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Letter to the editor

Precise and fast computation of complete elliptic integrals by piecewise minimax rational function approximation

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ABSTRACT

Piecewise minimax rational function approximations with the single and double precision accuracies are developed for (i) $K(m)$ and $E(m)$, the complete elliptic integral of the first and second kind, respectively, and (ii) $B(m) \equiv (E(m) - (1-m)K(m))/m$ and $D(m) \equiv (K(m) - E(m))/m$, two associate complete elliptic integrals of the second kind. The maximum relative error is one and 5 machine epsilons in the single and double precision computations, respectively. The new approximations run faster than the exponential function. When compared with the previous methods (Fukushima, 2009; Fukushima, 2011), which have been the fastest among the existing double precision procedures, the new method requires around a half of the memory and runs 1.7–2.2 times faster.

1. Introduction

The Legendre normal form complete elliptic integrals of the first and second kind, $K(m)$ and $E(m)$, and their linear combinations, $B(m)$ and $D(m)$, appear in various scenes of science and technology [1, Introduction]. They are defined [2, Section 4.4] as

$$K(m) \equiv \int_0^{\pi/2} \frac{d\theta}{\Delta(\theta|m)}, \qquad E(m) \equiv \int_0^{\pi/2} \Delta(\theta|m) d\theta, \tag{1}$$

$$B(m) \equiv \int_0^{\pi/2} \frac{\cos^2\theta d\theta}{\Delta(\theta|m)}, \qquad D(m) \equiv \int_0^{\pi/2} \frac{\sin^2\theta d\theta}{\Delta(\theta|m)}, \tag{2}$$

where m is the parameter [3, Section 3.2.11] and $\Delta(\theta|m)$ is Jacobi's Delta function defined as

$$\Delta(\theta|m) \equiv \sqrt{1 - m\sin^2\theta}. \tag{3}$$

Refer to Fig. 1 for the behavior of these integrals. The standard reference of elliptic integrals is [4, Chapter 19], which is freely accessible at http://dlmf.nist.gov/.

Among the existing methods to compute these integrals, $K(m)$, $E(m)$, $B(m)$, and $D(m)$, the fastest procedures are those based on the piecewise truncated Taylor series expansions [5,6] as shown in [2, Table 1]. Nevertheless, their relative error curves are far from being minimax because the adopted approximations are obtained by truncating the Taylor series expanded around the mid point of each sub interval. Also, the sub intervals are not optimally chosen. In fact, their separation points are naively set as 0.1, 0.2, . . . , 0.8, 0.85, and 0.9 [5,6]. This was just for simplicity.

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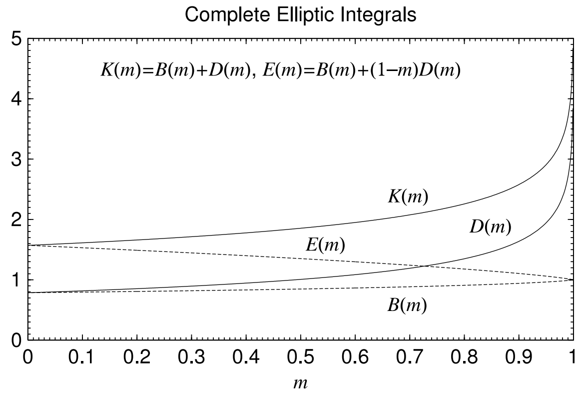

Fig. 1. Sketch of four complete elliptic integrals.

Table 1
Separation points of piecewise minimax rational approximations. Listed are the separation points of the sub intervals of approximation, m_j, such that the jth sub interval becomes $\left[m_{j-1}, m_j\right)$ while $m_0 \equiv 0$.

