 0$, then $K(m)$ and $E(m)$ coincide with each other as

$$K(0) = E(0) = \frac{\pi}{2}. \tag{4}$$

When $m \to 1$, on the other hand, $K(m)$ goes to infinity while $E(m)$ remains finite as

$$\lim_{m\to 1} K(m) = \infty, \quad E(1) = 1. \tag{5}$$

From a practical point of view, however, we can treat $K(m)$ as finite-valued. This is because the asymptotic expansion of $K(m)$ near $m = 1$ is as logarithmic as

$$K\left(1 - m'\right) \approx \frac{1}{2} \log\left(\frac{16}{m'}\right) \sim 1.3862943611198906 - 0.5 \log m', \tag{6}$$

for small values of the complementary parameter

$$m' \equiv 1 - m. \tag{7}$$

As a result, even when $m'$ is equal to the machine epsilon of double precision arithmetics, $\epsilon \equiv 2^{-53} \sim 1.11 \times 10^{-16}$, $K(m)$ takes not-so-large value as $K(1 - \epsilon) \sim 19.75$.

Thus the problem is reduced to find numerical values of the integrals for the parameter in the standard domain, $0 < m < 1$. The existing methods to solve the problem are classified into four categories in the chronological order (Abramowitz and Stegun 1964); (1) the methods using the descending Landen transformation, which is mathematically equivalent with those based on Gauss' arithmetic-geometric mean, represented by Bulirsch's `cel` routine (Bulirsch 1965a,b, 1969a,b; Press et al. 1986), (2) the methods using rapid convergent series in terms of Jacobi's nome, $q$, as Innes provided (Innes 1902), (3) the approximation formulae using the Chebyshev polynomial expansions of Hastings type to compute the integrals in a sort of the mini-max sense as presented by Cody (1965a,b, 1966), Moshier (2000), and (4) the methods using the duplication theorems in Carlson's standard form of elliptic integrals (Carlson 1977, 1978, 1979; Carlson and Notis 1981; Press et al. 2007).

The precisions of these four methods are all satisfactory, say with the maximum error being less than around a dozen machine epsilons. Then the main question of concern is their speed. See Table 1, which compares the CPU times of these four major methods to compute (1) $K(m)$ only, (2) $E(m)$ only, and (3) $K(m)$ and $E(m)$ simultaneously, respectively. The listed CPU times are those uniformly averaged with respect to $m$ in the standard domain, $0 < m < 1$, and normalized by that to compute sine function. The table clearly shows that Cody's polynomial approximation of Hastings type is the best for computing $K(m)$ only or $E(m)$ only. Meanwhile, Innes' $q$-expansion treatment is the fastest for the simultaneous computation of $K(m)$ and $E(m)$.

However, from the author's viewpoint, there seems a room to accelerate the computation of the integrals. The key idea is the usage of Taylor series expansions of $K(m)$ and $E(m)$ with respect to $m$. As a result, we obtain the piecewise approximation polynomials in term of $m$. The resulting method is so fast as to outperform any of the existing methods. See Table 1 again.

**Table 1** Averaged CPU time to compute complete elliptic integrals

| Method | $K(m)$ | $E(m)$ | $K(m)$ & $E(m)$ |
|---|---|---|---|
| Carlson | 16.50 | 35.91 | 35.91 |
| Bulirsch | 4.70 | 4.76 | 11.02 |
| Innes | 2.79 | 3.13 | 3.13 |
| Cody | 2.22 | 2.03 | 3.42 |
| New | 1.15 | 1.10 | 1.41 |

Listed are the CPU times to compute (1) $K(m)$ only, (2) $E(m)$ only, and (3) $K(m)$ and $E(m)$ simultaneously. They are the results uniformly averaged with respect to $m$ in its standard domain, $0 < m < 1$, and scaled in the unit of that to compute sine function. Compared methods are (a) Carlson's functions $R_F$ and $R_D$ (Carlson 1979; Carlson and Notis 1981), (b) Bulirsch's `cel` routine (Bulirsch 1969a,b), (c) Innes' procedure using Jacobi's nome (Innes 1902), (d) Cody's Chebyshev polynomial approximations of Hastings type (Cody 1965a,b, 1966), and (e) the new procedure presented in Sect. 3 of the main text

### 1.3 Structure of present article

In this article, we report the above achievements of ours. For the sake of easy implementation of the newly developed methods, we reverse the order of their presentations. Namely, in Sect. 2, we first summarize the four existing methods to compute the complete elliptic integrals. Then, in Sect. 3, we describe our method to compute them. Finally, in Sect. 4, we present our procedure to compute the elliptic functions.

## 2 Existing methods to compute complete elliptic integrals

### 2.1 Duplication theorem

Carlson (1977) created a novel expression of elliptic integrals by rewriting the polynomial in the square root of the integrand in a symmetric manner with respect to the newly introduced parameters. He then derived the duplication theorem which reduces the parameters by a factor of 4096 (Carlson 1978). Using this powerful technique, he presented a practical computation procedure of general elliptic integrals (Carlson 1979; Carlson and Notis 1981). His method is so simple and universal as to be regarded as the standard treatise on elliptic integrals (Wolfram 2003; Press et al. 2007).

In terms of his formulation, $K(m)$ and $E(m)$ are computed as

$$K(m) = R_F\left(1, m', 1\right), \quad E(m) = R_F\left(1, m', 1\right) - \left(\frac{1}{3}\right) R_D\left(1, m', 1\right), \tag{8}$$

where

$$R_F(x, y, z) \equiv \frac{1}{2}\int_0^\infty \frac{\mathrm{d}t}{\sqrt{(t+x)(t+y)(t+z)}}, \tag{9}$$

$$R_D(x, y, z) \equiv \frac{3}{2}\int_0^\infty \frac{\mathrm{d}t}{(t+z)\sqrt{(t+x)(t+y)(t+z)}}, \tag{10}$$

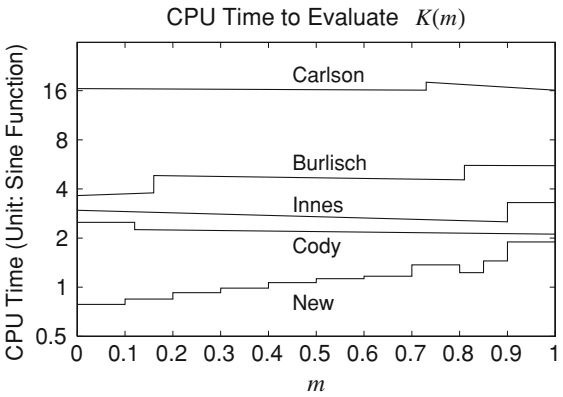


**Fig. 2** Shown are the CPU times of procedures to compute $K(m)$; Carlson's functions (Carlson 1979; Carlson and Notis 1981), Bulirsch's `cel` routine (Bulirsch 1969a,b), Innes' procedure using Jacobi's nome (Innes 1902), Cody's Chebyshev polynomial approximations of Hastings type (Cody 1965a,b, 1966), and the new procedure described in Sect. 3 of the main text. The CPU times are logarithmically plotted as functions of $m$ and measured in the unit of that to compute sine function

are two of four basic elliptic integrals introduced by Carlson. Sample programs of these functions are given in the latest version of Numerical Recipes (Press et al. 2007).

The duplication theorems are of linear convergence such that the resulting procedure is rather slow in terms of computational time. In fact, it turns out to be the slowest among the procedures we compared. See Figs. 2 through 4 for the $m$-dependence of the CPU times. The jump near $m \sim 0.73$ in the figure is due to the increase in the number of duplications required to assure the computational precision. Since the labor to compute $R_F$ and that to do $R_D$ are roughly the same, the ratio of the computational time of $K(m)$ only, that of $E(m)$ only, and that of both $K(m)$ and $E(m)$ is around 1:2:2. Refer to Table 1 again.

