Proceedings of the 35th
Conference on Decision and Control
Kobe, Japan • December 1996

Control Lyapunov Functions: New Ideas From an Old Source

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Abstract

A control design method for nonlinear systems based on control Lyapunov functions and inverse optimality is analyzed. This method is shown to recover the LQ optimal control when applied to linear systems. More generally, it is shown to recover the optimal control whenever the level sets of the control Lyapunov function match those of the optimal value function. The method can be readily applied to feedback linearizable systems, and the resulting inverse optimal control law is generally much different from the linearizing control law. Examples in two dimensions are given to illustrate both the strengths and the weaknesses of the method.

1 Control Lyapunov functions

We consider single-input, control-affine nonlinear systems of the form

$$\dot{x} \;=\; f(x) + g(x)\,u \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input, and f and g are known continuous functions. Our goal is to construct a continuous state feedback law $u = k(x)$ such that $x = 0$ is a globally asymptotically stable equilibrium point of the resulting closed-loop system.

Our control design will be based on knowledge of a *control Lyapunov function* (clf), that is, a C^1, proper, positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\inf_u \Big[L_f V(x) + L_g V(x)\,u \Big] \;<\; 0 \tag{2}$$

for all $x \neq 0$ [1, 2]. The existence of a clf for the system (1) is equivalent to the existence of a globally asymptotically stabilizing control law $u = k(x)$ which is smooth everywhere except possibly at $x = 0$ [1]. Moreover, one can calculate such a control law k explicitly from f, g, and V [3].

We will say that V as above is a *weak* clf when the inequality (2) is non-strict, namely, when

$$\inf_u \Big[L_f V(x) + L_g V(x)\,u \Big] \;\leq\; 0 \tag{3}$$

for all x. The existence of a weak clf does not guarantee global stabilizability as does the existence of a clf. Nevertheless, in many cases a weak clf can indeed be used to design a globally stabilizing control law as we will see in Section 4 below.

Given a general system of the form (1), it may be difficult to find a clf or even to determine whether or not one exists. Fortunately, there are significant classes of systems for which the systematic construction of a clf is possible. As we will see in Section 3, these include the class of (globally) feedback linearizable systems.

2 Inverse optimality

Once we have found a clf V, we can construct a control law $u = k(x)$ such that the Lyapunov derivative $\dot{V}$ is negative at every point in the state space. To prevent ourselves from making absurd choices in this construction, we will insist that the control law k be optimal with respect to some meaningful cost functional. A meaningful cost functional is one that places suitable penalty on both the state variable x and the control variable u so that useless conclusions like "every stabilizing control law is optimal" are avoided. The problem of associating some cost functional with a control law k or a clf V is known as an *inverse optimal control problem* [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

As shown in [15], every clf V is the value function for some meaningful cost functional. In other words, every clf solves the Hamilton-Jacobi-Bellman (HJB) equation associated with a meaningful cost. Moreover, one can compute the resulting (inverse) optimal control law k directly from V, f, and g without recourse to the HJB equation. Unfortunately, the construction of an inverse optimal control law k is not unique because V may be the value function for many different cost functionals, each of which may have a different optimal control. Nevertheless, the inverse optimal construction can be used to narrow down the choices for k to those which satisfy an optimality criterion.

One method for generating an inverse optimal control law k given a clf V is through the pointwise minimization of control effort. It is shown in [15] that control laws of the form

$$k(x) = \arg\min\Big\{|u| : L_fV(x)+L_gV(x)u \leq -\sigma(x)\Big\} \quad (4)$$

are inverse optimal, where $\sigma(x)$ is chosen to be continuous, positive definite, and such that $L_fV(x) \leq -\sigma(x)$ whenever $L_gV(x) = 0$. This design "parameter" σ represents the desired amount of negativity of the closed-loop Lyapunov derivative $\dot{V}$, and different choices for σ result in different inverse optimal control laws k.

