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## ABSTRACT

Periodic orbits are the key for understanding classical Hamiltonian systems. As we show here, they are the clue for understanding Bertrand's result relating the boundedness, flatness, and periodicity of orbits with the functional form of the potentials producing them. This result, which is known as Bertrand's theorem, was proved in 1883 using classical 19th century techniques. In this paper, we prove such a result using the relationship between the bounded plane and periodic orbits, constants of motion, and continuous symmetries in the Hamiltonian system.





## I. INTRODUCTION

The Bertrand theorem[1] states that the only central potentials in which the bounded trajectories are closed are the Newtonian, $-k/r$, and the potential of an isotropic three-dimensional harmonic oscillator, $kr^2$—discussions of this theorem may be found in some books or papers on classical mechanics; see, for example, Refs. 2–7. These problems are basic examples in classical mechanics since they help in developing intuition on the motion of bodies. On the other hand, they may also contribute to propagating erroneous beliefs on the properties of motion, which is not generally as regular and predictable as those examples may imply. In fact, it is seldom the case that they appear under other potentials. Those results have been recently related to the 250 years old curious but as yet unexplained geometric Titius–Bode rule for planetary orbits.[22] Bertrand's theorem has also been related to isochrone properties as explained in Ref. 8.

The Kepler and harmonic oscillator problems have common unique features that may be explained through the action of dynamical symmetries. These problems are examples of *superintegrable* systems,[9] since they possess more than three functionally independent, globally defined, and isolating constants of motion. This happens because the symmetry group of the problem is larger than radial invariance alone would suggest. Bertrand's result pinpointed the unique role of such particular radial systems by showing that they are the only 3D mechanical systems giving rise to both periodic and plane finite motions and, by imposing the extra requisite of the potential of going to zero as $r \to \infty$, also established that the only potential with three such features is the Newtonian one.[2,4] These two problems are examples of systems for which the Hamilton–Jacobi equation is separable in more than one system of coordinates and in which there are more than three constants of motion. In any $N$ degrees of freedom integrable system, it is necessary that at least $N$ of the constants of motion be in involution but, in any superintegrable system, exist more than $N$ but less than $2N$ of these constants, not all of them in involution.[9–13,23] It is also needed that they are functionally independent from each other, otherwise some might be only trivial variations of the remaining ones. A set of $s$ constants of motion $I_i$, $i = 1, \ldots, s$, where $N \le s < 2N$, is said to be functionally independent if the rank of the $s \times 2N$ Jacobian,

$$J = \frac{\partial(I_1, \ldots, I_n)}{\partial(x_i, p_i)}, \tag{1}$$

is precisely equal to $s$. If an $N$-dimensional Hamiltonian possesses $2N - 1$ of such isolating constants, it is called a maximally superintegrable system.[9]

In this paper, we discuss Bertrand's problem of finding the 3D systems in which all bounded orbits are plane and periodic—and these are the only facts about their motion that we need to know. Using the Hamiltonian formalism of classical mechanics and the realization of the associated symmetry algebra through Poisson brackets, we obtain a proof of Bertrand's theorem. In contrast to previous proofs,[1–3] we do not require comparing the Hamiltonian with similar ones. Let us mention the remarkable proof of Bertrand's result given in Arnold's book,[14] which was rewritten for non-mathematicians in Ref. 4. A quite different proof of Bertrand's theorem is discussed in this work.

## II. ORBITAL SYSTEMS WITH PLANAR PERIODIC BOUNDED ORBITS

What are the 3D mechanical systems having planar orbits? If in a mechanical system all motions lay on a plane,[24] Urbankte[15] has shown that the system has to be one with a potential energy term depending only on the radial distance to the center of force, that is, $V(\mathbf{r})$ should be a $SO(3)$ invariant system. Therefore, the Hamiltonian of the system should be

