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ON THE CONSTRUCTION OF LYAPUNOV FUNCTIONS*

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1. Introduction. Let A be an $n \times n$ matrix of real-valued functions of a real variable t which are piecewise continuous for $0 \leqq t < \infty$. Consider a system of linear differential equations

$$x' = Ax \qquad (' = d/dt), \tag{1}$$

where x is a column vector and X is the unknown principal solution matrix of (1); that is, $X(0) = E$ the identity matrix.

We show that (1) can be mapped into a system $y' = By$ such that all solutions of a system $w' = 2Bw$ can be found. Then we establish that, under special conditions, the solutions of the last two systems have stability properties which are essentially the same. These results are used to obtain new instability results for the Hill equation which extend the classical results of Lyapunov and Haupt. Finally, we show that a Lyapunov function for $w' = 2Bw$ can be used to *algorithmically* obtain a Lyapunov function for $y' = By$ along with certain verifiable conditions from which stability properties of $y' = By$ (and hence of $x' = Ax$) may be obtained. No such algorithm may be found in the present literature.

2. Preliminary results. The matrix A can be written as a sum of piecewise continuous matrices

$$A = A_1 + A_2 \tag{2}$$

so that both

$$h' = 2A_1h \tag{3}$$

and

$$z' = 2A_2z \tag{4}$$

can be integrated for principal solution matrices H and Z respectively. For example, A_1 and A_2 may be chosen as triangular matrices.

THEOREM 1. *For H defined by* (3), *the transformation $x = Hy$ maps* (1) *into*

$$y' = By \tag{5}$$

and $z = Hw$ maps (4) *into*

$$w' = 2Bw, \tag{6}$$

where $B = H^{-1}[A - H'H^{-1}]H$. Therefore, if H and Z are known, then $W = H^{-1}Z$ is known.

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Proof. We first calculate B and then show that B is the same in (5) and (6). Now $x = Hy$ implies that $x' = H'y + Hy' = [H'H^{-1} + HBH^{-1}]Hy$. Since $x' = Ax$ we have $A = H'H^{-1} + HBH^{-1}$ or $B = H^{-1}[A - H'H^{-1}]H$. But $A = A_1 + A_2$ while $A_1 = \frac{1}{2}H'H^{-1}$ and so $2B = H^{-1}[2A_2 - H'H^{-1}]H$. This verifies (5) and (6).

Now elementary considerations indicate that there may be strong connections between (5) and (6). For example, if B is constant, then $Y(2t) = W(t)$, where Y is the principal solution matrix for (5).

We shall assume henceforth that H is a Lyapunov matrix and hence that X is equivalent to Y in the sense of Lyapunov. Then stability properties of (1) will be determined whenever connections between (5) and (6) are determined. We refer to [4, pp. 140–143] for properties of Lyapunov transformations and Lyapunov matrices. Stability definitions may be found in [5, pp. 1–9].

THEOREM 2. *If there exists a matrix M satisfying $W = MM$ and $M'M = MM'$, then $Y = M$.*

Proof. If $W = MM$, then $W' = M'M + MM' = 2M'M = 2M'M^{-1}(MM)$. Thus $B = M'M^{-1}$ and $M' = BM$ or $Y = M$.

Now Theorem 2 holds, for example, if B commutes with its integral.

THEOREM 3. *If $W(t)$ is bounded as t tends to ∞ and if the sequence of partial sums in Picard's successive approximations for $W(t)$ are uniformly bounded, then $Y(t)$ is bounded for $0 \leqq t < \infty$.*

Proof. A calculation shows that if the partial sums for $Y(t)$ are denoted by $Y_m(t) = E + \sum_{n=0}^{n=m} C_n(t)$, then the partial sums for $W(t)$ are given by $W_m(t) = E + \sum_{n=0}^{n=m} 2^n C_n(t)$. The result then follows from Abel's lemma for sequences.

The hypotheses of Theorem 3 hold, for example, if B is odd and periodic.

