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HIDDEN SYMMETRIES OF SPECIAL FUNCTIONS*

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Abstract. Several familiar special functions possess a hidden permutation symmetry which accounts for some of their transformation properties. For example the transformation of the elliptic integral $K(k)$ into $K(ik/k')$ expresses the symmetry of an integral representing Gauss' arithmetic-geometric mean of two variables. Likewise, a transformation of the hypergeometric function ${}_2F_1$ with argument z into the same function with argument $z/(z-1)$ can be viewed as a symmetry under interchange of two variables, but in this case two parameters must be interchanged as well. A similar remark applies to Kummer's transformation of the confluent hypergeometric function. The five transformations which change the modulus k of an incomplete elliptic integral into k', $1/k$, $1/k'$, ik/k', and k'/ik are equivalent to the five nontrivial permutations of three variables, and the theory of elliptic integrals can be simplified by choosing standard integrals which are explicitly symmetric. Appell's double hypergeometric function F_1 has a partly concealed symmetry under simultaneous permutations of three parameters and three variables. The feature common to all these examples is a function $F(b_1, b_2, \cdots, b_k; z_1, z_2, \cdots, z_k)$ which is symmetric in the indices $1, 2, \cdots, k$ of the parameters b and variables z and represents a weighted average of a function of one variable over the convex hull of $\{z_1, z_2, \cdots, z_k\}$. The advantage of using explicitly symmetric functions is illustrated by generalizing Borchardt's iterative algorithm for computing an inverse cosine to an algorithm for computing an inverse elliptic cosine.

1. Introduction. To illustrate how a function of a single variable may be said to have permutation symmetry, we recall that the elliptic integral $K(k)$ can be computed by forming two sequences of successive arithmetic and geometric means. If x and y are positive numbers, the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_0 = x, \qquad y_0 = y,$$
$$x_{n+1} = \tfrac{1}{2}(x_n + y_n), \qquad y_{n+1} = (x_n y_n)^{1/2}, \qquad n = 0, 1, 2, \cdots, \tag{1.1}$$

converge to a common limit $M(x, y)$ called the arithmetic-geometric mean. Since x_1 and y_1 are symmetric in x and y, so is $M(x, y)$. One hundred seventy years ago, on May 30, 1799, Gauss discovered a special case of the integral representation

$$\frac{1}{M(x, y)} = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{(x^2 \sin^2 \theta + y^2 \cos^2 \theta)^{1/2}}. \tag{1.2}$$

The symmetry of this elliptic integral can be verified by taking $\pi/2 - \theta$ as a new variable of integration. More important, one can get some feeling for the source of the symmetry by noting that during the integration $z(\theta) = x^2 \sin^2 \theta + y^2 \cos^2 \theta$ traverses a line segment with endpoints x^2 and y^2, which may in general be complex. The right-hand side of (1.2) is an integral average of $z^{-1/2}$ over this line segment in the complex plane, and the symmetry arises because the averaging process makes no intrinsic distinction between the two endpoints.

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At the time of Gauss' discovery, elliptic integrals had already been studied extensively by Legendre, who had chosen certain standard integrals including

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{(1 - k^2 \sin^2 \theta)^{1/2}}. \tag{1.3}$$

Putting $\cos^2 \theta = 1 - \sin^2 \theta$ in (1.2), we find that

$$\frac{1}{M(x, y)} = \frac{2}{\pi y} K[(1 - x^2/y^2)^{1/2}]. \tag{1.4}$$

The symmetry of M now implies that

$$\frac{1}{y} K[(1 - x^2/y^2)^{1/2}] = \frac{1}{x} K[(1 - y^2/x^2)^{1/2}] \tag{1.5}$$

or, putting $x^2/y^2 = 1 - k^2 = k'^2$,

$$K(k) = \frac{1}{k'} K(ik/k'). \tag{1.6}$$

In the last equation the integral is being treated as a function of one variable instead of a homogeneous function of two variables, and the symmetry property (1.5) is disguised as a transformation of the modulus k which no longer has any appearance of symmetry. This is the simplest example of what we shall call a hidden symmetry. In (1.5) the symmetry is out in the open, while in (1.6) it has been covered up by the choice of notation.

A surprising number of special functions possess transformations of this sort, as we shall see by choosing several examples and uncovering their permutation symmetry. It will then be pertinent to consider what the various examples have in common and to discuss briefly a wider class of functions possessing the same type of symmetry, which even in the general case has its source in an averaging process. At the end we shall return to another algorithm involving arithmetic and geometric means, again one invented by Gauss, and show with the help of symmetry considerations how to extend it so that it can be used to compute elliptic integrals. Perhaps this practical application will suggest the value of keeping symmetry properties out in the open even at the expense of carrying an extra variable.

