Elliptic functions and elliptic integrals for celestial mechanics and dynamical astronomy

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1 Introduction

The elliptic functions and the elliptic integrals are one of the most complicated special functions [68]. Their textbooks are Akhiexer [1], Bowman [3], Cayley [25], Hancock [57, 58], Lawden [60], Thompson [66] and their references are Abramowitz and Stegun [2], Byrd and Friedman [12], Oldham et al. [61], Olver et al. [62], Wolfram [69]. The last reference [62] is freely accessible from its Website: //http:/dlmf.nist.gov/. For practical purposes, we limit ourselves with the case where all the input arguments and the function/integral values are real-valued hereafter.

Among the various forms of elliptic functions and elliptic integrals, the Jacobian elliptic functions and Legendre's form of elliptic integrals are most popular. In celestial mechanics and dynamical astronomy, the Jacobian elliptic functions frequently appear in the analytical expressions of the variables of some dynamical systems. For example, the rapidly convergent series required in the orbital dynamics of planets and satellites are constructed by using the Jacobian elliptic functions [4–6, 26, 59, 65, 67]. This is because the basic orbit is an ellipse, and therefore, the elliptic functions naturally appear in its coordinate description, especially when the independent variable is set as the arc length of an ellipse, which is represented by Legendre's form of incomplete elliptic integral of the second kind.

Also, the Jacobian elliptic functions are essentially needed in describing the torque-free rotation of the rigid body [31, 39–41, 43, 55]. In fact, a Gaussian formulation for non canonical elements of rotational dynamics requires their evaluation frequently in the forward and backward transformation between the elements and the standard variables such as the combination of the moving coordinate triad and the body-fixed components of the angular velocity vector [40]. Similarly, a canonical element formulation calls the functions repeatedly [41]. Indeed, some complicated problems on rotational dynamics are exactly solved in terms of the Jacobian elliptic functions [29].

On the other hand, the elliptic integrals naturally appear in the computation of gravitational field of a celestial body with a sort of symmetry. This is because the Green kernel of the Newtonian gravitational field is the reciprocal of the mutual dis-

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tance which is expressed as the square root of a certain function of coordinates, and therefore, the resulting integral reduces to the elliptic integrals when the inside of the square root is a simple function, say a polynomial of the integral variable of degree 4 at most. Good examples are that of a homogeneous ring or disk [45], and of homogeneous triaxial ellipsoid [12, Introduction].

Another application of the incomplete elliptic integrals is the analytical solution expression of angle variables of the torque-free rotation [39–41, 43]. Indeed, Encke's method for rotational dynamics assumes an easy access to computing routines of the reference solution as a function of time, which is usually taken as the torque-free solution osculating at the initial epoch [31]. These situations recall us that the solution of Kepler's equations of various kinds plays the key role in the perturbed two-body problems in orbital dynamics [30, 32–36]. These applications are not only important in astronomy but also in physics and chemistry, especially in the symplectic integration of the rotational dynamics of molecules.

However, it is also true that the elliptic functions and the elliptic integrals are difficult to approach, understand, and utilize. Therefore, in order to enhance their accessibility, we first present their numerous notations and remark some pitfalls to understand the written materials in Section 2. Next, we explain their properties, key formulas, and internal relationships for the elliptic functions in Section 3 and for the elliptic integrals in Section 4, respectively. Finally, in Section 5, we present the essence of the new procedures of their numerical computation recently developed by us [42, 44–54].

2 Notations

The elliptic functions and the elliptic integrals are classic subjects of mathematics and have a long history of researches. Since there are quite a few different notations in the literature [69], we first summarize them and then give some remarks.

2.1 Glossary

- a: the second argument of Jacobi's form of incomplete elliptic integral of the third kind.
- $\mathrm{am}(u|m)$: Jacobi's amplitude function.
- b: the main variable in the duplication of Jacobian elliptic functions.
- $c = \mathrm{cn}\, u = \mathrm{cn}(u|m)$: the cosine amplitude function.
- `cel`: Bulirsch's general form of complete elliptic integral of all kinds.
- `cel1`: Bulirsch's form of complete elliptic integral of the first kind.
- `cel2`: Bulirsch's general form of complete elliptic integral of the second kind.
- `cel3`: Bulirsch's form of complete elliptic integral of the third kind.

- $d = \mathrm{dn}\ u = \mathrm{dn}(u|m)$: the delta amplitude function.
- `el`: Bulirsch's general form of incomplete elliptic integral of all kinds.
- `el1`: Bulirsch's form of incomplete elliptic integral of the first kind.
- `el2`: Bulirsch's general form of incomplete elliptic integral of the second kind.
- `el3`: Bulirsch's form of incomplete elliptic integral of the third kind.
- g_2, g_3: the invariants of Weierstrass elliptic function.
- h: a parameter in the addition theorem of incomplete elliptic integral of the third kind.
- k: the modulus.
- k_c: the complementary modulus.
- m: the parameter.
- m_c: the complementary parameter.
- n: the characteristic.
- n_c: the complementary characteristic.
- $\wp$: Weierstrass elliptic function.
- $q(m)$: Jacobi' nome.
- $s = \mathrm{sn}\ u = \mathrm{sn}(u|m)$: the sine amplitude function.
- $t(u, v)$: a common factor in the addition theorems of incomplete elliptic integrals.
- u: the argument.
- x: the second main variable of the half-argument transformation.
- y: the first main variable of the half-argument transformation.
- $B(m)$: the first associate complete elliptic integral of the second kind.
- $B(\varphi|m)$: the first associate incomplete elliptic integral of the second kind.
- $B_u(u|m)$: Jacobi's form of first associate incomplete elliptic integral of the second kind.
- $D(m)$: the second associate complete elliptic integral of the second kind.
- $D(\varphi|m)$: the second associate incomplete elliptic integral of the second kind.
- $D_u(u|m)$: Jacobi's form of second associate incomplete elliptic integral of the second kind.
- $E(m)$: Legendre's form of complete elliptic integral of the second kind.
- $E(\varphi|m)$: Legendre's form of incomplete elliptic integral of the second kind.
- $E_u(u|m)$: Jacobi's form of incomplete elliptic integral of the second kind.
- $F(\varphi|m)$: Legendre's form of incomplete elliptic integral of the first kind.
- $J(n|m)$: the associate complete elliptic integral of the third kind.
- $J(\varphi, n|m)$: the associate incomplete elliptic integral of the third kind.
- $J_u(u, n|m)$: Jacobi's form of associate incomplete elliptic integral of the third kind.
- $K(m)$: Legendre's form of complete elliptic integral of the first kind.
- R_D: Carlson's symmetric form of elliptic integral of the second kind.
- R_F: Carlson's symmetric form of elliptic integral of the first kind.
- R_J: Carlson's symmetric form of elliptic integral of the third kind.
- $S(m)$: the special complete elliptic integral of the second kind.
- $T(t, h)$: the universal arc tangent function.

- $\alpha \equiv \sin^{-1} k$: the modular angle.
- $\alpha \equiv \sqrt{n}$: the alternative characteristic.
- $\varepsilon(u|m)$: Jacobi's Epsilon function.
- φ: the amplitude.
- ν: the negative characteristic.
- $\Delta(\theta|m)$: Jacobi's delta function.
- $\Pi(n|m)$: Legendre's form of complete elliptic integral of the third kind.
- $\Pi(\varphi, n|m)$: Legendre's form of incomplete elliptic integral of the third kind.
- $\Pi_u(u, n|m)$: Jacobi's form of incomplete elliptic integral of the third kind.
- $\Pi_1(a|m)$: Jacobi's original complete elliptic integral of the third kind.
- $\Pi_1(u, a|m)$: Jacobi's original incomplete elliptic integral of the third kind.

2.2 First input argument: φ, u, and x

A source of confusion is the difference in the first argument of the elliptic functions and the elliptic integrals: (1) the amplitude φ in Legendre's notation, (2) the argument $u \equiv F(\varphi|m)$ in Jacobi's notation, and (3) the tangent amplitude $x \equiv \tan\varphi$ in Bulirsch's notation. In the followings, we discriminate the first two carefully by explicitly attaching the subscript u to Jacobi's notation such as

$$E_u(u|m) = E(\varphi|m) . \tag{2.1}$$

2.3 Second input argument: k, m, and α

Another confusion comes from the different notations in describing the second input argument. Traditionally, it is written as k and called the modulus [12]. For example, the classic expression of the complete elliptic integral of the first kind is

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} . \tag{2.2}$$

Of course, if all numbers are regarded to be complex valued, this is acceptable. However, if only real numbers are dealt with as in the practical applications, k is not appropriate since the integral is real-valued even when k is a pure imaginary. Therefore, the parameter $m \equiv k^2$ is preferred recently [69]. The third option is the modular angle α [2] defined such that

$$\alpha \equiv \sin^{-1} k , \tag{2.3}$$

assuming that k is real and $0 \le k < 1$. In order to discriminate whether k, m, or α is used, the different separators are selected in front of them such as

$$F(\varphi, k) = F(\varphi|m) = F(\varphi \setminus \alpha) . \tag{2.4}$$

Namely, (1) if the separator is the comma “,” the following input argument means k, (2) if the separator is the vertical bar “|,” the following input argument means m, and (3) if the separator is the backslash “\,” the following input argument means α.

Still, there remains a source of confusion in expressing single-argument quantities such as the complete integrals K, E, B, and D. In this case, there is no apparent difference among $K(k)$, $K(m)$, and $K(\alpha)$, especially if the argument is expressed numerically or by the symbols other than k, m, and α.

2.4 Sign of n

A notorious confusion is in the sign of n appearing in the incomplete elliptic integral of the third kind:

$$\Pi = \int_0^{\varphi} \frac{d\theta}{(1 \pm n \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}} . \tag{2.5}$$

Traditionally, the plus sign “+” is adopted [2, 11, 62]. This is probably because the rotation angle of a triaxial rigid body in the short axis mode requires the integral of the third kind with the plus sign in the place of n [43]. Some authors use different symbol ν for the plus sign:

$$\Pi = \int_0^{\varphi} \frac{d\theta}{(1 + \nu \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}} . \tag{2.6}$$

In the followings, we adopt n with the negative sign “−” [69].

2.5 Third input argument: n and α

Quite confusingly, the same symbol α as the modular angle also appears as an alternative of the characteristic n as

$$\alpha^2 \equiv \nu = \pm n \geq 0 . \tag{2.7}$$

Again, there are both choices of the sign of n, which makes the problem more complicated.

2.6 Ordering of arguments

Another type of confusion is brought by different conventions in the ordering of the input arguments, especially in expressing the incomplete elliptic integral of the third kind. The usual convention is the order of (1) the argument (or its alternative) first,

(2) the characteristic (or its alternative) second, and (3) the parameter (or its alternative) last. An example is

$$\Pi(\varphi, -\alpha^2, k) = \Pi_u(u, n|m) \ . \tag{2.8}$$

On the other hand, Mathematica [69] puts the characteristic first as

$$\texttt{EllipticPi}[\texttt{n}, \texttt{phi}, \texttt{m}]$$

2.7 Omission of parameters

In the literature, some input arguments like k, m, or n are frequently omitted such as (1) K in place of $K(k)$ or $K(m)$, and (2) $\mathrm{sn}\ u$ in place of $\mathrm{sn}(u, k)$ or $\mathrm{sn}(u|m)$. This is just a convention to simplify the expression of lengthy formulas and equations. Nevertheless, one should be careful that hidden parameters exist behind the scene. This becomes important especially when conducting the partial differentiation of these abbreviated quantities.