The continuity of the control law given by the formula (4) depends on the choice for $\sigma(x)$. If σ is chosen so that the strict inequality $L_fV(x) < -\sigma(x)$ holds whenever $L_gV(x) = 0$ and $x \neq 0$, then k will be continuous except possibly at $x = 0$ [15]. A choice for σ resulting in a non-strict inequality could lead to a discontinuous control law k, so such functions σ should be chosen with care. Also, in some cases we may want to consider a positive *semi*definite σ, especially when V is only a weak clf.

We will now specify a particularly interesting choice for $\sigma(x)$. Suppose that we wish to minimize a cost functional of the form

$$J \;=\; \int_0^\infty \Big[q(x) + u^2\Big]\, dt \quad (5)$$

where q is a continuous, positive semidefinite function. Let us choose σ as follows:

$$\sigma(x) \;=\; \sqrt{\big[L_fV(x)\big]^2 + q(x)\big[L_gV(x)\big]^2} \quad (6)$$

When substituted into the formula (4), this σ yields the control law

$$k(x) \;=\; \begin{cases} -\dfrac{L_fV + \sqrt{[L_fV]^2 + q\cdot[L_gV]^2}}{L_gV} & \text{when } L_gV(x) \neq 0 \\ 0 & \text{when } L_gV(x) = 0 \end{cases} \quad (7)$$

which was originally proposed in [3]. This inverse optimal control law is continuous everywhere except possibly at $x = 0$. More importantly, it reduces to the optimal control for the cost (5) whenever the clf V has the same level sets as the value function. To see this, let $V^\star(x)$ be the value function associated with (5), and assume that it satisfies the HJB equation

$$\begin{aligned} 0 \;&=\; \min_u \Big[q(x) + u^2 + L_fV^\star(x) + L_gV^\star(x)\,u\Big] \\ &=\; q(x) + L_fV^\star(x) - \tfrac{1}{4}\big[L_gV^\star(x)\big]^2 \end{aligned} \quad (8)$$

Suppose that $V = \kappa(V^\star)$ for some smooth class $\mathcal{K}$ function κ (in other words, suppose that V and $V^\star$ have the same level sets). Because the derivative κ' is always positive, from (7) and (8) we have

$$\begin{aligned} k(x) \;&=\; -\frac{L_fV^\star + \sqrt{[L_fV^\star]^2 + q\cdot[L_gV^\star]^2}}{L_gV^\star} \\ &=\; -\frac{L_fV^\star + \sqrt{\big[q + \frac{1}{4}[L_gV^\star]^2\big]^2}}{L_gV^\star} \\ &=\; -\frac{L_fV^\star + q + \frac{1}{4}[L_gV^\star]^2}{L_gV^\star} \\ &=\; -\tfrac{1}{2}L_gV^\star(x) \end{aligned} \quad (9)$$

when $L_gV(x) \neq 0$, which is exactly the optimal control for the cost (5). When $L_gV(x)$ is zero, then $k(x) = 0$ which still matches the optimal control because $L_gV^\star(x)$ is also zero. To summarize, if the level sets of the clf match those of the value function, and if σ is chosen as in (6), then the inverse optimal control is in fact the optimal control.

As a special case, suppose that the system is linear and the cost is quadratic:

$$\dot{x} \;=\; Ax + Bu \quad (10)$$

$$J \;=\; \int_0^\infty \Big[x^{\mathrm{T}}Qx + u^2\Big]\, dt \quad (11)$$

If standard stabilizability and detectability assumptions are satisfied, then there exists a unique symmetric positive definite solution P to the Riccati equation

$$A^{\mathrm{T}}P + PA - PBB^{\mathrm{T}}P + Q \;=\; 0 \quad (12)$$

One can verify that $V(x) = x^{\mathrm{T}}Px$ is a clf for this linear system. If we choose σ as in (6), namely,

$$\begin{aligned} \sigma(x) &= \sqrt{[x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x]^2 + 4[x^{\mathrm{T}}Qx][x^{\mathrm{T}}PBB^{\mathrm{T}}Px]} \\ &= \sqrt{[x^{\mathrm{T}}(Q - PBB^{\mathrm{T}}P)x]^2 + 4[x^{\mathrm{T}}Qx][x^{\mathrm{T}}PBB^{\mathrm{T}}Px]} \\ &= x^{\mathrm{T}}\big[Q + PBB^{\mathrm{T}}P\big]\,x \end{aligned} \quad (13)$$

then the formula (7) generates the standard LQ linear optimal feedback law $u = k(x) = -B^{\mathrm{T}}Px$.

