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Sbornik: Mathematics **204**:7 1028–1055 *Matematicheskiĭ Sbornik* **204**:7 97–126

DOI 10.1070/SM2013v204n07ABEH004329

Approximation of Müntz-Szász type in weighted spaces

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Abstract. The paper looks at whether a system of exponentials $\exp(-\lambda_n t)$, $\operatorname{Re}\lambda_n > 0$, is complete in various function spaces on the half-line $\mathbb{R}_+$. Wide classes of Banach spaces E and F of functions on $\mathbb{R}_+$ are described such that this system is complete in E and F simultaneously. A test is established to determine when this system is complete in the weighted spaces C_0 and L^p with weight $(1+t)^\alpha$ on $\mathbb{R}_+$, for $\alpha < 0$ and $\alpha < -1$, respectively.

Bibliography: 18 titles.

Keywords: Müntz and Szász theorems, complete system of exponentials, spaces with combined norm, weighted spaces, Laplace transform.

§ 1. Introduction

The problem of describing complete systems of power functions

$$(x^{\mu_n})_{n=1}^{\infty}, \quad \text{where } \operatorname{Re}\mu_n > -\frac{1}{p}, \tag{1.1}$$

in the spaces $L^p(0,1)$ with $0 < p < \infty$ and $C_0[0,1] = (f \in C[0,1] : f(0) = 0)$ (in the case of C_0 we set $p = \infty$ in (1.1)) was prompted by Weierstrass's theorem. This problem is equivalent to describing systems of exponentials

$$e(\Lambda) = (e^{-\lambda_n t})_{n=1}^{\infty}, \quad \text{where } \operatorname{Re}\lambda_n > 0, \quad \Lambda = (\lambda_n)_{n=1}^{\infty}, \tag{1.2}$$

which are complete in the spaces

$$L^p = L^p(\mathbb{R}_+) \quad \text{and} \quad C_0 = C_0[0,\infty) = (f \in C[0,\infty)\colon f(t) \to 0,\ t \to +\infty)$$

(we make the substitution $x = e^{-t}$ and set $\lambda_n = \mu_n + 1/p$). The reason for going from (1.1) to (1.2) is to open a way to use complex analytic methods of investigation, when everything reduces to the problem of how the zeros of Laplace transforms of a certain special form are distributed in the right half-plane, although we have to limit ourselves to $1 \leqslant p < \infty$ in this case. In this paper we shall only deal with systems (1.2).

The fundamental results for this problem are due to Müntz [1] and Szász [2] (see also [3]); we state their theorems for the system (1.2).

This research was carried out with the support of the Russian Foundation for Basic Research (grant no. 13-01-00281).

AMS 2010 *Mathematics Subject Classification*. Primary 30B60; Secondary 46E15.

Theorem 1 (Müntz). *Let*

$$0 < \lambda_1 < \cdots < \lambda_n < \cdots . \tag{1.3}$$

Then the system (1.2) *is complete in* $L^2(\mathbb{R}_+)$ *(or* $C_0[0,\infty)$*) if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = +\infty.$$

Theorem 2 (Szász). *The system* (1.2) *is complete in* $L^2(\mathbb{R}_+)$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{\operatorname{Re}\lambda_n}{1+|\lambda_n|^2} = +\infty. \tag{1.4}$$

By Müntz-Szász type approximation authors usually mean the theory of the approximation properties of the system (1.2) in various function spaces on the half-line.

It is known (see, for instance, [4]) that for the system (1.2) to be complete in L^p, condition (1.4) (referred to here as the Szász condition) is necessary for $1 \leqslant p \leqslant 2$ and sufficient for $p \geqslant 2$; it is also sufficient in the space C_0. For $1 \leqslant p < 2$ it is not sufficient (see [5] and [6]) and it is not necessary in the space C_0 (see [7]); the question of whether it is necessary for $p > 2$ is open.

Necessary conditions for the system (1.2) to be complete in C_0 are due to Siegel [7] and Levinson [8]. It is claimed in [8] that if (1.2) is a complete system in C_0, then

$$\sum_{n=1}^{\infty} \frac{\operatorname{Re}\lambda_n + \exp(-\theta(|\lambda_n|))}{1+|\lambda_n|^2} = +\infty,$$

where $\theta(t)$, $t > 0$, is a positive nondecreasing function such that $\theta(t)/(1+t^2) \in L^1(\mathbb{R}_+)$.

The main problem consists in giving an adequate characterization of the behaviour of the λ_n close to the imaginary axis, which bounds the region where the Laplace transforms above are analytic. In this connection we can say that condition (1.3) in Müntz's theorem presents the most simple (but still important) pattern of distribution for the λ_n. In this case they are bounded away from the imaginary axis.

Gram [5] attempted to find the positions of the points λ_n in the right half-plane such that, if condition (1.4) holds, the system (1.2) will be complete in L^p, $1 \leqslant p < 2$. The position put forward in [5] is as follows: the λ_n lie in the union of the half-plane $\operatorname{Re} z \geqslant \delta > 0$ with a finite number of discs tangent to the imaginary axis.

In [9] we proposed a much more general condition:

$$\int_{\mathbb{R}} \frac{\log \operatorname{dist}(iy,\Lambda)}{1+y^2}\, dy > -\infty \tag{1.5}$$

and showed that if it holds, then (1.4) is necessary for the system (1.2) to be complete in L^p, $p > 2$, and C_0. Slightly later, Ladygin [10] proved that if (1.5) holds, then (1.4) ensures completeness in L^p, $1 \leqslant p < 2$ (see also [11] for a discussion of these results).

It was noted in [12] that the integral in (1.5) can only diverge to $-\infty$, so the statement in [9] can be formulated as follows: if the system (1.2) is complete in L^p, $p>2$, or C_0, then

$$\sum_{n=1}^{\infty} \frac{\operatorname{Re}\lambda_n}{1+|\lambda_n|^2} + \int_{\mathbb{R}} \frac{\log(1/\operatorname{dist}(iy,\Lambda))}{1+y^2}\,dy = +\infty. \tag{1.6}$$

The aim of this paper is to investigate conditions for the system (1.2) to be complete in weighted spaces on the half-line $\mathbb{R}_+$. One of our main results says that (1.6) *is a necessary and sufficient condition for the system* (1.2) *to be complete in the spaces* $C_{0,\alpha}$ with $\alpha<0$ *and* $L^{p,\alpha}$ with $1\leqslant p<\infty$ and $\alpha<-1$, that is, in the spaces C_0 and L^p with power-like weight $(1+t)^\alpha$ on the right half-line.

Note that the series in (1.6) diverges if there is a 'sufficient number' of points λ_n, but the integral diverges if the λ_n lie 'fairly close' to the imaginary axis.

Now we shall describe how we have distributed the material between the sections.

In § 2 we introduce the objects we shall consider in what follows and state and prove auxiliary results concerning these objects. In particular, starting from two Banach spaces $E_1=E_1(0,1)$ and $E_2=E_2(1,\infty)$ of functions on $(0,1)$ and $(1,\infty)$, respectively, we define a space (E_1,E_2) with combined norm, namely the space of functions f on the half-line $\mathbb{R}_+$ with the norm

$$\|f|_{(0,1)}\|_{E_1} + \|f|_{(1,\infty)}\|_{E_2}.$$

The closure of the linear span of the system of all exponentials $\exp(-\lambda t)$ with $\operatorname{Re}\lambda>0$ in the norm of (E_1,E_2) may turn out to be a proper subspace; in this case there is no sense in discussing whether the system (1.2) is complete in (E_1,E_2). So that we can discuss this even so, we distinguish a subspace of (E_1,E_2) satisfying some special conditions which coordinate the components $f|_{(0,1)}$ and $f|_{(1,\infty)}$ at the common end point 1 of the intervals $(0,1)$ and $(1,\infty)$; we denote this subspace by $(E_1,E_2)_{\mathrm{c}}$.

In § 3 we produce a broad class of spaces $E_1(0,1)$, $E_2(1,\infty)$ and weighted spaces $F=F(\mathbb{R}_+)$ on the half-line $\mathbb{R}_+$ such that the system (1.2) is complete in $(E_1,E_2)_{\mathrm{c}}$ if and only if it is complete in F (Theorem 3). In many cases this lets us look at the completeness problem for (1.2) in spaces which are more convenient to analyse. The proofs of Theorem 3 and its auxiliary lemmas in § 2 are based on the machinery of convolutions. Both the statements of problems and the approaches we use in this part of the paper develop [12], where the question of approximation by exponentials in spaces with combined norm was first considered. In § 3 we also prove that there are no bases of the form (1.2) in the spaces described by Theorem 3.

In § 4 we prove the result mentioned above (Theorem 5), that condition (1.6) holds if and only if (1.2) is complete in $C_{0,\alpha}$ with $\alpha<0$ (or in $L^{p,\alpha}$ with $1\leqslant p<\infty$ and $\alpha<-1$). With the help of Theorem 3 this can be extended to a broad class of spaces with combined norm. One special feature of the proof of Theorem 5 is the use of Bessel potentials. It seems they have not previously cropped up in Müntz-Szász type approximation.

Are the constraints on α essential in this test? In this respect we show that condition (1.6) is not sufficient for a system (1.2) to be complete in the spaces

$$L^{2,\alpha}\quad\text{with } \alpha\geqslant -1,\qquad L^{p,\alpha}\quad \text{with } p>2,\quad \alpha>-1,\qquad C_{0,\alpha}\quad\text{with } \alpha>0.$$

In § 5 we return to (1.4) and show that in neither the space L^p for $p \geqslant 2$ nor C_0 with weight Ω does this condition ensure that the system (1.2) is complete if $0 < \Omega(t) \to +\infty$ as $t \to +\infty$.

§ 2. Function spaces, functionals, and their Laplace transforms

2.1. Weighted spaces. Let $-\infty < a < b \leqslant +\infty$, $1 \leqslant p < \infty$. Let $\Omega(t)$ be a weight on (a,b), that is, a measurable function which is positive almost everywhere on (a,b). Then

$$L^p((a,b),\Omega(t)\,dt) = L^p_\Omega(a,b)$$

denotes the Banach space of functions with finite norm

$$\|f\|_{p,\Omega} = \|f\Omega^{1/p}\|_{L^p(a,b)}, \qquad 1 \leqslant p < \infty.$$

If b is finite we assume that $\Omega \in L^1(a,b)$; this ensures the topological embedding $C[a,b] \hookrightarrow L^p_\Omega(a,b)$, so that the functions in (1.2) belong to $L^p_\Omega(a,b)$.

