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A universal approach to the intermediate anomaly of Keplerian motion

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Abstract. Within the framework of the elliptic two-body problem, Nacozy defined and integrated a Sundman-type transformation of the time variable. The new independent variable τ introduced through this device was termed the intermediate anomaly of Keplerian motion. We propose a general and systematic derivation of this kind of anomaly within a universal formulation and representation of the two-body problem, and Nacozy's developments are thus generalized and adapted to yield a uniform treatment of Kepler motion. To this end, an essential analytical tool is provided by certain classes of special functions, the so-called universal functions and Stumpff c-functions.

1. Introduction

Within the framework of the elliptic two-body problem, Nacozy (1977) studied a *Sundman-type transformation* (Sundman 1912, pp 127, 174) of the time variable given by a differential relation

$$\mathrm{d}t = c_\alpha r^\alpha \mathrm{d}\tau \qquad \text{with } \alpha = \tfrac{3}{2} \text{ and } c_\alpha = 1/\sqrt{\mu} \tag{1}$$

where t stands for the physical time, r is the norm of the radius vector, and μ denotes the gravitational parameter of a two-body system.

This differential relation was analytically integrated for pure Kepler motion in terms of an incomplete elliptic integral of the first kind whose modulus (or parameter) k is related to the numerical eccentricity e of the orbit, the argument (or amplitude) being half the true anomaly. The new independent variable τ introduced through this device was termed as the *intermediate anomaly* of Keplerian motion.

In practical cases for space research, apart from theoretical investigations concerning perturbed motion, the improvement of numerical integrations is also to be considered. In general, the integration accuracy along the orbit is not homogeneous. For instance, integrating highly eccentric orbits involves fast varying functions; nevertheless, some transformations of the independent variable can smooth out a part of the variations of the functions in such a way that a constant step-size method can also be used.

In other words: to ensure a sufficiently uniform distribution of orbital points for equidistant values of the independent argument, and avoid an unreasonable accumulation of integration steps in the neighbourhood of certain points and their excessive dissemination along other parts of the orbit, i.e. to achieve an adequate *analytical step-size regulation*, a change of independent variable can be invoked.

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In order to obtain the solutions of various problems involving gravitational dynamical systems, defining an analytical step-size regulation was one of the purposes of introducing the reparametrization of time leading to the intermediate anomaly.

To be more precise, we are mainly concerned with analytical aspects of orbital dynamics of artificial Earth satellites, and leave to our colleagues the task of computing highly eccentric Earth satellite orbits with special perturbation methods.

Roughly speaking, for the numerical computation of orbits of artificial satellites, the exponent $\alpha = \frac{3}{2}$ achieves an analytical step-size regulation and is revealed to be a good exponent for the oblateness perturbation problem and for Earth oblateness plus third-body gravitational perturbations. It also exhibits a slightly advantageous behaviour over other exponents, even when some additional perturbations are taken into account. This same exponent has also been employed in analytical and qualitative studies.

In this paper we aim to show that the approach taken by Nacozy to defining the intermediate anomaly is not limited to the case of elliptic motion. Indeed, as an attempt towards the extension and generalization of previous results by Nacozy to the case of hyperbolic and parabolic orbits in the Kepler motion, the present research is devoted to an investigation of the possibility of obtaining *a general and systematic derivation* of this kind of anomaly within a *universal formulation* of the two-body problem, which leads to a unified treatment and representation of the motion. This approach is intended in the following sense: irrespective of the nature of the specific Keplerian orbit at hand, Nacozy's developments will be generalized and adapted to yield a uniform treatment of Kepler motion (also in line with the contents of Battin 1987, sections 4.5 and 4.6, Ferrándiz and Floría 1990, Floría 1993b, ch 9, Stiefel and Scheifele 1971, section 11, Stumpff 1959, ch V, section 41), which suggests a non-singular transition between different types of two-body orbits (Herrick 1965).

