Kepler's Equation: Derivation, Implementation, Validation

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Abstract

Kepler's law is a cornerstone of orbital mechanics.

Contents

1 Overview

Kepler's law is a cornerstone of orbital mechanics. Kepler's equation

$$M = E - e \sin E \tag{1}$$

Kepler's Law III

$$n = \mu^{\frac{1}{2}} a^{\frac{-3}{2}} \tag{2}$$

2 Derivation

2.1 Orbit Equation

$$\frac{du^2}{d^2\lambda} + u = \frac{\mu}{h}, \quad u(0) = u_0, u'(0) = v_0 \tag{3}$$

A second order linear partial differential equation Boyce, DiPrima, and Meade 2021 First find the solution for the homogenous equation

$$\frac{du^2}{d^2\lambda} + u = 0 \tag{4}$$

which is

$$u(\theta) = A\cos\theta + B\sin\theta \tag{5}$$

Using the boundary conditions, $u(\theta) = u_0\cos\theta + v_0\sin\theta$.

2.2 Ellipse

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1 \tag{6}$$

$$\xi = ae + r\cos f, \eta = r\sin f, b^2 = a^2(1-e^2) \tag{7}$$

$$(1-e^2)\xi^2 + \eta^2 - a^2(1-e^2) = 0 \tag{8}$$

$$f(r) = \alpha r^2 + \beta r + \gamma = 0 \tag{9}$$

$$\alpha = 1 - e^2\cos^2 f, \quad \beta = 2ae(1-e^2)\cos f, \quad \gamma = a^2(1-e^2)^2 \tag{10}$$

$$r_\pm = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \tag{11}$$

$$r_\pm = \pm\frac{a(1-e^2)}{1 \pm e\cos f} \tag{12}$$

3 Geometry of Kepler's Law

Disagreement with this YouTuber True Anomaly vs. Mean Anomaly

4 Mathematics

4.1 Definitions

Definition 1 (The ellipse)**.** *Given $\theta \in [0, 2\pi)$, and parameters $a, b \in \mathbb{R}^+$ with $a > b$ the following parametric form defines an ellipse.*

$$\epsilon(\theta) = (a\cos\theta, b\sin\theta) \tag{13}$$

Definition 2 (Eccentricity of the ellipse)**.** *The eccentricity is a scalar parameter $e \in (0, 1)$ and can be expressed in terms of fundamental parameters of the ellipse where $a > b$ as*

$$e = \frac{c}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2} \tag{14}$$

Definition 3 (Mean anomaly)**.** *Kepler's Law*[1] *defines the mean anomaly as the angular measure $M(e, E)\colon (0, 1) \times [0, 2\pi) \mapsto [0, 2\pi)$ as*

$$M(e, E) = E - e\sin E \tag{15}$$

[1] Bate et al. 2020, eq 4.5 Moulton 1970, p.159 Vallado 2022, §2.2, Kaula 2013, pp. 3–19

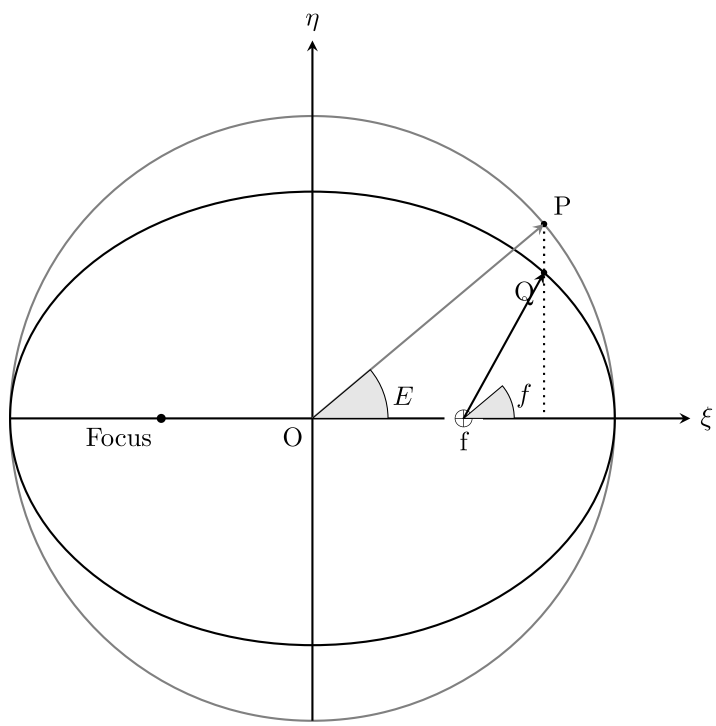

Figure 1: Orbit trajectory and the auxiliary circle showing the angles for eccentric anomaly, E, and the true anomaly, ν. The true anomaly indicates position relative to the Earth, $\oplus$, while the eccentric anomaly points to an echo point on the auxiliary circle.

Theorem 4 (Continuity of the mean anomaly)**.** *The mean anomaly as defined in definition 15 is a continuous function.*

Proof. To prove continuity show that for any two points p and q in the domain there exists a majorization constant K such that

$$M(p) - M(q) \leq K|p - q| \tag{16}$$

Spoiler alert: the majorization constant is 2π. $\square$

Observation: given the continuity of the mean anomaly, one may use Newton's method Gautschi 2011, §4.6 to solve the nonlinear equation.

