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# Search for periodic Hamiltonian flows: A generalized Bertrand's theorem

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A complete classification is given of the two-dimensional Hamiltonian systems (whose Hamilton–Jacobi equation separates in Cartesian or polar coordinates) which admit strictly periodic motions for open sets of initial conditions (completely degenerate systems). Any of the systems which are separable in Cartesian coordinates turn out to be canonically equivalent to some anisotropic harmonic oscillator. In the polar case our results provide a generalization of a celebrated theorem of Bertrand. It is proven that all the completely degenerate systems fall into two families. These families are characterized by the semiclassical inverse "spectral functions"

$n_1J_1+n_2J_2=\mathcal{J}(H)=\alpha(-H)^{-1/2}-\beta$,

$n_1J_1+n_2J_2=\mathcal{J}(H)=\alpha H-\beta$

($\alpha$, $\beta$ real positive constants) and contain, as central symmetric cases, the Kepler system and the isotropic harmonic oscillator, respectively. Qualitative and higher symmetry properties of these systems are also discussed.

## 1. INTRODUCTION

In this paper we present a method to characterize strictly periodic Hamiltonian dynamical systems in classical mechanics. It is well known, since the last century, that the only strictly periodic systems with rotational symmetry are the isotropic harmonic oscillator and the Kepler problem (Bertrand's theorem).[1,2] The extension to noncentral systems has been considered in the recent literature,[3,4] the aim being to classify systems possessing a maximal dynamical symmetry. The technique employed to determine such systems was based on the alleged connection between complete degeneracy of the motion (i.e., strict periodicity) and the separability of the Hamilton—Jacobi equation in more than one system of coordinates. As we stressed in previous papers,[5,6] this connection is not entirely rigorous and the results which stem from this principle must be checked. This is due, essentially, to the local character of curvilinear coordinates: The separation constants in such coordinates[7] may fail to be isolating in phase space.[5] Only if the motion is confined to the regularity domain of the local chart can one be sure that such constants of the motion are isolating, but this fact can be checked only on the explicit solutions of the equations of motion. On the other hand, there exist completely degenerate systems which are separable in only one system of coordinates (anisotropic oscillators, for instance) and these are overlooked in this approach.

We shall adopt a more direct attack on the problem. Starting from the existence of action-angle variables in a suitable region in phase space, we shall impose the degeneracy condition directly in the form

$$\mathcal{J}(H)=n_1J_1+n_2J_2, \tag{1.1}$$

where $H$ is the Hamiltonian, $(J_1,J_2)$ are action variables, $(n_1,n_2)$ are positive integers without common factors whose meaning is well known from analytical mechanics, and the function $\mathcal{J}(H)$ is *a priori* arbitrary—it is one of the unknown functions in our problem. This line has been followed by Enriotti and Faccini[4]; at a certain point in their analysis, however, they introduce an "ansatz" which severely limits the generality of their results. We shall be able, instead, to transform Eq. (1.1) into a system of integral equations where the potentials defining $H$ enter as unknown functions. These equations can be reduced to linear integral equations of the simplest type (Abel's equation); as such they are easily solved. This appears to be a simple extension to the multidimensional case of the "inverse problem" studied in Ref. 2 (Sec. 12). It is well known that the classical inverse scattering problem can also be solved in this way.[8]

We shall limit ourselves to systems with two degrees of freedom which are separable in Cartesian coordinates or in polar coordinates. In these cases we obtain an exhaustive characterization of all the potentials which admit strictly periodic motions. Unlike the previous known examples which depend only on a finite number of parameters, our potentials are defined in terms of one (polar case) or two (Cartesian case) arbitrary functions. It is nevertheless a simple task to check whether or not a given system is completely degenerate using our classification. The Cartesian case is essentially trivial, being completely reducible to the calculation of Landau and Lifshitz. The result in this case is that $H$ is canonically isomorphic to some harmonic oscillator. A more refined analysis is required for the polar case. We shall prove that the integral equations which follow from Eq. (1.1) admit solutions if and only if $\mathcal{J}$ is of the form

$$\mathcal{J}(H)=\begin{cases}\alpha(-H)^{-1/2}-\beta,\\ \alpha H-\beta\end{cases} \tag{1.2}$$

($\alpha,\beta$ real positive constants). We find two families of strictly periodic Hamiltonian systems, then; the Kepler problem belongs to the first family, the harmonics oscillator to the second one. This is the *generalized Bertrand's theorem* to which the title of this paper refers. All these Hamiltonians (except for $\beta=0$ which correspond to the Kepler problem and to the harmonic oscillator) have roughly the same qualitative properties. The motion takes place around a minimum of the potential which is at a distance $r_0 \propto \beta^2$ from the origin with a depth $V_0 \propto \beta^{-2}$ for the first family and $r_0 \propto \sqrt{\beta}$, $V_0 \propto \beta$ for the second one.

Little is known in general about the global symmetry in phase space of these Hamiltonians. In particular instances we have proved the existence of a global SU(2) symmetry for the members of the first (Kepler) family with $n_1 = n_2 = 1$; the Schrödinger equation of the corresponding quantum systems has been solved and it shows the characteristic higher degeneracy of a SU(2)-symmetric Hamiltonian. The existence of a dynamical (noninvariance) group and the "geometric quantization" of these Hamiltonians are being investigated.[9]

The paper is organized as follows: In Sec. 2 we discuss the Cartesian case, essentially to show how Abel's equation works in the simplest case. In Sec. 3 the more complex problem of systems which are separable in polar coordinates is dealt with and we prove our generalized Bertrand's theorem. In Sec. 4 we summarize the results and discuss the qualitative features and the symmetry properties of these Hamiltonians. Finally, the elementary properties of Abel's equation (Euler transform) are quoted in the Appendix, for the reader's convenience. Three tables at the end of the paper summarize the results.

## 2. "RECTANGULAR" SYSTEMS

Let us consider a particle of mass $m$ moving in a two-dimensional rectangular potential $U(x_1, x_2) = U_1(x_1) + U_2(x_2)$ so that the Hamiltonian is

$$H(x_1, x_2, p_1, p_2) = \frac{1}{2m}(p_1^2 + p_2^2) + U_1(x_1) + U_2(x_2). \qquad (2.1)$$

