A MORE ELEMENTARY PROOF OF BERTRAND'S POSTULATE

MANOJ VERMA

Abstract. Bertrand's Postulate is the statement that there is a prime between n and $2n$ for $n > 1$. It was proved first by Chebyshev in 1850 and a simple elementary proof not requiring even calculus was given by Erdős [1] in 1932. We make some changes to obtain a proof that, in addition, does not require the binomial theorem, knowing about logarithms or e or any infinite series, or a prime number beyond 29 to verify the postulate by hand for small n.

Mathematics Subject Classification 2000: 11A41.

1. Introduction

I have not read Erdős's original proof [1] but it is described in numerous articles on various websites, e.g., [2] and books, e.g., [3]. Main ideas in our article are essentially those of Chebyshev and Erdős (as presented in [2] or [3]). I have used slightly better bounds (derived from induction alone) to avoid the use of logarithms. Our proof does not require a calculator and the largest prime needed to verify the postulate by hand is 29.

2. The Proof

For a positive integer r and a prime p, we shall denote by $\mu_p(r)$ the unique integer k satisfying $p^k \leq r < p^{k+1}$ and by and $\nu_p(r)$ the largest non-negative integer l such that r is divisible by p^l. For a positive rational number r/s with r and s positive integers, we let $\nu_p(r/s)$=$\nu_p(r) - \nu_p(s)$. For a real number x, we shall denote by $\pi(x)$ the number of primes less than or equal to x.

For positive integers m, n let $A(m,n) := \frac{(m+n)!}{m!n!}$. Then

$$\nu_p(A(m,n)) = \nu_p((m+n)!) - \nu_p(m!) - \nu_p(n!) = \sum_{i=1}^{\mu_p(m+n)} \left(\left\lfloor \frac{m+n}{p^i} \right\rfloor - \left\lfloor \frac{m}{p^i} \right\rfloor - \left\lfloor \frac{n}{p^i} \right\rfloor \right). \tag{2.1}$$

Since $0 \leq \lfloor x+y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor \leq 1$ for any real numbers x and y, each term in the sum on the right is 0 or 1. Hence $A(m,n)$ is an integer. It follows from (2.1) for any m, n that

$$\nu_p(A(m,n)) = 1 \text{ for } \max(m,n) < p \leq m+n \tag{2.2}$$

and on taking $m = n$ that $p^{\nu_p(A(n,n))} \le p^{\mu_p(2n)} \le 2n$ for any prime p, $\nu_p(A(n,n)) \le 1$ for $\sqrt{2n} < p \le \frac{2n}{3}$, $\nu_p(A(n,n)) = 0$ for $\frac{2n}{3} < p \le n$, and $\nu_p(A(n,n)) = 1$ for $n < p \le 2n$. Thus

$$\begin{aligned} A(n,n) = \prod_{p \le 2n} p^{\nu_p(A(n,n))} &\le \prod_{p \le \sqrt{2n}} p^{\nu_p(A(n,n))} \prod_{\sqrt{2n} < p \le \frac{2n}{3}} p \prod_{n < p \le 2n} p \\ &\le (2n)^{\pi(\sqrt{2n})} \prod_{\sqrt{2n} < p \le \frac{2n}{3}} p \prod_{n < p \le 2n} p. \end{aligned}$$

Bertrand's postulate can be verified for $n < 29$ using the primes 2, 3, 5, 7, 13, 23, 29. Now we proceed to prove the postulate for $n \ge 29$ by the method of contradiction. If there were no prime number between n and $2n$, we would have

$$A(n,n) \le (2n)^{\pi(\sqrt{2n})} \prod_{p \le \frac{2n}{3}} p. \tag{2.3}$$

To obtain a contradiction for $n \ge 29$ we need some inequalities that we prove now.

Lemma 2.1. *We have*

$$\frac{4^n}{\sqrt{3n+1}} \ge A(n,n) \ge \frac{4^n}{\sqrt{4n}}$$

with equality iff $n = 1$.

Proof. By induction on n. Consider the expression $\frac{4^n}{\sqrt{xn+y}}$. It equals $A(n,n)$ when $n = 1$ if $x + y = 4$. For $n \ge 1$, on increasing n by 1 it gets multiplied by $\frac{4\sqrt{xn+y}}{\sqrt{xn+x+y}}$ while $A(n,n)$ gets multiplied by $\frac{(2n+1)(2n+2)}{(n+1)^2} = \frac{2(2n+1)}{n+1}$. Now

$$\frac{4\sqrt{xn+y}}{\sqrt{xn+x+y}} > \text{ or } = \text{ or } < \frac{2(2n+1)}{n+1}$$

according as

$$4(xn+y)(n+1)^2 > \text{ or } = \text{ or } < (2n+1)^2(xn+x+y).$$

The left hand side equals $4xn^3 + (8x+4y)n^2 + (4x+8y)n + 4y$ while the right hand side equals $4xn^3 + (8x+4y)n^2 + (5x+4y)n + x + y$. If we take $x = 3, y = 1$ the left hand side is greater than the right hand side for $n \ge 1$. If we take $x = 4, y = 0$ the left hand side is less than the right hand side for $n \ge 1$. □

Lemma 2.2. *The inequality* $\pi(x) \le 0.4x + 1$ *holds for any real number* $x \ge 7.5$.

