Fourier and Fourier-Like Functions: Examples in Spectral Decomposition

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Abstract

Spectral decomposition is a powerful method for the approximation of functions. Here we start with a decomposition using Fourier's basic functions, the sine and cosine, over an interval and then extend the analysis by using the Zernike polynomials which extend the domain to the unit disk using polynomial weighting. In a similar process using the spherical harmonic functions, the Fourier series extends the domain to the unit sphere

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1 Introduction

Trigonometric functions (like sine and cosine) are initially defined on the interval $[0, 2\pi] .Zernikepolynomialsextendthese$

1.1 A formula by de Moivre

We begin the Taylor series for the exponential functions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{1}$$

Let $x \to i\theta$ and using the cyclic property $i^2 = ii = -1$, $i^3 = ii^2 = -i$, $i^4 = ii^3 = 1$, we find that after separating real and imaginary components, we are left with

$$e^{i\theta} = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...\right) + i\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) \tag{2}$$

$$\begin{array}{cccccc} e^{i\theta} & = & \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...\right) & + & i\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) \\ & & \cos\theta & + & i\sin\theta \end{array} \tag{3}$$

Trigonometric functions, such as sine and cosine, are fundamental tools in mathematics, especially in the study of periodic phenomena. However, these functions are typically limited to a one-dimensional interval. In this work, we explore how these basic trigonometric functions can be extended into higher dimensions. First, we look at how they can be extended to the unit disk using Zernike polynomials, and later to the unit sphere using spherical harmonics.

2 Trigonometric Functions on the Unit Interval

Consider the standard trigonometric functions, sine and cosine, defined on the interval $[0, 2\pi]$:

$$\begin{aligned} \cos(\theta) &= \frac{e^{i\theta} + e^{-i\theta}}{2}, \\ \sin(\theta) &= \frac{e^{i\theta} - e^{-i\theta}}{2i}. \end{aligned} \tag{4}$$

These functions describe oscillations and periodic phenomena in one-dimensional space. However, for many applications, such as those in higher-dimensional spaces, we need to extend these functions to handle the geometry of a disk or a sphere.

3 Extension to the Unit Disk: Zernike Polynomials

The first extension we consider is from the unit circle (i.e., the interval $[0, 2\pi]$) to the unit disk. The **Zernike polynomials** are a family of orthogonal polynomials that are defined on the unit disk. These polynomials provide a way to express functions over the unit disk and can be considered an extension of trigonometric functions.

The general form of the Zernike polynomials is:

$$Z_{n,m}(r,\theta) = R_{n,m}(r)\cos(m\theta), \tag{5}$$

where $R_{n,m}(r)$ are the radial components and m is an integer that determines the angular frequency. The radial components $R_{n,m}(r)$ are polynomials that ensure the orthogonality over the unit disk.

For example, the lowest order Zernike polynomials include:

$$\begin{aligned}
Z_{0,0}(r,\theta) &= 1,\\
Z_{1,1}(r,\theta) &= r\cos(\theta),\\
Z_{2,2}(r,\theta) &= r^2\cos(2\theta).
\end{aligned} \tag{6}$$

These polynomials capture the radial and angular behavior of a function over the disk. The extension from trigonometric functions to Zernike polynomials involves not just the angular components (like $\cos(m\theta)$) but also radial components, which account for the geometry of the disk.

4 Extension to the Unit Sphere: Spherical Harmonics

Next, we extend our study from the unit disk to the unit sphere. The **spherical harmonics** provide a natural extension of trigonometric functions to two dimensions of angular dependence: latitude and longitude on the sphere.

Spherical harmonics $Y_\ell^m(\theta,\phi)$ are defined on the sphere and have the form:

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)(\ell-|m|)!}{4\pi(\ell+|m|)!}}\,P_\ell^m(\cos\theta)e^{im\phi}, \tag{7}$$

where ℓ is the degree (a non-negative integer), m is the order ($-\ell \leq m \leq \ell$), θ is the polar angle (latitude), and ϕ is the azimuthal angle (longitude). The associated Legendre polynomials $P_\ell^m(\cos\theta)$ represent the radial part, while the exponential term captures the angular part.

For example, the lowest-order spherical harmonics are:

$$\begin{aligned}
Y_0^0(\theta,\phi) &= \frac{1}{\sqrt{4\pi}},\\
Y_1^0(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos(\theta),\\
Y_1^1(\theta,\phi) &= -\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{i\phi}.
\end{aligned} \tag{8}$$

The spherical harmonics allow the representation of functions on the surface of the sphere. The angular components include **sinusoidal** terms (just like in the unit disk), but in this case, they depend on both θ and ϕ. The spherical harmonics thus provide a rich set of basis functions for expanding functions on the sphere.

5 Conclusion

We have explored how basic trigonometric functions, such as sine and cosine, can be extended to higher-dimensional spaces. The extension to the **unit disk** via Zernike polynomials adds radial components to the angular functions, allowing for the representation of functions over the disk. Further, spherical harmonics provide an elegant extension to the **unit sphere**, where functions are described in terms of both latitude and longitude.

These extensions provide a powerful framework for representing functions on curved domains, with applications in physics, engineering, and mathematics, especially in the fields of approximation theory and spectral decomposition.

6 Example: The Heaviside Step Function

7 Decomposition of the Heaviside Step Function

The Heaviside step function $H(x)$ defined on the interval $(-\pi, \pi)$ can be written as a Fourier series:

$$H(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n} \sin\left(\frac{n\pi}{2}\right) e^{inx}, \tag{9}$$

where the Fourier coefficients are given by:

$$c_n = \frac{1}{\pi i n}(1 - (-1)^n). \tag{10}$$

8 Zernike Polynomials on the Top-Hat Function

The top-hat function on the unit disk can be expanded using Zernike polynomials:

$$\text{Top-Hat}(r) = \sum_{n,m} a_{n,m} Z_{n,m}(r, \theta), \tag{11}$$

with the coefficients given by:

$$a_{n,m} = \int_0^{2\pi} \int_0^1 \text{Top-Hat}(r) Z_{n,m}(r, \theta) r \, dr \, d\theta. \tag{12}$$

9 Weierstrass Approximation Theorem

The Weierstrass approximation theorem states that any continuous function defined on a closed interval can be uniformly approximated by polynomials:

$$\sup_{x\in[a,b]} |f(x) - P(x)| < \epsilon. \tag{13}$$

10 Pointwise and Uniform Convergence

- **Pointwise Convergence**: A sequence $f_n(x)$ converges pointwise to $f(x)$ if for each x, $\lim_{n\to\infty} f_n(x) = f(x)$. - **Uniform Convergence**: A sequence $f_n(x)$ converges uniformly to $f(x)$ if:

$$\sup_{x\in[a,b]} |f_n(x) - f(x)| \to 0 \quad \text{as} \quad n \to \infty. \tag{14}$$