$$H = \frac{|\mathbf{p}|^2}{2m} + V(|\mathbf{r}|), \tag{2}$$

where $\mathbf{p}$ is the linear momentum, $\mathbf{r}$ is the position vector, $|\mathbf{v}|$ stands for the magnitude of $\mathbf{v}$, and $V(|\mathbf{r}|)$ is the potential energy function. Both the energy, $E = H(r)$, and the angular momentum, $\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}$, are constants of motion. Any Hamiltonian giving rise to plane orbits must be radial and therefore integrable. The three constants of motion in involution needed are: (1) any specific component of the angular momentum, let us say $L_z$, (2) the energy, $E$, which is the Hamiltonian value, and (3) the squared magnitude of the angular momentum, $L^2$. The periodicity of the orbits means that $\mathbf{r}(t) = \mathbf{r}(t + T)$, where $T$ is the period; this is valid in any coordinate system, although, with angular coordinates, there is also an implicit $2\pi$ periodicity.

For any two dynamical quantities, $A(q, p)$ and $B(q, p)$, the Poisson bracket (PB) is defined as[16]

$$\{A, B\} \equiv \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}, \tag{3}$$

where $q$ and $p$ stand for any set of canonically conjugated variables, and we are using the summation convention for repeated indices. The constancy of angular momentum may be written as

$$\{\mathbf{L}, H\} = 0. \tag{4}$$

The orbital plane is characterized by its normal $\mathbf{L}$. Furthermore, the SO(3)-symmetry properties may be summarized as

$$\{L_a, L_b\} = \epsilon_{abc} L_c, \quad a, b, c = 1, 2, 3, \tag{5}$$

i.e., the PB between the infinitesimal generators of rotations, that is, the components of angular momentum. In (5), the components of the Levi-Civita antisymmetric symbol, $\epsilon_{abc}$, are the structure constants of SO(3). The PB above is a canonical realization of an abstract Lie bracket.

After the basic consequences of the planar nature of the orbits have been mentioned, we need to ask for the consequences of the periodicity of bounded orbits. Any bounded periodic orbit, that is, an orbit such that $\mathbf{r}(t) = \mathbf{r}(t + T)$ where the constant period is $T$, is necessarily closed, and therefore, any point on it must be precisely located in a certain reference frame. This may be accomplished using a constant vector lying on the orbital plane. Hence, an extra vectorial conserved quantity should exist, and a Hamiltonian giving rise to only such orbits should be superintegrable. The independent components of the extra constants, together with those of $\mathbf{L}$, must play the role of the infinitesimal generators of its dynamical symmetry group,[14] as we will exhibit for the cases discussed here. The symmetry property of the extra constant determines the functional form of the potential needed for having periodic bounded orbits and hence is crucial for our proof of Bertrand's theorem.

## III. MORE CONSTANTS OF MOTION

As we have just mentioned, the extra conserved quantity, let us call it $\mathcal{A}$, determines the position of one of the points in the periodic bounded orbit in a certain reference frame. Then, consequently, if we select the origin of our coordinate system in the orbital plane, $\mathcal{A}$ must lie on such a plane, and it has to be orthogonal to the angular momentum, $\mathcal{A} \cdot \mathbf{L} = 0$. Furthermore, being a constant, it must comply with

$$\{\mathcal{A}, H\} = 0. \tag{6}$$

We must ask now what the precise nature of the extra conserved quantity is. To investigate such matter, we should be aware of the only property we can use to determine the constant, its ability for "nailing down" the orbit in its place. Any geometrical quantity orthogonal to $\mathbf{L}$ and able to fix the position of at least one point in the bounded orbit—in a certain frame of reference—should be enough for our purposes. Let us analyze the general form such quantities might have. To begin with, we might choose the simplest possible geometrical quantity, a vector, enough for pointing toward a single point. However, there are further possibilities, we may choose a rank-2 tensor quantity, enough for marking two points on the orbit. Such points are in directions orthogonal to each other along the principal axes of the tensor. This further requires the tensor to be diagonalizable. We could select tensor quantities of any higher rank for the same purpose. However, any tensor of rank greater than two is bound to be a useless complication, because the number of independent directions in a plane is two. Hence, any tensor of higher rank would give redundant information. We have to conclude, therefore, that any useful additional conserved quantity may only be either a vector or a symmetric rank-2 tensor. We call any of these quantities, the Laplace tensor of the corresponding problems.

