Reconstructing the sphere and the cone: a homotopy in the Zernike basis, an equivalence between static and inertial light sources, simplification of some quadratic integrals and Pell's equation

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Abstract

The problem of interest compare the amplitudes of a hemisphere and a right circular cone in terms of decomposition in terms a Zernike polynomial basis. The results were unexpectedly rich. The amplitudes have the exact same magnitude with a cubic decay. The error in the reconstruction is exactly the same for both surfaces order by order, creating a homotopy between these shapes in the Zernike basis. This homotopy corresponds to an equivalence between a point light source and a inertial light source emitting Cerenkov radiation. Because of this equivalence, certain quadratic integrals may be reduced to linear integrals. Finally, we relate the decay amplitudes in the Zernike basis to the solutions to Pell's equation.

1 Introduction

Because of the nature of other research projects, we were concerned studying the behavior of the disk polynomials of Zernike as a tool to describe surfaces. Our interest was to focus on using the combinations of modes of these polynomials to describe measurements. This allows us to take a test object and reduce its description to a list of 15 or 21 numbers. But perhaps more numbers are needed. This leads to the questions of how many numbers are needed and one of the many ways to study this problem was to ponder the following question.

What if we took a very smooth surface and compared that to a very smooth surface with a point? The questions are deceptively promising. What are these magical surfaces and what is this point? The devil is in the details. Eventually, simplicity reigned and the control surface, the smooth surface was a hemisphere. This seems like an easy choice, but adding a point to a hemisphere was not something we figured out even after much contemplation. In the interest of progress and simplicity, we choose the right circular cone. It is a smooth surface that naturally comes to a point. That is, there are no higher order derivative discontinuities.

The point of the exercise is to do a least squares decomposition of both surfaces and compare the amplitudes and watch the fit error. The point of a modal reconstruction is to see how well the decomposition approximates the target system. The expectation was that the sharp peak would stress the reconstruction. Of course, the surfaces are different, so the expectation was that the comparison would be qualitative at best.

Before proceeding with method and results, some background may help. The wonderful polynomials of Zernike suffer from a folklore that belies their fine mathematical heritage. If one reads Prof. Zernike's original paper[1] about setting forth these orthogonal polynomials, or the cogent explanation in Born and Wolf[2], there is is little room for confusion. However, over the years, a Homeric tradition of telling and retelling the story of these functions seems to have shifted even there very definitions. So before we turn to the tightly constrained problem we addressed, we would like to briefly sketch the properties of these special functions.

2 The disk polynomials of Zernike

The birth of and childhood of the polynomials was simple and unclouded. Sometime after Born and Wolf, the folk lore of the polynomials began. Perhaps it is because the lack of understanding of complex variables. That was left unstated in the original paper and in some sense breakdowns can be stated in terms of a problem with establishing a complex coordinate system. The goal here is not to unravel the jumbling of the notation, but rather to continue in a manner that we hope would appeal to Professor Zernike.

The one deviation that we think Zernike would agree to is that the modern term is disk polynomials as opposed to circle polynomials. The explosion of research[3] into polynomials defined on the circle $|z| = 1$ makes it vital to distinguish between polynomials defined on the disk $|z| \leq 1$ where we have deliberately omitted the equality. We are already on the path that some many readers depart from: these polynomials spring from complex value functions. Incredibly, the resolution of the complex valued functions into their real and imaginary counterparts has led to confounding problems in standardization. Rest assured that in this paper we will follow the conventions of Zernike, Euler, Cartan and the rest of the mathematical community.

Zernike created his polynomials to describe optical aberrations of circular mirrors. As such, the polynomials reside on the unit disk D_2 and they span the space of monomial functions. This means that they live on a disk and can describe any surface the covers the disk as long as it has a finite number of finite discontinuities. While we cannot say for certain what Zernike was thinking, the utility of his namesake polynomials is clear. They have three vital properties; the Zernike polynomials

1. are orthogonal over the unit disk

2. span the space of monomial functions,

[1] Zernikes English paper
[2] Born and Wolf, Chapter 9, Appendix on Zernike
[3] See for example Barry Simons 2 vols

3. span the space order by order,

While concise mathematical descriptions follow, it is more important to understand these four properties first. They provide the background for answering the two most fundamental questions about the Zernike polynomials: what are they and why were they created.

The first property is that the polynomials are orthogonal on the unit disc. While it makes mathematical sense to speak of the value of a polynomial away from the unit disk, we can't compute quantities like amplitudes in closed form outside of D_2 without the property of orthogonality.

The next property is that the polynomials span the space of monomial functions. We have to have this property to guarantee that we can build any surface, even those with discontinuities. Asking the polynomials to span the space is another way of asking that the polynomials be complete. Going on to the next property, we want the polynomials to span the space on an order by order basis. In other words, we want all the third order terms $\left(x^3, x^2y, xy^2, y^3\right)$ to be be in the third order polynomials and not spread about different orders. This allows for the polynomials to connect with the Seidel aberrations and optic aberrations of increasing orders.

When we through in normalization that the functions achieve the maximum of unity on the unit circle, we have sufficient criteria to uniquely specify Zernike's polynomials.

2.1 A look at the polynomials

Before starting with the mathematical forms of the polynomials, let's proceed to the next level of specification and look at how the polynomials are indexed or arranged.