The Hamilton–Jacobi equation is then separable in Cartesian coordinates and the constants of separation may be chosen to be the partial energies $E_1$, $E_2$ $(E_1 + E_2 = E)$. Assuming that the functions $U_1$, $U_2$ are sufficiently well behaved and possess a local minimum at $x_{0_1}$, $x_{0_2}$ respectively, it follows that for certain intervals of values of $E_1$, $E_2$ the surface $\sum(E_1, E_2)$,

$$\sum(E_1, E_2) \equiv \left\{\frac{1}{2m} p_i^2 + U_i(x_i) = E_i, \quad i = 1, 2\right\}, \qquad (2.2)$$

is compact and sufficiently regular. Actually in this case $\sum(E_1, E_2)$ is a two-dimensional torus[10] and it is possible to define the action-angle variables

$$J_i = \frac{1}{2\pi} \oint \{2m[E_i - U_i(x)]\}^{1/2} dx. \qquad (2.3)$$

It is well known that the Hamiltonian is a function of these action variables only, and not of the conjugate angle variables $w_1$, $w_2$. It is also well known that the system admits strictly periodic motions if and only if the Hamiltonian depends on the action variables through the expression $n_1 J_1 + n_2 J_2$ with positive integers $n_1$, $n_2$,[11] i.e., if a continuous function $\mathcal{J}$ exists such that

$$\mathcal{J}(E) = n_1 J_1 + n_2 J_2 = \mathcal{J}(E_1 + E_2). \qquad (2.4)$$

Let us choose the additive constant of the energy in such a way that $U_i(x_{0_i}) = 0$. Since $J_i$ is a function of $E_i$ alone and $J_i(0) = 0$, it follows that

$$\mathcal{J}(E_1 + E_2) = n_1 J_1(E_1) + n_2 J_2(E_2) = \mathcal{J}(E_1) + \mathcal{J}(E_2), \qquad (2.5)$$

which, because of the continuity of $\mathcal{J}$, implies

$$\mathcal{J}(\lambda E) = \lambda \mathcal{J}(E), \qquad (2.6)$$

i.e., $\mathcal{J}$ is linear,

$$\mathcal{J}(E) = E/\omega \qquad (2.7)$$

($\omega$ real positive constant).

Then we have

$$J_i(E_i) = \frac{1}{2\pi} \oint \sqrt{2m[E_i - U_i(x)]}\, dx = E_i/(n_i \omega). \qquad (2.8)$$

We shall now regard Eq. (2.8) as a system of integral equations defining $U_i(x_i)$. The two cases $i = 1, 2$ can be solved separately; to simplify the notation, we drop the index $i$, for the moment, and rewrite the equation as

$$\oint \sqrt{2m[E - U(x)]}\, dx = \frac{2\pi}{n\omega} E. \qquad (2.8')$$

This equation can be linearized[2] by introducing the inverse function $x = x(U)$; since $U(x)$ has a minimum at $x = x_0$, $x(U)$ has two branches $x_+(U)$, $x_-(U)$ which merge at $U = 0$. Let $\delta(U) = x_+(U) - x_-(U)$ be the discontinuity along the cut. Equation (2.8′) is now transformed as follows,

$$\begin{aligned} \frac{2\pi}{n\omega} E &= 2\int_0^E \sqrt{2m(E-U)}\left(\frac{dx_+}{dU} - \frac{dx_-}{dU}\right) dU \\ &= 2\sqrt{2m}\int_0^E \sqrt{E-U}\, \frac{d\delta(U)}{dU}\, dU \\ &= \sqrt{2m}\int_0^E \frac{\delta(U)}{\sqrt{E-U}}\, dU. \end{aligned} \qquad (2.9)$$

After an integration by parts, we have used the fact that $\delta(0) = 0$. The equation

$$\int_0^E \frac{\delta(U)}{\sqrt{E-U}}\, dU = \frac{2\pi}{\sqrt{2m}\,n\omega} E \qquad (2.10)$$

is of Abel's type and the solution is found at once [see Eq. (A3)],

$$\delta(U) = \left(\frac{2}{m}\right)^{1/2} \frac{1}{n\omega} \int_0^U \frac{dE}{\sqrt{U-E}} = \frac{2}{n\omega}\left(\frac{2U}{m}\right)^{1/2}. \qquad (2.11)$$

The most general solution $U(x)$ of Eq. (2.8′) is then obtained by writing

$$x(U) = \pm\tfrac{1}{2}\delta(U) + G(U), \qquad (2.12)$$

where $G(U)$ is a single-valued function of $U$ near $U = 0$. The simplest choice, $G = x_0 =$ constant, corresponds to the harmonic oscillator potential

$$U(x) = \frac{m}{2}(n\omega)^2 (x - x_0)^2. \qquad (2.13)$$

In order that $U(x)$ has the required properties (single valuedness, a local minimum at $x_0$), $G(U)$ must be restricted to verify

$$|G'(U)| < (n\omega\sqrt{2mU})^{-1}. \qquad (2.14)$$

As an example, let us consider

$$G(U)=\alpha(U+\gamma)^{1/2} \tag{2.15}$$

with $\gamma>0$, $|\alpha|<\alpha_0\equiv(1/n\omega)(2/m)^{1/2}$. By inverting we find

$$U(x)=(\alpha_0^2-\alpha^2)^{-2}\{\alpha_0 x-\alpha[x^2+\gamma(\alpha_0^2-\alpha^2)]^{1/2}\}^2. \tag{2.16}$$

The choice $\alpha=\alpha_0$ gives rise to an infinite barrier at the origin $x=0$; the potential is

$$U(x)=(2\alpha_0)^{-2}\left(x-\frac{\alpha_0^2\gamma}{x}\right)^2, \tag{2.17}$$

which is well known to give oscillations with a period independent of the energy.

In conclusion, the most general Hamiltonian of the form (2.1) which satisfies Eq. (2.4) is constructed as follows: We choose any two functions $G_1(U_1)$, $G_2(U_2)$ satisfying Eq. (2.14); these are inserted into Eq. (2.12) which defines $U_1(x_1)$ and $U_2(x_2)$. Notice that all these Hamiltonians can be transformed into some anisotropic harmonic oscillator through a global canonical transformation.[10]

## 3. "POLAR" SYSTEMS

### A. The classical inverse bound state problem

Let us now consider a particle of mass $m$ moving in a potential $U(r,\varphi)$ so that the Hamiltonian is

$$H(r,\varphi,p_r,p_\varphi)=\frac{1}{2m}\left(p_r^2+\frac{p_\varphi^2}{r^2}\right)+V(r)+\frac{U(\varphi)}{r^2}. \tag{3.1}$$

The Hamilton–Jacobi equation is then separable in polar coordinates. We assume that for certain intervals of values of the separation constants $E,\lambda$ the surface

$$\textstyle\sum(E,\lambda)\equiv\begin{cases} H(r,p_r,\varphi,p_\varphi)=E,\\ p_\varphi^2+2mU(\varphi)=\lambda^2,\end{cases} \tag{3.2}$$

is compact and sufficiently regular. In this case, it is a two-dimensional torus and it is possible to define the action variables

$$\begin{aligned} J_r(E,\lambda)&=\frac{1}{2\pi}\oint\{2m[E-V(r)]-\lambda^2/r^2\}^{1/2}dr,\\ J_\varphi(\lambda)&=\frac{1}{2\pi}\oint[\lambda^2-2mU(\varphi)]^{1/2}d\varphi.\end{aligned} \tag{3.3}$$

