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THEOREMS OF PALEY-WIENER AND MÜNTZ-SZÁSZ TYPE

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Abstract. In the paper a new integral representation of the well-known function classes $H_2[\alpha]$ $(0 < \alpha < 1)$ is established, which in the limiting case $\alpha = 1$ goes into the Paley-Wiener theorem. A Hilbert space metric is introduced into the classes $H_2[\alpha]$ $(0 < \alpha < +\infty)$, and a criterion for closedness of certain systems of functions in these spaces is established. In particular, a theorem of Müntz-Szász type in the complex domain is proved, and a complete intrinsic description is given of the corresponding nonclosed systems.

Bibliography: 9 titles.

1°. The celebrated Paley-Wiener theorem (cf. [1] and also [2], p. 143) on the integral representation of the class H_2^+, i.e. of functions analytic in the halfplane $\operatorname{Re} z > 0$, allows one to construct the isometric operator which maps the space $L_2(0, +\infty)$ onto H_2^+. The Paley-Wiener theorem found effective applications in solving various problems of closedness. In particular, by means of this theorem Paley and Wiener proved in [1] the theorem of Szász [3]; and in [4] Džrbašjan established the theorem of Szász in the following, generalized form:

Consider the system of functions $\{e^{-\lambda_k x}x^{s_k - 1}\}_1^\infty$ *where* $\operatorname{Re} \lambda_k > 0$ *and* s_k *is the multiplicity of the occurrence of the number* λ_k *on the segment* $\{\lambda_j\}_1^k$. *This system is closed in* $L_2(0, +\infty)$ *if and only if*

$$\sum_{k=1}^{\infty} \operatorname{Re} \lambda_k (1 + |\lambda_k|^2)^{-1} = +\infty.$$

A substantial generalization of the Paley-Wiener theorem for the classes $H_2[\alpha; \omega]$ $(\frac{1}{2} < \alpha < +\infty, -1 < \omega < 1)$ (for such classes as well as for the classes $H_2[\alpha]$ $(0 < \alpha < +\infty)$, cf. Chapters VII and VIII of the monograph [2], which also contains a detailed bibliography) was proved in the joint paper [5] of Džrbašjan and Avetisjan.

Starting from this theorem, Džrbašjan [6] obtained a substantial generalization of the Szász theorem. In [6], for any arbitrary sequence $\{\lambda_k\}_1^\infty$ of complex numbers from the angular region

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$$\Delta(\alpha)=\left\{z;\ |\arg z|<\frac{\pi}{2\alpha},\ 0<|z|<+\infty\right\}\ \left(\frac{1}{2}<\alpha<+\infty\right)$$

one introduces the system of functions

$$\{E_\rho^{(s_k-1)}(-\lambda_k x;\mu)\,x^{s_k-1}\}_{k=1}^{\infty}\quad\left(\rho=\frac{\alpha}{2\alpha-1},\quad -1<\omega<1,\quad \mu=\frac{1+\omega+\rho}{2\rho}\right) \tag{1}$$

($s_k \geqslant 1$ is the multiplicity of the appearance of the number λ_k on the segment $\{\lambda_j\}_1^k$), generated by the entire function

$$E_\rho(z;\mu)=\sum_{k=1}^{\infty}\frac{z^k}{\Gamma(\mu+k\rho^{-1})}, \tag{2}$$

of Mittag-Leffler type; and it is shown that *system* (1) *is closed in* $L_{2,\omega}(0,+\infty)$ *if and only if*

$$\sum_{k=1}^{\infty}\operatorname{Re}\lambda_k^{\alpha}(1+|\lambda_k|^{2\alpha})^{-1}=+\infty. \tag{3}$$

In connection with the problem of describing the closure of system (1) in case when this system is not closed in the corresponding space, Akopjan and Hačatrjan [7] introduced the system of simple rational fractions $\{r(\zeta;\lambda_k)\}_1^\infty$, where

$$r(\zeta;\lambda_k)=\frac{(s_k-1)!}{(\zeta+\lambda_k)^{s_k}}\quad(1\leqslant k<+\infty). \tag{4}$$

They also obtained (without formulating it explicitly) a complete intrinsic description of the closure of system (4) in the metric of the space $L_{2,\omega}(L_\rho)$ (L_ρ is the boundary of the angular region $\Delta(\rho)$) under the assumption

$$\sum_{k=1}^{\infty}\operatorname{Re}\lambda_k^{\alpha}(1+|\lambda_k|^{2\alpha})^{-1}<+\infty. \tag{5}$$

2°. In this article we establish a new integral representation of classes $H_2[\alpha]$ $(0<\alpha<1)$ of functions analytic in the region of the angle

$$\Delta(\alpha)=\left\{z\in G_\infty;\ |\operatorname{Arg} z|<\frac{\pi}{2\alpha},\ 0<|z|<+\infty\right\}, \tag{6}$$

lying on the Riemann surface G_∞ of the function $\operatorname{Ln} z$. Moreover, we establish criteria for the closedness of certain systems of functions in the corresponding spaces.

§1 is of preliminary character. We state there some known results which are necessary in the sequel, and we also establish Lemma 1.1, which allows us to introduce a Hilbert space metric into the classes $H_2[\alpha]$ $(0<\alpha<+\infty)$.

In §2 we first prove the important Lemmas 2.1, 2.2 and 2.2^{I}.

The substance of Lemma 2.1 consists in the following: If a function $f(t)\in L_2(0,+\infty)$ extends analytically into the angular region $\Delta(\beta)$ $(0<\beta<+\infty)$ as a function from $H_2[\beta]$, then the Laplace transform of $f(t)$ extends analytically into the angular region $\Delta(\alpha)$ $(\alpha^{-1} = \beta^{-1} + 1)$ as a function from $H_2[\alpha]$. This lemma also establishes an integral representation of the analytic continuation of the Laplace transform of $f(t)$ in terms of its analytic continuation $f(z)$.

In Lemma 2.2 one constructs a transformation of Fourier-Plancherel type of the function $F(\zeta) \in H_2[\alpha]$ $(0 < \alpha < 1)$. It is shown that this transform $f(z)$ belongs to the class $H_2[\beta]$, where $\beta^{-1} = \alpha^{-1} - 1$.

From Lemma 2.2 there follows Lemma 2.2[I] which states that the Fourier-Plancherel transform of the function

$$F(e^{i\frac{\pi}{2}\operatorname{sign} y}|y|) \quad (-\infty<y<+\infty),$$

where $F(\zeta) \in H_2[\alpha]$ $(0 < \alpha < 1)$, extends analytically from the half-axis $(0, +\infty)$ into the angular region $\Delta(\beta)$ $(\beta^{-1} = \alpha^{-1} - 1)$ as a function from $H_2[\beta]$.

Furthermore, starting from Lemmas 2.1 and 2.2, we prove the main result of §2, namely Theorem 1 (a theorem of Paley-Wiener type). This theorem yields an integral representation of functions from $H_2[\alpha]$ $(0 < \alpha < 1)$ by means of functions from $H_2[\beta]$, where $\beta^{-1} = \alpha^{-1} - 1$. Theorem 1 allows us to define a bounded linear, invertible operator which maps $H_2[\beta]$ on $H_2[\alpha]$. In the limiting case when $\beta = +\infty$ $(\alpha = 1)$ Theorem 1 goes into the Paley-Wiener theorem, provided we identify the class $H_2[+\infty]$ with $L_2(0, +\infty)$.