In particular, we have in mind the universal-variable formulation and analytical treatment of perturbed Keplerian dynamical systems (e.g. the problem of perturbed highly eccentric elliptic orbits of artificial satellites), and the transition between reference orbits of different nature while performing perturbation studies, especially when a universal-like anomaly angle is put in the place of the independent variable. In this respect, it should be kept in mind that *the type of orbit is occasionally changed by perturbing forces acting during a finite interval of time* (Stiefel and Scheifele 1971, section 11, p 42).

Some possible applications in astrodynamics, to be considered elsewhere, have to do with theoretical estimations concerning mission analysis, orbital manoeuvres, orbital transfer problems and the trajectory optimization, two-body boundary-value problem, but also other practical aspects of the general problem of two bodies (not only in celestial mechanics). In any case, further analytical or semianalytical developments and numerical studies will be carried out by other members of the Grupo de Mecánica Celeste at the Departamento de Matemática Aplicada a la Ingeniería (Universidad de Valladolid).

This approach and the subsequent developments in terms of universal variables and parameters may be modified and find application in other domains of physics and applied mathematics. As far as we know, although the universal description of two-body motion originated from astrodynamical problems, there is no reason why the formulation should be restricted to such problems. An example of where this kind of treatment might be useful can be found in the naive classical mechanics description of scattering processes of particles in a potential force field (Goldstein 1980, section 3.10), as a preliminary step toward more accurate and rigorous quantum considerations, especially when simplifying assumptions concerning symmetries of the interaction are suppressed. In line with a remark by Hori (1961, section 10), and extending his comments, the study could be adapted to the case of scattering problems, also for hyperbolic orbits of repulsive force fields. At a later stage, transitions between attractive and repulsive forces (as in certain models of intermolecular potentials used in kinetic theory of gases) could be analysed in terms of appropriate future modifications of our formulations.

With this general aim in view, an essential analytical tool is provided by certain classes of special functions, the so-called *Stumpff c-functions* (Stumpff 1959, vol I, sections 37 and 41, Stiefel and Scheifele 1971, section 11, pp 43–5) and *universal U-functions* (Battin 1987, sections 4.5 and 4.6, Shepperd, 1985). These functions can be contemplated as generalizations of the standard and hyperbolic trigonometric functions, and their application is intended to avoid having to distinguish between elliptic, parabolic or hyperbolic motion. Elliptic integrals and functions will also play a significant role in the subsequent developments.

In what follows, use will be made of the customary set of symbols (a, e, p) for referring to the well known Keplerian elements, regardless of the type of orbit; on the other hand, f will denote the true anomaly in the (not necessarily elliptic) Keplerian orbit, and E the eccentric anomaly in the elliptic motion. Additional notation will be introduced in the next section.

2. Some remarks on universal-like functions

Linear differential equations can be considered to constitute the more refined, complete and developed area within the field of differential equations. The knowledge of fundamental properties and powerful solution techniques is wider and more fundamental than that in other domains of the theory of differential equations. This is particularly true when referring to linear equations with constant coefficients.

A classical procedure to introduce certain families of special functions is based on the study of power series solutions to linear differential equations. The Stumpff c-functions (see the above mentioned references to the books by Stumpff (1959), Stiefel and Scheifele (1971) and Battin (1987)) constitute a family of transcendental functions whose first members integrate, under a unified treatment, the model of second-order linear differential equations with constant coefficients

$$\frac{\mathrm{d}^2 y}{\mathrm{d}s^2} + \varrho y = 0 \tag{2}$$

irrespective of the sign of the parameter ϱ. It should be borne in mind that, after appropriate changes of the dependent and independent variables, this is just the type of equation to which the differential equations of motion governing the Kepler problem can be reduced, the parameter ϱ being then related to the value of the energy of the two-body system. This reduction is an essential aspect of the linear and regular formulation of celestial mechanics problems.

By using the notation $z = \varrho s^2$, the general solution to the above equation can be represented, in terms of the single parameter z, as a linear combination of the Stumpff c-functions $c_0(z)$ and $c_1(z)$, the representation of the solution being independent of the sign and value of ϱ. In general, these functions obey the *defining relation*

$$c_n(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^k}{(2k+n)!} \qquad n = 0, 1, 2 \ldots$$

the power series being absolutely convergent for all values of the complex variable z (whence the series converge for all s regardless of ϱ.) In particular, they are real-valued functions for real values of z.