Let

$$J_r(E,\lambda)=\mathcal{J}(E,J_\varphi(\lambda)). \tag{3.4}$$

We shall solve the following inverse problem: Given the functions $\mathcal{J}(E,J_\varphi)$ and $J_\varphi(\lambda)$, determine the potentials $V(r)$ and $U(\varphi)$. Let us recall that $\mathcal{J}$ *and* $J_\varphi(\lambda)$ *determine the semiclassical spectrum of* $H$, so that the problem may have an independent interest in itself. The problem will be solved separately for $V(r)$ and $U(\varphi)$. The effective potential is defined as usual,

$$V(\lambda,r)=V(r)+\frac{\lambda^2}{2mr^2}. \tag{3.5}$$

In order to have bounded orbits on a compact $\sum(E,\lambda)$ the effective potential must have a minimum at some $r_0(\lambda)$; we define

$$V_0(\lambda)=V(\lambda,r_0(\lambda)). \tag{3.6}$$

In a neighborhood of $r_0(\lambda)$ we define the inverse function $r(\lambda,V)$ with two branches $r_1(\lambda,V)\geq r_2(\lambda,V)$. We have

$$\begin{aligned} J_r(E,\lambda)&=\frac{1}{\pi}\int_{V_0}^{E}\sqrt{2m(E-V)}\,\frac{d}{dV}\,(r_1(\lambda,V)-r_2(\lambda,V))\,dV\\ &=\frac{\sqrt{2m}}{\pi}\int_{V_0}^{E}\sqrt{E-V}\,\delta'(\lambda,V)\,dV\\ &=\frac{\sqrt{2m}}{2\pi}\int_{V_0}^{E}\frac{\delta(\lambda,V)}{\sqrt{E-V}}\,dV,\end{aligned} \tag{3.7}$$

where $\delta(\lambda,V)=r_1-r_2$. We are led to an Abel integral equation also in this case. Solving for $\delta$ [see Eq. (A3)] we obtain

$$\delta(\lambda,V)=\left(\frac{2}{m}\right)^{1/2}\int_{V_0(\lambda)}^{V}\frac{\mathcal{J}_E'(E,J_\varphi)}{\sqrt{V-E}}\,dE. \tag{3.8}$$

Now let us differentiate the first Eq. (3.3) with respect to $\lambda$; it follows that

$$\frac{\lambda}{2\pi}\oint\frac{d(1/r)}{\sqrt{2m(E-V)}}=\frac{\partial}{\partial\lambda}J_r(E,\lambda). \tag{3.9}$$

Then, by defining $\eta(\lambda,V)=1/r_2-1/r_1$ we have

$$\frac{\lambda}{\pi}\int_{V_0}^{E}\frac{\eta'(\lambda,V)}{\sqrt{2m(E-V)}}\,dV=\frac{\partial}{\partial\lambda}J_r(E,\lambda). \tag{3.10}$$

As usual, we obtain

$$\eta(\lambda,V)=\frac{\sqrt{2m}}{\lambda}\int_{V_0(\lambda)}^{V}\frac{\mathcal{J}_\lambda'(E,J_\varphi(\lambda))}{\sqrt{V-E}}\,dE. \tag{3.11}$$

Finally, from the knowledge of $\delta$ and $\eta$ we can find $r(\lambda,V)$, namely

$$r=r(\lambda,V)=\pm\tfrac12\delta(\lambda,V)+\left(\frac14\delta^2(\lambda,V)+\frac{\delta(\lambda,V)}{\eta(\lambda,V)}\right)^{1/2}. \tag{3.12}$$

Equations (3.8), (3.11), and (3.12) completely solve the problem for the radial potential $V(r)$. Actually the function $V_0(\lambda)$ is readily obtained from the data $\mathcal{J}(E,J_\varphi)$ and $J_\varphi(\lambda)$, since it holds that

$$\mathcal{J}(V_0(\lambda),J_\varphi(\lambda))=0. \tag{3.13}$$

*Notice that, unlike the Cartesian case,* $V(r)$ *is uniquely determined.*

We can find $U(\varphi)$ by a similar procedure. Let $U_0$ be the minimum of $U(\varphi)$. In a neighborhood of $U_0$ we can define the inverse function $\varphi(U)$ with two branches $\varphi_1(U)\geq\varphi_2(U)$. For $\lambda^2$ sufficiently close to $2mU_0$ it must hold that

$$\begin{aligned} &\sqrt{2m}\int_{U_0}^{\lambda^2/2m}\left(\frac{\lambda^2}{2m}-U\right)^{1/2}\frac{d}{dU}[\varphi_1(U)-\varphi_2(U)]\,dU=\pi J_\varphi(\lambda),\\ &J_\varphi((2mU_0)^{1/2})=0.\end{aligned} \tag{3.14}$$

This is again an Abel equation which can be solved to give

$$\delta\varphi=\varphi_1(U)-\varphi_2(U)=\left(\frac{2}{m}\right)^{1/2}\int_{U_0}^{U}\frac{(d/d\Lambda)J_\varphi(\sqrt{2m\Lambda})}{\sqrt{U-\Lambda}}\,d\Lambda. \tag{3.15}$$

As in Sec. 2, the most general $U(\varphi)$ is then obtained by solving for $U$ in the equation

$$\varphi(U)=\pm\tfrac{1}{2}\delta\varphi(U)+G(U), \qquad (3.16)$$

where $G$ is any regular function in a neighborhood of the real positive $U$ axis ($U \geq U_0$).

This formulation of the problem is correct in the assumption that $U_0$ is an isolated minimum, but not every choice of $J_\varphi(\lambda)$ is compatible with this assumption. For example, $J_\varphi(\lambda)=\lambda$ does not make sense when inserted in Eq. (3.15); we would obtain $\delta\varphi(U)=\pi$ for every $U$ which is incompatible with a local minimum of $U$. This fact raises a serious problem, which must also be solved for $J_r(E,\lambda)$, namely to determine the class of admissible initial data $J_r(E,\lambda)$, $J_\varphi(\lambda)$ for which a solution to the inverse problem exists. What may happen is that the potential $V(\lambda, r)$ which we obtain by solving for $V$ in Eq. (3.12), fails to be of the form $V(r)+\lambda^2/(2mr^2)$. In this case all of the formalism breaks down, and the only conclusion would be that the data is incompatible with the particular model given by Eq. (3.1). This problem will be solved in Sec. 3C in the special case of completely degenerate systems.

