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CONSTRUCTIVE APPROXIMATION

The "Full Müntz Theorem" Revisited

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Abstract. We give an elementary proof of the "convergent sum part" of the full Müntz Theorem in $L_p(A)$ and in $C(A)$, together with the "Clarkson–Erdős–Schwartz phenomenon" for all $p \in (0, \infty)$, and for all compact $A \subset [0, \infty)$ with positive lower density at 0. This extends earlier results of Müntz [8], Szász [14], Clarkson and Erdős [5], L. Schwartz [13], P. Borwein and Erdélyi [2] and [3], and Operstein [10], and offers an arguably shorter and more elementary approach to reprove a large part of the result W. B. Johnson achieved with me in [7]. This approach does not require the usage of Bastero's extension of the Krivine–Maurey stable theory. It requires only a standard undergraduate level familiarity with real and complex analysis.

1. Introduction

Denote by $\operatorname{span}\{f_1, f_2, \ldots\}$ the collection of all finite linear combinations of the functions $f_1, f_2, \ldots$ over $\mathbf{R}$. For the sake of brevity we introduce the notation

$$\sigma_\lambda := \sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j^2 + 1}$$

and

$$\sigma_{\lambda,p} := \sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1},$$

which will be used throughout the paper. Extending earlier results of Müntz, Szász, Clarkson, Erdős, L. Schwartz, P. Borwein, Erdélyi, and Operstein, in [7] we proved the result below.

Theorem 1.1 ("Full Müntz Theorem" in $L_p[0, 1]$ for $p \in (0, \infty)$). *Let $p \in (0, \infty)$. Suppose $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct real numbers greater than $-(1/p)$. Then*

$$\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$$

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is dense in $L_p[0,1]$ *if and only if* $\sigma_{\lambda,p} = \infty$. *Moreover, if* $\sigma_{\lambda,p} < \infty$, *then every function from the* $L_p[0,1]$ *closure of* $\mathrm{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ *can be represented as an analytic function on* $\{z \in \mathbf{C}\backslash(-\infty, 0] : |z| < 1\}$ *restricted to* $(0, 1)$.

In handling the "convergent sum part," that is, when $\sigma_{\lambda,p} < \infty$, in [7] we use Bastero's [1] extension of the Krivine–Maurey stable theory. In [7] the authors were not able to include the case $p = \infty$ in their discussion. The right result when $p = \infty$ is proved in [6].

Theorem 1.2 ("Full Clarkson–Erdős–Schwartz Theorem" in $C[0, 1]$)**.** *Let* $(\lambda_j)_{j=1}^{\infty}$ *be a sequence of distinct positive numbers. Then* $\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ *is dense in* $C[0,1]$ *if and only if* $\sigma_\lambda = \infty$. *Moreover, if* $\sigma_\lambda < \infty$, *then every function from the* $C[0,1]$ *closure of* $\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ *can be represented as an analytic function on* $\{z \in \mathbf{C}\backslash(-\infty, 0] : |z| < 1\}$ *restricted to* $(0, 1)$.

This result improves an earlier result by P. Borwein and Erdélyi (see [2] and [3]) stating that if $\sigma_\lambda < \infty$, then every function from the $C[0,1]$ closure of $\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ is in $C^\infty(0, 1)$.

The purpose of this paper is to replace Bastero's [1] extension of the Krivine–Maurey stable theory in the proof of the above "full Müntz Theorem" in $L_p[0,1]$ for $p \in (0, \infty)$ by more elementary textbook methods. It turns out that this is possible. In Section 2 we show this in the case when $p \geq 1$ by combining the "full Clarkson–Erdős–Schwartz Theorem" in $C[0,1]$ above and a guided exercise from [2] (see E.7 on page 216). We formulate this guided exercise as well before presenting our new "textbook proof" of the "full Müntz Theorem" in $L_p[0,1]$ for $p \in [1, \infty)$ in the "convergent sum part," that is, when $\sigma_{\lambda,p} < \infty$. In this "convergent sum part" a "Clarkson–Erdős–Schwartz phenomenon" is shown to hold as well. That is, we prove that every function in the $L_p[0,1]$ closure of $\mathrm{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ can be represented as an analytic function on

$$\{z \in \mathbf{C}\backslash(-\infty, 0] : |z| < 1\}$$

restricted to $(0, 1)$. As far as the completeness of this paper is concerned, Section 2 may as well be disregarded; in Sections 4–7 we formulate and prove more general results. In Sections 4 and 5 we will discuss the full Müntz Theorem in $C(A)$, while Sections 6 and 7 focus on the full Müntz Theorem in $L_p(A)$ when $p \in (0, \infty)$. A large part of Sections 6 and 7 is similar to the discussion in Sections 4 and 5, however, it would be a bit of an overstatement to call the presentation in the case of $L_p(A)$ as a "simple modification" of the arguments handling the case of $C(A)$.

Compared with Section 2, in Sections 6 and 7 the main challenge is to include the case $p \in (0, 1)$, although replacing $[0, 1]$ with a compact set $A \subset [0, \infty)$ with positive lower density at 0 is also remarkable, where the lower density of a measurable set $A \subset [0, \infty)$ at 0 is defined by

$$d(A) := \liminf_{y \to 0+} \frac{m(A \cap [0, y])}{y}.$$

Note that the Guided Exercise E.7 from [2] on page 216, that is used in Section 2, offers a bounded Nikolskii-type inequality only for $p \in [1, \infty)$, so one is unable to refer to this to handle the $p \in (0, 1)$ case in the proof of Theorem 3.1.

We conclude the Introduction by expressing the opinion shortly that this paper offers another, arguably more elementary and more elegant approach to handle the "convergent sum part" of the "full Müntz Theorem" in $L_p(A)$, together with the "Clarkson–Erdős–Schwartz phenomenon," for all $p \in (0, \infty)$ and for all compact sets $A \subset [0, \infty)$ with positive lower density at 0. In addition, the paper extends the Erdős–Clarkson–Schwartz phenomenon from the case of [0, 1] discussed in [6] to the case of a compact set $A \subset [0, \infty)$ with positive lower density at 0. This is the content of Theorem 3.2.

2. The "Full Müntz Theorem" in $L_p[0, 1]$ Together with the "Clarkson–Erdős–Schwartz Phenomenon" for $p \in [1, \infty)$

Our main tool in this section is from [2, see E.7 on page 216].