Finally, we show that Theorem 1 contains, as a special case, Theorem 1[I] whose formulation resembles the Paley-Wiener theorem in its classical formulation, and coincides with it in the limiting case $\alpha = 1$ $(\beta = +\infty)$.

In Theorem 2 of the concluding §3 we establish a criterion for closedness of system (4) in the Hilbert space $H_2[\rho]$. The proof of Theorem 2 essentially uses the crierion of closedness of system (1) (for $\omega = 0$) in $L_2(0, +\infty)$.

Moreover, for an arbitrary sequence $\{\lambda_k\}_1^\infty$ of complex numbers from the angular region $\Delta(\alpha)$ $(1 < \alpha < \infty)$ we introduce the system of functions

$$\omega(z; \lambda_k) = e^{-\lambda_k z} z^{s_k - 1} \quad (1 \leqslant k < +\infty), \quad z \in \Delta(\gamma), \tag{7}$$

where $\gamma^{-1} = 1 - \alpha^{-1}$, and $s_k \geqslant 1$ is the multiplicity of the appearance of the number λ_k on the segment $\{\lambda_j\}_1^k$. Furthermore, using Theorem 1 we deduce from Theorem 2 Theorem 3 (a theorem of Müntz-Szász type in the complex domain) which contains a criterion of closedness of system (7) in $H_2[\gamma]$. In the limiting case when $\alpha = 1$ $(\gamma = +\infty)$ this theorem goes into the Szász theorem in its generalized formulation, provided we identify $H_2[+\infty]$ with $L_2(0, +\infty)$.

In conclusion we formulate the above-mentioned result from [7] in a form suitable for us. From it, using Theorem 1 and Lemma 2.2, we prove Theorem 4. This theorem gives a complete intrinsic description of the closure of system (7) in the metric of $H_2[\gamma]$ in the case when system (7) is not closed in this space.

§1. Preliminary facts and lemmas

(a) For every α $(0 < \alpha < +\infty)$ let us denote by

$$\Delta(\alpha) = \left\{ z \in G_\infty;\ |\operatorname{Arg} z| < \frac{\pi}{2\alpha},\ 0 < |z| < +\infty \right\} \tag{1.1}$$

the angular region of opening π/α on the Riemann surface G_∞ of the function $\operatorname{Ln} z$:

$$G_\infty = \{z;\ |\operatorname{Arg} z| < +\infty,\ 0 < |z| < +\infty\}. \tag{1.2}$$

$H_2[\alpha]$ $(0 < \alpha < +\infty)$ will denote the class of functions $F(z)$ analytic in the angular region $\Delta(\alpha) \subset G_\infty$ (which is many-sheeted for $0 < \alpha < ½$) and satisfying the condition

$$\sup_{|\varphi| < \frac{\pi}{2\alpha}} \left\{ \int_0^{+\infty} |F(re^{i\varphi})|^2\, dr \right\} < +\infty. \tag{1.3}$$

The next theorem [5] (cf. also [2], p. 444, Corollary 2) shows that the classes $H_2[\alpha]$ are natural generalizations to arbitrary angular domains $\Delta(\alpha) \subset G_\infty$ of the well-known class H_2^+ of functions analytic in the halfplane $\Delta(1) = \{z;\ \operatorname{Re} z > 0\}$.

THEOREM A. *The class $H_2[1]$ coincides with the class H_2^+ of functions which are holomorphic in the halfplane $\Delta(1)$ and satisfy the condition*

$$\sup_{0 < x < +\infty} \left\{ \int_{-\infty}^{+\infty} |\Phi(x + iy)|^2\, dy \right\} < +\infty.$$

Concerning functions of class $H_2(\alpha)$, one has the following lemma (cf. [2], p. 507, Lemma 8.6):

LEMMA A. *For every function*

$$F(z) \in \mathscr{H}_2[\alpha] \quad (0 < \alpha < +\infty)$$

the following statements hold:

1°. *There exist boundary values $F(re^{i\pi/2\alpha})$ and $F(re^{-i\pi/2\alpha})$ of class $L_2(0, +\infty)$ such that on the halfaxis $(0, +\infty)$ one has*

$$\begin{aligned} F\left(re^{i\frac{\pi}{2\alpha}}\right) &= \operatorname*{l.i.m.}_{\varphi \to \frac{\pi}{2\alpha} - 0} F(re^{i\varphi}), \\ F\left(re^{-i\frac{\pi}{2\alpha}}\right) &= \operatorname*{l.i.m.}_{\varphi \to -\frac{\pi}{2\alpha} + 0} F(re^{i\varphi}). \end{aligned} \tag{1.4}$$

2°. *For every φ_0, $0 < \varphi_0 < \pi/2\alpha$,*

$$\lim_{r \to +0} \{ \max_{|\varphi| \leqslant \varphi_0} [\, |F(re^{i\varphi})|\, r^{\frac{1}{2}}] \} = \lim_{r \to +\infty} \{ \max_{|\varphi| \leqslant \varphi_0} [\, |F(re^{i\varphi})|\, r^{\frac{1}{2}}] \} = 0. \tag{1.5}$$

Later we shall need the Paley-Wiener theorem, which we now state in a form suitable for us.

THEOREM B. 1°. *The class* $H_2[1] \equiv H_2^+$ *coincides with the family of functions which can be represented in the form*

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-zt} \varphi(t)\, dt, \quad z \in \Delta(1), \tag{1.6}$$

where $\varphi(t) \in L_2(0, +\infty)$ *is arbitrary.*

2°. *In representation* (1.6), *for every fixed function* $\Phi(z) \in H_2^+$ *the function* $\varphi(t) \in L_2(0, +\infty)$ *is unique, and almost everywhere on* $(-\infty, +\infty)$ *one has*

$$\frac{1}{\sqrt{2\pi}} \frac{d}{dt} \int_{-\infty}^{+\infty} \frac{e^{ity}-1}{iy} \Phi(iy)\, dy = \begin{cases} \varphi(t), & t \in (0, +\infty), \\ 0, & t \in (-\infty, 0), \end{cases} \tag{1.7}$$

where $\Phi(iy)$ *are boundary values of* $\Phi(z)$ *on the imaginary axis.*

3°. *If* $\Phi(z) \in H_2^+$ *and* $\varphi(t) \in L_2(0, +\infty)$ *are related by* (1.6), *then the Parseval equality holds*:

$$\|\Phi\|_2 = \left\{ \int_0^{+\infty} |\varphi(t)|^2 dt \right\}^{\frac{1}{2}}, \tag{1.8}$$

where

$$\|\Phi\|_2 = \left\{ \int_{-\infty}^{+\infty} |\Phi(iy)|^2 dy \right\}^{\frac{1}{2}} = \sup_{0<x<+\infty} \left\{ \int_{-\infty}^{+\infty} |\Phi(x+iy)|^2 dy \right\}^{\frac{1}{2}}. \tag{1.9}$$

This theorem easily implies the well-known fact about the closedness of H_2^+, namely that for an arbitrary sequence of functions $\{\Phi_k(z)\}_1^\infty$, $\Phi_k(z) \in H_2^+$, the relation

$$\lim_{n,m\to+\infty} \|\Phi_n - \Phi_m\|_2 = 0,$$

implies the existence of a function $\Phi(z) \in H_2^+$ such that

$$\lim_{n\to+\infty} \|\Phi - \Phi_n\|_2 = 0.$$

Finally, we shall state a theorem which, as was shown in the note [8], follows immediately from well-known results of the monograph [2] on harmonic analysis in the complex domain:

THEOREM C. *If* $\Phi(z) \in H_2[1] \equiv H_2^+$ *and*

$$\|\Phi\|_{2,1}^* = \sup_{|\varphi|<\frac{\pi}{2}} \left\{ \int_0^{+\infty} |\Phi(re^{i\varphi})|^2 dr \right\}^{\frac{1}{2}},$$

$$\|\Phi\|_{2,1} = \left\{ \int_0^{+\infty} |\Phi(re^{i\frac{\pi}{2}})|^2 dr + \int_0^{+\infty} |\Phi(re^{-i\frac{\pi}{2}})|^2 dr \right\}^{\frac{1}{2}},$$

then

$$\|\Phi\|_{2,1}^{*} \leqslant \sqrt{2}\,\|\Phi\|_{2,1} \leqslant 2\|\Phi\|_{2,1}^{*}.$$

(b) It is obvious that the classes $H_2[\alpha]$ $(0<\alpha<+\infty)$ with the standard operations of addition and multiplication of functions by scalars are complex linear spaces. We shall prove a lemma which allows us to turn them into normed spaces.