THEOREM 4. *If there exists an absolutely continuous function $f(t)$ on $[0, \infty)$ satisfying $f'(t)B(f(t)) = 2B(t)$ for almost all t, then $Y(f(t)) = W(t)Y(f(0))$.*

This theorem can be proved in exactly the same manner as Theorem 1 of [10, p. 23].

THEOREM 5. *If W and B are symmetric, then $W = YY^T$, where T denotes the transpose.*

Proof. Consider the matrix $V = Y^T W^{-1} Y$. Then

$$V' = Y^T[B^T W^{-1} - 2B^T W^{-1} + W^{-1}B]Y = Y^T[-B^T W^{-1} + W^{-1}B]Y.$$

Since B and W^{-1} are symmetric, it follows that $V' = 0$ and so $V = E$.

THEOREM 6. *Let the conditions of Theorem 5 hold and let W be a Lyapunov matrix. Then $V = y^T W^{-1} y$ is a Lyapunov function for $y' = By$ with an infinitely small upper bound. Since $V' = 0$, the null solution is Lyapunov stable and every nonzero solution is bounded strictly away from zero. If W is periodic, then Y is almost periodic.*

Proof. We have seen that $V' = 0$. Since W^{-1} is symmetric, the characteristic roots of W^{-1} are real and bounded strictly away from zero by the nonsingularity of W and the fact that W is a Lyapunov matrix. Hence, there exist positive constants k and K such that $ky^T y \leqq y^T W^{-1} y \leqq K y^T y$ and so V is positive definite with an infinitely small upper bound. This proves that the solution $y = 0$ is stable and that no solution tends to zero. The almost periodic part follows in exactly the same manner as in [1, pp. 327–328].

Remark. If W is known, then a Lyapunov function for $w' = 2Bw$ can be constructed proving stability or instability. (For this and other facts about Lyapunov's direct method see [3, pp. 11–31], [5, pp. 1–35] and [8].) Notice that if $V(w)$ is an autonomous Lyapunov function for $w' = 2Bw$ proving stability or instability of the solution $w = 0$, then $V(y)$ is a Lyapunov function for $y' = By$ proving the same properties about $y = 0$. The difficulty here is that the constructable Lyapunov function for (6) often turns out to be nonautonomous.

3. Instability in the Hill equation. Consider the equation

$$u'' + c(t)u = 0, \tag{7}$$

where $c(t)$ is piecewise continuous and periodic of least period $T > 0$. In particular we require that $c(t) = -a^2 + q(t)$, where $a > 0$ and the integral of $q(t)$ over a period is zero.

It would be most difficult to cite the extensive literature on this problem, but much of it is summarized in [2, pp. 59–66] and in [6]. Under our assumption, (7) was investigated by Haupt [7] and Lyapunov [9, p. 404] with the following pertinent results.

THEOREM (Lyapunov). *If $c(t)$ is continuous, nonpositive, and not identically zero, then* (7) *has unbounded solutions.*

THEOREM (Haupt). *If $q(t) = kr(t)$, then for k sufficiently small* (7) *has unbounded solutions.*

Our main interest is in Haupt's theorem. It is known (see [2, p. 66]) that, for certain a^2 and certain $kr(t)$, (7) has only bounded solutions. We wish to give definite relations between a^2 and $q(t)$ to insure that (7) has unbounded solutions.