### 2.2 Arithmetic-geometric mean

By means of the well-known descending Landen transformations, which are based on Gauss' arithmetic-geometric mean, Bulirsch (1965a,b) provided three algorithms to compute the complete elliptic integrals; namely (1) that of the first kind, *cel1*, (2) that of a linear combination of the first and second kinds, *cel2*, and (3) that of the third kind, *cel3*, respectively. Later, in order to describe uniformly the complete elliptic integral of all three kinds, Bulirsch (1969a) introduced a general form of complete elliptic integral,

$$\mathrm{cel}\left(k', p, a, b\right) \equiv \int_0^\infty \frac{a + bt^2}{\left(1 + pt^2\right)\sqrt{\left(1 + t^2\right)\left[1 + \left(k'\right)^2 t^2\right]}}\,\mathrm{d}t, \tag{11}$$

and provided an algorithm to compute it by means of an extension of Bartkey's transformation, which is a sort of extension of the descending Landen transformation. Using this routine, one can compute $K(m)$ and $E(m)$ as

$$K(m) = \mathrm{cel}\left(k', 1, 1, 1\right), \quad E(m) = \mathrm{cel}\left(k', 1, 1, m'\right). \tag{12}$$

where

$$k' \equiv \sqrt{m'} = \sqrt{1 - m}, \tag{13}$$

is the complementary elliptic modulus. An implementation of the routine `cel` is originally given in Bulirsch (1969b). It is also found in the first version of Numerical Recipes (Press et al. 1986).

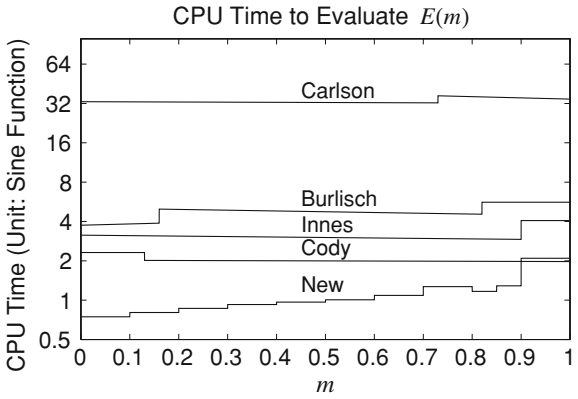


**Fig. 3** Same as Fig. 2 but for the computation of $E(m)$

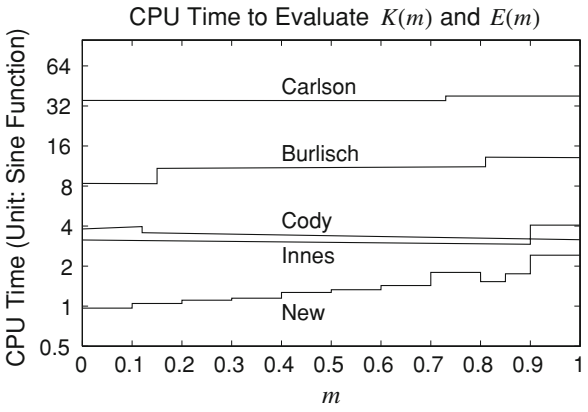


**Fig. 4** Same as Fig. 2 but for the simultaneous computation of $K(m)$ and $E(m)$

Bartkey transformation is of quadratic convergence. Therefore, `cel` runs significantly faster than Carlson's procedures. See Fig. 2 again. The jumps at $m \sim 0.16$ and $m \sim 0.81$ in the graph are due to the increases in the number of Bartkey transformations required to assure the computational precision. Since the computational time of `cel` weakly depends on its input variables, the ratio of the CPU time of $K(m)$ only, that of $E(m)$ only, and that of both $K(m)$ and $E(m)$ becomes around 1:1:2. See Figs. 3 and 4.

### 2.3 Nome expansion

Jacobi's nome, $q$, is defined as a function of $m$ as

$$q = q(m) \equiv \exp\left(\frac{-\pi K(m)}{K'(m)}\right). \tag{14}$$

where

$$K'(m) \equiv K\left(m'\right), \tag{15}$$

is the associate complete elliptic integral of the first kind. The numerical value of $q(m)$ corresponding to $m$ is significantly smaller than $m$ when $m$ is in its standard domain, $0 < m < 1$. Many quantities related to elliptic integrals are expanded in polynomial or rational functions of $q$. Some of them converge much faster than the similar expansions in terms of $m$ as will be

shown below. For example, Innes (1902) provided an efficient procedure to compute $K(m)$ and $E(m)$ in terms of $q(m)$. Since his treatment is not well advertised, we will explain its details below.

### 2.3.1 Case of small parameter

First, Innes introduced an auxiliary parameter[1] defined as a function of $m$ as

$$\lambda = \lambda(m) \equiv \frac{1}{2}\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right), \tag{16}$$

where $k'$ is the complementary modulus defined as a function of $m$ in Eq. 13. Due to a possible cancellation in the numerator, the definition formula itself is not suitable for numerical computation when $m$ is small, and therefore $k'$ is close to unity. In order to avoid the loss of information, we compute $\lambda$ as

$$\lambda = \frac{m}{2\,(1+k')\left(1+\sqrt{k'}\right)^2}, \tag{17}$$

as many authors suggested. Second, he computes $q$ from $\lambda$ by a series expansion formula as

$$q = \lambda + 2\lambda^5 + 15\lambda^9 + 150\lambda^{13} + 1707\lambda^{17} + 20910\lambda^{21} + \cdots. \tag{18}$$

The coefficients of the series expansion are easily obtained by formula processors. For example, one can obtain it in Mathematica (Wolfram 2003) as

```
Series[EllipticNomeQ [1-((1-2x)/(1+2x))^4], {x,0,30}]
```

where `x` stands for $\lambda$. Finally, Innes computes $K(m)$ and $E(m)$ by the formulae

$$K(m) = \left(\frac{\pi}{2}\right) A^2(q(m)), \quad E(m) = \left(\frac{\pi}{2}\right) \frac{B(q(m))}{A^2(q(m))C(q(m))}, \tag{19}$$

where $A(q)$, $B(q)$, and $C(q)$ are auxiliary functions of $q$ defined as

$$A(q) \equiv 1 + \Delta A(q), \quad B(q) \equiv 1 + \Delta B(q), \quad C(q) \equiv 1 + \Delta C(q), \tag{20}$$

$$\Delta A(q) \equiv 2\sum_{j=1}^{\infty} q^{(j^2)} \approx 2q + 2q^4 + 2q^9 + 2q^{16} + 2q^{25}, \tag{21}$$

$$\Delta B(q) \equiv \sum_{j=1}^{\infty} (2j+1)^2 q^{j(j+1)} \approx 9q^2 + 25q^6 + 49q^{12} + 81q^{20}, \tag{22}$$

$$\Delta C(q) \equiv \sum_{j=1}^{\infty} q^{j(j+1)} \approx q^2 + q^6 + q^{12} + q^{20}. \tag{23}$$

The series expressions of the auxiliary functions converge very rapidly. Thus the resulting expressions of $K(m)$ and $E(m)$ converge much faster than the $q$-expansion formulae given in Abramowitz and Stegun (1964) as

[1] Actually, he used $l \equiv 2\lambda$ instead.