3 Elliptic functions

3.1 General elliptic function

Historically, an elliptic function was introduced as the inverse of a standard elliptic integral [12, p.18]:

$$I(x) = \int_a^x \frac{dz}{\sqrt{f(z)}} \ , \tag{3.1}$$

where $f(z)$ is a cubic or quartic polynomial of z with distinct roots. If the argument of the integral x is regarded as a function of the integral value denoted by

$$u \equiv I(x) \ , \tag{3.2}$$

then x is called an elliptic function of argument u and is expressed as

$$x = p(u) \ . \tag{3.3}$$

We explicitly write $f(z)$ as

$$f(z) = a_0 z^4 + 4a_1 z^3 + 6a_2 z^2 + 4a_3 z + a_4 \ , \tag{3.4}$$

under the condition

$$a_0 \neq 0 \quad \text{and/or} \quad a_1 \neq 0 \ . \tag{3.5}$$

Thus, by definition, the first-order derivative of p with respect to u is expressed as

$$p' \equiv \frac{dp}{du} = 1 \Big/ \frac{du}{dp} = \sqrt{f(p)} = \sqrt{a_0 p^4 + 4a_1 p^3 + 6a_2 p^2 + 4a_3 p + a_4} \,. \qquad (3.6)$$

Namely, if a certain dynamical variable p is governed by a first-order ordinary differential equation (ODE) and if the right-hand side of ODE is expressed as the square root of a polynomial of p of degree 4 at most, then the variable is analytically expressed in terms of a sort of elliptic function. This is the first-type application of elliptic functions.

Square the above first-order derivative expression, differentiate both sides of it with respect to u, and divide them by the common factor, $2p'$, which is assumed to be nonzero. Then, the expression of the second-order derivative of p is obtained as

$$p'' \equiv \frac{d^2 p}{du^2} = 2a_0 p^3 + 6a_1 p^2 + 6a_2 p + 2a_3 \,. \qquad (3.7)$$

This means that, if the right-hand side of the equation of motion of a certain dynamical variable, p, is expressed as its polynomial of degree 3 at most, then the variable is analytically given by a kind of elliptic function. This is the second type application of elliptic functions.

3.2 Weierstrass elliptic function

Weierstrass's $\wp$ function [62, Chapter 23] is a special case of $p(u)$ such that

$$u = \int_{\wp}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}} \,. \qquad (3.8)$$

This is the case when

$$a_0 = a_2 = 0 \,, \quad a_1 = 1 \,, \quad a_3 = -\left(\frac{g_2}{4}\right) , \quad a_4 = -g_3 \,, \qquad (3.9)$$

where the newly introduced coefficients, g_2 and g_3, are called the invariants. Thus, its derivatives are expressed [62, Section 23.3(ii)] as

$$\wp' = \sqrt{4\wp^3 - g_2\wp - g_3} \,, \qquad (3.10)$$

$$\wp'' = 6\wp^2 - \frac{g_2}{2} \,. \qquad (3.11)$$

It is rare to use $\wp$ in the practical applications. This is true also in celestial mechanics and dynamical astronomy.

3.3 Jacobian elliptic functions

Move to the Jacobian elliptic functions [62, Section 22.1]. The three principal Jacobian elliptic functions,

$$s(u) \equiv \mathrm{sn}(u|m) , \tag{3.12}$$

$$c(u) \equiv \mathrm{cn}(u|m) , \tag{3.13}$$

$$d(u) \equiv \mathrm{dn}(u|m) , \tag{3.14}$$

are special cases of $p(u)$ when $f(z)$ is a quadratic polynomial of z^2 [62, Section 22.15(ii)] as

$$u = \int_0^s \frac{dz}{\sqrt{(1-z^2)(1-mz^2)}} , \tag{3.15}$$

$$u = \int_c^1 \frac{dz}{\sqrt{(1-z^2)(m_c+mz^2)}} , \tag{3.16}$$

$$u = \int_d^1 \frac{dz}{\sqrt{(1-z^2)(z^2-m_c)}} , \tag{3.17}$$

where m is a parameter simply called the parameter, and

$$m_c \equiv 1 - m , \tag{3.18}$$

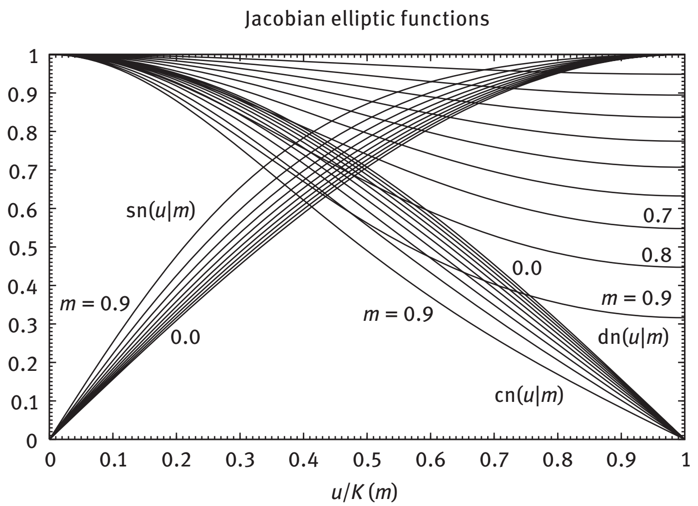

Fig. 1. Sketch of Jacobian elliptic functions: local. Illustrated are the graphs of three principal Jacobian elliptic functions, $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$ for u in its standard domain $0 \le u \le K(m)$ for various values of m as $m = 0.0, 0.1, \dots, 0.9$.

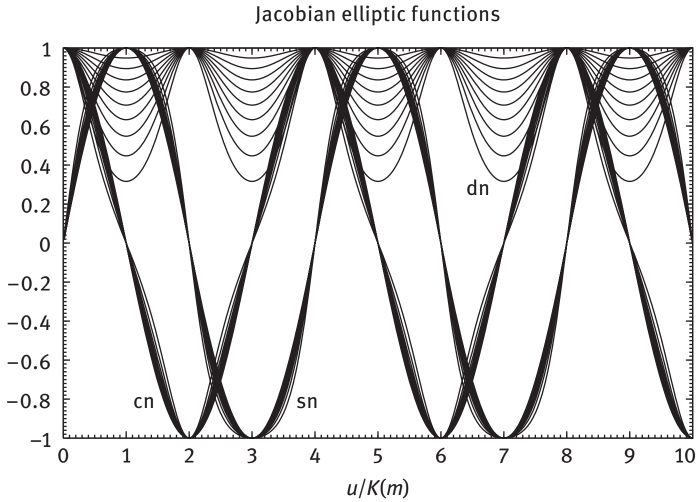

Fig. 2. Sketch of Jacobian elliptic functions: global. Same as Figure 1 but for a longer period.

is the complementary parameter. See Carlson [23] for the hidden symmetry in these elliptic functions. Traditionally, in place of m, another parameter called the modulus and defined as

$$k \equiv \sqrt{m}\,, \tag{3.19}$$

is frequently used.

Refer to Figures 1 and 2 for the behavior of the functions. It is noteworthy that $\operatorname{sn}(u|m)$ and $\operatorname{cn}(u|m)$ resemble $\sin[(\pi u)/(2K(m))]$ and $\cos[(\pi u)/(2K(m))]$ unless $m \approx 1$.

Classically, their numerical values are computed by the descending Landen transformation [8]. A faster algorithm using their conditional duplication is recently established [51].

3.4 Jacobi's amplitude function

Consider the partial derivative of the principal Jacobian elliptic functions with respect to the argument u while the parameter m is fixed. Using the general formula, Equation (3.6), we immediately obtain their expressions such as

$$s' \equiv \left(\frac{\partial s}{\partial u}\right)_m = \sqrt{\left(1-s^2\right)\left(1-ms^2\right)}\,. \tag{3.20}$$

However, these are not so useful. Instead, Jacobi noticed the similarity of the implicit definition of $s(u)$, Equation (3.15), with that of the sine function,

$$u = \int_0^{\sin u} \frac{dz}{\sqrt{1-z^2}} \, . \tag{3.21}$$

Then, he wrote the arc sine of $s(u)$ as

$$\varphi(u) \equiv \sin^{-1}[s(u)] \, , \tag{3.22}$$

and called it the amplitude. In the modern notation, it is written as

$$\varphi(u) = \mathrm{am}(u|m) \, . \tag{3.23}$$

At any rate, using the amplitude, Jacobi redefined $s(u)$ and introduced two other principal functions as its trigonometric function values as

$$s(u) \equiv \sin\varphi(u) \, , \tag{3.24}$$

$$c(u) \equiv \cos\varphi(u) \, , \tag{3.25}$$

$$d(u) \equiv \Delta(\varphi(u)|m) = \sqrt{1 - m\sin^2\varphi(u)} \, , \tag{3.26}$$

where $\Delta(\varphi|m)$ is Jacobi's delta function. These definitions immediately lead to the following identity relations:

$$[s(u)]^2 + [c(u)]^2 = 1 \, , \tag{3.27}$$

$$m\,[s(u)]^2 + [d(u)]^2 = 1 \, . \tag{3.28}$$

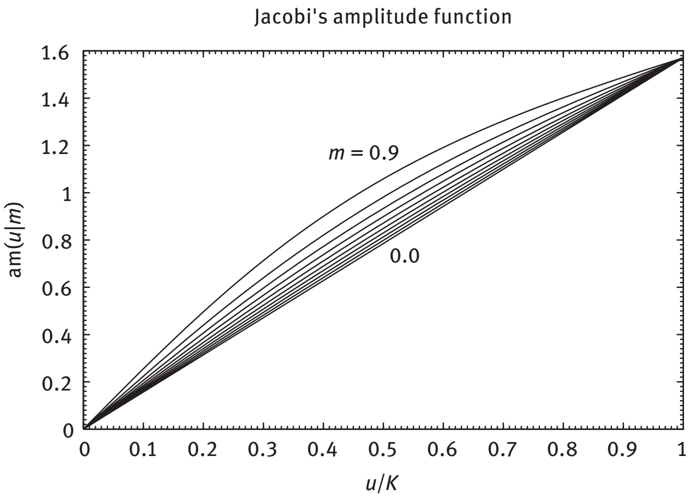

Fig. 3. Sketch of Jacobi's amplitude function: local. Same as Figure 1 but for $\mathrm{am}(u|m)$.

Jacobi called $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$ the sine amplitude function, the cosine amplitude function, and the delta amplitude function, respectively. Figure 3 illustrates Jacobi's amplitude function in the standard domain of the argument, $0 \leq u \leq K(m)$, for various values of m.