Theorem 2.1 (Bounded Nikolskii-Type Inequality). *Suppose $p \in [1, \infty)$ and $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct real numbers greater than $-(1/p)$. If $\sigma_{\lambda,p} < \infty$, then for every $\gamma > 0$ there is a constant c_γ depending only on $\gamma > 0$ and $(\lambda_j)_{j=1}^{\infty}$ such that*

$$|x^{1/p} Q(x)| \leq c_\gamma \|Q\|_{L_p[0,1]}$$

for every $Q \in \operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ and for every $x \in [0, 1 - \gamma]$.

Now we are ready to present our new "textbook proof" of the "convergent sum part" of the "full Müntz Theorem" in $L_p[0, 1]$ together with the "Clarkson–Erdős–Schwartz phenomenon" for $p \in [1, \infty)$.

A new proof of the "convergent sum part" of Theorem 1.1 *when* $p \in [1, \infty)$. Suppose $p \in [1, \infty)$ and $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct real numbers greater than $-(1/p)$. Let $\sigma_{\lambda,p} = \infty$. Suppose f is in the $L_p[0, 1]$ closure of $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$, that is, there are

$$Q_n \in \operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$$

such that

$$\lim_{n\to\infty} \|Q_n - f\|_{L_p[0,1]} = 0.$$

Then the sequence (Q_n) is Cauchy in $L_p[0, 1]$, and by Theorem 2.1 the sequence (S_n) with

$$S_n(x) := x^{1/p} Q_n(x), \qquad n = 1, 2, \ldots,$$

is uniformly Cauchy on $[0, 1 - \gamma]$. Now we can apply a linearly scaled version of Theorem 1.2 to the interval $[0, 1 - \gamma]$. We obtain that

$$\lim_{n\to\infty} \|S_n - g\|_{L_\infty[0,1-\gamma]} = 0,$$

where g can be represented as an analytic function on $\{z \in \mathbf{C}\backslash(-\infty, 0] : |z| < 1 - \gamma\}$ restricted to $(0, 1 - \gamma)$ and by a well-known theorem in complex analysis, called the "Normal Family Principle" (see Theorem 14.6 in [12]) we have

$$\lim_{n\to\infty} \|S_n - g\|_{L_\infty(A)} = 0$$

for any compact set $A \subset \{z \in \mathbf{C}\backslash(-\infty, 0] : |z| < 1\}$. Therefore $Q_n(z) := z^{-1/p} S_n(z)$ converges uniformly on any compact set $A \subset \{z \in \mathbf{C}\backslash(-\infty, 0] : |z| < 1\}$, and the "new proof" is finished by a well-known theorem of complex analysis (see Theorem 10.28 in [12]).

3. "Full Müntz Theorem" in $L_p(A)$, Together with the "Clarkson–Erdős–Schwartz Phenomenon" for all $p \in (0, \infty)$, and for all Compact $A \subset [0, \infty)$ with Positive Lower Density at 0

Our first theorem is an extension of the main result in [6] from the case of [0, 1] to the case of an arbitrary compact set $A \subset [0, \infty)$ with positive lower density at 0. The reasonably straightforward changes required in the proof of this extension, which can be made by following the method of the proof in [6] closely (see Lemma 4.1) while keeping in mind how the "convergent sum part" of a compact set $A \subset [0, \infty)$ with positive lower density at 0 was handled in [7] (see Lemmas 4.2, 4.3, and 4.4 in [7]), are worked out carefully. The key lemma to prove Theorem 3.1 is Lemma 4.1. Based on some ideas from [6], [7], and [4] we give a short proof of Lemma 4.1 in Section 5. The proof of the "convergent sum part" of Theorem 3.1 is presented in Section 5 as well. It requires no more than a familiarity with complex analysis at an introductory level. An elementary proof of the fact that $\sigma_\lambda = \infty$ implies the denseness of $\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ in $C(A)$ when $A = [0, 1]$ may be found in both [3] and [2]. However, this fact extends easily to the case when $A \subset [0, \infty)$ is an arbitrary compact set by using a linear scaling and the Tietze extension theorem.

Theorem 3.1 ("Full Clarkson–Erdős–Schwartz Theorem" in $C(A)$)**.** *Let $A \subset [0, \infty)$ be a compact set with positive lower density at* 0. *Suppose $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct positive numbers. Then* $\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ *is dense in $C(A)$ if and only if $\sigma_\lambda = \infty$. Moreover, if $\sigma_\lambda < \infty$, then every function from the $C(A)$ closure of* $\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ *can be represented as an analytic function on* $\{z \in \mathbf{C}\backslash(-\infty, 0] : |z| < r_A\}$ *restricted to* $A \cap (0, r_A)$, *where*

$$r_A := \sup\{y \in \mathbf{R} : m(A \cap [y, \infty)) > 0\}$$

($m(\cdot)$ *denotes the one-dimensional Lebesgue measure*).

In [7] not only $0 < p < 1$ has been allowed, but even the underlying set has been extended in the "full Müntz Theorem." Namely the interval [0, 1] has been replaced by an arbitrary compact set $A \subset [0, \infty)$ with positive lower density at 0. In Sections 6 and 7 we give a new simple proof of the "convergent sum part" of the main result of [7] below. That is, we offer a proof of the "convergent sum part" of the result below in its complete form without using Bastero's [1] extension of the Krivine–Maurey stable theory. It is nice to see this difficult case of the full result from two quite different points of view.

Theorem 3.2 ("Full Müntz Theorem" in $L_p(A)$ for $p \in (0, \infty)$ and for Compact Sets $A \subset [0, \infty)$ with Positive Lower Density at 0)**.** *Let $A \subset [0, \infty)$ be a compact set with* *positive lower density at* 0. *Let* $p \in (0, \infty)$. *Suppose* $(\lambda_j)_{j=1}^{\infty}$ *is a sequence of distinct real numbers greater than* $-(1/p)$. *Then* $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ *is dense in* $L_p(A)$ *if and only if* $\sigma_{\lambda,p} = \infty$. *Moreover, if* $\sigma_{\lambda,p} < \infty$, *then every function from the* $L_p(A)$ *closure of* $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ *can be represented as an analytic function on* $\{z \in \mathbf{C}\backslash(-\infty, 0] : |z| < r_A\}$ *restricted to* $A \cap (0, r_A)$, *where*

$$r_A := \sup\{y \in \mathbf{R} : m(A \cap [y, \infty)) > 0\}$$

($m(\cdot)$ *denotes the one-dimensional Lebesgue measure*).