2.1.1 Indexing the polynomials

Zernike defined a complex-valued function over the plane which he resolved into a radial function and complex polar function

$$V_n^m\left(r,\theta\right) = R_n^m\left(r\right)e^{im\theta} \tag{2.1}$$

where n is called the *order* and m is the $angular frequency$. The integers n, m are ordered so that $m < n$ and the difference is always an even number. Because of this, the indices (n, m) which specify a polynomial are either both even or both odd. A look at the indexing is perhaps the quickest way to see this.

As mentioned earlier, these polynomials span the space of polynomial function order by order just as the minimal spanning set does. Table (2) relates the Zernike polynomials to the minimal spanning set on an order by order basis. Why not use the minimal spanning set? Because when we resolve a surface into a series of monomial modes, the weighting factors would change every time the order of the expansion changes. These weighting factors are called the amplitudes or expansion coefficients or spectrum, are the key commodity we are after. If we choose an orthogonal basis set for the expansion, then these amplitudes do not change as the order of the expansion changes. This allows us to think of the basis functions as fundamental states.

	0	1	2	3	4	5	6	7
0	V_0^0							
1		V_1^1						
2	V_2^0		V_2^2					
3		V_3^1		V_3^3				
4	V_4^0		V_4^2		V_4^4			
5		V_5^1		V_5^3		V_5^5		
6	V_6^0		V_6^2		V_6^4		V_6^6	
7		V_7^1		V_7^3		V_7^5		V_7^7

Table 1: The indexing and arrangement of the Zernike polynomials through sixth order. On the left, we see the polynomial order, n, on the top we see the angular frequency, m.

monomials	Zernike polynomials
1	V_0^0
(x, y)	V_1^1
(x^2, xy, y^2)	V_2^0, V_2^2
(x^3, x^2y, xy^2, y^3)	V_3^1, V_3^3
$(x^4, x^3y, x^2y^2, xy^3, y^4)$	V_4^0, V_4^2, V_4^4
$(x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5)$	V_5^1, V_5^3, V_5^5
$(x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6)$	V_6^0, V_6^2, V_6^4, V_6^6
$(x^7, x^6y, x^5y^2, x^4y^3, x^3y^4, x^2y^5, xy^6, y^7)$	V_7^1, V_7^3, V_7^5, V_7^7

Table 2: Relating the Zernike polynomials to the minimal spanning set of monomials.

2.1.2 A first look at the polynomials

These polynomials, being complex-valued functions have an elegant and suggestive form when written as a function of the complex variable z where of course $z = x + iy$. In the following two tables ((??) and (??)) we present the Zernike polynomials through order seven. The first table shows the even orders, the second the odd. Notice that because of the evident offsets seen in table

	0	2	4	6
0	1			
2	$2z\bar{z} - 1$	z^2		
4	$6(z\bar{z})^2 - 6z\bar{z} + 1$	$z^2(4z\bar{z} - 3)$	z^4	
6	$20(z\bar{z})^3 - 30(z\bar{z})^2 + 12z\bar{z} - 1$	$z^2\left(15(z\bar{z})^2 - 20z\bar{z} + 6\right)$	$z^4(6z\bar{z} - 5)$	z^6

Table 3: The first Zernike polynomials of even order.

(1) it is more appealing to collect the odd and even orders into separate tables,

When we look at these tables we see the three general cases when dealing with functions of a complex variable:

1. the function is analytic; these are the $V_n^n(r, \theta) = z^n$ terms;

	1	3	5	7
1	z			
3	$2z(3z\bar{z}-2)$	z^3		
5	$z\left(10(z\bar{z})^2-12z\bar{z}+3\right)$	$z^3(5z\bar{z}-4)$	z^5	
7	$z\left(35(z\bar{z})^3-60(z\bar{z})^2+30z\bar{z}-4\right)$	$z^3\left(21(z\bar{z})^2-30z\bar{z}+10\right)$	$z^5(7z\bar{z}-6)$	z^7

Table 4: The first Zernike polynomials of odd order.

2. the function is real; these are the $V_{2k}^0(r)$ terms;

3. the function is strictly complex; these are the excluded terms.

Just from looking at these 20 complex functions we see a pattern to how the different species are distributed. Table (5) makes the pattern explicit. The functions that we will be using are the

	0	1	2	3	4	5	6	7
0	R							
1		H						
2	R		H					
3		C		H				
4	R		C		H			
5		C		C		H		
6	R		C		C		H	
7		C		C		C		H

Table 5: The species of Zernike functions: R is real, H is analytic or harmonic and C is complex.

purely real functions which are rotationally invariant. More formally

$$V_{2k}^0(\rho) = constant$$

for constant ρ.

2.1.3 Grouping the polynomials into families

When the polynomials are written in complex form like this we can quickly see family relationships. In these two tables, we see four distinct families, each increasingly complex. The obvious choice are the power series terms which form the first family. Collected together, they are shown in table (6) below. We can write the family I is given as $V_n^n(z) = z^n$ for $n \geq 0$.

What characterizes the family is the difference between the indices n and m. In the first family, there is no difference. In the next family the difference is two, which also forces $n \geq 4$; in the third family the difference is four and $n \geq 4$. The four families that we have are shown schematically in table (8).