3 Feedback linearizable systems

Let us illustrate an inverse optimal design for the class of (globally) feedback linearizable systems (see [16]). Suppose there exists a diffeomorphism $\xi = \Phi(x)$ with $\Phi(0) = 0$ which transforms our system into

$$\dot{\xi} \;=\; A\xi + B\Big[b(\xi) + a(\xi)\,u\Big] \quad (14)$$

where the matrix pair (A, B) is stabilizable and the smooth functions a and b are such that $b(0) = b'(0) = 0$, $a(0) = 1$, and $a(\xi) \neq 0$ for all ξ (we have normalized a and b so that (A, B) represents the Jacobian linearization of the system). Let Q be such that $\xi^TQ\xi$ approximates $q(x)$ in the cost functional (5) around $x = 0$, and let P be the symmetric positive definite solution to the Riccati equation (12). Then the function $V(x) = \Phi(x)^{\mathrm{T}}P\Phi(x) = \xi^{\mathrm{T}}P\xi$ is a clf for this system, and the inverse optimal control law (4) is

$$k(x) \;=\; \begin{cases} -\dfrac{\psi(\xi)}{2\xi^{\mathrm{T}}PB\,a(\xi)} & \text{when } \psi(\xi) > 0 \\ 0 & \text{when } \psi(\xi) \leq 0 \end{cases} \quad (15)$$

where the function ψ is given by

$$\psi(\xi) = \xi^{\mathrm{T}}\big[A^{\mathrm{T}}P + PA\big]\xi + 2\xi^{\mathrm{T}}PBb(\xi) + \sigma(\xi) \quad (16)$$

If σ is chosen as in (6), then the control law (15) will locally approximate the LQ optimal control $-B^{\mathrm{T}}P\xi$ for the linearized system.

We can compare the inverse optimal control law (15) with the feedback linearizing control law given by

$$k(x) = -\frac{b(\xi) + B^{\mathrm{T}}P\xi}{a(\xi)} \quad (17)$$

Although both control laws (15) and (17) globally asymptotically stabilize the system (14) and locally minimize the cost (5), the inverse optimal control law (15) is (globally) optimal with respect to a meaningful cost functional, whereas the feedback linearizing control law (17) is not (in general). For example, the feedback linearizing control law for the system

$$\dot{x} = -x^3 + u \quad (18)$$

would cancel the stabilizing nonlinearity $-x^3$, but the inverse optimal control law would not because such a cancellation is contrary to meaningful optimality. Unfortunately, one-dimensional examples are not rich enough to illustrate potential pitfalls of the clf design method, primarily because all clf's for a scalar system possess essentially the same level sets.

4 Examples in two dimensions

We will first consider the feedback linearizable system

$$\dot{x}_1 = x_2 \quad (19)$$
$$\dot{x}_2 = -x_1 + x_2 \sinh(x_1^2 + x_2^2) + u \quad (20)$$

One can verify that the control law

$$u^\star = -x_2 e^{x_1^2 + x_2^2} \quad (21)$$

minimizes the cost functional

$$J = \int_0^\infty \Big[x_2^2 + u^2\Big]\, dt \quad (22)$$

and that the associated value function is

$$V^\star(x) = e^{x_1^2 + x_2^2} - 1 \quad (23)$$

The Riccati equation (12) yields $P = I$, which means the feedback linearizing control law (17) is

$$u_{FL} = -x_2\Big[1 + \sinh(x_1^2 + x_2^2)\Big] \quad (24)$$

Although this control law is not the same as the optimal control law (21), it has the same qualitative shape.