If B is a Banach space we let B' denote the dual space, the space of functionals on B (by a 'functional' we always mean a 'continuous linear functional'). Then

$$(L^p_\Omega(a,b))' = L^{p'}(a,b), \qquad 1 \leqslant p < \infty, \quad \frac{1}{p'} + \frac{1}{p} = 1,$$

as is seen in the following realization of a functional $\varphi \in (L^p_\Omega(a,b))' = L^{p'}(a,b)$ (we identify a functional with the function or measure representing it):

$$(\varphi, f) = \int_a^b f(t)\Omega^{1/p}(t)\varphi(t)\,dt \quad \forall\, f \in L^p_\Omega(a,b).$$

Let $m \in \mathbb{Z}_+$ and $a > 0$. Then $W^m_{p,\Omega}(a,\infty)$ denotes the *weighted Sobolev space* on (a,∞), which consists of the functions f such that the functions $f^{(j)}$, $j = 0,\dots,m-1$, are absolutely continuous on $[a,b]$ for any $b \in (a,\infty)$, with the norm defined by

$$\|f\|_{W^m_{p,\Omega}(a,\infty)} = \sum_{j=0}^m \|f^{(j)}\|_{L^p_\Omega(a,\infty)}.$$

Note that $W^0_{p,\Omega}(a,\infty) = L^p_\Omega(a,\infty)$ and $W^m_{p,\Omega}(a,\infty)$ is a Banach space. We use the notation $W^m_{p,\Omega}$ in place of $W^m_{p,\Omega}(\mathbb{R}_+)$ and W^m_p in place of $W^m_{p,\Omega}$ in the weight-free case $\Omega \equiv 1$.

Now we look at the Cartesian $(m+1)$-fold product of spaces

$$L^p_\Omega(a,\infty) \times \dots \times L^p_\Omega(a,\infty). \tag{2.1}$$

Its dual is equal to the Cartesian $(m+1)$-fold direct product

$$(L^p(a,\infty))' \times \dots \times (L^p(a,\infty))',$$

and a general functional φ on (2.1) is determined by a tuple

$$\varphi = (\varphi_0,\dots,\varphi_m), \qquad \varphi_j \in (L^p(a,\infty))' = L^{p'}(a,\infty), \quad j = 0,\dots,m, \tag{2.2}$$

and can be realized as follows:

$$(\varphi, f) = \sum_{j=0}^{m} \int_a^\infty f_j(t)\Omega^{1/p}(t)\varphi_j(t)\,dt$$

for each $f = (f_0, \dots, f_m)$ in the space (2.1) (see, for instance [13], Ch. 1, § 4.6).

However, $W^m_{p,\Omega}(a,\infty)$ is a closed subspace of (2.1); its elements are tuples $(f, f', \dots, f^{(m)})$. Hence each functional on $W^m_{p,\Omega}(a,\infty)$ can be realized by some tuple of the form (2.2):

$$(\varphi, f) = \sum_{j=0}^{m} \int_a^\infty f^{(j)}(t)\Omega^{1/p}(t)\varphi_j(t)\,dt, \qquad f \in W^m_{p,\Omega}(a,\infty). \tag{2.3}$$

Now we introduce weighted spaces of continuous and continuously differentiable functions on $[a,\infty)$, $a > -\infty$, in the case when the weight Ω is continuous on $[a,\infty)$.

Let $C_{0,\Omega}[a,\infty)$ denote the space of continuous functions on $[a,\infty)$ such that

$$f(t)\Omega(t) \to 0, \qquad t \to \infty,$$

with the norm

$$\|f\|_{C_{0,\Omega}} = \sup(|f(t)|\Omega(t) : t \in [a,\infty)).$$

We also let $C^m_{0,\Omega}[a,\infty)$ denote the space of functions on $[a,\infty)$ with m continuous derivatives, with the norm

$$\|f\|_{C^m_{0,\Omega}} = \sum_{j=0}^{m} \|f^{(j)}\|_{C_{0,\Omega}}.$$

This is a Banach space. When $\Omega(t) \equiv 1$ we use the notation $C^m_{0,\Omega}$ in place of $C^m_{0,\Omega}[0,\infty)$ and C^m_0 in place of $C^m_{0,\Omega}$.

Let $V[a,\infty)$ denote the space of functions with bounded variation on $[a,\infty)$. Using the general form of functionals on C and arguing as we did in deducing (2.3) we conclude that a functional $d\sigma$ on $C^m_{0,\Omega}[a,\infty)$ can be realized by a tuple

$$\sigma = (\sigma_0, \dots, \sigma_m), \qquad \sigma_j \in V[a,\infty), \quad j = 0, \dots, m,$$

in accordance with the formula

$$(d\sigma, f) = \sum_{j=0}^{m} \int_a^\infty f^{(j)}(t)\Omega(t)\,d\sigma_j(t), \qquad f \in C^m_{0,\Omega}[a,\infty). \tag{2.4}$$

The identity

$$\int_{\mathbb{R}_+} f'(t)te^{-t}\,dt = -\int_{\mathbb{R}_+} f(t)\,d(te^{-t}),$$

where $f \in C^1_0[0,\infty)$ or $f \in W^1_p(\mathbb{R}_+)$, shows that for $m > 0$ such a realization of a functional on $C^m_{0,\Omega}$ or $W^m_{p,\Omega}$ is not unique (in the case of W^1_p, in integrating by parts we have taken account of the following property: if $f \in W^1_p(\mathbb{R}_+)$, then $f(t) \to 0$ as $t \to \infty$; see [12]). In this connection we say that a realization (2.4) or (2.3) of a functional is *regular* if we cannot reduce the order of any term in it (which is the order of the derivative appearing in this term).

2.2. Function spaces with combined norm on a half-line and their coordinated subspaces. Let $E_1 = E_1(0,1)$ and $E_2 = E_2(1,\infty)$ be Banach spaces of functions on $(0,1)$ and $(1,\infty)$, respectively. Then we let $E = (E_1, E_2)$ denote the space of functions f on the half-line $\mathbb{R}_+$ such that

$$f_1 = f|_{(0,1)} \in E_1, \qquad f_2 = f|_{(1,\infty)} \in E_2. \tag{2.5}$$

When endowed with the norm

$$\|f\|_E = \|f_1\|_{E_1} + \|f_2\|_{E_2}$$

E becomes a Banach space. Bearing (2.5) in mind we shall write $f = (f_1, f_2)$.

To introduce the next concept in a natural way we look at an example. Let $E_1 = C^s[0,1]$ and $E_2 = C^m[1,\infty)$, where $s, m, \in \mathbb{Z}_+$, and let $l = \min(s,m)$. Then the space $E = (E_1, E_2)$ contains functions f such that the equality

$$f_1^{(j)}(1-0) = f_2^{(j)}(1+0), \qquad j = 0, \dots, l, \tag{2.6}$$

fails for some j. However, if linear combinations of functions in the system $e(\Lambda)$ converge in $E = (E_1, E_2)$, then they and their derivatives of order up to l converge on both $[0,1]$ and $[1,2]$, so that they converge on $[0,2]$, that is, we have convergence on $[0,2]$ in the weakest of the norms in C^s and C^m. But this means that the functions $f^{(j)}(t)$ are continuous at $t = 1$ and the discussion of whether $e(\Lambda)$ can be complete in E makes no sense.

So to investigate the completeness of $e(\Lambda)$ we must single out the subspace of functions satisfying (2.6) in E. Condition (2.6) implies some suitable coordination between the components f_1 and f_2, and gives us some grounds to call this subspace *coordinated*.

Now we look at the general notion of a coordinated subspace. Let B_1 and B_2 be Banach spaces; if $B_1 \subseteq B_2$ and $\|f\|_{B_2} \leqslant M\|f\|_{B_1}$ for each $f \in B_1$, then we say that B_1 is *topologically embedded* in B_2 and write $B_1 \hookrightarrow B_2$. We also say in this case that the *norm in B_2 is weaker than the norm in B_1*. Let $B(a,b)$ be a Banach space of functions on (a,b). Let $B(a+h, b+h)$ denote the space of functions f on $(a+h, b+h)$ with the norm $\|f(t)\| = \|f(t-h)\|_{B(a,b)}$, and for $a < a' < b' < b$ let $B(a',b')$ denote the subspace of $B(a,b)$ consisting of the functions vanishing outside (a',b').

We say that two spaces $E_1(0,1)$ and $E_2(1,\infty)$ are *topologically comparable relative to the point* 1 if for some $\varepsilon \in (0,1)$ one of the spaces

$$E_1(1-\varepsilon, 1+\varepsilon) \quad \text{and} \quad E_2(1-\varepsilon, 1+\varepsilon)$$

is topologically embedded in the other; let E_w denote the space with the weaker norm. Then the *coordinated subspace* of $E = (E_1, E_2)$ is the subspace of E that is closed with respect to the norm

$$\|f_1\|_{E_1} + \|f_2\|_{E_2} + \|f|_{(1-\varepsilon,1+\varepsilon)}\|_{E_w}, \qquad f = (f_1, f_2).$$

We denote the coordinated subspace of E by $E_{\mathrm{c}} = (E_1, E_2)_{\mathrm{c}}$.

The space E_{c} is not necessarily a proper subspace of E. For example, if $E_1 = L^1(0,1)$ and $E_2 = L^2(1,\infty)$, then $E_{\mathrm{c}} = E$. Nevertheless, to unify the statements we shall use the term 'coordinated subspace' and the same notation in this case too.

2.3. Laplace transforms of functionals. Let φ ($d\sigma$) be a functional on $W^m_{p,\Omega}(\mathbb{R}_+)$ (on $C^m_{0,\Omega}[0,\infty)$, respectively). The *Laplace transform* of φ (of $d\sigma$) is the function

$$(\varphi(t), e^{-zt}) \quad ((d\sigma(t), e^{-zt}), \quad \text{respectively}), \qquad \operatorname{Re} z > 0.$$

If

$$\varphi = (\varphi_0, -\varphi_1, \dots, (-1)^m \varphi_m)$$
$$(d\sigma = (d\sigma_0, -d\sigma_1, \dots, (-1)^m d\sigma_m)),$$

then for $\operatorname{Re} z > 0$, from (2.3) and (2.4) we obtain

$$\Phi(z) := (\varphi(t), e^{-zt}) = \sum_{j=0}^{m} z^j \int_0^\infty e^{-zt} \Omega^{1/p}(t) \varphi_j(t)\, dt, \qquad \varphi_j \in L^{p'}(\mathbb{R}_+), \tag{2.7}$$

$$\Phi(z) := (d\sigma(t), e^{-zt}) = \sum_{j=0}^{m} z^j \int_0^\infty e^{-zt} \Omega(t)\, d\sigma_j(t), \qquad \sigma_j \in V[0,\infty). \tag{2.8}$$

Obviously, the condition

$$\limsup_{t\to\infty} \frac{\log \Omega(t)}{t} \leqslant 0 \tag{2.9}$$

ensures that the integrals in (2.7) and (2.8) converge uniformly in each half-plane of the form $\operatorname{Re} z \geqslant \delta > 0$ and therefore $\Phi(z)$ is analytic in the half-plane $\operatorname{Re} z > 0$.

Since a system of elements of a Banach space B is incomplete if and only if some nontrivial functional on B annihilates this system, we have the following result.

Lemma 1. *Let E be one of the spaces $W^m_{p,\Omega}(\mathbb{R}_+)$, $p \geqslant 1$, or $C^m_{0,\Omega}[0,\infty)$, where Ω is a weight satisfying* (2.9).