The key lemma to prove Theorem 3.2 is Lemma 6.1. We give a short proof of Lemma 6.1 in Section 7. The proof of the "convergent sum part" of Theorem 3.2 is presented in Section 7 as well. Similarly to the proof of the "convergent sum part" of Theorem 3.1 it requires no more than a familiarity with complex analysis at an introductory level. An elementary proof of the fact that $\sigma_{\lambda,p} = \infty$ implies the denseness of $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ in $L_p(A)$ may be found in [7].

4. Lemmas to Theorem 3.1

The proof of the "convergent sum part" in Theorem 3.2 follows from Lemmas 4.1, 4.2, and 4.3 below. The proof of Lemma 4.1 can be given by a simple modification of a key observation in [6], by replacing the interval [0, 1] by an arbitrary compact set A with positive lower density at 0. This will be done in Section 5. For the sake of brevity we will use the notation

$$\|f\|_A = \sup_{x \in A} |f(x)|$$

for real-valued functions f defined on a set A.

Lemma 4.1. *Let* $A \subset [0, \infty)$ *be a compact set with positive lower density at* 0. *Suppose* $(\lambda_j)_{j=1}^{\infty}$ *is a sequence of distinct positive numbers satisfying* $\sigma_\lambda < \infty$. *Suppose that the positive numbers* β_j, γ_j, *and* δ_j *are distinct and satisfy*

$$\{\lambda_j : j = 1, 2, \ldots\} = \{\beta_j : j = 1, 2, \ldots\} \cup \{\gamma_j : j = 1, 2, \ldots\} \cup \{\delta_j : j = 1, 2, \ldots, k\},$$

where $(\beta_j)_{j=1}^{\infty}$ *is decreasing*, $(\gamma_j)_{j=1}^{\infty}$ *and* $(\delta_j)_{j=1}^{k}$ *are increasing*, $\gamma_1 \geq 1$, *and*

$$\sum_{j=1}^{\infty} \beta_j \leq \eta \qquad \text{and} \qquad \sum_{j=1}^{\infty} 1/\gamma_j < \infty.$$

Let

$$H_\beta := \operatorname{span}\{x^{\beta_1}, x^{\beta_2}, \ldots\}, \qquad H_\gamma := \operatorname{span}\{1, x^{\gamma_1}, x^{\gamma_2}, \ldots\},$$

and

$$H_\delta := \operatorname{span}\{x^{\delta_1}, x^{\delta_2}, \ldots, x^{\delta_k}\}.$$

Suppose $Q \in H_\beta + H_\gamma$ is written as $Q = Q_\beta + Q_\gamma$ with some $Q_\beta \in H_\beta$ and $Q_\gamma \in H_\gamma$. Suppose $\eta > 0$ is sufficiently small. Then there are constants C_β and C_γ depending only on on H_β and H_γ, respectively, so that

$$\|Q_\beta\|_A \le C_\beta \|Q\|_A \tag{4.1}$$

and

$$\|Q_\gamma\|_A \le C_\gamma \|Q\|_A \tag{4.2}$$

for every $Q \in H_\beta + H_\gamma$.

To prove Lemma 4.1 one can follow the method used in [6] to prove the weaker statement in the case $A := [0, 1]$, and replace the interval $[0, 1]$ by an arbitrary compact set $A \subset [0, \infty)$ with positive lower density at 0. To record a proof of Theorem 4.1, one can modify the arguments in [6]. This can be done simply with the help of Lemmas 4.2–4.5 below.

The proof of the next lemma is similar to that of Corollary 2.8 in [7]. It is a straightforward combination of the Mean Value Theorem and D. J. Newman's Markov-type inequality formulated in Theorem 2.6 of [7] (see also [2, Theorem 6.1.1 on page 276] or [9]). Newman's result is formulated by Lemma 4.5 of this section.

Lemma 4.2. *Let $p \in (0, \infty)$. Let $B \subset [0, b]$ be a measurable set satisfying*

$$m(B \cap [0, \beta]) \ge \delta\beta$$

for every $\beta \in [0, b]$ with some $\delta \in (0, 1]$. Let $\beta_1, \beta_2, \dots, \beta_n$ be distinct positive real numbers. Suppose that

$$\sum_{j=1}^{n} \beta_j =: \eta \le \frac{\delta}{18}.$$

Then

$$\|Q\|_{[0,b]} \le 2\delta^{-1}\|Q\|_B,$$

and hence

$$\|Q\|_K \le c_1(K, b, \delta)\|Q\|_B$$

for every $Q \in \mathrm{span}\{x^{\beta_1}, x^{\beta_2}, \dots, x^{\beta_n}\}$ and for every compact set $K \subset \mathbf{C}\backslash\{0\}$, where the constant $c_1(K, b, \delta)$ depends only on K, b, and δ.

Proof of Lemma 4.2. Let $Q \in \mathrm{span}\{x^{\beta_1}, x^{\beta_2}, \dots, x^{\beta_n}\}$. Without loss of generality we may assume that Q is not identically 0. Choose a value $\beta \in [0, b]$ such that

$$|Q(\beta)| = M := \|Q\|_{[0,b]}.$$

Note that since $Q(0) = 0$ and Q is not identically 0, we have $\beta > 0$. Then the set $[\delta\beta/2, \beta] \cap B$ is not empty. Choose a point $\gamma \in [\delta\beta/2, \beta] \cap B$. Then, combining the Mean Value Theorem and D. J. Newman's Markov-type inequality formulated in Lemma 4.5, we obtain

$$\begin{aligned} M - |Q(\gamma)| &\leq |Q(\beta) - Q(\gamma)| \leq (\beta - \gamma)\|Q'\|_{[\gamma,\beta]} \\ &\leq (\beta - \gamma)\|Q'\|_{[\delta\beta/2,b]} \leq (1-\delta/2)\beta\,\frac{9\eta}{\delta\beta/2}\,M \\ &\leq (1-\delta/2)\beta\,\frac{\delta/2}{\delta\beta/2}\,M \leq (1-\delta/2)M, \end{aligned}$$

that is, $|Q(\gamma)| \geq (\delta/2)M$, and hence

$$\|Q\|_{[0,b]} \leq 2\delta^{-1}\|Q\|_B.$$

The second statement of the lemma can be obtained as follows. Using a linear scaling if it is necessary, we may assume that $b = 1$. Repeated applications of Lemma 4.5 with the substitution $x = e^{-t}$ imply that

$$\|(Q(e^{-t}))^{(m)}\|_{[0,\infty)} \leq (9\eta)^m\|Q(e^{-t})\|_{[0,\infty)}, \qquad m = 1, 2, \ldots,$$