The families we see are

$V_n^m(z)$	function
$V_0^0(z)$	1
$V_1^1(z)$	z
$V_2^2(z)$	z^2
$V_3^3(z)$	z^3
$V_4^4(z)$	z^4
$V_5^5(z)$	z^5
$V_6^6(z)$	z^6
$V_7^7(z)$	z^7

Table 6: The first family of functions within the Zernike polynomials: the power series.

	0	1	2	3	4	5	6	7
0	1							
1		1						
2	2		1					
3		2		1				
4	3		2		1			
5		3		2		1		
6	4		3		2		1	
7		4		3		2		1

Table 7: The family structure. This table shows the distribution of the four families discussed in the narrative.

family	$V_n^m(z,\bar{z})$	$n-m$	functional form
I	$V_n^n(z)$	0	z^n
II	$V_n^{n-2}(z,\bar{z})$	2	$z^{n-2}\left(n\left(z\bar{z}\right)-n+1\right)$
III	$V_n^{n-4}(z,\bar{z})$	4	$z^{n-4}\left(\frac{1}{2}\left(n^2-n\right)\left(z\bar{z}\right)^2+\left(-n^2+3n-2\right)\left(z\bar{z}\right)+\frac{1}{2}\left(n^2-5n+6\right)\right)$
IV	$V_n^{n-6}(z,\bar{z})$	6	$z^{n-6}\left(\right)$

Table 8: default

2.2 Generating the polynomials

There are a variety of ways to generate these functions and each provides a different perspective into the application of the polynomials. What exactly do we mean by generating the polynomials? Finding the complex function $V_n^m(x_1, x_2)$ in the (x_1, x_2) coordinate system, which is assumed, given the input integers n and m.

For example, if we are in polar coordinates what is the the function $V_{10}^8(x_1, x_2)$? We know this function is

$$\begin{aligned} V_{10}^8(r,\theta) &= r^8\left(10r^2 - 9\right)e^{i8\theta} \\ V_{10}^8(z,\bar{z}) &= z^8(10z\bar{z} - 9) \end{aligned} \tag{2.2}$$

In practice we want to use the measurements to describe measured phenomena, so we will want to resilve the complex functions into real and imaginary componet. In doing so, the