Precision	j	$K(m)$	$E(m)$	$B(m)$	$D(m)$
Single	1	0.734599	0.791217	0.810113	0.721197
	2	0.931503	0.968607	0.974420	0.924036
	3	0.982854	0.997566	0.998303	0.979874
Double	1	0.407010	0.433362	0.444927	0.400091
	2	0.649244	0.684847	0.697633	0.640820
	3	0.793076	0.828645	0.838948	0.785426
	4	0.878266	0.909330	0.916478	0.872125
	5	0.928558	0.953547	0.958034	0.923993
	6	0.958230	0.977087	0.979687	0.954948
	7	0.975640	0.989191	0.990592	0.973374
	8	0.985835	0.995159		0.984311
	9	0.991787			0.990784

In general, rational function approximations are better than polynomial ones including the truncated Taylor series in the sense to have a higher computational cost performance [7, Chapter 5]. Also, the recent computer chips such as the Intel Core series are capable to execute at least two multiply-and-add operations in parallel [8]. This fact enhances the cost performance of the evaluation of rational functions, especially of the even type, since the numerator and denominator polynomials can be evaluated by Horner's method simultaneously.

This short report provides a set of new formulas to compute these complete integrals by using the minimax rational function approximation. Section 2 describes the process of their construction, the summary of the obtained results, and the computational cost and performance of the new formulas.

2. Results

Construct a piecewise minimax approximation of the integrals, $K(m)$, $E(m)$, $B(m)$, and $D(m)$, when $0 \le m < 1$. Following the previous methods [5,6], the standard interval, $0 \le m < 1$, is split into several regions as shown in Table 1. The number of the sub intervals and the separation points are experimentally determined in order to take a balance between the memory saving and the computational speed-up while letting the relative error curves globally minimax as will be shown later.

Except the last sub interval, the integrals are approximated by rational functions of an even type as

$$K(m) \approx K_j(m) \equiv \frac{\sum_{n=0}^{N} P_n t^n}{\sum_{n=0}^{N} Q_n t^n}, \quad \left(m_{j-1} \le m < m_j;\ j = 1, \ldots, J\right) \tag{4}$$

where N is set as 3 and 5 for the single and double precision environments, respectively, and

$$t \equiv \frac{(m_j - 1) + m_c}{m_j - m_{j-1}}, \tag{5}$$

Table 2
Determined degrees of minimax rational approximations: nearly singular case. Listed are (N, M), the type of rational function adopted for the minimax approximation of auxiliary functions $K_X(m_c)$ through $D_0(m_c)$ in the single and double precision, respectively. The case $M = 0$ means the degree N polynomial approximation.

Function	Single	Double
$K_X(m_c)$	(2, 0)	(2, 2)
$K_0(m_c)$	(1, 0)	(2, 2)
$E_X(m_c)$	(1, 0)	(2, 2)
$E_0(m_c)$	(2, 0)	(2, 2)
$B_X(m_c)$	(2, 2)	(2, 2)
$B_0(m_c)$	(2, 2)	(2, 2)
$D_X(m_c)$	(3, 0)	(3, 3)
$D_0(m_c)$	(3, 0)	(2, 2)

where $m_c \equiv 1 - m$ is the complimentary parameter [3]. Meanwhile, in the last sub interval, the approximations are expressed [5,6] as