in particular,

$$|Q(e^{-t}))^{(m)}(0)| \leq (9\eta)^m\|Q(e^{-t})\|_{[0,\infty)}, \qquad m = 1, 2, \ldots,$$

for every $Q \in \operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$. By using the Taylor series expansion of $Q(e^{-t})$ around 0, we obtain that

$$|Q(z)| \leq c_2(K,\delta)\|Q\|_{[0,1]}, \qquad z \in K,$$

for every $Q \in \operatorname{span}\{x^{\beta_1}, x^{\beta_2}, \ldots\}$ and for every compact $K \subset \mathbf{C}\setminus\{0\}$, where, recalling that $9\eta \leq \delta/2$, we have

$$c_2(K,\delta) := \sum_{m=0}^{\infty}\frac{(\delta/2)^m(\max_{z\in K}|\log z|)^m}{m!} = \exp((\delta/2)\max_{z\in K}|\log z|)$$

is a constant depending only on K and δ. ∎

The next lemma is the key result in [4] (see Theorem 6.1 there). It gives a Clarkson–Erdős–Schwartz-type result for Müntz spaces $\operatorname{span}\{1, x^{\gamma_1}, x^{\gamma_2}, \ldots\}$ on sets $A \subset [0,\infty)$ of positive Lebesgue measure when the exponents γ_j are positive and satisfy $\sum_{j=1}^{\infty} 1/\gamma_j < \infty$.

Lemma 4.3. *Suppose* $\sum_{j=1}^{\infty} 1/\gamma_j < \infty$ *and* $A \subset [0,\infty)$ *is a set of positive Lebesgue measure. Then* $\operatorname{span}\{1, x^{\gamma_1}, x^{\gamma_2}, \ldots\}$ *is not dense in* $C(A)$. *Moreover, if the gap condition*

$$\inf\{\gamma_{j+1} - \gamma_j : j \in \mathbf{N}\} > 0 \tag{4.3}$$

holds, then every function $f \in C(A)$ *from the uniform closure of* $\operatorname{span}\{1, x^{\gamma_1}, x^{\gamma_2}, \ldots\}$ *on* A *is of the form*

$$f(x) = \sum_{j=0}^{\infty} a_j x^{\gamma_j}, \qquad x \in A \cap [0, r_A),$$

where

$$r_A := \sup\{x \in [0, \infty) : m(A \cap (x, \infty)) > 0\}$$

is the essential supremum of A.

If the gap condition (4.3) *does not hold, then every function* $f \in C(A)$ *from the uniform closure of* $\operatorname{span}\{1, x^{\gamma_1}, x^{\gamma_2}, \dots\}$ *on* A *can still be extended analytically throughout the region*

$$\{z \in \mathbf{C} \setminus (-\infty, 0] : |z| < r_A\}.$$

In the proof of Lemma 4.1 we also need the following well-known fact the proof of which is pretty standard (see Theorem 1.42 in [11], in fact, the result is true for any topological vector space).

Lemma 4.4. *Let* $p \in (0, \infty)$ *and suppose* $A \subset \mathbf{R}$ *is a compact set. Let* X *denote either* $L_p(A)$ *or* $C(A)$. *Let* $U \subset X$ *be a closed linear subspace and let* $V \subset X$ *be a finite-dimensional* (*hence closed*) *linear subspace. Then* $U + V$ *is closed in* X.

The following Markov-type inequality for Müntz polynomials is due to Newman [2, Theorem 6.1.1 on page 276] (see also [9]). It is used in the proof of Lemma 4.2.

Lemma 4.5. *Suppose that* $\beta_1, \beta_2, \dots, \beta_n$ *are distinct nonnegative numbers. Then*

$$\|xQ'(x)\|_{[0,1]} \le 9 \left(\sum_{j=1}^{n} \beta_j \right) \|Q\|_{[0,1]}$$

for every $Q \in \operatorname{span}\{x^{\beta_1}, x^{\beta_2}, \dots, x^{\beta_n}\}$.

The bounded Bernstein-type inequality below (see the Guided Exercise E.5b on page 182 of [2]) for certain Müntz spaces on [0, 1] is needed in the proof of Lemma 4.1 as well.

Lemma 4.6. *Suppose* $\Gamma := (\gamma_j)_{j=1}^{\infty}$ *is a sequence of distinct positive numbers satisfying* $\gamma_1 \ge 1$ *and* $\sum_{j=1}^{\infty} 1/\gamma_j < \infty$. *Then*

$$\|Q'\|_{[0,x]} \le c_3(\Gamma, x) \|Q\|_{[0,1]}$$

for every $Q \in \operatorname{span}\{1, x^{\gamma_1}, x^{\gamma_2}, \dots\}$ *and for every* $x \in [0, 1)$, *where* $c_3(\Gamma, x)$ *depends only on* Γ *and* x.

The following bounded Remez-type inequality due to P. Borwein and Erdélyi [4] is also an important ingredient of the proof of Lemma 4.1. In the proof of Theorem 3.1 it can be exploited similarly to the treatment of the case $A := [0, 1]$ in [6].

Lemma 4.7. *Suppose* $(\gamma_j)_{j=1}^{\infty}$ *is a sequence of distinct positive numbers satisfying*

$$\sum_{j=1}^{\infty} 1/\gamma_j < \infty.$$

Let $s > 0$. Then there exists a constant $c_4(\Gamma, s)$ depending only on $\Gamma := (\gamma_j)_{j=1}^{\infty}$ and s (and not on ϱ, B, or the number of terms in Q) so that

$$\|Q\|_{[0,\varrho]} \le c_4(\Gamma, s)\,\|Q\|_B$$

for every $Q \in \operatorname{span}\{1, x^{\gamma_1}, x^{\gamma_2}, \ldots\}$ and for every set $B \subset [\varrho, 1]$ of Lebesgue measure at least s.

5. Proof of Lemma 4.1 and Theorem 3.2

Proof of Lemma 4.1. Without loss of generality we may assume that

$$1 = r_A := \sup\{y \in \mathbf{R} : m(A \cap [y, \infty)) > 0\}.$$

It is sufficient to prove only (4.1); (4.2) follows from (4.1).