2.3 Essential properties of the polynomials

Orthogonality

Real, analytic, harmonic, mixed

2.4 Functional forms

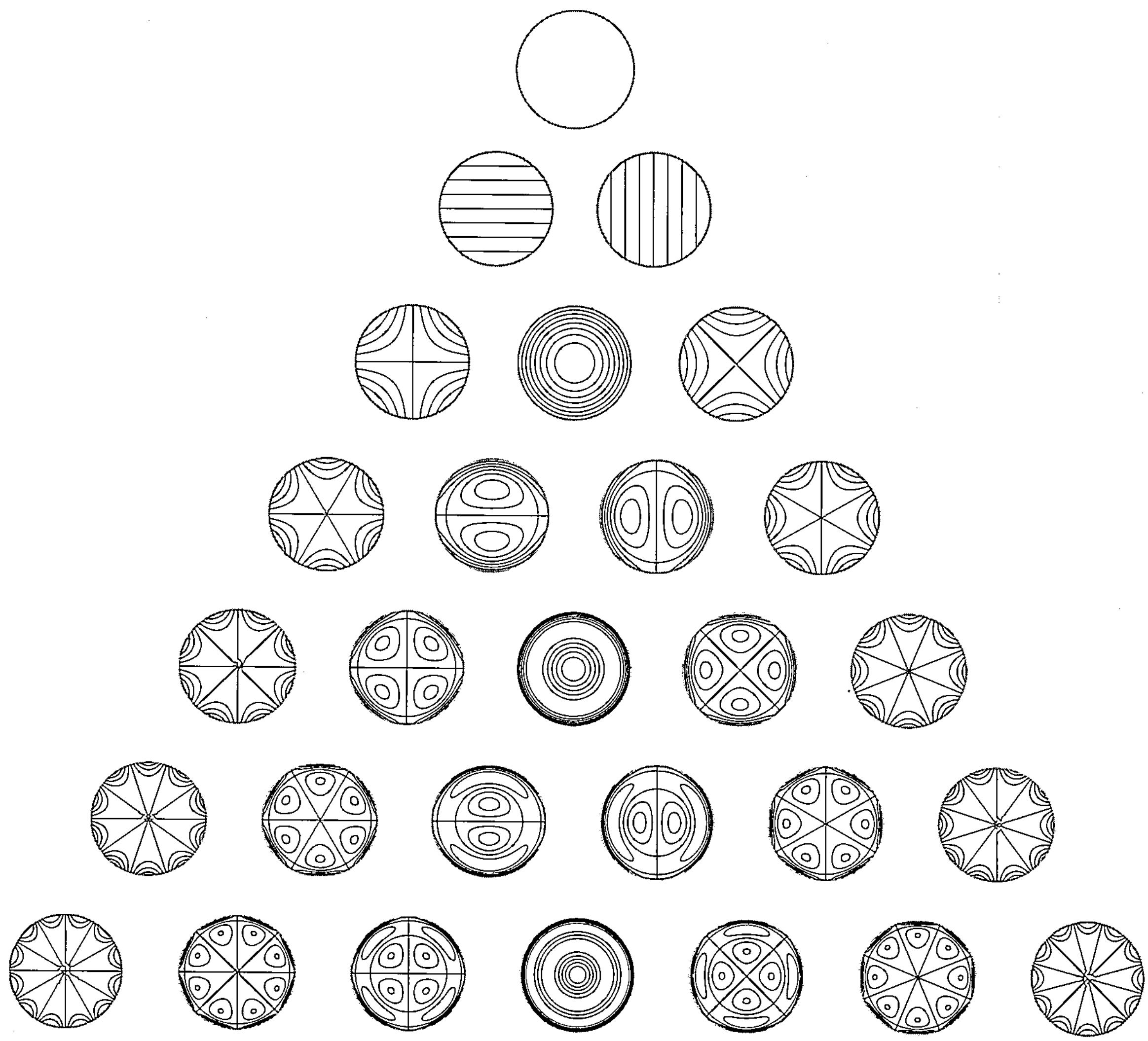

Figure 1: The first seven orders of the Zernike polynomials shown as contour plots without shading. In this form the relationship between the complex conjugate states is quite clear.

n	m	$V_n^m(z,\bar{z})$		Polar forms		Cartesian forms
0	0	1	Re:	1	Re:	1
			Im:	0	Im:	0
1	1	z	Re:	$r\cos(\theta)$	Re:	x
			Im:	$r\sin(\theta)$	Im:	y
2	0	$2z\bar{z}-1$	Re:	$2r^2-1$	Re:	$2x^2+2y^2-1$
			Im:	0	Im:	0
2	2	z^2	Re:	$r^2\cos(2\theta)$	Re:	x^2-y^2
			Im:	$r^2\sin(2\theta)$	Im:	$2xy$
3	1	$z(3z\bar{z}-2)$	Re:	$3r^3\cos(\theta)-2r\cos(\theta)$	Re:	$3x^3+3y^2x-2x$
			Im:	$3r^3\sin(\theta)-2r\sin(\theta)$	Im:	$3y^3+3x^2y-2y$
3	3	z^3	Re:	$r^3\cos(3\theta)$	Re:	x^3-3xy^2
			Im:	$r^3\sin(3\theta)$	Im:	$3x^2y-y^3$
4	0	$6(z\bar{z})^2-6z\bar{z}+1$	Re:	$6r^4-6r^2+1$	Re:	$6x^4+12y^2x^2-6x^2+6y^4-6y^2+1$
			Im:	0	Im:	0
4	2	$z^2(4z\bar{z}-3)$	Re:	$4r^4\cos(2\theta)-3r^2\cos(2\theta)$	Re:	$4x^4-3x^2-4y^4+3y^2$
			Im:	$4r^4\sin(2\theta)-3r^2\sin(2\theta)$	Im:	$8yx^3+8y^3x-6yx$
4	4	z^4	Re:	$r^4\cos(4\theta)$	Re:	$x^4-6y^2x^2+y^4$
			Im:	$r^4\sin(4\theta)$	Im:	$4x^3y-4xy^3$
5	1	$z\left(10(z\bar{z})^2-12z\bar{z}+3\right)$	Re:	$10r^5\cos(\theta)-12r^3\cos(\theta)+3r\cos(\theta)$	Re:	$10x^5+20y^2x^3-12x^3+10y^4x-12y^2x+3x$
			Im:	$10r^5\sin(\theta)-12r^3\sin(\theta)+3r\sin(\theta)$	Im:	$10y^5+20x^2y^3-12y^3+10x^4y-12x^2y+3y$
5	3	$z^3(5z\bar{z}-4)$	Re:	$5r^5\cos(3\theta)-4r^3\cos(3\theta)$	Re:	$5x^5-10y^2x^3-4x^3-15y^4x+12y^2x$
			Im:	$5r^5\sin(3\theta)-4r^3\sin(3\theta)$	Im:	$-5y^5+10x^2y^3+4y^3+15x^4y-12x^2y$
5	5	z^5	Re:	$r^5\cos(5\theta)$	Re:	$x^5-10y^2x^3+5y^4x$
			Im:	$r^5\sin(5\theta)$	Im:	$y^5-10x^2y^3+5x^4y$

Table 1: The Zernike polynomials through order five shown in three different coordinate systems.

3 Data reduction

This section details the why we constructed this problem and how we analyzed it. We conclude with the basic results. Detailed discussion of the findings is deferred until the following section.

3.1 The rotationally invariant Zernike polynomials

The Zernike polynomials quickly become complicated as the order increases. At nth order we will have $n + 1$ polynomials. To streamline the analysis, we decided to restrict the basis set to the centerline polynomials. For a given even order n we will have $n/2$ terms instead of $n(n + 1)/2$ terms in the expansion. The price for this computational simplicity is that we are now restricted to surfaces and aberrations that are rotationally invariant.

3.2 The hemisphere and the cone

The goal is to compare a modal reconstruction for two very different surfaces: the hemisphere, a smooth surface of continuous curvature; and a cone, a flat surface with a point discontinuity.

We are looking at the effect of a strong, isolated point on a reconstruction. To do that, you want to take to comparably smooth surfaces and the simpler the better. The thinking was to start with a hemisphere as the control case and the then look at the right circular cone as the peaked case. The cone is absolutely smooth with linear linear sides. Is essence, it is a point carrier.