2. Examples of symmetry in two variables. The elliptic integral K is a special case of the hypergeometric series

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \cdots \tag{2.1}$$

which Gauss introduced in 1812. Expanding the integrand of (1.3) in powers of k, one finds

$$\frac{2}{\pi} K(k) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2), \tag{2.2}$$

and the transformation (1.6) takes the form

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right) = (1-k^2)^{-1/2}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{k^2}{k^2-1}\right). \tag{2.3} $$

Now this is only a special case of a transformation discovered by Gauss [8, p. 218], [7, p. 64],

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right), \tag{2.4} $$

and it seems possible that the general case also hides a symmetry property similar to (1.5). Since the symmetry of (1.5) was lost by putting $x^2/y^2 = 1 - k^2$, we should obviously try to reverse the process by putting $z = 1 - x^2/y^2$. The substitution $z = 1 - x/y$ works just as well and gives

$$ y^{-a}{}_2F_1\left(a,b;c;1-\frac{x}{y}\right) = x^{-a}{}_2F_1\left(a,c-b;c;1-\frac{y}{x}\right). \tag{2.5} $$

This is a statement of symmetry under interchange of x and y accompanied by interchange of b and $c - b$. Formal symmetry in the symbols x and y can be achieved by writing $b = b_x$ and $c = b_x + b_y$:

$$ y^{-a}\,{}_2F_1\left(a,b_x;b_x+b_y;1-\frac{x}{y}\right) = x^{-a}\,{}_2F_1\left(a,b_y;b_x+b_y;1-\frac{y}{x}\right). \tag{2.6} $$

In the case of the elliptic integral we had $b_x = b_y$ and the symmetry under simultaneous interchange of variables and parameters reduced to symmetry in the variables alone.

The confluent hypergeometric function

$$ {}_1F_1(b;c;z) = 1 + \frac{b}{c}z + \frac{b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots \tag{2.7} $$

is obtained from (2.1) by replacing z by z/a and letting a tend to infinity. The same procedure, applied to the transformation (2.4), yields Kummer's transformation,

$$ {}_1F_1(b;c;z) = e^z\,{}_1F_1(c-b;c;-z). \tag{2.8} $$

Putting $z = x - y$, $b = b_x$, and $c = b_x + b_y$, we have

$$ e^y\,{}_1F_1(b_x;b_x+b_y;x-y) = e^x\,{}_1F_1(b_y;b_x+b_y;y-x). \tag{2.9} $$

This statement of symmetry in x and y can alternatively be obtained from (2.6) by replacing x by $1 - x/a$ and y by $1 - y/a$ and letting a tend to infinity.

3. Examples of symmetry in three variables. Let us return to the starting point and try a generalization in a different direction. The complete elliptic integral K is the special case $\varphi = \pi/2$ of Legendre's incomplete integral

$$ F(\varphi,k) = \int_0^\varphi \frac{d\theta}{(1-k^2\sin^2\theta)^{1/2}}, \tag{3.1} $$

and the transformation (1.6) is a special case of

$$F(\varphi, k) = \frac{1}{k'} F(\psi, ik/k'), \qquad \cos\psi = (1 - k^2 \sin^2 \varphi)^{-1/2} \cos\varphi. \tag{3.2}$$

There are five such transformations of F into itself, but only this one contains a transformation of K into itself. All five are expressions of a hidden symmetry, for they are known to be deducible [12, pp. 207–211] from Weierstrass' theory of elliptic functions, in which they correspond to the five nontrivial permutations of the Weierstrass constants e_1, e_2, e_3.

To expose the symmetry of $F(\varphi, k)$ without invoking the Weierstrass theory, we express in terms of F the explicitly symmetric integral

$$\int_0^\infty [(t + x)(t + y)(t + z)]^{-1/2}\, dt = 2(z - x)^{-1/2} F\left[\cos^{-1}\left(\frac{x}{z}\right)^{1/2}, \left(\frac{z - y}{z - x}\right)^{1/2}\right] \tag{3.3}$$

by substituting $t + z = (z - x)\csc^2\theta$ and comparing with (3.1). It follows that the right-hand side must be symmetric in x, y, z. In particular, interchanging y and z gives

$$\begin{aligned}(z - x)^{-1/2} & F\left[\cos^{-1}\left(\frac{x}{z}\right)^{1/2}, \left(\frac{z - y}{z - x}\right)^{1/2}\right] \\ &= (y - x)^{-1/2} F\left[\cos^{-1}\left(\frac{x}{y}\right)^{1/2}, \left(\frac{y - z}{y - x}\right)^{1/2}\right],\end{aligned} \tag{3.4}$$

and if we put $x/z = \cos^2\varphi$ and $y/z = 1 - k^2 \sin^2\varphi$, this becomes (3.2). All five transformations of F into itself can be obtained in the same way by permuting x, y, z. From the relation $k^2 = (z - y)/(z - x)$ it is easy to see that k is transformed into k', $1/k$, $1/k'$, ik/k', and k'/ik.

If Legendre had chosen (3.3) as a standard integral and denoted it by $A(x, y, z)$, say, the five transformations of F would have been replaced by the explicit symmetry of A. Because A is homogeneous in x, y, z, it depends primarily on two ratios of these variables, and indeed the variables φ and k of Legendre's actual choice depend only on two such ratios. However, economy in the number of variables seems to be incompatible with economy in the number of transformations. In the present instance two variables and five transformations are the alternative to three variables and no transformations.