Suppose to the contrary that inequality (4.1) fails for a sufficiently small $\eta > 0$ (we will tell later in the proof how small η should be to get a contradiction). Then there are Müntz polynomials $Q_{\beta,n} \in H_\beta$ and $Q_{\gamma,n} \in H_\gamma$ so that

$$\|Q_{\beta,n}\|_A = 1, \qquad \lim_{n\to\infty} \|Q_{\gamma,n}\|_A = 1, \tag{5.1}$$

and

$$\lim_{n\to\infty} \|Q_{\beta,n} + Q_{\gamma,n}\|_A = 0. \tag{5.2}$$

Choose a number $\alpha > 0$ so that the set $[\alpha, 1-\alpha] \cap A$ has a positive measure s depending only on A. Let $\eta > 0$ be sufficiently small. Then by Lemmas 4.2 and 4.5 $\{Q_{\beta,n} : n = 1, 2, \ldots\}$ is a family of bounded, equicontinuous functions on $[\alpha, 1]$, while by Lemmas 4.7 and 4.6 $\{Q_{\gamma,n} : n = 1, 2, \ldots\}$ is a family of bounded, equicontinuous functions on $[0, 1-\alpha]$. So by the Arzela–Ascoli Theorem there are a subsequence of $(Q_{\beta,n})$ (without loss of generality we may assume that this is $(Q_{\beta,n})$ itself) and a subsequence of $(Q_{\gamma,n})$ (without loss of generality we may assume that this is $(Q_{\gamma,n})$ itself) so that

$$\lim_{n\to\infty} \|Q_{\beta,n} - f\|_{[\alpha,1]} = 0 \tag{5.3}$$

and

$$\lim_{n\to\infty} \|Q_{\gamma,n} - g\|_{[0,1-\alpha]} = 0 \tag{5.4}$$

with some continuous functions f on $[\alpha, 1]$ and g on $[0, 1-\alpha]$. By (5.2), (5.3), and (5.4) we have $f = -g$ on $[\alpha, 1-\alpha] \cap A$, so the function

$$h(x) := \begin{cases} f(x), & x \in [\alpha, 1] \cap A, \\ -g(x), & x \in [0, 1-\alpha] \cap A, \end{cases} \tag{5.5}$$

is well-defined on A. By (5.2)–(5.5) we can deduce that

$$\lim_{n\to\infty} \|Q_{\beta,n} - h\|_A = 0 \tag{5.6}$$

and

$$\lim_{n\to\infty} \| - Q_{\gamma,n} - h\|_A = 0. \tag{5.7}$$

Let $\eta \in (0, \delta/18)$, where

$$0 < \delta := \inf_{y\in(0,1]} \frac{m(A \cap [0, y])}{y}.$$

Using (5.1), (5.6), Lemma 4.5, $\sum_{j=1}^{\infty} \beta_j \le \eta$, Lemma 4.2, and (5.7), we can deduce that

$$|h(x) - h(1)| \le 18\eta\|h\|_{[0,1]} \le 18\eta 2\delta^{-1}\|h\|_A = 36\eta\delta^{-1}, \qquad x \in [\tfrac{1}{2}, 1] \cap A.$$

Note that (5.1), (5.5), and (5.6) imply that $\|h\|_A = 1$ and $h(0) = 0$. Now observe that the function $h - h(1)$ is in the uniform closure of

$$H_\gamma = \mathrm{span}\{1, x^{\gamma_1}, x^{\gamma_2}, \ldots\}$$

on A, hence Lemma 4.7 implies with $s := c_5(A) := m([\frac{1}{2}, 1] \cap A)$ that

$$\|h - h(1)\|_A \le c_4(\Gamma, s)\, \|h - h(1)\|_{[1/2,1]\cap A} \le c_4(\Gamma, c_5(A))\, 36\eta\delta^{-1} < \tfrac{1}{2}$$

whenever η is sufficiently small, more precisely, whenever, in addition to $\eta \in (0, \delta/18)$, we have

$$\eta < \frac{\delta}{72 c_4(\Gamma, c_5(A))}.$$

This contradicts the facts that $h(0) = 0$ and $\|h\|_A = 1$ (note that $\|h - h(1)\|_A < \frac{1}{2}$ implies that $h(x) \in (h(1) - \frac{1}{2}, h(1) + \frac{1}{2})$ for every $x \in A$, and $0 \in A$). Hence the proof of (4.1) is finished for all sufficiently small $\eta > 0$. ■

Proof of Theorem 3.1. An elementary proof of the fact that $\sigma_\lambda = \infty$ implies the denseness of $\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ in $C(A)$ when $A = [0, 1]$ may be found in both [3] and [2]. However, this fact extends easily to the case when $A \subset [0, \infty)$ is an arbitrary compact set by using a linear scaling and the Tietze extension theorem. To extend the full result from the interval $[0, 1]$ to an arbitrary compact set $A \subset [0, \infty)$ with positive lower density at 0, here we consider the case when $\sigma_\lambda = \infty$. To handle this case, suppose $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct positive numbers satisfying $\sigma_\lambda < \infty$. Then there are positive numbers η, β_j, γ_j, and δ_j such that

$$\{\lambda_j : j = 1, 2, \ldots\} = \{\beta_j : j = 1, 2, \ldots\} \cup \{\gamma_j : j = 1, 2, \ldots\} \cup \{\delta_j : j = 1, 2, \ldots, k\},$$

where $\gamma_1 \ge 1$, $(\beta_j)_{j=1}^{\infty}$ is decreasing, $(\gamma_j)_{j=1}^{\infty}$ and $(\delta_j)_{j=1}^{k}$ are increasing, and

$$\sum_{j=1}^{\infty} \beta_j \le \eta, \qquad \sum_{j=1}^{\infty} 1/\gamma_j < \infty,$$

and $\eta > 0$ is as small as in Lemma 4.1. Without loss of generality we may assume that

$$1 = r_A := \sup\{y \in \mathbf{R} : m(A \cap [y, \infty)) > 0\}.$$

Let $\overline{H}$ denote the uniform closure of a subspace $H \subset C(A)$. Using the notation introduced in Lemma 4.1, we want to prove that restricted to A we have $\overline{H_\beta + H_\gamma + H_\delta} \subset \mathcal{A}$, where $\mathcal{A} \subset C(A)$ denotes the collection of functions $f \in C(A)$, which can be represented as an analytic function on $\{z \in \mathbf{C}\backslash(-\infty, 0] : |z| < r_A\}$ restricted to A. Since H_δ is finite-dimensional, Lemma 4.4 implies that

$$\overline{H_\beta + H_\gamma + H_\delta} \subset \overline{H_\beta + H_\gamma} + H_\delta,$$

so it is sufficient to prove that

$$\overline{H_\beta + H_\gamma} \subset \mathcal{A}. \tag{5.8}$$

However, (4.1) and (4.2) imply that

$$\overline{H_\beta + H_\gamma} \subset \overline{H_\beta} + \overline{H_\gamma},$$

where $\overline{H_\beta} \subset \mathcal{A}$ by Lemma 4.2 and $\overline{H_\gamma} \subset \mathcal{A}$ by Lemma 4.3. Hence (5.8) holds, indeed, and the proof of the theorem is finished. ■

6. Lemmas to Theorem 3.2

The proof of the "convergent sum part" in Theorem 3.1 follows from Lemmas 6.1, 6.2, and 6.3 below. The proof of Lemma 6.1 can be given by a simple modification of the method used in the proof of Lemma 4.1. This will be done in Section 7.

