Precise and Fast Computation of Elliptic Integrals and Elliptic Functions

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***Abstract*—Summarized is the recent progress of the new methods to compute Legendre's complete and incomplete elliptic integrals of all three kinds and Jacobian elliptic functions. Also reviewed are the entirely new methods to (i) compute the inverse functions of complete elliptic integrals, (ii) invert a general incomplete elliptic integral numerically, and (iii) evaluate the partial derivatives of the elliptic integrals and functions recursively. In order to avoid the information loss against small parameter and/or characteristic, newly introduced are the associate complete and incomplete elliptic integrals. The main techniques used are (i) the piecewise approximation for single variable functions, and (ii) a systematic utilization of the half and double argument transformations and the truncated Maclaurin series expansions for the others. The new methods are of the errors of 5 ulps at most without any chance of cancellation against small input arguments. They run significantly faster than the existing methods: (i) slightly faster than Bulirsch's procedure for the incomplete elliptic integral of the first kind, (ii) 1.5 times faster than Bulirsch's procedure for Jacobian elliptic functions, (iii) 2.5 times faster than Cody's and Bulirsch's procedures for the complete elliptic integrals, and (iv) 3.5 times faster than Carlson's procedures for the incomplete elliptic integrals of the second and third kind. Their Fortran programs are available at https://www.researchgate.net/profile/Toshio_Fukushima/**

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I. Introduction

The elliptic integrals and elliptic functions are widely used in science and technology [1]. Their textbooks and standard references are [2]–[15]. See [16] for a concise tutorial including some cautions in reading the existing literature. The modern procedures to compute the elliptic integrals and elliptic functions are established by Bulirsch's pioneer works [17]–[20] based on the quadratic transformations invented by Euler, Landen, and Gauss and extended by Bartkey [19]. However, Bulirsch's procedures computing the general incomplete elliptic integrals of the second and third kind face with severe cancellation problems [21]. This was not resolved even after a further generalization by the author [22]. Almost at the same time, Carlson published a series of monumental works on the elliptic integrals by introducing their symmetric forms [23]–[26]. His duplication method is a linear transformation method and free from the loss of information [27]. Nevertheless, Carlson's procedures run slow when compared with Bulirsch's surviving ones as clearly seen from Table I.

Recently developed are a series of new methods to compute the elliptic functions and integrals precisely and quickly [28]–[37] as well as entirely new methods to obtain their inverses and partial derivatives [38]–[40]. Table I has already shown that the newly developed procedures are significantly faster than the existing procedures. Below, they will be reviewed.

TABLE I. Experimentally determined latency at Intel Sandy Bridge CPU. The CPU times are obtained by averaging over 2^{24} sample points uniformly distributed in $0 \le m < 1$, $0 \le n < 1$, $0 \le \varphi < \pi/2$, and/or $0 \le u < K(m)$. The CPU time in ns is multiplied by the CPU clock rate in GHz to mimic overall latency.

EllipticIntegrals/Functions	Bulirsch	Carlson	Cody	new
$K(m)$	288	1020	192	74
$B(m), D(m)$	577	2860		90
$B(m), D(m), J(n\|m)$	1190	5800		493
$F(\varphi\|m)$	552	970		536
$B(\varphi\|m), D(\varphi\|m)$		2210		660
$B(\varphi\|m), D(\varphi\|m), J(\varphi, n\|m)$		4420		1440
$\mathrm{sn}(u\|m), \mathrm{cn}(u\|m), \mathrm{dn}(u\|m)$	512			360
$m_K(K)$				171
$m_E(E)$				180

II. Associate Elliptic Integrals

A general elliptic integral, $G(\varphi, n|m)$ of amplitude φ, characteristic n, and parameter m, is expressed [1] as

$$G(\varphi, n|m) = a(n,m)F(\varphi|m) + b(n,m)E(\varphi|m) + c(n,m)\Pi(\varphi, n|m) + R(\varphi, n, m), \tag{1}$$

where (i) $a(n,m)$, $b(n,m)$, and $c(n,m)$ are arbitrary functions of n and m, (ii) $R(\varphi, n, m)$ is a general elementary function, which is usually expressed as a rational function of the trigonometric functions of φ with the coefficients being arbitrary functions of n and m, and (iii) $F(\varphi|m)$, $E(\varphi|m)$, and $\Pi(\varphi, n|m)$ are Legendre's normal form incomplete elliptic integral of the first, second, and third kind defined [1] as

$$F(\varphi|m) \equiv \int_0^{\varphi} \frac{d\theta}{\Delta(\theta|m)}, \quad E(\varphi|m) \equiv \int_0^{\varphi} \Delta(\theta|m)\, d\theta,$$

$$\Pi(\varphi, n|m) \equiv \int_0^{\varphi} \frac{d\theta}{\left(1 - n\sin^2\theta\right)\Delta(\theta|m)}, \tag{2}$$

where $\Delta(\theta|m) \equiv \sqrt{1 - m\sin^2\theta}$ is Jacobi's Delta function [1].

