Kepler's Law

Daniel Topa
daniel.topa@hii-tsd.com

Mission Technologies
Huntington Ingalls Industries
Kirtland AFB, NM

October 15, 2024

Abstract

Kepler's law is a cornerstone of orbital mechanics.

Contents

1 Overview

Kepler's law is a cornerstone of orbital mechanics.

1. `env`
2. `ldd`

2 Derivation

$$\frac{du^2}{d^2\lambda} + u = \frac{\mu}{h}, \quad u(0) = u_0, u'(0) = v_0 \tag{1}$$

First find the solution for the homogenous equation

$$\frac{du^2}{d^2\lambda} + u = 0 \tag{2}$$

which is

$$u(\theta) = A\cos\theta + B\sin\theta \tag{3}$$

Using the boundary conditions, $u(\theta) = u_0\cos\theta + v_0\sin\theta$.

3 Geometry of Kepler's Law

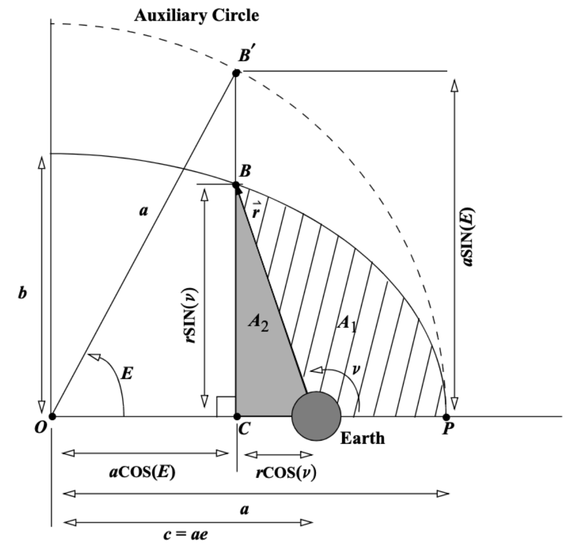

Figure 2-2. **Geometry of Kepler's Equation.** The eccentric anomaly uses an auxiliary circle as shown. The ultimate goal is to determine the area, A_1, which allows us to calculate the time.

Figure 1: Vallado's figure 2-2 showing E and ν.

Disagreement with this YouTuber True Anomaly vs. Mean Anomaly

4 Mathematics

4.1 Definitions

Definition 1 (The ellipse)**.** *Given* $\theta \in [0, 2\pi)$*, and parameters* $a, b \in \mathbb{R}^+$ *with* $a > b$ *the following parametric form defines an ellipse.*

$$\epsilon(\theta) = (a \cos\theta, b \sin\theta) \tag{4}$$

Definition 2 (Eccentricity of the ellipse)**.** *The eccentricity is a scalar parameter* $e \in (0, 1)$ *and can be expressed in terms of fundamental parameters of the ellipse where* $a > b$ *as*

$$e = \frac{c}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2} \tag{5}$$

Definition 3 (Mean anomaly)**.** *Kepler's Law*[1] *defines the mean anomaly as the angular measure* $M(e, E)\colon (0, 1) \times [0, 2\pi) \mapsto [0, 2\pi)$ *as*

$$M(e, E) = E - e \sin E \tag{6}$$

[1] Bate et al. 2020, eq 4.5 Moulton 1970, p.159 Vallado 2022, §2.2, Kaula 2013, pp. 3–19

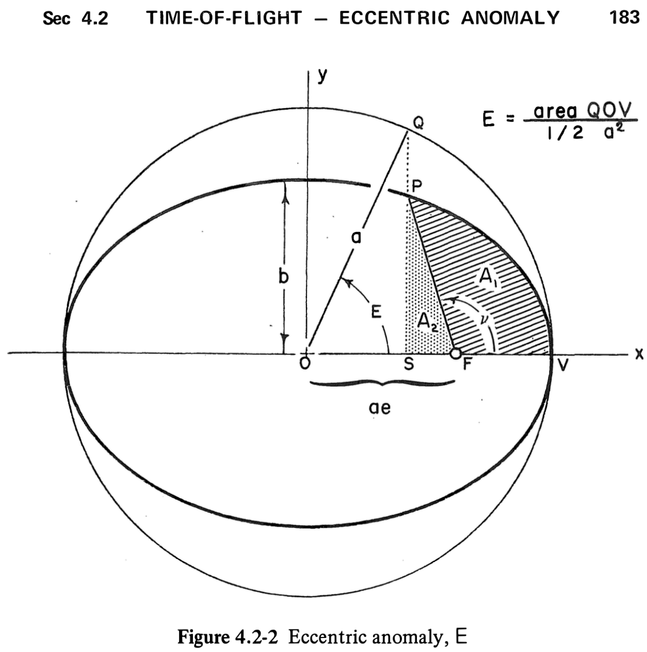

Figure 2: Figure 4-4 in Bate *et al.* showing E and ν.

