ISSN 0010-9525, Cosmic Research, 2014, Vol. 52, No. 4, pp. 304–317. © Pleiades Publishing, Ltd., 2014.
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Quaternion Regularization and Trajectory Motion Control in Celestial Mechanics and Astrodynamics: II[1]

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Received March 20, 2012

Abstract—Problems of regularization in celestial mechanics and astrodynamics are considered, and basic regular quaternion models for celestial mechanics and astrodynamics are presented. It is shown that the effectiveness of analytical studies and numerical solutions to boundary value problems of controlling the trajectory motion of spacecraft can be improved by using quaternion models of astrodynamics. In this second part of the paper, specific singularity-type features (division by zero) are considered. They result from using classical equations in angular variables (particularly in Euler variables) in celestial mechanics and astrodynamics and can be eliminated by using Euler (Rodrigues–Hamilton) parameters and Hamilton quaternions. Basic regular (in the above sense) quaternion models of celestial mechanics and astrodynamics are considered; these include equations of trajectory motion written in nonholonomic, orbital, and ideal moving trihedrals whose rotational motions are described by Euler parameters and quaternions of turn; and quaternion equations of instantaneous orbit orientation of a celestial body (spacecraft). New quaternion regular equations are derived for the perturbed three-dimensional two-body problem (spacecraft trajectory motion). These equations are constructed using ideal rectangular Hansen coordinates and quaternion variables, and they have additional advantages over those known for regular Kustaanheimo–Stiefel equations.

DOI: 10.1134/S0010952514030022

1. ELIMINATING SINGULARITIES IN EQUATIONS OF CELESTIAL MECHANICS AND ASTRODYNAMICS DUE TO USING ANGULAR VARIABLES IN ORBITAL MOTION MODELS

In the first part of this work [2], the problem of regularizating differential equations was considered for a perturbed three-dimensional two-body problem: How best to eliminate a singularity that arises in two-body problem equations when the second body collides with the central body (distance r between them being equal to zero). Such a singularity at the origin of the coordinates creates both theoretical and practical (computing) difficulties.

Along with the above singularity, there are other peculiarities of the singularity type (division by zero) in classical models of celestial mechanics and astrodynamics that are written in rotating coordinate systems and use Euler angles to describe the angular motion of these coordinate systems. The same is true of models describing in angular variables the instantaneous orientation of an orbit or the orbital plane of a celestial body or spacecraft.

Note that classical models of orbital motion written in Cartesian coordinates or in vector form have no such singularities. These models are also clearer and more compact. In many cases, however, they turn out to be ineffective for both analytical and numerical investigations of the motion of celestial bodies and spacecraft, or for solving optimum control problems of spacecraft trajectory motion.

The effectiveness of solving problems of celestial mechanics and astrodynamics can in many cases be improved by using equations of motion written in one rotating coordinate system or another, along with such concepts as the shape, dimensions, and orientation of the instantaneous orbit of a moving body. In equations of motion of this type, variables emerge that characterize the angular motion of the rotating coordinate system or the orientation of the instantaneous orbit (orbit plane) of the moving body. Euler angles or direction cosines are traditionally used as such variables in mechanics and astrodynamics.

The use of Euler angles results in cumbersome trigonometric expressions appearing in the equations, along with additional singular points at which the equations degenerate. For example, the widely used Newton–Euler equations for osculating elements (slowly changing variables) include differential equa-

[1] This work is a review based on materials of the plenary session report *Quaternion Regularization in Trajectory Motion Control and Astrodynamics*, presented at the 10th National Meeting on Fundamental Problems of Theoretical and Applied Mechanics [1] (Section I, "General and Applied Mechanics").

tions for angular elements: ascending node longitude Ω_u, orbit tilt (inclination) I, and angular pericenter distance from the node $\omega_{\pi\mathrm{tr}}$. These equations take the form [3, 4]

$$\begin{gathered}\dot{\Omega}_u = (r/c)p_3\sin\Sigma\operatorname{cosec}I, \quad \dot{I} = (r/c)p_3\cos\Sigma, \\ \dot{\omega}_{\pi ir} = -(c/(\mu e_{or}))[p_1\cos\varphi_{\mathrm{tr}} - (1+\mu r/c^2)p_2\sin\varphi_{\mathrm{tr}}] \\ -(r/c)p_3\sin\Sigma\cot I, \quad \dot{\varphi}_{\mathrm{tr}} = c/r^2 + (c/(\mu e_{\mathrm{or}})) \\ \times[p_1\cos\varphi_{\mathrm{tr}} - (1+\mu r/c^2)p_2\sin\varphi_{\mathrm{tr}}]; \\ c = (\mu p_{\mathrm{or}})^{1/2}, \quad r = p_{\mathrm{or}}/(1+e_{\mathrm{or}}\cos\varphi_{\mathrm{tr}}), \\ \mu = f(m+M),\end{gathered} \tag{1.1}$$

where $\Sigma = \omega_{\pi\mathrm{tr}} + \varphi_{\mathrm{tr}}$ is the argument of latitude, φ_{tr} is the true anomaly, f is the gravitational constant, M and m are masses of the first (central) and second (under study) bodies, respectively; r is the distance between the bodies; p_{or} and e_{or} are the parameter and eccentricity of instantaneous orbit of the second body; and p_i are projections of the vector of perturbing or controlling acceleration of the center of mass of the second body (or of their sum) onto axes of the moving (orbital) coordinate system, the dot denoting the derivatives with respect to time.

Equations (1.1) are degenerate when angle I of the inclination of the second body's instantaneous orbit is equal to zero or π, and at $e_{\mathrm{or}} = 0$ when the orbit is circular.

The use of direction cosines allows us to eliminate this feature of the equations of motion for the second body. However, it leads to a considerable increase in the dimensionality of the system of equations of motion and to the loss of geometrical clarity. We can avoid these drawbacks of using Euler angles and direction cosines if the Euler (Rodrigues–Hamilton) parameters are chosen as parameters of orientation of the rotating coordinate system. It is then convenient to use a hypercomplex variable to describe the orientation of this coordinate system: a quaternion of turn whose components are the Euler parameters. Among the equations of trajectory motion, there then appears a differential quaternion equation of angular motion for the rotating coordinate system (or instantaneous orbit (orbital plane) of the second body). This equation has a compact, symmetrical, and nonsingular structure.

Regular models of orbital motion in which Euler parameters were used were considered by, e.g., A. Deprit (1976) [5], V.A. Brumberg (1980) [6], and by A.F. Bragazin, V.N. Branets, and I.P. Shmyglevsky (1986, 1992) [7, 8]. In this work, regular (in the above sense) quaternion models of celestial mechanics and astrodynamics proposed by the author are considered. These equations have no singular points produced by the use of Euler angles, and they are convenient for solving certain problems of celestial mechanics and astrodynamics.

2. ORIGINAL EQUATIONS OF A PERTURBED SPATIAL TWO BODY PROBLEM AND THE TRAJECTORY (ORBITAL) MOTION OF A SPACECRAFT

The basis of celestial mechanics is formed by the vector Newton differential equation of the perturbed three-dimensional two body problem

$$d^2\mathbf{r}/dt^2 + \mu r^{-3}\mathbf{r} = \mathbf{p}(t, \mathbf{r}, d\mathbf{r}/dt), \quad r = |\mathbf{r}|, \tag{2.1}$$

where $\mathbf{r}$ is the radius vector for the center of mass of the body under study, the origin of which lies at the center of mass of the central body; $\mu = f(m + M)$; and $\mathbf{p}$ is the vector of perturbing acceleration of the second body.

In space flight mechanics, a spacecraft is considered a material point B of variable mass $m = m(t)$. The spacecraft's motion is studied in the inertial coordinate system $OX_1X_2X_3$ (X) with origin O at the center of attraction. Controllable motion of a spacecraft in a Newtonian central field of gravitational forces is described by a vector differential equation similar to Eq. (2.1) [4, 9]:

$$\begin{gathered}d^2\mathbf{r}/dt^2 + \mu r^{-3}\mathbf{r} = \mathbf{p}; \quad r = |\mathbf{r}|, \\ \mathbf{p} = \mathbf{p}_R + \mathbf{p}_F, \quad \mathbf{p}_R = (T/m)\mathbf{e} = \mathbf{T}/m,\end{gathered} \tag{2.2}$$

where $\mathbf{r}$ is the radius vector of the spacecraft's center of mass, the origin of which lies at the center of attraction; $\mu = fM$, M is the attracting body mass; $\mathbf{p}_R$ is the vector of acceleration of the spacecraft's center of mass due to rocket thrust (control), $\mathbf{T} = T\mathbf{e}$ is the rocket thrust vector, T and $\mathbf{e}$ are the thrust magnitude and unit vector, $\mathbf{p}_F$ is the vector of perturbing acceleration of the spacecraft's center of mass from other external forces (including those caused by the noncentrality of the gravitational field and the gravitational forces resulting from a third body).

Second-order vector differential equation (2.1) ((2.2)) is equivalent to two vector differential equations of the first order:

$$d\mathbf{v}/dt + \mu r^{-3}\mathbf{r} = \mathbf{p}, \quad d\mathbf{r}/dt = \mathbf{v}, \tag{2.3}$$

where $\mathbf{v}$ is the vector of absolute velocity for the center of mass of the second body (spacecraft) in coordinate system X.

Let us introduce coordinate system $\eta_1\eta_2\eta_3$ (η) with its origin in the center of mass of the second body (point B). We direct axis η_1 of this coordinate system along radius vector $\mathbf{r}$ of the center of mass of the second body (spacecraft). The angular position of coordinate system η in coordinate system X is specified by the normalized quaternion of turn [10, 11]

$$\begin{gathered}\boldsymbol{\lambda} = \lambda_0 + \lambda_1\mathbf{i}_1 + \lambda_2\mathbf{i}_2 + \lambda_3\mathbf{i}_3, \\ \|\boldsymbol{\lambda}\|^2 = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1,\end{gathered}$$

where $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ are Hamiltonian imaginary units; λ_j ($j = \overline{0, 3}$) are components of the orientation quaternion $\boldsymbol{\lambda}$ (Rodrigues–Hamilton (Euler) parameters [10–12]), identical in bases X and η.

We assume that vectors $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{p}$ are determined by their projections in coordinate system η:

$$\mathbf{r} = r\eta_1, \quad \mathbf{v} = v_1\eta_1 + v_2\eta_2 + v_3\eta_3,$$
$$\mathbf{p} = p_1\eta_1 + p_2\eta_2 + p_3\eta_3.$$

Here, η_i is the unit vector of axis $B\eta_i$; v_i and p_i are the projections of vectors $\mathbf{v}$ and $\mathbf{p}$ onto axis η_i.

Moving from absolute to local derivatives, calculated in coordinate system η in Eqs. (2.3), we get

$$\left(\frac{d\mathbf{v}}{dt}\right)_\eta + \omega \times \mathbf{v} + \mu r^{-3}\mathbf{r} = \mathbf{p}, \quad \left(\frac{d\mathbf{r}}{dt}\right)_\eta + \omega \times \mathbf{r} = \mathbf{v}. \quad (2.4)$$

Here, $\boldsymbol{\omega} = \omega_1\eta_1 + \omega_2\eta_2 + \omega_3\eta_3$ is the vector of absolute angular velocity of the η coordinate system; ω_i is the projection of vector ω onto the coordinate axis η_i; and

$$(d\mathbf{v}/dt)_\eta = \dot{v}_1\eta_1 + \dot{v}_2\eta_2 + \dot{v}_3\eta_3,$$
$$(d\mathbf{r}/dt)_\eta = \dot{r}\eta_1$$

are local derivatives of vectors $\mathbf{v}$ and $\mathbf{r}$.