When dealing with these functions, some calculations are simplified if the alternative *universal functions* introduced by Battin are used. To this end, for each $n = 0, 1, 2, \ldots,$ put (Stiefel and Scheifele 1971, section 11, formula (36))

$$U_n(s, \varrho) \equiv s^n c_n(\varrho s^2) = \sum_{k=0}^{\infty} (-1)^k \varrho^k \frac{s^{2k+n}}{(2k+n)!}. \tag{3}$$

Other equivalent defining relations can be found, e.g. in Battin (1987, section 4.5), or in Shepperd (1985, section 4).

2.1. Basic mathematical and mechanical properties

Many useful identities and relations involving universal-like functions can be found in the literature (see the previously mentioned references, Battin (1987), Shepperd (1985), Stiefel and Scheifele (1971), Stumpff (1959)). We restrict ourselves to summarizing some of them here, especially those required for future reference.

For convenience, according to the usual practice in studies related to the DS (Delaunay–Similar) variables introduced by Scheifele (Scheifele 1970, Scheifele and Graf 1974, Scheifele and Stiefel 1972), the quantity

$$L = \frac{\mu(1-e)}{2q} \tag{4}$$

is the *negative of the total energy* of the Keplerian orbit under consideration (see Stiefel and Scheifele 1971, p 50, formula (64)).

When needed, we shall take $\varrho = 2L$. For further purposes, borrowing the notation from Stiefel and Scheifele (1971), section 11, pp 47 and 50, the abbreviation q will represent the *distance of the pericentre*, and we consider the set of formulae and notations (Stiefel and Scheifele 1971, pp 50–1, Battin 1987, sections 4.5 and 4.6):

$$r = q + \mu e s^2 c_2\left(2Ls^2\right) = q + \mu e U_2(s, 2L) \tag{5}$$

$$\mathrm{d}t = r\,\mathrm{d}s \qquad \text{Sundman's transformation} \tag{6}$$

$$t = qs + \mu e U_3(s, 2L) \qquad \text{Kepler's equation.} \tag{7}$$

Observe that the fictitious time parameter s, a *universal eccentric-like anomaly* proportional to the classical eccentric anomaly in the cases of elliptic and hyperbolic motion, is introduced through Stumpff's generalization (1959, section 41) of Sundman's regularizing transformation (Sundman 1912, p 127), which defines s by means of the differential relation given in (6), where s vanishes at the chosen reference time. As a general rule, this variable is chosen so that the pericentre is reached for $s = 0$. Notice also that s only occurs implicitly in the equation for the radial distance r and, in Kepler's equation for the physical time t, through the transcendental Stumpff c_n and Battin U_n universal functions.

On the other hand, taking into account the universal character of the Keplerian true anomaly f, the parameters s and f are related to each other via the expressions (Stiefel and Scheifele 1971, p 51, formulae (67) and (68))

$$\sqrt{r}\cos\frac{f}{2} = \sqrt{q}\,c_0\left(\frac{L}{2}s^2\right) \tag{8}$$

$$\sqrt{r}\sin\frac{f}{2} = \frac{\sqrt{\mu(1+e)}}{2} s c_1\left(\frac{L}{2}s^2\right). \tag{9}$$

3. A glance at some Nacozy results

In order to put this research in context, before undertaking the presentation of the proposed generalizations, Nacozy's (1977) developments will be briefly reviewed. Here, some printing mistakes detected in that paper have been amended.

As stated above, he considered a Sundman-type transformation $t \longrightarrow \tau$ of the time parameter, from the physical time t to a pseudo-time τ, as given by a differential change of the time variable $\mathrm{d}t = c_\alpha r^\alpha \mathrm{d}\tau$, where $\alpha = \frac{3}{2}$; for the sake of dimensional homogeneity between the new anomaly angle and the classical ones, he chose the coefficient $c_\alpha = 1/\sqrt{\mu}$, although he also pointed out that other possible choices result in some simplification throughout the required calculations. (Remember that, as a general rule, c_α is usually taken either as a constant or a function of the orbital elements, mainly as a function of the semi-major axis a and the numerical eccentricity e.)