$$K(m) \approx K_X(m_c) X + m_c K_0(m_c), \tag{6}$$

$$E(m) \approx m_c E_X(m_c) X + E_0(m_c), \tag{7}$$

$$B(m) \approx m_c B_X(m_c) X + B_0(m_c), \tag{8}$$

$$D(m) \approx D_X(m_c) X + D_0(m_c), \tag{9}$$

where

$$X \equiv -\ln\left(\frac{m_c}{16}\right), \tag{10}$$

and $K_X(m_c)$ through $D_0(m_c)$ are auxiliary functions defined as

$$K_X(m_c) \equiv \frac{K(m_c)}{\pi}, \tag{11}$$

$$K_0(m_c) \equiv K_X(m_c) \ln\left(\frac{m_c}{16q(m_c)}\right), \tag{12}$$

$$E_X(m_c) \equiv \frac{L(m_c) K_X(m_c)}{m_c}, \tag{13}$$

$$E_0(m_c) \equiv \frac{1}{2K_X(m_c)} + L(m_c) K_0(m_c), \tag{14}$$

$$B_X(m_c) \equiv \frac{E_X(m_c) - m_c K_X(m_c)}{m m_c}, \tag{15}$$

$$B_0(m_c) \equiv \frac{E_0(m_c) - m_c K_0(m_c)}{m}, \tag{16}$$

$$D_X(m_c) \equiv \frac{K_X(m_c) - E_X(m_c)}{m}, \tag{17}$$

$$D_0(m_c) \equiv \frac{K_0(m_c) - E_0(m_c)}{m}, \tag{18}$$

while $q(m)$ is Jacobi's nome [1, Formula 901.00] defined as

$$q(m) \equiv \exp\left(\frac{-\pi K(1-m)}{K(m)}\right), \tag{19}$$

and

$$L(m) \equiv 1 - \frac{E(m)}{K(m)}. \tag{20}$$

These auxiliary functions are approximated as rational functions like Eq. (4). The adopted types of these rational functions are given in Table 2.

At any rate, the coefficients of rational functions are determined by using `MiniMaxApproximation` command of Mathematica Version 10, which employs the Remez algorithm [9]. Its sample usage is

```
mmK=MiniMaxApproximation[EllipticK[m],{m,{m0,m1},NK,NK}];
```

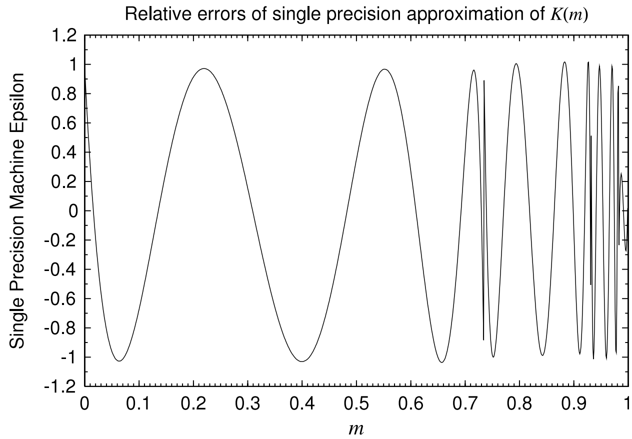

Fig. 2. Relative error curve of $K(m)$. Plotted is the relative error curve of the single precision minimax rational function approximation of $K(m)$ obtained by the new method in the standard interval, $[0, 1)$. The errors are measured by comparing with the reference values computed by the quadruple precision extension of Bulirsch's `cel` [10].

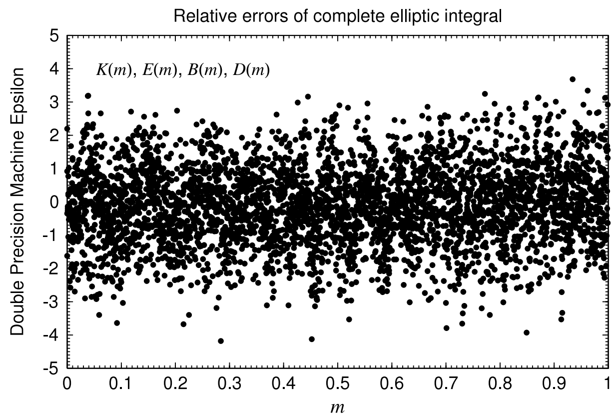

Fig. 3. Relative errors of four complete elliptic integrals. Plotted are the relative errors of the double precision minimax rational function approximations of the four complete elliptic integrals, $K(m)$, $E(m)$, $B(m)$, and $D(m)$, as a function of the parameter m in the standard interval, $[0, 1)$. Overlapped are the errors of four integrals since no significant difference in their distributions is observed. The displayed errors are all due to rounding-off in the evaluation of rational functions.

for $K(m)$ where $\mathtt{NK} = N$. When the minimax optimization process converges successfully, the determined rational approximation function is given as mmK⟦2, 1⟧ while the maximum relative error is provided as mmK⟦2, 2⟧.

These formulas are designed such that their maximum relative errors do not exceed the machine epsilon. As an illustration, Fig. 2 shows the relative error curve of the single precision approximation of $K(m)$. Omitted are the figures for the other integrals, $E(m)$, $B(m)$, and $D(m)$, since they are all similar. At any rate, the errors do not exceed the single precision machine epsilons and show the standard minimax feature.