$$K(m) = \frac{\pi}{2} + 2\pi \sum_{s=1}^{\infty} \frac{q^s}{1+q^{2s}}, \tag{24}$$

$$E(m) = K(m)\left[\frac{1+m'}{3} + \left(\frac{\pi}{K(m)}\right)^2 \left\{\frac{1}{12} - 2\sum_{s=1}^{\infty}\left(\frac{q^s}{1-q^{2s}}\right)^2\right\}\right]. \tag{25}$$

*2.3.2 Case of large parameter*

The convergence of the expansions of auxiliary functions becomes slow when $m$ is close to 1, say $m > 0.9$. In that case, Innes suggested[2] to compute the complementary nome

$$q' \equiv q\left(m'\right) = \exp\left(\frac{-\pi K'(m)}{K(m)}\right) \approx \left(\lambda'\right) + 2\left(\lambda'\right)^5 + 15\left(\lambda'\right)^9 + 150\left(\lambda'\right)^{13} + 1707\left(\lambda'\right)^{17} + 20910\left(\lambda'\right)^{21}, \tag{26}$$

by way of the complementary auxiliary parameter defined as

$$\lambda' \equiv \lambda\left(m'\right) = \frac{1}{2}\left(\frac{1-\sqrt{k}}{1+\sqrt{k}}\right) = \frac{m'}{2\left(1+k\right)\left(1+\sqrt{k}\right)^2}, \tag{27}$$

and obtain $K'(m)$ and

$$E'(m) \equiv E\left(m'\right), \tag{28}$$

by using a similar procedure as above. Then, Innes computes $K(m)$ by a formula

$$K(m) = \left(-\log q'\right)\left(\frac{K'(m)}{\pi}\right), \tag{29}$$

which is a rewriting of the definition of complementary nome. Also, he calculates $E(m)$ by another formula,

$$E(m) = K(m) + \frac{1}{K'(m)}\left(\frac{\pi}{2} - E'(m)K(m)\right), \tag{30}$$

which is obtained by solving the well-known Legendre's identity relation,

$$E(m)K'(m) + E'(m)K(m) - K(m)K'(m) = \frac{\pi}{2}, \tag{31}$$

with respect to $E(m)$.

*2.3.3 Care for reduction of round-off*

The above expression of $E(m)$ suffers round-off errors when $m'$ is small. In fact, its right hand side contains a subtraction of quantities of similar magnitude as

$$K(m) - \left(\frac{E'(m)K(m)}{K'(m)}\right)$$

[2] Innes and many other authors recommend to shift to the usage of the complementary nome when $m > 0.5$. From the viewpoint to minimize the computational labor, however, we find that the critical value for the shift should be larger, say $m = 0.9$. This is because the usage of complementary nome requires one additional call of the logarithmic function, which significantly increases the total CPU time.

since

$$E'(m) = E\left(m'\right) \approx \frac{\pi}{2}, \quad K'(m) = K\left(m'\right) \approx \frac{\pi}{2}, \tag{32}$$

if $m' \ll 1$. In order to reduce the resulting loss of precision, we rewrite Eq. 30 as

$$E(m) = \frac{1}{K'(m)}\left(\frac{\pi}{2} + K(m)H'\right), \tag{33}$$

where we introduced an auxiliary complete elliptic integral,

$$H(m) \equiv K(m) - E(m), \tag{34}$$

and the associate integral,

$$H' \equiv H\left(m'\right). \tag{35}$$

The magnitude of $H'$ is around two digits smaller than that of either $K\left(m'\right)$ or $E\left(m'\right)$. See Tables 2 and 13 later. As a result, the associated round-off errors would be reduced by a factor of 100 or so. The integral $H(m)$ is computed without a loss of information as

$$H(m) = \frac{\pi}{2}\left(\frac{\Delta A_4(q(m)) + A^4(q(m))\Delta C(q(m)) - \Delta B(q(m))}{A^2(q(m))C(q(m))}\right), \tag{36}$$

where

$$\Delta A_4(q) \equiv A^4(q) - 1 = \left(A^3(q) + A^2(q) + A(q) + 1\right)\Delta A(q). \tag{37}$$

Figure 5 illustrates that this treatment greatly reduces the round-off errors.

#### *2.3.4 CPU time comparison*

At any rate, all the above expansions in terms of $q$ or $q'$ converge quite rapidly. As a result, the formulation presented here significantly outperforms Bulirsch's `cel` routine based on Bartkey transformation. See Fig. 2 for example. The jump at $m = 0.9$ in the figure is due to the shift to use the complementary nome. This time, the main part of computational labor is the computation of $q$. It is common to all the cases. Therefore, the computational times of $K(m)$ only, of $E(m)$ only, and of both $K(m)$ and $E(m)$ are almost the same. See Figs. 3 and 4.

### 2.4 Chebyshev approximation

Following Hastings' idea (Hastings 1955), Cody (1965a,b, 1966) presented the Chebyshev polynomial approximations of the integrals as

$$K(m) \approx P\left(m'\right) - Q\left(m'\right)\log m', \quad E(m) \approx R\left(m'\right) - S\left(m'\right)\log m', \tag{38}$$

where $P\left(m'\right)$, $Q\left(m'\right)$, $R\left(m'\right)$, and $S\left(m'\right)$ are polynomials of the complementary parameter, $m'$. In the double precision environment, all the orders of the polynomials are 10. An implementation of the procedures in the double precision environment is found in the Cephes mathematical library (Moshier 2000).

The above representations of $K(m)$ and $E(m)$ are so simple and call only one transcendental function, $\log m'$. Thus the routines to compute them run fairly fast. See Fig. 2 for example. The jump at $m \sim 0.12$ in the graph is due to the change of internal algorithm of the logarithmic function in the standard mathematical library. The computational time of one of

these four polynomials is roughly the half of that of the logarithmic function. Therefore, the ratio of the computational time of $K(m)$ only, that of $E(m)$ only, and that of both $K(m)$ and $E(m)$ becomes around 1:1:1.5. See Figs. 3 and 4. As a result, this method runs faster than any of the methods described in the previous subsections for the computation of $K(m)$ only and of $E(m)$ only. However, for the case of the simultaneous computation of $K(m)$ and $E(m)$, the Chebyshev polynomial approximations run a little slower than the nome expansion. Refer to Table 1.

## 3 New method to compute complete elliptic integrals

### 3.1 Case of small parameter

The success of Chebyshev polynomial approximations of Hastings type viewed in the previous section naturally let us try to approximate the complete elliptic integrals by a piecewise polynomial of the elliptic parameter, $m$. To obtain such polynomial approximations simply, we use Taylor series expansions. By truncating the expansions around $m = m_0$ at certain orders, we obtain the approximate polynomials as expressed as

$$K(m) \approx \sum_{j=0}^{J_K} K_j \, (m - m_0)^j \,, \quad E(m) \approx \sum_{j=0}^{J_E} E_j \, (m - m_0)^j \,, \tag{39}$$

where the coefficients are defined as

$$K_j \equiv \frac{1}{j!} \left[ \left( \frac{\mathrm{d}}{\mathrm{d}m} \right)^j K(m) \right]_{m=m_0} \,, \quad E_j \equiv \frac{1}{j!} \left[ \left( \frac{\mathrm{d}}{\mathrm{d}m} \right)^j E(m) \right]_{m=m_0} \,. \tag{40}$$

It is easy to show that all $K_j$ and $E_0$ are positive definite. Meanwhile $E_j$ is negative definite as long as $j \neq 0$. Practically, the coefficients $K_j$ and $E_j$ are computed by formula processors. In case of Mathematica (Wolfram 2003), the series expansions are obtained by issuing a command such as

```
Series [EllipticK[m],{m,0.050000000000000000,25}]
```

Except the case when $m$ is close to 1, say when $m \geq 0.9$, we experimentally learn that this approach works well.