The set of Keplerian formulae holding in elliptic motion

$$r = \frac{a(1-e^2)}{1+e\cos f} \qquad \mathrm{d}t = \frac{r^2}{\sqrt{\mu a(1-e^2)}}\mathrm{d}f$$

introduced into the above time transformation, gives

$$\begin{aligned}\mathrm{d}\tau &= \frac{df}{\sqrt{1+e\cos f}} = \frac{\mathrm{d}f}{\sqrt{(1+e)\left[1-\frac{2e}{(1+e)}\sin^2(f/2)\right]}} \\ &= \frac{2\mathrm{d}\theta}{\sqrt{1+e}\sqrt{1-k^2\sin^2\theta}}\end{aligned}$$

provided that, *for convenience in writing, the following abbreviations had been used:*

$$\theta = \frac{1}{2}f \qquad k^2 = \frac{2e}{1+e}.$$

Then, performing the required quadratures

$$\int_0^\tau \frac{\sqrt{1+e}}{2}\mathrm{d}\tau = \int_0^\theta \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}$$

one has

$$\frac{\sqrt{1+e}}{2}\tau = F\left(\frac{f}{2}, k\right). \tag{10}$$

Note that, just at this point, a printing mistake occurs in Nacozy's 1977 article, in which one reads (Nacozy 1977, p 310, formula (5))

$$\sqrt{\frac{1+e}{2}} \qquad \text{instead of} \qquad \frac{\sqrt{1+e}}{2}.$$

The preceding formula provides the direct relationship between the intermediate anomaly τ and the true anomaly f in elliptic motion, with the usual symbol $F(\vartheta, k)$ for the *normal incomplete elliptic integral of the first kind with modulus k and argument ϑ.*

By taking into account the classical relations between the true and the eccentric anomaly in elliptic motion, and the definition and basic properties of the Jacobian elliptic functions (Byrd and Friedman 1971, formulae (110.02), (120.01) and (280.00), Abramowitz and

Stegun 1972, sections 16.1 and 17.2, Gradshteyn, and Ryzhik 1980, section 8.14), the set of formulae connecting these anomalies with each other is

$$\frac{f}{2} = \operatorname{am}\left(\frac{\sqrt{1+e}}{2}\tau, k\right) \qquad \sin\frac{f}{2} = \operatorname{sn}\left(\frac{\sqrt{1+e}}{2}\tau, k\right)$$

$$\sqrt{\frac{(1+e)(1-\cos E)}{2(1-e\cos E)}} = \operatorname{sn}\left(\frac{\sqrt{1+e}}{2}\tau, k\right)$$

where the symbols am and sn are the usual notations for the Jacobian elliptic functions amplitude and sine-amplitude, respectively.

4. Generalization of Nacozy's developments

The results outlined in Nacozy (1977), and summarized in the preceding section, are easily extended to the case of non-elliptic orbits by realizing that the relations between the true and the intermediate anomaly are formally valid irrespective of the nature of the motion at hand. Indeed, bearing in mind the universally valid formulae (Stiefel and Scheifele 1971 section 11, formulae (57) and (66)), and the usual change $t \longrightarrow f$

$$r = \frac{q(1+e)}{1+e\cos f} \qquad \mathrm{d}t = \frac{r^2}{\sqrt{\mu q(1+e)}}\mathrm{d}f$$

along with the differential relation (1),

$$\mathrm{d}t = \frac{r^{3/2}}{\sqrt{\mu}}\mathrm{d}\tau$$

the same conclusion is attained:

$$\mathrm{d}\tau = \frac{\mathrm{d}f}{\sqrt{1+e\cos f}} \Longrightarrow \frac{\sqrt{1+e}}{2}\tau = F\left(\frac{f}{2}, k\right). \tag{11}$$

Attention must be paid to appropriately interpreting the incomplete elliptic integral of the first kind when the modulus is not a quantity less than unity.