In (7) let $u' = av$ to obtain the equivalent system

$$\begin{aligned} u' &= av, \\ v' &= [a - q(t)/a]u, \end{aligned} \tag{8}$$

which we denote by $x' = Ax$. Then let

$$A_1 = \begin{pmatrix} 0 & q/(4a) \\ -q/(4a) & 0 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & a - [q/(4a)] \\ a - [3q/(4a)], & 0 \end{pmatrix}.$$

Then

$$H = \begin{pmatrix} \cos \displaystyle\int_{t_0}^{t} \frac{q(s)}{2a}\, ds & \sin \displaystyle\int_{t_0}^{t} \frac{q(s)}{2a}\, ds \\ -\sin \displaystyle\int_{t_0}^{t} \frac{q(s)}{2a}\, ds & \cos \displaystyle\int_{t_0}^{t} \frac{q(s)}{2a}\, ds \end{pmatrix}.$$

Also,

$$B = H^{-1}[A - H'H^{-1}]H = H^{-1}[A - 2A_1]H = \begin{pmatrix} -f_1(t) & f_2(t) \\ f_2(t) & f_1(t) \end{pmatrix},$$

where

$$f_1(t) = 2\left[a - \frac{q}{2a}\right]\left[\sin \int_{t_0}^{t} \frac{q}{2a}\,ds\right]\left[\cos \int_{t_0}^{t} \frac{q}{2a}\,ds\right]$$

and

$$f_2(t) = \left[a - \frac{q}{2a}\right]\left[2\cos^2 \int_{t_0}^{t} \frac{q}{2a}\,ds - 1\right].$$

Write $y' = By$ as

$$\begin{aligned} g' &= -f_1(t)g + f_2(t)r, \\ r' &= f_2(t)g + f_1(t)r. \end{aligned} \tag{9}$$

Denote z by $(z_1, z_2)^T$ and observe that for $z' = 2A_2z$ we have the first quadrant of the z-plane as an invariant set R and a Lyapunov function $V(z) = z_1z_2$ with $V'(z) > 0$ as specified below. In this case we obtain a Lyapunov function for $w' = 2Bw$ which is autonomous.

THEOREM 7. *Suppose that* $|q(t)| < 2a^2$ *for all* t *and* $\cos^2 \int_{t_0}^{t} [q(s)/(2a)]\,ds > \frac{1}{2}$ *for some* t_0 *and* $t_0 \leqq t \leqq t_0 + T$. *Then* (7) *has unbounded solutions.*

Proof. Consider the function $V = gr$. Then $V' = f_2(t)[r^2 + g^2]$. Now V vanishes on $g = 0$ and $r = 0$, while $V' > 0$ inside the first quadrant. Using standard Lyapunov arguments we see that the null solution of (9) is unstable. But $X = HY$ while H is a Lyapunov matrix and so the null solution of (7) is unstable. It follows from Floquet theory that $X(t)$ becomes unbounded as t tends to ∞.

4. A general construction and stability result. We return to (5) and (6) where we make the following assumptions:

(a) $w = 0$ is asymptotically stable;

(b) there is a Lyapunov function for $w' = 2Bw$, say

$$V(w, t) = w^TC(t)w,$$

such that C is a continuously differentiable positive definite matrix and

$$V'(w, t) \leqq -f(t)V(w, t)$$

for all (w, t) where f is continuous and positive.

If our construction is to be accomplished in a finite number of steps, then we ask also that

(c) C and f be periodic of period $T > 0$.

Remark. In case B is periodic and (a) holds, then assumptions (b) and (c) are satisfied and we have $W(t) = P(t) \exp Rt$, where P is T-periodic, R is constant, and the function defined by

$$V = w^T(P^{-1})^TMP^{-1}w,$$

with M a constant positive definite matrix such that $R^TM + MR$ is negative definite, suffices as a Lyapunov function. For the details of obtaining f, see [8, p. 57]. If R is not real, other converse theorems may be used.