Lemma 6.1. *Let $p \in (0, \infty)$. Let $A \subset [0, \infty)$ be a compact set with positive lower density at 0. Suppose $(\lambda_j)_{j=1}^\infty$ is a sequence of distinct numbers greater than $-1/p$ satisfying $\sigma_{\lambda,p} < \infty$. Suppose that the positive numbers β_j, γ_j, and δ_j are distinct and satisfy*

$$\{\lambda_j : j = 1, 2, \ldots\} = \{\beta_j : j = 1, 2, \ldots\} \cup \{\gamma_j : j = 1, 2, \ldots\} \cup \{\delta_j : j = 1, 2, \ldots, k\},$$

where $(\beta_j)_{j=1}^\infty$ is decreasing, $(\gamma_j)_{j=1}^\infty$ and $(\delta_j)_{j=1}^k$ are increasing, $\gamma_1 \geq 1$, and

$$\sum_{j=1}^\infty (\beta_j + (1/p)) \leq \eta \qquad \text{and} \qquad \sum_{j=1}^\infty 1/\gamma_j < \infty.$$

Let

$$H_\beta := \operatorname{span}\{x^{\beta_1}, x^{\beta_2}, \ldots\}, \qquad H_\gamma := \operatorname{span}\{1, x^{\gamma_1}, x^{\gamma_2}, \ldots\},$$

and

$$H_\delta := \operatorname{span}\{x^{\delta_1}, x^{\delta_2}, \ldots, x^{\delta_k}\}.$$

Suppose $Q \in H_\beta + H_\gamma$ is written as $Q = Q_\beta + Q_\gamma$ with some $Q_\beta \in H_\beta$ and $Q_\gamma \in H_\gamma$. Suppose $\eta > 0$ is sufficiently small. Then there are constans C_β and C_γ depending only on H_β and H_γ, respectively, so that

$$\|Q_\beta\|_{L_p(A)} \leq C_\beta \|Q\|_{L_p(A)} \tag{6.1}$$

and

$$\|Q_\gamma\|_{L_p(A)} \le C_\gamma \|Q\|_{L_p(A)} \tag{6.2}$$

for every $Q \in H_\beta + H_\gamma$.

To prove Lemma 6.1 we need to combine the bounded Nikolskii-type inequality below with Lemmas 4.3, 4.4, and 4.5.

Lemma 6.2. *Let* $p \in (0,\infty)$. *Let* $B \subset [0,b]$ *be a measurable set satisfying* $m(B \cap [0,y]) \ge \delta y$ *for every* $y \in [0,b]$. *Let* $\beta_1, \beta_2, \dots, \beta_n$ *be distinct real numbers greater than* $-(1/p)$. *Suppose that*

$$\sum_{j=1}^{n} (\beta_j + (1/p)) =: \eta \le \delta b/36,$$

where $\delta \in (0,1]$. *Then*

$$\|x^{1/p} Q(x)\|_{L_\infty[0,b]} \le \left(\frac{2^{p+1}}{\delta b}\right)^{1/p} \|Q\|_{L_p(B)},$$

and hence

$$\max_{z\in K} |Q(z)| \le c_6(K,p,b,\delta)\|Q\|_{L_p(B)}$$

for every $Q \in \operatorname{span}\{x^{\beta_1}, x^{\beta_2}, \dots, x^{\beta_n}\}$ *and for every compact* $K \subset \mathbf{C}\backslash\{0\}$, *where the constant* $c_6(K,p,b,\delta)$ *depends only on* K, p, b, *and* δ.

Lemma 6.2 is borrowed from [7], see Lemma 2.8 there. For the sake of completeness we present its short proof here as well.

Proof of Lemma 6.2. The proof of the lemma is easy. By using a linear scaling if necessary, without loss of generality we may assume that $b = 1$. Let

$$Q \in \operatorname{span}\{x^{\beta_1}, x^{\beta_2}, \dots, x^{\beta_n}\},$$

and pick a point $y \in (0,1]$ for which

$$|y^{1/p} Q(y)| = \max_{t\in[0,1]} |t^{1/p} Q(t)|.$$

Then using the Mean Value Theorem and applying Lemma 4.5 to

$$x^{1/p} Q(x) \in \operatorname{span}\{x^{\beta_1+(1/p)}, x^{\beta_2+(1/p)}, \dots, x^{\beta_n+(1/p)}\},$$

we obtain for $x \in [(\delta/2)y, y]$ that

$$\begin{aligned}\left(\max_{t\in[0,1]} |t^{1/p} Q(t)|\right) - |x^{1/p} Q(x)| &\le |y^{1/p} Q(y)| - |x^{1/p} Q(x)| \\ &\le |y^{1/p} Q(y) - x^{1/p} Q(x)|\end{aligned}$$

$$
\begin{aligned}
&\le (y-x)\max_{t\in[x,y]}|(t^{1/p}Q(t))'| \\
&\le y\frac{1}{x}\max_{t\in[x,y]}|t(t^{1/p}Q(t))'| \\
&\le \frac{2}{\delta}x\frac{9\eta}{x}\max_{t\in[0,1]}|t^{1/p}Q(t)| \\
&\le \frac{18\eta}{\delta}\max_{t\in[0,1]}|t^{1/p}Q(t)| \le \tfrac{1}{2}\max_{t\in[0,1]}|t^{1/p}Q(t)|.
\end{aligned}
$$