Let us write Eqs. (2.4) in scalar form:

$$\begin{aligned} \dot{v}_1 + \omega_2 v_3 - \omega_3 v_2 + \mu r^{-2} &= p_1, \\ \dot{v}_2 + \omega_3 v_1 - \omega_1 v_3 &= p_2, \\ \dot{v}_3 + \omega_1 v_2 - \omega_2 v_1 &= p_3; \end{aligned} \quad (2.5)$$

$$\dot{r} = v_1, \quad r\omega_3 = v_2, \quad -r\omega_2 = v_3. \quad (2.6)$$

Inserting relations (2.6) into Eqs. (2.5) and supplementing the resulting equations with kinematic equations for the rotational motion of coordinate system η in Rodrigues–Hamilton parameters λ_j [10–12], we have [13–19]:

$$d^2r/dt^2 - r(\omega_2^2 + \omega_3^3) + \mu r^{-2} = p_1; \quad (2.7)$$

$$\begin{aligned} 2\omega_3 dr/dt + r d\omega_3/dt + r\omega_1\omega_2 &= p_2, \\ 2\omega_2 dr/dt + r d\omega_2/dt - r\omega_1\omega_3 &= -p_3; \end{aligned} \quad (2.8)$$

$$\begin{aligned} 2d\lambda_0/dt &= -\omega_1\lambda_1 - \omega_2\lambda_2 - \omega_3\lambda_3, \\ 2d\lambda_1/dt &= \omega_1\lambda_0 + \omega_3\lambda_2 - \omega_2\lambda_3, \\ 2d\lambda_2/dt &= \omega_2\lambda_0 - \omega_3\lambda_1 + \omega_1\lambda_3, \\ 2d\lambda_3/dt &= \omega_3\lambda_0 + \omega_2\lambda_1 - \omega_1\lambda_2. \end{aligned} \quad (2.9)$$

Equations of motion (2.7)–(2.9) of the second body (spacecraft), written in rotating coordinate system η, have the following variables: distance r from the center of mass of the second body (spacecraft) to the center of attraction; derivative $\dot{r}$ (projection v_1 of velocity vector $\mathbf{v}$ of the center of mass of the second body onto the direction of radius vector $\mathbf{r}$); projections of the angular velocity ω_2 and ω_3 onto the axes of coordinate system η; and Rodrigues–Hamilton parameters λ_j, characterizing the orientation of coordinate system η in the inertial coordinate system X. Projection ω_1 of angular velocity vector ω onto the direction of radius vector $\mathbf{r}$ is an arbitrarily specified parameter. Quantities p_k in these equations are projections of vector $\mathbf{p}$ of perturbing acceleration for the center of mass of the second body onto axes of rotating coordinate system η, or projections of vector sum $\mathbf{p} = \mathbf{p}_R + \mathbf{p}_F$ of the controlling and perturbing accelerations of the spacecraft's center of mass onto axes of coordinate system η.

Cartesian coordinates x_k of the center of mass of the second body (spacecraft) in inertial coordinate system X and projections v_k of vector of absolute velocity $\mathbf{v}$ for the center of mass of the second body (spacecraft) onto the axes of coordinate system η can be found with the above variables using the formulas

$$\begin{aligned} x_1 &= r(\lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^3), \quad x_2 = 2r(\lambda_1\lambda_2 + \lambda_0\lambda_3), \\ x_3 &= 2r(\lambda_1\lambda_3 - \lambda_0\lambda_2); \end{aligned} \quad (2.10)$$

$$v_1 = \dot{r}, \quad v_2 = r\omega_3, \quad v_3 = -r\omega_2. \quad (2.11)$$

Notice that relations (2.10) are those of the reprojection of radius vector $\mathbf{r}$ from coordinate system η into coordinate system X [10–12], while relations (2.11) are the scalar notation of vector relations (2.4).

Projections p_k^+ of vector $\mathbf{p}$ onto the axes of coordinate system X are connected with its projections p_k onto the axes of coordinate system η by the following reprojection relations

$$\begin{aligned} p_1^+ &= (\lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2)p_1 + 2(\lambda_1\lambda_2 - \lambda_0\lambda_3)p_2 \\ &\quad + 2(\lambda_1\lambda_3 + \lambda_0\lambda_2)p_3, \\ p_2^+ &= 2(\lambda_1\lambda_2 + \lambda_0\lambda_3)p_1 + (\lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2)p_2 \\ &\quad + 2(\lambda_2\lambda_3 - \lambda_0\lambda_1)p_3, \\ p_3^+ &= 2(\lambda_1\lambda_3 - \lambda_0\lambda_2)p_1 + 2(\lambda_2\lambda_3 + \lambda_0\lambda_1)p_2 \\ &\quad + (\lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2)p_3; \\ p_1 &= (\lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2)p_1^+ + 2(\lambda_1\lambda_2 + \lambda_0\lambda_3)p_2^+ \\ &\quad + 2(\lambda_1\lambda_3 - \lambda_0\lambda_2)p_3^+, \\ p_2 &= 2(\lambda_1\lambda_2 - \lambda_0\lambda_3)p_1^+ + (\lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2)p_2^+ \\ &\quad + 2(\lambda_2\lambda_3 + \lambda_0\lambda_1)p_3^+, \\ p_3 &= 2(\lambda_1\lambda_3 + \lambda_0\lambda_2)p_1^+ + 2(\lambda_2\lambda_3 - \lambda_0\lambda_1)p_2^+ \\ &\quad + (\lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2)p_3^+. \end{aligned} \quad (2.12)$$

Equations (2.9) and relations (2.10), (2.12) have the following form in quaternion notation:

$$\begin{aligned} 2d\boldsymbol{\lambda}/dt &= \boldsymbol{\lambda} \circ \omega_\eta, \\ \boldsymbol{\lambda} = \lambda_0 + \lambda_1\mathbf{i}_1 + \lambda_2\mathbf{i}_2 + \lambda_3\mathbf{i}_3, &\quad \omega_\eta = \omega_1\mathbf{i}_1 + \omega_2\mathbf{i}_2 + \omega_3\mathbf{i}_3; \end{aligned} \quad (2.13)$$

$$\mathbf{r}_x = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3 = \boldsymbol{\lambda} \circ \mathbf{r}_\eta \circ \bar{\boldsymbol{\lambda}} = r\boldsymbol{\lambda} \circ \mathbf{i}_1 \circ \bar{\boldsymbol{\lambda}},$$
$$\mathbf{p}_x = \overset{+}{p_1}\mathbf{i}_1 + \overset{+}{p_2}\mathbf{i}_2 + \overset{+}{p_3}\mathbf{i}_3 = \boldsymbol{\lambda} \circ \mathbf{p}_\eta \circ \bar{\boldsymbol{\lambda}}, \qquad (2.14)$$
$$\mathbf{p}_\eta = p_1\mathbf{i}_1 + p_2\mathbf{i}_2 + p_3\mathbf{i}_3 = \bar{\boldsymbol{\lambda}} \circ \mathbf{p}_x \circ \boldsymbol{\lambda}.$$

Here and below, the notation of the form $\mathbf{a}_\xi$ signifies the mapping of vector $\mathbf{a}$ onto basis $\xi(\xi = X, \eta)$, determined as the quaternion $\mathbf{a}_\xi = a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3$, where a_k is the projection of vector $\mathbf{a}$ onto axis ξ_k; the quaternion is differentiated under the assumption that unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ are invariable; symbol $\circ$ represents quaternion multiplication; and the overlining denotes quaternion conjugation: $\bar{\boldsymbol{\lambda}} = \lambda_0 - \lambda_1\mathbf{i}_1 - \lambda_2\mathbf{i}_2 - \lambda_3\mathbf{i}_3$.

Projections $\overset{+}{v_k}$ of absolute velocity vector $\mathbf{v}$ onto the axes of inertial coordinate system X are connected with its projections v_k onto axes of rotating coordinate system η by the quaternion equalities

$$\mathbf{v}_x = \overset{+}{v_1}\mathbf{i}_1 + \overset{+}{v_2}\mathbf{i}_2 + \overset{+}{v_3}\mathbf{i}_3 = \boldsymbol{\lambda} \circ \mathbf{v}_\eta \circ \bar{\boldsymbol{\lambda}},$$
$$\mathbf{v}_\eta = v_1\mathbf{i}_1 + v_2\mathbf{i}_2 + v_3\mathbf{i}_3 = \dot{r}\mathbf{i}_1 + r\omega_3\mathbf{i}_2 - r\omega_2\mathbf{i}_3. \qquad (2.15)$$

We can use different equations of motion following from (2.7)–(2.9) and (2.13) to solve problems of celestial mechanics and controlling a spacecraft's motion. Let us consider the most basic of these.

3. EQUATIONS OF A TWO-BODY PROBLEM AND SPACECRAFT TRAJECTORY MOTION CONTROL, WRITTEN IN A NONHOLONOMIC (AZIMUTHALLY FREE) ACCOMPANYING COORDINATE TRIHEDRON

We consider vector $\mathbf{c}$ of the orbital velocity moment of a second body (spacecraft) relative to center of attraction O: $\mathbf{c} = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{r} \times \mathbf{v}$.

Projections c_k of vector $\mathbf{c}$ onto the axes of coordinate system η are determined by the relations

$$c_1 = 0, \quad c_2 = r^2\omega_2, \quad c_3 = r^2\omega_3. \qquad (3.1)$$

We redefine the motion of trihedron η by assuming that arbitrarily specified projection ω_1 of its absolute velocity vector ω onto the direction of radius vector $\mathbf{r}$ (axis η_1) is equal to zero:

$$\omega_1 = 2(-\lambda_1\dot{\lambda}_0 + \lambda_0\dot{\lambda}_1 + \lambda_3\dot{\lambda}_2 - \lambda_2\dot{\lambda}_3) = 0. \qquad (3.2)$$

As follows from (3.1) and (3.2), coordinate system η rotates with the following absolute velocity, collinear to the vector of the velocity moment of the second body (spacecraft): $\omega = r^{-2}c$. This coordinate system is referred to as a nonholonomic (azimuthally free) accompanying coordinate trihedron.

With (3.1) and (3.2), equations of motion (2.7)–(2.9) for the second body (spacecraft) take the form

$$\dot{v}_1 = (c_2^2 + c_3^2)/r^3 - \mu/r^2 + p_1, \quad \dot{r} = v_1; \qquad (3.3)$$

$$c_1 = 0, \quad \dot{c}_2 = -rp_3, \quad \dot{c}_3 = rp_2; \qquad (3.4)$$

$$2\dot{\lambda}_0 = -r^{-2}(c_2\lambda_2 + c_3\lambda_3), \quad 2\dot{\lambda}_1 = r^{-2}(c_3\lambda_2 - c_2\lambda_3),$$
$$2\dot{\lambda}_2 = r^{-2}(c_0\lambda_0 - c_3\lambda_1), \quad 2\dot{\lambda}_3 = r^{-2}(c_3\lambda_0 + c_2\lambda_1). \qquad (3.5)$$

Let us write subsystem (3.5) in quaternion form:

$$2\dot{\lambda} = r^{-2}\lambda \circ \mathbf{c}_\eta,$$
$$\boldsymbol{\lambda} = \lambda_0 + \lambda_1\mathbf{i}_1 + \lambda_2\mathbf{i}_2 + \lambda_3\mathbf{i}_3, \quad \mathbf{c}_\eta = c_2\mathbf{i}_2 + c_3\mathbf{i}_3. \qquad (3.6)$$

Equations (3.3)–(3.5) (or (3.3), (3.4), and (3.6)) are equations of motion for the second body (spacecraft) written in nonholonomic accompanying coordinate trihedron η. The variables in these equations are distance r; derivative $\dot{r}$, projections c_2 and c_3 of the vector of the orbital velocity moment for the second body (spacecraft) onto the axes of coordinate system η; and Rodrigues–Hamilton parameters λ_j, characterizing the orientation of trihedron η in inertial coordinate system X.