Bulirsch [17]–[20] already noticed that $F(\varphi|m)$, $E(\varphi|m)$, and $\Pi(\varphi, n|m)$ are not suitable when n and/or m are small. In

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order to overcome this issue, newly introduced by the author are the associate incomplete elliptic integrals defined as

$$B(\varphi|m) \equiv \int_0^{\varphi} \frac{\cos^2\theta \, d\theta}{\Delta(\theta|m)} = \frac{E(\varphi|m) - m_c F(\varphi|m)}{m},$$

$$D(\varphi|m) \equiv \int_0^{\varphi} \frac{\sin^2\theta \, d\theta}{\Delta(\theta|m)} = \frac{F(\varphi|m) - E(\varphi|m)}{m},$$

$$J(\varphi,n|m) \equiv \int_0^{\varphi} \frac{\sin^2\theta \, d\theta}{\left(1-n\sin^2\theta\right)\Delta(\theta|m)} = \frac{\Pi(\varphi,n|m) - F(\varphi|m)}{n}, \tag{3}$$

where $m_c \equiv 1-m$ is the complementary parameter [11]. Indeed, the computation of the associate integrals from the normal ones faces with the loss of information when $n \ll 1$ and/or $m \ll 1$ while the reverse process are free from it as

$$F(\varphi|m) = B(\varphi|m) + D(\varphi|m),$$

$$E(\varphi|m) = B(\varphi|m) + m_c D(\varphi|m),$$

$$\Pi(\varphi,n|m) = B(\varphi|m) + D(\varphi|m) + nJ(\varphi,n|m). \tag{4}$$

The same thing is said about the complete elliptic integrals as

$$K(m) \equiv F(\pi/2|m) = B(m) + D(m),$$

$$E(m) \equiv E(\pi/2|m) = B(m) + m_c D(m),$$

$$\Pi(n|m) \equiv \Pi(\pi/2,n|m) = B(m) + D(m) + nJ(n|m), \tag{5}$$

where $B(m)$, $D(m)$, and $J(n|m)$ are the associate complete elliptic integrals defined as

$$B(m) \equiv B(\pi/2|m), \; D(m) \equiv D(\pi/2|m),$$

$$J(n|m) \equiv J(\pi/2,n|m). \tag{6}$$

III. Methods of Computation

A. Piecewise Approximation

The single variable functions such as $K(m)$, $E(m)$, $B(m)$, and $D(m)$ as well as a special complete elliptic integral of the second kind [31] defined as

$$S(m) \equiv \left[(2-m)K(m) - 2E(m)\right]/m^2 = \left[D(m) - B(m)\right]/m, \tag{7}$$

are computed by the piecewise approximation. Except the care for the logarithmic singularity of these integrals at $m=1$ [42], it is essentially a brute-force method.

First, the standard domain, $0 \le m < 1$, is cut into around a dozen pieces. Next, except the last sub interval, an appropriately-truncated Taylor series expansion [29] or a globally-minimax rational function approximation [37] are developed with help of `Series` and `MiniMaxApproximation` commands of Mathematica [11]. The latest work [37] is further accelerated by Dorn's 2nd order vectorization of Horner's rule [41], which is effective at Intel CPUs based on the Haswell micro architecture thanks to the fast FMA3 instructions.

Meanwhile, in the last sub interval, which is closest to the logarithmic singularity of the integrals, $m=1$ or $m_c=0$, the integrals are written as the linear functions of a variable representing the logarithmic singularity,

$$X \equiv -\ln(m_c/16), \tag{8}$$

such that

$$K(m) = K_X(m_c)X + m_c K_0(m_c),$$

$$E(m) = m_c E_X(m_c)X + E_0(m_c),$$

$$B(m) = m_c B_X(m_c)X + B_0(m_c),$$

$$D(m) = D_X(m_c)X + D_0(m_c). \tag{9}$$

Here the coefficient functions are defined as

$$K_X(m) \equiv \frac{K(m)}{\pi}, \; K_0(m) \equiv K_X(m)\ln\left(\frac{m}{16q(m)}\right),$$

$$E_X(m) \equiv \frac{L(m)K_X(m)}{m}, \; E_0(m) \equiv \frac{1}{2K_X(m)} + L(m)K_0(m),$$

$$B_X(m) \equiv \frac{E_X(m) - mK_X(m)}{m(1-m)}, \; B_0(m) \equiv \frac{E_0(m) - mE_X(m)}{1-m},$$

$$D_X(m) \equiv \frac{K_X(m) - E_X(m)}{1-m}, \; D_0(m) \equiv \frac{K_0(m) - E_0(m)}{1-m}, \tag{10}$$

while $q(m)$ is Jacobi's nome [1] defined as

$$q(m) \equiv \exp\left(\frac{-\pi K(m_c)}{K(m)}\right), \tag{11}$$

and $L(m)$ is an auxiliary function expressed as

$$L(m) \equiv 1 - E(m)/K(m), \tag{12}$$

being related to Legendre's relation [1]

$$E(m)K(m_c) + K(m)E(m_c) - K(m)K(m_c) = \pi/2. \tag{13}$$

These coefficient functions are all regular around $m_c = 0$. For example, $K_X(m_c)$ and $K_0(m_c)$ are expanded as

$$K_X(m_c) = \frac{1}{2} + \frac{m_c}{8} + \frac{9m_c^2}{128} + \cdots,$$

$$K_0(m_c) = -\frac{1}{4} - \frac{21m_c}{128} - \frac{185m_c^2}{1536} - \cdots, \tag{14}$$

Therefore, the coefficient functions can be well approximated by truncated Taylor series or minimax rational function approximations. At any rate, the adopted strategy of 'divide and rule' is so effective that the resulting procedures are sufficiently accurate and more than twice faster than the existing standard methods utilizing the Chebyshev polynomial approximations invented by Hastings [42] and extended by Cody [43]–[45].