### B. Example: The "Kepler family"

An interesting example (which we shall prove to exhaust almost all the possibilities for the periodic case) is given by the following choice of initial data:

$$\begin{aligned}
&J_r(E,\lambda)=\mathcal{J}(E)-qJ_\varphi(\lambda),\\
&\mathcal{J}(E)=mk(-2mE)^{-1/2}-\beta \quad (k,\beta \text{ real positive constants}),\\
&J_\varphi(\lambda)=J(\lambda), \quad \text{(to be determined)}\\
&\quad q=\text{rational positive number}.
\end{aligned} \qquad (3.17)$$

By inserting this data into Eqs. (3.8), (3.11), and (3.13), we easily obtain:

$$\begin{aligned}
&\delta(\lambda, V)=-\frac{k}{V}\left(1-\frac{V}{V_0}\right)^{1/2},\\
&\eta(\lambda, V)=2\sqrt{2m}\,q[J'(\lambda)/\lambda](V-V_0(\lambda))^{1/2},\\
&mk[-2mV_0(\lambda)]^{-1/2}=qJ(\lambda)+\beta.
\end{aligned} \qquad (3.18)$$

From Eq. (3.12) we then obtain

$$r(\lambda, V)=\pm\frac{k}{2V}\left(1-\frac{V}{V_0}\right)^{1/2}+\left[\left(-\frac{k}{2V}\right)^2\left(1-\frac{V}{V_0}\right)-\frac{k\lambda}{2\sqrt{-2mV_0}\,qJ'V}\right]^{1/2} \qquad (3.19)$$

and, after some algebraic manipulations,

$$r^2V^2+2\left(\frac{\lambda(qJ+\beta)}{2mqJ'}-\frac{(qJ+\beta)^2}{m}\right)-k^2+\left(\frac{\lambda}{2mr}\right)^2\left(\frac{qJ+\beta}{qJ'}\right)^2=0. \qquad (3.20)$$

This shows that the solution $V=V(\lambda, r)$ is of the form $V(r)+\lambda^2/(2mr^2)$ *if and only if* $J(\lambda)$ *satisfies the differential equation*

$$-\frac{\lambda(qJ+\beta)}{qJ'}+2(qJ+\beta)^2=\lambda^2+c_1, \qquad (3.21)$$

whose general solution is

$$J(\lambda)=\frac{1}{\sqrt{2q}}\{\lambda^2+c_1\pm[(\lambda^2+c_1)^2+c_2]^{1/2}\}^{1/2}-\frac{\beta}{q}. \qquad (3.22)$$

Correspondingly we obtain the following potential,

$$V(r)=c_1/2mr^2-(k^2r^2-c_2(2m)^{-2})^{1/2}\,r^{-2}. \qquad (3.23)$$

The constant $c_1$ is irrelevant since it can always be associated with $U(\varphi)$; it will be adjusted in order to have $U(\varphi)\geq 0$, as we have tacitly assumed in Eq. (3.2). Equation (3.22) gives the most general $J(\lambda)$ for which the radial inverse problem has solutions: only the plus sign survives when we solve for the angular potential. To simplify the calculation, let $c_1=0$, $c_2=-\gamma^4$, then it holds that

$$J(\lambda)=\frac{1}{2q}[(\lambda^2+\gamma^2)^{1/2}\pm(\lambda^2-\gamma^2)^{1/2}]-\frac{\beta}{q}. \qquad (3.24)$$

This is to be inserted into Eq. (3.15) to yield

$$\delta\varphi(U)=q^{-1}\tan^{-1}\left(\frac{U-U_0}{U_0+\gamma^2/(2m)}\right)^{1/2}+q^{-1}\tan^{-1}\left(\frac{U-U_0}{U_0-\gamma^2/(2m)}\right)^{1/2}, \qquad (3.25)$$

where $U_0=(1/2m)[\beta^2+(\gamma^2/2\beta)^2]$. The minus sign in Eq. (3.22) would give a negative $\delta\varphi(U)$ and must be discarded. According to Eq. (3.16) we have now to choose a function $G(U)$; the simplest choice is $G=0$ which gives

$$U(\varphi)=\frac{\beta^2}{m}\,\frac{1+\alpha^2\cos^2(2q\varphi)}{1+\cos(2q\varphi)[1-\alpha^2\sin^2(2q\varphi)]^{1/2}} \qquad (3.26)$$

which is defined for $|\varphi|<\pi/2q$, $\alpha=\frac{1}{2}(\gamma/\beta)^2<1$. The limiting case $\alpha\to 1$ gives a nonanalytic potential for which $U_0$ is not an isolated minimum; apart from that, the potential is continuous, with continuous gradient and it gives a strictly periodic motion,

$$U(\varphi)\big|_{\alpha=1}=\frac{\beta^2}{2m}\,\frac{1+\cos^2(2q\varphi)}{1+\cos(2q\varphi)\,|\cos(2q\varphi)|}. \qquad (3.27)$$

### C. The generalized Bertrand's theorem

We are now prepared to examine the basic problem of classifying all the Hamiltonians which admit separation in polar coordinates and whose time evolution is periodic for an open region in phase space. We shall make use of the formalism of Sec. 3 A and we shall choose

$$J_r(E,\lambda)\equiv\mathcal{J}(E)-qJ(\lambda), \quad J_\varphi(\lambda)\equiv J(\lambda), \qquad (3.28)$$

where $q$ is any positive rational number. The example of Sec. 3B shows that *not every input* $\{\mathcal{J}(E), J(\lambda), q\}$ *is compatible with the structure of the Hamiltonian*. Therefore, we shall now determine the most general admissible input. Equations (3.8), (3.11), and (3.13) assume the form

$$\begin{aligned}
&\text{(i) } \delta(\lambda, V)=\left(\frac{2}{m}\right)^{1/2}\int_{V_0(\lambda)}^{V}\frac{\mathcal{J}'(E)\,dE}{\sqrt{V-E}},\\
&\text{(ii) } \eta(\lambda, V)=2\sqrt{2m}\,q\,\frac{J'(\lambda)}{\lambda}\sqrt{V-V_0(\lambda)},\\
&\text{(iii) } \mathcal{J}(V_0(\lambda))=qJ(\lambda).
\end{aligned} \qquad (3.29)$$

By differentiation we find

$$\delta'_\lambda = -\left(\frac{2}{m}\right)^{1/2} \frac{qJ'(\lambda)}{\sqrt{V - V_0(\lambda)}} \tag{3.30}$$

and Eq. (3.12) is transformed into

$$r(\lambda, V) = \pm\tfrac{1}{2}\delta + \left(\frac{\delta^2}{4} - \frac{\lambda\delta\delta'_\lambda}{(2qJ')^2}\right)^{1/2}. \tag{3.31}$$

The condition that $V(\lambda, r)$, implicitly defined by Eq. (3.31), be of the form $V(r) + \lambda^2/(2mr^2)$ can be stated in differential terms as follows:

$$\left.\frac{\partial V}{\partial \lambda}\right|_{r=\mathrm{const}} = -\frac{\partial r}{\partial \lambda}\bigg/\frac{\partial r}{\partial V} = \frac{\lambda}{mr^2}, \tag{3.32}$$

i.e.,

$$mr^2 \frac{\partial r}{\partial \lambda} + \frac{\partial r}{\partial V} = 0. \tag{3.33}$$

By inserting $r(\lambda, V)$ in terms of $\delta$ [Eq. (3.31)], we obtain *two* partial differential equations in the single unknown $\delta$, namely

$$\delta''_{\lambda V} = -\Omega\delta'^3_\lambda, \tag{3.34}$$

$$\delta''_{\lambda\lambda} = \frac{\delta'_V}{\Omega\delta^2} - 2\frac{\delta'^2_\lambda}{\delta} - \frac{\Omega - 1 + \lambda\Omega'}{\lambda\Omega}\delta'_\lambda, \tag{3.35}$$

where $\Omega = [2qJ'(\lambda)]^{-2}$. [Notice that Eq. (3.34) is precisely the differential version of Eq. (3.30). In order to simplify the notation, we put $m = 1$ until Eq. (3.60)].