Cartesian coordinates and projections of the velocity vector for the center of mass of the second body (spacecraft) onto the axes of coordinate system η can be found with these variables using formulas (2.10) and relations $v_1 = \dot{r}$, $v_2 = r^{-1}c_3$, $v_3 = -r^{-1}c_2$, which in quaternion notation take the form of the second relation in (2.14) and relation

$$\mathbf{v}_\eta = v_1\mathbf{i}_1 + v_2\mathbf{i}_2 + v_3\mathbf{i}_3 = \dot{r}\mathbf{i}_1 + r^{-1}c_3\mathbf{i}_2 - r^{-1}c_2\mathbf{i}_3.$$

Projections $\overset{+}{v_k}$ of the velocity vector for the center of mass of the second body (spacecraft) onto the axes of inertial coordinate system X are determined by quaternion relation (2.15).

Quantities p_k in Eqs. (3.3)–(3.5) are (for two body problem) the projections of vector $\mathbf{p}$ of perturbing acceleration of the center of mass of the second body onto axes of nonholonomic coordinate system η or (for spacecraft) the projections of vector sum $\mathbf{p} = \mathbf{p}_R + \mathbf{p}_F$ of controlling and perturbing accelerations of the spacecraft center of mass onto axes of the same coordinate system η. They are connected with projections $\overset{+}{p_k}$ of this vector onto the axes of inertial coordinate system by relations (2.12) (or by third quaternion relation (2.14)).

Note that Eqs. (3.3)–(3.5) were derived in [13, 14]. Other variants of the derivation and some examples of using these equations were presented in [15–19]. As was already noted in the first part of this work, we can easily derive from Eqs. (3.3)–(3.5) quaternion regular equations for a three-dimensional two-body problem in Kustaanheimo–Stiefel variables.

4. EQUATIONS OF A TWO-BODY PROBLEM AND SPACECRAFT TRAJECTORY MOTION WRITTEN IN AN ORBITAL COORDINATE TRIHEDRON

Assuming as before that axis η_1 of coordinate system η is directed along radius vector $\mathbf{r}$, let us direct axis η_3 of this system along vector $\mathbf{c} = \mathbf{r} \times \mathbf{v}$ of the moment of velocity for the central mass of the second body (spacecraft).

Projections c_i of vector $\mathbf{c}$ onto the system's axes then have the form

$$c_1 = c_2 = 0, \quad c_3 = |\mathbf{c}| = c.$$

Projecting vector equality $\mathbf{c} = \mathbf{r} \times \mathbf{v}$ onto the axes of orbital coordinate system η, we get

$$c_1 = 0, \quad c_2 = -rv_3 = 0, \quad c_3 = c = rv_2.$$

In the case under consideration, projections of absolute velocity vector $\mathbf{v}$ for the center of mass of the second body (spacecraft) onto the axes of orbital coordinate system η are thus determined by relations

$$v_1 = \dot{r}, \quad v_2 = c/r, \quad v_3 = 0. \tag{4.1}$$

From (2.11) and (4.1), we find

$$\omega_2 = 0, \quad \omega_3 = c/r^2. \tag{4.2}$$

Inserting equalities (4.2) into Eqs. (2.7), (2.8), we get

$$\ddot{r} - c^2 r^{-3} + \mu r^{-2} = p_1; \tag{4.3}$$

$$\dot{c} = rp_2, \quad r\omega_1\omega_3 = p_3. \tag{4.4}$$

From the second relations of (4.2) and (4.4), we have $\omega_1 = (r/c)p_3$.

Thus, projections of the absolute angular velocity vector of orbital coordinate system η onto its own coordinate axes take the form

$$\omega_1 = (r/c)p_3, \quad \omega_2 = 0, \quad \omega_3 = c/r^2, \quad c = |\mathbf{r} \times \mathbf{v}|. \tag{4.5}$$

With (4.5), equations of motion for the second body (spacecraft) (4.3), (4.4), and (2.9) (or (2.13)) have the form

$$\dot{v}_1 = r^{-3}c^2 - \mu r^{-2} + p_1, \quad \dot{r} = v_1; \tag{4.6}$$

$$\dot{c} = rp_2; \tag{4.7}$$

$$\begin{gathered} 2\dot{\lambda}_0 = -\omega_1\lambda_1 - \omega_3\lambda_3, \quad 2\dot{\lambda}_1 = \omega_1\lambda_0 + \omega_3\lambda_2, \\ 2\dot{\lambda}_2 = -\omega_3\lambda_1 + \omega_1\lambda_3, \quad 2\dot{\lambda}_3 = \omega_3\lambda_0 - \omega_1\lambda_2, \\ \omega_1 = (r/c)p_3, \quad \omega_3 = c/r^2. \end{gathered} \tag{4.8}$$

Let us write subsystem (4.8) in quaternion form:

$$\begin{gathered} 2\dot{\lambda} = \lambda \circ \omega_\eta, \\ \boldsymbol{\lambda} = \lambda_0 + \lambda_1\mathbf{i}_1 + \lambda_2\mathbf{i}_2 + \lambda_3\mathbf{i}_3, \\ \omega_\eta = \omega_1\mathbf{i}_1 + \omega_3\mathbf{i}_3 = rc^{-1}p_3\mathbf{i}_1 + r^{-2}c\mathbf{i}_3. \end{gathered} \tag{4.9}$$

Equations (4.6)–(4.8) (or (4.6), (4.7), and (4.9)) are equations of motion for the second body (spacecraft) written in orbital trihedron η. The variables in these equations are distance r; derivative $\dot{r}$, magnitude of the vector of orbital velocity moment of the second body (spacecraft) $c = |\mathbf{r} \times \mathbf{v}|$; and Rodrigues–Hamilton parameters λ_j, characterizing the orientation of orbital trihedron η in inertial coordinate system X. Quantities p_k in these equations are (for a two-body problem) projections of vector $\mathbf{p}$ of perturbing accelerations of the second body's center of mass onto the axes of orbital coordinate system η or (for spacecraft) projections of vector sum $\mathbf{p} = \mathbf{p}_R + \mathbf{p}_F$ of controlling and perturbing accelerations of the spacecraft's center of mass onto the axes of coordinate system η.

The Cartesian coordinates (in an inertial system) and projections of the velocity vector for the center of mass of the second body (spacecraft) onto the axes of the orbital coordinate system can be found with the above variables using formulas (2.10) and the relations $v_1 = \dot{r}, \quad v_2 = c/r, \quad v_3 = 0$. Projections v_k^+ of the velocity vector for the center of mass of the second body (spacecraft) onto the axes of inertial coordinate system X are determined by the quaternion relation

$$\begin{aligned} \mathbf{v}_x &= v_1^+\mathbf{i}_1 + v_2^+\mathbf{i}_2 + v_3^+\mathbf{i}_3 \\ &= \boldsymbol{\lambda} \circ \mathbf{v}_\eta \circ \bar{\boldsymbol{\lambda}} = \boldsymbol{\lambda} \circ (\dot{r}\mathbf{i}_1 + r^{-1}c\mathbf{i}_2) \circ \bar{\boldsymbol{\lambda}}, \end{aligned} \tag{4.10}$$

while connections of projections p_k^+ of vector $\mathbf{p}$ onto the axes of the inertial coordinate system with their projections p_k onto the axes of the orbital coordinate system are determined by relations of reprojection (2.12) (or by the second quaternion relation of (2.14)).

Note that projection p_3 of the acceleration vector onto the direction orthogonal to the plane of instantaneous orbit of the second body (spacecraft) appears in the equations of motion of accompanying (orbital) trihedron η (4.8) (in quaternion equation (4.9)).

Components λ_j of quaternion $\boldsymbol{\lambda}$ of the orbital coordinate system's orientation are connected with orbital elements of the second body (spacecraft) by the relations

$$\begin{aligned} \lambda_0 &= \cos(I/2)\cos((\Omega_u + \Sigma)/2), \\ \lambda_1 &= \sin(I/2)\cos((\Omega_u - \Sigma)/2), \\ \lambda_2 &= \sin(I/2)\sin((\Omega_u - \Sigma)/2), \\ \lambda_3 &= \cos(I/2)\sin((\Omega_u + \Sigma)/2). \end{aligned} \tag{4.11}$$

Quaternion $\boldsymbol{\lambda}$ of the orbital coordinate system's orientation characterizes both the orientation of the instantaneous orbit of the second body (spacecraft) in the inertial system and the position of the second body (spacecraft) on this orbit.

Notice that equations (4.6)–(4.8) were derived in [14]. Different variants of the derivation and use of these equations were presented in [16, 18, 19]. A quaternion equation for an orbital coordinate system orientation similar to (4.9) was also obtained and used to describe orbital spacecraft motion by A.F. Bragazin, V.N. Branets, and I.P. Shmyglevsky [7, 8]. Equations of perturbed Keplerian motion, written in an orbital coordinate system and consisting of equations for distance r, magnitude of orbital velocity moment c, and equations for angular variables describing the motion of an orbital system, were obtained by H. Andoyer [5].

5. EQUATIONS OF A TWO-BODY PROBLEM AND SPACECRAFT TRAJECTORY MOTION WRITTEN IN ORBITAL AND IDEAL COORDINATE SYSTEMS WITH USE OF THE FIRST QUATERNION OSCULATING ELEMENT

Let us introduce angular variables φ and ω_π, connected by relation $\Sigma = \omega_\pi + \varphi$ and determined by differential equations $\dot{\varphi} = c/r^2$, $\dot{\omega}_\pi = -(r/c)p_3\sin\Sigma\cot I$ and by the initial conditions

$$\varphi(0) = \varphi_{\mathrm{tr}}(0) = \varphi^0_{\mathrm{tr}}, \quad \omega_\pi(0) = \omega_{\pi\mathrm{tr}}(0) = \omega^0_{\pi\mathrm{tr}},$$

where φ^0_{tr} and $\omega^0_{\pi\mathrm{tr}}$ are the initial values of a true anomaly and the angular distance of a pericenter from a node.

Variable φ coincides with the true anomaly, and variable ω_π coincides with the angular distance of the pericenter from the node when there is unperturbed (uncontrollable) motion, when $\mathbf{p} = 0$, and when $p_1 = p_2 = 0$. When $p_3 = 0$, variable $\omega_\pi = \omega^0_{\pi\mathrm{tr}} = \mathrm{const}$, and variable $\varphi = \Sigma - \omega^0_{\pi\mathrm{tr}}$.