LEMMA 1.1. *The following statements hold for an arbitrary* α $(0<\alpha<+\infty)$:

1°. *The space* $H_2[\alpha]$ *with the norm*

$$\|F\|_{2,\alpha}^{*} = \sup_{|\varphi|<\frac{\pi}{2\alpha}} \left\{ \int_0^{+\infty} |F(re^{i\varphi})|^2\, dr \right\}^{\frac{1}{2}} \tag{1.10}$$

is a Banach space.

2°. *The space* $H_2[\alpha]$ *with the scalar product*

$$(F, G)_{2,\alpha} = \int_0^{+\infty} F\left(re^{i\frac{\pi}{2\alpha}}\right)\overline{G\left(re^{i\frac{\pi}{2\alpha}}\right)}\, dr + \int_0^{+\infty} F\left(re^{-i\frac{\pi}{2\alpha}}\right)\overline{G\left(re^{-i\frac{\pi}{2\alpha}}\right)}\, dr \tag{1.11}$$

is a Hilbert space.

3°. *If $F(z)$ is an arbitrary function from* $H_2[\alpha]$ *and*

$$\|F\|_{2,\alpha} = \sqrt{(F, F)_{2,\alpha}}, \tag{1.12}$$

then

$$\|F\|_{2,\alpha}^{*} \leqslant \sqrt{2}\,\|F\|_{2,\alpha} \leqslant 2\|F\|_{2,\alpha}^{*}. \tag{1.13}$$

PROOF. 1°. Let $\{F_n(z)\}_1^{\infty}$ be an arbitrary sequence of functions from $H_2[\alpha]$ such that

$$\lim_{n,m\to+\infty} \|F_n - F_m\|_{2,\alpha}^{*} = 0. \tag{1.14}$$

Together with $\{F_n(z)\}_1^{\infty}$, we shall consider the sequence of functions $\{\Phi_n(w)\}_1^{\infty}$ defined by

$$\Phi_n(w) = F_n\left(w^{\frac{1}{\alpha}}\right) w^{\frac{1-\alpha}{2\alpha}} \quad (1 \leqslant n < +\infty),\ w \in \Delta(1).$$

It is easy to check that the functions $\Phi_n(w)$ $(1 \leqslant n < +\infty)$ belong to $H_2[1] \equiv H_2^{+}$. Moreover, by a direct verification one can show that

$$\|\Phi_n - \Phi_m\|_{2,1}^{*} = \sqrt{\alpha}\,\|F_n - F_m\|_{2,\alpha}^{*} \quad (1 \leqslant n,\ m < +\infty).$$

Hence by (1.14) we obtain

$$\lim_{n,m\to+\infty} \|\Phi_n-\Phi_m\|^*_{2,1}=0. \tag{1.15}$$

Next we observe that in $H_2^+ \equiv H_2[1]$ the norm defined by (1.9) coincides with the norm $\|\cdot\|_{2,1}$. Taking this remark into account, we conclude on the basis of Theorem C that in view of the completeness of $H_2^+ \equiv H_2[1]$ with norm (1.9), the space $H_2[1] \equiv H_2^+$ with the norm $\|\cdot\|^*_{2,1}$ is also complete.

Consequently, in view of (1.15) there exists a function $\Phi(w) \in H_2[1]$ such that

$$\lim_{n\to+\infty} \|\Phi-\Phi_n\|^*_{2,1}=0. \tag{1.16}$$

Set[1]

$$F(z)=\Phi(z^\alpha)z^{\frac{\alpha-1}{2}}, \quad z\in\Delta(\alpha).$$

It is easy to see that $F(z) \in H_2[\alpha]$. In addition,

$$\|F-F_n\|^*_{2,\alpha}=\frac{1}{\sqrt{\alpha}}\|\Phi-\Phi_n\|^*_{2,1} \quad (1\leqslant n<+\infty).$$

This together with (1.16) gives the limit relation

$$\lim_{n\to+\infty}\|F-F_n\|^*_{2,\alpha}=0,$$

which shows that the space $H_2[\alpha]$ with the norm $\|\cdot\|^*_{2,\alpha}$ is complete.

Finally, a direct verification shows that $H_2[\alpha]$ with the norm $\|\cdot\|^*_{2,\alpha}$ satisfies the remaining axioms of a Banach space.

2°. The completeness of $H_2[\alpha]$ with the norm $\|\cdot\|_{2,\alpha}$ follows from statements 1° and 3° of the lemma.

Since the remaining axioms of a Hilbert space are easy to verify for $H_2[\alpha]$ with the scalar product $(\cdot,\cdot)_{2,\alpha}$, we shall turn to the proof of part 3° of the lemma.

3°. Let $F(z) \in H_2[\alpha]$. Set

$$\Phi(w)=F(w^{\frac{1}{\alpha}})w^{\frac{1-\alpha}{2\alpha}}, \quad w\in\Delta(1).$$

It is easy to see that $\Phi(w) \in H_2[1]$; moreover,

$$\|\Phi\|_{2,1}=\sqrt{\alpha}\|F\|_{2,\alpha}, \quad \|\Phi\|^*_{2,1}=\sqrt{\alpha}\|F\|^*_{2,\alpha}. \tag{1.17}$$

By Theorem C we also have

$$\|\Phi\|^*_{2,1}\leqslant\sqrt{2}\|\Phi\|_{2,1}\leqslant 2\|\Phi\|^*_{2,1}.$$

Hence, by means of (1.17), we obtain (1.13). The lemma is proved.

§2. A theorem of Paley-Wiener type for the spaces $H_2[\alpha]$

2.1 For any φ $(-\infty<\varphi<+\infty)$,

(1)Here and below, by z^α we mean its principal branch.

$$D(\varphi)=\left\{z\in G_\infty;\ |\operatorname{Arg} z-\varphi|<\frac{\pi}{2},\ 0<|z|<+\infty\right\} \tag{2.1}$$

denotes the region of the type of a halfplane on the Riemann surface G_∞ of the function $\operatorname{Ln} z$.

If $f(z)\in H_2[\beta]$ $(0<\beta<+\infty)$, then for every $\varphi\in(-\pi/2\beta,\pi/2\beta)$ we form a function

$$g_\varphi(\zeta;f)=\frac{1}{\sqrt{2\pi}}e^{-i\varphi}\int_0^{+\infty}e^{-\zeta te^{-i\varphi}}f(te^{-i\varphi})\,dt, \tag{2.2}$$

and consider it in the domain $D(\varphi)$.