Since we have more equations than unknown functions, some compatibility conditions must be satisifed. We proceed as follows: By differentiating Eq. (3.34) with respect to $\lambda$, and Eq. (3.35) with respect to $V$ we find a third equation, independent from the previous ones, namely

$$\delta''_{VV} = 2\frac{\delta'^2_V}{\delta} + 2\Omega^2\delta\delta'^4_\lambda - 5\Omega\delta'_V\delta'^2_\lambda + 2\Omega\left(\frac{\Omega-1}{\lambda} + \frac{\Omega'}{2}\right)\delta^2\delta'^3_\lambda. \tag{3.36}$$

There is a second integrability condition between this latter and Eq. (3.34) which gives

$$\begin{aligned}&12(\Omega\delta\delta'^2_\lambda - \delta'_V)^2 + 16\left(\frac{\Omega-1}{\lambda} + \frac{\Omega'}{2}\right)\delta'_\lambda\delta^2(\Omega\delta\delta'^2_\lambda - \delta'_V)\\ &\quad - \delta^4\delta'^2_\lambda\left(\Omega\Omega'' - 2\Omega'^2 + \lambda^{-1}(7\Omega' - 5\Omega\Omega') + \frac{(\Omega-1)(6-8\Omega)}{\lambda^2}\right)\\ &\quad = 0.\end{aligned} \tag{3.37}$$

This implies that for some function $G(\lambda)$ it must hold that

$$\delta'_V = \Omega\delta\delta'^2_\lambda + G(\lambda)\,\delta'_\lambda\delta^2. \tag{3.38}$$

By comparing this latter equation with Eq. (3.34) we find

$$\begin{aligned}&\delta\delta'^2_\lambda\left[3G - 2\left(\frac{\Omega-1}{\lambda} + \frac{\Omega'}{2}\right)\right]\\ &\quad + \delta'_\lambda\delta^2\left[G' + \frac{G^2}{\Omega} - \frac{G}{\Omega}\left(\frac{\Omega-1}{\lambda} + \Omega'\right)\right] = 0.\end{aligned} \tag{3.39}$$

We are forced now to conclude that both quantities in square brackets in Eq. (3.39) must vanish, since otherwise we would obtain that

$$\delta'_\lambda/\delta = \Phi(\lambda), \tag{3.40}$$

i.e., $\delta(\lambda, V)$ would be separated in a product of a function of $\lambda$ times a function of $V$ which in turn would imply [through Eq. (3.30)] that

$$V_0(\lambda) = \mathrm{const} \Rightarrow J(\lambda) = \mathrm{const} \tag{3.41}$$

which is nonsense. In conclusion we can go on with Eq. (3.38) where $G(\lambda)$ is a solution of the following equations:

$$\begin{aligned}G(\lambda) &= \tfrac{1}{3}\left(\Omega' + 2\frac{\Omega-1}{\lambda}\right),\\ G'(\lambda) &= \frac{G}{3\Omega}\left(2\Omega' + \frac{\Omega-1}{\lambda}\right).\end{aligned} \tag{3.42}$$

It follows that the function $\Omega(\lambda)$, which is related to our input function $J(\lambda)$, must satisfy the following second order equation

$$3\Omega\Omega'' - 2\Omega'^2 + \lambda^{-1}\Omega'(\Omega+5) - 2\lambda^{-2}(\Omega-1)(4\Omega-1) = 0. \tag{3.43}$$

The general solution of this equation can be found to be

$$\Omega(\lambda) = \frac{(\lambda^2 + c_1)^2 + c_2}{2\lambda^2[\lambda^2 + c_1 \pm ((\lambda^2 + c_1)^2 + c_2)^{1/2}]}. \tag{3.44}$$

Actually it will be more convenient to use Eq. (3.42) as the formal definition of $G(\lambda)$. From Eq. (3.38) it follows that

$$\Delta'_V = \frac{1}{3}\left(5\Omega' + 4\frac{\Omega-1}{\lambda}\right)\Delta^2 + 2\Omega\Delta\Delta'_\lambda, \tag{3.45}$$

where $\Delta = \delta\delta'_\lambda$; we can also derive an equation for $\Delta$ directly from Eq. (3.35); it holds that

$$\Delta'_V = \left(\Omega' + \frac{\Omega-1}{\lambda}\right)\Delta^2 + \Omega\Delta\Delta'_\lambda. \tag{3.46}$$

Let us solve Eq. (3.45), (3.46) with respect to $\Delta'_\lambda$ and $\Delta'_V$:

$$\Delta'_\lambda = -\frac{G'}{G}\Delta, \quad \Delta'_V = G\Delta^2. \tag{3.47}$$

The general solution is then

$$\Delta(\lambda, V) = -G(\lambda)^{-1}(V - a)^{-1} \tag{3.48}$$

The only integration constant $a$ may be dropped, since it is an additive constant to the energy. The singular case $G(\lambda) \equiv 0$ must be treated separately (see below). From Eq. (3.48) we obtain

$$\delta = \frac{\Delta}{\delta'_\lambda} = \frac{\sqrt{2\Omega}\sqrt{V - V_0(\lambda)}}{VG(\lambda)}. \tag{3.49}$$

By differentiating with respect to $\lambda$ and comparing to Eq. (3.30), it follows that

$$\frac{V'_0}{V_0} = \frac{G(\lambda)}{\Omega(\lambda)} = \frac{d}{d\lambda}\ln\frac{G(\lambda)^2}{\Omega}, \tag{3.50}$$

where the last step is a consequence of Eq. (3.42). In conclusion we have

$$V_0(\lambda) \propto G^2/\Omega. \tag{3.51}$$

Now by computing the derivative of $(-V_0)^{-1/2}$ we find

$$\frac{d}{d\lambda}[-V_0(\lambda)]^{-1/2} \propto \frac{d}{d\lambda}\left(\frac{\sqrt{\Omega}}{G}\right) = -\frac{1}{2\sqrt{\Omega}} \propto qJ', \tag{3.52}$$

hence

$$[-V_0(\lambda)]^{-1/2} \propto qJ(\lambda) + \text{const.} \tag{3.53}$$

By recalling Eq. (3.29iii) we conclude that the only admissible $\mathcal{J}(E)$ is given by

$$\mathcal{J}(E) = \alpha(-E)^{-1/2} - \beta. \tag{3.54}$$

As a consequence, the potentials found in Sec. 3B) exhaust all the possibilities, except for the singular case $G \equiv 0$, which we shall now examine.