The same strategy works for the inverse functions, $m_K(K)$ and $m_E(E)$, defined as the functions to satisfy the relations

$$m_K(K(m)) = m_E(E(m)) = m. \tag{15}$$

Of course, the integral value domain of K is semi-infinite as $\pi/2 \le K < \infty$. Since K grows only logarithmically, nevertheless, a practical consideration can limit the upper bound to be finite as $K_C \equiv -\ln\varepsilon$ where ε is the machine epsilon in the adopted computing precision. Then, the integral value domain, $\pi/2 \le K < K_C$ or $1 < E \le \pi/2$, is split into 8 pieces. For the first 4 sub intervals being far from the logarithmic singularity, $K \to \infty$ and $E = 1$, an appropriately truncated Taylor series expansion [39] is developed again. On the other hand, in the last 4 sub intervals, the variable to be solved for is altered from m to

$$p \equiv -\ln m_c. \tag{16}$$

Also, in the case of $m_E(E)$, the integral value to be used as the input value is changed from $E \equiv E(m)$ to a new argument

$$r \equiv -\ln(E-1). \tag{17}$$

This is because the usage of p and r results simpler asymptotic expansions around the logarithmic singularity as

$$K = \ln 4 + \frac{p}{2} + \cdots, \; r = p - \ln\left(\frac{p-1}{4} + \ln 2\right) + \cdots. \tag{18}$$

At any rate, an appropriately truncated Taylor series expansions of p with respect to K and r are similarly obtained. Once p is determined, it is straightforward to compute m as

$$m = 1 - \exp(-p). \tag{19}$$

The resulting procedures [39] are (i) entirely new, (ii) sufficiently precise, and (iii) as fast as those computing $K(m)$ and/or other single variable complete elliptic integrals described in the above as already shown in Table I.

B. Double Argument Transformation

The multiple variable functions, namely (i) the incomplete integrals, $F(\varphi|m)$, $B(\varphi|m)$, $D(\varphi|m)$, and $J(\varphi,n|m)$, (ii) three Jacobian elliptic functions, $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$, and (iii) the associate complete integral of the third kind, $J(n|m)$, are computed by utilizing their half and double argument transformations [30], [33]–[36]. These methods are completely anew and developed by the author from the addition theorems with respect to the argument, $u \equiv F(\varphi|m)$. In order to show the essence of these transformation methods, their special case applied to the trigonometric functions will be given first.

Consider the simultaneous computation of the sine and cosine functions, $\sin\theta$ and $\cos\theta$, for the angle θ in the standard domain, $0 \le \theta < \pi/2$. If θ is sufficiently large, say greater than 0.1, it is reduced by the repeated application of the half-angle transformation, $\theta \to \theta/2$. This process can be efficiently conducted if the exponent manipulation of a binary floating point number is sufficiently fast. At any rate, once θ becomes sufficiently small, say less than 0.1, the corresponding trigonometric function values can be precisely and quickly evaluated by (i) truncating the Maclaurin series expansions of the sine function as

$$\sin\theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots, \tag{20}$$

and (ii) computing $\cos\theta$ from $\sin\theta$ as

$$\cos\theta = \sqrt{1-\sin^2\theta}. \tag{21}$$

Then, the function values are recovered by repeatedly applying the double-angle formulas

$$\sin 2\theta = 2\sin\theta\cos\theta, \; \cos 2\theta = \cos^2\theta - \sin^2\theta. \tag{22}$$

Notice that, except for one call of the square root in computing $\cos\theta$, this algorithm needs only addition/subtraction and multiplication only. Thus, it runs fairly fast. Also, easy is its extension to the multiple precision computations if $\cos\theta$ is also computed by truncating its Maclaurin series expansion.

Anyhow, the same approach is applied to Jacobian elliptic functions [29] by using (i) the Maclaurin series expansion of $\mathrm{sn}(u|m)$ [28] expressed as

$$s \approx \sum_{j=0}^{6} (-1)^j s_{2j+1} u^{2j+1}, \tag{23}$$

where the series coefficients are explicitly given as

$$s_1 = 1, \; s_3 = (1+m)/6, \; s_5 = \left(1+14m+m^2\right)/120,$$
$$s_7 = \left(1+135m+135m^2+m^3\right)/5040,$$
$$s_9 = \left(1+1228m+5478m^2+1228m^3+m^4\right)/362880,$$
$$s_{11} = \frac{1+11069m+165826m^2+165826m^3+11069m^4+m^5}{39916800},$$
$$s_{13} = \left(1+99642m+4494351m^2+13180268m^3 \right.$$
$$\left. +4494351m^4+99642m^5+m^6\right)\Big/6227020800, \tag{24}$$

(ii) the identity relations to compute other elliptic functions as

$$c \equiv \mathrm{cn}(u|m) = \sqrt{1-s^2}, \; d \equiv \mathrm{dn}(u|m) = \sqrt{1-ms^2}, \tag{25}$$

and (iii) the double argument formulas [1] written as

$$\mathrm{sn}(2u|m) = \frac{2scd}{1-ms^4}, \; \mathrm{cn}(2u|m) = \frac{c^2-s^2d^2}{1-ms^4},$$
$$\mathrm{dn}(2u|m) = \frac{d^2-ms^2c^2}{1-ms^4}. \tag{26}$$

Since the trio of s, c, and d must satisfy the two constraints

$$s^2+c^2 = ms^2+d^2 = 1, \tag{27}$$

a better algorithm [35] is developed by focusing on the half and double argument transformations of only one main variable, $y \equiv s^2$ and/or $x \equiv c^2$.