3.5 Differential equations of Jacobian elliptic functions

Return to the derivative expressions of the Jacobian elliptic functions. From the not-so-useful expression of s', Equation (3.20), the derivative expression of Jacobi's amplitude function is obtained as

$$\varphi' = d \,. \tag{3.29}$$

This naturally leads to the derivative expression of the sine amplitude function and the cosine amplitude function [62, Table 22.13.1] as

$$s' = cd \,, \tag{3.30}$$

$$c' = -sd \,. \tag{3.31}$$

Also, from the identity relation of $s(u)$ and $d(u)$, the derivative expression of $d(u)$ is obtained as

$$d' = -msc \,. \tag{3.32}$$

Indeed, s, c, and d can be defined as the fundamental solution of a system of first order ODEs of three components:

$$\frac{dx}{du} = yz \,, \tag{3.33}$$

$$\frac{dy}{du} = -xz \,, \tag{3.34}$$

$$\frac{dz}{du} = -mxy \,, \tag{3.35}$$

where m is a constant being independent on u. This is the third-type application of elliptic functions.

It is noteworthy that the introduction of Jacobi's amplitude function φ can be regarded as a sort of regularization of the rotational dynamics described by the above system of ODE equations [37, 38].

3.6 Addition theorem of Jacobian elliptic functions

Among many properties of the elliptic functions, the most important one is the availability of the addition theorems with respect to their argument. Indeed, the existence of the addition theorems is a key feature to characterize the elliptic function among many special functions.

In the case of the Jacobian elliptic functions, the theorems are written [12, Formula 123.01] as

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}u\ \mathrm{cn}v\ \mathrm{dn}v + \mathrm{sn}v\ \mathrm{cn}u\ \mathrm{dn}u}{1 - m\ \mathrm{sn}^2u\ \mathrm{sn}^2v}, \tag{3.36}$$

$$\mathrm{cn}(u+v) = \frac{\mathrm{cn}u\ \mathrm{cn}v - \mathrm{sn}u\ \mathrm{sn}v\ \mathrm{dn}u\ \mathrm{dn}v}{1 - m\ \mathrm{sn}^2u\ \mathrm{sn}^2v}, \tag{3.37}$$

$$\mathrm{dn}(u+v) = \frac{\mathrm{dn}u\ \mathrm{dn}v - m\ \mathrm{sn}u\ \mathrm{sn}v\ \mathrm{cn}u\ \mathrm{cn}v}{1 - m\ \mathrm{sn}^2u\ \mathrm{sn}^2v}, \tag{3.38}$$

where m as the second argument is omitted for it is common to all the functions. As a by-product, the double argument formulas are obtained as

$$\mathrm{sn}\ 2u = \frac{2\mathrm{sn}\ u\ \mathrm{cn}\ u\ \mathrm{dn}\ u}{1 - m\ \mathrm{sn}^4\ u}, \tag{3.39}$$

$$\mathrm{cn}\ 2u = \frac{\mathrm{cn}^2\ u - \mathrm{sn}^2\ u\ \mathrm{dn}^2\ u}{1 - m\ \mathrm{sn}^4\ u}, \tag{3.40}$$

$$\mathrm{dn}\ 2u = \frac{\mathrm{dn}^2\ u - m\ \mathrm{sn}^2\ u\ \mathrm{cn}^2\ u}{1 - m\ \mathrm{sn}^4\ u}, \tag{3.41}$$

which play an important rule in the numerical computation of the Jacobian elliptic functions.

The double argument formulas are inverted as

$$\mathrm{sn}\frac{u}{2} = \sqrt{\frac{1 - \mathrm{cn}\ u}{1 + \mathrm{dn}\ u}}, \tag{3.42}$$

$$\mathrm{cn}\frac{u}{2} = \sqrt{\frac{\mathrm{cn}\ u + \mathrm{dn}\ u}{1 + \mathrm{dn}\ u}}, \tag{3.43}$$

$$\mathrm{dn}\frac{u}{2} = \sqrt{\frac{\mathrm{cn}\ u + \mathrm{dn}\ u}{1 + \mathrm{cn}\ u}}, \tag{3.44}$$

which are called the half-argument formulas. Since the first of them suffers from the cancellation problem when $|u|$ is small, it is rewritten as

$$\mathrm{sn}^2\frac{u}{2} = \frac{\mathrm{sn}^2\ u}{(1 + \mathrm{cn}\ u)(1 + \mathrm{dn}\ u)}, \tag{3.45}$$

which becomes important in the numerical computation of all the incomplete elliptic integrals.

3.7 Jacobi's form of incomplete elliptic integrals

Jacobi regarded incomplete elliptic integrals as functions of Legendre's form of incomplete elliptic integral of the first kind,

$$u \equiv F(\varphi|m) \,. \tag{3.46}$$

In his notation, some incomplete elliptic integrals are expressed as the integrals of rational functions of Jacobian elliptic functions [12, Formulas 110.02 through 110.04]:

$$E_u(u|m) \equiv E(\mathrm{am}(u|m)|m) = \int_0^u \mathrm{dn}^2(v|m)dv \,, \tag{3.47}$$

$$\Pi_u(u,n|m) \equiv \Pi(\mathrm{am}(u|m),n|m) = \int_0^u \frac{dv}{1-n\mathrm{sn}^2(v|m)} \,, \tag{3.48}$$

$$B_u(u|m) \equiv B(\mathrm{am}(u|m)|m) = \int_0^u \mathrm{cn}^2(v|m)dv \,, \tag{3.49}$$

$$D_u(u|m) \equiv D(\mathrm{am}(u|m)|m) = \int_0^u \mathrm{sn}^2(v|m)dv \,, \tag{3.50}$$

$$J_u(u,n|m) \equiv J(\mathrm{am}(u|m),n|m) = \int_0^u \frac{\mathrm{sn}^2(v|m)\; dv}{1-n\mathrm{sn}^2(v|m)} \,, \tag{3.51}$$

where n is called the characteristic.

Although the standard notation is without the suffix u, we dare attach it in order to discriminate them from Legendre's form of incomplete elliptic integrals, $E(\varphi|m)$, $\Pi(\varphi,n|m)$, $B(\varphi|m)$, $D(\varphi|m)$, and $J(\varphi,n|m)$, which will be discussed in Section 4 later, since the meaning of the first argument is different. Among them, $E_u(u|m)$ is known as Jacobi's Epsilon function and sometimes denoted by $\varepsilon(u)$ [62, Section 22.16(ii)]. See Figure 4 plotting its curves in the standard domain of the argument, $0 \le u \le K(m)$, for various values of m. Also, $J_u(u,n|m)$ is tightly related with Jacobi's original incomplete elliptic integral of the third kind $\Pi_1(u)$ [12, p.233]. See Figure 5 for its sketch as a function of u for various values of m when $n = 0.5$. The numerical value of Jacobi's form of incomplete elliptic integrals are efficiently computed by the procedure in our work [42].

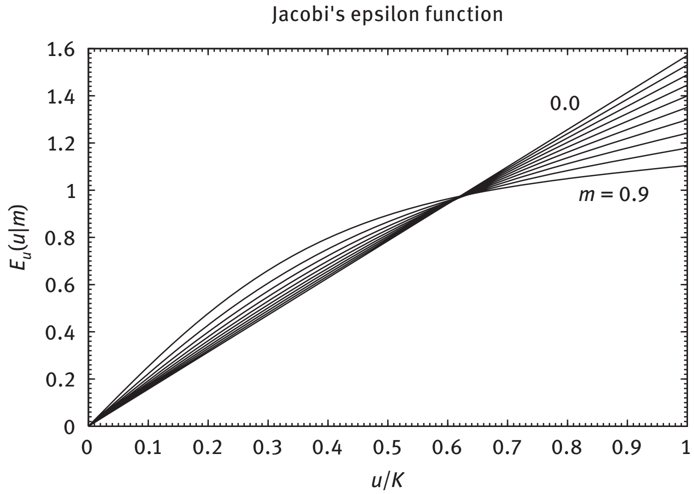

Fig. 4. Sketch of Jacobi's Epsilon function: local. Same as Figure 1 but for $E_u(u|m)$.

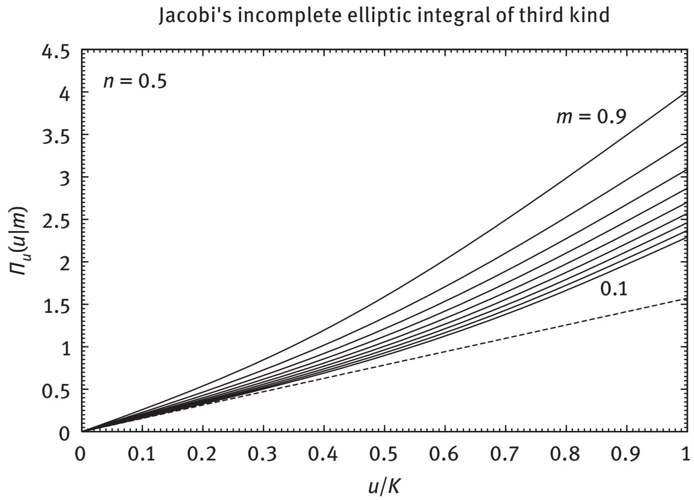

Fig. 5. Sketch of Jacobi's form of incomplete elliptic integral of the third kind: local. Same as Figure 1 but for $\Pi_u(u, n|m)$ when $n = 0.5$. Added is the straight line u plotted by a broken line for reference.

When a set of coordinate variables, $p(u)$, are expressed in terms of the Jacobian elliptic functions, their conjugate momentum variables are sometimes expressed by their integral with respect to u. In that case, these Jacobi's form of incomplete elliptic integrals appear as the fourth-type application of elliptic functions.

3.8 Addition theorem of incomplete elliptic integrals

The addition theorems of Jacobi's form of incomplete elliptic integrals are also available as

$$E_u(u+v) = E_u(u) + E_u(v) - m \,\mathrm{sn}u \,\mathrm{sn}v \,\mathrm{sn}(u+v)\,, \tag{3.52}$$

$$\Pi_u(u+v) = \Pi_u(u) + \Pi_u(v) + nT(t(u,v),h)\,, \tag{3.53}$$

$$B_u(u+v) = B_u(u) + B_u(v) - \,\mathrm{sn}u \,\mathrm{sn}v \,\mathrm{sn}(u+v)\,, \tag{3.54}$$

$$D_u(u+v) = D_u(u) + D_u(v) + \,\mathrm{sn}u \,\mathrm{sn}v \,\mathrm{sn}(u+v)\,, \tag{3.55}$$

$$J_u(u+v) = J_u(u) + J_u(v) + T(t(u,v),h)\,, \tag{3.56}$$

where m and/or n are omitted, for they being common to all the integrals,

$$t(u,v) \equiv \frac{\mathrm{sn}\; u\; \mathrm{sn}\; v\; \mathrm{sn}(u+v)}{1-n\left[\mathrm{sn}^2\;(u+v) - \mathrm{sn}\; u\; \mathrm{sn}\; v\; \mathrm{cn}(u+v)\; \mathrm{dn}(u+v)\right]}\,, \tag{3.57}$$

$$h \equiv nn_c(n-m)\,, \tag{3.58}$$

and $T(t,h)$ is the universal arc tangent function defined [49, Equation (25)] as

$$T(t,h) \equiv t\sum_{j=0}^{\infty} \frac{\left(-ht^2\right)^j}{2j+1} = \begin{cases} \tan^{-1}\left(t\sqrt{h}\right)/\sqrt{h} & (h>0) \\ t & (h=0) \\ \tanh^{-1}\left(t\sqrt{-h}\right)/\sqrt{-h} & (h<0) \end{cases}\,, \tag{3.59}$$

the fast computation of which is described in [49, § 3.7].