The symmetric elliptic integral (3.3) can be expressed in terms of another well-known special function by a different substitution. Putting $t + z = z/u$, we find

$$\begin{aligned}\int_0^\infty & [(t + x)(t + y)(t + z)]^{-1/2}\, dt \\ &= z^{-1/2} \int_0^1 u^{-1/2}\left[1 - u\left(1 - \frac{x}{z}\right)\right]^{-1/2}\left[1 - u\left(1 - \frac{y}{z}\right)\right]^{-1/2} du \\ &= 2z^{-1/2} F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; 1 - \frac{x}{z}, 1 - \frac{y}{z}\right),\end{aligned} \tag{3.5}$$

where F_1 is the first of Appell's double hypergeometric functions [1]. We have used the integral representation

$$F_1(a, b_x, b_y; c; x', y') = \frac{1}{B(a, c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux')^{-b_x}(1-uy')^{-b_y}\,du, \tag{3.6}$$

where B denotes the beta function and we assume $\operatorname{Re} c > \operatorname{Re} a > 0$. (If the integrand is expanded in powers of x' and y', termwise integration gives a double hypergeometric series.) Beyond the obvious symmetry of (3.6) in x and y, we suspect a hidden symmetry in three variables like that of the special case (3.5). Since the symmetry of the special case was partly concealed by a substitution, we proceed in reverse by putting $x' = 1 - x/z$, $y' = 1 - y/z$, and $u = z/(t + z)$ in (3.6) to obtain

$$F_1\left(a, b_x, b_y; c; 1 - \frac{x}{z}, 1 - \frac{y}{z}\right) = \frac{z^a}{B(a, c-a)} \int_0^\infty t^{c-a-1}(t+x)^{-b_x}(t+y)^{-b_y}(t+z)^{b_x+b_y-c}\,dt. \tag{3.7}$$

With $c = b_x + b_y + b_z$ it is plain that

$$z^{-a}F_1\left(a, b_x, b_y; b_x + b_y + b_z; 1 - \frac{x}{z}, 1 - \frac{y}{z}\right) \tag{3.8}$$

is indeed symmetric in x, y, z. In particular, interchanging y and z gives

$$z^{-a}F_1\left(a, b_x, b_y; c; 1 - \frac{x}{z}, 1 - \frac{y}{z}\right) = y^{-a}F_1\left(a, b_x, b_z; c; 1 - \frac{x}{y}, 1 - \frac{z}{y}\right), \tag{3.9}$$

and if we put $x/z = 1 - x'$ and $y/z = 1 - y'$, this becomes the transformation [1, p. 30], [7, p. 239]

$$F_1(a, b_x, b_y; c; x', y') = (1 - y')^{-a}F_1\left(a, b_x, c - b_x - b_y; c; \frac{y' - x'}{y' - 1}, \frac{y'}{y' - 1}\right). \tag{3.10}$$

A similar transformation expresses the underlying symmetry in x and z.

In the course of uncovering the symmetries of $F(\varphi, k)$ and F_1, we have incidentally verified that the elliptic integral is a special case of F_1, as one sees by comparing (3.3) and (3.5), and that its transformation (3.2) is a special case of (3.10), since both express the hidden symmetry in y and z. Legendre's integral of the second kind,

$$E(\varphi, k) = \int_0^\varphi (1 - k^2 \sin^2 \theta)^{1/2}\,d\theta, \tag{3.11}$$

also is a special case of Appell's F_1, but the parameters b_x, b_y, b_z are all different [4] instead of all equal as in the case of $F(\varphi, k)$. Hence E has no symmetry at all in the variables x, y, z and any permutation changes E into a different function.

As a standard integral of the second kind it is better to choose a case with $b_x = b_y = b_z$, say

$$
\begin{aligned}
&2z^{1/2}F_1\left(-\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{2};1-\frac{x}{z},1-\frac{y}{z}\right)\\
&\qquad = (z-x)^{1/2}E\left[\cos^{-1}\left(\frac{x}{z}\right)^{1/2},\left(\frac{z-y}{z-x}\right)^{1/2}\right]\\
&\qquad\quad + x(z-x)^{-1/2}F\left[\cos^{-1}\left(\frac{x}{z}\right)^{1/2},\left(\frac{z-y}{z-x}\right)^{1/2}\right] + \left(\frac{xy}{z}\right)^{1/2}.
\end{aligned}
\tag{3.12}
$$

It is not E alone but rather this combination of E, F, and an elementary function that is symmetric in x, y, z.