Let us introduce the quaternion of turn $\boldsymbol{\Lambda}$, defined by the relations

$$\begin{aligned}\boldsymbol{\Lambda} &= \lambda \circ [\cos(\varphi/2) - \mathbf{i}_3\sin(\varphi/2)]\\ &= [\cos(\Omega_u/2) + \mathbf{i}_3\sin(\Omega_u/2)] \circ\\ &\circ [\cos(I/2) + \mathbf{i}_1\sin(I/2)] \circ [\cos(\omega_\pi/2)\\ &+ \mathbf{i}_3\sin(\omega_\pi/2)].\end{aligned} \tag{5.1}$$

Connections between components of quaternions $\boldsymbol{\Lambda}$ and $\boldsymbol{\lambda}$ are determined by the relations

$$\begin{aligned}\Lambda_0 &= \lambda_0\varphi_0 + \lambda_3\varphi_3, \quad \Lambda_1 = \lambda_1\varphi_0 - \lambda_2\varphi_3,\\ \Lambda_2 &= \lambda_2\varphi_0 + \lambda_1\varphi_3, \quad \Lambda_3 = -\lambda_0\varphi_3 + \lambda_3\varphi_0;\\ \lambda_0 &= \Lambda_0\varphi_0 - \Lambda_3\varphi_3, \quad \lambda_1 = \Lambda_1\varphi_0 + \Lambda_2\varphi_3,\\ \lambda_2 &= \Lambda_2\varphi_0 - \Lambda_1\varphi_3, \quad \lambda_3 = \Lambda_0\varphi_3 + \Lambda_3\varphi_0;\\ \varphi_0 &= \cos(\varphi/2), \quad \varphi_3 = \sin(\varphi/2).\end{aligned} \tag{5.2}$$

Components Λ_j of quaternion $\boldsymbol{\Lambda}$ are expressed through angular orbital elements Ω_u, I, and variable ω_π, using the following relations similar to (4.10):

$$\begin{aligned}\Lambda_0 &= \cos(I/2)\cos((\Omega_u + \omega_\pi)/2),\\ \Lambda_1 &= \sin(I/2)\cos((\Omega_n - \omega_\pi)/2),\\ \Lambda_2 &= \sin(I/2)\sin((\Omega_u - \omega_\pi)/2),\\ \Lambda_3 &= \cos(I/2)\sin((\Omega_u + \omega_\pi)/2).\end{aligned} \tag{5.3}$$

Quaternion $\boldsymbol{\Lambda}$ characterizes orientation in the inertial coordinate system of a new rotating coordinate system ξ whose origin is located at point Π on the instantaneous orbit of a second body (spacecraft). Point Π coincides with the instantaneous pericenter of an orbit for an uncontrollable (unperturbed) motion, and when $p_1 = p_2 = 0$. When $p_3 = 0$, point Π coincides with the initial position of the orbit's instantaneous pericenter over the entire interval of motion. In the general case of controllable (perturbed) motion, when all acceleration components p_1, p_2, and p_3 are nonzero, this point does not coincide with the instantaneous pericenter of the orbit. We therefore refer to point Π as the generalized pericenter of the instantaneous orbit of the second body (spacecraft).

Axis ξ_1 of coordinate system ξ is directed along the radius vector of the generalized pericenter, while axis ξ_3 is parallel to vector $\mathbf{c}$ of the velocity moment of the center of mass of the second body (spacecraft) (axis η_3). The angular position of coordinate system η can be obtained (the second body's center of mass being superimposed on point Π) from coordinate system ξ by its rotation around axis ξ_3 at angle φ, referred to below as a generalized true anomaly of the second body (spacecraft). Note that the angular position of coordinate system ξ in the inertial coordinate system can also be specified by angles Ω_u, I, and ω_π.

Vector $\boldsymbol{\Omega}$ of absolute angular velocity of coordinate system ξ is parallel to radius vector $\mathbf{r}$ of the center of mass of the second body (spacecraft). It is defined by the formula

$$\boldsymbol{\Omega} = \omega_1\boldsymbol{\eta}_1 = (p_3/c)\mathbf{r},$$

where $\boldsymbol{\eta}_1$ is the unit vector of coordinate axis η_1 (we assume that vector Ω is applied at generalized pericenter Π).

Projections Ω_i of vector $\mathbf{\Omega}$ onto the axes of coordinate system ξ take the form

$$\Omega_1 = (r/c)p_3\cos\varphi = (\Xi_1/c)p_3,$$
$$\Omega_2 = (r/c)p_3\sin\varphi = (\Xi_2/c)p_3,$$
$$\Omega_3 = 0,$$

where Ξ_i is the projection of radius vector $\mathbf{r}$ onto axis ξ_i.

The introduced coordinate system ξ is an ideal coordinate system if the projection of its angular velocity vector $\Omega_3 = 0$ [5].

By switching to new variables Λ_j and $\mathbf{\Lambda}$ in Eqs. (4.8) and (4.9) according to formulas (5.2) and (5.1), and considering relation $\dot{\varphi} = c/r^2$, we obtain the equations [14, 19–21]

$$\dot{v}_1 = c^2/r^3 - fM/r^2 + p_1, \quad \dot{r} = v_1, \quad \dot{c} = rp_2; \quad (5.4)$$

$$\dot{\varphi} = c/r^2; \quad (5.5)$$

$$\begin{gathered} 2\dot{\Lambda}_0 = -\Omega_1\Lambda_1 - \Omega_2\Lambda_2, \quad 2\dot{\Lambda}_1 = \Omega_1\Lambda_0 - \Omega_2\Lambda_3, \\ 2\dot{\Lambda}_2 = \Omega_2\Lambda_0 + \Omega_1\Lambda_3, \quad 2\dot{\Lambda}_3 = \Omega_2\Lambda_1 - \Omega_1\Lambda_2, \end{gathered} \quad (5.6)$$

$$\Omega_1 = (r/c)p_3\cos\varphi, \quad \Omega_2 = (r/c)p_3\sin\varphi. \quad (5.7)$$

$$\begin{aligned} 2\dot{\mathbf{\Lambda}} &= \mathbf{\Lambda} \circ \mathbf{\Omega}_\xi, \quad \mathbf{\Omega}_\xi = \Omega_1\mathbf{i}_1 + \Omega_2\mathbf{i}_2 \\ &= (r/c)p_3(\cos\varphi\mathbf{i}_1 + \sin\varphi\mathbf{i}_2). \end{aligned} \quad (5.8)$$

Equation (5.8) is the quaternion notation for a scalar system of differential equations (5.6). Quaternion variable $\mathbf{\Lambda}$ in this equation denotes the quaternion osculating orbital element of a second body (spacecraft): when component p_3 of perturbing acceleration vector (or the vector sum of controlling and perturbing accelerations) $\mathbf{p}$, perpendicular to the plane passing through vectors $\mathbf{r}$ and $\mathbf{v}$ (i.e., the plane of instantaneous orbit of the second body (spacecraft)), is equal to zero, then quaternion $\mathbf{\Lambda} = \text{const}$. We refer to quaternion variable $\mathbf{\Lambda}$ as the first quaternion osculating element of the second body's (spacecraft's) orbit.

Equations (5.4)–(5.7) or (5.4), (5.5), and (5.8) are equations of motion for the second body (spacecraft), written in two rotating coordinate systems: orbital coordinate system η and ideal coordinate system ξ. Equations (5.4) and (5.5) are written in the orbital coordinate system, while (5.6), (5.7), and (5.8) are written in the ideal coordinate system. The variables in these equations are distance r; derivative $\dot{r}$, magnitude of the vector of orbital velocity moment of the second body (spacecraft) $c = |\mathbf{r} \times \mathbf{v}|$; generalized true anomaly φ; and Rodrigues–Hamilton (Euler) parameters Λ_j, characterizing the orientation of ideal coordinate system ξ in the inertial coordinate system X. Quantities p_k in these equations are (for two body problem) projections of vector $\mathbf{p}$ of perturbing acceleration of the center of mass of the second body onto axes of orbital coordinate system η or (for spacecraft) projections of vector sum $\mathbf{p} = \mathbf{p}_R + \mathbf{p}_F$ of controlling and perturbing accelerations of the spacecraft's center of mass onto the axes of the coordinate system η.

The Cartesian coordinates (in the inertial coordinate system) and projections of the velocity vector of the center of mass of the second body (spacecraft) onto the axes of the orbital coordinate system can be found with the above variables using formulas (2.10) and the relations $v_1 = \dot{r}$, $v_2 = c/r$, $v_3 = 0$, where parameters λ_j are predetermined with variables Λ_j using formulas (5.2).

Projections v_k^+ of the velocity vector of the center of mass of the second body (spacecraft) onto the axes of inertial coordinate system X are determined by quaternion relation (4.10), while connections of projections p_k^+ of vector $\mathbf{p}$ onto the axes of the inertial coordinate system with its projections p_k onto the axes of the orbital coordinate system are determined by the relations of reprojection (2.12) (or by quaternion relations (2.14)). Quaternion $\boldsymbol{\lambda}$ in these relations must be preliminarily determined with quaternion $\mathbf{\Lambda}$ using the formula

$$\boldsymbol{\lambda} = \mathbf{\Lambda} \circ [\cos(\varphi/2) + \mathbf{i}_3\sin(\varphi/2)].$$

It should be noted that H. Andoyer (1923) and P. Musen (1959) derived equations of perturbed Keplerian motion written in the ideal coordinate system and consisting of equations for polar coordinates and equations for angular variables describing the orientation of the ideal coordinate system [5].

Let us switch to a new independent variable in Eqs. (5.4)–(5.7): polar angle φ. We obtain the system of equations [14, 19–21]

$$\rho'' + \rho = (1/c^2)[\mu - (1/2)(c^2)'\rho' - (1/\rho^2)p_1], \quad (5.9)$$

$$(c^2)' = (2/\rho^3)p_2, \quad (5.10)$$

$$2\mathbf{\Lambda}' = (p_3/(\rho^3c^2))\mathbf{\Lambda} \circ (\cos\varphi\mathbf{i}_1 + \sin\varphi\mathbf{i}_2), \quad (5.11)$$

$$t' = 1/(\rho^2c), \quad (5.12)$$

where a new variable $\rho = 1/r$ is introduced instead of r, and the notation prime designates derivatives with respect to variable φ.

In system (5.9)–(5.12), we can use the following equation [20] instead of quaternion equation (5.11):

$$2\lambda' = \lambda \circ [(p_3/(\rho^3c^2))\mathbf{i}_1 + \mathbf{i}_3].$$

For unperturbed Keplerian motion we obtain from (5.9)–(5.12) the equations

$$\rho'' + \rho = \mu/c^2, \quad c = \text{const},$$
$$\mathbf{\Lambda} = \text{const}, \quad t' = 1/(\rho^2c).$$

It is therefore convenient to use Eqs. (5.9)–(5.12) when constructing a complete system of equations for the orbital motion of a second body (spacecraft) in osculating elements that include a differential equation for quaternion osculating element $\mathbf{\Lambda}$.