Then a system $e(\Lambda)$ is incomplete in E if and only if there exists a nontrivial functional on E such that its Laplace transform $\Phi(z)$ vanishes at the points Λ.

It is well known that *if $m = 0$ and $\Omega \equiv 1$, then a functional on E is trivial if and only if $\Phi(z) \equiv 0$.* Going over from $\Phi(z)$ to $\Phi(z+\epsilon)$ for an arbitrary $\epsilon > 0$ we see that this equivalence holds for $m = 0$ if the weight satisfies (2.9). We shall also need this fact for $m > 0$, but we need to prove this.

Lemma 2. *Let $h \in L^1(0,a)$ with $a > 0$, let $h \equiv 0$ outside $(0,a)$, and assume that the weight $\Omega(t)$, $t \geqslant 0$, satisfies the following condition: there exists $M > 0$ such that*

$$\Omega(t) \leqslant M\Omega(x), \qquad \max(0, x-a) < t < x, \quad x > 0. \tag{2.10}$$

Then the following results hold:

1) *if $\sigma \in V[0,\infty)$ and $\Omega \in C[0,\infty)$, then*

$$\frac{h * (\Omega\, d\sigma)}{\Omega} \in L^1(\mathbb{R}_+);$$

2) *if $\varphi \in L^{p'}(\mathbb{R}_+)$, $1 \leqslant p' \leqslant \infty$, then*

$$\frac{h * (\Omega^{1/p}\varphi)}{\Omega^{1/p}} \in L^{p'}(\mathbb{R}_+).$$

Proof. We shall use (2.10) and the following two well-known properties of convolutions. Let $f \in L^1(\mathbb{R})$; then

(a) if $\sigma \in V[0,\infty)$, then $f * d\sigma \in L^1(\mathbb{R})$;

(b) if $g \in L^q(\mathbb{R})$, where $1 \leqslant q \leqslant \infty$, then $f * g \in L^q(\mathbb{R})$.

1) Let $v(t) = \operatorname{var}(\sigma(x) : 0 \leqslant x \leqslant t)$. Then

$$(h * (\Omega\, d\sigma))(x) = \int_{\mathbb{R}_+} h(x-t)\Omega(t)\, d\sigma(t), \qquad x > 0, \tag{2.11}$$

where (since $\operatorname{supp} h \subset [0,a]$) in fact we integrate over the interval $[\max(0, x-a), x]$. Using this observation and condition (2.10) we obtain

$$|(h * (\Omega\, d\sigma))(x)| \leqslant M\Omega(x) \int_{\mathbb{R}_+} |h(x-t)|\, dv(t). \tag{2.12}$$

However, the last integral is the convolution $|h| * dv$, where $|h| \in L^1(\mathbb{R}_+)$ and $v \in V[0,\infty)$. By property (a) of convolutions this integral belongs to $L^1(\mathbb{R}_+)$, which proves 1).

2) The only difference from the above consists in using property (b) of convolutions in place of (a). We have

$$(h * (\Omega^{1/p}\varphi))(x) = \int_{\mathbb{R}_+} h(x-t)\Omega^{1/p}(t)\varphi(t)\, dt, \qquad \varphi \in L^{p'}(\mathbb{R}_+),$$

where the set of integration is in fact the same as in (2.11). Hence by (2.10) we have an estimate similar to (2.12):

$$|(h * (\Omega^{1/p}\varphi))(x)| \leqslant M\Omega^{1/p}(x) \int_{\mathbb{R}_+} |h(x-t)\varphi(t)|\, dt.$$

The last integral is the convolution $|h| * |\varphi|$, where $|h| \in L^1(\mathbb{R}_+)$ and $|\varphi| \in L^{p'}(\mathbb{R}_+)$. By property (b) this convolution lies in $L^{p'}(\mathbb{R}_+)$, which yields 2). The proof is complete.

Lemma 3. *Let $\Phi(z)$ be the Laplace transform (2.8) of a functional $d\sigma$ acting on $C^m_{0,\Omega} = C^m_{0,\Omega}[0,\infty)$, $m \in \mathbb{Z}_+$ (or the Laplace transform (2.7) of a functional φ on $W^m_{p,\Omega} = W^m_{p,\Omega}(\mathbb{R}_+)$ with $1 \leqslant p < \infty$), where the positive continuous weight Ω satisfies (2.9) and (2.10).*

Then the following results hold:

1) *if $m \geqslant 1$, then for each $a > 0$ the function*

$$\Phi_a(z) := \Phi(z)\frac{1-e^{-az}}{z}, \qquad \operatorname{Re} z > 0,$$

is the Laplace transform of a functional on $C^{m-1}_{0,\Omega}$ (on $W^{m-1}_{p,\Omega}$, respectively);

2) *the condition*

$$\Phi(z) \equiv 0, \qquad \operatorname{Re} z > 0 \tag{2.13}$$

holds if and only if $d\sigma$ (φ, respectively) is the trivial functional.

Proof. 1) We look at the case of $C^m_{0,\Omega}$. Then

$$\Phi(z) = \sum_{j=0}^{m} z^j \int_0^\infty e^{-zt}\Omega(t)\, d\sigma_j(t), \qquad \operatorname{Re} z > 0, \quad \sigma_j \in V[0,\infty). \tag{2.14}$$

Multiply (2.14) by

$$\frac{1-e^{-az}}{z}, \qquad a > 0,$$

which is the Laplace transform of the indicator $\chi_a(t)$ of the interval $(0,a)$. By the theorem on the Laplace transform of a convolution

$$\Phi_a(z) = \int_{\mathbb{R}_+} e^{-zt}(\chi_a * (\Omega\, d\sigma_0))\, dt + (1-e^{-az}) \sum_{j=1}^{m} z^{j-1} \int_{\mathbb{R}_+} e^{-zt}\Omega(t)\, d\sigma_j(t). \tag{2.15}$$

Next we observe that if $F(z)$ is the Laplace transform of the measure $\Omega\, d\sigma$, then

$$F(z)e^{-az} = \int_a^\infty e^{-zt}\Omega(t-a)\, d\sigma(t-a)$$

is the Laplace transform of the measure

$$\Omega(t)\, d\sigma^a(t), \tag{2.16}$$

where $d\sigma^a(t) \equiv 0$ on $[0,a)$ and

$$d\sigma^a(t) = g_a(t)\, d\sigma(t-a), \quad t \geqslant a, \quad \text{where} \quad g_a(t) = \frac{\Omega(t-a)}{\Omega(t)}.$$

In view of (2.10), since Ω is continuous and positive, we have

$$g_a(t) \in L^\infty(\mathbb{R}_+) \cap C[a,\infty).$$

Hence the measure (2.16) defines a functional on $C^0_{0,\Omega}$.

Using the above notation we can write (2.15) as follows:

$$\begin{aligned}\Phi_a(z) = \int_{\mathbb{R}_+} e^{-zt}\big((\chi_a * (\Omega\, d\sigma_0))(t)\, dt + \Omega(t)\, d\sigma_1(t) - \Omega(t)\, d\sigma_1^a(t)\big) \\ + \sum_{j=1}^{m-1} z^j \int_{\mathbb{R}_+} e^{-zt}\Omega(t)(d\sigma_{j+1}(t) - d\sigma^a_{j+1}(t)), \qquad \operatorname{Re} z > 0. \end{aligned}\tag{2.17}$$

By assertion 1) in Lemma 2

$$(\chi_a * (\Omega\, d\sigma_0))(t) = \Omega(t) g(t), \qquad g \in L^1(\mathbb{R}_+).$$

Hence the measure $(\chi_a * (\Omega\, d\sigma_0))\, dt$ defines a functional on $C^0_{0,\Omega}$. Thus, (2.17) is the Laplace transform of a functional on $C^{m-1}_{0,\Omega}$, which proves 1) for the space $C^m_{0,\Omega}$.

The case of the space $W^m_{0,\Omega}$ is similar. The only modification is that assertion 2) in Lemma 2 is used in place of 1).

The following observation will be important in what follows. If the functional in (2.14) has a regular realization, then the functional (2.17) also has a regular realization.

Now we prove this. We look at the term with index j in (2.17):

$$z^j \int_{\mathbb{R}_+} e^{-zt}\Omega(t)\, d\sigma_{j+1}(t) - z^j \int_{\mathbb{R}_+} e^{-zt}\Omega(t)\, d\sigma^a_{j+1}(t). \tag{2.18}$$

The first term in (2.18) appears in (2.14) multiplied by z. However, by assumption its order in (2.14) cannot be reduced, and therefore the order of the first term in (2.18) cannot be reduced. Hence, whatever the second term in (2.18) may be, the order of the term with index j in (2.17) cannot be reduced. As j is arbitrary, the functional in (2.17) has a regular realization.

2) Since the function e^{-zt} is an element of the space under consideration for each z with $\operatorname{Re} z > 0$, it follows that if the functional if trivial, then (2.13) holds.

Suppose (2.13) holds. Without loss of generality we assume that the functional $d\sigma$ (or φ) has a regular realization. We will show that

$$\begin{gathered} d\sigma_j(t) \equiv 0 \text{ for } t \geqslant 0 \\ (\text{respectively, } \varphi_j(t) \equiv 0 \text{ for } t > 0), \\ j = 0, \dots, m. \end{gathered} \tag{2.19}$$

This will complete the proof of the lemma.

We prove this using induction on m. First we look at $C^m_{0,\Omega}$. For $m = 0$ the required result is well-known. We assume that it holds for the index $m-1$ and prove it for the index m.

Thus $\Phi(z) \equiv 0$, where $\Phi(z)$ has the form (2.14). Then $\Phi_a(z) \equiv 0$ too, and the right-hand side of (2.17) is identically equal to zero. We have already shown that the realization of a functional on $C^{m-1}_{0,\Omega}$ which appears in this formula is also regular, so by the inductive assumption

$$(\chi_a * (\Omega\, d\sigma_0))(t)\, dt + \Omega(t)\, d\sigma_1(t) - \Omega(t)\, d\sigma^a_1(t) \equiv 0, \qquad t \geqslant 0, \tag{2.20}$$

$$\Omega(t)(d\sigma_{j+1}(t) - d\sigma^a_{j+1}(t)) \equiv 0, \qquad t \geqslant 0, \quad j = 1, \dots, m-1. \tag{2.21}$$

By the definition of $d\sigma^a$ the second term in (2.21) vanishes identically on $[0, a)$, and therefore

$$d\sigma_j(t) \equiv 0, \qquad t \in [0, a), \quad j = 2, \dots, m.$$

However, $a > 0$ is arbitrary, so we have proved (2.19) in the case of $C^m_{0,\Omega}$ for the indices $j = 2, \dots, m$. It remains to prove it for $j = 0, 1$.