We now establish a lemma about properties of the function $g_\varphi(\zeta;f)$:

LEMMA 2.1.(2) *Let $f(z)\in H_2[\beta]$ $(0<\beta<+\infty)$. Then the following statements hold*:

1°. *For every*

$$\varphi\in\left(-\frac{\pi}{2\beta},\frac{\pi}{2\beta}\right)$$

the function (2.2) *is analytic in $D(\varphi)$.*

2°. *There exists a function $G(\zeta;f)$ analytic in $\Delta(\alpha)$, where*

$$\frac{1}{\alpha}=\frac{1}{\beta}+1, \tag{2.3}$$

such that for any $\varphi\in(-\pi/2\beta,\pi/2\beta)$

$$g_\varphi(\zeta;f)\equiv G(\zeta;f),\quad \zeta\in D(\varphi). \tag{2.4}$$

3°. *$G(\zeta;f)$ belongs to $H_2[\alpha]$ and satisfies*

$$\|G(\zeta;f)\|^*_{2,\alpha}\leqslant\|f\|^*_{2,\beta}. \tag{2.5}$$

PROOF. 1°. Let $\varphi\in(-\pi/2\beta,\pi/2\beta)$, and let Ω be an arbitrary bounded closed domain of the Riemann surface G_∞ lying inside $D(\varphi)$. Taking into account that

$$\min_{\zeta\in\Omega}\operatorname{Re}(\zeta e^{-i\varphi})=\nu_0>0,$$

from (2.2) we obtain for $\zeta\in\Omega$

$$|g_\varphi(\zeta;f)|\leqslant\frac{1}{\sqrt{2\pi}}\left\{\int_0^{+\infty}e^{-2\nu_0 t}\,dt\right\}^{\frac{1}{2}}\left\{\int_0^{+\infty}|f(te^{-i\varphi})|^2\,dt\right\}^{\frac{1}{2}}\leqslant\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\nu_0}}\|f\|^*_{2,\beta}.$$

(2)For the function classes $H_2[\alpha;\omega]$ $(\frac{1}{2}<\alpha<+\infty,\ -1<\omega<1)$ a more general statement holds (cf. [5], and also [2], Theorem 7.6).

This implies that (2.2) converges absolutely and uniformly in an arbitrary domain Ω of the above-mentioned type, and $g_\varphi(\zeta; f)$ is analytic in $D(\varphi)$.

2°. Consider now two functions $g_{\varphi_1}(\zeta; f)$ and $g_{\varphi_2}(\zeta; \varphi)$ corresponding to two different values φ_1 and φ_2 such that $0 < \varphi_1 - \varphi_2 < \pi$. Then the domains $D(\varphi_1)$ and $D(\varphi_2)$, of the type of a halfplane, in which these functions are respectively defined, have a nonempty intersection $D(\varphi_1; \varphi_2) = D(\varphi_1) \cap D(\varphi_2)$.

We shall show that

$$g_{\varphi_1}(\zeta; f) \equiv g_{\varphi_2}(\zeta; f), \quad \zeta \in D(\varphi_1; \varphi_2). \tag{2.6}$$

Let $C(\epsilon; R)$ $(0 < \epsilon < R < +\infty)$ be a closed contour on the Riemann surface G_∞ which represents the boundary of the region

$$G(\varepsilon; R) = \{z \in G_\infty - \varphi_1 < \operatorname{Arg} \zeta < -\varphi_2;\ \varepsilon < |z| < R\}.$$

Since $G(\epsilon; R)$ together with its boundary $C(\epsilon; R)$ lies in the domain $\Delta(\beta)$ of analyticity of $f(z)$, for every $\zeta \in D(\varphi_1; \varphi_2)$ the Cauchy theorem yields

$$\int\limits_{C(\varepsilon;R)} e^{-\zeta z} f(z)\, dz = 0.$$

This identity can be rewritten in the following way:

$$\begin{aligned} & e^{-i\varphi_1} \int\limits_{\varepsilon}^{R} e^{-\zeta t e^{-i\varphi_1}} f(te^{-i\varphi_1})\, dt - e^{-i\varphi_2} \int\limits_{\varepsilon}^{R} e^{-\zeta t e^{-i\varphi_2}} f(te^{-i\varphi_2})\, dt \\ &= i\varepsilon \int\limits_{-\varphi_1}^{-\varphi_2} e^{-\zeta\varepsilon e^{i\varphi}} f(\varepsilon e^{i\varphi})\, e^{i\varphi}\, d\varphi - iR \int\limits_{-\varphi_1}^{-\varphi_2} e^{-\zeta R e^{i\varphi}} f(Re^{i\varphi})\, e^{i\varphi} d\varphi \\ &\equiv U_1(\varepsilon) + U_2(R), \quad \zeta \in D(\varphi_1; \varphi_2). \end{aligned} \tag{2.7}$$

Since $f(z) \in H_2[\beta]$ and $[-\varphi_1, -\varphi_2] \subset (-\pi/2\beta, \pi/2\beta)$, Lemma A (2°) gives the estimate

$$\max_{-\varphi_1 \leqslant \varphi \leqslant -\varphi_2} |f(re^{i\varphi})| \leqslant A_1 r^{-\frac{1}{2}}, \quad 0 < r < +\infty, \tag{2.8}$$

where A_1 does not depend on r.

For every $\zeta \in D(\varphi_1; \varphi_2)$ we have

$$\min_{-\varphi_1 \leqslant \varphi \leqslant -\varphi_2} \operatorname{Re}(\zeta e^{i\varphi}) = \nu(\zeta) > 0.$$

Then from the definition (2.7) of the integrals $U_1(\epsilon)$ and $U_2(R)$ we obtain

$$|U_1(\varepsilon)| \leqslant A_2 \varepsilon^{\frac{1}{2}}, \quad |U_2(R)| \leqslant A_3 R^{\frac{1}{2}} e^{-\nu(\zeta)R}, \tag{2.9}$$

where A_2 and A_3 do not depend on ϵ and R.

Using (2.9) and taking the limit in (2.7) as $\epsilon \longrightarrow +0$ and $R \longrightarrow +\infty$, we obtain

$$e^{-i\varphi_1}\int_0^{+\infty} e^{-\zeta te^{-i\varphi_1}} f(te^{-i\varphi_1})\,dt \equiv e^{-i\varphi_2}\int_0^{\infty} e^{-\zeta te^{-i\varphi_2}} f(te^{-i\varphi_2})\,dt, \quad \zeta \in D(\varphi_1;\varphi_2).$$

Multiplying by $1/\sqrt{2\pi}$, we arrive at the required statement (2.6).

We now vary φ in the interval $(-\pi/2\beta, \pi/2\beta)$. Then by means of (2.6) we obtain the analytic continuation of $g_\varphi(\zeta; f)$ in the whole domain $\Delta(\alpha)$, where $1/\alpha = 1/\beta + 1$. The resulting function $G(\zeta; f)$ obviously satisfies (2.4).