For simplicity, we fix the intervals of the piecewise polynomials as [0, 0.1), [0.1, 0.2), [0.2, 0.3), [0.3, 0.4), [0.4, 0.5), [0.5, 0.6), [0.6, 0.7), [0.7, 0.8), [0.8, 0.85), and [0.85, 0.9). Then, we set $m_0$, the zero point of the expansions, as the center of each intervals; 0.05, 0.15, 0.25, 0.35, 0.45, 0.55, 0.65, 0.75, 0.825, and 0.875, respectively. The reason why the last two intervals are of the half length is to reduce the order of polynomials, $J_K$ and $J_E$, at reasonably small numbers, say less than 20. Finally, we seek for the minimum order, $J_K$ or $J_E$, satisfying the condition that the truncation error of Taylor expansions is less than the error tolerance, which we set 5 machine epsilons. The resulting sets of coefficients are listed in Tables 2–11.

### 3.2 Case of large parameter

When $m$ is large, say $m \geq 0.9$, we basically follow Innes' approach. Namely, we first compute the associate integrals, $K'(m)$ and $E'(m)$, by the approximate polynomials described in the previous subsection, namely the coefficient of which are listed in Table 2. Then, we transform them into $K(m)$ and $E(m)$ by using Eqs. 29 and 33. This time, we accelerate the

**Table 2** Coefficients of Taylor expansion polynomials of $K(m)$ and $E(m)$ around $m = 0.05$

| $j$ | $K_j$ | $E_j$ |
|---|---|---|
| 0 | 1.591003453790792180 | +1.550973351780472328 |
| 1 | 0.416000743991786912 | −0.400301020103198524 |
| 2 | 0.245791514264103415 | −0.078498619442941939 |
| 3 | 0.179481482914906162 | −0.034318853117591992 |
| 4 | 0.144556057087555150 | −0.019718043317365499 |
| 5 | 0.123200993312427711 | −0.013059507731993309 |
| 6 | 0.108938811574293531 | −0.009442372874146547 |
| 7 | 0.098853409871592910 | −0.007246728512402157 |
| 8 | 0.091439629201749751 | −0.005807424012956090 |
| 9 | 0.085842591595413900 | −0.004809187786009338 |
| 10 | 0.081541118718303215 | |

For the parameter $m$ in the domain $0 \le m < 0.1$, the complete elliptic integral of the first and second kinds are approximately computed as

$$K(m) \approx \sum_{j=0}^{J_K} K_j \, (m - m_0)^j \,, \quad E(m) \approx \sum_{j=0}^{J_E} E_j \, (m - m_0)^j \,,$$

where $m_0 = 0.05$, $J_K = 10$, and $J_E = 9$

**Table 3** Coefficients of Taylor expansion polynomials of $K(m)$ and $E(m)$ around $m = 0.15$

| $j$ | $K_j$ | $E_j$ |
|---|---|---|
| 0 | 1.635256732264579992 | +1.510121832092819728 |
| 1 | 0.471190626148732291 | −0.417116333905867549 |
| 2 | 0.309728410831499587 | −0.090123820404774569 |
| 3 | 0.252208311773135699 | −0.043729944019084312 |
| 4 | 0.226725623219684650 | −0.027965493064761785 |
| 5 | 0.215774446729585976 | −0.020644781177568105 |
| 6 | 0.213108771877348910 | −0.016650786739707238 |
| 7 | 0.216029124605188282 | −0.014261960828842520 |
| 8 | 0.223255831633057896 | −0.012759847429264803 |
| 9 | 0.234180501294209925 | −0.011799303775587354 |
| 10 | 0.248557682972264071 | −0.011197445703074968 |
| 11 | 0.266363809892617521 | |

Same as Table 2 but for the domain $0.1 \le m < 0.2$. This time, $m_0 = 0.15$, $J_K = 11$, and $J_E = 10$

computation of complementary nome by using its Maclaurin series expansion with respect to the complementary parameter as

$$q' \equiv q\left(m'\right) \approx \sum_{j=1}^{J_q} q_j \left(m'\right)^j \,. \tag{41}$$

The coefficients, $q_j$, are obtained by formula processors. In case of Mathematica (Wolfram 2003), the Maclaurin series expansion is derived by issuing a command

**Table 4** Coefficients of Taylor expansion polynomials of $K(m)$ and $E(m)$ around $m = 0.25$

| $j$ | $K_j$ | $E_j$ |
|---|---|---|
| 0 | 1.685750354812596043 | +1.467462209339427155 |
| 1 | 0.541731848613280329 | −0.436576290946337775 |
| 2 | 0.401524438390690257 | −0.105155557666942554 |
| 3 | 0.369642473420889090 | −0.057371843593241730 |
| 4 | 0.376060715354583645 | −0.041391627727340220 |
| 5 | 0.405235887085125919 | −0.034527728505280841 |
| 6 | 0.453294381753999079 | −0.031495443512532783 |
| 7 | 0.520518947651184205 | −0.030527000890325277 |
| 8 | 0.609426039204995055 | −0.030916984019238900 |
| 9 | 0.724263522282908870 | −0.032371395314758122 |
| 10 | 0.871013847709812357 | −0.034789960386404158 |
| 11 | 1.057652872753547036 | |

Same as Table 2 but for the domain $0.2 \leq m < 0.3$. This time, $m_0 = 0.25$, $J_K = 11$, and $J_E = 10$

**Table 5** Coefficients of Taylor expansion polynomials of $K(m)$ and $E(m)$ around $m = 0.35$

| $j$ | $K_j$ | $E_j$ |
|---|---|---|
| 0 | 1.744350597225613243 | +1.422691133490879171 |
| 1 | 0.634864275371935304 | −0.459513519621048674 |
| 2 | 0.539842564164445538 | −0.125250539822061878 |
| 3 | 0.571892705193787391 | −0.078138545094409477 |
| 4 | 0.670295136265406100 | −0.064714278472050002 |
| 5 | 0.832586590010977199 | −0.062084339131730311 |
| 6 | 1.073857448247933265 | −0.065197032815572477 |
| 7 | 1.422091460675497751 | −0.072793895362578779 |
| 8 | 1.920387183402304829 | −0.084959075171781003 |
| 9 | 2.632552548331654201 | −0.102539850131045997 |
| 10 | 3.652109747319039160 | −0.127053585157696036 |
| 11 | 5.115867135558865806 | −0.160791120691274606 |
| 12 | 7.224080007363877411 | |

Same as Table 2 but for the domain $0.3 \leq m < 0.4$. This time, $m_0 = 0.35$, $J_K = 12$, and $J_E = 11$

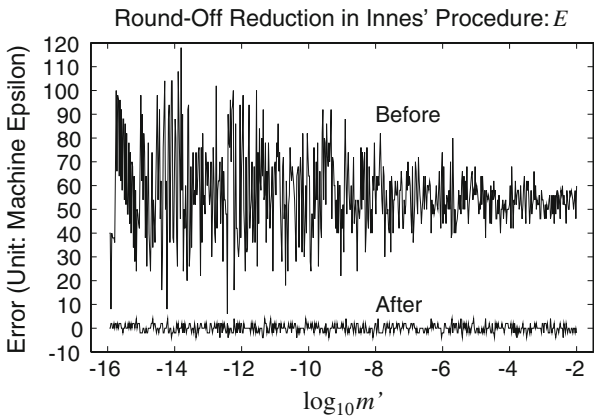


**Fig. 5** Shown are the error of $E(m)$ before and after the round-off reduction in Innes' procedure. The errors are counted as the deviation from the results of Cody's Chebyshev polynomial expressions and measured in the unit of $\epsilon$, the machine epsilon of double precision computation. In order to avoid the overlap of two figures, the errors before the reduction are shifted by $50\epsilon$