If $k = 1$ (i.e. for *parabolic orbits*, in which case $e = 1$), the preceding relation (11) between f and τ can be readily expressed in terms of elementary functions, since in that case

$$\frac{\sqrt{1+e}}{2}\tau = \frac{\tau}{\sqrt{2}}$$

and the relations (see, e.g. Byrd and Friedman 1971, formulae (111.04) and (132.01), Gradshteyn and Ryzhik 1980, formula (2.526.9), Abramowitz and Stegun 1972, formulae (4.3.117) and (17.4.21))

$$F(\vartheta, 1) = \ln\tan\left(\frac{\pi}{4} + \frac{\vartheta}{2}\right) = \ln(\tan\vartheta + \sec\vartheta) = \ln\sqrt{\frac{1+\sin\vartheta}{1-\sin\vartheta}} \tag{12}$$

imply that

$$\sin\left(\frac{f}{2}\right) = \frac{\exp\{\sqrt{2}\tau\} - 1}{\exp\{\sqrt{2}\tau\} + 1}. \tag{13}$$

If $k > 1$ (case of *hyperbolic orbits*), the well known reciprocal modulus transformation (Byrd and Friedman 1971, formulae (114.01), (162.02) and (283.00), Gradshteyn and Ryzhik 1980, formulae (2.571.4), (2.595.1) and (8.127), Abramowitz and Stegun 1972, section 16.11

and formula (17.4.15)) can be used to bring the elliptic integral with the modulus greater than one into the form corresponding to another one with a modulus less than unity:

$$\frac{\sqrt{1+e}}{2}\tau = F(\vartheta, k) = \frac{1}{k}F\left(\arcsin(k\sin\vartheta), \frac{1}{k}\right)$$

whence

$$\arcsin\left(k\sin\left(\frac{f}{2}\right)\right) = \mathrm{am}\left(\frac{k\sqrt{1+e}}{2}\tau, \frac{1}{k}\right) \Longrightarrow \sin\left(\frac{f}{2}\right) = \frac{1}{k}\mathrm{sn}\left(\sqrt{\frac{e}{2}}\tau, \frac{1}{k}\right).$$

On the other hand, from the well known geometrical and dynamical relations holding in hyperbolic Keplerian motion, in terms of the hyperbolic eccentric anomaly F, one concludes that

$$\sin\frac{f}{2} = \sqrt{\frac{(1+e)(1-\cosh F)}{2(1-e\cosh F)}} = \frac{1}{k}\mathrm{sn}\left(\sqrt{\frac{e}{2}}\tau, \frac{1}{k}\right). \tag{14}$$

As for these kind of relations ((12), (13), (14) and those at the end of section 3) between the intermediate anomaly and some type of eccentric-like anomaly, the auxiliary variable s and the general representation of Keplerian motion under the uniform treatment presented in Stiefel and Scheifele (1971, section 11), resorting to Stumpff and universal functions (see section 2), provide some analytical tools to reach this goal. Thus, advantage can be taken of the relation given in formula (3) of our previous section 2 and of the general formula, derived from (9) and (5),

$$\sin\frac{f}{2} = \frac{\sqrt{\mu(1+e)}sc_1\left((L/2)s^2\right)}{2\sqrt{q+\mu es^2c_2\left(2Ls^2\right)}} = \frac{\sqrt{\mu(1+e)}U_1\left((s/2), 2L\right)}{\sqrt{q+\mu eU_2\left(s, 2L\right)}}. \tag{15}$$

Leaving aside the requirement of dimensional compatibility between the intermediate anomaly and the Keplerian true and eccentric anomalies, slight simplifications in the reckoning work are obtained if one opts for other choices of c_α. Thus, the option

$$c_\alpha = \frac{1}{\sqrt{\mu(1+e)}} \Longrightarrow \tau = 2F\left(\frac{f}{2}, k\right)$$

while the possibility suggested by Nacozy yields

$$c_\alpha = \sqrt{\frac{1+e}{\mu}} \Longrightarrow \tau = \frac{2}{1+e}F\left(\frac{f}{2}, k\right).$$

5. Interpretation and redundancy of some alternative relations

After some additional reckoning work, we might conclude that the search for new alternative direct relations between the intermediate anomaly τ and the auxiliary variable s occurring in the universal-like functions leads us to the results presented in the preceding sections 3 and 4, which can be contemplated as the *essential and basic* set of formulae.