Applying the same arguments to (2.20) we see that the measure

$$(\chi_a * (\Omega\, d\sigma_0))(t)\, dt + \Omega(t)\, d\sigma_1(t)$$

vanishes on $[0, a)$, and as $a > 0$ is arbitrary,

$$\left(\int_0^t \Omega(u)\, d\sigma_0(u)\right) dt = -\Omega(t)\, d\sigma_1(t), \qquad t \geqslant 0. \tag{2.22}$$

If the measures $d\sigma_0$ and $d\sigma_1$ are trivial, then there is nothing to prove. We claim that the assumption that they are nontrivial (which, in view of (2.22), they can only be simultaneously) contradicts the regular realization of the functional.

As we have already proved (2.19) for $j = 2, \dots, m$, the functional corresponding to the Laplace transform (2.14) has the following form:

$$(d\sigma, f) = \int_{\mathbb{R}_+} f(t)\Omega(t)\, d\sigma_0(t) - \int_{\mathbb{R}_+} f'(t)\Omega(t)\, d\sigma_1(t), \qquad f \in C^m_{0,\Omega}. \tag{2.23}$$

Using (2.22) we integrate the second integral in (2.23) by parts. Assuming that $f \equiv 0$ outside some interval $[0, b]$ we obtain

$$\begin{aligned} \int_{\mathbb{R}_+} f'(t)\Omega(t)\, d\sigma_1(t) &= -\int_{\mathbb{R}_+} \left(\int_0^t \Omega(u)\, d\sigma_0(u)\right) df(t) \\ &= \int_{\mathbb{R}_+} f(t)\Omega(t)\, d\sigma_0(t). \end{aligned} \tag{2.24}$$

However, the functions in $C^m_{0,\Omega}$ which vanish for all sufficiently large t are dense in $C^m_{0,\Omega}$, and therefore (2.24) holds for all $f \in C^m_{0,\Omega}$. Thus (2.24) shows that we can reduce the order of the second term in the realization (2.23), which contradicts the regular realization of the functional and finishes our analysis of the case $C^m_{0,\Omega}$.

The case of $W^m_{0,\Omega}$ is analogous, except that on the way, in place of assertion 1) in Lemma 2, we must use assertion 2). The proof is complete.

Combining Lemma 1 and assertion 2) in Lemma 3 we obtain the following result.

Lemma 4. *Let E be one of the spaces $W^m_{p,\Omega}(\mathbb{R}_+)$, $p \geqslant 1$, or $C^m_{0,\Omega}[0, \infty)$, where Ω is a positive continuous weight satisfying (2.9) and (2.10).*

Then a system $e(\Lambda)$ is incomplete in E if and only if sone functional on E has a nontrivial Laplace transform $\Phi(z)$ vanishing at the points in Λ.

Remark 1. Throughout what follows we let $e(\Lambda)$ denote a more general system of exponentials:

$$e(\Lambda) = ((t^k e^{-\lambda_n t})_{k=0}^{m_n - 1})_{n=1}^{\infty}, \quad \text{where } \Lambda = (\lambda_n; m_n)_{n=1}^{\infty}, \quad \operatorname{Re}\lambda_n > 0, \quad m_n \in \mathbb{N}.$$

Then in Lemmas 4 and 1 we mean that the function $\Phi(z)$ vanishes with multiplicity m_n at each point λ_n.

§ 3. Spaces in which exponentials are simultaneously complete

3.1. Pairs of spaces. We say that a space B_1 is *topologically densely embedded* in B_2 and write $B_1 \looparrowright B_2$ if $B_1 \hookrightarrow B_2$ and B_1 is dense in B_2.

Let Ω_q denote the class of weights on $\omega \in L^1(0, 1)$ each of which satisfies the following conditions for some $\delta \in [0, 1]$:

$$m(t) := \inf(\omega(x) : 0 < t \leqslant x < \delta) > 0, \qquad 0 < t < \delta,$$
$$\omega^{-1/q}(t) \in L^{q'}(\delta, 1).$$

If $\delta = 0$ (or $\delta = 1$) then the first (second) condition is omitted.

Theorem 3. *Let $\Omega(t)$, $t \geqslant 0$, be a positive continuous weight satisfying* (2.9) *and* (2.10). *For some $s \in \mathbb{Z}_+$, $p, q \in [1, \infty)$, and $\omega \in \Omega_q$ let $E_1(0,1)$ be an intermediate space between $W_p^s(0,1)$ and $L_\omega^q(0,1)$ in the sense that*

$$W_p^s(0,1) \hookrightarrow E_1(0,1) \hookrightarrow L_\omega^q(0,1). \tag{3.1}$$

Then any system $e(\Lambda)$ is simultaneously complete in the spaces

$$(E_1(0,1), W_{p,\Omega}^m(1,\infty))_{\mathrm{c}}, \quad m \in \mathbb{Z}_+, \qquad \text{and} \quad L_\Omega^p(\mathbb{R}_+), \tag{3.2}$$

and it is also simultaneously complete in the spaces

$$(E_1(0,1), C_{0,\Omega}^m[1,\infty))_{\mathrm{c}}, \quad m \in \mathbb{Z}_+, \qquad \text{and} \quad C_{0,\Omega}[0,\infty). \tag{3.3}$$

Proof. Throughout, we shall use the following obvious result: let $B_1 \hookrightarrow B_2$ and assume that B_1 contains some system of elements; if this system is incomplete in B_2, then it is also incomplete in B_1.

As $C^s[0,1] \hookrightarrow W_p^s(0,1)$, the embeddings in (3.1) imply that

$$C^s[0,1] \hookrightarrow E_1(0,1) \hookrightarrow L_\omega^q(0,1). \tag{3.4}$$

The positive continuous function $\Omega(t)$ is therefore bounded away from zero and infinity on $[0,1]$, so that

$$(W_p^l(0,1), W_{p,\Omega}^l(1,\infty))_{\mathrm{c}} = W_{p,\Omega}^l(\mathbb{R}_+), \tag{3.5}$$

$$(C^l[0,1], C_{0,\Omega}^l[1,\infty))_{\mathrm{c}} = C_{0,\Omega}^l[0,\infty). \tag{3.6}$$

For the same reason, in view of the left-hand embeddings in (3.1) and (3.4), the components of the left-hand spaces in (3.2) and (3.3) are topologically comparable relative to $t = 1$. Hence the corresponding coordinated subspaces are well defined.

The right-hand embeddings in (3.1) and (3.4) and condition (2.9) ensure that elements of $e(\Lambda)$ belong to the spaces in (3.2) and (3.3).

1) We claim that if a system $e(\Lambda)$ is incomplete in the left-hand space in (3.2) (in (3.3)), then it is also incomplete in the right-hand space in (3.2) (in (3.3), respectively). In essence, the proof is obtained by applying Lemma 3.

Assume that $e(\Lambda)$ is incomplete in the left-hand space in (3.2) (in (3.3)). Let $l = \max(s, m)$. Taking account of the left-hand embedding in (3.1) (in (3.4), respectively), we obtain

$$W_p^l(0,1) \hookrightarrow W_p^s(0,1) \hookrightarrow E_1, \qquad W_{p,\Omega}^l(1,\infty) \hookrightarrow W_{p,\Omega}^m(1,\infty)$$
$$\bigl(C^l[0,1] \hookrightarrow C^s[0,1] \hookrightarrow E_1, \quad C_{0,\Omega}^l[1,\infty) \hookrightarrow C_{0,\Omega}^m[1,\infty), \quad \text{respectively}\bigr).$$

Thus in view of (3.5) and (3.6),

$$W_{p,\Omega}^l(\mathbb{R}_+) \hookrightarrow (E_1, W_{p,\Omega}^m(1,\infty))_{\mathrm{c}}$$
$$\bigl(C_{0,\Omega}^l[0,\infty) \hookrightarrow (E_1(0,1),\, C_{0,\Omega}^m[1,\infty))_{\mathrm{c}}, \quad \text{respectively}\bigr).$$

Since $e(\Lambda)$ is incomplete in the left-hand space in (3.2) (in (3.3)) by assumption, the observation at the beginning of the proof shows that it is incomplete in the space (3.5) (in the space (3.6), respectively).

Thus the system is incomplete in $W^l_{p,\Omega}(\mathbb{R}_+)$ (in $C^l_{0,\Omega}[0,\infty)$, respectively). Then by Lemma 4 there exists a nontrivial Laplace transform $\Phi(z)$ of some functional on $W^l_{p,\Omega}(\mathbb{R}_+)$ (of a functional on $C^l_{0,\Omega}[0,\infty)$, respectively) that vanishes at the points in Λ. Next we go over to

$$\left(\frac{1-e^{-z}}{z}\right)^l \Phi(z). \tag{3.7}$$

By assertion 1) in Lemma 3 the function (3.7) is the Laplace transform of a functional on $L^p_\Omega(\mathbb{R}_+)$ (on $C_{0,\Omega}[0,\infty)$, respectively). However, this function is nontrivial (as is $\Phi(z)$) and vanishes at the points in Λ. By Lemma 4 the system $e(\Lambda)$ is incomplete in the right-hand space in (3.2) (in (3.3), respectively).

2) To prove the converse result, in using the condition $\omega \in \Omega_q$ we can also assume that $\omega^{p/q} \in C^1[0,\delta]$, $\omega(0)=0$ and $\omega^{p/q}(t)$ increases on $[0,\delta]$ together with its derivative. Indeed, set

$$\omega_1^{p/q}(x) = \int_0^x m^{p/q}(t)\,dt, \qquad 0 \leqslant x \leqslant \delta.$$

Then $\omega_1^{p/q} \in C[0,\delta]$, $\omega_1(0)=0$ and $\omega_1^{p/q}(x)$ is increasing. Now, as $m(t)$ is a positive nondecreasing minorant of the weight $\omega(t)$ on $(0,\delta)$, it follows that

$$0 < \omega_1(x) \leqslant x^{q/p} m(x) < \omega(x), \qquad 0 < x < \delta.$$

Letting $\omega_2^{p/q}$ denote the primitive function of $\omega_1^{p/q}$ we see that $\omega_2^{p/q} \in C^1[0,\delta]$, $\omega_2(0)=0$, $\omega_2^{p/q}(x)$ is increasing together with its derivative and

$$0 < \omega_2(x) < \omega_1(x) < \omega(x), \qquad 0 < x < \delta.$$

If we set $\omega_2(t)=\omega(t)$, $\delta < t < 1$, then in view of the last inequality and right-hand embedding in (3.1),

$$E_1(0,1) \hookrightarrow L^q_\omega(0,1) \hookrightarrow L^q_{\omega_2}(0,1),$$

and we can indeed say that the weight ω has the additional properties we claimed.