THEOREM 8. *If assumptions* (a) *and* (b) *hold, then from C there can be constructed a positive definite piecewise continuous and piecewise smooth matrix D together with a piecewise continuous positive scalar function g such that*

$$L(y, t) = y^T D(t) y$$

satisfies

$$L'(y, t) \leqq -g(t) L(y, t)$$

for all (y, t), *where* $L'(y, t)$ *denotes the right-hand derivative along trajectories of* (5). *If* (c) *holds, then the construction can be accomplished in a finite number of steps and both g and D are periodic.*

We shall delay the proof until after the next theorem. For the next result we need the following notation. Let $0 = e_1 < e_2 < \cdots < e_n = T$ be the points of discontinuity of D on an interval $[0, T]$, where T is arbitrary if (c) does not hold and T is the period if (c) holds. Denote by $D(e_i^-)$ and $D(e_i^+)$ the left- and right-hand limits of D at e_i. Set

$$G(e_{i+1}, e_i) = \int_{e_i}^{e_{i+1}} g(t)\, dt.$$

Finally, let the following numbers k_i be the minimal positive real numbers for which the following set inclusions hold.

THEOREM 9. *If in addition to the assumptions of the previous theorem there are constants* $k_2, k_3, \cdots, k_n$ *such that*

$$\{y : y^T D(e_2^-) y \leqq \exp - G(e_2, e_1)\} \subseteq \{y : y^T D(e_2^+) y \leqq k_2\},$$

and in general

$$\{y : y^T D(e_i^-) y \leqq k_{i-1} \exp - G(e_i, e_{i-1})\} \subseteq \{y : y^T D(e_i^+) y \leqq k_i\},$$

then

(i) $y = 0$ *is Lyapunov stable if* $k_n \leqq 1$ *and* (c) *holds,*
(ii) $y = 0$ *is asymptotically stable if* $k_n < 1$ *and* (c) *holds, and*
(iii) $y = 0$ *is Lyapunov stable if* $k_n \leqq 1$ *on arbitrarily large intervals* $[0, T]$.

Proof of Theorem 8. We shall construct D on an arbitrary compact interval, but for emphasis of the result when (c) holds we denote it by $[0, T]$. By (b) we have

$$V'(w, t) = w^T[2B^T(t)C(t) + C'(t) + 2C(t)B(t)]w \leqq -f(t)V(w, t).$$

Now define a matrix function by

$$D(t, s) = C(t) + C(s).$$

For each fixed s, $D(t, s)$ is differentiable and positive definite. Define

$$L(y, t, s) = y^T D(t, s) y$$

so that for fixed s,

$$L'(y, t, s) = y^T[B^T(t)C(t) + B^T(t)C(s) + C'(t) + C(t)B(t) + C(s)B(t)]y.$$

Then for $t = s$,

$$L'(y, s, s) = y^T[2B^T(s)C(s) + C'(s) + 2C(s)B(s)]y,$$

and so

$$L'(y, s, s) \leqq -\tfrac{1}{2} f(s) L(y, s, s)$$

with $f(s) > 0$. By continuity of $L'(y, t, s)$ for fixed s, there exists $\delta(s) > 0$ such that

$$L'(y, t, s) < 0 \quad \text{for } s - \delta(s) < t < s + \delta(s).$$

Then for each t in this interval we can find a positive number $g(t)$ [8, p. 57] such that

$$L'(y, t, s) \leqq -g(t) L(y, t, s).$$

Now the collection of intervals

$$\{(s - \delta(s), s + \delta(s)) : s \text{ is in } [0, T]\}$$

forms an open cover of $[0, T]$, so there exists a finite subcover, say

$$\{(s_i - \delta(s_i), s_i + \delta(s_i)) : i = 1, \cdots, n\}.$$

From this subcover we may obtain a set of closed intervals with endpoints e_i satisfying $0 = e_1 < e_2 < \cdots < e_n = T$ so that the sectionally smooth matrix D defined by

$$D(t) = D(t, e_i) \quad \text{for } e_i \leqq t < e_{i+1},$$

whose derivative at points of discontinuity shall be understood to be the right-hand derivative, makes the function defined by

$$L(y, t) = y^T D(t) y$$

have the property that

$$L'(y, t) \leqq -g(t) L(y, t)$$

for every t in $[0, T]$. Then $g(t)$ is positive and can be chosen so that g is sectionally continuous. If (c) holds, then the asserted periodicity is obvious.