Similar remarks apply to Legendre's third integral,

$$
\Pi(\varphi, \alpha^2, k) = \int_0^{\varphi} \frac{d\theta}{(1-\alpha^2\sin^2\theta)(1-k^2\sin^2\theta)^{1/2}}.
\tag{3.13}
$$

It is a triple hypergeometric function of the type known as Lauricella's F_D, which has a hidden symmetry in four variables and four parameters [3], but in the case of Legendre's integral only two of the parameters are equal. One would like to choose all four parameters equal in order to achieve complete symmetry in four variables, but such a choice does not yield an elliptic integral of the third kind and one must settle for symmetry in three variables. An integral [15] with this symmetry is

$$
\begin{aligned}
&\int_0^{\infty}[(t+x)(t+y)(t+z)]^{-1/2}(t+\rho)^{-1}\,dt\\
&\qquad = 2(z-\rho)^{-1}(z-x)^{-1/2}\\
&\qquad\quad \cdot\left\{\Pi\left[\cos^{-1}\left(\frac{x}{z}\right)^{1/2},\frac{z-\rho}{z-x},\left(\frac{z-y}{z-x}\right)^{1/2}\right]\right.\\
&\qquad\quad \left. - F\left[\cos^{-1}\left(\frac{x}{z}\right)^{1/2},\left(\frac{z-y}{z-x}\right)^{1/2}\right]\right\},
\end{aligned}
\tag{3.14}
$$

where $F(\varphi, k)$ is the integral of the first kind.

The five nontrivial permutations of x, y, z correspond to five transformations of E and Π in which the modulus undergoes the same changes as in the case of F, but now there is an additional complication. Whereas F is transformed essentially into itself with changed modulus and amplitude, as in (3.2), it is plain from the symmetry of (3.12) that E is transformed into itself plus further terms in F and an elementary function. Likewise one sees from (3.14) that Π is transformed into a mixture of Π and F. Even for the integral of the first kind Legendre's notation has the effect of producing five superfluous transformations, which generate others when they combine with transformations of higher order such as Landen and Gauss transformations. As standard integrals of the second and third kinds, however, Legendre chose functions which have in addition an intrinsic lack of symmetry. At the end of the eighteenth century $E(\varphi, k)$ was an attractive choice because of its historical importance in the problem of rectifying an arc of an ellipse. Now that the role of symmetry is better understood, there is good reason to adopt explicitly symmetric standard functions and to modernize various practices of tabulation and computation. Some ways and benefits of doing so have been demonstrated elsewhere [4], [5], [10], [15].

4. Integral averages of elementary functions. As we observed at the outset, the reciprocal of Gauss' arithmetic-geometric mean is an integral average of $z^{-1/2}$ over a line segment in the complex plane, and its symmetry comes from treating the two ends of the line segment on an equal footing. We shall find that all the symmetric functions discussed in preceding sections result from averaging either a power of z or e^z over the convex hull of two or more points in the plane.

We begin with the ${}_2F_1$ and ${}_1F_1$ functions and apply their familiar integral representations [7, pp. 59, 255] to the symmetric functions in (2.6) and (2.9). If the real parts of b_x and b_y are positive, we have

$$\begin{aligned} &y^{-a}\,{}_2F_1\left(a, b_x; b_x + b_y; 1 - \frac{x}{y}\right) \\ &\qquad = \frac{\Gamma(b_x + b_y)}{\Gamma(b_x)\Gamma(b_y)} \int_0^1 u^{b_x - 1}(1 - u)^{b_y - 1}[ux + (1 - u)y]^{-a}\, du, \end{aligned} \tag{4.1}$$

where the real parts of x and y also are positive, and

$$e^y\,{}_1F_1(b_x; b_x + b_y; x - y) = \frac{\Gamma(b_x + b_y)}{\Gamma(b_x)\Gamma(b_y)} \int_0^1 u^{b_x - 1}(1 - u)^{b_y - 1}\, e^{ux + (1-u)y}\, du. \tag{4.2}$$

Symmetry in x and y is easily verified by taking $1 - u$ as a new variable. If we write

$$d\mu_b(u) = \frac{\Gamma(b_x + b_y)}{\Gamma(b_x)\Gamma(b_y)} u^{b_x - 1}(1 - u)^{b_y - 1}\, du \tag{4.3}$$

and let E denote the unit interval, then

$$\int_E d\mu_b(u) = 1. \tag{4.4}$$

Since $ux + (1 - u)y$ with $0 \leqq u \leqq 1$ is on the line segment joining the points x and y in the complex plane, the integrals in (4.1) and (4.2) represent weighted averages of the functions z^{-a} and e^z, respectively, over the line segment which is the convex hull of $\{x, y\}$. The averaging process can plainly be applied to any function $f(z)$ which is analytic on a neighborhood of this line segment (or even piecewise continuous on the segment itself), and the average will be denoted by

$$F(b_x, b_y; x, y) = \int_E f[ux + (1 - u)y]\, d\mu_b(u). \tag{4.5}$$

Symmetry in x and y can be verified as before. Intuitively the cause of symmetry is this: to each end of the line segment is attached a variable (x or y) and an associated parameter (b_x or b_y), and the averaging process makes no intrinsic distinction between the two ends of the line segment.