A Survey

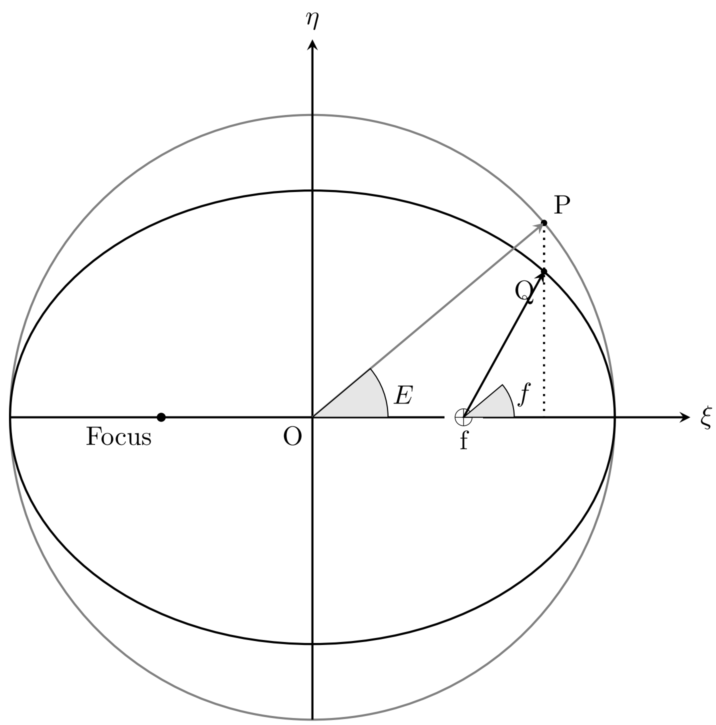

Figure 2: Orbit trajectory and the auxiliary circle showing the angles for eccentric anomaly, E, and the true anomaly, ν. The true anomaly indicates position relative to the Earth, $\oplus$, while the eccentric anomaly points to an echo point on the auxiliary circle.

Disagreement with this YouTuber True Anomaly vs. Mean Anomaly

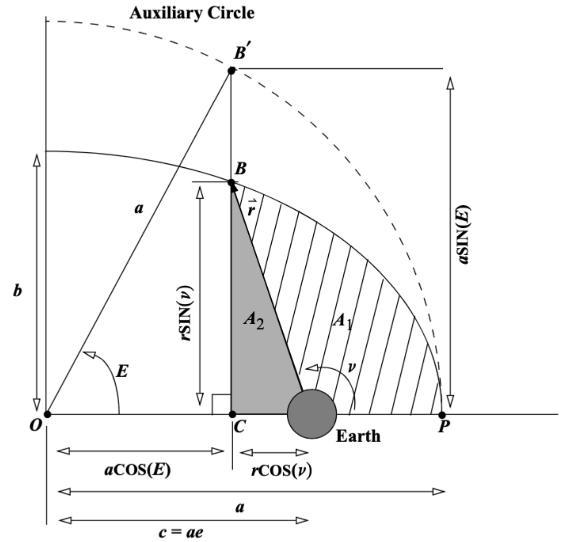

Figure 2-2. **Geometry of Kepler's Equation.** The eccentric anomaly uses an auxiliary circle as shown. The ultimate goal is to determine the area, A_1, which allows us to calculate the time.

Figure 3: Vallado's figure 2-2 showing E and ν.

Sec 4.2 TIME-OF-FLIGHT — ECCENTRIC ANOMALY 183

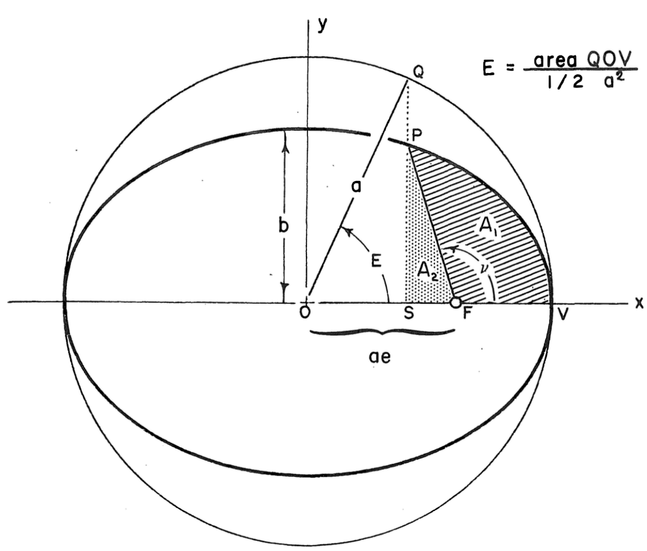

Figure 4.2-2 Eccentric anomaly, E

Figure 4: Figure 4-4 in Bate *et al.* showing E and ν.

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ACQ will be defined as the eccentric anomaly, E, and it will be shown that the relation between M and E is given by Kepler's equation.

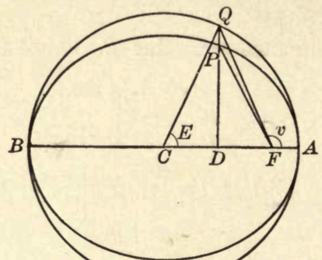

Fig. 28.

From the law of areas and the properties of the auxiliary circle, it follows that

$$\frac{M}{2\pi} = \frac{\text{area } AFP}{\text{area ellipse}} = \frac{\text{area } AFQ}{\text{area circle}}.$$

Area AFQ = area ACQ − area $FCQ = \dfrac{a^2E}{2} - \dfrac{a}{2}ae \sin E$.

Therefore

$$\frac{M}{2\pi} = \frac{a^2}{2}\frac{(E - e \sin E)}{\pi a^2};$$

or,

$$\begin{cases} M = E - e \sin E, \\ FP = r = \dfrac{a(1-e^2)}{1 + e \cos v} = \sqrt{\overline{PD}^2 + \overline{FD}^2} = a(1 - e \cos E), \end{cases}$$

Figure 5: Moulton's figure 28 showing E and ν.

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5. Moulton, Forest Ray (1970). *An introduction to celestial mechanics*. Dover.
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