**Table 6** Coefficients of Taylor expansion polynomials of $K(m)$ and $E(m)$ around $m = 0.45$

| $j$ | $K_j$ | $E_j$ |
|---|---|---|
| 0 | 1.813883936816982644 | +1.375401971871116291 |
| 1 | 0.763163245700557246 | −0.487202183273184837 |
| 2 | 0.761928605321595831 | −0.153311701348540228 |
| 3 | 0.951074653668427927 | −0.111849444917027833 |
| 4 | 1.315180671703161215 | −0.108840952523135768 |
| 5 | 1.928560693477410941 | −0.122954223120269076 |
| 6 | 2.937509342531378755 | −0.152217163962035047 |
| 7 | 4.594894405442878062 | −0.200495323642697339 |
| 8 | 7.330071221881720772 | −0.276174333067751758 |
| 9 | 11.87151259742530180 | −0.393513114304375851 |
| 10 | 19.45851374822937738 | −0.575754406027879147 |
| 11 | 32.20638657246426863 | −0.860523235727239756 |
| 12 | 53.73749198700554656 | −1.308833205758540162 |
| 13 | 90.27388602940998849 | |

Same as Table 2 but for the domain $0.4 \leq m < 0.5$. This time, $m_0 = 0.45$, $J_K = 13$, and $J_E = 12$

**Table 7** Coefficients of Taylor expansion polynomials of $K(m)$ and $E(m)$ around $m = 0.55$

| $j$ | $K_j$ | $E_j$ |
|---|---|---|
| 0 | 1.898924910271553526 | +1.325024497958230082 |
| 1 | 0.950521794618244435 | −0.521727647557566767 |
| 2 | 1.151077589959015808 | −0.194906430482126213 |
| 3 | 1.750239106986300540 | −0.171623726822011264 |
| 4 | 2.952676812636875180 | −0.202754652926419141 |
| 5 | 5.285800396121450889 | −0.278798953118534762 |
| 6 | 9.832485716659979747 | −0.420698457281005762 |
| 7 | 18.78714868327559562 | −0.675948400853106021 |
| 8 | 36.61468615273698145 | −1.136343121839229244 |
| 9 | 72.45292395127771801 | −1.976721143954398261 |
| 10 | 145.1079577347069102 | −3.531696773095722506 |
| 11 | 293.4786396308497026 | −6.446753640156048150 |
| 12 | 598.3851815055010179 | −11.97703130208884026 |
| 13 | 1228.420013075863451 | |
| 14 | 2536.529755382764488 | |

Same as Table 2 but for the domain $0.5 \leq m < 0.6$. This time, $m_0 = 0.55$, $J_K = 14$, and $J_E = 12$

```
Series [EllipticNomeQ [m], {m, 0, 20}]
```

Table 12 lists the first 14 coefficients, which are sufficient to calculate $q'$ in the double precision environment for the interval, $0 \leq m' \leq 0.1$.

Also we directly provide the Taylor expansion of $H(m)$ for small parameters as

$$H(m) \approx \sum_{j=0}^{J_H} H_j \, (m - m_0)^j \,, \tag{42}$$

where the coefficients are defined as

$$H_j \equiv K_j - E_j, \tag{43}$$

**Table 8** Coefficients of Taylor expansion polynomials of $K(m)$ and $E(m)$ around $m = 0.65$

| $j$ | $K_j$ | $E_j$ |
|---|---|---|
| 0 | 2.007598398424376302 | +1.270707479650149744 |
| 1 | 1.248457231212347337 | −0.566839168287866583 |
| 2 | 1.926234657076479729 | −0.262160793432492598 |
| 3 | 3.751289640087587680 | −0.292244173533077419 |
| 4 | 8.119944554932045802 | −0.440397840850423189 |
| 5 | 18.66572130873555361 | −0.774947641381397458 |
| 6 | 44.60392484291437063 | −1.498870837987561088 |
| 7 | 109.5092054309498377 | −3.089708310445186667 |
| 8 | 274.2779548232413480 | −6.667595903381001064 |
| 9 | 697.5598008606326163 | −14.89436036517319078 |
| 10 | 1795.716014500247129 | −34.18120574251449024 |
| 11 | 4668.381716790389910 | −80.15895841905397306 |
| 12 | 12235.76246813664335 | −191.3489480762984920 |
| 13 | 32290.17809718320818 | −463.5938853480342030 |
| 14 | 85713.07608195964685 | −1137.380822169360061 |
| 15 | 228672.1890493117096 | |
| 16 | 612757.2711915852774 | |

Same as Table 2 but for the domain $0.6 \leq m < 0.7$. This time, $m_0 = 0.65$, $J_K = 16$, and $J_E = 14$

**Table 9** Coefficients of Taylor expansion polynomials of $K(m)$ and $E(m)$ around $m = 0.75$

| $j$ | $K_j$ | $E_j$ |
|---|---|---|
| 0 | 2.156515647499643235 | +1.211056027568459525 |
| 1 | 1.791805641849463243 | −0.630306413287455807 |
| 2 | 3.826751287465713147 | −0.387166409520669145 |
| 3 | 10.38672468363797208 | −0.592278235311934603 |
| 4 | 31.40331405468070290 | −1.237555584513049844 |
| 5 | 100.9237039498695416 | −3.032056661745247199 |
| 6 | 337.3268282632272897 | −8.181688221573590762 |
| 7 | 1158.707930567827917 | −23.55507217389693250 |
| 8 | 4060.990742193632092 | −71.04099935893064956 |
| 9 | 14454.00184034344795 | −221.8796853192349888 |
| 10 | 52076.66107599404803 | −712.1364793277635425 |
| 11 | 189493.6591462156887 | −2336.125331440396407 |
| 12 | 695184.5762413896145 | −7801.945954775964673 |
| 13 | 2567994.048255284686 | −26448.19586059191933 |
| 14 | 9541921.966748386322 | −90799.48341621365251 |
| 15 | 35634927.44218076174 | −315126.0406449163424 |
| 16 | 133669298.4612040871 | −1104011.344311591159 |
| 17 | 503352186.6866284541 | |
| 18 | 1901975729.538660119 | |
| 19 | 7208915015.330103756 | |

Same as Table 2 but for the domain $0.7 \leq m < 0.8$. This time, $m_0 = 0.75$, $J_K = 19$, and $J_E = 16$

**Table 10** Coefficients of Taylor expansion polynomials of $K(m)$ and $E(m)$ around $m = 0.825$.

| $j$ | $K_j$ | $E_j$ |
|---|---|---|
| 0 | 2.318122621712510589 | +1.161307152196282836 |
| 1 | 2.616920150291232841 | −0.701100284555289548 |
| 2 | 7.897935075731355823 | −0.580551474465437362 |
| 3 | 30.50239715446672327 | −1.243693061077786614 |
| 4 | 131.4869365523528456 | −3.679383613496634879 |
| 5 | 602.9847637356491617 | −12.81590924337895775 |
| 6 | 2877.024617809972641 | −49.25672530759985272 |
| 7 | 14110.51991915180325 | −202.1818735434090269 |
| 8 | 70621.44088156540229 | −869.8602699308701437 |
| 9 | 358977.2665825309926 | −3877.005847313289571 |
| 10 | 1847238.263723971684 | −17761.70710170939814 |
| 11 | 9600515.416049214109 | −83182.69029154232061 |
| 12 | 50307677.08502366879 | −396650.4505013548170 |
| 13 | 265444188.6527127967 | −1920033.413682634405 |
| 14 | 1408862325.028702687 | |
| 15 | 7515687935.373774627 | |