A similar interpretation of the symmetric function in (3.8) can be drawn from a representation of Appell's F_1 by a double integral [7, p. 230]. If the real parts of x, y, z, b_x, b_y, b_z are positive, we have

$$z^{-a}F_1\left(a, b_x, b_y; b_x + b_y + b_z; 1 - \frac{x}{z}, 1 - \frac{y}{z}\right)$$

$$= \frac{\Gamma(b_x + b_y + b_z)}{\Gamma(b_x)\Gamma(b_y)\Gamma(b_z)} \tag{4.6}$$

$$\cdot \underset{u+v \leqq 1}{\int_0^1 \int_0^1} u^{b_x - 1} v^{b_y - 1} (1 - u - v)^{b_z - 1} [ux + vy + (1 - u - v)z]^{-a} \, du \, dv.$$

The region of integration is now a triangle and its interior, again denoted by E, with vertices $(u, v) = (1, 0)$, $(0, 1)$, and $(0, 0)$. The function $\zeta(u, v) = ux + vy + (1 - u - v)z$ maps these vertices onto the points x, y, z, respectively, and maps E onto the convex hull of $\{x, y, z\}$. The right-hand side is an integral average of ζ^{-a} over this convex hull. Symmetry in x, y, z becomes more conspicuous if one puts $1 - u - v = w$ and replaces the double integral by a triple integral with respect to u, v, w over the unit cube, with a Dirac delta function $\delta(1 - u - v - w)$ inserted in the integrand to insure that the only contributions to the integral occur when $w = 1 - u - v$.

The pattern is now clear and we can proceed to any number of dimensions. Let $u = (u_1, \cdots, u_k)$ be a k-tuple of real nonnegative weights with $\sum_{i=1}^k u_i = 1$. Since $u_1, \cdots, u_{k-1}$ determine $u_k = 1 - u_1 - \cdots - u_{k-1}$, the set of all such k-tuples forms a simplex $E \subset R^{k-1}$. If $z = (z_1, \cdots, z_k)$ is a k-tuple of complex variables, the function

$$\zeta(u_1, \cdots, u_{k-1}) = u \cdot z = \sum_{i=1}^{k} u_i z_i \tag{4.7}$$

maps E onto the convex hull of $\{z_1, \cdots, z_k\}$. Let $b = (b_1, \cdots, b_k)$ be a k-tuple of complex parameters with positive real parts; define

$$d\mu_b(u) = \Gamma\left(\sum_{i=1}^{k} b_i\right) \prod_{i=1}^{k} \frac{u_i^{b_i - 1}}{\Gamma(b_i)} du_1 \cdots du_{k-1}; \tag{4.8}$$

and note that

$$\int_E d\mu_b(u) = 1. \tag{4.9}$$

Let f be analytic on a neighborhood of the convex hull of $\{z_1, \cdots, z_k\}$ and denote the integral average [6] of f over the convex hull by

$$F(b, z) = \int_E f(u \cdot z) \, d\mu_b(u). \tag{4.10}$$

The function $F(b, z) = F(b_1, \cdots, b_k; z_1, \cdots, z_k)$ is then symmetric under simultaneous permutations of the parameters b and variables z (i.e., under permutations of the subscripts $1, 2, \cdots, k$). The symmetry of the integral is more conspicuous if one replaces

$$\int_E du_1 \cdots du_{k-1} \quad \text{by} \quad \int_0^1 \cdots \int_0^1 \delta\left(1 - \sum_{i=1}^{k} u_i\right) du_1 \cdots du_k,$$

where δ is the Dirac delta function. However, the significant point is the basic reason for the symmetry: to each vertex of the convex hull (more precisely, to each vertex of the simplex E) is attached a variable z_i and an associated parameter b_i, and the averaging process makes no intrinsic distinction between the several vertices.

The cases most important in practice are the integral average of the exponential function, denoted by

$$S(b, z) = \int_E e^{u \cdot z} \, d\mu_b(u), \tag{4.11}$$

and the average of a general (possibly complex) power of z, denoted by

$$R_t(b, z) = \int_E (u \cdot z)^t \, d\mu_b(u). \tag{4.12}$$

In the latter case it is assumed that the convex hull of $\{z_1, \cdots, z_k\}$ does not contain the origin. The function R_t is plainly homogeneous of degree t in $z_1, \cdots, z_k$.

For example the explicitly symmetric form of the confluent hypergeometric function is seen from (4.2) to be

$$e^y \, {}_1F_1(b_x; b_x + b_y; x - y) = S(b_x, b_y; x, y). \tag{4.13}$$

All the other functions that we have considered are special cases of R_t. The symmetric versions of Legendre's K and Gauss' hypergeometric function are averages over a line segment:

$$\frac{1}{M(x, y)} = \frac{2}{\pi y} K[(1 - x^2/y^2)^{1/2}] = R_{-1/2}\left(\frac{1}{2}, \frac{1}{2}; x^2, y^2\right), \tag{4.14}$$

$$y^{-a} \, {}_2F_1(a, b_x; b_x + b_y; 1 - x/y) = R_{-a}(b_x, b_y; x, y). \tag{4.15}$$