Equation (3.47) with $G(\lambda) \equiv 0$ becomes

$$\Delta'_V = 0, \quad \Delta'_\lambda = -\frac{\Omega'}{2\Omega}\,\Delta \tag{3.55}$$

which imply $\Delta(\lambda, V) = \text{const} \times \Omega^{-1/2}$. It follows that

$$\delta(\lambda, V) = -\sqrt{2}\,C\sqrt{V - V_0} \quad (C \text{ a real constant}). \tag{3.56}$$

An argument similar to that used above shows that in this case

$$V_0'(\lambda) \propto \Omega^{-1/2} = 2qJ'(\lambda), \tag{3.57}$$

hence

$$\mathcal{J}(E) = \alpha E - \beta. \tag{3.58}$$

In conclusion the only admissible input functions are

$$\mathcal{J}(E) = \begin{cases} \alpha(-E)^{-1/2} - \beta, \\ \alpha E - \beta. \end{cases} \tag{3.59}$$

This is the result we refer to as a "generalized Bertrand's theorem."

We are left to consider in detail the case $\mathcal{J}(E) = \alpha E - \beta$. We have

$$\alpha V_0(\lambda) - \beta = qJ(\lambda),$$

$$\delta = \left(\frac{2}{m}\right)^{1/2} 2\alpha\sqrt{V - V_0},$$

$$r(\lambda, V) = \pm\alpha\left(\frac{2}{m}\right)^{1/2}\sqrt{V - V_0} + \left(\frac{2\alpha^2}{m}(V - V_0) + \frac{\alpha\lambda}{mqJ'}\right)^{1/2}. \tag{3.60}$$

By inverting we find

$$V(\lambda, r) = V_0 + \frac{m}{8\alpha^2}\,r^2 - \frac{\lambda}{4\alpha qJ'} + \frac{\lambda^2}{2m(2qJ')^2 r^2}. \tag{3.61}$$

The admissible $J(\lambda)$ are given by

$$(2qJ')^{-2} = 1 + c_1\lambda^{-2}. \tag{3.62}$$

Correspondingly the potential $V(r)$ is given by

$$V(r) = \frac{m}{8\alpha^2}\,r^2 + \frac{c_1}{2mr^2}. \tag{3.63}$$

The second term is trivial, since it can be reabsorbed into the angular part $U(\varphi)$; therefore, a linear $\mathcal{J}(E)$ corresponds to a harmonic oscillator radial potential. $U(\varphi)$ can be obtained as a special case of Eq. (3.25) with $\gamma = 0$; we do not find anything new with respect to the potentials already known, except for the arbitrariness in the choice of $G(U)$.

The discussion of the angular part $U(\varphi)$ is not complete until we have discussed the following situation. Suppose that $U(\varphi)$ is bounded and let

$$U_0 \le U \le U_1, \quad \lambda^2 > 2mU_1. \tag{3.64}$$

This means that the particle rotates around the origin. The equation defining $J_\varphi(\lambda)$ is the following,

$$J_\varphi(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi[\lambda^2 - 2mU(\varphi)]^{1/2}, \tag{3.65}$$

which *is not* equivalent to an integral equation of Abel's type. In general we do not have an explicit inversion formula for this case, however if we require that the flow is periodic for $\lambda^2 > 2mU_1$, we know that $J_\varphi(\lambda)$ is restricted to the form given in Eqs. (3.22) and (3.24). For this particular input we can solve Eq. (3.65). Let

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi[\lambda^2 - 2mU(\varphi)]^{1/2} = \frac{1}{2q}(\lambda^2 + \gamma^2)^{1/2} \pm \frac{1}{2q}(\lambda^2 - \gamma^2)^{1/2} - \frac{\beta}{q}. \tag{3.66}$$

An asymptotic estimate as $\lambda \to \infty$ shows that $\beta = 0$, $q = 1$, and only the plus sign is admissible. The equation

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi[\lambda^2 - 2mU(\varphi)]^{1/2} = \frac{1}{2q}(\lambda^2 + \gamma^2)^{1/2} + \frac{1}{2q}(\lambda^2 - \gamma^2)^{1/2} \tag{3.67}$$

admits an infinite number of solutions obtained as follows: Let $A_1$, $A_2$ be a measurable partition of the interval $(0, \ldots, 2\pi)$ such that $\mu(A_1) = \mu(A_2) = \pi$. The potential

$$U(\varphi) = \begin{cases} -\gamma^2/2m, & \varphi \in A_1, \\ +\gamma^2/2m, & \varphi \in A_2, \end{cases} \tag{3.68}$$

is a solution of Eq. (3.67). It is still to be shown that this is the general solution. To this aim, let us make an expansion in $\lambda^{-1}$,

$$(2\pi)^{-1}\int_0^{2\pi} d\varphi \sum_n \binom{1/2}{n}(-2mU(\varphi))^n\lambda^{-2n} = \tfrac{1}{2}\sum_n \binom{1/2}{n}[\gamma^{2n} + (-\gamma)^{2n}]\lambda^{-2n}, \tag{3.69}$$

which implies ($a = \gamma^2/2m$)

$$(2\pi)^{-1}\int_0^{2\pi} U(\varphi)^n\,d\varphi = \tfrac{1}{2}[a^n + (-a)^n]. \tag{3.70}$$

It follows that

$$(2\pi)^{-1}\int_0^{2\pi}[U(\varphi)^2 - a^2]^2\,d\varphi = 0. \tag{3.71}$$

$(U^2 - a^2)^2$ being nonnegative, it must vanish, which shows that Eq. (3.68) is the general solution. The other choice of $c_2$ [i.e., $J(\lambda)$ given by Eq. (3.22) with $c_2 > 0$] does not admit solutions of this type, since we would obtain an equation

$$(2\pi)^{-1}\int_0^{2\pi} U^{2n}\,d\varphi = (-1)^n a^{2n}, \tag{3.72}$$

which is impossible.

## 4. CONCLUDING REMARKS

Let us summarize the results obtained in previous sections and briefly comment on them. The most general Hamiltonian in two degrees of freedom with strictly periodic time evolution has been determined by assuming separability in Cartesian coordinates *or* polar coordinates. The result is given in Table I (Cartesian case) and Table II′–II″ (polar case). It will be noted that previously known potentials are contained as special cases.[3,4] We have not listed discontinuous potentials which arise for some special choice of parameters (see Sec. 3B) nor the potentials given by Eq. (3.68) which also are discontinuous.

It would take too long to study the properties of these Hamiltonians in detail. We shall limit ourselves to a few remarks.

First of all, since all the Hamiltonians belonging to the same family have in common the same function $\mathcal{J}(H)$ apart from the additive constant $\beta$, one may ask whether it is possible to identify them by means of a canonical transformation. The simplest candidate is of course a translation on the action variables,

$$\begin{aligned} J \to \tilde{J} &= J + \text{const}, \\ w \to \tilde{w} &= w. \end{aligned} \tag{4.1}$$

However such a transformation cannot be everywhere defined since the action variables are positive definite. Even if we allow for more general transformations, it can be proven that Hamiltonians with different values of $\beta$ are canonically inequivalent from a global point of view.[10] This fact is obvious in the case $\beta = 0$ for the "Kepler" family ($H$ is unbounded from below) as compared to the case $\beta \neq 0$ ($H$ is bounded from below). Still a transformation like Eq. (4.1) may be used *locally* to find the solution of Hamilton's equations for a Hamiltonian with $\beta > 0$ starting from the known solution of the Kepler problem [see for example the Hamiltonian defined by Eq. (4.3) below].