This assertion can be verified by establishing the corresponding relation starting from the differential expressions connecting these variables, τ and s, with each other. To this end, remember that, according to the Sundman transformation (6) and the Nacozy choice (1),

$$\mathrm{d}t = r\mathrm{d}s \qquad \text{and} \qquad \mathrm{d}t = \frac{r^{3/2}}{\sqrt{\mu}}\mathrm{d}\tau \Longrightarrow \mathrm{d}\tau = \sqrt{\frac{\mu}{r}}\mathrm{d}s.$$

Now, taking into account the expression (5) for the magnitude r of the radius vector in terms of s, we obtain

$$\int_0^\tau \mathrm{d}\tau = \sqrt{\mu} I \qquad \text{where } I = \int_0^s \frac{\mathrm{d}s}{\sqrt{q + \mu e U_2(s, 2L)}}. \tag{16}$$

Unfortunately, we have not been able to integrate this formal relation by subsuming the quadrature under a unified treatment. To perform this calculation, we must limit ourselves to the evaluation of this quadrature after distinguishing between the three main cases encountered in the study of Kepler problems. As shown in Battin (1987, p 180; see also Stiefel and Scheifele 1971, p 43, formulae (35) and (36), and p 45), the expression of the universal function $U_2(s, 2L)$ is

$$\begin{aligned} &\text{Parabola:} && L = 0 : U_2(s, 2L) = \frac{s^2}{2} \\ &\text{Ellipse:} && L > 0 : U_2(s, 2L) = \frac{1 - \cos\left(s\sqrt{2L}\right)}{2L} \\ &\text{Hyperbola:} && L < 0 : U_2(s, 2L) = \frac{\cosh\left(s\sqrt{-2L}\right) - 1}{-2L}. \end{aligned}$$

In the case of *elliptic motion*, for convenience in writing we introduce the notation $2L = \alpha > 0$, and so we obtain (Byrd and Friedman 1971, formula (291.00), Gradshteyn and Ryzhik 1980, formula (2.571.5))

$$I = \int_0^s \frac{\mathrm{d}s}{\sqrt{A - B\cos(s\sqrt{\alpha})}} = \frac{1}{\sqrt{\alpha}} \frac{2}{\sqrt{A + B}} F(\delta, \kappa) \tag{17}$$

with the abbreviations

$$\begin{aligned} A &= \frac{\mu}{\alpha} > B = \frac{\mu e}{\alpha} > 0 \qquad \kappa^2 = \frac{2B}{A + B} = \frac{2e}{1 + e} \equiv k^2 \\ \delta &= \arcsin\sqrt{\frac{B\left[1 - \cos\left(s\sqrt{\alpha}\right)\right]}{\kappa^2\left[A - B\cos\left(s\sqrt{\alpha}\right)\right]}} \\ &= \arcsin\sqrt{\frac{1 + e}{2} \frac{1 - \cos\left(s\sqrt{\alpha}\right)}{1 - e\cos\left(s\sqrt{\alpha}\right)}}. \end{aligned}$$

In principle, the formula

$$\tau = \sqrt{\mu} I = \frac{2}{\sqrt{1 + e}} F(\delta, \kappa) \tag{18}$$

could be understood as giving the relation between τ and s via δ. However, in view of the special meaning of the variable s in elliptic motion (proportional to the eccentric anomaly E through the expression $E = s\sqrt{2L}$, as seen in Stiefel and Scheifele (1971, pp 38 and 50, formula (64)), we conclude that $\delta = f/2$, from which *we recover the formula obtained by Nacozy* (1977) (and also recorded in section 3).