As a natural consequence, the double argument formulas are derived from these as

$$E_u(2u) = 2E_u(u) - m\;\mathrm{sn}^2 u\;\mathrm{sn}2u\,, \tag{3.60}$$

$$\Pi_u(2u) = 2\Pi_u(u) + nT(t(u,u),h)\,, \tag{3.61}$$

$$B_u(2u) = 2B_u(u) - \;\mathrm{sn}^2 u\;\mathrm{sn}2u\,, \tag{3.62}$$

$$D_u(2u) = 2D_u(u) + \;\mathrm{sn}^2 u\;\mathrm{sn}2u\,, \tag{3.63}$$

$$J_u(2u) = 2J_u(u) + T(t(u,u),h)\,, \tag{3.64}$$

where $t(u,u)$ is simplified as

$$t(u,u) = \frac{\mathrm{sn}^2\; u\; \mathrm{sn}2u}{1-n\left[\mathrm{sn}^2\; 2u - \mathrm{sn}^2\; u\; \mathrm{cn}2u\; \mathrm{dn}2u\right]}\,, \tag{3.65}$$

These formulas are useful in the numerical computation of these integrals. Among the above formulas, the most fundamental are those of J_u since (1) those of B_u are obtained from those of E_u by letting $m = 1$, (2) those of J_u are obtained from those of Π_u by letting $n = 1$, (3) those of D_u are obtained from those of J_u by letting $n = 0$. and (4) those of B_u and D_u are practically the same since $B_u(u) + D_u(u) = u$.

3.9 Jacobi's original form of incomplete elliptic integral of the third kind

In place of $\Pi_u(u, n|m)$, Jacobi preferred a different form as the incomplete elliptic integral of the third kind [58, p.420]:

$$\Pi_1(u, a|m) \equiv m \operatorname{sn}(a|m) \operatorname{cn}(a|m) \operatorname{dn}(a|m) \int_0^u \frac{\operatorname{sn}^2(v|m)\, dv}{1 - m \operatorname{sn}^2(a|m) \operatorname{sn}^2(v|m)}, \quad (3.66)$$

where a is a variable replacing n such that

$$n = m \operatorname{sn}^2(a|m) . \quad (3.67)$$

Although it is denoted by Π_1, the integral is essentially the same as $J_u(u, n|m)$ since

$$\Pi_1(u, a|m) = m \operatorname{sn}(a|m) \operatorname{cn}(a|m) \operatorname{dn}(a|m)\ J_u(u, m\operatorname{sn}^2(a|m)|m) . \quad (3.68)$$

The complete elliptic integral is similarly written as

$$\Pi_1(a|m) \equiv \Pi_1(K(m), a|m) = m \operatorname{sn}(a|m) \operatorname{cn}(a|m) \operatorname{dn}(a|m)\ J_u(m\operatorname{sn}^2(a|m)|m) . \quad (3.69)$$

The usefulness of this form is in the existence of the not-so-popular addition theorem with respect to a [58, p.428]:

$$\Pi_1(u, a + b|m) = \Pi_1(u, a|m) + \Pi_1(u, b|m) - m\, u \operatorname{sn}(a|m) \operatorname{sn}(b|m) \operatorname{sn}(a + b|m) - \frac{1}{2} \log\left(\frac{1 - m \operatorname{sn}(a|m) \operatorname{sn}(b|m) \operatorname{sn}(u|m) \operatorname{sn}(a + b - u|m)}{1 + m \operatorname{sn}(a|m) \operatorname{sn}(b|m) \operatorname{sn}(u|m) \operatorname{sn}(a + b + u|m)} \right) . \quad (3.70)$$

The complete version becomes

$$\Pi_1(a+b|m) = \Pi_1(a|m) + \Pi_1(b|m) - m\, K(m) \operatorname{sn}(a|m) \operatorname{sn}(b|m) \operatorname{sn}(a+b|m) . \quad (3.71)$$

As a result, the double argument formula with respect to a is derived as

$$\Pi_1(2a|m) = 2\Pi_1(a|m) - m\, K(m) \operatorname{sn}^2(a|m) \operatorname{sn}(2a|m) . \quad (3.72)$$

This becomes useful in computing $J(n|m)$ efficiently [53].

4 Elliptic integrals

4.1 General elliptic integral

An elliptic integral was naturally introduced in discussing an integral of a general function $g(z)$ as

$$G(x) = \int_a^x g(z)\, dz . \quad (4.1)$$

If $g(z)$ is a rational function, $G(x)$ becomes a sum of a rational function and a product of the logarithm and another rational function. The next simplest case is to add a single square root of a polynomial as its irrational part:

$$G(x) = \int_a^x g\left(z; \sqrt{f(z)}\right) dz \,. \tag{4.2}$$

If $f(z)$ is a polynomial of degree 2 at most, the resulting integral contains the inverse trigonometric function and the logarithm.

The difficulty increases for higher degrees such as when encountered in investigating an arc length of a general quadratic curve including an ellipse or a hyperbola. If $f(z)$ is a polynomial of degree 4 at most as described in Equation (3.4), $G(x)$ is called a general elliptic integral. Since g is a rational function of z when f is regarded as a constant, $G(x)$ can be split into a sum of a rational function $R(x)$ and a special kind of elliptic integral as

$$G(x) = R(x) + \int_a^x \frac{P(z)}{Q(z)\sqrt{f(z)}} dz \,, \tag{4.3}$$

where $P(z)$ and $Q(z)$ are polynomials.

4.2 Legendre's form of incomplete elliptic integrals

It is very complicated to simplify the general elliptic integral, $G(x)$. See the compact reduction tables in terms of the symmetric integrals [17–21]. Legendre finally succeeded to resolve its elliptic part into a linear combination of three fundamental integrals [62, Section 19.2]:

$$F(\varphi|m) \equiv \int_0^\varphi \frac{d\theta}{\Delta(\theta|m)} \,, \tag{4.4}$$

$$E(\varphi|m) \equiv \int_0^\varphi \Delta(\theta|m)\, d\theta \,, \tag{4.5}$$

$$\Pi(\varphi, n|m) \equiv \int_0^\varphi \frac{d\theta}{(1 - n\sin^2\theta)\,\Delta(\theta|m)} \,, \tag{4.6}$$

where n is a parameter called the characteristic. They are called Legendre's form of incomplete elliptic integral of the first, the second, and the third kind, respectively. These are termed "incomplete" since the upper end point of the integral is general. Figures 6 and 7 show the curves of $F(\varphi|m)$ and $E(\varphi|m)$ as functions of φ for various values of m.

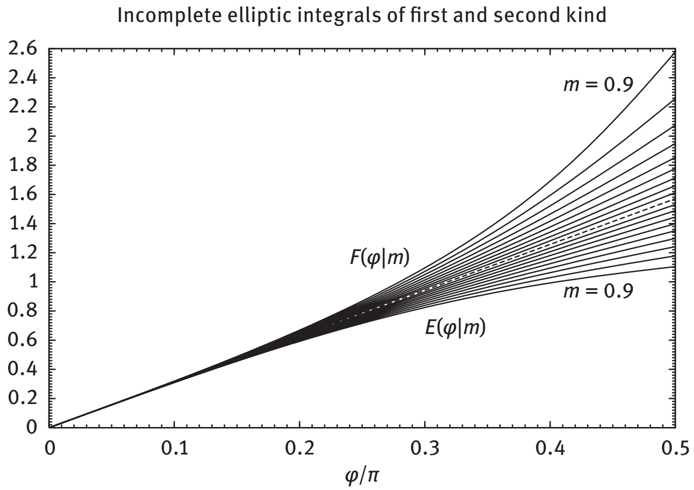

Fig. 6. Sketch of Legendre's form of incomplete elliptic integrals: local. Plotted are the graphs of $F(\varphi|m)$ and $E(\varphi|m)$ as functions of φ in its standard domain, $0 \le \varphi \le \pi/2$ for various values of m as $m = 0.1, 0.2, \ldots, 0.9$. Notice inequalities $E(\varphi|0.9) < E(\varphi|0.8) < \cdots < F(\varphi|0.8) < F(\varphi|0.9)$. The results when $m = 0$ become the same straight line, $F(\varphi|0) = E(\varphi|0) = \varphi$, which is shown by a broken line.

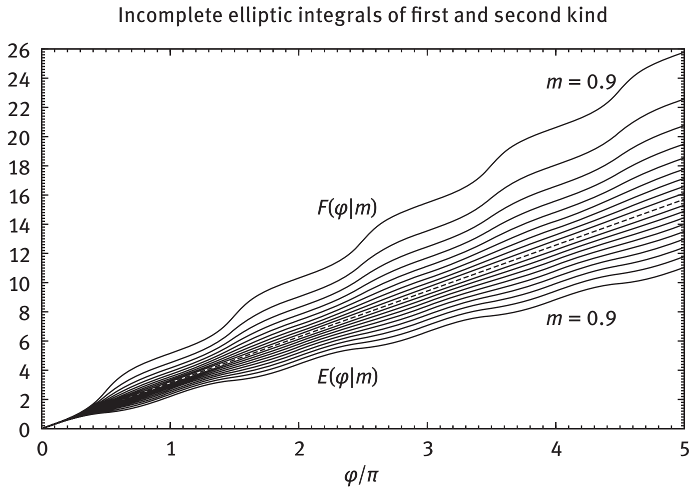

Fig. 7. Sketch of Legendre's form of incomplete elliptic integrals: global. Same as Figure 6 but for a longer period.

4.3 Associate incomplete elliptic integrals

In many applications, Legendre's form of incomplete elliptic integrals, $F(\varphi|m)$, $E(\varphi|m)$, and/or $\Pi(\varphi, n|m)$ are used in combination. For example, the length of hyperbola requires both of $F(\varphi|m)$ and $E(\varphi|m)$ [12, Introduction]. Also, the rotation angle of the torque-free rotation of a rigid body is described by a linear combination of $F(\varphi|m)$ and $\Pi(\varphi, n|m)$ [39]. Further, both of $F(\varphi|m)$ and $E(\varphi|m)$ are needed in computing their partial derivatives with respect to m [62]. These partial derivatives are necessary in computing the force field of potentials expressed using $F(\varphi|m)$. Similar needs arise in evaluating the partial derivatives of Jacobian elliptic functions with respect to u and/or m [12].