Another interesting point is the existence of a global symmetry [SO(3) or SU(2)] for the Hamiltonians we have classified. In the Cartesian case, with $n_1 = n_2$, it is fairly obvious that a suitable choice of $G_i(U_i)$ such that $J_1$ and $J_2$ are still defined in the whole range $(0, +\infty)$ gives a Hamiltonian essentially equivalent to an isotropic harmonic oscillator. We conclude, in this case, that the Hamiltonian is SU(2)-symmetric, the realiza-

TABLE I. Hamiltonian in Cartesian coordinates $(x_1, x_2, p_1, p_2)$.

| |
|---|
| $H = \frac{1}{2m}(p_1^2 + p_2^2) + U_1(x_1) + U_2(x_2)$ |
| $\mathcal{J}(H) = n_1 J_1 + n_2 J_2 = \frac{H}{\omega}$ |
| $x_i(U_i) = \left(\frac{2}{m}\right)^{1/2} U_i^{1/2}(n_i\omega)^{-1} + G_i(U_i) \quad (i = 1, 2)$ |
| $G_i(U_i)$ single valued in a neighborhood of $U_i \geq 0$ |
| $\lvert G_i' \rvert < (2m\, n_i^2 \omega^2 U_i)^{-1/2}$ |
| Special cases: |
| $G_i = 0$ Harmonic oscillator |
| $G_i = \alpha(U_i + \gamma)^{1/2}$ Eq. (2.16) |
| $\alpha = (1/n_i\omega)\left(\frac{2}{m}\right)^{1/2}$ Eq. (2.17) |

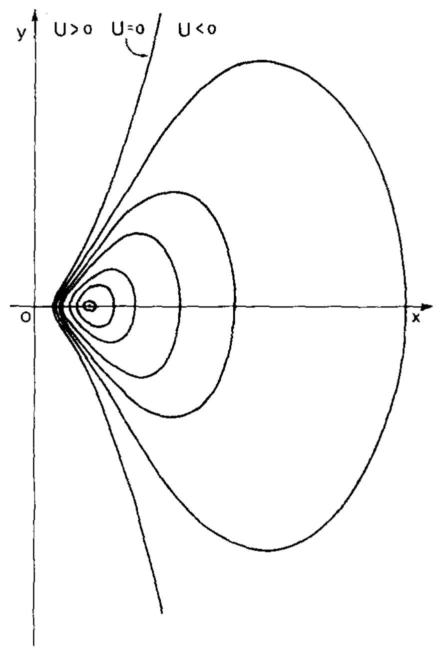


FIG. 1. $U(r, \varphi)$. $\gamma = \kappa = 1$, $m = 1/2$, $q = 1$, $\beta = (3)^{-1/4}$, $r_0 = (3)^{-1/2}$.

tion of the Lie algebra in terms of Poisson brackets being given by

$$\begin{aligned} M_1 &= (J_1 J_2)^{1/2} \cos(w_1 - w_2), \\ M_2 &= (J_1 J_2)^{1/2} \sin(w_1 - w_2), \\ M_3 &= \tfrac{1}{2}(J_1 - J_2), \\ I &= (M_1^2 + M_2^2 + M_3^2)^{1/2} = H/(2\omega). \end{aligned} \tag{4.2}$$

Consider now the polar case. Let $H$ belong to the first family, $\beta > 0$, $G(U) = 0$, $q = 1$. The potential has a minimum at $\varphi = 0$, $r = r_0 \propto \beta^2$ with a depth $V_0 \propto \beta^{-2}$ (see Fig. 1). Unlike the Kepler problem, both $J_r$ and $J_\varphi$ are defined in the range $(0, +\infty)$. The motion takes place around the minimum and for a sufficiently low energy the system can be approximated by an isotropic harmonic oscillator. So we expect a SU(2) symmetry instead of the SO(3) symmetry of the regularized Kepler problem. If we defined $M_i$ as in Eq. (4.2) with $J_1 = J_\varphi$ and $J_2 = J_r$ we have a realization of the Lie algebra of SU(2); in some cases it is easy to prove that this realization can be integrated to give a global realization of the group SU(2). This is the case of the Hamiltonian (first given in Ref. 3)[12]

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p^2}{r^2}\right) - \frac{k}{r} + \beta^2(2mr^2\cos^2\varphi)^{-1}. \tag{4.3}$$

A direct calculation of $M_i(J, w)$ shows that these functions are differentiable throughout the energy surface $H = E(-mk^2/(2\beta^2) < E < 0)$; this fact, together with the compactness of the energy surface, implies the integrability of the Lie algebra realization to a realization of SU(2) (according to a well-known theorem by Palais[13]). It is plausible that the other Hamiltonians in both families (at least for $q = 1$) also share the same SU(2) symmetry. The isotropic oscillator ($q = \frac{1}{2}$) and the Kepler problem emerge here as rather exceptional cases.

Finally, it may be interesting to investigate whether

TABLE II′. Hamiltonian in polar coordinates $(r, \varphi, p_\varphi)$—"Kepler" family.

| |
|---|
| $H=\frac{1}{2m}(p_r^2+p_\varphi^2/r^2)+V(r)+U(\varphi)/r^2$ |
| $\mathcal{J}(H)=J_r+q\,J_\varphi=\frac{mk}{\sqrt{-2mH}}-\beta$ |
| $J(\lambda)=\frac{1}{\sqrt{2}\,q}\{\lambda^2+c_1+[(\lambda^2+c_1)^2+c_2]^{1/2}\}^{1/2}-\frac{\beta}{q}$ |
| $V(r)=\frac{c_1}{2mr^2}-\frac{1}{r^2}\left(k^2r^2-\frac{c_2}{(2m)^2}\right)^{1/2}$ |
| $\varphi(U)=\pm\frac{1}{2q}\tan^{-1}\left(\frac{2\beta\sqrt{2m(U-U_0)}}{\Gamma-2m(U-U_0)}\right)+G(U)\quad(\beta>0),\quad U(\varphi)=0 \text{ if } \beta=0$ |
| $\beta\geqslant 0,\quad q=n_1/n_2>0,\quad k>0\quad (c_1, c_2)$ real constants, $\beta=0\Longrightarrow c_1=c_2=0,\quad q=1$ (Kepler) |
| $U_0=(2m)^{-1}[\beta^2-c_2(2\beta)^{-2}-c_1]\geqslant 0,\quad \Gamma=\beta^2+c_2(2\beta)^{-2}$ |
| $\lvert G'(U)\rvert<\beta\left(U+\frac{c_1}{2m}+\frac{c_2}{4m\beta^2}\right)[2m(U-U_0)]^{1/2}\left[\left(U+\frac{c_1}{2m}+\frac{c_2}{4m\beta^2}\right)^2+\frac{c_2}{4m\beta^2}\right]^{-1}$ |
| $c_1=0,\quad c_2=-\gamma^4,\quad G=0$: $V=-\frac{1}{r}\left[k^2+\left(\frac{\gamma^2}{2mr}\right)^2\right]^{1/2},\quad U(\varphi)=\frac{\beta^2}{m}\frac{1+\alpha^2\cos^2(2q\varphi)}{1+\cos(2q\varphi)[1-\alpha^2\sin^2(2q\varphi)]^{1/2}}$ |
| $c_1=c_2=0, G=q^{-1}\tan^{-1}(U/U_1-1)^{1/2},\quad V=-\frac{k}{r},\quad U(\varphi)=\frac{\beta_1^2}{\cos^2(\frac{1}{2}q\varphi)}+\frac{\beta_2^2}{\sin^2(\frac{1}{2}q\varphi)}$ |
| where |
| $\alpha=\frac{\gamma^2}{2\beta^2}\quad \beta_1=\frac{\sqrt{U_0}-\sqrt{U_1}}{2},\quad \beta_2=\frac{\sqrt{U_0}+\sqrt{U_1}}{2}$ |