In the case of a Hamiltonian having an extra vector constant of motion, the orbits must have a single dynamical symmetry axis, whereas in the case of a Hamiltonian having a rank-2 tensor conserved quantity, the orbits must have two dynamical symmetry axes.

In other words, in the former case, only one of the orbit's symmetry axes passes through the origin, whereas in the latter, two of the orbit's symmetry axes pass through the origin, and thus, the center of force must coincide with the geometrical center of the orbit.[25]

With such ingredients, we proceed to find, one at a time, the Hamiltonian systems giving rise only to periodic and plane bounded orbits.

## IV. A VECTORIAL CONSTANT QUANTITY

Let us first assume that the additional conserved quantity is a vector, therefore, its PB with **L** has to be[17]

$$\{\mathcal{A}_a, L_b\} = \epsilon_{abc}\mathcal{A}_c. \tag{7}$$

We need to employ the most general vectorial expression built using the known vectors associated with the problem, that is, **r** and **p**. We choose to write such an expression, with the explicit use of the angular momentum, as

$$\mathcal{A} = \alpha_1(r,p)\mathbf{r} + \alpha_2(r,p)(\mathbf{L}\times\mathbf{p}), \tag{8}$$

where $r \equiv |\mathbf{r}|$, $p \equiv |\mathbf{p}|$, and $\alpha_1$ and $\alpha_2$ stand for two at yet unknown phase-space functions. Though this last expression is rather general, why we not try other expressions which could come as easily to the mind as expression (8)? We shall address this point at the end of this section. If we use expression (8) to evaluate the PBs, using (7), it is possible to obtain expressions for the phase-space functions $\alpha_1$ and $\alpha_2$, and for the potential energy function $V(r)$. Applying the mentioned equations, consequently, $\beta \equiv \alpha_2$ must be a constant, and we additionally obtained the following two differential equations:

$$\alpha_1(r,p) - \beta m r\frac{dV(r)}{dr} = 0 \tag{9}$$

and

$$p\frac{\partial\alpha_3}{\partial r} - m\frac{dV}{dr}\left(\frac{\partial\alpha_3}{\partial p} - \beta p\right) = 0, \tag{10}$$

where, for the sake of conciseness, we have defined the function $\alpha_3(r, p) \equiv \alpha_1(r, p) - \beta p^2$. From these equations, we conclude that $\alpha_1$ is a function of $r$ and not of $p$, and that it has to be of the form

$$\alpha_1(r) = \frac{c}{r}, \tag{11}$$

with $c$ being an integration constant. Using (9), we found the explicit form of the 3D potential giving rise to only plane and periodic bounded orbits as solutions,

$$V(r) = -\frac{k}{r}, \tag{12}$$

where $k \equiv -c/\beta m$ is assumed to be positive. This is the Newtonian gravitational potential, which is thus the only system with periodic plane bounded orbits and an extra vector constant of motion. The elliptic bounded orbits of the system have only one dynamical symmetry axis.[6]

We can write down the explicit expression for the extra vector conserved quantity, i.e., the rank-1 Laplace tensor,

$$\mathcal{A} = \frac{1}{mk}(\mathbf{p}\times\mathbf{L}) - \frac{\mathbf{r}}{r}. \tag{13}$$

This constant quantity is known as the Laplace–Runge–Lenz vector though the original idea seems to date back to Bernoulli (1742). As it is well known, the vector $\mathcal{A}$ points from the origin toward the pericenter of the orbit, and its magnitude, $|\mathcal{A}| = \sqrt{1 + 2EL^2/mk^2}$, may be made to coincide with the orbit's eccentricity. The constants of motion for the Kepler problem are hence the Hamiltonian and the four—only four out of six possible ones—independent components of **L** and of $\mathcal{A}$. Hence, the Kepler problem is explicitly found to be maximally superintegrable.