6. EQUATIONS OF A TWO-BODY PROBLEM AND SPACECRAFT TRAJECTORY MOTION, WRITTEN IN THE IDEAL COORDINATE SYSTEM USING THE FIRST QUATERNION OSCULATING ELEMENT AND HANSEN'S IDEAL RECTANGULAR COORDINATES

Let us introduce Hansen's ideal rectangular coordinates Ξ_i representing projections of radius vector $\mathbf{r}$ of the center of mass of a second body (spacecraft) onto the axes of ideal coordinate system ξ. They are associated with variables r and φ by the relations

$$\Xi_1 = r\cos\varphi, \quad \Xi_2 = r\cos\varphi, \quad \Xi_3 = 0. \tag{6.1}$$

Differentiating relations (6.1) twice with respect to time and using Eqs. (5.4) and (5.5), we get

$$\ddot{\Xi}_1 + \mu r^{-3}\Xi_1 = p_{\xi 1}, \quad \ddot{\Xi}_2 + \mu r^{-3}\Xi_2 = p_{\xi 2}, \quad r^2 = \Xi_1^2 + \Xi_2^2; \tag{6.2}$$

$$\begin{aligned} 2\dot{\Lambda}_0 &= -\Omega_1\Lambda_1 - \Omega_2\Lambda_2, \quad 2\dot{\Lambda}_1 = \Omega_1\Lambda_0 - \Omega_2\Lambda_3, \\ 2\dot{\Lambda}_2 &= \Omega_2\Lambda_0 + \Omega_1\Lambda_3, \quad 2\dot{\Lambda}_3 = \Omega_2\Lambda_1 - \Omega_1\Lambda_2, \end{aligned} \tag{6.3}$$

$$\Omega_1 = (\Xi_1/c)p_3, \quad \Omega_2 = (\Xi_2/c)p_3, \quad c = \Xi_1\dot{\Xi}_2 - \Xi_2\dot{\Xi}_1, \quad p_3 = p_{\xi 3}. \tag{6.4}$$

Cartesian coordinates x_k of the center of mass of the second body (spacecraft) in inertial coordinate system X and projections v_k^+ of absolute velocity vector $\mathbf{v}$ of the center of mass of the second body (spacecraft) onto the axes of the inertial coordinate system can be found with Hansen's ideal coordinates Ξ_i and their derivatives $\dot{\Xi}_i$ using the formulas

$$\begin{aligned} x_1 &= (\Lambda_0^2 + \Lambda_1^2 - \Lambda_2^2 - \Lambda_3^2)\Xi_1 + 2(\Lambda_1\Lambda_2 - \Lambda_0\Lambda_3)\Xi_2, \\ x_2 &= 2(\Lambda_1\Lambda_2 + \Lambda_0\Lambda_3)\Xi_1 + (\Lambda_0^2 - \Lambda_1^2 + \Lambda_2^2 - \Lambda_3^2)\Xi_2, \\ x_3 &= 2(\Lambda_1\Lambda_3 - \Lambda_0\Lambda_2)\Xi_1 + 2(\Lambda_2\Lambda_3 + \Lambda_0\Lambda_1)\Xi_2; \end{aligned} \tag{6.5}$$

$$\begin{aligned} v_1^+ &= \dot{x}_1 = (\Lambda_0^2 + \Lambda_1^2 - \Lambda_2^2 - \Lambda_3^3)\dot{\Xi}_1 + 2(\Lambda_1\Lambda_2 - \Lambda_0\Lambda_3)\dot{\Xi}_2; \\ v_2^+ &= \dot{x}_2 = 2(\Lambda_1\Lambda_2 + \Lambda_0\Lambda_3)\dot{\Xi}_1 + (\Lambda_0^2 - \Lambda_1^2 + \Lambda_2^2 - \Lambda_3^2)\dot{\Xi}_2, \\ v_3^+ &= \dot{x}_3 = 2(\Lambda_1\Lambda_3 - \Lambda_0\Lambda_2)\dot{\Xi}_1 + 2(\Lambda_2\Lambda_3 + \Lambda_0\Lambda_1)\dot{\Xi}_2. \end{aligned} \tag{6.6}$$

Projections $p_{\xi k}$ of vector $\mathbf{p}$ onto the axes of ideal coordinate system ξ are connected with its projections p_k^+ onto axes of inertial coordinate system X by the relations of reprojection

$$\begin{aligned} p_{\xi 1} &= (\Lambda_0^2 + \Lambda_1^2 - \Lambda_2^2 - \Lambda_3^2)p_1^+ + 2(\Lambda_1\Lambda_2 + \Lambda_0\Lambda_3)p_2^+ + 2(\Lambda_1\Lambda_3 - \Lambda_0\Lambda_2)p_3^+, \\ p_{\xi 2} &= 2(\Lambda_1\Lambda_2 - \Lambda_0\Lambda_3)p_1^+ + (\Lambda_0^2 - \Lambda_1^2 + \Lambda_2^2 - \Lambda_3^2)p_2^+ + 2(\Lambda_2\Lambda_3 + \Lambda_0\Lambda_1)p_3^+, \\ p_{\xi 3} &= 2(\Lambda_1\Lambda_3 + \Lambda_0\Lambda_2)p_1^+ + 2(\Lambda_2\Lambda_3 - \Lambda_0\Lambda_1)p_2^+ + (\Lambda_0^2 - \Lambda_1^2 - \Lambda_2^2 + \Lambda_3^2)p_3^+. \end{aligned} \tag{6.7}$$

Equations (6.3) and relations (6.5)–(6.7) in quaternion notation take the forms

$$\begin{aligned} 2\dot{\mathbf{\Lambda}} &= \mathbf{\Lambda} \circ \mathbf{\Omega}_\xi, \\ \mathbf{\Lambda} &= \Lambda_0 + \Lambda_1\mathbf{i}_1 + \Lambda_2\mathbf{i}_2 + \Lambda_3\mathbf{i}_3, \\ \mathbf{\Omega}_\xi &= \Omega_1\mathbf{i}_1 + \Omega_2\mathbf{i}_2 = p_{\xi 3}(\Xi_1\dot{\Xi}_2 - \Xi_2\dot{\Xi}_1)^{-1} \times (\Xi_1\mathbf{i}_2 + \Xi_2\mathbf{i}_2). \end{aligned} \tag{6.8}$$

$$\mathbf{r}_x = \mathbf{\Lambda} \circ \mathbf{r}_\xi \circ \overline{\mathbf{\Lambda}}, \quad \mathbf{r}_x = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3, \quad \mathbf{r}_\xi = \Xi_1\mathbf{i}_1 + \Xi_2\mathbf{i}_2; \tag{6.9}$$

$$\mathbf{v}_x = \mathbf{\Lambda} \circ \mathbf{v}_\xi \circ \overline{\mathbf{\Lambda}}, \quad \mathbf{v}_x = v_1^+\mathbf{i}_1 + v_2^+\mathbf{i}_2 + v_3^+\mathbf{i}_3, \quad \mathbf{v}_\xi = \dot{\Xi}_1\mathbf{i}_1 + \dot{\Xi}_2\mathbf{i}_2; \tag{6.10}$$

$$\mathbf{p}_\xi = \overline{\mathbf{\Lambda}} \circ \mathbf{p}_x \circ \Lambda, \quad \mathbf{p}_\xi = p_{\xi 1}\mathbf{i}_1 + p_{\xi 2}\mathbf{i}_2 + p_{\xi 3}\mathbf{i}_3, \quad \mathbf{p}_x = p_1^+\mathbf{i}_1 + p_2^+\mathbf{i}_2 + p_3^+\mathbf{i}_3. \tag{6.11}$$

Equations (6.2)–(6.4) and (6.2), (6.8) are equations of motion of the second body (spacecraft) written in ideal coordinate system ξ. The variables in these equations are Hansen's ideal rectangular coordinates Ξ_i; their first time derivatives $\dot{\Xi}_i$; and Rodrigues–Hamilton (Euler) parameters Λ_j, characterizing the orientation of ideal coordinate system ξ in inertial coordinate system X. Quantities $p_{\xi k}$ in these equations are (for a two-body problem) projections of vector $\mathbf{p}$ of perturbing acceleration of the second body`s center of mass onto the axes of ideal coordinate system $\xi$ or (for spacecraft) projections of vector sum $\mathbf{p} = \mathbf{p}_R + \mathbf{p}_F$ of the controlling and perturbing accelerations of the spacecraft`s center of mass onto axes of the same coordinate system ξ.

Note that scalar equations of perturbed Keplerian motion in Hansen coordinates and Euler parameters, written in the ideal coordinate system, were derived in other forms and by other methods by A. Deprit (1976) [5] and V.A. Brumberg (1980) [6].

7. REGULAR EQUATIONS OF A THREE-DIMENSIONAL THREE BODY PROBLEM AND SPACECRAFT TRAJECTORY MOTION, CONSTRUCTED USING HANSEN'S IDEAL RECTANGULAR COORDINATES

Let us introduce rotating coordinate system $O\xi$, the origin of which lies at the center of attraction (the center of mass of the first body), with coordinate axes parallel to those of our ideal coordinate system ξ. Orientation of the orbital coordinate system η in coordinate system $O\xi$ is characterized by a quaternion of turn Φ with the form

$$\Phi = \cos(\varphi/2) + \sin(\varphi/2)\mathbf{i}_3. \tag{7.1}$$

Components Φ_j of this quaternion are determined by the relations

$$\begin{gathered}\Phi_0 = \cos(\varphi/2), \quad \Phi_1 = \Phi_2 = 0, \\ \Phi_3 = \sin(\varphi/2).\end{gathered} \tag{7.2}$$

Hansen's ideal rectangular coordinates Ξ_i defined by (6.1) are Cartesian coordinates of the center of mass of the second body (spacecraft) in coordinate system $O\xi$, and they are connected with variables r and Φ_j by the relations

$$\begin{gathered}\Xi_1 = r\cos\varphi = r(\Phi_0^2 - \Phi_3^2), \\ \Xi_2 = r\sin\varphi = 2r\Phi_0\Phi_3, \quad \Xi_3 = 0.\end{gathered} \tag{7.3}$$

Let us introduce the Levi–Civita variables

$$U_0 = r^{1/2}\Phi_0, \quad U_3 = -r^{1/2}\Phi_3, \tag{7.4}$$

associated with Hansen's coordinates by the relations

$$\Xi_1 = U_0^2 - U_3^2, \quad \Xi_2 = -2U_0U_3. \tag{7.5}$$

Projections of the velocity vector of the center of mass of the second body (spacecraft) onto the axes of the ideal coordinate system are associated with the time derivatives of the Levi–Civita variables by the relations

$$\begin{aligned}v_{\xi 1} &= v_1\cos\varphi - v_2\sin\varphi = \dot{\Xi}_1 = 2(U_0\dot{U}_0 - U_3\dot{U}_3), \\ v_{\xi 2} &= v_1\sin\varphi + v_2\cos\varphi = \dot{\Xi}_2 \\ &= -2(U_3\dot{U}_0 + U_0\dot{U}_3), \\ v_{\xi 3} &= v_3 = \dot{\Xi}_3 = 0.\end{aligned} \tag{7.6}$$

Formulas for Keplerian energy h and magnitude c of the vector of an orbital velocity moment of the second body (spacecraft), determined by the expressions

$$\begin{aligned}h &= (1/2)v^2 - \mu r^{-1} = (1/2)(\dot{\Xi}_1^2 + \dot{\Xi}_2^2) \\ &\quad - \mu(\Xi_1^2 + \Xi_2^2)^{-1/2}, \\ c &= |\mathbf{r}\times\mathbf{v}| = \Xi_1\dot{\Xi}_2 - \Xi_2\dot{\Xi}_1,\end{aligned} \tag{7.7}$$

in our new Levi–Civita variables take the form

$$\begin{gathered}h = 2r(\dot{U}_0^2 + \dot{U}_3^2) - \mu r^{-1}, \quad r = U_0^2 + U_3^2, \\ c = 2(U_0^2 + U_3^2)(U_3\dot{U}_0 - U_0\dot{U}_3).\end{gathered} \tag{7.8}$$