Hence, for $x \in [(\delta/2)y, y]$, we have

$$|x^{1/p}Q(x)| \ge \tfrac{1}{2}\max_{t\in[0,1]}|t^{1/p}Q(t)|.$$

Using the assumption on the set B, we conclude that

$$m(B\cap[(\delta/2)y, y]) \ge \delta y - (\delta/2)y = (\delta/2)y$$

and hence

$$
\begin{aligned}
\|Q\|^p_{L_p(B)} &= \int_B |Q(t)|^p\,dt \ge \int_{B\cap[(\delta/2)y,y]}|Q(t)|^p\,dt \\
&\ge (\delta/2)y2^{-p}(y^{-(1/p)})^p\left(\max_{t\in[0,1]}|t^{1/p}Q(t)|\right)^p \\
&\ge (\delta/2)2^{-p}\left(\max_{t\in[0,1]}|t^{1/p}Q(t)|\right)^p
\end{aligned}
$$

This finishes the proof of the first inequality of the lemma when $b = 1$. As we have already remarked, the case of an arbitrary $b > 0$ follows by a linear scaling. The second inequality of the lemma follows from the first one combined with Lemma 4.2. ■

7. Proof of Lemma 6.1 and Theorem 3.2

Proof of Lemma 6.1. Without loss of generality we may assume that

$$1 = r_A := \sup\{y \in \mathbf{R} : m(A\cap[y,\infty)) > 0\}.$$

It is sufficient to prove only (6.1); (6.2) follows from (6.1). Suppose to the contrary that inequality (6.1) fails for a sufficiently small $\eta > 0$. Then there are Müntz polynomials $Q_{\beta,n} \in H_\beta$ and $Q_{\gamma,n} \in H_\gamma$ so that

$$\|Q_{\beta,n}\|_{L_p(A)} = 1, \qquad \lim_{n\to\infty}\|Q_{\gamma,n}\|_{L_p(A)} = 1, \tag{7.1}$$

and

$$\lim_{n\to\infty}\|Q_{\beta,n}+Q_{\gamma,n}\|_{L_p(A)} = 0. \tag{7.2}$$

Choose a number $\alpha > 0$ so that the set $[\alpha, 1-\alpha]\cap A$ has a positive measure depending only on A. Let $\eta > 0$ be sufficiently small. Then, by Lemmas 6.2 and 4.5,

$$\{S_{\beta,n}(x) := x^{1/p}Q_{\beta,n}(x) : n = 1, 2, \ldots\}$$

is a family of bounded, equicontinuous functions on $[\alpha, 1]$, while by Lemmas 4.7 and 4.6 $\{Q_{\gamma,n} : n = 1, 2, \ldots\}$ is a family of bounded, equicontinuous functions on $[0, 1-\alpha]$ (note that (7.1) ensures that there are measurable sets $B_n \subset [1-\alpha/2, 1] \cap A$ with $M(B_n) \geq \frac{1}{2}m([1-\alpha/2, 1] \cap A)$ such that

$$\begin{aligned}\|Q_{\gamma,n}\|_{B_n} &\leq \left(\tfrac{1}{2}m([1-\alpha/2, 1] \cap A)\right)^{-1/p} \|Q_{\gamma,n}\|_{L_p(A)} \\ &\leq \left(\tfrac{1}{2}m([1-\alpha/2, 1] \cap A)\right)^{-1/p} \sup_k \|Q_{\gamma,k}\|_{L_p(A)} < \infty,\end{aligned}$$

hence by Lemma 4.7, we have $\sup_n \|Q_{\gamma,n}\|_{[0,1-\alpha/2]} < \infty$). So by the Arzela–Ascoli Theorem there are a subsequence of $(Q_{\beta,n})$ (without loss of generality we may assume that this is $(Q_{\beta,n})$ itself) and a subsequence of $(Q_{\gamma,n})$ (without loss of generality we may assume that this is $(Q_{\gamma,n})$ itself) so that

$$\lim_{n\to\infty} \|Q_{\beta,n} - f\|_{[\alpha,1]} = 0 \tag{7.3}$$

and

$$\lim_{n\to\infty} \|Q_{\gamma,n} - g\|_{[0,1-\alpha]} = 0 \tag{7.4}$$

with some continuous functions f on $[\alpha, 1]$ and g on $[0, 1-\alpha]$. By (7.2), (7.3), and (7.4) we have $f = -g$ on $[\alpha, 1-\alpha] \cap A$, so the function

$$h(x) := \begin{cases} f(x), & x \in [\alpha, 1] \cap A, \\ -g(x), & x \in [0, 1-\alpha] \cap A, \end{cases} \tag{7.5}$$

is well-defined on A. By (7.2)–(7.5) we can deduce that

$$\lim_{n\to\infty} \|Q_{\beta,n} - h\|_{L_p(A)} = 0 \tag{7.6}$$

and

$$\lim_{n\to\infty} \|-Q_{\gamma,n} - h\|_{L_p(A)} = 0. \tag{7.7}$$

Together with (7.1) either (7.6) or (7.7) shows that

$$\|h\|_{L_p(A)} = 1. \tag{7.8}$$

Let $H(x) := x^{1/p}h(x)$. We claim that

$$\|H\|_A \geq c_7(\Gamma, \alpha, A, p) > 0. \tag{7.9}$$

Indeed, we have

$$\|h\|_{[\alpha,1]\cap A} \leq \alpha^{-1/p}\|H\|_A. \tag{7.10}$$

Also, using (7.4), (7.5), $m([\alpha, 1-\alpha] \cap A) > 0$, and Lemma 4.7, we deduce that

$$\|h\|_{[0,\alpha]\cap A} \leq c_8(\Gamma, \alpha, A)\|h\|_{[\alpha,1-\alpha]\cap A} \leq c_8(\Gamma, \alpha, A)\alpha^{-1/p}\|H\|_A$$

with a constant $c_8(\Gamma, \alpha, A) \geq 1$ depending only on Γ, α, and A. This, together with (7.10), gives

$$\|h\|_A \leq c_8(\Gamma, \alpha, A)\alpha^{-1/p}\|H\|_A. \tag{7.11}$$

As a consequence of (7.8) we have

$$m(A)^{-1/p} = m(A)^{-1/p}\|h\|_{L_p(A)} \le \|h\|_A,$$

which, together with (7.11), yields (7.9). Let $\eta \in (0, \delta/36)$, where

$$0 < \delta := \inf_{y\in(0,1]} \frac{m(A \cap [0, y])}{y}.$$

Observe that Lemma 6.2 and (7.6) imply that the sequence $(S_{\beta,n})$ with

$$S_{\beta,n}(x) := x^{1/p} Q_{\beta,n}(x), \qquad n = 1, 2, \ldots, \tag{7.12}$$

is uniformly Cauchy on [0, 1], and by (7.6) it has a subsequence that converges to H almost everywhere on A. This, together with Lemma 6.2, yields that H has a unique continuous extension from A to [0, 1] (denote this extension by H as well) for which