Let us now try an inverse optimal design using the clf $V(x) = x^{\mathrm{T}}Px = x_1^2 + x_2^2$. This is actually a weak clf for this system because the inequality in (2) is non-strict when $x_2 = 0$. Nevertheless, we can proceed with the inverse optimal design provided we do the following: first, we must choose σ to be positive *semi*definite and check that the resulting control law is continuous; second, we must make sure that LaSalle's theorem applies so that we can conclude global *asymptotic* stability. Keeping these cautions in mind, we see that ψ in (16) is

$$\psi(x) = 2x_2^2 \sinh(x_1^2 + x_2^2) + \sigma(x) \quad (25)$$

One can verify that the choice $\sigma = 2x_2^2$ recovers the feedback linearizing control law (24). Also, the choice $\sigma = 2x_2^2\cosh(x_1^2 + x_2^2)$ from (6) recovers the optimal control law (21); this was to be expected because the value function $V^\star$ and the clf V both have circles as level sets.

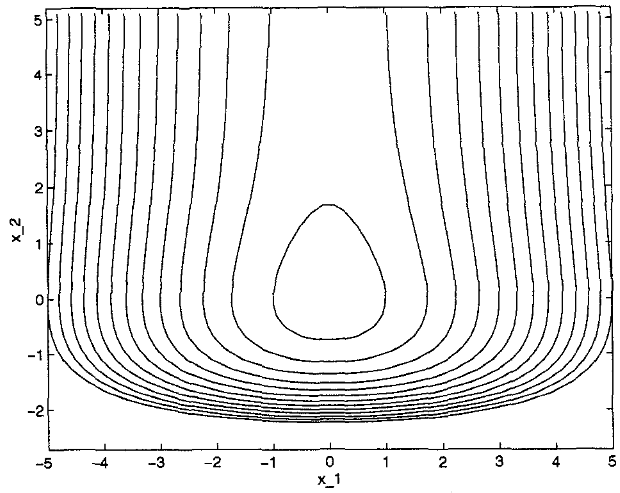

Figure 1: The value function $V^\star(x)$ from (29).

We next consider an example for which the level sets of the value function are far from being ellipsoid:

$$\dot{x}_1 = x_2 \quad (26)$$
$$\dot{x}_2 = -e^{x_2}(x_1 + \tfrac{1}{2}x_2) + \tfrac{1}{2}x_2 e^{4x_1 + 3x_2} + e^{2x_1 + 2x_2}u \quad (27)$$

One can verify that the control law

$$u^\star = -x_2 e^{2x_1 + x_2} \quad (28)$$

minimizes the cost functional (22) and that the associated value function is

$$V^\star(x) = x_1^2 + 2 - 2e^{-x_2}(1 + x_2) \quad (29)$$

This value function is smooth and positive definite, but it is not proper as can be seen from the noncompactness of some of its level sets (Figure 1). Thus no clf will have the same level sets as $V^\star$, and it remains to be seen whether or not some clf design can recover the optimal control.

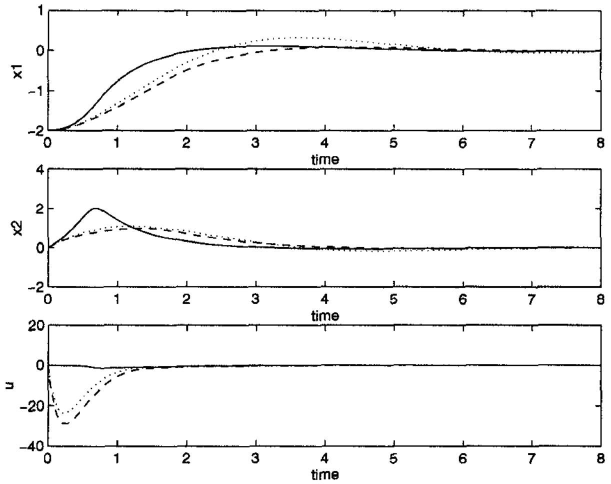

Figure 2: Solutions to (26)–(27) from initial condition $(-2,0)$ with optimal control (28) (solid), inverse optimal control (34) (dashed), and feedback linearizing control (17) (dotted).