Proof of Theorem 9. It is clear that since D is not necessarily continuous we can not conclude from $L' \leqq 0$ that $y = 0$ is Lyapunov stable. In order to prove stability we consider for fixed t in $[e_i, e_{i+1}]$ the set

$$S(t, e_i, k) = \{y : y^T D(t, e_i) y \leqq k \quad \text{for constant } k > 0\}$$

which is homeomorphic to a closed n-sphere. Consider $S(t, e_1, 1)$ and pick k_2 so that

$$S(e_2, e_1, \exp -G(e_2, e_1)) \subseteq S(e_2, e_2, k_2)$$

but inclusion holds for no smaller value of k_2. Pick k_3 so that

$$S(e_3, e_2, k_2 \exp -G(e_3, e_2)) \subseteq S(e_3, e_3, k_3)$$

but inclusion holds for no smaller k_3. Inductively, pick the minimal k_i such that

$$S(e_i, e_{i-1}, k_{i-1} \exp -G(e_i, e_{i-1})) \subseteq S(e_i, e_i, k_i).$$

It then follows by Gronwall's inequality applied to each interval $[e_i, e_{i+1}]$ that if $k_n < 1$, then under assumption (c), $y = 0$ is asymptotically stable, while $k_n \leqq 1$ implies that $y = 0$ is Lyapunov stable. If (c) does not hold, and if $k_n \leqq 1$ on arbitrarily large intervals $[0, T]$, then $y = 0$ is Lyapunov stable. This completes the proof, but for computational reasons some additional observations should be made.

Now the Heine-Borel theorem was used to obtain the e_i. We should show that suitable e_i may be found computationally. Let $e_1 = 0$ and consider $L(y, t, e_1)$ so that $L'(y, t, e_1) \leqq -g(t)L(y, t, e_1)$ can be solved for $g(t) > 0$ on an interval $e_1 \leqq t \leqq e_2$ for some $e_2 > 0$ by continuity of L' and L and the fact that $g(0) = \frac{1}{2}f(0)$. For details of solution see Krasovskii [8, p. 57]. Then consider $L(y, t, e_2)$ and, in the same manner as before, obtain $e_3 > e_2$. Inductively we obtain a sequence $\{e_i\}$. Using continuity of L' and the $D(t)$ of the previous theorem one can show that for any fixed T there exists i such that $e_i > T$.

Remark. It is obvious that the above result can be extended to complete instability by asking that $V' \geqq f(t)V$ for $f(t)$ positive. It is perhaps not so obvious, but also true, that the theorem may be modified to include instability theorems of the Chetayev variety (cf. Hahn [5, pp. 18–19]). The only essential innovation is to restrict the discussion to

$$\{(y, t) : 0 \leqq t \leqq T \text{ and } \|y\| \leqq \exp \max_{t \in [0,T]} \|B(t)\| T\}.$$

5. Critique. The only indefinite part about our method is the choice of A_1 and A_2. As we have mentioned before, a decomposition of A can always be made such that (3) and (4) can be solved. However, the choice of A_1 and A_2 as triangular matrices may not yield $H(t)$ as a Lyapunov matrix. Here, some ingenuity must be used to insure that (3) may be solved for $H(t)$ and that $H(t)$ is a Lyapunov matrix. We may observe that for Theorems 8 and 9 we would not need to solve (4) for $Z(t)$, but a suitable Lyapunov function for (4) would suffice.

It is certainly the case that finding the e_i, the k_i and g for a particular problem is tedious when D has many discontinuities on $[0, T]$. In fact, if the order of the differential equation were higher than two, then one might need a computer for the computation.

The method itself is of interest since it is the first algorithmic method of constructing Lyapunov functions for a completely arbitrary linear system. It may eventually be refined to the point that $D(t)$ can be chosen continuous along lines analogous to those used when the characteristic roots of B have negative real parts for every t (cf. Hahn [5, pp. 29–30]). The author has had little success in that direction.

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