The symmetric elliptic integrals of the first two kinds and Appell's double hypergeometric function F_1 are averages over a triangle:

$$(z - x)^{-1/2} F\left[\cos^{-1}\left(\frac{x}{z}\right)^{1/2}, \left(\frac{z - y}{z - x}\right)^{1/2}\right] = R_{-1/2}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x, y, z\right), \tag{4.16}$$

$$\begin{aligned} &\frac{1}{2}(z - x)^{1/2} E\left[\cos^{-1}\left(\frac{x}{z}\right)^{1/2}, \left(\frac{z - y}{z - x}\right)^{1/2}\right] \\ &\qquad + \frac{x}{2(z - x)^{1/2}} F\left[\cos^{-1}\left(\frac{x}{z}\right)^{1/2}, \left(\frac{z - y}{z - x}\right)^{1/2}\right] \\ &\qquad + \frac{1}{2}\left(\frac{xy}{z}\right)^{1/2} = R_{1/2}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x, y, z\right), \end{aligned} \tag{4.17}$$

$$z^{-a} F_1(a, b_x, b_y; b_x + b_y + b_z; 1 - x/z, 1 - y/z) = R_{-a}(b_x, b_y, b_z; x, y, z). \tag{4.18}$$

Finally, the symmetric elliptic integral (3.14) of the third kind is an average of $z^{-3/2}$ over a quadrilateral:

$$\begin{aligned}\frac{3}{(z-\rho)(z-x)^{1/2}}\Bigg\{\Pi\left[\cos^{-1}\left(\frac{x}{z}\right)^{1/2}, \frac{z-\rho}{z-x}, \left(\frac{z-y}{z-x}\right)^{1/2}\right] \\ - F\left[\cos^{-1}\left(\frac{x}{z}\right)^{1/2}, \left(\frac{z-y}{z-x}\right)^{1/2}\right]\Bigg\} \\ = R_{-3/2}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},1;x,y,z,\rho\right).\end{aligned} \tag{4.19}$$

It is clear from the notation what symmetry is possessed by a particular function; for example the last function has three equal b-parameters and so is symmetric in x, y, z but not in ρ. When a function such as (4.16) is used frequently, it is convenient to replace the array of numerical parameters by a subscript, as in

$$R_F(x,y,z) = R_{-1/2}(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};x,y,z), \tag{4.20}$$

where the subscript F is a reminder that the function is equivalent to Legendre's $F(\varphi, k)$.

5. Computation by arithmetic and geometric means. The use of explicitly symmetric functions sometimes suggests a fresh approach to a problem, and symmetry often carries with it a genuine simplification. By way of illustration we shall apply the symmetric elliptic integral R_F to a problem which grew out of Gauss' algorithm (1.1) of the arithmetic-geometric mean. In 1880, in a paper [2] completed less than two months before his death, C. W. Borchardt modified the algorithm by replacing x_n by x_{n+1} in the geometric mean:

$$\begin{aligned} x_{n+1} &= \tfrac{1}{2}(x_n + y_n), \\ y_{n+1} &= (x_{n+1}y_n)^{1/2} = \left(\frac{x_n+y_n}{2}y_n\right)^{1/2}, \qquad n = 0,1,2,\cdots. \end{aligned} \tag{5.1}$$

Actually the same modification had been considered eighty years earlier in correspondence between Gauss and Pfaff [9, pp. 234–237, 284–285] which was not published until many years after Borchardt's death. The replacement of x_n by x_{n+1} spoils the symmetry of the algorithm, slows the convergence, and has the effect that the common limit of x_n and y_n is no longer a complete elliptic integral but rather an inverse cosine:

$$\begin{aligned} \lim_{n\to\infty}\frac{1}{x_n} &= (y_0^2 - x_0^2)^{-1/2}\cos^{-1}(x_0/y_0), \qquad 0 \leqq x_0 < y_0, \\ \lim_{n\to\infty}\frac{1}{x_n} &= (x_0^2 - y_0^2)^{-1/2}\cosh^{-1}(x_0/y_0), \qquad 0 < y_0 < x_0. \end{aligned} \tag{5.2}$$

Since the circular and hyperbolic inverse cosines are degenerate cases of an incomplete elliptic integral with $k = 0$ and $k = 1$, respectively, it is reasonable to hope that a suitable generalization of Borchardt's algorithm would have as its limit an incomplete elliptic integral with a general value of k.

I am indebted to Professor John Todd for calling my attention not only to Borchardt's algorithm [11] but also to a generalization proposed by Tricomi [13], [14] in 1965, which stimulated my interest in the problem:

$$(5.3)\qquad \begin{aligned} x_{n+1} &= \frac{x_n + (k^2x_n^2 + k'^2y_n^2)^{1/2}}{y_n + (k^2x_n^2 + k'^2y_n^2)^{1/2}} y_n, & k'^2 &= 1 - k^2, \\ y_{n+1} &= (x_{n+1}y_n)^{1/2}, & n &= 0, 1, 2, \cdots. \end{aligned}$$

This algorithm, which plainly reduces to Borchardt's if $k = 0$, was suggested by the bisection formula for the elliptic cosine. Tricomi succeeded in expressing the common limit of x_n and y_n as an infinite product but not as an elliptic integral.