C. Half Argument Transformation

Reversely, the half argument transformation, defined as the inverse of the double argument transformation, is effective in computing $F(\varphi|m)$ [30]. Before going further, take a simple example to compute the inverse trigonometric function, $\theta \equiv \sin^{-1}\sigma$, when σ satisfies the condition, $0 < \sigma < 1/\sqrt{2}$. The half-angle formula of the sine function is translated into a transformation of $\psi \equiv \sigma^2 = \sin^2\theta$ as

$$\psi \to \psi_H \equiv \sin^2\frac{\theta}{2} = \frac{\psi}{2\left(1+\sqrt{1-\psi}\right)}, \tag{28}$$

where the suffix H stands for the half angle. This is a contraction mapping. Thus, its repetition makes ψ sufficiently small, say less than 0.1. Once ψ becomes small, $\sin^{-1}\sqrt{\psi}$ is obtained by truncating its Maclaurin series expansion

$$\sin^{-1}\sqrt{\psi} = \sqrt{\psi}\left(1+\frac{\psi}{6}+\frac{3\psi^2}{40}+\cdots\right). \tag{29}$$

Finally, the inverse sine function value of the original input argument is recovered by multiplying a power of 2 as

$$\theta = 2^N \sin^{-1}\sqrt{\psi}, \tag{30}$$

where N is the number of applications of the half-angle formula. The last process runs quickly at binary computers.

Again, the same formulation is applied to evaluate $u \equiv F(\varphi|m)$ when $y \equiv \sin^2\varphi$ and m are specified [30]. By definition, u is regarded as a sort of inversion of y in the sense

$$\mathrm{sn}(u|m) = \sqrt{y}, \tag{31}$$

This results the half argument transformation of y as

$$y \to y_H \equiv \left[\mathrm{sn}\left(\frac{u}{2}\middle|m\right)\right]^2 = \frac{y}{\left(1+\sqrt{1-y}\right)\left(1+\sqrt{1-my}\right)}. \tag{32}$$

A repeated application of this transformation reduces y sufficiently small, say less than 0.1. Then, $\mathrm{sn}^{-1}\sqrt{y}$ is obtained by truncating the Maclaurin series expansion as

$$\mathrm{sn}^{-1}\sqrt{y} \approx \sqrt{y}\sum_{j=0}^{6} u_{2j+1}y^j, \tag{33}$$

where the series coefficients are explicitly given as

$$\begin{aligned}
&u_1 = 1,\ u_3 = (1+m)/6,\ u_5 = \left(3+2m+3m^2\right)\big/40,\\
&u_7 = \left(5+3m+3m^2+5m^3\right)\big/112,\\
&u_9 = \left(35+20m+18m^2+20m^3+35m^4\right)\big/1152,\\
&u_{11} = \left(63+35m+30m^2+30m^3+35m^4+63m^5\right)\Big/2816,\\
&u_{13} = \big(231+126m+105m^2+100m^3+105m^4\\
&\qquad +126m^5+231m^6\big)\Big/13312,
\end{aligned} \tag{34}$$

which are obtained by inverting the Maclaurin series expansion of $\mathrm{sn}(u|m)$ by means of `InverseSeries` command of Mathematica [11].

The forward transformation $y \to y_H$ has a chance of information loss when y and m are not so small initially. In that case, selected is another main variable, $x \equiv \cos^2\varphi$, which is sufficiently small. Then, the forward transformation becomes

$$x \to x_H = (\sqrt{x}+d)/(1+d), \tag{35}$$

where $d \equiv \sqrt{m_c+mx}$. This is an expansive mapping. Its repetition soon magnifies x such that the corresponding y becomes less than $1/2$. At that point, the main variable is switched from x to y and the reduction in terms of y is going on. This conditional switching is in order to keep the relative accuracy during the process of transformation [30]. At any rate, u is finally obtained by multiplying a power of 2 as

$$u = 2^N \mathrm{sn}^{-1}\sqrt{y}, \tag{36}$$

where N is the total number of applications of the half argument transformation.

D. Combination of Half and Double Argument Transformations

A similar approach is effective in evaluating the three associate incomplete elliptic integrals, $B \equiv B(\varphi|m)$, $D \equiv D(\varphi|m)$, and $J \equiv J(\varphi,n|m)$. This time, the first process, namely the half argument transformation on u, is common and the same as given in the previous subsection. Rather, the difference is in the Maclaurin series expansions and in the reverse process, i.e. the double argument transformation. Before going further, examine the necessary minimum integrals among B, D, and J. Notice a relation between B and D as

$$B + D = u. \tag{37}$$

Thus, the pair of J and either B or D is the more fundamental. From the robustness against the smallness of m, D is preferred to B. Meanwhile, by definition, the computation of D is regarded as a special case of that of J as

$$D = D(\varphi|m) = J(\varphi,0|m). \tag{38}$$

In this sense, J is the most fundamental integral.