From a practical viewpoint, however, rather important is not $F(\varphi|m)$, $E(\varphi|m)$, and $\Pi(\varphi, n|m)$ but a trio of associate incomplete elliptic integrals [8, 48, 49]:

$$B(\varphi|m) \equiv \int_0^{\varphi} \frac{\cos^2\theta \, d\theta}{\Delta(\theta|m)} \,, \tag{4.7}$$

$$D(\varphi|m) \equiv \int_0^{\varphi} \frac{\sin^2\theta \, d\theta}{\Delta(\theta|m)} \,, \tag{4.8}$$

$$J(\varphi, n|m) \equiv \int_0^{\varphi} \frac{\sin^2\theta \, d\theta}{(1 - n\sin^2\theta)\,\Delta(\theta|m)} \,. \tag{4.9}$$

The practical usefulness of $B(\varphi|m)$, $D(\varphi|m)$, and $J(\varphi, n|m)$ are well described [8, 47–49, 52, 53]. See Figures 8 and 9 for the plots of $J(\varphi, n|m)$ as function of φ for various values of m when n is fixed as $n = 0.5$.

The numerical values of these incomplete elliptic integrals are effectively computed by the following procedures: (1) $F(\varphi|m)$ by `elf` [45], (2) $B(\varphi|m)$ and $D(\varphi|m)$ simultaneously by `elbd` [48], and (3) $B(\varphi|m)$, $D(\varphi|m)$, and $J(\varphi, n|m)$ simultaneously by `elbdj` [49]. The last procedure, `elbdj`, is the most general one since $F(\varphi|m)$, $E(\varphi|m)$, and $\Pi(\varphi, n|m)$ are computed from $B(\varphi|m)$, $D(\varphi|m)$, and $J(\varphi, n|m)$ without suffering from the precision loss as

$$F(\varphi, n|m) = B(\varphi|m) + D(\varphi|m) \,, \tag{4.10}$$

$$E(\varphi, n|m) = B(\varphi|m) + m_c D(\varphi|m) \,, \tag{4.11}$$

$$\Pi(\varphi, n|m) = B(\varphi|m) + D(\varphi|m) + nJ(\varphi, n|m) \,, \tag{4.12}$$

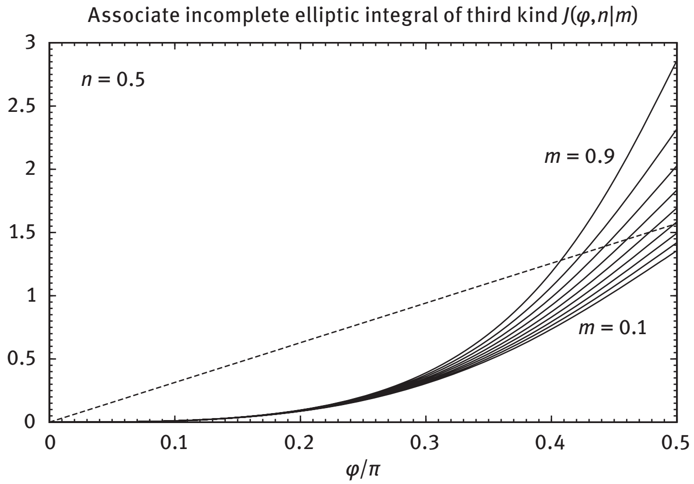

Fig. 8. Sketch of associate incomplete elliptic integral of the third kind: local. Same as Figure 6 but for $J(\varphi, n|m)$ when $n = 0.5$. For reference, a straight line φ is shown by a broken line again.

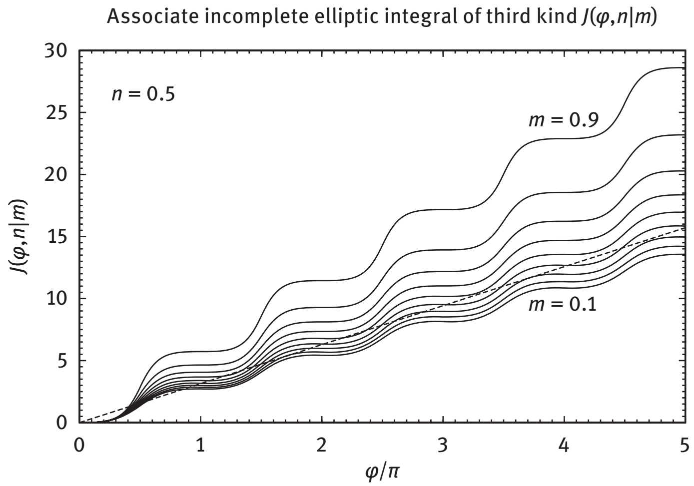

Fig. 9. Sketch of associate incomplete elliptic integral of the third kind: global. Same as Figure 8 but for a longer period.

respectively. The reverse procedures face with the problem of small divisors, m and n as

$$B(\varphi|m) = \frac{E(\varphi|m) - m_c F(\varphi|m)}{m}, \tag{4.13}$$

$$D(\varphi|m) = \frac{F(\varphi|m) - E(\varphi|m)}{m}, \tag{4.14}$$

$$J(\varphi, n|m) = \frac{\Pi(\varphi, n|m) - F(\varphi|m)}{n}. \tag{4.15}$$

This is problematic in the actual applications since

$$m \approx +1 \times 10^{-7}, \quad n \approx -7 \times 10^{-4}, \tag{4.16}$$

in the case of the Earth rotation [40, 41].

4.4 Complete elliptic integrals

The incomplete elliptic integrals are called "complete" when $\varphi = \pi/2$:

$$K(m) \equiv F(\pi/2|m) = \int_0^{\pi/2} \frac{d\theta}{\Delta(\theta|m)}, \tag{4.17}$$

$$E(m) \equiv E(\pi/2|m) = \int_0^{\pi/2} \Delta(\theta|m)\, d\theta, \tag{4.18}$$

$$\Pi(n|m) \equiv \Pi(\pi/2, n|m) = \int_0^{\pi/2} \frac{d\theta}{(1 - n\sin^2\theta)\,\Delta(\theta|m)}, \tag{4.19}$$

$$B(m) \equiv B(\pi/2|m) = \int_0^{\pi/2} \frac{\cos^2\theta\, d\theta}{\Delta(\theta|m)}, \tag{4.20}$$

$$D(m) \equiv D(\varphi|m) = \int_0^{\pi/2} \frac{\sin^2\theta\, d\theta}{\Delta(\theta|m)}, \tag{4.21}$$

$$J(n|m) \equiv J(\pi/2, n|m) = \int_0^{\pi/2} \frac{\sin^2\theta\, d\theta}{(1 - n\sin^2\theta)\,\Delta(\theta|m)}. \tag{4.22}$$

The numerical values of these complete integrals are effectively computed by the following procedures: (1) $K(m)$ and $E(m)$ by `celk` and `cele`, respectively [45], (2) $B(m)$ and $D(m)$ simultaneously by `celbd` [48], and (3) $B(m)$, $D(m)$, and $J(n|m)$ simultaneously by `celbdj` [51]. The last procedure `celbdj` is the most general one since $K(m)$, $E(m)$, and $\Pi(n|m)$ are computed from $B(m)$, $D(m)$, and $J(n|m)$ without suffering from the loss of information as

$$K(m) = B(m) + D(m), \tag{4.23}$$

$$E(m) = B(m) + m_c D(m), \tag{4.24}$$

$$\Pi(n|m) = B(m) + D(m) + nJ(n|m). \tag{4.25}$$

During the course of investigation of the gravitational acceleration field due to a uniform ring, we faced with the round-off errors even if using $B(m)$ and $D(m)$. To resolve

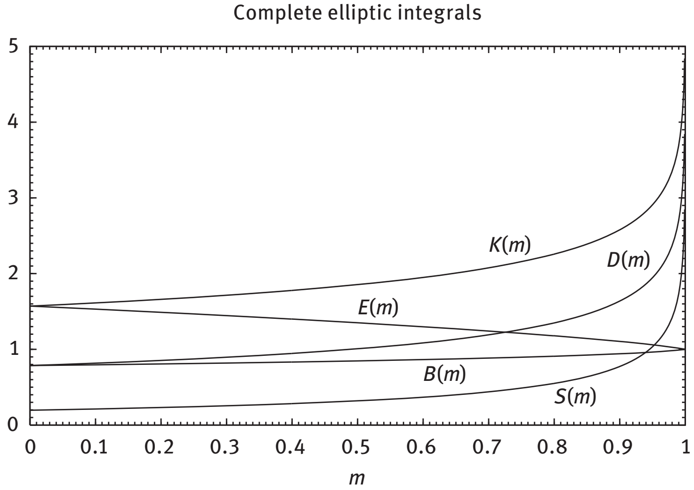

Fig. 10. Sketch of five complete elliptic integrals. Illustrated are the graphs of five complete elliptic integrals, $K(m)$, $E(m)$, $B(m)$, $D(m)$, and $S(m)$ for m in its standard domain $0 \le m < 1$.

it, a special complete elliptic integral is introduced [46] as

$$S(m) \equiv \frac{-1}{m} \int_0^{\pi/2} \frac{\cos 2\theta \, d\theta}{\Delta(\theta|m)} = \frac{D(m) - B(m)}{m} = \frac{(2-m)K(m) - 2E(m)}{m^2} . \qquad (4.26)$$

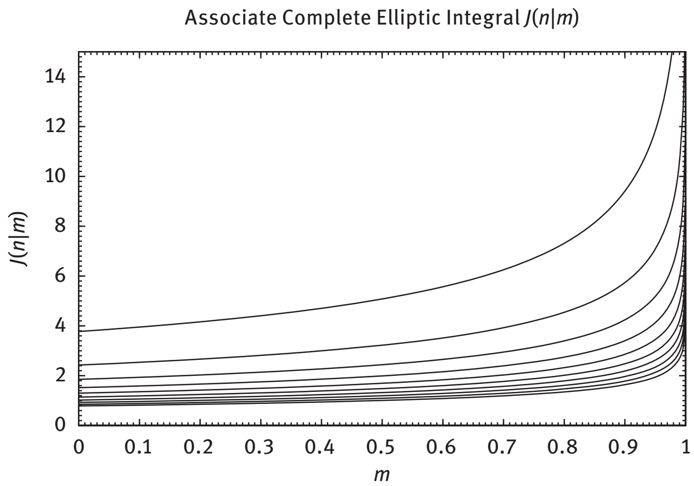

Fig. 11. Sketch of $J(n|m)$. Same as Figure 10 but of $J(n|m)$ for various values of n as $n = 0.0, 0.1, \ldots, 0.9$.

The pair of $B(m)$ and $S(m)$ is more fundamental than that of $B(m)$ and $D(m)$ since $D(m)$ is computed from them without information loss as

$$D(m) = B(m) + mS(m) . \tag{4.27}$$

Figure 10 illustrates the behavior of these five complete elliptic integrals as functions of m in the standard domain, $0 \le m < 1$. Also, Figure 11 shows the curves of $J(n|m)$ as functions of m for various values of n in the standard domain, $0 \le n < 1$.