Is there any further meaning in the extra constant? To answer this, let us evaluate the Poisson brackets between all the problem constants, two of them have been already given in Eqs. (3) and (5), for the third one, we obtain

$$\{A_s, A_k\} = -\mathrm{sgn}(E)\epsilon_{skm}L_m, \tag{14}$$

where the function $\mathrm{sgn}(x)$ takes the values +1, −1 depending on whether $x$ is a positive or a negative number, and $E$ is the value of the Hamiltonian, $H$, that is, the energy. Equations (3), (5), and (14) show that the hidden symmetry of the Kepler problem is SO(4), the rotation group in four-dimensions, but only in the case of bounded orbits with $E < 0$. For the scattering orbits, the motions with $E > 0$, we have the SO(3,1) group, the Lorentz group, as their symmetry group.[26]

Since in all the calculations just done, the form assumed for $\mathcal{A}$ was chosen on the basis of Eq. (8) only, we may wonder what would happen if, as a different starting point, another expression for $\mathcal{A}$ is assumed. To see what happens in such a case, let us try the ansatz

$$\mathcal{A}' = \alpha_1'(r,p)\mathbf{p} + \alpha_2'(r,p)(\mathbf{L}\times\mathbf{r}), \tag{15}$$

which is as consistent with the required conditions as expression (8). If we repeat our calculations, but using expression (15) instead of expression (8), we obtain the same conclusion regarding the potential energy function, that is, we obtain the Newtonian potential but, for the extra constant of motion, we get

$$\mathcal{A}' = \mathbf{p} + \frac{mk}{rL^2}(\mathbf{L}\times\mathbf{r}), \tag{16}$$

a quantity known as the Hamilton vector.[19] This vector is closely related to the hodograph of the Kepler problem. Let us note that $\mathcal{A}'$ is not independent of $\mathcal{A}$ since the relationship $\mathcal{A}' = mk(\mathbf{L}\times\mathcal{A})/L^2$ exists. For a discussion of these facts, see Ref. 18.

## V. AN ADDITIONAL TENSOR CONSTANT

Let us now consider a system with a rank-2 tensor as the additional constant of motion. The most general tensor we can define using the basic variables of the problem, **p** and **r**, both diagonalizable and compatible with (8), can be given as the exterior product given next (written down in components),

$$A_{ij} = \alpha_1(r,p)p_ip_j + \alpha_2(r,p)(x_ip_j + p_ix_j) + \alpha_3(r,p)x_ix_j, \tag{17}$$

where $i$ and $j$ run from 1 to 3, $\alpha_1$, $\alpha_2$, and $\alpha_3$ are functions of $p$ and $q$ to be determined, as in Sec. IV. This tensor must transform appropriately under rotations. Such transformation is embodied in the following PB:[17]

$$\{A_{ij}, L_s\} = \epsilon_{isn}A_{nj} + \epsilon_{jsn}A_{in}. \tag{18}$$

We may employ a method similar to that used in Sec. IV, but using Eqs. (17) and (18) instead, to get that $\alpha_1$ and $\alpha_3$ are constants and that $\alpha_2$ must vanish. We also get the differential equation,

$$\frac{1}{r}\frac{dV(r)}{dr} = \frac{\delta}{m\alpha_1}, \tag{19}$$

whose solution gives us the harmonic oscillator potential,

$$V(r) = \frac{1}{2}kr^2, \tag{20}$$

where $k \equiv \alpha_3/m\alpha_1$. Therefore, the assumption of a Hamiltonian system allowing both periodic and plane bounded orbits, together with having an extra rank-2 tensor constant of motion, has led us to the harmonic oscillator, which is thus the only system with periodic plane bounded orbits and an extra tensor constant of motion. The explicit form of the Laplace tensor is

$$\mathcal{A}_{ij} = \frac{1}{2}\left(\frac{1}{m}p_i p_j + kx_i x_j\right). \tag{21}$$

As we have mentioned, and it is known for the harmonic oscillator, the geometrical center of the orbits coincides with the location of the center of force. Again, taking into consideration all the relationships between the components of the Laplace tensor—for example, that the Hamiltonian is just the trace of (21), we found only five constants of motion in the system. Therefore, the harmonic oscillator is also a maximally superintegrable problem. To investigate the hidden symmetry of the harmonic oscillator, we need to know the PB between the components of the Laplace tensor $\mathcal{A}$,

$$\{\mathcal{A}_{ij}, \mathcal{A}_{km}\} = (\delta_{jk}\epsilon_{ims} + \delta_{jm}\epsilon_{iks} - \delta_{ik}\epsilon_{mjs} - \delta_{im}\epsilon_{kjs})L_s; \tag{22}$$

this equation together with Eqs. (5) and (18) is one of the various forms in which the Lie algebra of the SU(3) group can be expressed.