3°. It clearly suffices to verify the inequality

$$\sup_{|\vartheta|<\frac{\pi}{2\alpha}} \left\{\int_0^{+\infty} |G(re^{i\vartheta}; f)|^2\,dr\right\}^{\frac{1}{2}} \leqslant \|f\|^*_{2,\beta}. \tag{2.10}$$

For this let us first observe that if $|\vartheta| < \pi/2\alpha$, then $\vartheta - (\pi/2)\operatorname{sign}\vartheta \in (-\pi/2\beta, \pi/2\beta)$. Hence, in view of the definition (2.4) of $G(\zeta; f)$, on the boundary L_ϑ of $D(\vartheta - (\pi/2)\operatorname{sign}\vartheta)$, its values coincide with the boundary values of the function $g_\varphi(\zeta; f)$, where $\varphi = \vartheta - (\pi/2)\operatorname{sign}\vartheta$. Consequently,

$$\left\{\int_0^{+\infty} |G(re^{i\vartheta}; f)|^2\,dr\right\}^{\frac{1}{2}} < \left\{\int_{L_\vartheta} |G(\zeta; f)|^2\,|d\zeta|\right\}^{\frac{1}{2}} = \left\{\int_{L_\vartheta} |g_{\vartheta-\frac{\pi}{2}\operatorname{sign}\vartheta}(\zeta; f)|^2\,|d\zeta|\right\}^{\frac{1}{2}}. \tag{2.11}$$

Moreover, in view of (2.2)

$$g_\varphi(\zeta; f) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi}\int_0^{+\infty} e^{-\zeta te^{-i\varphi}} f(te^{-i\varphi})\,dt, \quad \zeta \in D(\varphi), \quad \varphi = \vartheta - \frac{\pi}{2}\operatorname{sign}\vartheta,$$

and hence the Parseval equality holds:

$$\left\{\int_{L_\vartheta} |g_\varphi(\zeta; f)|^2\,|d\zeta|\right\}^{\frac{1}{2}} = \left\{\int_0^{+\infty} |f(te^{-i\varphi})|^2\,dt\right\}^{\frac{1}{2}} \quad \left(\varphi = \vartheta - \frac{\pi}{2}\operatorname{sign}\vartheta\right).$$

In view of (2.11), this implies that for any ϑ, $|\vartheta| < \pi/2\alpha$, and $\varphi = \vartheta - (\pi/2)\operatorname{sign}\vartheta$ one has

$$\left\{\int_0^{+\infty} |G(re^{i\vartheta}; f)|^2\,dr\right\}^{\frac{1}{2}} \leqslant \left\{\int_0^{+\infty} |f(te^{-i\varphi})|^2\,dt\right\}^{\frac{1}{2}} \leqslant \|f\|^*_{2,\beta},$$

which yields (2.10). The proof of the lemma is now complete.

We now prove another lemma:

LEMMA 2.2. *If $F(\zeta) \in H_2[\alpha]$ $(0 < \alpha < 1)$, then the following statements hold:*
1°. *The function*

$$f(re^{i\varphi}) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi}\frac{d}{dr}\int_{-\infty}^{+\infty} \frac{e^{iry}-1}{iy} F\left(e^{i\left(\frac{\pi}{2}\operatorname{sign} y - \varphi\right)}|y|\right)dy \tag{2.12}$$

exists at every point $z = re^{i\varphi}$ of the angular domain $\Delta(\beta) \subset G_\infty$, where

$$\frac{1}{\beta} = \frac{1}{\alpha} - 1, \tag{2.13}$$

and is analytic in this domain.

2°. *The function $f(z)$ belongs to $H_2[\beta]$ and satisfies*

$$\|f\|^*_{2,\beta} \leqslant \sqrt{2}\,\|F\|^*_{2,\alpha}. \tag{2.14}$$

PROOF. 1°. Consider for every ϑ, $-\infty < \vartheta < \infty$, the ray

$$L(\vartheta) = \{\zeta \in G_\infty;\ \operatorname{Arg}\zeta = \vartheta,\ 0 < |\zeta| < +\infty\} \tag{2.15}$$

on the Riemann surface G_∞, and in the angular domain $\Delta(\beta) \subset G_\infty$ define the function $G(z)$ by setting

$$G(re^{-i\varphi}) = \int\limits_{L\left(\varphi+\frac{\pi}{2}\right)} \frac{e^{\zeta re^{-i\varphi}} - 1}{\zeta} F(\zeta)\,d\zeta - \int\limits_{L\left(\varphi-\frac{\pi}{2}\right)} \frac{e^{\zeta re^{-i\varphi}} - 1}{\zeta} F(\zeta)\,d\zeta, \quad z = re^{-i\varphi} \in \Delta(\beta). \tag{2.16}$$

For arbitrary φ and γ $(-\infty < \varphi < +\infty,\ 0 < \gamma < +\infty)$ let

$$\Delta(\varphi;\ \gamma) = \{z \in G_\infty;\ |\operatorname{Arg} z - \varphi| < \gamma,\ 0 < |z| < +\infty\} \tag{2.17}$$

be the angular domain on the Riemann surface G_∞.

Setting for a given φ, $|\varphi| < \pi/2\beta$,

$$\gamma_\varphi = \frac{1}{2}\min\left\{\frac{\pi}{2},\ \frac{\pi}{2\beta} - |\varphi|\right\}, \tag{2.18}$$

one can easily see that

$$\Delta(\beta) = \bigcup_{|\varphi| < \frac{\pi}{2\beta}} \Delta(-\varphi;\ \gamma_\varphi). \tag{2.19}$$

We shall show that if the point $z = re^{-i\varphi}$ varies in the fixed domain $\Delta(-\varphi_0;\ \gamma_{\varphi_0})$ $(|\varphi_0| < \pi/2\beta)$, then the integration in (2.16) can be carried out along certain fixed rays. More exactly, we shall show that

$$G(re^{-i\varphi}) = \int\limits_{L\left(\varphi_0+\gamma_{\varphi_0}+\frac{\pi}{2}\right)} \frac{e^{\zeta re^{-i\varphi}}-1}{\zeta} F(\zeta)\,d\zeta - \int\limits_{L\left(\varphi_0-\gamma_{\varphi_0}-\frac{\pi}{2}\right)} \frac{e^{\zeta re^{-i\varphi}}-1}{\zeta} F(\zeta)\,d\zeta, \quad z = re^{-i\varphi} \in \Delta(-\varphi_0;\ \gamma_{\varphi_0}). \tag{2.20}$$

By (2.16) it suffices to prove the identities

$$\int\limits_{L\left(\varphi+\frac{\pi}{2}\right)} \frac{e^{\zeta re^{-i\varphi}}-1}{\zeta} F(\zeta)\,d\zeta = \int\limits_{L\left(\varphi_0+\gamma_{\varphi_0}+\frac{\pi}{2}\right)} \frac{e^{\zeta re^{-i\varphi}}-1}{\zeta} F(\zeta)\,d\zeta, \qquad z = re^{-i\varphi} \in \Delta(-\varphi_0;\ \gamma_{\varphi_0}), \tag{2.21}$$

$$\int\limits_{L\left(\varphi-\frac{\pi}{2}\right)} \frac{e^{\zeta re^{-i\varphi}}-1}{\zeta} F(\zeta)\,d\zeta = \int\limits_{L\left(\varphi_0-\gamma_{\varphi_0}-\frac{\pi}{2}\right)} \frac{e^{\zeta re^{-i\varphi}}-1}{\zeta} F(\zeta)\,d\zeta, \qquad z = re^{-i\varphi} \in \Delta(-\varphi_0;\ \gamma_{\varphi_0}). \tag{2.22}$$

Let us prove (2.21).