a SU(2,1) transitive realization can be defined in the negative energy portion of phase space; this would allow a geometric quantization[9] of these Hamiltonians.

For the time being, we checked that the Schrödinger quantization of the Hamiltonian of Eq. (4.3) shows the characteristic degeneracy of a SU(2)-symmetric Hamiltonian and the spectrum is the semiclassical one; both results should be confirmed in a geometric quantization.

## APPENDIX: ABEL'S EQUATION

The equation

$$\int_a^x \frac{f(y)\,dy}{\sqrt{x-y}}=g(x) \tag{A1}$$

is a special case of Abel's equation (or Euler transform).[14] When $g(x)$ is absolutely continuous, the solution is given by

$$\int_a^x f(t)\,dt=\frac{1}{\pi}\int_a^y \frac{g(x)}{\sqrt{y-x}}\,dx \tag{A2}$$

or, equivalently, by

$$f(y)=\frac{1}{\pi}\frac{g(a)}{\sqrt{y-a}}+\frac{1}{\pi}\int_a^y \frac{g'(x)\,dx}{\sqrt{y-x}}\ . \tag{A3}$$

In order that $f(y)$ be bounded in a neighborhood of $a$ it is necessary that

$$g(a)=0. \tag{A4}$$

In fact, from $\lvert f(y)\rvert<M\ (a\leqslant y\leqslant x)$, it follows that

$$\lvert g(x)\rvert=\left\lvert\int_a^x \frac{f(y)\,dy}{\sqrt{x-y}}\right\rvert<2M\sqrt{x-a} \tag{A5}$$

and $g(a)=0$.

If moreover we require that $f(a)=0$, as in most of our applications, then $g(x)$ must behave like $(x-a)^\alpha$ with $\alpha>+\frac{1}{2}$ as $x\to a$. In fact

$$\begin{aligned} f(y)&\approx\frac{1}{\pi}\int_a^y \frac{\alpha(x-a)^{\alpha-1}}{\sqrt{y-x}}\\ &=\frac{\alpha}{\pi}\int_0^1 dt(y-a)^{\alpha-1/2}(1-t)^{-1/2}t^{\alpha-1}\\ &=\frac{\alpha}{\pi}(y-a)^{\alpha-1/2}B(\alpha,1/2). \end{aligned} \tag{A6}$$

TABLE II″. Hamiltonian in polar coordinates—"Oscillator" family.

| |
|---|
| $\mathcal{J}(H)=J_r+qJ_\varphi=\frac{H}{2\omega}-\beta$ |
| $J(\lambda)=\frac{1}{q}(\lambda-\beta)$ |
| $V(r)=\frac{1}{2}m\omega^2r^2$ |
| $\varphi(U)=\pm q^{-1}\tan^{-1}\left(\frac{U}{U_0}-1\right)^{1/2}+G(U)\quad(\beta>0),\quad U=0 \text{ if } \beta=0$ |
| See Table II′ with $c_1=c_2=0;\ \omega>0$ |
| $G=0$: $U(\varphi)=U_0/\cos^2(q\varphi)$ |
| $G=q^{-1}\tan^{-1}\left(\frac{U}{U_1}-1\right)^{1/2}$: $U(\varphi)$ as in Table II′ |

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[9]See for instance D. Simms, "Proceedings of the 4th International Colloquium on Group-Theoretical Methods in Physics, 1975," University of Nijmegen, The Netherlands (Lecture Notes in Physics, Vol. 50, Springer, 1976).

[10]V.I. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics* (Benjamin, New York, 1968), Appendixes 26, 32, and 33.

[11]As is well known, $n_i$ represents the number of times that the projection of an orbit of the system into the coordinate plane $(q_i/p_i)$ is covered during one period of the motion.

[12]It is worthwhile to notice that the transformation

$$\tilde{r}=r,$$

$$\tilde{p}_r=p_r,$$

$$\tilde{\varphi}=\cos^{-1}\left[\left(\frac{p_\varphi^2+\beta^2/\sin^2\varphi}{p_\varphi^2+\beta^2\cot^2\varphi}\right)^{1/2}\cos\varphi\right],$$

$$\tilde{p}_\varphi=\left(p_\varphi^2+\frac{\beta^2}{\sin^2\varphi}\right)^{1/2},$$

maps the system defined by Eq. (4.3) into a subsystem of the Kepler problem corresponding to $\tilde{p}_\varphi>\beta$. In terms of action-angle variables this transformation reads

$$J_\varphi \to J_\varphi=\tilde{J}_\varphi+\beta,$$
$$J_r \to \tilde{J}_r=J_r,$$
$$w_\varphi \to \tilde{w}_\varphi=w_\varphi,$$
$$w_r \to \tilde{w}_r=w_r,$$

and thus is of the form given by Eq. (4.1). Of course it does not represent a global canonical transformation and therefore it cannot relate the symmetry properties of the two systems. Actually the generators of the Kepler system[7] are transformed into

$$M_x=\frac{M_z}{(-2mH)^{1/2}(M_z^2-\beta^2)^{1/2}}\times[-p_rp_\varphi\sin\varphi+(mk-M_z^2/r)\cos\varphi],$$

$$M_y=\frac{1}{(-2mH)^{1/2}(M_z^2-\beta^2)^{1/2}}\times[M_z^2p_r\cos\varphi+(mk-M_z^2/r)p_\varphi\sin\varphi],$$

$$M_z=(p_\varphi^2+\beta^2/\sin^2\varphi)^{1/2},$$

which are clearly singular at $\tilde{p}_\varphi=M_z=\beta$.

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