We have found that the Laplace tensor and the angular momentum vector comprise the infinitesimal generators of SU(3), the group of special unitary transformations in three dimensions, which is thus the dynamical symmetry group of the harmonic oscillator.

## VI. CONCLUSIONS

Taken together, the results of Secs. IV and V finish a proof of Bertrand's theorem. We have exhibited the maximally superintegrable nature of both the harmonic oscillator and the Kepler problems, which is what makes them so special among the rest of the 3D radial problems. The difference, from the point of view of Bertrand's theorem, between these two problems has to be found in the symmetry properties of the extra constant $\mathcal{A}$, i.e., on whether it is a 3D-vector or a 3D-second rank tensor.

In both cases, that is, the isotropic harmonic oscillator or the Kepler problem, we have explicitly calculated the extra conserved quantities needed for the superintegrability and have stated the hidden symmetry group responsible for it. An interesting fact we have not mentioned yet is that all the bounded orbits of Bertrand's Hamiltonians are always ellipses. We can speculate that perhaps there is a certain connection between superintegrability and the geometric properties of this particular conic section, or perhaps this could have something to do with the as yet unproven fact that every superintegrable system separates in at least one of the confocal ellipsoidal coordinates or their degenerations. In fact, Bertrand's result emphasizes the peculiarity of the Keplerian and harmonic oscillator problems, a peculiarity that put them apart from the vast majority of systems one may encounter in classical mechanics. Bertrand's theorem contributes to explaining both the enormous differences between such problems and the overwhelming majority of classical mechanical systems and the easiness in finding their solutions at difference with the impossibility of finding them in most mechanical problems. We have also pinpointed the suspected connection between the Bertrand result and the Titius–Bode rule and with the isochrone potentials of interest in potential theory. We think that the importance of the proof presented here has to be found in the method used for accomplishing it. Such a method should be applicable to a wider class of problems, as, for example, in the search of superintegrable systems but with a different set of symmetry properties. Using it, we should in principle be able to answer the question, what is the set of Hamiltonians with bounded closed orbits but with nonvanishing torsion? However, we should point out that such properties pinpoint precisely to the features that make these problems so useful in classical mechanics courses. Otherwise, the students would think that such features are common to the majority of mechanical problems, despite the 19th century results of H. Poincaré on the problem of three bodies[20] and the evidence to the contrary that surrounds us—just consider the random behavior expected from the simple act of tossing a coin.[21]

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[21]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Menlo Park, USA, 1963), Vol. I.
[22]The Titius-Bode rule says that the semi-major axes of each planet's orbit appear as a simple function of planetary sequence: let us consider three consecutive planets with semimajor axes $a_1$, $a_2$, and $a_3$. According to this rule the intermediate axis, $a_2$, is approximately $a_2 \simeq (a_1 a_3)^{1/2}$, i.e., the geometric mean of the neighboring planet axes. Such a relationship is just an approximate numerical relationship with no verified physical basis, but see Ref. 10.
[23]To be in involution means that any two of the constants, $I_j$, $I_k$, should comply with $\{I_j, I_k\} = 0$.
[24]We are assuming that the orbits lay completely on a plane. We are not referring to the plane projection of 3D orbits like those possibly occurring in certain helical trajectories.
[25]To offer proofs of the preceding statements, we must explicitly use the associated potentials, since those statements are features of the motion under precisely those two potentials. Such proofs are outside the scope of this paper. We do not know the facts about the details of the motion under such potentials.
[26]This symmetry does not imply any relationship between our problem and relativity.