Let $C(\epsilon; R; \varphi)$ $(0 < \epsilon < R < +\infty,\ |\varphi - \varphi_0| < \gamma_{\varphi_0})$ be the closed contour on the Riemann surface G_∞ which bounds the region

$$G(\varepsilon; R; \varphi) = \left\{\zeta \in G_\infty;\ \varphi + \frac{\pi}{2} < \operatorname{Arg}\zeta < \varphi_0 + \gamma_{\varphi_0} + \frac{\pi}{2},\ \varepsilon < |\zeta| < R\right\}.$$

Since $G(\epsilon; R; \varphi)$ together with its boundary $C(\epsilon; R; \varphi)$ lies in the domain $\Delta(\alpha)$ of analyticity of $F(\zeta)$, for every $z = re^{-i\varphi} \in \Delta(-\varphi_0; \gamma_{\varphi_0})$ the Cauchy theorem yields

$$\int\limits_{C(\varepsilon;\ R;\ \varphi)} \frac{e^{\zeta re^{-i\varphi}}-1}{\zeta} F(\zeta)\,d\zeta = 0.$$

This identity can be rewritten as follows:

$$\int\limits_{L_1(\varepsilon;\ R)} \frac{e^{\zeta re^{-i\varphi}}-1}{\zeta} F(\zeta)\,d\zeta - \int\limits_{L_2(\varepsilon;\ R)} \frac{e^{\zeta re^{-i\varphi}}-1}{\zeta} F(\zeta)\,d\zeta = i \int\limits_{\varphi+\frac{\pi}{2}}^{\varphi_0+\gamma_{\varphi_0}+\frac{\pi}{2}} F(\varepsilon e^{i\vartheta})\left(e^{\varepsilon re^{i(\vartheta-\varphi)}}-1\right)d\vartheta - i \int\limits_{\varphi+\frac{\pi}{2}}^{\varphi_0+\gamma_{\varphi_0}+\frac{\pi}{2}} F(Re^{i\vartheta})\left(e^{Rre^{i(\vartheta-\varphi)}}-1\right)d\vartheta \equiv U_1(\varepsilon) + U_2(R), \tag{2.23}$$

where

$$L_1(\varepsilon; R) = \left\{\zeta \in G_\infty;\ \operatorname{Arg}\zeta = \varphi + \frac{\pi}{2},\ \varepsilon \leqslant |\zeta| \leqslant R\right\}, \tag{2.24}$$

$$L_2(\varepsilon; R) = \left\{\zeta \in G_\infty;\ \operatorname{Arg}\zeta = \varphi_0 + \gamma_{\varphi_0} + \frac{\pi}{2},\ \varepsilon \leqslant |\zeta| \leqslant R\right\}. \tag{2.25}$$

Since $F(\zeta) \in H_2[\alpha]$ and $[\varphi + \pi/2, \varphi_0 + \gamma_{\varphi_0} + \pi/2] \subset (-\pi/2\alpha, \pi/2\alpha)$, Lemma A(2°) implies the estimate

$$\max_{\varphi + \frac{\pi}{2} \leqslant \vartheta \leqslant \varphi_0 + \gamma_{\varphi_0} + \frac{\pi}{2}} |F(re^{i\vartheta})| \leqslant A_1 r^{-\frac{1}{2}}, \quad 0 < r < +\infty, \tag{2.26}$$

where A_1 does not depend on r.

If $|\varphi - \varphi_0| < \gamma_{\varphi_0}$ and $\varphi + \pi/2 \leqslant \vartheta \leqslant \varphi_0 + \gamma_{\varphi_0} + \pi/2$, then by (2.18) we have

$$\frac{\pi}{2} \leqslant \vartheta - \varphi \leqslant \varphi_0 - \varphi + \gamma_{\varphi_0} + \frac{\pi}{2} < 2\gamma_{\varphi_0} + \frac{\pi}{2} < \pi.$$

Hence

$$\max_{\varphi + \frac{\pi}{2} \leqslant \vartheta \leqslant \varphi_0 + \gamma_{\varphi_0} + \frac{\pi}{2}} |e^{Rre^{i(\vartheta - \varphi)}} - 1| \leqslant 2 \quad (|\varphi - \varphi_0| < \gamma_{\varphi_0},\ 0 \leqslant R,\ r < +\infty). \tag{2.27}$$

It is not difficult to see that

$$\sup_{-\infty < \vartheta,\ \varphi < +\infty} |e^{\varepsilon re^{i(\vartheta - \varphi)}} - 1| \leqslant \varepsilon r e^{\varepsilon r} \quad (0 \leqslant \varepsilon,\ r < +\infty). \tag{2.28}$$

In view of (2.26)–(2.28) and the definition (2.23) of $U_1(\epsilon)$ and $U_2(R)$ we obtain

$$|U_1(\varepsilon)| \leqslant A_2 \varepsilon^{-\frac{1}{2}} \varepsilon r e^{\varepsilon r}, \quad |U_2(R)| \leqslant A_3 R^{-\frac{1}{2}}, \tag{2.29}$$

where A_2 and A_3 do not depend on ϵ and R.

In view of (2.29), by taking the limit in (2.23) for $\epsilon \to +0$ and $R \to +\infty$ we obtain

$$\lim_{\substack{\varepsilon \to +0 \\ R \to +\infty}} \int_{L_1(\varepsilon;\, R)} \frac{e^{\zeta re^{-i\varphi}} - 1}{\zeta} F(\zeta)\, d\zeta$$

$$= \lim_{\substack{\varepsilon \to +0 \\ R \to +\infty}} \int_{L_2(\varepsilon;\, R)} \frac{e^{\zeta re^{-i\varphi}} - 1}{\zeta} F(\zeta)\, d\zeta, \quad z = re^{-i\varphi} \in \Delta(-\varphi_0;\ \gamma_{\varphi_0}).$$

By (2.24) and (2.25) this identity goes into (2.21).

Similarly one proves (2.22).

Therefore (2.16) implies (2.20).

Furthermore, using the Cauchy-Schwarz-Bunjakovskiĭ inequality it is not difficult to prove that the integrals on the right-hand side of (2.20) converge absolutely and uniformly inside $\Delta(-\varphi_0;\gamma_{\varphi_0})$, and represent analytic functions there. Hence $G(z)$ is also analytic in $\Delta(-\varphi_0;\gamma_{\varphi_0})$, which in view of (2.19) implies that $G(z)$ is analytic in the entire domain $\Delta(\beta)$.

Observe that (2.16) can be rewritten as follows:

$$\frac{1}{i\sqrt{2\pi}}G(re^{i\varphi})=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{e^{iry}-1}{iy}F\left(e^{i\left(\frac{\pi}{2}\operatorname{sign} y-\varphi\right)}|y|\right)dy,\quad z=re^{i\varphi}\in\Delta(\beta). \tag{2.30}$$

Since for every $\varphi\in(-\pi/2\beta,\pi/2\beta)$ the function $G(re^{i\varphi})$ is differentiable with respect to r on the halfaxis $(0,+\infty)$, the function (2.12) exists by (2.30) at each point $z=re^{i\varphi}$ of the angular domain $\Delta(\beta)$.

On the other hand, by (2.30) we have

$$\frac{1}{i\sqrt{2\pi}}\frac{d}{dz}G(z)\Big|_{z=re^{i\varphi}}=\frac{1}{i\sqrt{2\pi}}e^{-i\varphi}\frac{d}{dr}G(re^{i\varphi})$$

$$=\frac{1}{\sqrt{2\pi}}e^{-i\varphi}\frac{d}{dr}\int_{-\infty}^{+\infty}\frac{e^{iry}-1}{iy}F\left(e^{i\left(\frac{\pi}{2}\operatorname{sing} y-\varphi\right)}|y|\right)dy,\quad z=re^{i\varphi}\in\Delta(\beta),$$

which shows that the function (2.12) is analytic in $\Delta(\beta)$.