It's worth a few moments to quickly review the orientation and features of these two surfaces are shown below in figure (2). The grayscale shading is done according to height. What we see is that the cone has a linear slope and the sphere has a more complicated shape. Of course the curvature, given by

$$\kappa = \frac{|y''|}{(1 + y'^2)^{3/2}} \tag{3.1}$$

The expectation was that we would watch the familiar case of the sphere gradually take form as we had so often seen. Then we expected a to stress the reconstructor to pick up the fine details of the point. Before the numerics of course we were able to solve this analytically. With a closed form solution available, we eagerly started there to have another baseline to calibrate the numeric codes.

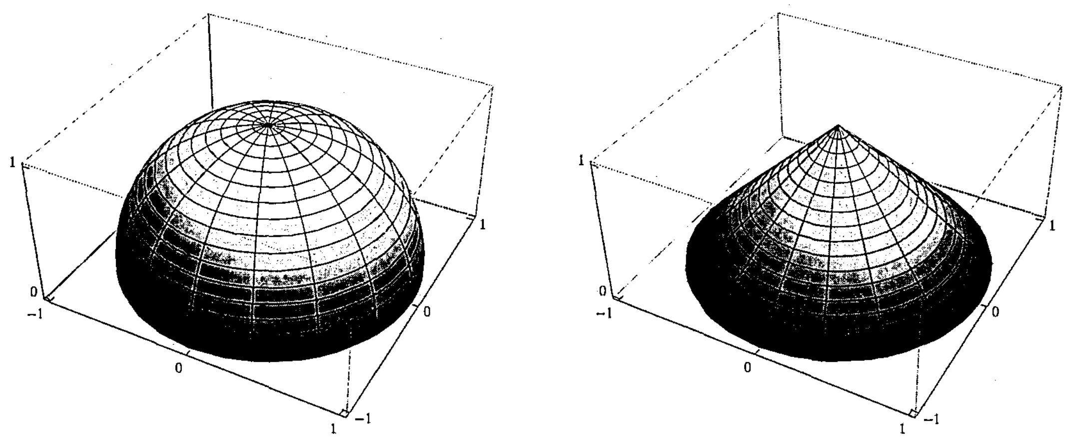

Figure 2: Three dimensional renderings shaded according to height. Both surfaces have pathological first derivatives. The cone has the obvious point and is discontinuous there. The reconstructed surface will of course have zero slope there like the hemisphere. One the other hand, the hemisphere is surrounded by an infinite slope.

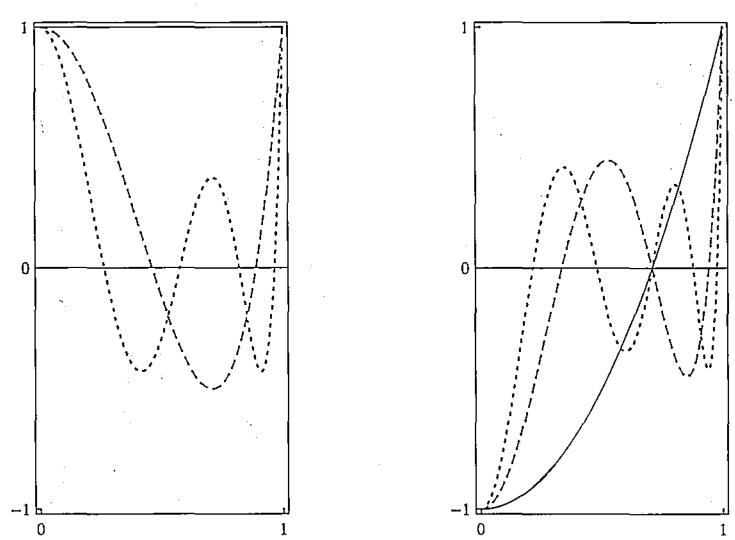

Figure 3: The first six rotationally invariant Zernike polynomials show in equation (??) below. The sign of the constant term makes them easy to identify. They all are flat at the origin and the slope becomes increasingly sharp near the boundary $r = 1$

3.3 The Zernike basis

The first six rotationally invariant Zernike polynomials are these:

$$\begin{aligned}
V_0^0(r,\theta) &= R_0^0(r) = 1 \\
V_2^0(r,\theta) &= R_2^0(r) = 2r^2 - 1 \\
V_4^0(r,\theta) &= R_4^0(r) = 6r^4 - 6r^2 + 1 \\
V_6^0(r,\theta) &= R_6^0(r) = 20r^6 - 30r^4 + 12r^2 - 1 \\
V_8^0(r,\theta) &= R_8^0(r) = 70r^8 - 140r^6 + 90r^4 - 20r^2 + 1 \\
V_{10}^0(r,\theta) &= R_{10}^0(r) = 252r^{10} - 630r^8 + 560r^6 - 210r^4 + 30r^2 - 1
\end{aligned} \tag{3.2}$$

Note that the normally complex valued functions are completely described by the real valued radial polynomials. This is true for the entire series of rotationally invariant polynomials,

$$V_{2n}^0(r,\theta), k = 1,2,3,\ldots.$$

This patten is clear when looking at the table of the even order polynomials. The column for the angular frequency $m = 0$ shows that the complex combination $z\bar{z}$ is the basic unit so the polynomials are real.

Another issue we see from this brief list above is that the complexity of the polynomials is growing rapidly.