Anyhow, the Maclaurin series of J with respect to not φ but $u \equiv F(\varphi|m)$ [28] is written as

$$J \approx \sum_{j=1}^{6}(-1)^{j-1}J_{2j+1}u^{2j+1}, \tag{39}$$

where the series coefficients are explicitly given as

$$\begin{aligned}
&J_3 = 1/3,\ J_5 = (1+m-3n)/15,\\
&J_7 = \left[2+13m+2m^2-n(30+30m)+45n^2\right]/315,\\
&J_9 = \big[1+30m+30m^2+m^3-\left(63+252m+63m^2\right)n\\
&\qquad +(315+315m)n^2-315n^3\big]\big/2835,\\
&J_{11} = \big[2+251m+876m^2+251m^3+2m^4\\
&\qquad -\left(510+5850m+5850m^2+510m^3\right)n\\
&\qquad +\left(6615+21735m+6615m^2\right)n^2\\
&\qquad -(18900+18900m)n^3+14175n^4\big]\big/155925,\\
&J_{13} = \Big[2+1018m+9902m^2+9902m^3+1018m^4+2m^5\\
&\qquad -\left(2046+59268m+158103m^2+59268m^3+2046m^4\right)n\\
&\qquad +\left(63360+497475m+497475m^2+63360m^3\right)n^2\\
&\qquad -\left(395010+1164240m+395010m^2\right)n^3\\
&\qquad +(779625+779625m)n^4-467775n^5\Big]\Big/6081075.
\end{aligned} \tag{40}$$

Those of D are automatically obtained by setting $n=0$ in the above expressions.

Meanwhile, the double argument transformation of J in terms of u becomes

$$J = 2J_H + T(t,h), \tag{41}$$

where J_H is the integral value before the argument u is doubled in the sense $u = 2u_H$ while t and h are defined as

$$t \equiv s/c,\ \ h \equiv n(1-n)(n-m). \tag{42}$$

Notice that $T(t,h)$ is a new function connecting the arctangent and inverse hyperbolic tangent functions seamlessly as

$$T(t,h) \equiv \begin{cases} \tan^{-1}\left(\sqrt{h}\,t\right)/\sqrt{h} & (h>0) \\ t & (h=0) \\ \tanh^{-1}\left(\sqrt{-h}\,t\right)/\sqrt{-h} & (h<0) \end{cases} \quad (43)$$

The double argument transformation of D is derived from that of J by setting $n=0$, then $h=0$, and therefore $T=t$ as

$$D = 2D_H + t. \quad (44)$$

Once u, D, and J are computed, the others are obtained as

$$E = u - mD,\ \Pi = u + nJ,\ B = u - D. \quad (45)$$

E. Computation of $J(n|m)$

The precise and fast computation of $J(n|m)$ is a difficult problem. Since it is a bivariate function, neither the series expansion nor the minimax approximation is easily applicable even if using the symmetric integral forms [46]. Fortunately, there exists a transformation formula with respect to n: the double argument formula with respect to a of $\Pi_1(a|m)$, Jacobi's original complete elliptic integral of the third kind [1]. This integral is essentially the same as $J(n|m)$ in the sense

$$\Pi_1(a|m) = m\,\mathrm{sn}(a|m)\mathrm{cn}(a|m)\mathrm{dn}(a|m)J\left(m\,\mathrm{sn}^2(a|m)|m\right). \quad (46)$$

Thus, the symbol J may stand for Jacobi. Anyhow, this formula is not so popular and mentioned only in a few references [2]. Thanks to the above formula, conducted is the same half argument transformation by adopting a new variable as

$$y \equiv n/m. \quad (47)$$

The Maclaurin series expansion of $J(n|m)$ with respect to y is written as

$$J(n|m) = D_0(m) + D_1(m)y + D_2(m)y^2 + \cdots, \quad (48)$$

where $D_j(m)$ are auxiliary complete elliptic integrals recursively computed as

$$D_j(m) = \frac{2j(1+m)D_{j-1}(m) - (2j-1)mD_{j-2}(m)}{2j+1}, \quad (49)$$

from two starting values computed as

$$D_0(m) \equiv D(m),\ D_1(m) \equiv [(1+2m)D(m) - B(m)]/3. \quad (50)$$

Meanwhile, the double argument transformation of $J(n|m)$ is obtained from the addition theorem of $\Pi_1(a|m)$ [2] as

$$J = \frac{2(c+d)J_D - yK}{c(1+c)(1+d)}, \quad (51)$$

where $c \equiv \sqrt{1-y}$ and $d \equiv \sqrt{1-my}$ while J and J_D are the integral values before and after duplicating the corresponding argument a defined in the sense $\sqrt{y} = \mathrm{sn}(a|m)$. Notice that K remains to be a constant throughout the transformations because m is kept the same in the half and double argument transformations. This constancy of K greatly accelerates the double argument transformation.

F. Inversion of General Elliptic Integral

The combination of the half and double argument transformations is also effective in inverting a general elliptic integral, $G(\varphi,n|m)$, with respect to φ. As the variable to be solved for, however, not φ but $u \equiv F(\varphi|m)$ is more convenient. This is because the reduction of the solution interval in terms of u is much simpler than that in terms of φ.

At any rate, once u is determined, or more precisely speaking, once $s \equiv \mathrm{sn}(u|m)$ and/or $c \equiv \mathrm{cn}(u|m)$ are obtained, φ is computed by the inverse trigonometric functions as

$$\varphi = \sin^{-1} s = \cos^{-1} c = \tan^{-1}(s/c). \quad (52)$$

Daringly extending the problem, investigated is a systematic procedure to solve a general nonlinear equation

$$G(u) \equiv G(u, B^*(u), D^*(u), J^*(u), s(u), c(u), d(u)) = 0, \quad (53)$$

where $G(u)$ is a general function containing (i) u, (ii) other elliptic integrals regarded as functions of u as

$$B^*(u) \equiv B(\mathrm{am}(u|m)|m),\ D^*(u) \equiv D(\mathrm{am}(u|m)|m),$$
$$J^*(u) \equiv J(\mathrm{am}(u|m),n|m), \quad (54)$$

and (iii) Jacobian elliptic functions,

$$s(u) \equiv \mathrm{sn}(u|m),\ c(u) \equiv \mathrm{cn}(u|m),\ d(u) \equiv \mathrm{dn}(u|m). \quad (55)$$