Theorem 4 (Continuity of the mean anomaly)**.** *The mean anomaly as defined in definition 3 is a continuous function.*

Proof. To prove continuity show that for any two points p and q in the domain there exists a majorization constant K such that

$$M(p) - M(q) \leq K|p - q| \tag{7}$$

Spoiler alert: the majorization constant is 2π. □

Observation: given the continuity of the mean anomaly, one may use Newton's method Gautschi 2011, §4.6 to solve the nonlinear equation.

References

1. Bate, Roger R et al. (2020). *Fundamentals of astrodynamics.* Courier Dover Publications.
2. Gautschi, Walter (2011). *Numerical analysis.* Springer Science & Business Media.
3. Kaula, William M (2013). *Theory of satellite geodesy: applications of satellites to geodesy.* Courier Corporation.
4. Moulton, Forest Ray (1970). *An introduction to celestial mechanics.* Dover.
5. Vallado, D.A. (2022). *Fundamentals of Astrodynamics and Applications.* 5th ed. Microcosm Press;

160 SOLUTION OF KEPLER'S EQUATION. [95

ACQ will be defined as the eccentric anomaly, E, and it will be shown that the relation between M and E is given by Kepler's equation.

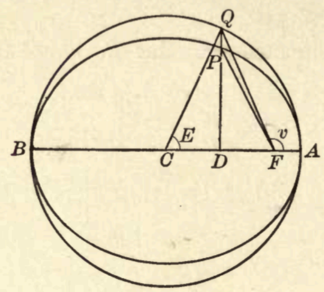

Fig. 28.

From the law of areas and the properties of the auxiliary circle, it follows that

$$\frac{M}{2\pi} = \frac{\text{area } AFP}{\text{area ellipse}} = \frac{\text{area } AFQ}{\text{area circle}}.$$

Area AFQ = area ACQ − area $FCQ = \dfrac{a^2E}{2} - \dfrac{a}{2} ae \sin E.$

Therefore

$$\frac{M}{2\pi} = \frac{a^2}{2} \frac{(E - e \sin E)}{\pi a^2};$$

or,

$$\begin{cases} M = E - e \sin E, \\ FP = r = \dfrac{a(1 - e^2)}{1 + e \cos v} = \sqrt{\overline{PD}^2 + \overline{FD}^2} = a(1 - e \cos E), \end{cases}$$

Figure 3: Moulton's figure 28 showing E and ν.

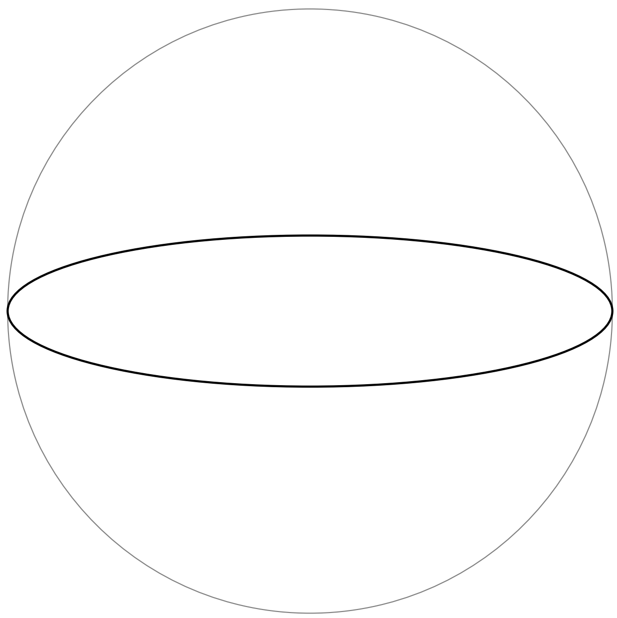

Figure 4: Geometry