Assume that the system $e(\Lambda)$ is incomplete in the right-hand space in (3.2) (in (3.3)). Then using the right-hand embedding in (3.1) and the obvious embeddings

$$W^m_{p,\Omega}(1,\infty) \hookrightarrow L^p_\Omega(1,\infty) \quad \text{and} \quad C^m_{0,\Omega}[1,\infty) \hookrightarrow C_{0,\Omega}[1,\infty),$$

we obtain

$$(E_1, W^m_{p,\Omega}(1,\infty))_{\mathrm{c}} \hookrightarrow (L^q_\omega(0,1), L^p_\Omega(1,\infty))_{\mathrm{c}} = (L^q_\omega(0,1), L^p_\Omega(1,\infty)),$$
$$(E_1, C^m_{0,\Omega}(1,\infty))_{\mathrm{c}} \hookrightarrow (L^q_\omega(0,1), C_{0,\Omega}[1,\infty))_{\mathrm{c}} = (L^q_\omega(0,1), C_{0,\Omega}[1,\infty)).$$

Hence, if we assume that $e(\Lambda)$ is incomplete in the right-hand space in (3.2) (in the right-hand space in (3.3), respectively), then it is sufficient to prove that it is incomplete in

$$(L^q_\omega(0,1), L^p_\Omega(1,\infty)) \quad (\text{in } (L^q_\omega(0,1), C_{0,\Omega}[1,\infty)), \text{ respectively}). \tag{3.8}$$

Assume that $e(\Lambda)$ is incomplete in $L^p_\Omega(\mathbb{R}_+)$, the right-hand space in (3.2). Then by Lemma 4 there exists a nontrivial Laplace transform

$$\Phi(z) = \int_{\mathbb{R}_+} e^{-zt}\Omega^{1/p}(t)\varphi(t)\,dt, \qquad \varphi \in L^{p'}(\mathbb{R}_+), \quad \operatorname{Re} z > 0,$$

which vanishes at the points in Λ.

We set $h \equiv 0$ outside $[0,\delta]$ and

$$h(t) = ((\omega^{p/q}(t))')^{1/p}, \qquad 0 < t < \delta. \tag{3.9}$$

By assumption $\omega^{p/q} \in C^1[0,\delta]$, so that $(\omega^{p/q})' \in C[0,\delta]$. As $\omega^{p/q}(t)$ is an increasing function, $(\omega^{p/q})' \geqslant 0$ and the function $h(t)$ defined by (3.9) is nonnegative and integrable on $[0,\delta]$. Furthermore,

$$\int_0^x h^p(t)\,dt = \omega^{p/q}(x), \qquad 0 < x < \delta. \tag{3.10}$$

As a possibility for the function ψ annihilating $e(\Lambda)$ (when regarded as a functional on the left-hand space in (3.8)), we look at the convolution

$$\psi = h * (\Omega^{1/p}\varphi).$$

Since h belongs to L^1, it follows from Lemma 2 (assertion 2)) that

$$\frac{\psi}{\Omega^{1/p}} \in L^{p'}(\mathbb{R}_+).$$

In particular,

$$g_2(t) := \frac{1}{\Omega^{1/p}}\psi|_{(1,\infty)} \in L^{p'}(1,\infty).$$

We claim that

$$\frac{1}{\omega^{1/q}(t)}\psi(t)|_{(0,\delta)} \in L^\infty(0,\delta) \quad \text{and} \quad \psi(t)|_{(\delta,1)} \in L^\infty(\delta,1). \tag{3.11}$$

We have

$$\psi(x) = \int_0^x h(t)\Omega^{1/p}(x-t)\varphi(x-t)\,dt, \qquad x > 0. \tag{3.12}$$

Hence using the properties

$$\varphi|_{(0,1)} \in L^{p'}(0,1), \qquad \Omega|_{(0,1)} \in L^\infty(0,1),$$

formula (3.10) and the fact that h is nonnegative we obtain

$$|\psi(x)| \leqslant M\|\varphi\|_{L^{p'}(0,\delta)}\left(\int_0^x h^p(t)\,dt\right)^{1/p} = C\omega^{1/q}(x), \qquad 0 < x < \delta.$$

The first condition in (3.11) is fulfilled. The second is an obvious consequence of (3.12).

Now we claim that

$$\psi(t) = \omega^{1/q}(t) g_1(t), \qquad 0 < t < 1, \quad \text{where } g_1 \in L^{q'}(0,1).$$

In fact, the property

$$\psi(t)|_{(0,\delta)} = \omega^{1/q}(t) g_1(t), \qquad 0 < t < \delta, \quad \text{where } g_1 \in L^{q'}(0,\delta),$$

follows from the first condition in (3.11), while the property

$$\psi(t)|_{(\delta,1)} = \omega^{1/q}(t) g_1(t), \qquad \delta < t < 1, \quad g_1 \in L^{q'}(\delta,1),$$

follows from the second condition in (3.11) and the second condition in the definition of the class Ω_q.

Thus the Laplace transform of ψ has the following form:

$$\Psi(z) = \int_{\mathbb{R}_+} e^{-zt}\psi(t)\,dt = \int_0^1 e^{-zt}\omega^{1/q}(t) g_1(t)\,dt + \int_1^\infty e^{-zt}\Omega^{1/p}(t) g_2(t)\,dt,$$

where $g_1 \in L^{q'}(0,1)$ and $g_2 \in L^{p'}(1,\infty)$. That is, $\Psi(z)$ is the Laplace transform of a functional on the left-hand space in (3.8). On the other hand, $\Psi(z)$ is the Laplace transform of a convolution and therefore

$$\Psi(z) = \Phi(z) H(z),$$

where $H(z)$ is the Laplace transform of the function h. Hence $\Psi(z)$ vanishes at the points in Λ.

By Lemma 1 the system $e(\Lambda)$ is incomplete in the left-hand space in (3.8), as required. Thus we have shown that if $e(\Lambda)$ is incomplete in the right-hand space in (3.2), then it is also incomplete in the left-hand space.

Now we must prove the same result for the spaces in (3.3).

Our arguments will basically remain the same as for (3.2). By assumption $e(\Lambda)$ is incomplete in $C_{0,\Omega}[0,\infty)$. Hence there exists a function $\sigma \in V[0,\infty)$ such that the Laplace-Stieltjes transform

$$\Phi(z) = \int_0^\infty e^{-zt}\Omega(t)\,d\sigma(t), \qquad \operatorname{Re} z > 0,$$

vanishes at the points in Λ. We set $h \equiv 0$ outside $[0,\delta]$ and

$$h(x) = \omega^{1/q}(x), \qquad 0 \leqslant x \leqslant \delta.$$

We can assume here that $h(t)$ is increasing and continuous on $[0,\delta]$ and $h(0) = 0$. We can show this by formally setting $p = 1$ in the argument at the beginning of part 2) of the proof.

Consider the convolution

$$\psi(t) = (h * (\Omega\,d\sigma))(t).$$

By assertion 1) in Lemma 2,

$$g_2(t) := \frac{1}{\Omega}\psi|_{(1,\infty)} \in L^1(1,\infty).$$

That is, if we can prove (3.11), then the function $\Psi(z)$ defined above is the Laplace transform of some functional on the right-hand space in (3.8). Since $\Psi(z) = 0$ for $z \in \Lambda$ (again, $\Psi = \Phi H$ by the properties of convolution), this will mean that the system $e(\Lambda)$ is incomplete in the right-hand space in (3.8) and will complete the proof.

We have

$$\psi(x) = \int_0^x h(x-t)\Lambda(t)\,d\sigma(t), \qquad x > 0.$$

Hence the second property in (3.11) follows directly from the continuity of $h(t)$ and $\Omega(t)$. On the other hand, the inequality

$$|\psi(x)| \leqslant M(\operatorname{var}\sigma)h(x) = C\omega^{1/q}(x), \qquad 0 < x < \delta,$$

which ensures the first property in (3.11), holds because $h(t)$ is positive and increasing.

The proof of Theorem 3 is complete.

Corollary 1. *Let $l, m \in \mathbb{Z}_+$, $1 \leqslant p, q < \infty$, and let $\Omega(t)$, $t \geqslant 0$, be a continuous positive weight satisfying (2.9) and (2.10). Let $E_1(0,1)$ be one of the spaces*

$$L^q_\omega(0,1), \quad \omega \in \Omega_q, \qquad W^l_q(0,1), \qquad C^l[0,1]. \tag{3.13}$$

Then the assertion of Theorem 3 holds.

In fact, in this case the spaces (3.13) satisfy the assumptions of Theorem 3.

Remark 2. The function $\Omega(t) \equiv 1$ satisfies the assumptions of Theorem 3. Hence the classes of spaces $W^m_{p,\Omega}$ and $C^m_{0,\Omega}$ covered by the theorem contain the weight-free spaces W^m_p and C^m_0.

Remark 3. Corollary 1 supports the view (a leitmotif of [12]) that the second component E_2 is more important in determining whether a system $e(\Lambda)$ is complete in the space $(E_1(0,1), E_2(1,\infty))_{\mathrm{c}}$.

3.2. The absence of bases of exponentials.

Theorem 4. *Let B be one of the spaces*

$$(E_1(0,1), W^m_{p,\Omega}(\mathbb{R}_+))_{\mathrm{c}} \quad \text{and} \quad (E_1(0,1), C^m_{0,\Omega}[0,\infty))_{\mathrm{c}},$$

where E_1, Ω, m and p are as in Theorem 3.

Then

1) *an incomplete system $e(\Lambda)$ remains so after the addition of the system*

$$e(M) \quad \text{with } M = (\mu_k)_{k=1}^\infty, \quad \operatorname{Re}\mu_k > 0, \quad \mu_k \notin \Lambda,$$

provided that

$$\sum_{k=1}^\infty \frac{1}{|\mu_k|} < +\infty; \tag{3.14}$$

2) *if $e(\Lambda)$ is a minimal system in B, then it is incomplete*;

3) *B has no bases of the form $e(\Lambda)$.*

Proof. 1) By Theorem 3 we can assume that B is L^p_Ω or $C_{0,\Omega}$. For definiteness let $B = L^p_\Omega$ and assume that a system $e(\Lambda)$ is incomplete in B. Then by Lemma 4 some nontrivial Laplace transform

$$\Phi(z) = \int_{\mathbb{R}_+} e^{-zt}\Omega^{1/p}(t)f(t)\,dt, \qquad f \in L^{p'}(\mathbb{R}_+), \quad \operatorname{Re} z > 0,$$

vanishes at the points in Λ.

We use the following fact (see [11], § 3.3). If (3.14) holds, then for each $a > 0$ there exists a nontrivial function

$$F(z) = \int_0^a e^{-zt}g(t)\,dt, \qquad g \in C[0,1],$$

which vanishes at the points μ_k, $k \in \mathbb{N}$.

Now we look at $\Phi(z)F(z)$ for some $a > 0$. This is a nontrivial function vanishing at the points in $(\lambda_n) \cup (\mu_k)$, and it is the Laplace transform of the convolution $g * (\Omega^{1/p}f)$. By Lemma 2 this convolution lies in $\bigl(L^p_\Omega(\mathbb{R}_+)\bigr)'$. By Lemma 4 the combined system $e(\Lambda) \cup e(M)$ is incomplete in B, which proves 1).