4.5 Generalized elliptic integrals

From a practical viewpoint of numerical computation, Bulirsch generalized Legendre's form of elliptic integrals by changing the main variable from φ to $x \equiv \tan\varphi$ and adding a few parameters [8, 11]. Bulirsch's form of incomplete elliptic integrals of the first, second, and third kind are defined as

$$\texttt{el1}\left(x, k_c\right) \equiv \int_0^x \frac{d\xi}{\sqrt{\left(1+\xi^2\right)\left(1+k_c^2\xi^2\right)}} = \int_0^\varphi \frac{d\theta}{\sqrt{\cos^2\theta + k_c^2\sin^2\theta}} , \tag{4.28}$$

$$\texttt{el2}\left(x, k_c, a, b\right) \equiv \int_0^x \frac{a+b\xi^2}{\sqrt{\left(1+\xi^2\right)\left(1+k_c^2\xi^2\right)}} d\xi = \int_0^\varphi \frac{a\cos^2\theta + b\sin^2\theta}{\sqrt{\cos^2\theta + k_c^2\sin^2\theta}} d\theta , \tag{4.29}$$

$$\begin{aligned}\texttt{el3}\left(x, k_c, p\right) &\equiv \int_0^x \frac{d\xi}{\left(1+p\xi^2\right)\sqrt{\left(1+\xi^2\right)\left(1+k_c^2\xi^2\right)}} \\ &= \int_0^\varphi \frac{d\theta}{\left(\cos^2\theta + p\sin^2\theta\right)\sqrt{\cos^2\theta + k_c^2\sin^2\theta}} ,\end{aligned} \tag{4.30}$$

where

$$k_c \equiv \sqrt{m_c} = \sqrt{1-m} , \tag{4.31}$$

is the complementary modulus. Their computing procedures are described in [8, 9, 11]. Originally, he intended to generalize these integrals further [10, Equation (4.1.14)]:

$$\begin{aligned}\texttt{el}\left(x, k_c, p, a, b\right) &\equiv \int_0^x \frac{a+b\xi^2}{\left(1+p\xi^2\right)\sqrt{\left(1+\xi^2\right)\left(1+k_c^2\xi^2\right)}} d\xi \\ &= \int_0^\varphi \frac{a\cos^2\theta + b\sin^2\theta}{\left(\cos^2\theta + p\sin^2\theta\right)\sqrt{\cos^2\theta + k_c^2\sin^2\theta}} ,\end{aligned} \tag{4.32}$$

since it covers all of the incomplete elliptic integrals of three kinds as

$$\mathtt{el1}\left(x, k_c\right) = \mathtt{el}\left(x, k_c, p, 1, p\right), \quad (p\colon \text{arbitrary}) \tag{4.33}$$

$$\mathtt{el2}\left(x, k_c, a, b\right) = \mathtt{el}\left(x, k_c, 1, a, b\right), \tag{4.34}$$

$$\mathtt{el3}\left(x, k_c, p\right) = \mathtt{el}\left(x, k_c, p, 1, 1\right). \tag{4.35}$$

Bulirsch did not provide its computing algorithm. It was later presented by us [56]. At any rate, Bulirsch's form of incomplete elliptic integrals are related with Legendre's and the associate incomplete elliptic integrals as

$$F(\varphi|m) = \mathtt{el1}\left(\tan\varphi, k_c\right), \tag{4.36}$$

$$E(\varphi|m) = \mathtt{el2}\left(\tan\varphi, k_c, 1, m_c\right), \tag{4.37}$$

$$B(\varphi|m) = \mathtt{el2}\left(\tan\varphi, k_c, 1, 0\right), \tag{4.38}$$

$$D(\varphi|m) = \mathtt{el2}\left(\tan\varphi, k_c, 0, 1\right), \tag{4.39}$$

$$\Pi(\varphi, n|m) = \mathtt{el3}\left(\tan\varphi, k_c, n_c\right), \tag{4.40}$$

$$J(\varphi, n|m) = \mathtt{el}\left(\tan\varphi, k_c, n_c, 0, 1\right), \tag{4.41}$$

where

$$n_c \equiv 1 - n, \tag{4.42}$$

is the complementary characteristic. Unfortunately, Bulirsch's algorithms to compute el2, el3, and el numerically suffer from the loss of precision when $|\varphi|$ is small [48, 49]. Therefore, we do not recommend their usage in the practical computations.

Bulirsch's general form of complete integrals of the first, the second, and the third kind are defined as

$$\mathtt{cel1}\left(k_c\right) \equiv \int_0^\infty \frac{d\xi}{\sqrt{\left(1+\xi^2\right)\left(1+k_c^2\xi^2\right)}}, \tag{4.43}$$

$$\mathtt{cel2}\left(k_c, a, b\right) \equiv \int_0^\infty \frac{a + b\xi^2}{\sqrt{\left(1+\xi^2\right)\left(1+k_c^2\xi^2\right)}} d\xi, \tag{4.44}$$

$$\mathtt{cel3}\left(k_c, p\right) \equiv \int_0^\infty \frac{d\xi}{\left(1+p\xi^2\right)\sqrt{\left(1+\xi^2\right)\left(1+k_c^2\xi^2\right)}} d\xi. \tag{4.45}$$

Bulirsch succeeded to generalize these as

$$\mathtt{cel}\left(k_c, p, a, b\right) \equiv \int_0^\infty \frac{a + b\xi^2}{\left(1+p\xi^2\right)\sqrt{\left(1+\xi^2\right)\left(1+k_c^2\xi^2\right)}} d\xi, \tag{4.46}$$

These Bulirsch's form of complete elliptic integrals are related with Legendre's and the associate complete elliptic integrals as

$$K(m) = \texttt{cel1}\left(k_c\right) , \tag{4.47}$$

$$E(m) = \texttt{cel2}\left(k_c, 1, m_c\right) , \tag{4.48}$$

$$B(m) = \texttt{cel2}\left(k_c, 1, 0\right) , \tag{4.49}$$

$$D(m) = \texttt{cel2}\left(k_c, 0, 1\right) , \tag{4.50}$$

$$\Pi(n|m) = \texttt{cel3}\left(k_c, n_c\right) , \tag{4.51}$$

$$J(n|m) = \texttt{cel}\left(k_c, n_c, 0, 1\right) . \tag{4.52}$$

4.6 Symmetric elliptic integrals

Carlson reconstructed the theory of elliptic integrals by introducing their symmetric forms [62, Section 19.16 (i)]:

$$R_F(x, y, z) \equiv \frac{1}{2} \int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}} . \tag{4.53}$$

$$R_D(x, y, z) \equiv \frac{3}{2} \int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)^3}} = R_J(x, y, z, z) , \tag{4.54}$$

$$R_J(x, y, z, p) \equiv \frac{3}{2} \int_0^\infty \frac{dt}{(t+p)\sqrt{(t+x)(t+y)(t+z)}} . \tag{4.55}$$

As its subprocedures, R_J requires R_F and a nonelliptic symmetric integral defined as

$$R_C(x, y) \equiv \frac{1}{2} \int_0^\infty \frac{dt}{(t+y)\sqrt{t+x}} = R_F(x, y, y) . \tag{4.56}$$

Legendre's form of incomplete elliptic integrals and the associate incomplete elliptic integrals are expressed in terms of these symmetric integrals as

$$F(\varphi|m) = sR_F\left(c^2, d^2, 1\right) , \tag{4.57}$$

$$B(\varphi|m) = \frac{1}{3} m_c s^3 R_D\left(c^2, 1, d^2\right) + \frac{sc}{d} , \tag{4.58}$$

$$D(\varphi|m) = \frac{1}{3} s^3 R_D\left(c^2, d^2, 1\right) , \tag{4.59}$$

$$J(\varphi, n|m) = \frac{1}{3} s^3 R_J\left(c^2, d^2, 1, 1 - ns^2\right) , \tag{4.60}$$

where

$$s \equiv \sin\varphi\,, \tag{4.61}$$

$$c \equiv \cos\varphi\,, \tag{4.62}$$

$$d \equiv \sqrt{1 - ms^2}\,. \tag{4.63}$$

are nothing but the three principal Jacobian elliptic functions if φ is regarded as Jacobi's amplitude function. Similarly, the complete elliptic integrals are expressed in terms of the symmetric integrals as

$$K(m) = R_F\left(0, m_c, 1\right)\,, \tag{4.64}$$

$$B(m) = \frac{1}{3} m_c R_D\left(0, 1, m_c\right)\,, \tag{4.65}$$

$$D(m) = \frac{1}{3} R_D\left(0, m_c, 1\right)\,, \tag{4.66}$$

$$J(n|m) = \frac{1}{3} R_J\left(0, m_c, 1, n_c\right)\,. \tag{4.67}$$

5 Numerical computation of elliptic functions and elliptic integrals

5.1 Overview

Although the detailed theoretical and some numerical aspects of the computation of elliptic functions and elliptic integrals had been well discussed from the early days of Euler, Landen, and Gauss, it is only after Bulirsch's pioneer works [8–11] when their precise numerical computation becomes available [63]. Nevertheless, some of his procedures computing elliptic integrals turned out to face with a loss of precision in practical applications.

Then, Carlson made a breakthrough in the theory of elliptic integrals by introducing their symmetric forms [13–16, 22]. His duplication method [24] is free from the precision loss, and therefore, has been regarded as the standard numerical method [64]. Nevertheless, Carlson's procedures are fairly slow as will be seen later.

Therefore, we have conducted a series of studies [42, 44, 45, 47–54] to develop the algorithms computing various elliptic functions, elliptic integrals, and their inverses. The obtained procedures `scd2`, `elf`, `elbd`, `elbdj`, `celk`, `celbd`, and `celbdj` are (1) greatly faster than Carlson's procedures `rf`, `rd`, and `rj`, (2) significantly faster than Bulirsch's procedures `sncndn`, `el1`, `cel1`, `cel2`, and `cel`, (3) without suffering from the precision loss experienced by Bulirsch's algorithms `el2`, `el3`, and our implementation of `el` [56], and (4) as precise as the best existing procedures.

5.2 Transformation method

All the methods of Bulirsch, Carlson, and ours belong to the same category of computing method: the transformation method. Its spirit is similar to the canonical transformation method in the analytical dynamics. Let us explain its essence.

Consider the computation of a given function

$$p = f(q) \,. \tag{5.1}$$

Regard the pair of the function value and the argument, (p, q), as a sort of "conjugate" pair of variables. Notice that the dimension of p and q are not the same in general. For example, in the case of the simultaneous computation of the principal Jacobian elliptic functions, the dimension of p is three as $p = (s, c, d)$ while that of q is two as $q = (u, m)$.