Here $\mathrm{am}(u|m)$ is Jacobi's amplitude function [1] and defined as a direct inverse of $F(\varphi|m)$ in the sense that

$$F(\mathrm{am}(u|m)|m) = u. \quad (56)$$

Once again, an example based on the trigonometric functions [47], [48] will be shown below for the illustrating purpose. Consider the solution of a following nonlinear equation,

$$f(\theta) \equiv \theta - e\sin\theta - \lambda = 0, \quad (57)$$

for the solution domain, $0 < \theta < \pi$, when the input arguments, e and λ, satisfy the conditions, $0 < e < 1$ and $0 < \lambda < \pi$, respectively. This is the elliptic Kepler equation, the first transcendental equation in the history of science [49].

Notice that $f(0) < 0 < f(\pi)$ and $f'(\theta) = 1 - e\cos\theta > 0$ in the solution domain, $(0,\pi)$. Thus, the equation has a unique solution there. The adopted approach is the combination of (i) the bisection method to diminish the solution interval starting from the initial bracketing, $0 < \theta < \pi$, and (ii) the Newton method, Halley's third-order method, or Schröder's higher order method utilizing the derivative information [50] to improve the approximate solution when the length of reduced solution interval becomes sufficiently small, say less than 0.2.

The actual process of the bisection is significantly accelerated by utilizing the half angle formula and the addition theorem of the trigonometric functions. The end point values of the trigonometric functions for the initial interval are known beforehand as $\sin 0 = \sin\pi = 0$, $\cos 0 = 1$, and $\cos\pi = -1$. In the bisection method, the end points of the reduced solution intervals are of the form of $\theta_{kj} \equiv k\pi/2^j$ where k is an odd positive integer and j is a positive integer. Thus, the trigonometric function values for θ_{kj} can be computed accurately by the repeated usage of the half-angle formulas and the addition theorems, which consist of the square root, the addition/subtraction, and the multiplication operations only.

In other words, there is no need of direct calling the transcendental functions as $\sin\theta$ or $\cos\theta$ in the bisection stage.

On the other hand, in the process of improving the approximate solution, employed is the combination of the addition theorem and the Maclaurin series of the trigonometric functions. The trigonometric functions of the incremented argument, $\sigma_+ \equiv \sin(\theta+\Delta\theta)$ and $\gamma_+ \equiv \cos(\theta+\Delta\theta)$, are exactly and quickly computed by the addition theorems as

$$\sigma_+ = \sigma\gamma_\Delta + \gamma\sigma_\Delta,\ \gamma_+ = \gamma\gamma_\Delta - \sigma\sigma_\Delta, \tag{58}$$

where $\sigma_\Delta \equiv \sin\Delta\theta$ and $\gamma_\Delta \equiv \cos\Delta\theta$ are effectively computed by their appropriately-truncated Maclaurin series since the incremental solution $\Delta\theta$ is assumed to be small while $\sigma \equiv \sin\theta$ and $\gamma \equiv \cos\theta$ are assumed to be already obtained in the previous step of improvement or at the end of the bisection search. As a result, this process of solution improvement requires no direct call of the trigonometric functions, too.

Return to the original equation, $G(u) = 0$. The exactly same approach is used in solving it. Assume that the solution interval is limited as $0 < u < K(m)$. The interval reduction becomes simple if noting the periodic relations and the reflection formulas of the associate incomplete elliptic integrals and Jacobian elliptic functions [1] as

$$\begin{aligned}
B^*(jK(m) \pm u) &= 2jB(m) \pm B^*(u),\\
D^*(jK(m) \pm u) &= 2jD(m) \mp D^*(u),\\
J^*(jK(m) \pm u) &= 2jJ(n|m) \mp J^*(u),\\
\mathrm{sn}(2jK(m) \pm u|m) &= \pm(-1)^j \mathrm{sn}(u|m),\\
\mathrm{cn}(2jK(m) \pm u|m) &= (-1)^j \mathrm{cn}(u|m),\\
\mathrm{dn}(2jK(m) \pm u|m) &= \mathrm{dn}(u|m).
\end{aligned} \tag{59}$$

Also assume that $G(u)$ is monotonically increasing. This is usually satisfied especially when $G(u)$ is defined as an indefinite integral of a positive definite function.

Notice that the initial values of the elliptic integrals and elliptic functions at the end points, $u = 0$ and $u = K(m)$, are computed from their special value formulas [1] as

$$\begin{aligned}
&B^*_0 = D^*_0 = J^*_0 = s_0 = c_K = 0,\ c_0 = d_0 = s_K = 1,\\
&B^*_K = B(m),\ D^*_K = D(m),\ J^*_K = J(n|m),\ d_K = \sqrt{m_c},
\end{aligned} \tag{60}$$

where the suffices 0 and K mean the values at $u = 0$ and $u = K(m)$, respectively. Introduce three auxiliary variables as

$$x \equiv c^2,\ y \equiv s^2,\ z \equiv d^2, \tag{61}$$

so as to simplify the following descriptions. Then, the half argument transformation from the variables without the suffix to those with the suffix H is described in a sequence of operations as