2) Let $e(\Lambda)$ be a minimal system in B. We shall prove that it is incomplete. If it is complete, then by assertion 1) the system

$$e(\Lambda_1) \quad \text{with } \Lambda_1 = \Lambda \setminus \lambda_1$$

is too. In particular, the function $\exp(-\lambda_1 t)$ can be approximated arbitrarily well by linear combinations of functions in this system, which contradicts the minimality.

3) As completeness and minimality are both necessary for a basis, assertion 3) follows from 2). The proof is complete.

Remark 4. Assertion 1) in Theorem 4 also holds when $\mu_k = \lambda_s$. In this case we treat the addition of an exponential $\exp(-\mu_k t)$ to the system $e(\Lambda)$ as increasing the multiplicity of the point λ_s by 1.

In the proof of assertion 2) we implicitly assumed for greater transparency that λ_1 is a simple point. If it is not, Λ_1 must be treated as the sequence obtained from Λ by reducing the multiplicity of λ_1 by 1, and we must take the function $t^{m_1-1}\exp(-\lambda_1 t)$ in place of $\exp(-\lambda_1 t)$.

3.3. Spaces with power-like weights. We look at the spaces

$$\begin{gathered} L^{p,\alpha} := L^p(\mathbb{R}_+, (1+t)^\alpha\,dt), \quad C_{0,\alpha} := C_{0,(1+t)^\alpha}[0,\infty), \qquad \alpha \in \mathbb{R}, \\ L^p_\alpha := L^p(\mathbb{R}_+, t^\alpha\,dt), \qquad \alpha > -1. \end{gathered} \tag{3.15}$$

(Do not confuse $C_{0,\alpha}$ and $C_{0,\Omega}$: Ω is a weight, while α is an exponent.) The constraint $\alpha > -1$ stems from the condition $\omega = t^\alpha \in L^1(0,1)$.

Corollary 2. *For $p \geqslant 1$ and $\alpha > -1$ the system $e(\Lambda)$ is complete (or incomplete) in the spaces $L^{p,\alpha}$ and L^p_α simultaneously.*

In fact, since the weights t^α and $(1+t)^\alpha$ are equivalent for $t \geqslant 1$, this result is obtained by applying Corollary 1 to the spaces (3.2) with $m = 0$, $\Omega = (1+t)^\alpha$

and $E_1 = L^p_{\omega_1}(0,1)$ with $\omega_1(t) = t^\alpha$, where we must set $\omega(t) = t^\alpha$ for $\alpha \geqslant 0$ and $\omega(t) \equiv 1$ for $\alpha < 0$.

This author [11] and Khabibullin [14] obtained many results concerning the completeness of systems $e(\Lambda)$ in the spaces L^p_α with $\alpha > -1$. Using Corollary 2 we can also state these results for the spaces $L^{p,\alpha}$ with $\alpha > -1$, but we shall not expound on this subject.

In what follows we require the following result.

Lemma 5. *The following topologically dense embeddings hold*:

$$C_{0,\beta} \hookrightarrow C_{0,\alpha}, \quad L^{p,\beta} \hookrightarrow L^{p,\alpha} \quad \textit{for} \ \ \alpha < \beta;$$
$$C_{0,\beta} \hookrightarrow L^{p,\alpha} \quad \textit{for} \ \ \alpha < p\beta - 1;$$
$$L^{p,\alpha} \hookrightarrow L^{2,-1} \quad \textit{for} \ \ p > 2, \quad \alpha > -1.$$

Proof. As the functions with compact support in $C^\infty[0,\infty)$ are dense in $C_{0,\alpha}$ and $L^{p,\alpha}$, the first two embedding are obvious, while the rest hold by the following inequalities.

If $\alpha < p\beta - 1$, then

$$\|g\|^p_{L^{p,\alpha}} = \int_0^\infty \frac{|g(t)|^p(1+t)^{p\beta}}{(1+t)^{p\beta-\alpha}}\,dt \leqslant \|g\|^p_{C_{0,\beta}} \int_0^\infty \frac{dt}{(1+t)^{p\beta-\alpha}} = C\|g\|^p_{C_{0,\beta}}.$$

Using the notation $r = p/2$ and $r' = p/(p-2)$, from Hölder's inequality we obtain

$$\begin{aligned}\|f\|^2_{L^{2,-1}} &= \int_{\mathbb{R}_+} |f(t)|^2(1+t)^{2\alpha/p} \frac{dt}{(1+t)^{1+2\alpha/p}} \\ &\leqslant \|f\|^2_{L^{p,\alpha}} \left(\int_{\mathbb{R}_+} \frac{dt}{(1+t)^{(1+\alpha/r)r'}}\right)^{1/r'} = C\|f\|^2_{L^{p,\alpha}},\end{aligned}$$

provided that

$$1 < \left(1 + \frac{\alpha}{r}\right) r' = \left(1 + \frac{2\alpha}{p}\right) \frac{p}{p-2} = \frac{p+2\alpha}{p-2},$$

that is, if $p > 2$ and $\alpha > -1$.

Taking the third embedding in Lemma 5, with $p = 2$ and $\alpha = 0$, in combination with Szász's theorem we obtain the following result.

Corollary 3. *The condition*

$$\sum_{n=1}^\infty \frac{\operatorname{Re}\lambda_n}{1 + |\lambda_n|^2} = +\infty$$

is necessary for the system $e(\Lambda)$ to be complete in the space $C_{0,\beta}$ with $\beta > 1/2$.

§ 4. A completeness test in spaces with power-like weight

4.1. Lemmas. Let $u(t) \in L^1(\mathbb{R}, (1+t^2)^{-1}\,dt)$. Then the Poisson integral of $u(t)$ (for the upper half-plane) is the convolution of u with the Poisson kernel $P_y(t) = y/(t^2+y^2)$:

$$U(x,y) = \frac{y}{\pi}\int_{\mathbb{R}} \frac{u(x-t)\,dt}{t^2+y^2}, \qquad x \in \mathbb{R}, \quad y > 0.$$

If $u \in L^\infty(\mathbb{R})$, then obviously

$$|U(x,y)| \leqslant \|u\|_\infty, \qquad x \in \mathbb{R}, \quad y > 0. \tag{4.1}$$

It is well known that if $u(t)$ is a continuous bounded function on the line, then $U(x,y)$ is continuous in the half-plane $x \in \mathbb{R}$, $y \geqslant 0$, and $u(x)$ is the function giving the boundary values of $U(x,y)$: $U(x,0) = u(x)$, $x \in \mathbb{R}$.

Let $0 < \beta < 1$. Then we define the space Λ_β by

$$\Lambda_\beta = (f : f \in L^\infty(\mathbb{R}), \|f(x+t) - f(x)\|_\infty \leqslant A|t|^\beta), \qquad A = A(f) > 0.$$

It is known that $\Lambda_\beta \subset C(\mathbb{R})$ (see [15], Ch. 5, § 4).

It is convenient to also view the Poisson integral as a function of the complex variable $z = x+iy$, that is, along with $U(x,y)$ we shall use the notation $U(x+iy) = U(z)$.

Lemma 6. *If* $u \in \Lambda_\beta$ *where* $0 < \beta < 1$, *then* $U(z)$ *belongs to the class* $\operatorname{Lip}\beta$ *in the half-plane* $y = \operatorname{Im} z \geqslant 0$, *that is,*

$$|U(z_1) - U(z_2)| \leqslant B|z_1 - z_2|^\beta, \qquad \operatorname{Im} z_1, \operatorname{Im} z_2 \geqslant 0.$$

Proof. It is sufficient to prove the estimates

$$|U(x,y_1) - U(x,y_2)| \leqslant K|y_1 - y_2|^\beta \qquad \forall\, x \in \mathbb{R}, \quad y_2 > y_1 \geqslant 0, \tag{4.2}$$

and

$$|U(x+\delta,y) - U(x,y)| \leqslant L\delta^\beta, \qquad x \in \mathbb{R}, \quad y \in \mathbb{R}_+, \quad \delta > 0. \tag{4.3}$$

Note that

$$U(x+\delta,y) - U(x,y) \tag{4.4}$$

is the Poisson integral of the function

$$u(t-\delta) - u(t).$$

So using property (4.1) and the condition $u \in \Lambda_\beta$ in turn, we derive inequality (4.3).

To prove (4.2) we use the following property (see [15], Ch. 5, § 4): if $u \in \Lambda_\beta$ for $0 < \beta < 1$, then

$$\left|\frac{\partial U(x,y)}{\partial y}\right| \leqslant Ay^{\beta-1} \quad \text{for} \quad y > 0, \quad x \in \mathbb{R}.$$

Applying this we obtain

$$|U(x,y_2)-U(x,y_1)| = \left|\int_{y_1}^{y_2} \frac{\partial U(x,y)}{\partial y}\,dy\right| \leqslant A\int_{y_1}^{y_2} y^{\beta-1}\,dy \leqslant A\int_0^{y_2-y_1} y^{\beta-1}\,dy = C(y_2-y_1)^{\beta},$$

which is just (4.2). The proof is complete.

By the Fourier transform and inverse Fourier transform of a function $f \in L^1(\mathbb{R})$ we shall mean the functions

$$\widehat{f}(x) = \int_{\mathbb{R}} e^{-ixt} f(t)\,dt, \quad x \in \mathbb{R}, \quad \text{and} \quad \widetilde{f}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixt} f(x)\,dx, \quad t \in \mathbb{R},$$

respectively, and by the Fourier-Stieltjes transform of $\sigma(t) \in V(\mathbb{R})$ (that is, of a function of bounded variation on $\mathbb{R}$) we shall mean the function

$$\widehat{d\sigma}(x) = \int_{\mathbb{R}} e^{-ixt}\,d\sigma(t), \qquad x \in \mathbb{R}.$$

Lemma 7. *Let $h \in L^1(\mathbb{R})$ and $\sigma \in V(\mathbb{R})$. Then*

$$\widehat{\widetilde{h}\,d\sigma} = \frac{1}{2\pi} h * \widehat{d\sigma}.$$

Proof. We have

$$\begin{aligned}\widehat{\widetilde{h}\,d\sigma} &= \int_{\mathbb{R}} e^{-ixt}\widetilde{h}(t)\,d\sigma(t) = \int_{\mathbb{R}} e^{-ixt}\,d\sigma(t)\frac{1}{2\pi}\int_{\mathbb{R}} e^{iyt}h(y)\,dy \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} h(y)\,dy \int_{\mathbb{R}} e^{-it(x-y)}\,d\sigma(t) = \frac{1}{2\pi} h * \widehat{d\sigma}. \end{aligned} \tag{4.5}$$

Here we can change the order of integration, because the repeated integral is absolutely convergent:

$$\int_{\mathbb{R}} |d\sigma(t)| \int_{\mathbb{R}} |h(y)|\,dy \leqslant \|h\|_1 \cdot \operatorname{var} \sigma.$$

The right-hand side of (4.5) is well-defined; as the convolution of an integrable and a bounded function it is bounded on the line. The proof is complete.