Assume that there exists a one-to-one transformation of (p, q) as

$$p^* = P^*(p, q) \,, \quad q^* = Q^*(p, q) \,, \tag{5.2}$$

$$p = P(p^*, q^*) \,, \quad q = Q(p^*, q^*) \,, \tag{5.3}$$

such that the transformed pair satisfies the same functional relation as

$$p^* = f(q^*) \,. \tag{5.4}$$

If the direct evaluation of $f(q)$ is more difficult than that of $f(q^*)$, a method of computation of $f(q)$ is constructed as a three-step-method based on the transformation as

(1) Forward transformation of the argument: Equation (5.2)
(2) Direct evaluation of the function value for the transformed argument: Equation (5.4)
(3) Backward transformation of the obtained function value: Equation (5.3)

In general, the difficulty of the direct computation is not sufficiently reduced by a single application of the transformation. In that case, a series of transformations are applied in both the forward and the backward directions such as

(1) Forward stage: repeat the execution of Equation (5.2) until Equation (5.4) is easily computed.
(2) Evaluation stage: compute Equation (5.4).
(3) Backward stage: repeat the execution of Equation (5.3) as many as in the forward stage.

From the computational viewpoint, the transformation method is effective only when the computational labor of the direct computation of the untransformed argument is significantly larger than the sum of the computational labor of these three stages.

5.3 Example of transformation method

Let us show a simple example of the transformation method of the function computation. Consider the computation of the exponential function, $p = e^q$. An example of suitable transformations in this case is the half-argument transformation:

$$p^* = \sqrt{p} , \quad q^* = q/2 , \tag{5.5}$$

$$p = (p^*)^2 , \quad q = 2q^* . \tag{5.6}$$

If $|q|$ is sufficiently small, e^q is precisely and quickly computed by its Maclaurin series

$$e^q = 1 + q + \frac{q^2}{2} + \frac{q^3}{3!} + \cdots \tag{5.7}$$

Thus, the resulting algorithm of computation becomes
(1) Halve the argument repeatedly until its magnitude becomes sufficiently small.
(2) Evaluate the function value of the reduced argument by the truncated Maclaurin series.
(3) Repeat squaring the obtained function value as many times as the number of half-argument transformations.

Notice that only the forward transformation of q and the backward transformation of p are needed. The computational labor of both of them are small: one multiplication each. Therefore, except the aspect of the computational error accumulation, the transformation method can be fairly efficient.

5.4 Simultaneous computation of Jacobian elliptic functions

Begin with the simultaneous computation of the three principal Jacobian elliptic functions,

$$s \equiv \mathrm{sn}(u|m) , \quad c \equiv \mathrm{cn}(u|m) , \quad d \equiv \mathrm{dn}(u|m) . \tag{5.8}$$

Once s and c are obtained, Jacobi's amplitude function can be evaluated from them as

$$\mathrm{am}(u|m) = \mathrm{atan2}(s, c) , \tag{5.9}$$

where $\mathrm{atan2}(y, x)$ is the two-argument arc tangent function computing $\tan^{-1}(y/x)$ while taking the sign of arguments into account. Therefore, its discussion will be omitted hereafter.

A simple forward transformation is the half-argument transformation [42, 44]. It is a transformation halving the argument u as

$$u^* = \frac{u}{2} , \tag{5.10}$$

while keeping the parameter m the same. When $|u|$ becomes sufficiently small, the function values s, c, and d are evaluated by the Maclaurin series expansion of the Jacobian elliptic functions as

$$s = u - \frac{1+m}{6}u^3 + \frac{1+14m+m^2}{120}u^5 - \cdots , \tag{5.11}$$

$$c = 1 - \frac{1}{2}u^2 + \frac{1+4m}{24}u^4 - \cdots , \tag{5.12}$$

$$d = 1 - \frac{m}{2}u^2 + \frac{4m+m^2}{24}u^4 - \cdots . \tag{5.13}$$

The expansion coefficients for up to the term u^{16} are explicitly given in Table 2 of Fukushima [42]. Those of higher order ones are easily obtained by recursion [54].

The corresponding backward transformation is the double argument transformation of s, c, and d, Equations (3.39) through (3.41), as

$$s = \frac{2s^* c^* d^*}{1 - m\,(s^*)^4} , \tag{5.14}$$

$$c = \frac{(c^*)^2 - (s^*)^2\,(d^*)^2}{1 - m\,(s^*)^4} , \tag{5.15}$$

$$d = \frac{(d^*)^2 - m\,(s^*)^2\,(c^*)^2}{1 - m\,(s^*)^4} . \tag{5.16}$$

Since the algorithm needs only the four arithmetic operations, it runs fairly fast. This is the reason why the present algorithm using a linear transformation is faster than Bulirsch's algorithm based on the Landen transformation, which must be fast in principle since it is quadratic but requires the costly operations such as the square root. The resulting procedure is named `scd` [44].

5.5 Better computation of Jacobian elliptic functions

Later, the simultaneous computation scheme described in the previous subsection turns out to suffer from the accumulation of round-off errors in the course of successive application of the backward transformation. Then, another method is developed [52]. As the new main variable, we adopted the one's complement of c defined as

$$b \equiv 1 - c . \tag{5.17}$$

Its Maclaurin series is practically the same as that of c as

$$b = \frac{1}{2}u^2 - \frac{1+4m}{24}u^4 + \cdots . \tag{5.18}$$

while its double argument transformation is expressed as

$$b = \frac{2y^* \left(1 - my^*\right)}{1 - m\left(y^*\right)^2} \,, \tag{5.19}$$

where y is an auxiliary variable defined and computed as

$$y \equiv s^2 = b(2 - b) \,. \tag{5.20}$$

When both of y and m become sufficiently large, the computation of b may face with the information loss in the process to compute $1-my^*$. In that case, the main variable is switched from b to c. Its double argument transformation is rewritten as

$$c = \frac{\left(mx^* + 2m_c\right)x^* - m_c}{m_c + mx^* \left(2 - x^*\right)} \,, \tag{5.21}$$

where x is another auxiliary variable defined as

$$x \equiv c^2 \,. \tag{5.22}$$

Once the backward transformation is finished, the corresponding value of s, c, and d are computed from b and y as

$$s = \sqrt{y} \,, \tag{5.23}$$

$$c = 1 - b \,, \tag{5.24}$$

$$d = \sqrt{1 - my} \,. \tag{5.25}$$

When the main variable is switched, s and d are computed from c and x as

$$s = \sqrt{1 - x} \,, \tag{5.26}$$

$$d = \sqrt{m_c + mx} \,. \tag{5.27}$$

Thus, revised procedure is named `scd2` [51]. Although it calls two square roots in the final step, the number of main variable is reduced from three to one when compared with `scd` such that the total amount of computational labor significantly reduced [51].

5.6 Computation of Jacobi's form of incomplete elliptic integrals

The double argument transformation is also effective in computing Jacobi's form of incomplete elliptic integrals [42]. Abbreviate them simply as

$$E \equiv E_u(u|m), \quad \Pi \equiv \Pi_u(u,n|m), \quad B \equiv B_u(u|m), \quad D \equiv D_u(u|m), \quad J \equiv J_u(u,n|m). \tag{5.28}$$

As remarked earlier, E and Π are computed from B, D, and J. Also, there is a relation between B and D as

$$B + D = u \,. \tag{5.29}$$

Therefore, the more fundamental are J and either B or D. Between B and D, we prefer D since it becomes the smaller when u is small, and therefore the less erroneous in computing the Maclaurin series. Notice that the computation of J includes that of D because

$$J_u(u, 0|m) = D_u(u|m) , \tag{5.30}$$

by definition. In this sense, the computation of J is the most fundamental. At any rate, the forward transformation is the same: $u^* = u/2$. The Maclaurin series of J becomes

$$J = \frac{1}{3}u^3 - \frac{1+m-3n}{15}u^5 + \cdots , \tag{5.31}$$

the coefficients of which are explicitly listed in Table 2 of Fukushima [42]. The higher order coefficients can be recursively obtained [54]. Meanwhile, the double argument transformation of J is expressed as

$$J = 2J^* + T(t, h) . \tag{5.32}$$

Once D and J are computed, the other integrals are obtained as

$$E = u - mD , \tag{5.33}$$

$$\Pi = u + nJ , \tag{5.34}$$

$$B = u - D . \tag{5.35}$$

5.7 Computation of Legendre's form of incomplete elliptic integral of the first kind

The computational algorithm of $F(\varphi|m)$ is easily derived from that of Jacobian elliptic functions by reversing the relationship of the argument and the function value [45]. Regard $u \equiv F(\varphi|m)$ as a function of the main variable

$$y \equiv \sin^2 \varphi , \tag{5.36}$$

while m is fixed. Then, the forward transformation becomes the half-argument transformation of y, Equation (3.45), as

$$y^* = \frac{y}{(1+c)(1+d)} , \tag{5.37}$$

where c and d are computed from y as

$$c = \sqrt{1-y} , \tag{5.38}$$

$$d = \sqrt{1-my} . \tag{5.39}$$

Meanwhile, the Maclaurin series of u with respect to $s = \sqrt{y}$ becomes

$$u = s + \frac{1+m}{6}s^3 + \frac{3+2m+3m^2}{40}s^5 + \cdots . \tag{5.40}$$

The analytical expression of the expansion coefficients are known and the first 13 coefficients are explicitly given [45, Section 2.3]. On the other hand, the backward transformation is as simple as

$$u = 2u^* . \tag{5.41}$$

Thus, the transformation method to compute $F(\varphi|m)$ is completed.

The above forward transformation has a chance of information loss when y and m are not so small initially. In that case, another main variable is adopted as

$$x \equiv \cos^2\varphi , \tag{5.42}$$

which is assumed to be sufficiently small. This time, the forward transformation becomes

$$x^* = \frac{c+d}{1+d} , \tag{5.43}$$

where c and d are computed from x as

$$c = \sqrt{x} , \tag{5.44}$$

$$d = \sqrt{m_c + mx} . \tag{5.45}$$

Since the half-argument transformation inflates the magnitude of x, soon the computation of x faces with the loss of information. In that case, the main variables is switched to $y = 1 - x$ and the half-argument transformation of y is continued.

5.8 Computation of other incomplete elliptic integrals

Once the computational algorithm of $u = F(\varphi|m)$ is established, it is automatic to obtain those of the other incomplete elliptic integrals [48, 49]. After the forward transformation to obtain u from the given φ and m as explained in the previous subsection, the corresponding values of these integrals are obtained by the same procedure described in Section 5.6 where the abbreviations are now read as

$$E = E(\varphi|m) , \quad \Pi = \Pi(\varphi, n|m) , \quad B = B(\varphi|m) , \quad D = D(\varphi|m) , \quad J = J(\varphi, n|m) . \tag{5.46}$$

5.9 Computation of complete elliptic integrals other than $\Pi(n|m)$ and $J(n|m)$

Consider the computation of Legendre's form of complete elliptic integrals. There exists a difference between (1) the complete integrals of the first kind and the second kind, $K(m)$, $E(m)$, $B(m)$, and $D(m)$, and (2) the complete integrals of the third kind, $\Pi(n|m)$ and $J(n|m)$. The former integrals are single variable functions while the latter ones are not. Let us begin with the first group [44, 47].