$$\begin{aligned}
&\xi := c + d,\ \eta := 1/(1+c),\ \zeta := 1/(1+d),\ x_H := \xi\zeta,\\
&y_H := \eta\zeta y,\ z_H := \xi\eta,\ c_H := \sqrt{x_H},\ s_H := \sqrt{y_H},\ d_H := \sqrt{z_H},\\
&W := s y_H,\ B^*_H := (B^* + W)/2,\ D^*_H := (D^* - W)/2,\\
&t := W/\left[1 - n\left(y - c d y_H\right)\right],\ J^*_H := (J^* - T(t,h))/2,
\end{aligned} \tag{62}$$

where (i) ξ, η, ζ, W, and t are working variables, (ii) $T(t,h)$ is the universal arc tangent function already defined in Eq. (43), the fast computing procedure of which is discussed in Appendix of [34], and (iii) $h \equiv n(1-n)(n-m)$ is a constant which can be computed at the beginning, stored and reused.

Similarly, the addition theorems computing the quantities corresponding to the summed argument, $w \equiv u + v$, from those corresponding to the arguments, u and v, are expressed as another sequence of operations as

$$\begin{aligned}
&\xi := c_u c_v,\ \eta := s_u s_v,\ \zeta := d_u d_v,\ \rho := 1/\left(1 - m y_u y_v\right),\\
&c_w := (\xi - \eta\zeta)\rho,\ s_w := \left(s_u c_v d_v + s_v c_u d_u\right)\rho,\\
&d_w := (\zeta - m\xi\eta)\rho,\ x_w := c_w^2,\ y_w := s_w^2,\ z_w := d_w^2,\\
&W := \eta s_w,\ B^*_w := B^*_u + B^*_v - W,\ D^*_w := D^*_u + D^*_v + W,\\
&t := W/\left[1 - n\left(y_w - c_w d_w \eta\right)\right],\ J^*_w := J^*_u + J^*_v + T(t,h),
\end{aligned} \tag{63}$$

where (i) the suffices, u, v, and w, mean that the quantities corresponding to the arguments, u, v, and w, respectively, and (ii) ξ, η, ζ, ρ, W, and t are again working variables. Notice the identity relations

$$\begin{aligned}
&y = s^2,\ x = 1 - y,\ z = 1 - my,\\
&c = \sqrt{x},\ d = \sqrt{z},\ B^* = u - D^*.
\end{aligned} \tag{64}$$

By using them, the Maclaurin series of the quantities needed to evaluate $G(u)$ are all derived from those of s, D^*, and J^*. Here those of s and J^* are already explicitly given in Eqs (23) and (39), respectively, while that of D^* is obtained from those of J^* by putting $n = 0$. Thus, the algorithm is completed.

G. Partial Derivatives with respect to Argument or Amplitude

The differential improvement of the approximate solution of $G(u) = 0$, a general equation containing the incomplete elliptic integrals and Jacobian elliptic functions, demands the evaluation of their partial derivatives with respect to the argument, u. Of course, the explicit expressions of the first few order derivatives are well known [1]. However, the resulting analytical expressions become lengthy and complicated when the order increases. This trouble is not resolved even by using `FullSimplify` command of Mathematica [11]. Therefore, newly developed is a recursive formulation to compute the derivatives systematically [40]. Its key point is an extensive usage of Leibniz's derivative formula

$$(PQ)_\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} P_j Q_{\ell-j}. \tag{65}$$

Here the suffix denotes the order of differentiation throughout this subsection only. In order to illustrate the idea plainly, show a sample using the trigonometric functions once again.

Consider the evaluation of arbitrary order derivatives of $\tau \equiv \tan\theta$. Introduce the auxiliary variables, $\gamma \equiv \cos\theta$ and $\sigma \equiv \sin\theta$. Then, τ becomes a variable satisfying the equation

$$\gamma\tau = \sigma. \tag{66}$$

The ℓ-th order derivative of its both hand sides becomes

$$(\gamma\tau)_\ell = \gamma\tau_\ell + \sum_{j=1}^{\ell} \binom{\ell}{j} \gamma_j \tau_{\ell-j} = \sigma_\ell. \tag{67}$$

This is solved for τ_ℓ, the ℓ-th order derivative of τ, as

$$\tau_\ell = \frac{1}{\gamma}\left[\sigma_\ell - \sum_{j=1}^{\ell}\begin{pmatrix}\ell\\ j\end{pmatrix}\gamma_j\tau_{\ell-j}\right], \tag{68}$$

which can be regarded as a recurrence formula to obtain τ_ℓ from the lower order derivatives, τ_j for $j=0,1,\cdots,\ell-1$, when the derivatives of auxiliary variables, γ and σ, are provided up to the order ℓ. The derivatives of γ and σ are well known as

$$\gamma_{2j} = (-1)^j\gamma,\ \ \gamma_{2j+1} = (-1)^{j+1}\sigma,$$
$$\sigma_{2j} = (-1)^j\sigma,\ \ \sigma_{2j+1} = (-1)^j\gamma. \tag{69}$$

Also the binomial coefficients can be generated recursively. Thus, τ_ℓ is recursively computed. Notice that, if γ^{-1} is computed once at the beginning, stored, and reused, the following recursions require multiply-and-add operations only. This fact does accelerate the derivative computation significantly.