Let us now switch to Levi–Civita variables U_i using formulas (7.5) and (7.6) in the equations of motion for the second body (spacecraft) (6.2)–(6.4), and to a new independent variable τ in accordance with differential relation $dt = rd\tau$. As an additional variable, we also introduce Keplerian energy h, defined by relations (7.7) and (7.8) and satisfying differential equation $dh/dt = \mathbf{p}\cdot\mathbf{v}$. As a result, we obtain the following regular equations of motion for the second body (spacecraft) (regular equations of a three-dimensional two-body problem):

$$\begin{aligned}d^2U_0/d\tau^2 - (h/2)U_0 &= (r/2)Q_0, \\ d^2U_3/d\tau^2 - (h/2)U_3 &= (r/2)Q_3;\end{aligned} \tag{7.9}$$

$$dh/d\tau = 2(Q_0(dU_0/d\tau) + Q_3(dU_3/d\tau)); \tag{7.10}$$

$$\begin{aligned}2d\Lambda_0/d\tau &= -r(\Omega_1\Lambda_1 + \Omega_2\Lambda_2), \\ 2d\Lambda_1/d\tau &= r(\Omega_1\Lambda_0 - \Omega_2\Lambda_3), \\ 2d\Lambda_2/d\tau &= r(\Omega_2\Lambda_0 + \Omega_1\Lambda_3), \\ 2d\Lambda_3/d\tau &= r(\Omega_2\Lambda_1 - \Omega_1\Lambda_2);\end{aligned} \tag{7.11}$$

$$dt/d\tau = r; \tag{7.12}$$

$$r = |\mathbf{r}| = U_0^2 + U_3^2, \tag{7.13}$$

$$c = 2(U_3(dU_0/d\tau) - U_0(dU_3/d\tau)), \tag{7.14}$$

$$\begin{gathered}\Omega_1 = c^{-1}(U_0^2 - U_3^2)p_3, \\ \Omega_2 = -2c^{-1}U_0U_3p_3, \quad p_3 = p_{\xi 3},\end{gathered} \tag{7.15}$$

$$Q_0 = U_0p_{\xi 1} - U_3p_{\xi 2}, \quad Q_3 = -U_3p_{\xi 1} - U_0p_{\xi 2}. \tag{7.16}$$

These equations must be supplemented with relations of reprojection (6.5)–(6.7).

Note that regular equations (7.9)–(7.16) include as a subsystem Eqs. (7.9), (7.10), (7.12), (7.13), and (7.16) in the form of regular Levi–Civita equations for a planar two-body problem. Note also that the above subsystem of equations can be formally derived if we set

$$\begin{gathered}u_0 = U_0, \quad u_1 = u_2 = 0, \quad u_3 = U_3; \\ q_0 = Q_0, \quad q_1 = q_2 = 0, \quad q_3 = Q_3,\end{gathered}$$

in the regular Kustaanheimo–Stiefel equations (1.4)–(1.6) presented in [2].

Let us write regular equations (7.9)–(7.16) in quaternion form:

$$d^2\mathbf{U}/d\tau^2 - (h/2)\mathbf{U} = (r/2)\mathbf{Q}, \tag{7.17}$$

$$dh/d\tau = 2\mathrm{scal}((d\bar{\mathbf{U}}/d\tau)\circ\mathbf{Q}), \tag{7.18}$$

$$2d\mathbf{\Lambda}/d\tau = r\mathbf{\Lambda}\circ\mathbf{\Omega}_\xi = (r/c)p_{\xi 3}\mathbf{\Lambda}\circ \circ[(U_0^2-U_3^2)\mathbf{i}_1 - 2U_0U_3\mathbf{i}_2], \tag{7.19}$$

$$dt/d\tau = r; \tag{7.20}$$

$$\mathbf{U} = U_0 + U_3\mathbf{i}_3,\quad \mathbf{\Lambda} = \Lambda_0 + \Lambda_1\mathbf{i}_1 + \Lambda_2\mathbf{i}_2 + \Lambda_3\mathbf{i}_3,$$
$$r = \|\mathbf{U}\|^2 = \mathbf{U}\circ\bar{\mathbf{U}} = \bar{\mathbf{U}}\circ\mathbf{U} = U_0^2 + U_3^2,$$
$$c = 2(U_3(dU_0/d\tau) - U_0(dU_3/d\tau)), \tag{7.21}$$
$$\mathbf{Q} = -\mathbf{i}_1\circ\mathbf{U}\circ\mathbf{P}_\xi,\quad \mathbf{P}_\xi = p_{\xi 1}\mathbf{i}_1 + p_{\xi 2}\mathbf{i}_2;$$
$$\mathbf{r}_\xi = \xi_1\mathbf{i}_1 + \xi_2\mathbf{i}_2 = \bar{\mathbf{U}}\circ\mathbf{i}_1\circ\mathbf{U},$$

$$\mathbf{v}_\xi = d\mathbf{r}_\xi/dt = \dot{\xi}_1\mathbf{i}_1 + \dot{\xi}_2\mathbf{i}_2 = 2\bar{\mathbf{U}}\circ\mathbf{i}_1\circ d\mathbf{U}/dt = 2r^{-1}\bar{\mathbf{U}}\circ\mathbf{i}_1\circ d\mathbf{U}/d\tau. \tag{7.22}$$

Equations (7.17)–(7.20) must be supplemented by quaternion relations of reprojection (6.9)–(6.11).

Equations (7.9)–(7.12) and (7.17)–(7.20) are new regular equations of a three-dimensional two body problem (the trajectory motion of a spacecraft) constructed using Hansen's ideal rectangular coordinates. Levi–Civita variables U_0 and U_3, describing the motion of the center of mass of the second body (spacecraft) in ideal coordinate system $O\xi$; Keplerian energy h; time t; and Rodrigues–Hamilton (Euler) parameters Λ_j, characterizing the orientation of an ideal coordinate system in inertial coordinate system X, are regular variables in scalar equations (7.9)–(7.12). In Eqs. (7.17)–(7.20), the regular variables are represented by quaternion $\mathbf{U}$, describing the motion of the second body (spacecraft) center of mass in ideal coordinate system $O\xi$; Keplerian energy h; time t; and quaternion (quaternion osculating element) Λ, characterizing the orientation of an ideal coordinate system in inertial coordinate system X.

Regular equations (7.9)–(7.12) and (7.17)–(7.20) form a tenth-order system of nonlinear nonstationary differential equations (regular Kustaanheimo–Stiefel equations have the same dimensionality) with all the advantages of Kustaanheimo–Stiefel equations:

—Unlike Newtonian equations, they are regular at the center of attraction.

—They are linear for unperturbed Keplerian motions and have in this case the form

$$d^2U_i/d\tau^2 - (h/2)U_i = 0\ (i = 0, 3),$$
$$h = \mathrm{const},\quad \Lambda_j = \mathrm{const}\ (j = 0, 1, 2, 3)$$

or the form

$$d^2\mathbf{U}/d\tau^2 - (h/2)\mathbf{U} = 0,$$
$$h = \mathrm{const},\quad \mathbf{\Lambda} = \mathrm{const}$$

(for elliptical Keplerian motion, these equations are equivalent to the equations of motion not of a four-dimensional single-frequency harmonic oscillator when Keplerian energy $h < 0$, as in the case of Kustaanheimo–Stiefel equations, but of a two-dimensional single-frequency harmonic oscillator whose squared frequency is equal to half the Keplerian energy with the opposite sign).

—They allow us to develop a common approach to studying all three types of Keplerian motion.

—They are close to linear equations for unperturbed Keplerian motions.

—They allow us to represent the right-hand sides of differential equations of motion for celestial and cosmic bodies in polynomial forms convenient when solving them with computers.

8. EQUATIONS OF A TWO-BODY PROBLEM AND SPACECRAFT TRAJECTORY MOTION, WRITTEN WITH A SECOND QUATERNION OSCULATING ORBIT ELEMENT

Instead of quaternion osculating element $\mathbf{\Lambda}$ characterizing the orientation of the plane of an instantaneous orbit of a second body (spacecraft) in an inertial coordinate system and the position of the generalized pericenter on the orbit, we introduce new (second) quaternion osculating element $\mathbf{\Lambda}_{\mathrm{or}}$ characterizing the orientation of an instantaneous orbit of a second body (spacecraft) in an inertial coordinate system. To do so, we switch to new quaternion variable $\mathbf{\Lambda}_{\mathrm{or}}$ in quaternion differential equation (4.9) using the formula

$$\mathbf{\Lambda}_{\mathrm{or}} = \boldsymbol{\lambda}\circ[\cos(\varphi_{\mathrm{tr}}/2) - \mathbf{i}_3\sin(\varphi_{\mathrm{tr}}/2)], \tag{8.1}$$

where true anomaly φ_{tr} is determined by the third differential equation from system (1.1).

Quaternion of turn $\mathbf{\Lambda}_{\mathrm{or}}$ characterizes the orientation in the inertial coordinate system of new coordinate system ζ, bound with the instantaneous orbit of the second body (spacecraft). Axis ζ_1 of this coordinate system is directed along the radius vector of an instantaneous pericenter of the second body's (spacecraft's) orbit, while axis ζ_3 is perpendicular to the instantaneous orbital plane (parallel to coordinate axes η_3 and ξ_3). Components $\Lambda_{\mathrm{or}j}$ of quaternion $\mathbf{\Lambda}_{\mathrm{or}}$ are connected with angular osculating elements of the second body (spacecraft) orbit by relations similar to (5.3):

$$\begin{aligned}
\Lambda_{\mathrm{or}0} &= \cos(I/2)\cos((\Omega_u + \omega_{\pi\mathrm{tr}})/2),\\
\Lambda_{\mathrm{or}1} &= \sin(I/2)\cos((\Omega_u - \omega_{\pi\mathrm{tr}})/2),\\
\Lambda_{\mathrm{or}2} &= \sin(I/2)\sin((\Omega_u - \omega_{\pi\mathrm{tr}})/2),\\
\Lambda_{\mathrm{or}3} &= \cos(I/2)\sin((\Omega_u + \omega_{\pi\mathrm{tr}})/2).
\end{aligned} \tag{8.2}$$

As a result of switching variables, the motion of the second body (spacecraft) in variables r, v_1, c, φ_{tr}, and Λ_{orj} are described by differential equations

$$\dot{v}_1 = c^2/r^3 - \mu/r^2 + p_1, \quad \dot{r} = v_1, \quad \dot{c} = rp_2; \tag{8.3}$$

$$\begin{aligned}\dot{\varphi}_{tr} = cr^{-2} + r(c^2 - \mu r)^{-1}\cos\varphi_{tr}(cp_1\cos\varphi_{tr} \\ - (c + \mu rc^{-1})p_2\sin\varphi_{tr});\end{aligned} \tag{8.4}$$

$$\begin{aligned}2d\Lambda_{or0}/dt &= -\Omega_{or1}\Lambda_{or1} - \Omega_{or2}\Lambda_{or2} - \Omega_{or3}\Lambda_{or3},\\ 2d\Lambda_{or1}/dt &= \Omega_{or1}\Lambda_{or0} + \Omega_{or3}\Lambda_{or2} - \Omega_{or2}\Lambda_{or3},\\ 2d\Lambda_{or2}/dt &= \Omega_{or2}\Lambda_{or0} - \Omega_{or3}\Lambda_{or1} + \Omega_{or1}\Lambda_{or3},\\ 2d\Lambda_{or3}/dt &= \Omega_{or3}\Lambda_{or0} + \Omega_{or2}\Lambda_{or1} - \Omega_{or1}\Lambda_{or2};\end{aligned} \tag{8.5}$$

$$\begin{aligned}\Omega_{or1} = rc^{-1}p_3\cos\varphi_{tr}, \quad \Omega_{or2} = rc^{-1}p_3\sin\varphi_{tr},\\ \Omega_{or3} = -r(c^2 - \mu r)^{-1}\cos\varphi_{tr}(cp_1\cos\varphi_{tr}\\ - (c + \mu rc^{-1})p_2\sin\varphi_{tr}).\end{aligned} \tag{8.6}$$

Equations (8.5) and (8.6) in quaternion notation assume the forms

$$2\boldsymbol{\Lambda}_{or} = \boldsymbol{\Lambda}_{or} \circ \boldsymbol{\Omega}_{or\zeta}, \tag{8.7}$$

$$\boldsymbol{\Lambda}_{or} = \Lambda_{or0} + \Lambda_{or1}\mathbf{i}_1 + \Lambda_{or2}\mathbf{i}_2 + \Lambda_{or3}\mathbf{i}_3,$$

$$\begin{aligned}\boldsymbol{\Omega}_{or\zeta} = rc^{-1}p_3(\cos\varphi_{tr}\mathbf{i}_1 + \sin\varphi_{tr}\mathbf{i}_2)\\ - r(c^2 - \mu r)^{-1}\cos\varphi_{tr}(cp_1\cos\varphi_{tr} - (c + \mu rc^{-1})\\ \times p_2\sin\varphi_{tr})\mathbf{i}_3,\end{aligned}$$

where $\boldsymbol{\Omega}_{or}$ is the vector of the absolute angular velocity of an instantaneous orbit (coordinate system ζ bound with the instantaneous orbit of the second body (spacecraft)).