A different approach to the problem of generalizing Borchardt's algorithm is suggested by the relation between the symmetric integral R_F and the Jacobian elliptic functions. It is easy to verify from (4.16) that

$$(5.4)\qquad u = R_F(\mathrm{cs}^2 u, \mathrm{ds}^2 u, \mathrm{ns}^2 u),$$

where, for example, $\mathrm{cs}\, u = (\mathrm{cn}\, u)/(\mathrm{sn}\, u)$ is Glaisher's notation for the elliptic cotangent. Replacing u by $u/2$ and recalling that R_F is homogeneous of degree $-\frac{1}{2}$, we have first

$$(5.5)\qquad u = R_F\left(\frac{1}{4}\mathrm{cs}^2\frac{u}{2}, \frac{1}{4}\mathrm{ds}^2\frac{u}{2}, \frac{1}{4}\mathrm{ns}^2\frac{u}{2}\right)$$

and then, by repeating the process n times,

$$(5.6)\qquad u = R_F[4^{-n}\,\mathrm{cs}^2(2^{-n}u), 4^{-n}\,\mathrm{ds}^2(2^{-n}u), 4^{-n}\,\mathrm{ns}^2(2^{-n}u)].$$

If we define

$$(5.7)\qquad x_n = 2^{-n}\,\mathrm{cs}\,(2^{-n}u), \quad y_n = 2^{-n}\,\mathrm{ds}\,(2^{-n}u), \quad z_n = 2^{-n}\,\mathrm{ns}\,(2^{-n}u),$$

it follows that

$$(5.8)\qquad R_F(x_0^2, y_0^2, z_0^2) = R_F(x_n^2, y_n^2, z_n^2), \qquad n = 0, 1, 2, \cdots.$$

The recurrence relations for x_n, y_n, z_n are found from the bisection formulas [12, p. 144] for the Jacobian elliptic functions, e.g.,

$$(5.9)\qquad \mathrm{cs}^2\frac{u}{2} = (\mathrm{cs}\, u + \mathrm{ds}\, u)(\mathrm{cs}\, u + \mathrm{ns}\, u).$$

This is the point at which symmetry brings in a simplification. The bisection formulas for the ds and ns functions are obtained by simply permuting the functions cs, ds, and ns in this bisection formula for the elliptic cotangent. Consequently y_{n+1} and z_{n+1} are found by permuting x, y, z in

$$(5.10)\qquad x_{n+1}^2 = \tfrac{1}{4}(x_n + y_n)(x_n + z_n).$$

The recurrence relations can be broken into two steps. One first takes the numbers x_n, y_n, z_n two at a time and forms their arithmetic means

$$(5.11)\qquad \alpha_n = \tfrac{1}{2}(y_n + z_n), \quad \beta_n = \tfrac{1}{2}(z_n + x_n), \quad \gamma_n = \tfrac{1}{2}(x_n + y_n).$$

One then takes the arithmetic means two at a time and forms their geometric means

$$x_{n+1} = (\beta_n \gamma_n)^{1/2}, \quad y_{n+1} = (\gamma_n \alpha_n)^{1/2}, \quad z_{n+1} = (\alpha_n \beta_n)^{1/2}. \tag{5.12}$$

If x_0, y_0, z_0 are real and nonnegative, it is to be expected that formation of successive means will lead to a common limit for x_n, y_n, z_n. To verify this we may assume without loss of generality that $0 \leqq x_0 \leqq y_0 \leqq z_0$, and we assume also $y_0 > 0$ to insure $x_1 > 0$. It then follows from the recurrence relations that $0 < x_n \leqq y_n \leqq z_n$, and hence it suffices to show that x_n and z_n approach a common limit. Now x_n has a limit because it increases with n and is bounded above by z_0, and z_n has a limit because it is decreasing and bounded below by x_0. Moreover we have

$$z_{n+1}^2 - x_{n+1}^2 = \tfrac{1}{4}(z_n^2 - x_n^2), \tag{5.13}$$

$$z_{n+1}^2 - y_{n+1}^2 = \tfrac{1}{4}(z_n^2 - y_n^2). \tag{5.14}$$

Letting n tend to infinity in (5.13), we find that the limits of x_n and z_n are indeed the same. For future reference we divide (5.14) by (5.13) to show that the quantity

$$k^2 = \frac{z_n^2 - y_n^2}{z_n^2 - x_n^2} \tag{5.15}$$

is independent of n.

By its definition as an integral average of $z^{-1/2}$ over a triangle, R_F is a continuous function of its arguments, provided they have positive real parts, and satisfies $R_F(1, 1, 1) = 1$. Denoting by L the common limit of x_n, y_n, z_n and using the homogeneity of R_F, we find from (5.8) that

$$R_F(x_0^2, y_0^2, z_0^2) = \lim_{n\to\infty} R_F(x_n^2, y_n^2, z_n^2) = R_F(L^2, L^2, L^2) = \frac{1}{L} R_F(1, 1, 1) = \frac{1}{L}. \tag{5.16}$$

This completes the proof of the following.