Return back to the elliptic cases and consider the derivative computation of three Jacobian elliptic functions, $s\equiv \mathrm{sn}(u|m)$, $c\equiv \mathrm{cn}(u|m)$, and $d\equiv \mathrm{dn}(u|m)$, with respect to u. It is well known [1] that they satisfy the following differential relations

$$s_1 = cd,\ \ c_1 = -sd,\ \ d_1 = -msc. \tag{70}$$

These immediately lead to the recurrence formulas

$$s_{\ell+1} = \sum_{j=0}^{\ell}\begin{pmatrix}\ell\\ j\end{pmatrix}c_j d_{\ell-j},\ \ c_{\ell+1} = -\sum_{j=0}^{\ell}\begin{pmatrix}\ell\\ j\end{pmatrix}s_j d_{\ell-j},$$
$$d_{\ell+1} = -m\sum_{j=0}^{\ell}\begin{pmatrix}\ell\\ j\end{pmatrix}s_j c_{\ell-j}. \tag{71}$$

Move to the derivatives of $B^*\equiv B(u)$, $D^*\equiv D(u)$, and $J^*\equiv J(u)$, the associate incomplete elliptic integrals regarded as functions of u. By definition, their first order derivatives with respect to u are the integrand of their representations as

$$B_1^* = 1-y,\ \ D_1^* = y,\ \ J_1^* = y/(1-ny), \tag{72}$$

where $y\equiv s^2 = [\mathrm{sn}(u|m)]^2$. Thus, if the derivative expressions of y are obtained, the evaluation of the derivatives of B^* and D^* is trivial. Also, the implicit relation of J_1^* written as

$$(1-ny)J_1^* = y, \tag{73}$$

immediately leads to the recurrence formula to compute the derivatives of J^* as

$$J_{\ell+1}^* = \frac{1}{1+nJ_1^*}\left[y_\ell + n\sum_{j=1}^{\ell}\begin{pmatrix}\ell\\ j\end{pmatrix}y_j J_{\ell+1-j}^*\right]. \tag{74}$$

If $1/(1+nJ_1^*)$ is computed once at the beginning, stored, and reused, all the rest procedures consist of the additions and multiplications only. Meanwhile, the first and second order derivatives of y are written in the form as

$$y_1 = 2scd,\ \ y_2 = 2-4(1+m)y+6my^2. \tag{75}$$

The differentiation of the expression of y_2 results a recurrence formula to compute the higher order derivatives of y as

$$y_{\ell+3} = -4(1+m)y_{\ell+1} + 12m\sum_{j=0}^{\ell}\begin{pmatrix}\ell\\ j\end{pmatrix}y_j y_{\ell+1-j}. \tag{76}$$

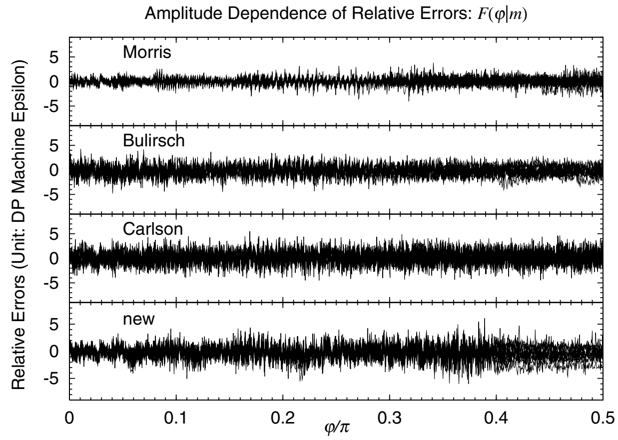

Fig. 1. Amplitude dependence of relative errors to compute $F(\varphi|m)$ in the double precision environment. The relative errors are measured by comparing with the quadruple precision extension of Bulirsch's `el1` and expressed in the unit of the double precision machine epsilon. Overlapped are the results for various values of m as $m = 0.0(0.1)1.0$.

In case of the partial derivatives of Legendre's normal form incomplete elliptic integrals with respect to φ, those of $\Delta\equiv\Delta(\varphi|m)$ play key roles. They are recursively computed as

$$\Delta_\ell = \frac{-1}{2\Delta}\left[m\psi_\ell + \sum_{j=1}^{\ell-1}\begin{pmatrix}\ell\\ j\end{pmatrix}\Delta_j\Delta_{\ell-j}\right], \tag{77}$$

where the derivatives of $\psi\equiv\sin^2\varphi$ are recursively obtained as

$$\psi_1 = 2\sqrt{\psi(1-\psi)},\ \ \psi_2 = 2-4\psi,\ \ \psi_\ell = -4\psi_{\ell-2}. \tag{78}$$

IV. Cost Performance

Examine the cost performance of the new methods. As an illustration of the error distributions, Fig. 1 compares the relative errors of the three existing and new methods to compute $F(\varphi|m)$. Here added is Morris's implementation [51] of the combination of various series expansions [52]. Obviously, there exist no significant differences among the computational errors. Meanwhile, the superiority of the new method in the computational speed is clear from Table I.

V. Conclusion

Reviewed here are a group of new methods to compute Legendre's complete and incomplete elliptic integrals and Jacobian elliptic functions. By means of the piecewise approximation and the repeated usage of the half and double argument transformations, the new methods are sufficiently accurate and run faster than the existing methods developed by Bulirsch, Carlson, and Cody. Also, summarized are the completely new methods to compute the inverse functions and partial derivatives of the elliptic integrals and functions.

All the published articles and the Fortran programs as well as their test drivers and output sample files are freely available from the following WEB site.

https://www.researchgate.net/profile/Toshio_Fukushima/

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