Equations (8.3)–(8.6) using the true anomaly are considerably more complicated than Eqs. (5.4)–(5.7) using the generalized anomaly. At the same time, they (as equations in classical osculating elements (1.1)) include singular point $e_{or} = 0$, at which these equations (or, more exactly, equations (8.4) and (8.5)) become degenerate. At this point, $c^2 - \mu r = 0$, and the instantaneous orbit takes circular form. Earlier equations for a generalized true anomaly and first quaternion osculating element did not have this singular point.

At the same time, Eqs. (8.3)–(8.6) ((8.3), (8.4), and (8.7)) are more descriptive, since they include differential equations (8.5), (8.6) (quaternion differential equation (8.7)) describing changes in the orientation of the instantaneous orbit of the second body (spacecraft).

As we can see from (8.7), vector $\boldsymbol{\Omega}_{or}$ of the absolute angular velocity of the instantaneous orbit of the second body (spacecraft), unlike the vector of angular velocity $\boldsymbol{\Omega}$ appearing in quaternion equation (5.8), depends in explicit form on all three projections of perturbation (control) vector $\mathbf{p}$.

Cartesian coordinates x_k of the second body (spacecraft) in inertial coordinate system X and projections v_k^+ of absolute velocity vector $\mathbf{v}$ of the second body (spacecraft) onto the axes of the inertial coordinate system are found with variables r, v_1, and c using the formulas presented in Section 4 in which Euler parameters λ_j must be preliminarily determined with parameters Λ_{orj} using the formula

$$\boldsymbol{\lambda} = \boldsymbol{\Lambda}_{or} \circ [\cos(\varphi_{tr}/2) + \mathbf{i}_3\sin(\varphi_{tr}/2)]. \tag{8.8}$$

Projections p_k of vector $\mathbf{p}$ onto the axes of orbital coordinate system η are found with its projections p_k^+ onto the axes of inertial coordinate system X using the quaternion relations of reprojection (2.14) with allowance for relations (8.8).

Note that Eqs. (8.3)–(8.6) ((8.3), (8.4), and (8.7)) were derived in [18, 19, 22].

9. EQUATIONS OF A TWO-BODY PROBLEM AND SPACECRAFT TRAJECTORY MOTION WHEN A PERTURBATION OR CONTROL ACCELERATION VECTOR (OR THEIR VECTOR SUM) THROUGHOUT THE TIME OF MOTION REMAINS ORTHOGONAL TO THE SECOND BODY'S (SPACECRAFT'S) ORBITAL PLANE

Differential equations describing form and dimensions of the second body (spacecraft) orbit are in this case integrable in elementary functions, yielding equations of conic sections. For us, $p_1 = p_2 = 0$, $p_3 = p$ and Eqs. (4.6) and (4.7) assume the form

$$\ddot{r} - c^2r^{-3} + \mu r^{-2} = 0, \quad c = \text{const}. \tag{9.1}$$

The form of quaternion equation (4.9) of the orientation of an orbital coordinate system does not change in this case. However, in this equation quantity $c = |\mathbf{r} \times \mathbf{v}| = \text{const}$ (i.e., it is a constant of areas).

Equation (9.1) is known to be integrable. In order to integrate it, let us switch to new variable

$$\rho = r^{-1} \tag{9.2}$$

and new independent variable φ, according to the formula

$$dt = (r^2/c)d\varphi = (1/(c\rho^2))d\varphi. \tag{9.3}$$

With (9.2) and (9.3), we find

$$\frac{d^2r}{dt^2} = -c^2\rho^2\frac{d^2\rho}{d\varphi^2}. \tag{9.4}$$

Substituting (9.2) and (9.4) into Eq. (9.1), we get

$$\frac{d^2\rho}{d\varphi^2} + \rho = \frac{\mu}{c^2}. \tag{9.5}$$

The position of the second body (spacecraft) on the orbit is characterized by true anomaly φ_{tr}. Angle φ_{tr} is measured counterclockwise in the plane of instantaneous orbit of the second body (spacecraft) from the pericenter, and it satisfies differential equation (8.4) or [3]

$$d\varphi_{tr}/dt = c/r^2 + (p_{or}/(ce_{or}))\cos\varphi_{tr}p_1$$
$$-(p_{or}/(ce_{or}))(1 + r/p_{or})\sin\varphi_{tr}p_2, \quad e \neq 0,$$

where p_{or} and e_{or} are the orbital parameter and eccentricity, respectively, and $c = |\mathbf{r} \times \mathbf{v}| = (\mu p_{or})^{1/2}$.

For us, $p_1 = p_2 = 0$, and the equation for the true anomaly takes the form

$$d\varphi_{tr}/dt = c/r^2. \tag{9.6}$$

Comparing Eq. (9.3) with (9.6) and assuming that at the initial moment in time $\varphi(t_0) = \varphi_{tr}(t_0)$, we conclude that φ is the true anomaly in (9.5): $\varphi(t) = \varphi_{tr}(t)$.

As is well known [3], the general solution to Eq. (9.5) can be written as

$$\rho = (1 + e_{or}\cos\varphi)/p_{or},$$
$$p_{or} = c^2/\mu.$$

Hence, the trajectory of the second body (spacecraft) in its orbit plane is a conic section described by the equation

$$r = p_{or}/(1 + e_{or}\cos\varphi), \quad p_{or} = c^2/\mu.$$

One focus of the conic section coincides with the attracting center; angle $\varphi = \varphi_{tr}$ is measured in the orbital plane from the pericenter, and its initial value $\varphi_0 = \varphi(t_0)$ and eccentricity e_{or} are found from the initial conditions of motion for the center of mass of the second body (spacecraft) (using values of r_0, v_0, and c_0) according to the familiar formulas [3, 23, 24]

$$\tan\varphi_0 = \pm c(v_0^2 - c^2\rho_0^2)^{1/2}/(c^2\rho_0 - \mu),$$
$$e_{or} = [1 + (c^2/\mu^2)(v_0^2 - 2\mu\rho_0)]^{1/2};$$
$$\sin\varphi_0 = (c/(\mu e_{or}))(x_{10}\dot{x}_{10} + x_{20}\dot{x}_{20} + x_{30}\dot{x}_{30})/r_0.$$

Here, $v_0 = |\mathbf{v}(t_0)|$, $\rho_0 = 1/r_0$, $r_0 = r(t_0)$, and x_{i0} and $\dot{x}_{i0}$ are the initial values for projections of radius vector $\mathbf{r}$ and velocity vector $\mathbf{v}$ of the center of mass of the second body (spacecraft) onto the axes of inertial coordinate system X.

The initial values of variables λ_j are found from those of angular orbital elements Ω_u, I, and $\omega_{\pi tr}$, along with that of true anomaly φ_{tr}, according to formulas (4.11). The initial values of variables λ_j can also be found with known projections of radius vector $\mathbf{r}$ and vector $\mathbf{c} = \mathbf{r} \times \mathbf{v}$ of the orbital velocity moment onto the axes of coordinate systems X and η using the formulas obtained in [20].

When the vectors of perturbing or control acceleration or their vector sum remain orthogonal to the orbital plane of the second body (spacecraft) throughout the time of motion, equations of motion for the second body (spacecraft) thus take the following forms in quaternion notation:

$$2d\boldsymbol{\lambda}/dt = \boldsymbol{\lambda} \circ \omega_\eta = \boldsymbol{\lambda} \circ [(r/c)p_3\mathbf{i}_1 + (c/r^2)\mathbf{i}_3]; \tag{9.7}$$

$$d\varphi/dt = c/r^2, \; c = \text{const}, \; r = p_{or}/(1 + e_{or}\cos\varphi), \quad \varphi = \varphi_{tr}. \tag{9.8}$$

Equations (9.7) and (9.8) are in fact equations of motion for an orbital coordinate system whose absolute velocity vector has projections determined by (4.5).

We can obtain the differential quaternion equation for the instantaneous orientation of the second body's (spacecraft's) orbit from Eqs. (5.8) or (8.7). It takes the form

$$2\frac{d\boldsymbol{\Lambda}}{dt} = \boldsymbol{\Lambda} \circ \boldsymbol{\Omega}_\xi, \quad \boldsymbol{\Omega}_\xi = \Omega_1\mathbf{i}_1 + \Omega_2\mathbf{i}_2 = (r/c)p_3(\cos\varphi \mathrm{i}_1 + \sin\varphi\mathbf{i}_2), \tag{9.9}$$

$$d\varphi/dt = c/r^2, \quad r = p_{or}/(1 + e_{or}\cos\varphi), \quad c = \text{const}, \quad \varphi = \varphi_{tr}. \tag{9.10}$$

In Eq. (9.9), quaternion $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_{or}$ characterizes the instantaneous orientation of the second body's (spacecraft's) orbit in inertial space; $\boldsymbol{\Omega}_\xi = \boldsymbol{\Omega}_{or\zeta}$ is the mapping of vector $\boldsymbol{\Omega} = \boldsymbol{\Omega}_{or}$ of the absolute angular velocity of the orbit onto coordinate basis ξ bound to it and coinciding with basis ζ.

Vector $\boldsymbol{\Omega}$ of instantaneous absolute angular velocity of the second body (spacecraft) orbit is directed along radius vector $\mathbf{r}$ and has the from $\boldsymbol{\Omega} = \omega_1\eta_1 = (p_3/c)\mathbf{r}$, where $\boldsymbol{\eta}_1$ is the unit vector of coordinate axis η_1.

The initial values of variables $\Lambda_j = \Lambda_{orj}$ are found with the initial values of angular orbital elements Ω_u, I, and $\omega_{\pi tr}$, according to formulas (8.2).

We can see from Eq. (9.9) that when projection p_3 of the vector of perturbing or control acceleration (or their vector sum) is equal to zero,

$$\boldsymbol{\Omega}_\xi = \boldsymbol{\Omega}_{or\zeta} = 0, \quad \boldsymbol{\Lambda} = \boldsymbol{\Lambda}_{or}(t_0) = \text{const}.$$

Hence, quaternion $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_{or}$ is a quaternion osculating element of the second body's (spacecraft's) orbit, and Eqs. (9.9) and (9.10) are differential equations for the orientation of a spacecraft's orbit in quaternion osculating elements.