2°. The definition (2.12) of $f(z)$ and the Parseval identity imply

$$\int_0^{+\infty}|f(re^{i\varphi})|^2dr=\int_0^{\infty}\left|F\left(e^{i\left(\frac{\pi}{2}\operatorname{sign} y-\varphi\right)}|y|\right)\right|^2dy\leqslant 2\{\|F\|^*_{2,\alpha}\}^2\quad\left(|\varphi|<\frac{\pi}{2\beta}\right). \tag{2.31}$$

Since the analyticity of $f(z)$ in $\Delta(\beta)$ is already established, (2.31) shows that $f(z)$ belongs to $H_2[\beta]$ and also that (2.31) holds. This completes the proof of the lemma.

Lemma 2.2 has an important corollary:

LEMMA 2.2$^{\mathrm{I}}$. *If $F(\zeta)\in H_2[\alpha]$ $(0<\alpha<1)$, then the function*

$$f(t)=\frac{1}{\sqrt{2\pi}}\frac{d}{dt}\int_{-\infty}^{+\infty}\frac{e^{ity}-1}{iy}F\left(e^{i\frac{\pi}{2}\operatorname{sign} y}|y|\right)dy \tag{2.32}$$

exists at each point $t\in(0,+\infty)$ and can be continued analytically into the angular region $\Delta(\beta)$ $(1/\beta=1/\alpha-1)$ as a function of class $H_2[\beta]$. Moreover,

$$\|f\|^*_{2,\beta}\leqslant\sqrt{2}\|F\|^*_{2,\alpha}. \tag{2.33}$$

Indeed, by Lemma 2.2 the function

$$f(re^{i\varphi}) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi} \frac{d}{dr} \int_{-\infty}^{+\infty} \frac{e^{iry}-1}{iy} F\left(e^{i\left(\frac{\pi}{2}\operatorname{sign} y-\varphi\right)}|y|\right) dy,$$

$$z = re^{i\varphi} \in \Delta(\beta),$$

belongs to $H_2[\beta]$, satisfies (2.33) and, for $\varphi = 0$, coincides with the function (2.32).

2.2. (a). The Paley-Wiener theorem allows us to construct an isometric operator mapping $L_2(0, +\infty)$ on H_2^+. For $H_2[\alpha]$ we shall establish the following theorem of Paley-Wiener type which will enable us to define a bounded, invertible, linear operator, mapping $H_2[\beta]$ $(0 < \beta < +\infty)$ on $H_2[\alpha]$ $(1/\alpha = 1/\beta + 1)$.

THEOREM 1. *If $0 < \alpha < 1$ and $1/\beta = 1/\alpha - 1$, then the following statements hold*:

1°. *$H_2[\alpha]$ coincides with the family of functions which can be represented in the form*

$$F(\zeta) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi} \int_0^{+\infty} e^{-\zeta t e^{-i\varphi}} f(te^{-i\varphi})\, dt, \quad \zeta \in D(\varphi), \tag{2.34}$$

where $\varphi \in (-\pi/2\beta, \pi/2\beta)$ is arbitrary and $f(z)$ is an arbitrary function of class $H_2[\beta]$.

2°. *In* (2.34), *for every fixed function $f(\zeta) \in H_2[\alpha]$ the function $f(z) \in H_2[\beta]$ is unique, and one has the inversion formula*

$$f(re^{i\varphi}) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi} \frac{d}{dr} \int_{-\infty}^{+\infty} \frac{e^{iry}-1}{iy} F\left(e^{i\left(\frac{\pi}{2}\operatorname{sign} y-\varphi\right)}|y|\right) dy, \quad z = re^{i\varphi} \in \Delta(\beta). \tag{2.35}$$

3°. *If $f(\zeta) \in H_2[\alpha]$ can be represented in* (2.34) *in terms of $f(z) \in H_2[\beta]$, then*

$$\|F\|_{2,\alpha} \leqslant 2\|f\|_{2,\beta} \leqslant 2\sqrt{2}\|F\|_{2,\alpha}. \tag{2.36}$$

PROOF. 1°. If $F(\zeta)$ has a representation (2.34), then in view of Lemma 2.1

$$F(\zeta) \equiv G(\zeta;\, f), \quad \zeta \in \Delta(\alpha).$$

Hence by the same lemma we conclude that $F(\zeta)$ belongs to $H_2[\alpha]$.

Conversely, let $F(\zeta)$ be an arbitrary function of class $H_2[\alpha]$. Set

$$f(re^{i\varphi}) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi} \frac{d}{dr} \int_{-\infty}^{+\infty} \frac{e^{iry}-1}{iy} F\left(e^{i\left(\frac{\pi}{2}\operatorname{sign} y-\varphi\right)}|y|\right) dy, \tag{2.37}$$

$$z = re^{i\varphi} \in \Delta(\beta).$$

By Lemma 2.2, $f(z) \in H_2[\beta]$.

Setting $\varphi = 0$ in (2.37), we obtain on the basis of Theorem B that $F(\zeta)$ can be represented in the half-plane $D(0) \equiv \Delta(1)$ as follows:

$$F(\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\zeta t} f(t)\, dt, \quad \zeta \in \Delta(1). \tag{2.38}$$

Furthermore, taking into account that $f(z) \in H_2[\beta]$, for every $\varphi \in (-\pi/2\beta, \pi/2\beta)$ we define

$$g_\varphi(\zeta; f) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi} \int_0^{+\infty} e^{-\zeta t e^{-i\varphi}} f(te^{-i\varphi})\, dt, \quad \zeta \in D(\varphi). \tag{2.39}$$

By (2.38),

$$F(\zeta) \equiv g_0(\zeta; f), \quad \zeta \in D(0) = \Delta(1). \tag{2.40}$$

It follows from (2.40) and (2.39), in view of Lemma 2.1, that

$$F(\zeta) \equiv g_\varphi(\zeta; f), \quad \zeta \in D(\varphi), \tag{2.41}$$

for arbitrary $\varphi \in (-\pi/2\beta, \pi/2\beta)$.

From (2.41) and (2.39) we obtain the representation (2.34) of $F(\zeta)$.

2°. By Lemma 2.2, the function

$$\Phi(re^{i\varphi}) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi} \frac{d}{dr} \int_{-\infty}^{+\infty} \frac{e^{iry} - 1}{iy} F\left(e^{i\left(\frac{\pi}{2}\operatorname{sign} y - \varphi\right)} |y|\right) dy,$$
$$z = re^{i\varphi} \in \Delta(\beta), \tag{2.42}$$

is analytic in $\Delta(\beta)$. Setting $\varphi = 0$ in (2.34) and (2.42), we conclude that

$$f(t) = \Phi(t), \quad t \in (0, +\infty).$$

Therefore, by the principle of analytic continuation,

$$f(z) \equiv \Phi(z), \quad z \in \Delta(\beta),$$

which in view of (2.42) gives the inversion formula (2.35).

The uniqueness of $f(z)$ in (2.34) follows from (2.35).

3°. If $F(\zeta) \in H_2[\alpha]$ can be represented by (2.34) in terms of a function $f(z) \in H_2[\beta]$, then by Lemma 2.1

$$F(\zeta) \equiv G(\zeta; f), \quad \zeta \in \Delta(\alpha),$$

which together with Lemma 2.1 yields the inequality

$$\|F\|^*_{2,\alpha} \leqslant \|f\|^*_{2,\beta}. \tag{2.43}$$

while the inversion formula (2.35), together with Lemma 2.2, yields

$$\|f\|^{*}_{2,\beta} \leq \sqrt{2}\,\|F\|^{*}_{2,\alpha}. \tag{2.44}$$

From (2.43), (2.44) and Lemma 1.1 we obtain (2.36). Hence the theorem is completely established.