The meaningful domain of the parameter m for the complete integrals is as limited as $m < 1$ since the integrals are no longer real-valued otherwise. By means of various kind of transformation formulas, this is further reduced to the so-called standard domain, $0 \le m < 1$. A single variable function can be quickly computed *if* the function is sufficiently smooth [27, 28]. The complete elliptic integrals, $K(m)$, $E(m)$, $B(m)$, and $D(m)$, are sufficiently smooth except near their common logarithmic singularity, $m = 1$. Following the policy of "divide and rule," we first split the standard domain into 11 subdomains as $[0, 0.1), [0.1, 0.2), \ldots, [0.7, 0.8), [0.8, 0.85), [0.85, 0.9)$, and $[0.9, 1.0)$. For the first 10 subintervals, the integrals are effectively approximated by their Taylor series expansion such as

$$K(m) \approx \sum_j K_j \left(m - m_0\right)^j \ , \tag{5.47}$$

where K_j is the jth order Taylor series coefficient of $K(m)$ at $m = m_0$, the mid point of the subinterval. The coefficients required in the double precision computation are explicitly listed in the tables of [44, 47]. If necessary, the higher order coefficients can be efficiently obtained by recursion [54].

In the last subinterval, $0.9 \le m < 1$, the integrals are expressed as a sum of two parts: the regular part and the logarithmically singular one [47]. For example, $K(m)$ is split into

$$K(m) = K_X X + K_0 \ , \tag{5.48}$$

where

$$X \equiv -\log\left(\frac{m_c}{16}\right) \ , \tag{5.49}$$

$$K_X \equiv \frac{K\left(m_c\right)}{\pi} \ , \tag{5.50}$$

$$K_0 \equiv K_X \left[-\log\left(\frac{16q\left(m_c\right)}{m_c}\right)\right] \ , \tag{5.51}$$

while $q(m)$ is Jacobi's nome defined as

$$q(m) \equiv \exp\left(\frac{-\pi K\left(m_c\right)}{K(m)}\right) \ . \tag{5.52}$$

Both of K_X and K_0 are regular around $m = 1$, namely around $m_c = 0$. Thus, they are well approximated by their Taylor series expansion even in the problematic interval, $0.9 \le m < 1$. Similar expansions are obtained for $E(m)$ as

$$E(m) = E_X X + E_0 \ , \tag{5.53}$$

where

$$E_X \left[1 - \left(\frac{E(m_c)}{K(m_c)} \right) \right] K_X \tag{5.54}$$

$$E_0 \frac{\pi}{2K(m_c)} + E_X K_0 \,, \tag{5.55}$$

In the case of $B(m)$ and $D(m)$, not themselves but their modifications are easily split as

$$B^*(m) \equiv \frac{B(m)}{m} = B_X^* X + B_0^* \,, \tag{5.56}$$

$$D^*(m) \equiv \frac{D(m)}{m} = D_X^* X + D_0^* \,. \tag{5.57}$$

The division by m is harmless near the singularity where $m \approx 1$. The corresponding coefficients are derived from those of $K(m)$ and $E(m)$ as

$$B_X^* = E_X - m_c K_X \,, \tag{5.58}$$

$$B_0^* = E_0 - m_c K_0 \,, \tag{5.59}$$

$$D_X^* = K_X - E_X \,, \tag{5.60}$$

$$D_0^* = K_0 - E_0 \,. \tag{5.61}$$

The tables of the Taylor series coefficients of the basic quantities, K_X, K_0, E_X, E_0, B_X^*, B_0^*, D_X^*, and D_0^*, are found in Fukushima [44, 47].

5.10 Computation of complete elliptic integrals of the third kind

On the other hand, the computation of the complete elliptic integral of the third kind is a difficult problem. Since it is a bivariate function, the technique of the series expansion is not easily applicable even if using the symmetric integral forms [50]. Fortunately, there exists a transformation with respect to n: the double argument formula with respect to a in Jacobi's original form of complete elliptic integral of the third kind, $\Pi_1(a|m)$, Equation (3.72). The resulting algorithm [53] is explained below.

First, by using various reduction formulas [53, Appendix A], the domain of n and m are reduced as

$$0 < m < 1 \,, \quad -k < n < \frac{m}{1 + k_c} \,. \tag{5.62}$$

Next, the forward transformation takes the same form as the half-argument transformation described in Section 5.7 by adopting a new variable

$$y \equiv \frac{n}{m} \,, \tag{5.63}$$

and its complement

$$x \equiv 1 - y = \frac{m-n}{m} \; . \tag{5.64}$$

Third, the Maclaurin series expansion with respect to y becomes

$$J(n|m) = D_0(m) + D_1(m)y + D_2(m)y^2 + \cdots \; , \tag{5.65}$$

where $D_j(m)$ are computed recursively as

$$D_j(m) = \left(\frac{2j}{2j+1}\right)(1+m)D_{j-1}(m) - \left(\frac{2j-1}{2j+1}\right) mD_{j-2}(m) \; , \tag{5.66}$$

while their starting values are computed from $D(m)$ and $B(m)$ as

$$D_0(m) = D(m) \; , \quad D_1(m) = \frac{(1+2m)D(m) - B(m)}{3} \; . \tag{5.67}$$

Finally, the backward transformation is the double argument transformation of $J(n|m)$, which is obtained by rewriting that of $\Pi_1(a|m)$ in a cancellation-error-free form as

$$J = \frac{2(c+d)J^* - yK}{cd(1+c)(1+d)} \; , \tag{5.68}$$

where K is the abbreviation of $K(m)$. It remains as a constant throughout the transformations because both the forward and backward transformations do not change the value of m, and therefore, $K(m)$ too. This is a benefit of the double argument transformation with respect to a although it is a linear transformation.

5.11 CPU time comparison

As already mentioned, there exist no significant difference in the computational errors of the existing procedures to compute the Jacobian elliptic integrals and Legendre's form of complete and incomplete elliptic integrals *except* Bulirsch's `el2`, `el3`, and our implementation of Bulirsch's `el`. Then, the question of concern is the computational speed.

We conducted the CPU time measurements of the existing precise procedures: (1) Bulirsch's `sncndn` and our `scd2` for the simultaneous computation of $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$, (2) Bulirsch's `cel1`, Carlson's R_F, and our `celk` for the computation of $K(m)$, (3) Bulirsch's `cel2`, Carlson's R_D, and our `celbd` for the simultaneous computation of $B(m)$ and $D(m)$, (4) Bulirsch's `cel`, Carlson's R_C, R_D, R_F, and R_J, and our `celbdj` for the simultaneous computation of $B(m), D(m)$, and $J(n|m)$, (5) Bulirsch's `el1`, Carlson's R_F, and our `elf` for the computation of $F(\varphi|m)$, (6) Carlson's R_D and our `elbd` for the simultaneous computation of $B(\varphi|m)$ and $D(\varphi|m)$, and (7) Carlson's R_C, R_D, R_F, and R_J, and our `elbdj` for the simultaneous computation of $B(\varphi|m)$, $D(\varphi|m)$, and $J(\varphi, n|m)$.

All the computation codes were (1) written in Fortran 77/90, (2) compiled by the Intel Visual Fortran Composer XE 2011 update 8 with the level 3 optimization, and (3) executed at a PC with an Intel Core i7-2675QM processor running at 2.20 GHz under Windows 7 OS. Therefore, the one machine clock cycle is 0.455 ns at the PC used for comparison.

Table 1 compares the averaged CPU times of the above 7 computations in the single and double precision environments, respectively. The averages were taken over the standard domain of the input arguments: $0 < \varphi < \pi/2$, $0 < m < 1$, $0 < n < 1$, and $0 < u < K(m)/2$. The sampling points are evenly distributed with the separations (1) $\Delta m = 2^{-14}$ and $\Delta u = 2^{-15}K(m)$ for the simultaneous computation of $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$, (2) $\Delta m = 2^{-28}$ for the computation of $K(m)$ and the simultaneous computation of $B(m)$ and $D(m)$, (3) $\Delta m = 2^{-14}$ and $\Delta\varphi = 2^{-15}\pi$ for the computation of $F(\varphi|m)$ and the simultaneous computation of $B(\varphi|m)$ and $D(\varphi|m)$, and (4) $\Delta m = \Delta n = 2^{-9}$ and $\Delta\varphi = 2^{-11}\pi$ for the simultaneous computation of $B(\varphi|m)$, $D(\varphi|m)$, and $J(\varphi, n|m)$, respectively. As a result, all the total execution times are of the comparable order. The results of comparison show the superiority of our computing procedures except the single precision computation of three Jacobian elliptic functions where Bulirsch's `sncndn` is the faster.

Table 1. Averaged CPU times to compute various elliptic functions and integrals. The unit of CPU time is ns at a PC with an Intel Core i7-2675QM processor running at 2.20 GHz where one machine clock cycle is 0.455 ns.

Functions/Integrals	Method	Procedure	Single	Double
$\mathrm{sn}(u\|m)$, $\mathrm{cn}(u\|m)$, $\mathrm{dn}(u\|m)$	Bulirsch	`sncndn`	99	165
	Fukushima	`scd2`	111	116
$K(m)$	Carlson	R_F	137	330
	Bulirsch	`cel1`	43	62
	Fukushima	`celk`	22	62
$B(m)$, $D(m)$	Carlson	R_D	384	922
	Bulirsch	`cel2`	109	186
	Fukushima	`celbd`	23	62
$B(m)$, $D(m)$, $J(n\|m)$	Carlson	R_C, R_D, R_F, R_J	697	1871
	Bulirsch	`cel`	192	383
	Fukushima	`celbdj`	121	198
$F(\varphi\|m)$	Carlson	R_F	119	313
	Bulirsch	`el1`	105	178
	Fukushima	`elf`	90	173
$B(\varphi\|m)$, $D(\varphi\|m)$	Carlson	R_D	553	712
	Fukushima	`elbd`	182	213
$B(\varphi\|m)$, $D(\varphi\|m)$, $J(\varphi, n\|m)$	Carlson	R_C, R_D, R_F, R_J	759	1426
	Fukushima	`elbdj`	349	464

5.12 Software

The computing procedures described in the previous subsections, namely `scd2`, `celk`, `celbd`, `celbdj`, `elf`, `elbd`, and `elbdj`, are available from the author's personal WEB page at ResearchGate:

https://www.researchgate.net/profile/Toshio_Fukushima/

with their test drivers and subprograms as well as sample output files.

6 Conclusion

Trigonometric functions and hyperbolic functions are daily used in the science and technology. Of course, there are many reasons of their popularity. Nevertheless, one of them is the easy availability of their functional values. Indeed, the functions are incorporated in the standard mathematical library of computers. Even some electronic calculators support them.

On the other hand, the elliptic functions and the elliptic integrals are not so popular despite that some scientific problems become simpler if they are deployed. Now that their fast and precise numerical procedures are available, we hope that these difficult-to-approach but deep-and-beneficial mathematical tools will be more utilized in celestial mechanics and dynamical astronomy.

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