The study of the *hyperbolic case* is similar to that of the elliptic one. We put $2L = \alpha = -\beta < 0$. Then (Byrd and Friedman 1971, formula (297.00), Gradshteyn and Ryzhik 1980, formula (2.464.41))

$$I = \int_0^s \frac{\mathrm{d}s}{\sqrt{B\cosh\left(s\sqrt{\beta}\right) - A}} = \frac{1}{\sqrt{\beta}}\sqrt{\frac{2}{B}} F(\xi, \kappa) \tag{19}$$

with the notations

$$B = \frac{\mu e}{\beta} > A = \frac{\mu}{\beta} > 0 \qquad \kappa^2 = \frac{A+B}{2B} = \frac{1+e}{2e} \equiv \frac{1}{k^2}$$

$$\xi = \arcsin\sqrt{\frac{B\left[\cosh\left(s\sqrt{\beta}\right)-1\right]}{B\cosh\left(s\sqrt{\beta}\right)-A}}$$

whence

$$\tau = \sqrt{\mu} I = \sqrt{\frac{2}{e}} F(\xi, \kappa). \tag{20}$$

Notice that here the quantity $\kappa < 1$, and $\kappa = 1/k$, where k is the same as in the preceding sections. Thus, the meaning of ξ can be elucidated if one introduces an angle γ such that, by virtue of the reciprocal modulus transformation,

$$F(\xi, \kappa) = kF(\gamma, k) \qquad \text{that is, } \xi = \arcsin(k \sin\gamma) \tag{21}$$

from which we are led to

$$\sin\xi = k\sin\gamma = \sqrt{\frac{e\left[\cosh\left(s\sqrt{\beta}\right)-1\right]}{e\cosh\left(s\sqrt{\beta}\right)-1}}$$

and finally

$$\sin\gamma = \sqrt{\frac{1+e}{2}\frac{1-\cosh\left(s\sqrt{\beta}\right)}{1-e\cosh\left(s\sqrt{\beta}\right)}}. \tag{22}$$

Now, by reason of the differential relations (6), $\mathrm{d}t = r\mathrm{d}s$ (Sundman's transformation), and

$$\mathrm{d}t = \frac{r}{\sqrt{-2L}}\mathrm{d}F$$

(Floría 1993a, p 1373), one sees that s is proportional to the hyperbolic eccentric anomaly F through the expression $F = s\sqrt{-2L}$, and we arrive at the certainty that $\gamma = f/2$, from which we reconstruct formula (14), mentioned in section 4.

As for the parabolic orbital motion, from Gradshteyn and Ryzhik (1980, formula (2.271.4)), we get

$$I = \int_0^s \frac{\sqrt{2}\mathrm{d}s}{\sqrt{2q+\mu s^2}} = \sqrt{\frac{2}{\mu}}\left[\ln\left(s\sqrt{\mu}+\sqrt{2q+\mu s^2}\right) - \ln\sqrt{2q}\right]$$

$$\Longrightarrow I = \sqrt{\frac{2}{\mu}}\ln\left(s\sqrt{\frac{\mu}{2q}}+\sqrt{1+\frac{\mu}{2q}s^2}\right). \tag{23}$$

The next step resorts to the specific meaning of the variable s for parabolic orbits. From Stiefel and Scheifele (1971), pp 49, 48 (formula (57)), or p 51 (formulae (65)–(68)), and remembering the expressions of the Stumpff c_2-function and the universal U_2-function when $L = 0$ (say, when $e = 1$), we obtain

$$r = q + \frac{\mu}{2}s^2 = \frac{q(1+e)}{1+e\cos f} \Longrightarrow \frac{\mu}{2}s^2 = qe\frac{1-e\cos f}{1+e\cos f}.$$

Consequently

$$s = \sqrt{\frac{2q}{\mu}}\tan\frac{f}{2} \Longrightarrow I = \sqrt{\frac{2}{\mu}}\ln\left[\tan\left(\frac{f}{2}\right)+\sec\left(\frac{f}{2}\right)\right]$$

the final relation being

$$\tau = \sqrt{\mu} I = \sqrt{2} \ln \left[\tan\left(\frac{f}{2}\right) + \sec\left(\frac{f}{2}\right) \right] \tag{24}$$

as stated in section 4, formula (12).

Final remark

The coauthor of this research, Mr Rafael Caballero, died tragically after a road accident. In his memory, L Floría has completed this part of a wider plan of joint work, and written the present version of the paper.

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