Same as Table 2 but for the domain $0.8 \leq m < 0.85$. This time, $m_0 = 0.825$, $J_K = 15$, and $J_E = 13$

**Table 11** Coefficients of Taylor expansion polynomials of $K(m)$ and $E(m)$ around $m = 0.875$

| $j$ | $K_j$ | $E_j$ |
|---|---|---|
| 0 | 2.473596173751343912 | +1.124617325119752213 |
| 1 | 3.727624244118099310 | −0.770845056360909542 |
| 2 | 15.60739303554930496 | −0.844794053644911362 |
| 3 | 84.12850842805887747 | −2.490097309450394453 |
| 4 | 506.9818197040613935 | −10.23971741154384360 |
| 5 | 3252.277058145123644 | −49.74900546551479866 |
| 6 | 21713.24241957434256 | −267.0986675195705196 |
| 7 | 149037.0451890932766 | −1532.665883825229947 |
| 8 | 1043999.331089990839 | −9222.313478526091951 |
| 9 | 7427974.817042038995 | −57502.51612140314030 |
| 10 | 53503839.67558661151 | −368596.1167416106063 |
| 11 | 389249886.9948708474 | −2415611.088701091428 |
| 12 | 2855288351.100810619 | −16120097.81581656797 |
| 13 | 21090077038.76684053 | −109209938.5203089915 |
| 14 | 156699833947.7902014 | −749380758.1942496220 |
| 15 | 1170222242422.439893 | −5198725846.725541393 |
| 16 | 8777948323668.937971 | −36409256888.12139973 |
| 17 | 66101242752484.95041 | |
| 18 | 499488053713388.7989 | |
| 19 | 37859743397240299.20 | |

Same as Table 2 but for the domain $0.85 \leq m < 0.9$. This time, $m_0 = 0.875$, $J_K = 19$, and $J_E = 16$

**Table 12** Coefficients of Maclaurin expansion of nome with respect to parameter

| $j$ | $q_j$ |
|---|---|
| 1 | 1/16 |
| 2 | 1/32 |
| 3 | 21/1024 |
| 4 | 31/2048 |
| 5 | 6257/524288 |
| 6 | 10293/1048576 |
| 7 | 279025/33554432 |
| 8 | 483127/67108864 |
| 9 | 435506703/68719476736 |
| 10 | 776957575/137438953472 |
| 11 | 22417045555/4398046511104 |
| 12 | 40784671953/8796093022208 |
| 13 | 9569130097211/2251799813685248 |
| 14 | 17652604545791/4503599627370496 |

Listed are the first 14 coefficients of Maclaurin expansion of Jacobi's nome, $q$, as a function of the elliptic parameter, $m$

**Table 13** Coefficients of Taylor expansion polynomial of $H(m)$ around $m = 0.05$

| $j$ | $H_j$ |
|---|---|
| 0 | 0.040030102010319852 |
| 1 | 0.816301764094985436 |
| 2 | 0.324290133707045355 |
| 3 | 0.213800336032498154 |
| 4 | 0.164274100404920649 |
| 5 | 0.136260501044421020 |
| 6 | 0.118381184448440078 |
| 7 | 0.106100138383995067 |
| 8 | 0.097247053214705841 |
| 9 | 0.090651779381423238 |
| 10 | 0.085627517951558365 |

Listed are Taylor expansion coefficients of $H(m) \equiv K(m) - E(m)$ for the parameter $m$ in the domain $0 \le m \le 0.1$. The function $H(m)$ is approximately computed as

$$H(m) \approx \sum_{j=0}^{J_H} H_j \, (m - m_0)^j \,,$$

where $m_0 = 0.05$ and $J_H = 10$

and are listed in Table 13 for the interval $0 \le m \le 0.1$.

### *3.2.1 Cost performance comparison*

Let us measure the cost and performance of the new method. Figure 6 illustrates the relative errors of $K$ and $E$ computed by the new method. The graphs show the deviation of the computed values from those obtained by Cody's Chebyshev polynomial approximations and normalized by the value of the integrals themselves. As seen there, the new method is sufficiently precise since the relative errors do not exceed five machine epsilons.

Let us move to the cost comparison. Table 1 and Figs. 2 through 4 have already shown that the new method is the fastest among the methods we compared. This is mainly due to the fact that no transcendental function is required for the main part of the parameter domain, $0 \le m < 0.9$.

These two results prove that the new method is of the highest cost performance in computing the complete elliptic integral of the first and/or second kinds.

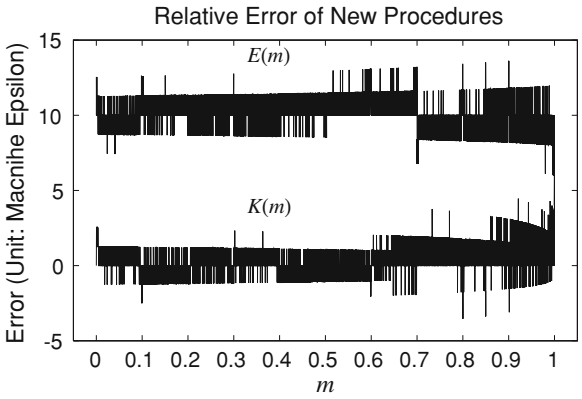


**Fig. 6** Shown are the relative error of the new procedure to compute $K(m)$ and $E(m)$. The errors are counted as the deviation from the results of Cody's Chebyshev polynomial expressions and measured in the unit of $\epsilon$, the machine epsilon of double precision computation. In order to avoid the overlap of two figures, the errors of $E(m)$ are shifted by $10\epsilon$

## 4 Fast computation of Jacobian elliptic functions

### 4.1 Reduction of parameter and argument domain

Let us consider to compute simultaneously the three Jacobian elliptic functions, $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$. Before going into the details, we explain how to reduce the domain of $m$ and $u$ into the standard one; $0 < m < 1$ and $0 < u < K(m)/4$, where $K(m)$ is the complete elliptic integral of the first kind.

First, we discuss the reduction in terms of $m$. When $m = 0$ or $m = 1$, the elliptic functions are expressed in terms of trigonometric or hyperbolic functions, respectively (Byrd and Friedman 1971). Meanwhile, the case when $m < 0$ or $m > 1$ can be reduced to the standard case by utilizing various transformations known in the literatures. An example of the reduction algorithm is provided in Sect. 3 of Fukushima (2009a).