3.4 Least squares

Least squares is the method of minimizing the square of the merit function. Underneath that we are saying that we are measuring difference between the input function and the trial function in the L_2 norm. In this form, a question is a bit more natural? Why the L_2 norm, why not for example the L_1 norm? What are these norms and what do they represent?

3.4.1 p-norms

The for $p \geq 1$ the p-norm of an n dimensional vector x is defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i| \right)^{1/p} \tag{3.3}$$

3.5 Method

The imput surfaces are ψ by

$$\begin{aligned}
\psi_s(r) &= \sqrt{1-r^2} \\
\psi_c(r) &= 1 - r
\end{aligned} \tag{3.4}$$

In general we will be working in a n dimensional subspace of a Hilbert space with a basis spanned by the Zernike polynomials. In the example here, we look at the first six equations and so go to order 12:

$$\begin{aligned} B_{12}(r) &= \left(R_0^0(r), R_2^0(r), R_4^0(r), R_6^0(r), R_8^0(r), R_{10}^0(r), R_{12}^0(r)\right) \\ &= \left(1, \left(2r^2-1\right), \left(6r^4-6r^2+1\right), \left(20r^6-30r^4+12r^2-1\right), \ldots\right). \end{aligned} \tag{3.5}$$

The output of the least squares fit is a vector of amplitudes a which describe how much of each mode of the the different polynomials contribute the decomposition. Hence the name modal reconstruction. Since this analysis is being done in the continuum, the modes or amplitudes a will not change as the order d is increased. This is a beauty of orthogonality.

The output will be the vector of amplitudes a which will generate the function creating the reconstructed surface

$$\phi_\zeta(r) = a_\zeta \cdot B_n(r) \tag{3.6}$$

where ζ is a dummy variable. The coefficients a are weight factors or amplitudes. They are the recipe that determine how much of each mode is used to descrbe the surface. This is the reason for the name modal fit.

This is just the first part of the output for the method of least squares is a power qualitative analytic tool. Unfortuantely in many communties, the method has become part folk lore and less analytic. This is a problem that needs urgent redress and we will make a modest effort to so here despite the fact that we are dealing with exact analytic results. As a general rule: when amplitudes for a least squares fit are computed, they should include the computational uncertainty.

Proceding, the basic quantity of interest in sculpting our fit is the residual error which the difference between the given surface and the constructed surface. Mathematically this is

$$\epsilon_\zeta(r) = \psi_s(r) - a_\zeta \cdot B_n(r). \tag{3.7}$$

$$\begin{aligned} M_\zeta(r) &= \int_0^{2\pi}\int_0^1 \epsilon_\zeta^2(r) r dr d\theta \\ &= \int_0^{2\pi}\int_0^1 \left(\psi_\zeta(r) - a_\zeta \cdot B_n(r)\right)^2 r dr d\theta \end{aligned} \tag{3.8}$$

$$\epsilon(r) = \psi_s(r) - a \cdot B_n(r). \tag{3.9}$$

$$\begin{aligned} M(r) &= \int_0^{2\pi}\int_0^1 \epsilon^2(r) r dr d\theta \\ &= \int_0^{2\pi}\int_0^1 \left(\psi(r) - a \cdot B_n(r)\right)^2 r dr d\theta \end{aligned} \tag{3.10}$$

$$
\begin{aligned}
\frac{\partial}{\partial a_k} M(r) &= 0 \\
&= -2 \int_0^{2\pi} \int_0^1 \left(\psi(r) - a \cdot \mathbf{B}_n(r)\right) B_k r dr d\theta \\
&= -4\pi \int_0^1 \left(\psi(r) - a \cdot \mathbf{B}_n(r)\right) B_k r dr
\end{aligned} \tag{3.11}
$$

$$
\int_0^{2\pi} \int_0^1 \psi(r) B_k r dr d\theta = \int_0^{2\pi} \int_0^1 \left(a \cdot \mathbf{B}_n(r)\right) B_k r dr d\theta \tag{3.12}
$$

$$
\begin{aligned}
\int_0^1 \psi(r) B_k r dr &= \int_0^1 \left(a \cdot \mathbf{B}_n(r)\right) B_k r dr \\
&= a_k \int_0^1 \left(B_k^2\right) r dr \\
&= a_k 2 \left(k+1\right)
\end{aligned} \tag{3.13}
$$

$$
a_k = \left(2k+2\right)^{-1} \int_0^1 \psi(r) B_k r dr \tag{3.14}
$$

3.6 Results

3.6.1 The computation of the amplitudes

The result vectors are given below. They are in the format

$$(a_0, a_1, a_2, \ldots, a_{15}.)$$

The amplitudes for the sphere are a_s and the amplitudes for the cone are a_c:

	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
a_s	$\frac{2}{5}$	$\frac{2}{21}$	$\frac{2}{45}$	$\frac{2}{77}$	$\frac{2}{117}$	$\frac{2}{165}$	$\frac{2}{221}$	$\frac{2}{285}$	$\frac{2}{357}$	$\frac{2}{437}$	$\frac{2}{525}$	$\frac{2}{621}$	$\frac{2}{725}$	$\frac{2}{837}$	$\frac{2}{957}$	$\frac{2}{1085}$
a_c	$\frac{2}{3}$	$-\frac{2}{21}$	$\frac{2}{45}$	$-\frac{2}{77}$	$\frac{2}{117}$	$-\frac{2}{165}$	$\frac{2}{221}$	$-\frac{2}{285}$	$\frac{2}{357}$	$-\frac{2}{437}$	$\frac{2}{525}$	$-\frac{2}{621}$	$\frac{2}{725}$	$-\frac{2}{837}$	$\frac{2}{957}$	$-\frac{2}{1085}$

Table caption: The first few amplitudes are computed by Mathematica: a_s represents the amplitudes for the hemisphere and a_c for the cone. Notice that because these computations were done analytically, we can see that modulo a multiplicative sign, they are exactly the same numbers.