On the other hand, Fig. 3 plots the relative errors of the double precision approximations of all the four integrals. The errors randomly scatter and no dependence on the integrals is seen. Also, Table 3 reports the statistics of the obtained relative errors. These support a conjecture that all the observed errors are due to rounding off in the evaluation of numerator and denominator polynomials by Horner's method.

An index of the complexity of approximations is the total number of polynomial coefficients. It well represents the word length of the computer programs to realize the approximation formulas. Table 4 compares this index for the previous and new methods. The numbers are roughly halved when moving from the truncated Taylor series expansion to the minimax rational function approximation. This results a significant simplification of the actual machine codes produced by the compilers.

Table 5 compares the averaged CPU times of the previous and new methods. The averages are taken over $2^{28} \approx 2.68 \times 10^8$ values of m uniformly distributed in the standard interval, $0 \le m < 1$. All the programs are coded in Fortran 90 and compiled by the Intel Visual Fortran Composer XE 2011 update 8 with the maximum optimization. The unit of CPU time is ns. The measurements are conducted at a PC with the Intel Core i7-4600U running at 2.10 GHz clock. The results shown here are after the exclusion of the overhead time to call functions in Fortran, which amounts to 12.2 ns in the same environment. As a reference, the averaged CPU time of the double precision exponential function in the same environment is 23.8 ns. This

Table 3
Statistics of relative errors of double precision minimax approximations of complete elliptic integrals. Listed are the mean, the sample standard deviation (SD), the maximum, and the minimum of the relative errors of the double precision minimax approximations of $K(m)$, $E(m)$, $B(m)$, and $D(m)$. The statistics are taken for 10^6 sample points of m evenly distributed in the standard interval, $[0, 1)$. The results are expressed in the unit of the double precision machine epsilon.

	Mean	SD	Max.	Min.
$K(m)$	0.0	1.2	5.3	−5.1
$E(m)$	−0.3	1.3	5.1	−5.7
$B(m)$	0.0	1.2	5.6	−5.3
$D(m)$	−0.2	1.2	5.0	−5.1

Table 4
Comparison of memory requirements. Shown are the total number of numerical coefficients, being other than ± 1, of a set of rational functions or polynomials used in the approximations adopted by the previous methods [5,6] and by the new method.

Method	Precision	$K(m)$	$E(m)$	$B(m)$ & $D(m)$
Previous	Double	197	171	382
New	Double	109	98	198
New	Single	26	26	59

Table 5
Comparison of CPU time. Shown are the averaged CPU times to compute complete elliptic integrals, (i) $K(m)$ solely, (ii) $E(m)$ solely, and (iii) $B(m)$ and $D(m)$ simultaneously. Listed are the results by the previous methods [5,6] and the new approximations given in the main text. The unit of CPU time is ns at a PC with the Intel Core i7-4600U running at 2.10 GHz clock. The averaged CPU time of the double precision exponential function in the same environment is 23.8 ns.

Method	Precision	$K(m)$	$E(m)$	$B(m)$ & $D(m)$
Previous	Double	45.0	42.0	46.6
New	Double	20.8	21.0	27.0
New	Single	17.9	18.1	22.6

means that the averaged CPU time of the new approximations is reduced down to the level of an elementary function such as the exponential function. As a result, the new method runs 1.6–2.8 times faster than the fastest procedures among the existing ones [5,6].

3. Conclusion

Piecewise minimax rational function approximations are developed for four complete elliptic integrals, $K(m)$, $E(m)$, $B(m)$, and $D(m)$, with the single and double precision accuracies, respectively. When compared with the previous procedures obtained by truncating the Taylor series [5,6], the new approximations (i) are more compact since the total number of numerical coefficients are roughly halved, (ii) achieve a better approximation in the sense that the relative truncation errors are minimax, and (iii) run 1.7–2.2 times faster.

The Fortran functions to compute the obtained minimax approximations of $K(m)$, $E(m)$, $B(m)$, and $D(m)$ are named as (i) `sceik`, `sceie`, `sceib`, and `sceid` for the single precision computation, respectively, and (ii) `ceik`, `ceie`, `ceib`, and `ceid` for the double precision computation, respectively. Their codes as well as the sample outputs are freely available from the following WEB site.

https://www.researchgate.net/profile/Toshio_Fukushima/

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