Next, we consider the reduction of the domain of $u$ under the condition that $0 < m < 1$. We first note that the function values of the zero argument are independent on the value of $m$ as

$$\mathrm{sn}(0|m) = 0, \quad \mathrm{cn}(0|m) = \mathrm{dn}(0|m) = 1. \tag{44}$$

Then the reduction up to an intermediate domain, $0 < u < K(m)/2$, is easily conducted by various folding formulae with respect to $u$. As for a practical process, refer to Sect. 2.4 of Fukushima (2009a). In the final reduction to the standard domain, $0 < u < K(m)/4$, we use the transformation formulae;

$$\mathrm{sn}(K^* - u|m) = \frac{\sqrt{1+k'}\left[\mathrm{cn}(u|m)\mathrm{dn}(u|m) - k'\mathrm{sn}(u|m)\right]}{1 + k' - m\mathrm{sn}^2(u|m)}, \tag{45}$$

$$\mathrm{cn}(K^* - u|m) = \frac{\sqrt{k'(1+k')}\left[\mathrm{cn}(u|m) + \mathrm{sn}(u|m)\mathrm{dn}(u|m)\right]}{1 + k' - m\mathrm{sn}^2(u|m)}, \tag{46}$$

$$\mathrm{dn}(K^* - u|m) = \frac{\sqrt{k'}\left[(1+k')\mathrm{dn}(u|m) + m\mathrm{sn}(u|m)\mathrm{cn}(u|m)\right]}{1 + k' - m\mathrm{sn}^2(u|m)}, \tag{47}$$

where

$$K^* \equiv \frac{K(m)}{2}, \tag{48}$$

and

$$k' \equiv \sqrt{1-m}, \tag{49}$$

is the complementary elliptic modulus. The above formulae are rewriting of the addition theorems for the arguments $K^*$ and $-u$,

$$\mathrm{sn}(K^*-u|m) = \frac{\mathrm{sn}(K^*|m)\mathrm{cn}(u|m)\mathrm{dn}(u|m) - \mathrm{cn}(K^*|m)\mathrm{dn}(K^*|m)\mathrm{sn}(u|m)}{1-m\mathrm{sn}^2(K^*|m)\mathrm{sn}^2(u|m)}, \tag{50}$$

$$\mathrm{cn}(K^*-u|m) = \frac{\mathrm{cn}(K^*|m)\mathrm{cn}(u|m) + \mathrm{sn}(K^*|m)\mathrm{dn}(K^*|m)\mathrm{sn}(u|m)\mathrm{dn}(u|m)}{1-m\mathrm{sn}^2(K^*|m)\mathrm{sn}^2(u|m)}, \tag{51}$$

$$\mathrm{dn}(K^*-u|m) = \frac{\mathrm{dn}(K^*|m)\mathrm{dn}(u|m) + m\mathrm{sn}(K^*|m)\mathrm{cn}(K^*|m)\mathrm{sn}(u|m)\mathrm{cn}(u|m)}{1-m\mathrm{sn}^2(K^*|m)\mathrm{sn}^2(u|m)}, \tag{52}$$

where the special values for $u = K^*$ are given (Wolfram 2003) as

$$\mathrm{sn}(K^*|m) = \frac{1}{\sqrt{1+k'}}, \quad \mathrm{cn}(K^*|m) = \sqrt{\frac{k'}{1+k'}}, \quad \mathrm{dn}(K^*|m) = \sqrt{k'}. \tag{53}$$

### 4.2 Strategy

Assume that the elliptic parameter and the elliptic argument are in the standard range as $0 < m < 1$ and $0 < u < K(m)/4$. Our strategy to compute $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$ simultaneously is to combine their double argument formulae (Byrd and Friedman 1971),

$$\mathrm{sn}(2u|m) = \frac{2\mathrm{sn}(u|m)\mathrm{cn}(u|m)\mathrm{dn}(u|m)}{1-m\mathrm{sn}^4(u|m)}, \tag{54}$$

$$\mathrm{cn}(2u|m) = \frac{\mathrm{cn}^2(u|m) - \mathrm{sn}^2(u|m)\mathrm{dn}^2(u|m)}{1-m\mathrm{sn}^4(u|m)}, \tag{55}$$

$$\mathrm{dn}(2u|m) = \frac{\mathrm{dn}^2(u|m) - m\mathrm{sn}^2(u|m)\mathrm{cn}^2(u|m)}{1-m\mathrm{sn}^4(u|m)}, \tag{56}$$

and their Maclaurin series approximations described in Fukushima (2009a).

More specifically speaking, for the given value of $u$, we halve it repeatedly such that the Maclaurin series expansion is effective for the function computation of the reduced argument, $\delta u$. In other words, we find the minimum nonnegative integer, $n$, satisfying the condition

$$\delta u \equiv 2^{-n}u \le u_L, \tag{57}$$

where $u_L$ is a certain small number assuring the precision of the Maclaurin series expansion truncated at the $L$-th terms as

$$\left|\mathrm{sn}(\delta u|m) - \sum_{\ell=0}^{L}(-1)^\ell S_\ell(m)\,(\delta u)^{2\ell+1}\right| \le \delta, \tag{58}$$

$$\left|\mathrm{cn}(\delta u|m) - \sum_{\ell=0}^{L}(-1)^\ell C_\ell(m)\,(\delta u)^{2\ell}\right| \le \delta, \tag{59}$$

$$\left|\mathrm{dn}(\delta u|m) - \sum_{\ell=0}^{L}(-1)^\ell D_\ell(m)\,(\delta u)^{2\ell}\right| \le \delta, \tag{60}$$

where $\delta$ is the error tolerance of the approximation. The expansion coefficients, $S_\ell(m)$, $C_\ell(m)$, and $D_\ell(m)$, are seen in Byrd and Friedman (1971). See also Table 2 of Fukushima (2009a).

Then, we apply $n$ times the above double argument formulae, Eqs. 54 through 56, to the function values of the reduced argument, $\mathrm{sn}(\delta u|m)$, $\mathrm{cn}(\delta u|m)$, and $\mathrm{dn}(\delta u|m)$, in order to obtain the function values for the original argument, $u$.

### 4.3 Encke's method

The approach described in the previous subsection is seriously degraded by accumulation of round-off errors in the process of repeated application of the double argument formulae. In order to reduce the round-off errors, we introduce Encke's method. The key idea of Encke's method is to transform the variables into their deviation from the reference solutions which are analytically computable (Fukushima 1996). Since the elliptic argument $u$ is in the limited range as $0 < u < K/4$, we adopt the lowest order polynomial approximations as such approximates. In other words, we introduce the finite deviations from constants or linear functions as new functions as

$$\Delta\mathrm{sn}(u|m) \equiv u - \mathrm{sn}(u|m), \tag{61}$$

$$\Delta\mathrm{cn}(u|m) \equiv 1 - \mathrm{cn}(u|m), \tag{62}$$

$$\Delta\mathrm{dn}(u|m) \equiv 1 - \mathrm{dn}(u|m). \tag{63}$$

Then, we rewrite the double argument formulae, Eqs. 54 through 56, in terms of the newly introduced functions as

$$\Delta\mathrm{sn}(2u|m) = \frac{2}{1 - m\mathrm{sn}^4(u|m)}\Big[\mathrm{cn}(u|m)\mathrm{dn}(u|m)\Delta\mathrm{sn}(u|m) + u\Big\{\Delta\mathrm{cn}(u|m) + \Delta\mathrm{dn}(u|m) - \Delta\mathrm{cn}(u|m)\Delta\mathrm{dn}(u|m) - m\mathrm{sn}^4(u|m)\Big\}\Big], \tag{64}$$

$$\Delta\mathrm{cn}(2u|m) = \frac{[1 + \mathrm{cn}(u|m)]\,\Delta\mathrm{cn}(u|m) + \left[1 - 2m\mathrm{sn}^2(u|m)\right]\mathrm{sn}^2(u|m)}{1 - m\mathrm{sn}^4(u|m)}, \tag{65}$$

$$\Delta\mathrm{dn}(2u|m) = \frac{[1 + \mathrm{dn}(u|m)]\,\Delta\mathrm{dn}(u|m) + m\left[1 - 2\mathrm{sn}^2(u|m)\right]\mathrm{sn}^2(u|m)}{1 - m\mathrm{sn}^4(u|m)}, \tag{66}$$

the original function values, for double argument, being computed from their deviations by using definitions, Eqs. (61) through (63). This technique significantly suppresses the accumulation of round-off errors in the process of repeated application of double argument formulae. See Fig. 7.

