Zernike Decomposition of the Tophat Function

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1 Problem Setup

The tophat function $g(r)$ is defined as:

$$g(r) = \begin{cases} 0 & \text{if } 0 \leq r < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq r \leq 1, \\ 0 & \text{if } r > 1. \end{cases} \tag{1}$$

Achates' idea for improvement: Add a plot of $g(r)$ generated in Mathematica.

The goal is to approximate $g(r)$ using a series of rotationally invariant Zernike polynomials:

$$f(r) = a_{0,0}Z_{0,0}(r) + a_{2,0}Z_{2,0}(r) + a_{4,0}Z_{4,0}(r) + \dots \tag{2}$$

where $Z_{k,0}(r)$ are the radial Zernike polynomials with even indices k, and $a_{k,0}$ are the amplitudes.

2 Objective Function

To find the amplitudes $a_{k,0}$, we minimize the least-squares difference:

$$\int_0^1 (g(r) - f(r))^2 \, r \, dr. \tag{3}$$

Expanding $f(r)$ in terms of the Zernike polynomials, we have:

$$f(r) = \sum_{k=0, k \text{ even}}^{\infty} a_{k,0} Z_{k,0}(r). \tag{4}$$

Achates' idea for improvement: Add an explanation of why Zernike polynomials are suitable for this problem.

3 Amplitude Calculation

The amplitudes $a_{k,0}$ are computed by:

$$a_{k,0} = \frac{\int_0^1 g(r) Z_{k,0}(r) r \, dr}{\int_0^1 Z_{k,0}(r)^2 r \, dr}. \tag{5}$$

This formula minimizes the least-squares error in the approximation.

Achates' idea for improvement: Add a plot illustrating the first few Zernike radial polynomials.

4 Future Work

- Extend the decomposition to higher-order terms and evaluate convergence.
- Explore 2D Zernike polynomials for functions defined over a disk.
- Add numerical results and graphs showing the approximation of $g(r)$.

5 Conclusion

The Zernike decomposition provides a systematic way to approximate rotationally symmetric functions like the tophat function. By computing the amplitudes $a_{k,0}$, we achieve an efficient representation using a compact basis of polynomials.

Achates' idea for improvement: Add numerical results and discuss error analysis.