4.2. A completeness test. As we already mentioned in § 1, the condition

$$\sum_{n=1}^{\infty} \frac{\operatorname{Re}\lambda_n}{1+|\lambda_n|^2} + \int_{\mathbb{R}} \frac{\log \operatorname{dist}(iy,\Lambda)}{1+y^2}\,dy = +\infty \tag{4.6}$$

is necessary for a system $e(\Lambda)$ to be complete in $L^p(\mathbb{R}_+)$, $p > 2$, or in $C_0[0,\infty)$. It turns out that this condition is the test for completeness of the system in the spaces $C_{0,\alpha}$ with $\alpha < 0$, and $L^{p,\alpha}$ with $p \geqslant 1$ and $\alpha < -1$, which we defined in (3.15).

Theorem 5. 1) *Condition* (4.6) *is necessary for a system* $e(\Lambda)$ *to be complete in* $C_{0,\alpha}$ *and in* $L^{p,\alpha}$ *with* $p \geqslant 1$, *for each* $\alpha \in \mathbb{R}$.

2) *Condition* (4.6) *is sufficient for a system* $e(\Lambda)$ *to be complete in* $C_{0,\alpha}$ *with* $\alpha < 0$, *and is sufficient for* $e(\Lambda)$ *to be complete in* $L^{p\alpha}$ *with* $p \geqslant 1$ *and* $\alpha < -1$.

3) *Condition* (4.6) *is necessary and sufficient for a system* $e(\Lambda)$ *to be complete in each of the spaces*

$$C_{0,\alpha} \quad \text{with } \alpha < 0 \quad \text{and} \quad L^{p,\alpha} \quad \text{with } p \geqslant 1, \quad \alpha < -1.$$

Proof. We only need prove 1) and 2).

1) In view of the third embedding in Lemma 5 we only need to consider the case of $L^{p,\alpha}$. For this space assertion 1) is a special case of the following result (see [12], § 7).

Theorem 6. *Let* $\Omega(t)$, $t > 0$, *be a weight such that* $\Omega(t)|_{(0,1)} \in \Omega_p$ *and assume that*

$$\frac{1}{\Omega(t)} = O(t^r), \qquad t > 1,$$

for some $r > 0$ *and moreover,*

$$e^{-at}\Omega(t)|_{(1,\infty)} \in L^1(1,\infty) \quad \forall\, a > 0.$$

Then (4.6) *is necessary for the system* $e(\Lambda)$ *to be complete in* $L^p(\mathbb{R}_+, \Omega(t)\,dt)$, $1 \leqslant p < \infty$.

In fact the weight $\Omega(t) = (1+t)^\alpha$ satisfies the assumptions of Theorem 6, so assertion 1) holds.

2) The case of the space $C_{0,\alpha}$. We must show that if $\alpha < 0$, then it follows from (4.6) that the system $e(\Lambda)$ is complete in $C_{0,\alpha}$. By the first embedding in Lemma 5 we need only consider $-1 < \alpha < 0$.

Assume the converse: (4.6) holds, but $e(\Lambda)$ is incomplete in $C_{0,\alpha}$ for some α, $-1 < \alpha < 0$.

For $t \geqslant 0$ the weights

$$(1+t)^\alpha \quad \text{and} \quad (1+t^2)^{\alpha/2}$$

are equivalent, so by Lemma 4 and formula (2.8) we can find a nontrivial analytic function of the form

$$\Phi(z) = \int_{\mathbb{R}_+} \frac{e^{-zt}\,d\sigma(t)}{(1+t^2)^{-\alpha/2}}, \qquad \operatorname{Re} z > 0, \quad \sigma \in V[0,\infty), \tag{4.7}$$

which vanishes at the points in Λ.

The function $\Phi(z)$ is bounded in the half-plane $\operatorname{Re} z > 0$, so its zeros and, in particular, the points λ_n satisfy the Blaschke condition in the half-plane:

$$\sum_{n=1}^{\infty} \frac{\operatorname{Re}\lambda_n}{1+|\lambda_n|^2} < +\infty. \tag{4.8}$$

In the theory of Bessel potentials (see [15], Ch. 5, § 3) it is proved that for $\beta > 0$ we have

$$(1+t^2)^{-\beta/2} = 2\pi\widetilde{G_\beta}$$

for some function $G_\beta(x) \in L^1(\mathbb{R})$. But then, since $\Phi(z)$ is continuous for $\operatorname{Re} z \geqslant 0$, (4.7) shows that

$$\Phi(iy) = 2\pi\widehat{\widetilde{G_\beta}\, d\sigma}, \qquad \beta = -\alpha > 0.$$

By Lemma 7

$$\Phi(iy) = G_\beta * \widehat{d\sigma},$$

so that $\Phi(iy)$ is the Bessel potential of order $\beta = -\alpha \in (0,1)$ of the function $\widehat{d\sigma}$. Since $\widehat{d\sigma} \in L^\infty$, it follows that $\Phi(iy) \in \Lambda_\beta$ (see [15], Ch. 5, §§ 6.16, 6.17).

The function $\Phi(z)$ can be represented by the Poisson integral (for the right half-plane) of its boundary values:

$$\Phi(z) = \frac{x}{\pi}\int_{\mathbb{R}} \frac{\Phi(it)\,dt}{x^2+(y-t)^2}, \qquad x = \operatorname{Re} z > 0, \quad y = \operatorname{Im} z \in \mathbb{R}.$$

By Lemma 6, $\Phi(z)$ belongs to the class $\operatorname{Lip}\beta$ in the half-plane $\operatorname{Re} z \geqslant 0$.

Now we use the following well-known property of a bounded analytic function in the half-plane $\operatorname{Re} z > 0$:

$$\int_{\mathbb{R}} \frac{\log|\Phi(iy)|}{1+y^2}\,dy > -\infty.$$

Since $\Phi(\lambda_n) = 0$, we can express this property as

$$\int_{\mathbb{R}} \frac{\log|\Phi(iy)-\Phi(\lambda_n)|}{1+y^2}\,dy \geqslant M > -\infty \quad \forall\, n,$$

where M is independent of n. However,

$$|\Phi(iy)-\Phi(\lambda_n)| \leqslant K|iy-\lambda_n|^\beta \quad \forall\, y \in \mathbb{R}, \quad \lambda_n \in \Lambda,$$

because $\Phi(z)$ belongs to $\operatorname{Lip}\beta$. Hence

$$\int_{\mathbb{R}} \frac{\log|iy-\lambda_n|}{1+y^2}\,dy \geqslant M_1 > -\infty.$$

Taking the infimum over $\lambda_n \in \Lambda$ here, we obtain

$$\int_{\mathbb{R}} \frac{\log \operatorname{dist}(iy,\Lambda)}{1+y^2}\,dy > -\infty,$$

or equivalently,

$$\int_{\mathbb{R}} \frac{\log(1/\operatorname{dist}(iy,\Lambda))}{1+y^2}\,dy < +\infty.$$

The last relation in combination with (4.8) contradicts condition (4.6). This completes our analysis of the case of $C_{0,\alpha}$.

The case of the space $L^{p,\alpha}$. Assume that $\alpha < -1$. Then we can find a negative β close to 0 such that $\alpha < p\beta - 1$, and the part of 2) relating to $L^{p,\alpha}$ follows from the part of 2) relating to $C_{0,\beta}$ (which we have already proved) and Lemma 5.

The proof of the theorem is complete.

Using Corollary 1 we can significantly increase the range of application of Theorem 5.

Corollary 4. *Let m, p and q be as in Corollary 1 and let E_1 be one of the spaces in (3.13).*

Then the assertions of Theorem 5 are still valid if the spaces $L^{p,\alpha}$ and $C_{0,\alpha}$ are replaced by the coordinated subspaces of

$$(E_1(0,1), W^m_{p,(1+t)^\alpha}(1,\infty)) \quad \text{and} \quad (E_1(0,1), C^m_{0,(1+t)^\alpha}[1,\infty)),$$

respectively.

4.3. The significance of the constraints on α.

Theorem 7. *The condition*

$$\int_{\mathbb{R}} \frac{\log(1/\operatorname{dist}(iy,\Lambda))}{1+y^2}\,dy = +\infty, \tag{4.9}$$

and therefore also condition (4.6), is not sufficient for the system $e(\Lambda)$ to be complete in any of the spaces

$$L^{2,\alpha} \quad \text{with} \quad \alpha \geqslant -1; \qquad L^{p,\alpha} \quad \text{with} \quad p>2, \quad \alpha > -1;$$
$$C_{0,\alpha} \quad \text{with} \quad \alpha > 0; \qquad L^{p,\alpha} \quad \text{with} \quad 1 \leqslant p < 2, \quad \alpha > -\frac{p}{2}.$$

Proof. As shown by Shapiro and Shields [16], if points z_n, $n \in \mathbb{N}$, in the unit disc satisfy

$$\sum \frac{1}{-\log(1-|z_n|)} < +\infty, \tag{4.10}$$

then the points z_n form the set of zeros of some power series

$$S(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{4.11}$$

such that

$$\sum_{n=1}^{\infty} |c_n|^2 n < +\infty. \tag{4.12}$$

Fix a sequence z_n, $0 < |z_n| < 1$, such that each point on the circle $|z| = 1$ is a limit point of this sequence and condition (4.10) holds (for example, it is sufficient to pick k points with step $2\pi/k$ on the circle $|z| = 1 - \exp(-k^3)$ and take the union of these sets for $k \in \mathbb{N}$).

Now fixing the coefficients c_n corresponding to this particular case we set

$$\varphi(t) = \sqrt{1+t}, \quad \text{for } 0<t<1; \qquad \varphi(t) = c_n\sqrt{1+t}, \quad \text{for } t \in (n, n+1), \quad n \in \mathbb{N}.$$

Then (4.12) shows that $\varphi(t)$ is a function in $L^2(\mathbb{R}_+)$, which therefore defines a non-trivial functional on the space $L^{2,-1}$ by formula (2.3). Taking the Laplace transform of this functional using (2.7) we obtain

$$\Phi(w) = \int_{\mathbb{R}_+} e^{-wt} \frac{\varphi(t)}{\sqrt{1+t}}\,dt = \frac{1-e^{-w}}{w}\biggl(1 + \sum_{n=1}^{\infty} c_n e^{-nw}\biggr), \qquad \operatorname{Re} w > 0. \tag{4.13}$$

Let Λ be the sequence of zeros of $\Phi(w)$. By Lemma 1 the system $e(\Lambda)$ is incomplete in $L^{2,-1}$. It remains to show that (4.9) is fulfilled.

Let (μ_n), $n \in \mathbb{N}$, be the part of Λ lying in the half-strip $\operatorname{Re} w > 0$, $0 \leqslant \operatorname{Im} w < 2\pi$. By (4.11) and (4.13)

$$\Phi(w) = \frac{1 - e^{-w}}{w} S(e^{-w}), \qquad \operatorname{Re} w > 0, \quad 0 \leqslant \operatorname{Im} w < 2\pi.$$

This means that

$$z_n = e^{-\mu_n}, \qquad n \in \mathbb{N}$$

(if the μ_n are numbered appropriately). Since each point in $|z| = 1$ is a limit point of the sequence z_n, it follows from this formula that each point in the line interval $\operatorname{Re} w = 0$, $0 \leqslant \operatorname{Im} w < 2\pi$, is a limit point of the sequence μ_n. Hence

$$\operatorname{dist}(iv, (\mu_n)) = \operatorname{dist}(iv, \Lambda) = 0, \qquad 0 \leqslant v \leqslant 2\pi,$$

and (4.9) is fulfilled. This proves the theorem for $L^{2,-1}$.