Ignore the first terms. These are the "piston" terms which describe the mean value of the surface and hold no information about the shape. This term encodes where the shape is relative to the coordinate system. All the subsequent terms encode shape information and all of these terms have

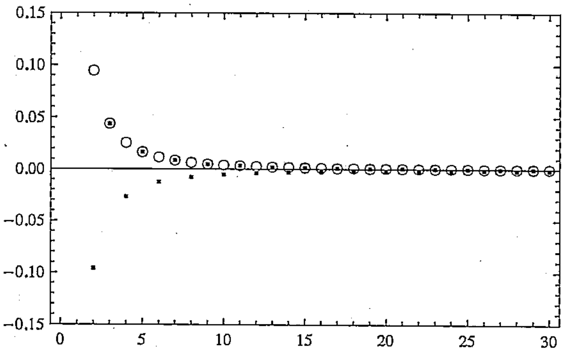

Figure 4: The amplitudes for the hemisphere (open circle) and the cone (filled circle). The amplitudes have the exact same magnitude and differ only by a sign oscillation. We see the same decay for both sets of coefficients.

the exact same magnitude. The sphere terms are always positive and the cone terms alternate in sign according to $(-1)^{n/2}$. In general, the magnitude of the amplitudes are exactly the same. This is one benefit of doing the computation analytically.

For this project, we computed the amplitudes out to order $n = 50,000$ for the cone because the computation is so much faster. We can peek at the first 250 amplitudes in figure (REF) below. To read this table realize that the amplitude is written as

$$a_s(n) = \frac{2}{d}.$$

For example, to get the amplitude for the 10th order term, we find $n = 10$ in the table and see that $d = 165$. The amplitude is then

$$a_s(10) = \frac{2}{165}.$$

n	d	n	d	n	d	n	d	n	d
0	5	100	10605	200	41205	300	91805	400	162405
2	21	102	11021	202	42021	302	93021	402	164021
4	45	104	11445	204	42845	304	94245	404	165645
6	77	106	11877	206	43677	306	95477	406	167277
8	117	108	12317	208	44517	308	96717	408	168917
10	165	110	12765	210	45365	310	97965	410	170565
12	221	112	13221	212	46221	312	99221	412	172221
14	285	114	13685	214	47085	314	100485	414	173885
16	357	116	14157	216	47957	316	101757	416	175557
18	437	118	14637	218	48837	318	103037	418	177237
20	525	120	15125	220	49725	320	104325	420	178925
22	621	122	15621	222	50621	322	105621	422	180621
24	725	124	16125	224	51525	324	106925	424	182325
26	837	126	16637	226	52437	326	108237	426	184037
28	957	128	17157	228	53357	328	109557	428	185757
30	1085	130	17685	230	54285	330	110885	430	187485
32	1221	132	18221	232	55221	332	112221	432	189221
34	1365	134	18765	234	56165	334	113565	434	190965
36	1517	136	19317	236	57117	336	114917	436	192717
38	1677	138	19877	238	58077	338	116277	438	194477
40	1845	140	20445	240	59045	340	117645	440	196245
42	2021	142	21021	242	60021	342	119021	442	198021
44	2205	144	21605	244	61005	344	120405	444	199805
46	2397	146	22197	246	61997	346	121797	446	201597
48	2597	148	22797	248	62997	348	123197	448	203397
50	2805	150	23405	250	64005	350	124605	450	205205
52	3021	152	24021	252	65021	352	126021	452	207021
54	3245	154	24645	254	66045	354	127445	454	208845
56	3477	156	25277	256	67077	356	128877	456	210677
58	3717	158	25917	258	68117	358	130317	458	212517
60	3965	160	26565	260	69165	360	131765	460	214365
62	4221	162	27221	262	70221	362	133221	462	216221
64	4485	164	27885	264	71285	364	134685	464	218085
66	4757	166	28557	266	72357	366	136157	466	219957
68	5037	168	29237	268	73437	368	137637	468	221837
70	5325	170	29925	270	74525	370	139125	470	223725
72	5621	172	30621	272	75621	372	140621	472	225621
74	5925	174	31325	274	76725	374	142125	474	227525
76	6237	176	32037	276	77837	376	143637	476	229437
78	6557	178	32757	278	78957	378	145157	478	231357
80	6885	180	33485	280	80085	380	146685	480	233285
82	7221	182	34221	282	81221	382	148221	482	235221
84	7565	184	34965	284	82365	384	149765	484	237165
86	7917	186	35717	286	83517	386	151317	486	239117
88	8277	188	36477	288	84677	388	152877	488	241077
90	8645	190	37245	290	85845	390	154445	490	243045
92	9021	192	38021	292	87021	392	156021	492	245021
94	9405	194	38805	294	88205	394	157605	494	247005
96	9797	196	39597	296	89397	396	159197	496	248997
98	10197	198	40397	298	90597	398	160797	498	250997

Figure caption: The denominators of the first 250 amplitudes showing the unusual properties mentioned in the discussion: all the numbers are odd and patterns are forming.