$$\lim_{n\to\infty} \|S_{\beta,n} - H\|_{[0,1]} = 0. \tag{7.13}$$

Using (7.1), Lemma 6.2, (7.12), and (7.13), we obtain

$$\|H\|_{[0,1]} \le \left(\frac{2^{p+1}}{\delta}\right)^{1/p} = c_9(A, p) \tag{7.14}$$

with a constant $c_9(A, p)$ depending only on A and p. Applying (7.12), (7.13), Lemma 4.5, $\sum_{j=1}^{\infty}(\beta_j + (1/p)) \le \eta$, and (7.14), we can deduce that

$$H(x) - H(1) \le 18\eta\|H\|_{[0,1]} \le c_{10}(A, p)\eta, \qquad x \in [\tfrac{1}{2}, 1] \cap A,$$

with a constant $c_{10}(A, p)$ depending only on A and p. Note that since each β_j is greater than $-1/p$, (7.13) implies that $H(0) = 0$. Since the sequence $(S_{\gamma,n})$ with

$$S_{\gamma,n}(x) := x^{1/p} Q_{\gamma,n}(x) - H(1)$$

converges to $H(x) - H(1)$ in $L_p(A)$, it has a subsequence (without loss of generality we may assume that it is the sequence $(S_{\gamma,n})$ itself) that converges almost everywhere on A. Therefore by Yegoroff's Theorem, for every $\varrho \in (0, 1)$, the sequence $(S_{\gamma,n})$ converges uniformly on a compact set $B_\varrho \subset [\varrho, 1] \cap A$ of a positive measure s_ϱ depending only on ϱ and A. Hence by Lemma 4.7 the sequence $(S_{\gamma,n})$ converges to $H(x) - H(1)$ uniformly on $[0, \varrho]$ and

$$\|H - H(1)\|_{[0,1/2]} \le c_{11}(\Gamma, A, p)\|H - H(1)\|_{[1/2,1]\cap A}$$

with a constant $c_{11}(\Gamma, A, p) \ge 1$ depending only on Γ, A, and p. We conclude that

$$\begin{aligned}\|H - H(1)\|_A &\le c_{11}(\Gamma, A, p)\,\|H - H(1)\|_{[1/2,1]\cap A} \le c_{11}(\Gamma, A, p)c_{10}(A, p)\,\eta \\ &< (1/2)c_7(\Gamma, \alpha, A, p)\end{aligned}$$

whenever $\eta \in (0, \delta/36)$ is sufficiently small, more precisely, whenever, in addition to $\eta \in (0, \delta/36)$, we have

$$\eta < \frac{c_7(\Gamma, \alpha, A, p)}{2c_{11}(\Gamma, A, p)c_{10}(A, p)}.$$

Therefore

$$H(x) \in (H(1) - \tfrac{1}{2}c_7(\Gamma, \alpha, A, p),\ H(1) + \tfrac{1}{2}c_7(\Gamma, \alpha, A, p)), \qquad x \in A.$$

Since $0 \in A$ and $H(0) = 0$, this contradicts (7.9). Hence the proof of (6.1) is finished for all sufficiently small $\eta > 0$. ∎

Proof of Theorem 3.2. An elementary proof of the fact that $\sigma_{\lambda,p} = \infty$ implies the denseness of $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ in $L_p(A)$ may be found in [7]. Suppose now that $\sigma_{\lambda,p} < \infty$. To handle this case, suppose $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct numbers greater than $-1/p$ satisfying $\sigma_{\lambda,p} < \infty$. Then there are positive numbers η, β_j, γ_j, and δ_j such that

$$\{\lambda_j : j = 1, 2, \dots\} = \{\beta_j : j = 1, 2, \dots\} \cup \{\gamma_j : j = 1, 2, \dots\} \cup \{\delta_j : j = 1, 2, \dots, k\},$$

where $\gamma_1 \ge 1$, $(\beta_j)_{j=1}^{\infty}$ is decreasing, $(\gamma_j)_{j=1}^{\infty}$ and $(\delta_j)_{j=1}^{k}$ are increasing, and

$$\sum_{j=1}^{\infty} (\beta_j + (1/p)) \le \eta, \qquad \sum_{j=1}^{\infty} 1/\gamma_j < \infty,$$

and $\eta > 0$ is as small as in Lemma 7.1. Without loss of generality we may assume that

$$1 = r_A := \sup\{y \in \mathbf{R} : m(A \cap [y, \infty)) > 0\}.$$

Let $\overline{H}$ denote the uniform closure of a subspace $H \subset L_p(A)$. Using the notation introduced in Lemma 6.1, we want to prove that restricted to A we have $\overline{H_\beta + H_\gamma + H_\delta} \subset \mathcal{A}$, where $\mathcal{A} \subset L_p(A)$ denotes the collection of functions $f \in L_p(A)$, which can be represented as an analytic function on $\{z \in \mathbf{C}\backslash(-\infty, 0] : |z| < r_A\}$ restricted to A. Since H_δ is finite-dimensional, Lemma 4.4 implies that

$$\overline{H_\beta + H_\gamma + H_\delta} \subset \overline{H_\beta + H_\gamma} + H_\delta,$$

so it is sufficient to prove that

$$\overline{H_\beta + H_\gamma} \subset \mathcal{A}.$$

However, (6.1) and (6.2) imply that

$$\overline{H_\beta + H_\gamma} \subset \overline{H_\beta} + \overline{H_\gamma},$$

where $\overline{H_\beta} \subset \mathcal{A}$ by Lemma 6.2. The fact that $\overline{H_\gamma} \subset \mathcal{A}$ can be seen as follows. Suppose that $Q_{n,\gamma} \in H_\gamma$ converges in $L_p(A)$. Let $\varrho \in (0, 1)$. Then there is a subsequence of $(Q_{n,\gamma})$ (without loss of generality we may assume that this is $(Q_{n,\gamma})$ itself) that converges on $[\varrho, 1] \cap A$ almost everywhere. Then by Yegoroff's Theorem, it converges uniformly on a compact set $B \subset [\varrho, 1] \cap A$ with positive Lebesgue measure. Then, by Lemma 4.7 it converges uniformly on $[0, \varrho]$, hence by Lemma 4.3 the limit function can be extended analytically throughout

$$\{z \in \mathbf{C}\backslash(-\infty, 0] : |z| < \varrho\}.$$

Since $\varrho \in (0, 1)$ is arbitrary and $r_A = 1$ the proof is finished. ∎

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