Hence, in view of Lemma 5, we can deduce the theorem holds for the spaces

$$L^{p,\alpha} \text{ with } p \geqslant 2 \text{ and } \alpha > -1 \quad \text{and} \quad C_{0,\alpha} \text{ with } \alpha > 0.$$

To consider the remaining case, when we have $L^{p,\alpha}$ with $1 \leqslant p < 2$, we use the following result in [11], § 10.5, which concerns the spaces L^p_α (see (3.15)).

If a system $e(\Lambda)$ is incomplete in $L^{p_1}_{\alpha_1}$ with $p_1 \geqslant 1$ and $\alpha_1 > -1$, then it is also incomplete in L^p_α for any p and α such that

$$1 \leqslant p \leqslant p_1 \quad \text{and} \quad \frac{\alpha}{p} \geqslant \frac{\alpha_1}{p_1}.$$

Corollary 2 shows that the same holds for the spaces $L^{p,\alpha}$, so that, setting $p_1 = 2$ and $\alpha_1 > -1$, it remains to cite the part of Theorem 7 relating to $L^{2,\alpha}$, which we have already proved.

§ 5. The Szász condition is not sufficient for completeness in spaces with increasing weight

Recall that condition (1.4) is sufficient for $E(\Lambda)$ to be complete in the spaces L^p with $p \geqslant 2$ and C_0. The following result is of interest in this connection.

Theorem 8. *Let $\Omega(t)$, $t \geqslant 0$, be a weight with the following properties*:

$$0 < \Omega(t) \to +\infty, \qquad t \to +\infty. \tag{5.1}$$

Then condition (1.4) *is not sufficient for the system $e(\Lambda)$ to be complete in any of the spaces*

$$L^p_\Omega(\mathbb{R}_+) \text{ with } p \geqslant 2 \quad \text{or} \quad C_{0,\Omega}[0,\infty). \tag{5.2}$$

Proof. We require Davydov's theorem (see [17], Ch. 2, § 13). We present a simplified version which is all we need.

Theorem 9. *For* $a_n \geqslant 0$, $n \in \mathbb{Z}_+$, *let*

$$\lim_{n\to\infty} a(n) = \infty, \qquad \limsup_{n\to\infty} (a(n))^{1/n} = 1. \tag{5.3}$$

Let n_k *be a sequence of positive integers such that*

$$a(n_{k+1}) > n_k \sum_{j=1}^{k} a(n_j) \quad \text{and} \quad a(n_j) > 1, \quad j \in \mathbb{N}. \tag{5.4}$$

If

$$A(z) = \sum_{k=1}^{\infty} a(n_k) z^{m_k}, \quad \text{where } m_k = n_k[a(n_k)], \quad |z| < 1, \tag{5.5}$$

then there exists a sequence r_s, $0 < r_s \uparrow 1-0$ *as* $s \to \infty$, *such that*

$$\lim_{r_s \to 1-0} A(r_s e^{i\theta}) = \infty$$

uniformly for $\theta \in [0, 2\pi]$.

If $(z_n)_1^\infty$ is the sequence of zeros of an analytic function $A(z)$ in the disc $|z| < 1$, then Blaschke's condition

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty$$

holds if and only if the integral

$$\int_0^{2\pi} \log |A(re^{i\theta})| \, d\theta, \qquad 0 < r < 1,$$

is bounded (see, for instance, [18], Ch. 9, § 3). Hence it follows from Theorem 9 that, under the assumptions there, the sequence of zeros (z_n) of the power series (5.5) fails to satisfy Blaschke's condition. This result underlies the proof of Theorem 8.

For each function Ω satisfying (5.1) we can find an increasing function $\omega(t)$ such that $0 < \omega(t) \uparrow \infty$ as $t \to +\infty$ and $\omega(t) \leqslant \Omega(t)$, $t \geqslant 0$. If the system $e(\Lambda)$ is incomplete in $L^p_\omega(\mathbb{R}_+)$ (in $C_{0,\omega}[0,\infty)$), it is incomplete in $L^p_\Omega(\mathbb{R}_+)$ (in $C_{0,\Omega}[0,\infty)$), and therefore we can assume that $\Omega(t)$ is a monotone function growing not too rapidly to ∞ in the following sense:

$$\lim_{t\to\infty} (\Omega(t))^{1/t} = 1. \tag{5.6}$$

We must find a nontrivial functional φ on the space (5.2) which annihilates $e(\Lambda)$ and such that the sequence of zeros (λ_n) of the Laplace transform of this functional $\Phi(w)$ satisfies condition (1.4).

a) *The case of* $L^p_\Omega(\mathbb{R}_+)$, $p \geqslant 2$.

Fixing $\gamma \in (0, 1/p)$ we set

$$a_n = \Omega^\gamma(n), \qquad n \in \mathbb{N}. \tag{5.7}$$

Then the a_n satisfy (5.3) by (5.6). Now we pick a sequence of positive integers n_k such that conditions (5.4) and

$$\sum_{k=1}^{\infty} \frac{1}{\Omega^{p'(1/p-\gamma)}(n_k)} < \infty \tag{5.8}$$

hold simultaneously (here condition (5.1) is essential).

Keeping the notation m_k introduced in (5.5) we define a function $\varphi(t)$ as follows:

$$\Omega^{1/p}\varphi(t) = a(n_k) \text{ for } t \in I_k = (m_k, m_k+1); \qquad \varphi(t) = 0 \text{ for } t \notin \bigcup_{k=1}^{\infty} I_k. \tag{5.9}$$

Then $\varphi \in L^{p'}(\mathbb{R}_+)$. In fact,

$$\|\varphi\|_{p'}^{p'} = \sum_{k=1}^{\infty} a^{p'}(n_k) \int_{m_k}^{m_k+1} \Omega^{-p'/p}(t)\, dt \leqslant \sum_{k=1}^{\infty} \frac{1}{\Omega^{p'(1/p-\gamma)}(n_k)} < \infty$$

since Ω is monotonic and (5.8) holds.

Using formula (2.7) for $m = 0$ we see that

$$\begin{aligned} \Phi(w) &= \int_0^{\infty} e^{-wt}\Omega^{1/p}(t)\varphi(t)\, dt = \sum_{k=1}^{\infty} a(n_k) \int_{m_k}^{m_k+1} e^{-wt}\, dt \\ &= \frac{1-e^{-w}}{w} \sum_{k=1}^{\infty} a(n_k) e^{-wm_k}, \qquad \operatorname{Re} w > 0, \end{aligned} \tag{5.10}$$

is the Laplace transform of a nontrivial functional on the space $L^p_\Omega(\mathbb{R}_+)$.

Let $\Lambda = (\lambda_n)$ denote the sequence of zeros of $\Phi(w)$ or, equivalently, the sequence of zeros of the Dirichlet series on the right-hand side of (5.10). By Lemma 1 the system $e(\Lambda)$ is incomplete in $L^p_\Omega(\mathbb{R}_+)$.

We claim that the sequence Λ satisfies (1.4). It is sufficient to verify that

$$\sum (\operatorname{Re}\lambda_n : 0 < \operatorname{Re}\lambda_n < 1,\, 0 < \operatorname{Im}\lambda_n \leqslant 2\pi) = \infty. \tag{5.11}$$

We look at the mapping $z = e^{-w}$ from the rectangle

$$0 < \operatorname{Re} w < 1, \qquad 0 < \operatorname{Im} w \leqslant 2\pi \tag{5.12}$$

onto the annulus $1/e < |z| < 1$. Since $m_k \in \mathbb{N}$, the Dirichlet series in (5.10) is taken to the power series

$$\sum_{k=1}^{\infty} a(n_k) z^{m_k},$$

which is just the function $A(z)$ in Theorem 9. The assumptions of this theorem hold by construction. Hence the sequence (z_n) of zeros of $A(z)$ satisfies

$$\sum (1 - |z_n|) = \infty. \tag{5.13}$$

Since the disc $|z| \leqslant 1/e$ contains finitely many zeros from the set (z_n), (5.13) also holds for the points z_n lying in the annulus $1/e < |z| < 1$. However, these are the images of the zeros λ_n of $\Phi(w)$ lying in the rectangle (5.12):

$$z_n = e^{-\lambda_n}, \qquad 0 < \operatorname{Re}\lambda_n < 1, \quad 0 < \operatorname{Im}\lambda_n \leqslant 2\pi.$$

Hence (5.13) is transformed into the condition

$$\sum (1 - e^{-\operatorname{Re}\lambda_n} : 0 < \operatorname{Re}\lambda_n < 1,\ 0 < \operatorname{Im}\lambda_n \leqslant 2\pi) = \infty,$$

which is obviously equivalent to (5.11).

This completes our analysis of the case of $L^p_\Omega(\mathbb{R}_+)$, $p \geqslant 2$.

b) *The case of the space* $C_{0,\Omega}[0,\infty)$.

We pick the a_n defined by formula (5.7) with $\gamma = 1/2$. Then the a_n satisfy (5.3). We also take n_k, $k \in \mathbb{N}$, such that both the conditions (5.4) and

$$\sum_{k=1}^{\infty} \frac{1}{\Omega^{1/2}(n_k)} < \infty \tag{5.14}$$

hold. We define the quantity m_k as in (5.5) and define the function φ by (5.9) with $p = 1$. Then $\varphi \in L^1(\mathbb{R}_+)$. In fact, since Ω is monotonic and (5.14) holds, we have

$$\|\varphi\|_1 = \sum_{k=1}^{\infty} a(n_k) \int_{m_k}^{m_k+1} \frac{1}{\Omega(t)}\,dt < \sum_{k=1}^{\infty} \frac{1}{\Omega^{1/2}(n_k)} < \infty.$$

Hence the function $\Phi(w)$ featuring in (5.10) with $p = 1$ is the Laplace transform of some functional on $C_{0,\Omega}[0,\infty)$. Now, repeating the arguments after formula (5.10) completes the proof of the theorem in case b).

The proof is complete.

Remark 5. Our proof for case a) holds for all $p > 1$, but we only include the case $p \geqslant 2$ in the statement since we have a stronger result for $1 \leqslant p < 2$, by contrast with $p \geqslant 2$ (see § 1): condition (1.4) is not sufficient for the system $e(\Lambda)$ to be complete in the weight-free space $L^p(\mathbb{R}_+)$.

The author is obliged to the referee for useful comments, which were helpful in formulating adequate terminology and the concomitant notation.

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Received 26/NOV/12 and 18/FEB/13
Translated by N. KRUZHILIN