Note that if differential equations (1.1) of instantaneous orbit orientation in angular osculating elements form a system of four nonlinear stationary first-order differential equations for angular variables Ω_u, I, $\omega_{\pi tr}$, and true anomaly φ_{tr}, Eqs. (9.9) and (9.10) form a system of five nonlinear stationary first-order differential equations for Euler parameters $\Lambda_j = \Lambda_{orj}$ and true anomaly $\varphi = \varphi_{tr}$. Unlike system (1.1), however, system (9.9), (9.10) has no singular points. In addition, by switching from time t to new independent variable φ in accordance with relation $dt = (r^2/c)d\varphi$, we get (at $p_3 =$

$p_3(\varphi)$) a system of four linear nonstationary differential equations for Euler parameters Λ_j, while system (1.1) remains essentially nonlinear upon such a change in the independent variable. Note also that Eqs. (1.1) and (9.9), (9.10) can be considered nonstationary systems of differential equations of the third and fourth orders, respectively, for variables Ω_u, I, $\omega_{\pi\mathrm{tr}}$, and Λ_j, since in these systems the last equation for the true anomaly is integrable by quadratures independent of the other equations; variable φ can therefore be considered a known function of time t. As a result, (1.1) is a nonlinear system, while (9.9), (9.10) is linear. The above makes using Eqs. (9.9), (9.10) more convenient and effective for solving certain problems of celestial mechanics and astrodynamics than using Eqs. (1.1).

Using Eqs. (9.7) and (9.8) describing the orientation of an orbital coordinate system in solving problems of celestial mechanics and astrodynamics has some advantages over using orbital orientation equations (9.9), (9.10). For example, quaternion equation (9.7) for a circular orbit (at $r = \mathrm{const}$ and $p_3 = \mathrm{const}$) is a linear differential equation with constant coefficients, while quaternion equation (9.9) is a linear differential equation with variable coefficients. From an analytical point of view, Eq. (9.7) is more convenient and efficient than (9.9). We should, however, bear in mind that from the standpoint of numerical solutions, quaternion equation (9.9) is more effective, since it contains a variable that is a quaternion osculating element of the orbit.

Let us consider a circular orbit in which the magnitude of radius vector of the center of mass of the second body (spacecraft) is constant: $r = r_0 = \mathrm{const}$. From equation (9.1), we have $c^2 = \mu r$, and from relations (4.1) we get

$$v = v_2 = c/r = (\mu/r)^{1/2} = \mathrm{const}. \qquad (9.11)$$

Polar coordinate $\varphi = \varphi_{\mathrm{tr}}$, measured in the orbital plane from point $\omega_\pi = \omega_{\pi\mathrm{tr}}$, can be introduced as before using the equation

$$d\varphi/dt = c/r^2 = \omega_3; \quad c, r = \mathrm{const}. \qquad (9.12)$$

By virtue of (9.11), with a circular orbit $d\varphi/dt = d\varphi_{\mathrm{tr}}/dt = v/r$. Differential equations (9.7), (9.8) for orbital coordinate system orientation and (9.9), (9.10) for orbital orientation of the second body (spacecraft) retain their forms. However, we must set $r = p_{\mathrm{or}} = \mathrm{const}$ and $e_{\mathrm{or}} = 0$ in them.

Angular coordinate $\varphi = \varphi_{\mathrm{tr}}$ determines the position of the center of mass of the second body (spacecraft) in a circular orbit. Introducing it via Eq. (9.12) and using the quaternion formula for the addition of finite turns,

$$\mathbf{\Lambda} = \mathbf{\Lambda}_{\mathrm{or}} = \boldsymbol{\lambda} \circ [\cos(\varphi_{\mathrm{tr}}/2) - \mathbf{i}_3 \sin(\varphi_{\mathrm{tr}}/2)]$$

we arrive at quaternion differential equation of orbital orientation (9.9) in this case too. This equation uses as a variable the quaternion osculating element of orbital orientation $\mathbf{\Lambda} = \mathbf{\Lambda}_{\mathrm{or}}$. At the initial moment in time, we can use the zero value of the angular distance of pericenter from node $\omega_\pi = \omega_{\pi\mathrm{tr}}$, and the initial value of angle $\varphi = \varphi_{\mathrm{tr}}$ can be found using the familiar relations [3, 4]

$$x_3 = r\sin\varphi_{\mathrm{tr}}\sin I,$$

$$c_1^+ = c\sin I\sin\Omega_u, \quad c_2^+ = -c\sin I\cos\Omega_u,$$

$$c_3^+ = c\cos I,$$

where c_i^+ are projections of vector $\mathbf{c} = \mathbf{r} \times \mathbf{v}$ onto the axes of the inertial coordinate system.

Note too that variable $\varphi = \varphi_{\mathrm{tr}}$ can be found from the quaternion relation

$$\cos(\varphi_{\mathrm{tr}}/2) + \mathbf{i}_3 \sin(\varphi_{\mathrm{tr}}/2) = \bar{\mathbf{\Lambda}} \circ \boldsymbol{\lambda} = \bar{\mathbf{\Lambda}}_{\mathrm{or}} \circ \boldsymbol{\lambda},$$

provided that parameters λ_j and $\Lambda_j = \Lambda_{\mathrm{or}j}$ are found first. The initial values of variables λ_j and $\Lambda_j = \Lambda_{\mathrm{or}j}$ can be found in the same way as for the elliptical orbit considered above.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, project no. 12-01-00165.

REFERENCES

1. Chelnokov, Yu.N., Quaternion regularization in astrodynamics and trajectory motion control, *Vestnik Nizhegorod. Univ. im. N.I. Lobachevskogo*, 2011, no. 4 (5), pp. 2583–2585.
2. Chelnokov, Yu.N., Quaternion regularization in celestial mechanics and astrodynamics and trajectory motion control. I, *Kosm. Issled.*, 2013, vol. 51, no. 5, p. 389–401. [*Cosmic Research*, pp. 353–364].
3. Duboshin, G.N., *Nebesnaya mekhanika. Osnovnye zadachi i metody* (Celestial Mechanics: Basic Problems and Methods), Moscow: Nauka, 1968.
4. Abalakin, V.K., Aksenov, E.P., Grebenikov, E.A., Demin, V.G., and Ryabov, Yu.A., *Spravochnoe rukovodstvo po nebesnoi mekhanike i astrodinamike* (Reference Manual on Celestial Mechanics and Astrodynamics), Moscow: Nauka, 1976.
5. Deprit, A., Ideal frames for perturbed Keplerian motions, *Celestial Mechanics*, 1976, vol. 13, no. 2, pp. 253–263.
6. Brumberg, V.A., *Analiticheskie algoritmy nebesnoi mekhaniki* (Analytical Algorithms of Celestial Mechanics), Moscow: Nauka, 1980.
7. Bragazin, A.F., Branets, V.N., and Shmyglevskii, I.P., Description of orbital motion using quaternions and velocity parameters, in *Ann. dokladov shestogo Vsesoyuznogo s'ezda po teoret. i prikladnoi mekhanike* (Abstracts of Reports at the 6th All-Union Congress on Theoret, and Applied Mechanics), Tashkent: Fan, 1986, p. 133.

8. Branets, V.N. and Shmyglevskii, I.P., *Vvedenie v teoriyu besplatformennykh inertsial'nykh navigatsionnykh sistem* (Introduction to Theory of Platformless Inertial Navigation Systems), Moscow: Nauka, 1992.
9. Il'in, V.A. and Kuzmak, G.V., *Optimal'nye perelety kosmicheskikh apparatov* (Optimal Transfers of Spacecraft), Moscow: Nauka, 1976.
10. Branets, V.N., and Shmyglevskii, I.P., *Primenenie kvaternionov v zadachakh orientatsii tverdogo tela* (Application of Quaternions in Problems of Solid Body Orientation), Moscow: Nauka, 1973.
11. Chelnokov, Yu.N., *Kvaternionnye i bikvaternionnye modeli i metody mekhaniki tverdogo tela i ikh prilozheniya: Geometriya i kinematika dvizheniya* (Quaternion and Bi-Quaternion Models and Methods of Solid Body Mechanics and Their Application: Geometry and Kinematics of Motion), Moscow: Fizmatlit, 2006.
12. Lur'e, A.I., *Analiticheskaya mekhanika* (Analytical Mechanics), Moscow: Fizmatgiz, 1961.
13. Chelnokov, Yu.N., *Kvaternionnye metody v zadachakh vozmushchennogo tsentral'nogo dvizheniya material'noi tochki. Ch. 1: Obshchaya teoriya. Prilozheniya k zadache regulyarizatsii i k zadache o dvizhenii ISZ* (Quaternion Methods in Problems of Perturbed Central Motion of a Material Point. Part 1: General Theory. Applications to Problem of Regularization and Problem of the Earth's Satellite Motion), Available from VINITI, 1985, no. 8628-B.
14. Chelnokov, Yu.N., Application of quaternions in the theory of orbital motion of an artificial satellite, I, *Kosm. Issled.*, 1992, vol. 30, no. 6, pp. 759–770. [*Cosmic Research*, pp. 612–621].
15. Chelnokov, Yu.N., Quaternion regularization and stabilization of perturbed central motion. Part 1, *Izv. Ross. Akad. Nauk, Mekh. Tverdogo Tela*, 1993, no. 1, pp. 20–30.
16. Chelnokov, Yu.N., Construction of optimum control and trajectories of spacecraft flight by employing quaternion description of orbit spatial orientation, *Kosm. Issled.*, 1997, vol. 35, no. 5, pp. 534–542. [*Cosmic Research*, pp. 499–507].
17. Chelnokov, Yu.N., Application of quaternions in space flight mechanics, *Giroskopiya i Navigatsiya*, 1999, no. 4 (27), pp. 47–66.
18. Chelnokov, Yu.N., Analysis of optimal motion control of a point in gravitational field with the use of quaternions, *Izv Ross. Akad. Nauk, Teor Sistemy Upravl.*, 2007, no. 5, pp. 18–44.
19. Chelnokov, Yu.N., *Kvaternnonnye modeli i metody dinamiki, navigatsii i upravleniya dvizheniem* (Quaternion Models and Methods for Dynamics, Navigation, and Motion Control), Moscow: Fizmatlit, 2011.
20. Chelnokov, Yu.N., Application of quaternions in the theory of orbital motion of an artificial satellite. II, *Kosm. Issled.*, 1993, vol. 31, no. 3, pp. 3–15. [*Cosmic Research*, pp. 400–418].
21. Chelnokov, Yu.N., The use of quaternions in the optimal control problems of motion of the center of mass of a spacecraft in a Newtonian gravitational field: I, *Kosm. Issled.*, 2001, vol. 39, no. 5, pp. 502–517. [*Cosmic Research*, pp. 470–484].
22. Chelnokov, Yu.N., The use of quaternions in the optimal control problems of motion of the center of mass of a spacecraft in a Newtonian gravitational field: II, *Kosm. Issled.*, 2003, vol. 41, no. 1, pp. 92–107. [*Cosmic Research*, pp. 85–99].
23. Bukhgolts, N.N., *Osnovnoi kurs teoreticheskoi mekhaniki (chast' pervaya)* (Fundamental Course of Theoretical Mechanics, First Part), Moscow: Nauka, 1972.
24. Alekseev, K.V., Bebenin, G.G., and Yaroshevskii, V.A., *Manevrirovanie kosmicheskikh apparatov* (Maneuvering of Spacecraft), Moscow: Mashinostroenie, 1970.

Translated by A. Lidvansky