Proof. First of all, the inequality holds for all x in the interval $7.5 \le x < 17$ since it holds at $x = 7.5$ and at the primes in that range. For $x > 3$, on increasing x by 6, $0.4x$ increases by 2.4 while $\pi(x)$ increases by at most 2. So the inequality holds for all $x \ge 7.5$. □

Lemma 2.3. *The inequality* $\prod_{p \le n} p < 4^n / \sqrt{54(n+1)^3}$ *holds for* $n \ge 8$.

Proof. The inequality is equivalent to $54(n+1)^3 (\prod_{p \le n} p)^2 \le 16^n$ and holds at $n = 8$ since $54 \cdot 729 \cdot 2^2 \cdot (3 \cdot 5)^2 \cdot 7^2 < 2^6 \cdot 2^{10} \cdot 2^2 \cdot (2^4)^2 \cdot 2^6 = 2^{32} = 16^8$. For $8 < n < 17$,

$$\frac{(n+1)^3}{9^3} \left(\prod_{8 < p \le n} p \right)^2 < 8 \cdot 16^{2 \cdot (\text{the number of primes } p \text{ with } 8 < p \le n)} < 16^{n-8}$$

(as the only primes between 8 and 16 are 11 and 13) so the inequality holds for $8 < n \leq 16$ as well. Assuming that the inequality holds for $n = m \geq 8$ we prove it for $n = 2m-1, 2m$ to complete the argument by induction. Taking $n = m-1$ in (2.2), $\nu_p(A(m, m-1)) = 1$ for $m+1 \leq p \leq 2m-1$. Thus $A(m, m-1)$ is divisible by any prime between $m+1$ and $2m-1$ and hence is greater than or equal to their product. Also, $A(m, m-1) = \frac{2m-1}{m}A(m-1, m-1) < \frac{2m-1}{m}\frac{4^{m-1}}{\sqrt{3m-2}}$. As $(2m+1)^3 < 16(2m)^3$ and $2m$ is not prime,

$$\begin{aligned}
\frac{54(2m+1)^3(\prod_{p\leq 2m} p)^2}{16^{2m}} &< \frac{54(2m)^3(\prod_{p\leq 2m-1} p)^2}{16^{2m-1}} \\
&= \frac{8m^3}{(m+1)^3} \cdot \frac{54(m+1)^3(\prod_{p\leq m} p)^2}{16^m} \cdot \frac{(\prod_{m+1\leq p\leq 2m-1} p)^2}{16^{m-1}} \\
&< \frac{8m^3}{(m+1)^3} \cdot 1 \cdot \frac{(A(m, m-1))^2}{16^{m-1}} \\
&= \frac{8m^3}{(m+1)^3} \cdot \frac{(2m-1)^2}{m^2} \cdot \frac{(A(m-1, m-1))^2}{16^{m-1}} \\
&< \frac{8m^3}{m^2(m+3)} \cdot \frac{m(4m-3)}{m^2} \cdot \frac{1}{3(m-1)+1} \\
&= \frac{32m-24}{(m+3)(3m-2)} \leq \frac{32m-24}{11(3m-2)} < 1. \quad \square
\end{aligned}$$

It follows from lemma 2.3 that for any real number $x \geq 8$,

$$\prod_{p\leq x} p = \prod_{p\leq \lfloor x\rfloor} p < 4^{\lfloor x\rfloor}/\sqrt{54(\lfloor x\rfloor + 1)^3} < 4^x/\sqrt{54x^3}.$$

Since $(7.5)^2/2 = 56.25/2 < 29$, for $n \geq 29$ we have $\sqrt{2n} > 7.5$ (and $2n/3 > 8$) so we can apply the inequality above and those in lemmas 2.1 and 2.2 in (2.3) to get

$$\frac{4^n}{\sqrt{4n}} < (2n)^{0.4\sqrt{2n}+1} \cdot \frac{4^{2n/3}}{\sqrt{54(2n/3)^3}} = (2n)^{0.4\sqrt{2n}} \cdot \frac{4^{2n/3}}{\sqrt{4n}}.$$

Dividing both sides by $4^{2n/3}/\sqrt{4n}$ and then multiplying the exponents by $6/\sqrt{2n}$,

$$4^{\sqrt{2n}} < (2n)^{2.4}, \text{ i.e., } 2^{\sqrt{2n}} < (\sqrt{2n})^{2.4}.$$

But $2^x > x^{2.4}$ for any real number $x \geq 7.5$ by induction! For $7.5 \leq x \leq 8$, $2^x \geq 2^{7.5} > 2^{7.2} = 8^{2.4} \geq x^{2.4}$. For $x \geq 7.5$, on increasing x by 0.5, 2^x gets multiplied by $\sqrt{2}$ while $x^{2.4}$ gets multiplied by $(1 + \frac{1}{2x})^{2.4} < (\frac{16}{15})^3 < \frac{16}{15} \cdot \frac{15}{14} \cdot \frac{14}{13} = \frac{16}{13} = \sqrt{\frac{256}{169}} < \sqrt{2}$. So the inequality $2^x > x^{2.4}$ holds for all real numbers $x \geq 7.5$.

References

[1] P. Erdős, Beweis eines Satzes von Tschebyschef, Acta Sci. Math. (Szeged) **5** (1930-1932), 194-198.
[2] D. Galvin, Erdős's proof of Bertrand's postulate, http://www3.nd.edu/d̃galvin1/pdf/bertrand.pdf
[3] I. Niven, H. S. Zuckerman and H. L Montgomery, An Introduction to the Theory of Numbers, 5th ed., Wiley & Sons, New York (1991).

MANOJ VERMA: Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600113, India

E-mail address: mverma@imsc.res.in