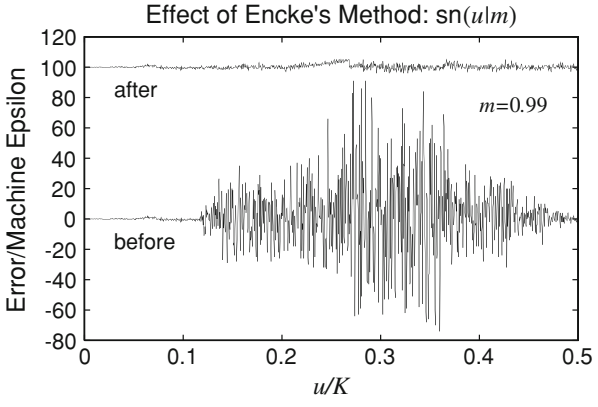


**Fig. 7** Illustrated are the errors before and after the application of Encke's method to reduce the round-off error accumulation. The graph of those after the application is shifted by 100 machine epsilons in order to avoid the overlap with those before the application. The errors are those of $\mathrm{sn}(u|m)$ for $m = 0.99$, measured as the difference from the function values obtained by Bulirsch's `sncndn`, and expressed in the unit of double precision machine epsilon, $\epsilon \sim 1.11 \times 10^{-16}$

### 4.4 Parameter tuning

The new procedure has two free parameters to be tuned; the order of approximate polynomial of the Maclaurin series expansion, $L$, and the critical value of the elliptic argument to apply the approximate polynomials, $u_L$. In general, the two parameters can be freely chosen. However, combinations of small $L$ and large $u_L$ lead to large approximation errors. In order to assure the precision of the new procedure, we impose a condition such that the resulting final errors of function values are of the order of a few machine epsilons such as illustrated in Fig. 8. For a fixed number of $L$, there is the maximum value of $u_L$ to satisfy the condition. After some trials, we found the optimal values as listed in Table 14.

On the other hand, pairs of large $L$ and small $u_L$ require longer computational times. To minimize the computational labor while keeping the precision of computed function values, we measured the averaged CPU time of elliptic functions by the new procedure as functions of $L$ while the relation between $u_L$ and $L$ is fixed as in Table 14. The result is shown in Fig. 9, from which we learn that $L = 4$ is the best value. Then the associated value of the critical argument is $u_L = 0.031$. Using this optimal pair of free parameters, we obtained the errors of the new procedure as illustrated in Fig. 8. Meanwhile, the averaged CPU time is already shown in Fig. 1.

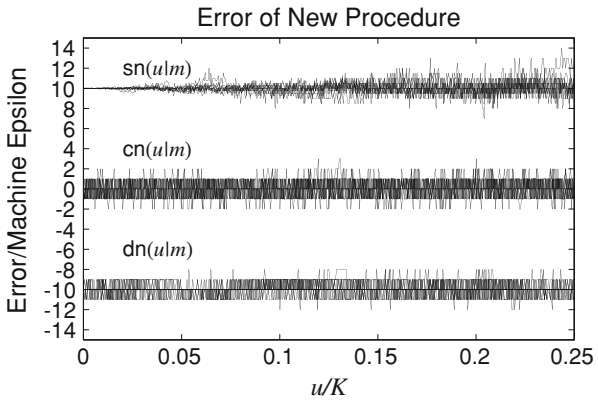


**Fig. 8** Drawn are the errors of the new procedure after the parameter tuning, namely $L = 4$ and $u_L = 0.031$. The results for the values of $m = 0, 0.1, 0.2, \ldots, 0.9, 0.99$ are overlapped. The graphs of $\mathrm{sn}(u|m)$ and $\mathrm{dn}(u|m)$ are shifted by $\pm 10\epsilon$. Again, the errors are measured as the difference from Bulirsch's `sncndn`, and expressed in the unit of double precision machine epsilon

**Table 14** Optimal values of critical elliptic argument for Maclaurin series approximates

| Order $L$ | Critical value $u_L$ |
|---|---|
| 1 | 0.000013 |
| 2 | 0.0013 |
| 3 | 0.010 |
| 4 | 0.031 |
| 5 | 0.031 |
| 6 | 0.029 |
| 7 | 0.029 |

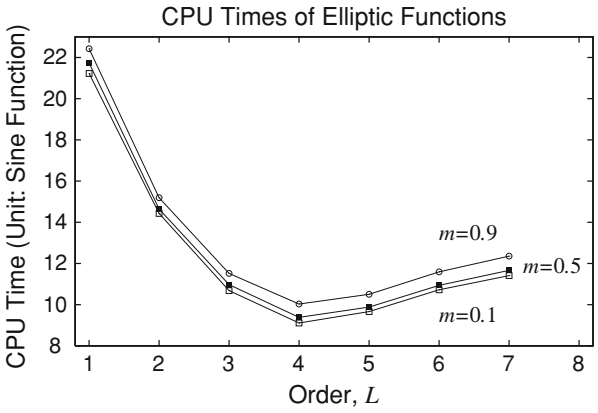


**Fig. 9** Graphed are the averaged CPU time of elliptic functions by the new procedure as functions of the order of approximate polynomial, $L$, while the relation between $L$ and the critical value to apply the approximate polynomial, $u_L$, is fixed as listed in Table 14. Insignificant is the dependence on the elliptic parameter, $m$

## 5 Conclusion

We developed a fast method to compute the complete elliptic integral of the first and/or second kinds, $K(m)$ and $E(m)$, for the standard domain of the elliptic parameter, $0 < m < 1$. This is done by using the combination of a piecewise polynomial obtained from truncated Taylor series expansions and the transformation from the associate integrals by means of Jacobi's nome and Legendre's identity relation. The new method is sufficiently precise such that the relative errors are less than five machine epsilon in the double precision arithmetics. Thanks to maximum utilization of polynomial approximations, the new method is more-than-twice faster than the existing procedures: (1) Cody's Chebyshev polynomial approximation of Hastings type, (2) Innes' expansion formulae in terms of Jacobi's nome, (3) Bulirsch's routine based on Gauss' arithmetic-geometric mean, and (4) Carlson's procedure using the duplication theorems.

Also, we created a new procedure to compute simultaneously three Jacobian elliptic functions, $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$. The procedure first reduces the domain of the input argument and parameter into their standard domain, $0 < u < K(m)/4$ and $0 < m < 1$, where $K(m)$ is computed by the new method described in the above. Next, it finds a nonnegative integer $n$ such that $\delta u \equiv 2^{-n}u \leq 0.031$. Third, it obtains the values of the deviation elliptic functions, $\Delta\mathrm{sn}(u|m) \equiv u - \mathrm{sn}(u|m)$, $\Delta\mathrm{cn}(u|m) \equiv 1 - \mathrm{cn}(u|m)$, and $\Delta\mathrm{dn}(u|m) \equiv 1 - \mathrm{dn}(u|m)$, for $\delta u$ by the fourth order polynomials derived from the Maclaurin series expansions. Fourth, the method computes the deviation elliptic function values for $u$ by applying $n$ times the double argument formulae of the deviation elliptic functions. Finally, the computed function values are transformed to the elliptic function values of the original input argument and parameter. The new method is sufficiently precise such that the errors of the computed function values are less than the machine epsilon in average. On the other hand, the new method runs fairly fast. In fact, its CPU time is 25–70% smaller than that of Bulirsch's routine, `sncndn`.

The Fortran 77 routines of the new method to compute $K(m)$ only, $E(m)$ only, and $K(m)$ and $E(m)$ together, and of the new procedure to compute $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$ simultaneously are available from the author on request.

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