ALGORITHM. Let x_0, y_0, and z_0 be real and nonnegative and assume that not more than one of them is zero. For $n = 0, 1, 2, \cdots$ define

$$\begin{aligned} x_{n+1} &= \left(\frac{x_n + y_n}{2} \frac{x_n + z_n}{2}\right)^{1/2}, \quad y_{n+1} = \left(\frac{y_n + z_n}{2} \frac{y_n + x_n}{2}\right)^{1/2}, \\ z_{n+1} &= \left(\frac{z_n + x_n}{2} \frac{z_n + y_n}{2}\right)^{1/2}. \end{aligned} \tag{5.17}$$

Then, as $n \to \infty$, x_n, y_n, and z_n approach a common limit L given by

$$\frac{1}{L} = R_F(x_0^2, y_0^2, z_0^2). \tag{5.18}$$

To see that this symmetrical algorithm includes Borchardt's algorithm as an unsymmetrical special case, let two of the starting values be equal, say $z_0 = y_0$. Because k^2 as defined by (5.15) is zero independent of n, the equality $z_n = y_n$ holds for all n and hence (5.17) reduces to (5.1).

Compared with Landen or Gauss transformations as a method of computing an incomplete elliptic integral of the first kind, the present algorithm has an advantage in simplicity and a disadvantage in rate of convergence. By (5.13) and (5.14) we have

$$\lim_{n\to\infty} \frac{z_{n+1} - x_{n+1}}{z_n - x_n} = \lim_{n\to\infty} \frac{z_{n+1} - y_{n+1}}{z_n - y_n} = \frac{1}{4}, \tag{5.19}$$

so that the spread of the values x_n, y_n, z_n ultimately decreases by a factor of four in each cycle of iteration. Ten-decimal accuracy in R_F requires approximately twenty cycles of iteration for a wide range of initial values. With Landen or Gauss transformations three or four cycles are sufficient, although they must be followed by separate computation of an inverse circular or hyperbolic function.

The three-dimensional algorithm can be made into a two-dimensional algorithm similar to (5.3) by using (5.15) to eliminate

$$y_n = (k^2 x_n^2 + k'^2 z_n^2)^{1/2} \tag{5.20}$$

in favor of k^2. The result is

$$\begin{aligned} x_{n+1} &= \tfrac{1}{2}(x_n + z_n)^{1/2}[x_n + (k^2 x_n^2 + k'^2 z_n^2)^{1/2}]^{1/2}, \\ z_{n+1} &= \tfrac{1}{2}(x_n + z_n)^{1/2}[z_n + (k^2 x_n^2 + k'^2 z_n^2)^{1/2}]^{1/2}, \end{aligned} \tag{5.21}$$

and for $0 \leqq k \leqq 1$ the common limit of x_n and z_n is given by

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{x_n} &= (z_0^2 - x_0^2)^{-1/2} \operatorname{cn}^{-1}(x_0/z_0, k), \qquad 0 \leqq x_0 < z_0, \\ \lim_{n\to\infty} \frac{1}{x_n} &= (x_0^2 - z_0^2)^{-1/2} \operatorname{cn}^{-1}(z_0/x_0, k'), \qquad 0 \leqq z_0 < x_0. \end{aligned} \tag{5.22}$$

If $k = 0$ or $k = 1$ these equations reduce once again to Borchardt's algorithm (5.1) and (5.2). The three-dimensional form of the general algorithm requires no more computational labor and on the whole seems preferable to the two-dimensional form.

Instead of using the bisection formulas for the Jacobian elliptic functions in the proof, one can alternatively use the duplication formula for R_F. This can be proved without use of Jacobian functions in several ways, one of which is given in [15].

6. Conclusion. We are accustomed to traditional notations for special functions and no one welcomes additional symbols in a field that already has too many. Nevertheless it is a fact that traditional notations for some of the commonest special functions hide an underlying permutation symmetry which has a good deal to do with their essential nature. Keeping symmetry out in the open is a practice to be recommended. For one thing, it sometimes provides a fresh point of view which gives the clue to a problem such as that of generalizing Borchardt's algorithm. For another, the use of explicitly symmetric functions avoids the superfluous transformations which arise when one covers up the symmetry by eliminating one variable. This is especially important for elliptic integrals, partly because more variables mean more permutations and hence more superfluous transformations and partly because these propagate by combining with transformations of higher order such as Landen and Gauss transformations. Substantial parts of the theory of elliptic integrals can be simplified by using explicitly symmetric standard integrals. They are convenient and economical also for practical use in integral tables and numerical work, including direct computation by recurrence relations and numerical estimation by inequalities. For numerical tables the extra variable presents no difficulty because the symmetric integrals are homogeneous and one variable can be standardized to unity without loss of generality.

The interpretation of explicitly symmetric special functions as integral averages of elementary functions provides a fresh view also of the functions themselves. Now that a mathematics student has much less time for special functions than his predecessor of fifty years ago, he may be appreciative to find that some of them are not far removed from the elementary functions. There are very striking analogies between the properties of an analytic function of one complex variable, elementary or not, and the properties of its average (4.10) over the convex hull of several variables. Taylor series, Laurent series, the Cauchy integral formula, and the domain of analyticity of a function of one variable have direct and simple analogues [6] which lose their transparency if the symmetry is covered up. It is a happy circumstance that symmetry and conceptual simplicity seem to go together.

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