Kepler's Law

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Abstract

Kepler's law is a cornerstone of orbital mechanics.

Contents

1 Overview

Kepler's law is a cornerstone of orbital mechanics.

1. `env`
2. `ldd`

2 Geometry of Kepler's Law

Disagreement with this YouTuber True Anomaly vs. Mean Anomaly

3 Mathematics

3.1 Definitions

Definition 1 (The ellipse)**.** *Given $\theta \in [0, 2\pi)$, and parameters $a, b \in \mathbb{R}^+$ with $a > b$ the following parametric form defines an ellipse.*

$$\epsilon(\theta) = (a \cos\theta, b \sin\theta) \tag{1}$$

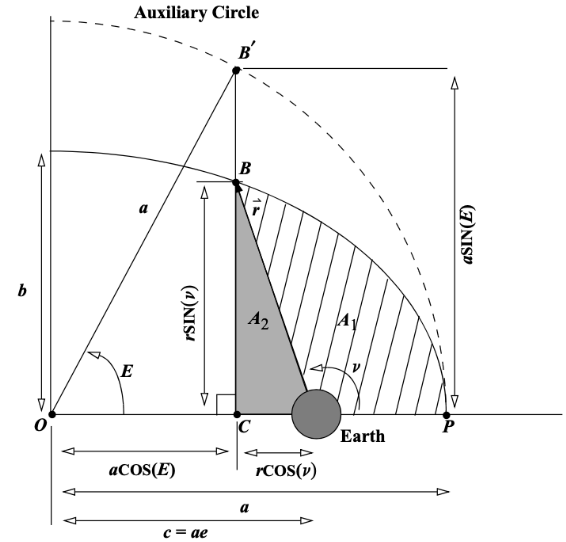

Figure 2-2. **Geometry of Kepler's Equation.** The eccentric anomaly uses an auxiliary circle as shown. The ultimate goal is to determine the area, A_1, which allows us to calculate the time.

Figure 1: Vallado's figure 2-2 showing E and ν.

Sec 4.2 TIME-OF-FLIGHT — ECCENTRIC ANOMALY 183

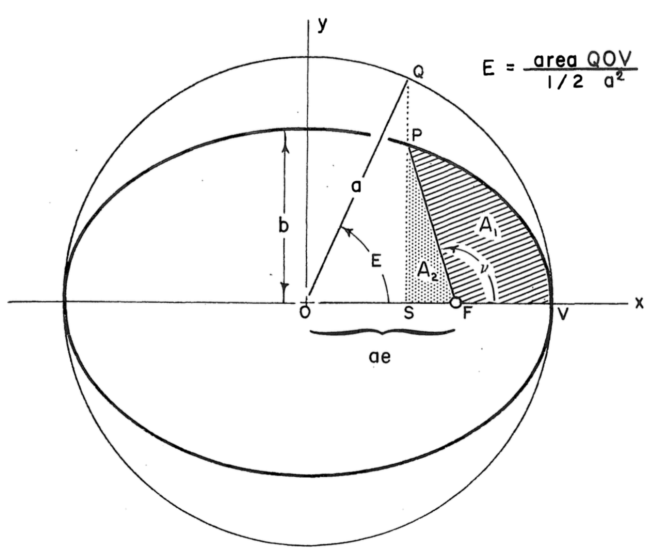

Figure 4.2-2 Eccentric anomaly, E

Figure 2: Figure 4-4 in Bate *et al.* showing E and ν.

160 SOLUTION OF KEPLER'S EQUATION. [95

ACQ will be defined as the eccentric anomaly, E, and it will be shown that the relation between M and E is given by Kepler's equation.

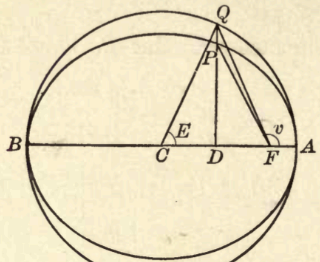

Fig. 28.

From the law of areas and the properties of the auxiliary circle, it follows that

$$\frac{M}{2\pi} = \frac{\text{area } AFP}{\text{area ellipse}} = \frac{\text{area } AFQ}{\text{area circle}}.$$

Area AFQ = area ACQ − area $FCQ = \frac{a^2E}{2} - \frac{a}{2} ae \sin E.$

Therefore

$$\frac{M}{2\pi} = \frac{a^2}{2} \frac{(E - e \sin E)}{\pi a^2};$$

or,

$$\begin{cases} M = E - e \sin E, \\ FP = r = \dfrac{a(1-e^2)}{1 + e \cos v} = \sqrt{\overline{PD}^2 + \overline{FD}^2} = a(1 - e \cos E), \end{cases}$$

Figure 3: Moulton's figure 28 showing E and ν.

Definition 2 (Eccentricity of the ellipse)**.** *The eccentricity is a scalar parameter $e \in (0,1)$ and can be expressed in terms of fundamental parameters of the ellipse where $a > b$ as*

$$e = \frac{c}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2} \tag{2}$$

Definition 3 (Mean anomaly)**.** *Kepler's Law*[1] *defines the mean anomaly as the angular measure $M(e, E)\colon (0,1) \times [0, 2\pi\,) \mapsto [0, 2\pi\,)$ as*

$$M(e, E) = E - e \sin E \tag{3}$$

Theorem 4 (Continuity of the mean anomaly)**.** *The mean anomaly as defined in definition 3 is a continuous function.*

Proof. To prove continuity show that for any two points p and q in the domain there exists a majorization constant K such that

$$M(p) - M(q) \leq K|p - q| \tag{4}$$

Spoiler alert: the majorization constant is 2π. □

Observation: given the continuity of the mean anomaly, one may use Newton's method [2, §4.6] to solve the nonlinear equation.

References

[1] Roger R Bate et al. *Fundamentals of astrodynamics.* Courier Dover Publications, 2020.

[2] Walter Gautschi. *Numerical analysis.* Springer Science & Business Media, 2011.

[3] Forest Ray Moulton. *An introduction to celestial mechanics.* Dover, 1970.

[4] D.A. Vallado. *Fundamentals of Astrodynamics and Applications.* 5th ed. Microcosm Press; 2022.

[1] [1, eq 4.5] [3, p.159] [4, §2.2], [**kaula2013theory**]