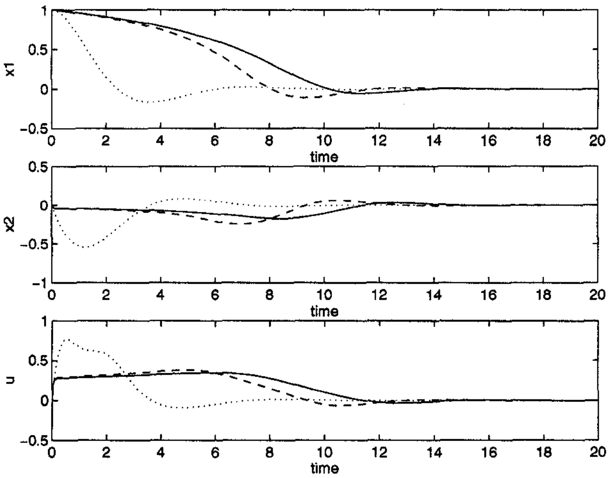

Figure 3: Solutions to (26)–(27) from initial condition $(1,0)$ with optimal control (28) (solid), inverse optimal control (34) (dashed), and feedback linearizing control (17) (dotted).

We will now try the clf design outlined in Section 3. We let (A, B) be the linearization of the system (26)–(27) about zero:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{30}$$

With Q chosen according to the cost (22), the solution to the Riccati equation (12) is $P = I$. Thus we will use the weak clf $V(x) = x^{\mathrm{T}}Px = x_1^2 + x_2^2$.

We will choose σ as in (6) so that our control law is given by the formula (7). For this example we have

$$L_fV(x) = x_2^2\Big[2x_1p(x_2) + e^{4x_1+3x_2} - e^{x_2}\Big] \tag{31}$$

$$L_gV(x) = 2x_2e^{2x_1+2x_2} \tag{32}$$

where $p(x_2)$ represents the smooth function

$$p(x_2) = \frac{1-e^{x_2}}{x_2} \tag{33}$$

(the apparent singularity at $x_2 = 0$ is removable). Upon substituting these expressions into (7) using $q(x) = x_2^2$, we obtain the following control law:

$$k(x) = -\tfrac{1}{2}x_2e^{-2x_1-2x_2}\Bigg[2x_1p + e^{4x_1+3x_2} - e^{x_2} \tag{34}$$
$$+\sqrt{4x_1^2p^2 + 4x_1p\big[e^{4x_1+3x_2} - e^{x_2}\big] + \big[e^{4x_1+3x_2} + e^{x_2}\big]^2}\,\Bigg]$$

This control law is continuous even though we used only a weak clf in the formula (7). Moreover, $k(x) = 0$ when $x_2 = 0$ which means LaSalle's theorem guarantees the global asymptotic stability of the closed-loop system. Note that this inverse optimal control (34) coincides with the optimal control (28) at points where either $x_1 = 0$ or $x_2 = 0$.

Figure 2 shows simulation results for the system (26)–(27) from the initial condition $(-2,0)$. Not surprisingly, the optimal control (28) (solid line), which generates a cost (22) of $J = 4$ from this initial condition, yields better results than either the inverse optimal control (34) (dashed line, $J = 390$) or the feedback linearizing control (17) (dotted line, $J = 238$). In fact, the inverse optimal control generates the highest cost from this initial condition. This does not contradict inverse optimality because this control (34) optimizes a *different, unspecified* cost functional. Figure 3 shows simulation results from the initial condition $(1,0)$; here the costs are $J = 1$ (optimal), $J = 1.02$ (inverse optimal), and $J = 1.51$ (feedback linearization). Note that all three controllers provide nearly the same performance for small initial conditions.