This is a fascinating group of numbers to work with and over the weeks it took to gather them, certain properties stood out. You notice that all of the numbers are odd; even more striking the sequence of ending digits is always 5..1..5..7..7. Furthermore, the basic sequences appear in the terminating digits. Here is a sample from farther out in the computation:

n	d
10000	10006000**5**
10001	1001000**21**
10002	1001400**45**
10003	1001800**77**
10004	100220**117**
10005	100260**165**
10006	100300**221**
10007	100340**285**
10008	100380**357**
10009	100420**437**
10010	100460**525**

Table caption: The initial sequence plays out over and over in the trailing digits of the higher order terms.

You notice that the distribution of the digits is not uniform; you observe patterns such as these:

n	d
10	165
100	10605
1000	1006005
10000	100060005

Table 9: An unusual sequence.

n	d
48	2597
498	250997
4998	25009997
49998	2500099997

Table 10: Another sequence.

To look at the distribution of digits, we took all 50,000 amplitudes and used Mathematica to plot the distribution show in figure 5 below. What mechanism is afoot here to keep these amplitudes from taking even values? Why are some digits favored? In the conclusions that follow we will address these questions and move closer to understanding them.

Figure 5: The distribution of digits in the first 50,000 amplitudes for the hemisphere. Some digits are clearly favored over others.

3.6.2 The series for the amplitudes

Given such a well behaved set of numbers for the denominators (figure big table of numbers), a natural question arises. Can they be described by a simple polynomial expression? Here the answer is yes. After doing a basic quadratic fit, we see that the denominator relates to the order according to

$$d(n) = (n+1)(n+5). \tag{3.15}$$

Besides esthetic pleasure, this relationship tells us exactly how the Zernike amplitudes decay since the coefficients are

$$\begin{aligned} a_s(n) &= \frac{2}{(n+1)(n+5)} \\ a_c(n) &= (-1)^n \frac{2}{(n+1)(n+5)} \end{aligned} \tag{3.16}$$

for $n > 0$. The decay is quadratic.

4 Conclusions

When we compared the cone and the hemisphere we found a treasure chest of interesting artifacts. The main finding is that the two surfaces have the same "smoothness" in the Zernike basis. That sharp little point atop the cone does little to distinguish the two surfaces. In the following narrative we will discuss this finding as well as all others.

4.1 A critical duality relationship

After seeing that the amplitudes differed by a sign, we were motivated to look for a transformation that would relate the two functions in the Zernike basis.

$$R_{2k}^{0}\left(\sqrt{1-r^2}\right) = (-1)^k R_{2k}^{0}(r) \tag{4.1}$$

With this relationship, we now see the analytic reason the amplitudes must be the same.

For example

$$R_2^0\left(\sqrt{1-\rho^2}\right) = 2\sqrt{1-\rho^2}^{\,2} - 1 = -2\rho^2 + 1 = -R_2^0(\rho) \tag{4.2}$$

4.2 Homotopy in the Zernike basis

The most shocking finding is that the the cone and the hemisphere have the same smoothness in the Zernike basis where the smoothness is measured by the decay of the amplitudes. The smoothness equivalence of these two shapes is a homotopy in the Zernike basis and is the first discovered as far as we can tell.

4.2.1 The amplitudes are a convex set

Are these amplitudes a convex set? We can tell by inspection of figure (4) that they are. Still, we will show the formalities.

A set is convex if

$$a_{n+1} + a_{n-1} - 2a_n \geq 0. \tag{4.3}$$

The we are begging the question is

$$\frac{2}{(n+2)(n+6)} + \frac{2}{n(n+4)} - \frac{4}{(n+1)(n+5)} \geq 0? \tag{4.4}$$

This is true for all values of $n > 0$ since the left-hand side of the inequality is

$$\frac{12\left(n^2+6n+10\right)}{n^6+18n^5+121n^4+372n^3+508n^2+240n} \tag{4.5}$$

which shows a positive numerator over a positive denominator. Hence, the decay amplitudes form a convex set.

4.2.2 Smoothness and the decay of the amplitudes

4.2.3 A perspective on intuition

At first, the result was completely shocking: how a smooth ball resemble a sharp cone. In a collaborative light, we realiz both shapes are pathological. The cone clearly a $\mathbf{C}^1$ curve, discontinuous in the first derivative at the apex. However, the hemisphere also suffers from having the first derivatives approach infinity at the boundary. If we exclude the boundary the sphere is $\mathbf{C}^\infty$; however with the boundary the sphere becomes $\mathbf{C}^1$.

Would you expect the coefficients to have equal amplitudes? No, but you would realize that the two surfaces are not so dissimilar.

4.2.4 Equivalence of static and inertial forms

Until now, we have used the generic word "surface" for the shapes we are building over the unit disk.

4.3 Computational simplicity of certain integral forms

4.4 A rare instance of where the Zernike basis is superior

In general the Zernike basis is a computational disaster that serves as a tar pit serving to to trap.

4.5 Pell's equation

Pell's equation is a Diophantine equation of the form

$$x^2 + jy^2 = 1 \tag{4.6}$$

where j is a nonsquare integer and x and y are integers.

5 Bibliography