We conclude this section by showing how an alternative clf design can recover the optimal control (28) for this system (26)–(27). Recall that we cannot use the value function (29) as our clf because it is not a proper function (some of its level sets are not compact). However, we should be able to find a valid clf whose level sets look more like the ones in Figure 1 than like the circular level sets of the clf used above.

Rather than choose our clf through feedback linearization as above, we will take the system nonlinearities into account during our construction of the clf. We first rewrite the system (26)–(27) as follows:

$$\dot{x}_1 = x_2 \tag{35}$$

$$\dot{x}_2 = -e^{x_2}(x_1 + \tfrac{1}{2}x_2) + e^{2x_1+2x_2}(u + \tfrac{1}{2}x_2e^{2x_1+x_2}) \tag{36}$$

We next observe that the system obtained by dropping the second term in (36), namely,

$$\dot{x}_1 = x_2 \tag{37}$$

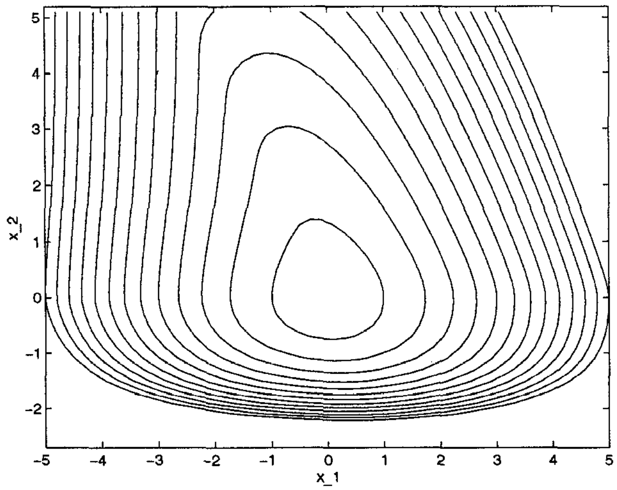

Figure 4: The clf $V(x)$ from (42) with $c = 1$.

$$\dot{x}_2 \quad = \quad -e^{x_2}(x_1 + \tfrac{1}{2}x_2) \qquad (38)$$

is already globally asymptotically stable. Any Lyapunov function for this truncated system (37)–(38) will be a clf for the complete system (35)–(36); our strategy is to find such a function.

Let the function $W(x)$ be given by

$$W(x) \quad = \quad x_2(x_1 + \tfrac{1}{2}x_2) \qquad (39)$$

We define a C^1 function $\phi : I\!R \to I\!R_+$ with derivative ϕ' as follows:

$$\phi(s) \quad = \quad \begin{cases} \frac{s^2}{1+s} & \text{when } s \geq 0 \\ 0 & \text{when } s < 0 \end{cases} \qquad (40)$$

$$\phi'(s) \quad = \quad \begin{cases} \frac{s(2+s)}{(1+s)^2} & \text{when } s \geq 0 \\ 0 & \text{when } s < 0 \end{cases} \qquad (41)$$

The derivative satisfies $\phi'(s) = 0$ for $s \leq 0$ and $0 < \phi'(s) < 1$ for $s > 0$. Furthermore, we have $\phi(s) \to \infty$ as $s \to \infty$. It follows that the function V defined by

$$V(x) \quad = \quad (c+1)V^\star(x) \; + \; \phi(W(x)) \qquad (42)$$

is C^1, positive definite, and radially unbounded, where $c > 0$ is a design parameter. One can verify that V is in fact a Lyapunov function for the truncated system (37)–(38), which means we may use V as a clf in our control design. The level sets of V, shown in Figure 4, are similar in shape to those of the value function $V^\star$.

The first step in the construction of the control law from the clf V is to compute the derivative of V along solutions to the original system (35)–(36):

$$\begin{aligned} \dot{V} &= L_fV(x) \; + \; L_gV(x)\,u \\ &= -x_2^2\,[c+1-\phi'(W)] \; - \; \phi'(W)e^{x_2}(x_1 + \tfrac{1}{2}x_2)^2 \\ &\quad - \tfrac{1}{2}\phi'(W)e^{x_2}W \; + \; e^{2x_1+2x_2}(u + \tfrac{1}{2}x_2e^{2x_1+x_2}) \cdot \\ &\quad [2(c+1)x_2e^{-x_2} + \phi'(W)(x_1+x_2)] \end{aligned} \qquad (43)$$

From this expression we can conclude that V is a weak clf: the control $u = -\frac{1}{2}x_2e^{2x_1+x_2}$ renders $\dot{V}$ negative semidefinite.

We have left to choose the function σ in the formula (4) for the inverse optimal control law. Choosing σ as in (6) should produce a near-optimal control law because the level sets of our new clf (42) closely match those of the value function (29). Indeed, simulations from the initial condition $(-2, 0)$ show that the cost (22) generated by this new inverse optimal control is $J = 4.0002$ which is only slightly higher than the optimal cost $J = 4$. A different choice for σ, namely,

$$\begin{aligned} \sigma(x) &= -L_fV(x) \; + \; x_2e^{2x_1+x_2}L_gV(x) \\ &= x_2^2\,[c+1-\phi'(W)] \; + \; \phi'(W)e^{x_2}(x_1 + \tfrac{1}{2}x_2)^2 \\ &\quad + \tfrac{1}{2}\phi'(W)e^{x_2}W \; + \; \tfrac{1}{2}x_2e^{2x_1+x_2}L_gV(x) \end{aligned} \qquad (44)$$

will exactly recover the optimal control (28). One can verify that this choice (44) is positive semidefinite and is therefore a valid choice in the inverse optimal design.

The conclusion drawn from these examples is that the quality of the clf design with inverse optimality can depend heavily on the choice for the clf V and the desired size σ of its derivative. Finding the best clf for a given cost would require solving an HJB equation, a task which the inverse optimal design is meant to avoid. What we need, therefore, are methods for improving the choice for the clf when the controller designs it generates are unsatisfactory.

5 Discussion

The Lyapunov design discussed in this paper consists of two basic steps:

Step #1: find a clf V
Step #2: use the clf to construct an inverse optimal control law k

While a clf will always exist whenever the system is stabilizable, the task of constructing a clf in Step #1 may not be feasible for a general nonlinear system. For this reason, Lyapunov design is best suited to systems having special structures which we know how to exploit (e.g. the feedback linearizable systems of Section 3). If a clf cannot be found for the complete system, it may be possible to "decentralize" the problem, find clf's for the separate subsystems, and use various interconnection/small-gain theorems [17, 18, 19, 20] to prove the stability of the complete system.

For those classes of systems for which methods for constructing clf's are available, great care should be taken to exploit design flexibilities in Step #1 so that good choices for the control law k are possible in Step #2. We have shown that if the level sets of the clf match those of the value function, then the inverse optimal design, with σ chosen as in (6), is in fact optimal. Moreover, for feedback linearizable systems we can always find a clf whose level sets match those of the value function in a neighborhood of the equilibrium, and the resulting inverse optimal design is locally optimal. In most cases, however, the level sets will not match globally, and the simplest choices in the clf design method may generate control laws which are far from optimal for large deviations from the equilibrium. In such cases, non-obvious choices in the clf design may be required to approach global optimality.

Perhaps the greatest advantage of the Lyapunov design method is its ability to account for uncertainty in the functions f and g in the system description (1). Indeed, the inverse optimal design discussed in this paper was originally developed for uncertain nonlinear systems admitting "robust" clf's (rclf's) [15]. Although feedback linearization methods do not easily yield rclf's for uncertain systems, more powerful backstepping methods can often be successfully applied [21]. However, backstepping design methods are extremely flexible, and again the simplest design choices in Step #1 may exclude all good control laws in Step #2 [22]. Much has yet to be discovered about how to make smart choices in Lyapunov design.

Please see `http://hot.caltech.edu/ doyle` for further details and a comparison of results on several examples.

Acknowledgments: The authors would like to thank John Doyle for his contributions to discussions which revealed that (6) is a particularly interesting choice in the proposed design method.

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