In the limit case $\alpha = 1$ ($\beta = -\infty$) Theorem 1 goes into the Paley-Wiener theorem, provided we identify $H_2[+\infty]$ with $L_2(0, +\infty)$. In particular, Theorem 1 contains the following theorem whose formulation recalls the classical version of the Paley-Wiener theorem [1] (cf. also [2], p. 413) and which coincides with it in the limit case $\alpha = 1$ ($\beta = \infty$).

THEOREM 1^{I}. *If $0 < \alpha < 1$ and $1/\beta = 1/\alpha - 1$, then $H_2[\alpha]$ coincides with the class of functions which can be represented in the halfplane $\Delta(1)$ in the form*

$$F(\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\zeta t} f(t)\,dt, \quad \zeta \in \Delta(1), \tag{2.45}$$

where $f(z)$ is an arbitrary function from $H_2[\beta]$.

Indeed, let $f(z)$ be an arbitrary function from $H_2[\beta]$. Set

$$F(\zeta) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi} \int_0^{+\infty} e^{-\zeta t e^{-i\varphi}} f(te^{-i\varphi})\,dt, \quad \zeta \in D(\varphi), \tag{2.46}$$

for arbitrary $\varphi \in (-\pi/2\beta, \pi/2\beta)$. In view of Theorem 1, $F(\zeta) \in H_2[\alpha]$. Setting $\varphi = 0$ in (2.46), we conclude that the Laplace transform (2.45) of $f(t)$ can be continued analytically into the angular domain $\Delta(\alpha)$ as a function of class $H_2[\alpha]$.

Conversely, let $F(\zeta)$ be an arbitrary function from $H_2[\alpha]$. Setting

$$f(re^{i\varphi}) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi} \frac{d}{dr} \int_{-\infty}^{+\infty} \frac{e^{iry} - 1}{iy} F\left(e^{i\left(\frac{\pi}{2}\operatorname{sign} y - \varphi\right)} |y|\right) dy,$$
$$z = re^{i\varphi} \in \Delta(\beta), \tag{2.47}$$

we conclude by Theorem 1 that $f(z) \in H_2[\beta]$. In view of (2.47) and Theorem 1 we see that

$$F(\zeta) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi} \int_0^{+\infty} e^{-\zeta t e^{-i\varphi}} f(te^{-i\varphi})\,dt, \quad \zeta \in D(\varphi), \tag{2.48}$$

for any $\varphi \in (-\pi/2\beta, \pi/2\beta)$. Setting $\varphi = 0$ in (2.48), we obtain the following representation of $F(\zeta)$ in the halfplane $D(0) = \Delta(1)$:

$$F(\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\zeta t} f(t)\,dt, \quad \zeta \in \Delta(1).$$

This representation coincides with the representation of type (2.45).

§3. A theorem of Müntz-Szász type in the complex domain

3.1. (a) Let $\{\lambda_k\}_1^\infty$ be an arbitrary sequence of complex numbers from the angular domain $\Delta(\alpha)$, $\frac{1}{2} < \alpha < +\infty$. For an arbitrary k $(1 \leqslant k < +\infty)$, let $s_k \geqslant 1$ be the multiplicity of the appearance of the number λ_k on the segment $\{\lambda_j\}_1^k$ of the sequence.

For a given value of ω $(-1 < \omega < 1)$ let $L_{2,\omega}(0, +\infty)$ be the class of functions $f(x)$, measurable on $(0, +\infty)$, such that

$$\|f\|_{L_{2,\omega}(0,+\infty)} = \left\{\int_0^{+\infty} |f(x)|^2 x^\omega dx\right\}^{\frac{1}{2}} < +\infty. \tag{3.1}$$

Setting

$$\rho = \frac{\alpha}{2\alpha - 1}, \quad \mu = \frac{1 + \omega + \rho}{2\rho}, \tag{3.2}$$

let us define the system of functions

$$\{E_\rho^{(s_k-1)}(-\lambda_k x; \mu)\, x^{s_k-1}\}_{k=1}^\infty, \tag{3.3}$$

generated by an entire function of Mittag-Leffler type

$$E_\rho(z; \mu) = \sum_{k=0}^\infty \frac{z^k}{\Gamma\left(\mu + \dfrac{k}{\rho}\right)}.$$

In [6] there was established a criterion for closedness of the system (3.3) in the metric of $L_{2,\omega}(0, +\infty)$. Since later we shall make an essential use of this criterion, let us state this result, representing a substantial generalization of the Szász theorem [3]:

THEOREM D. *In order that system* (3.3) *be closed in the metric of* $L_{2,\omega}(0, +\infty)$ *it is necessary and sufficient that*

$$\sum_{k=1}^\infty \frac{\operatorname{Re} \lambda_k^\alpha}{1 + |\lambda_k|^{2\alpha}} = +\infty. \tag{3.4}$$

3.1. (b) Now we shall establish a criterion for closedness of systems of functions in the corresponding Hilbert spaces $H_2[\alpha]$.

First we shall prove a lemma.

LEMMA 3.1. *If* $\frac{1}{2} < \rho < +\infty$, *and* $u = (1 + \rho)/2\rho$, *then the following statements hold*:

1°. *The class* $H_2[\rho]$ *coincides with the system of functions which can be represented in the form*

$$F(\zeta)=\frac{\rho}{\sqrt{2\pi}}\zeta^{\mu\rho-1}\int_0^{+\infty}e^{-\zeta^\rho t^\rho}t^{\mu\rho-1}\varphi(t)\,dt,\quad \zeta\in\Delta(\rho), \tag{3.5}$$

where $\varphi(t)$ is any function of class $L_2(0,+\infty)$.

2°. *In* (3.5), *for every fixed function $F(\zeta)\in H_2[\rho]$ the function $\varphi(t)\in L_2(0,+\infty)$ is unique and almost everywhere on the axis $(-\infty,+\infty)$ one has*

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}t^{1-\mu}\frac{d}{dt}\int_{-\infty}^{+\infty}\frac{e^{ity}-1}{iy}F\left(e^{i\frac{\pi}{2\rho}\operatorname{sign}y}|y|^{\frac{1}{\rho}}\right)\left(e^{i\frac{\pi}{2}\operatorname{sign}y}|y|\right)^{\mu-1}dy\\ =\begin{cases}\varphi(t^{\frac{1}{\rho}}) & \text{for } t\in(0,+\infty),\\ 0 & \text{for } t\in(-\infty,0).\end{cases}\end{aligned}\tag{3.6}$$

where $F(e^{\pm i\pi/2\rho}y)$ is the boundary value of $F(\zeta)$.

3°. *If $F(\zeta)\in H_2[\rho]$ and $\varphi(t)\in L_2(0,+\infty)$ are related by* (3.5), *then the Parseval equality holds*:

$$\|F\|_{2,\rho}=\|\varphi\|_{L_2(0,+\infty)}. \tag{3.7}$$

Proof. If $F(\zeta)$ is defined by (3.5), then after the substitutions $\zeta^\rho=w$, $t^\rho=\tau$ we conclude that the function

$$\Phi(w)=F(w^{\frac{1}{\rho}})w^{\mu-1},\quad w\in\Delta(1), \tag{3.8}$$

has a representation

$$\Phi(w)=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}e^{-w\tau}\tau^{\mu-1}\varphi(\tau^{\frac{1}{\rho}})\,d\tau,\quad w\in\Delta(1), \tag{3.9}$$

where $\tau^{\mu-1}\varphi(\tau^{1/\rho})\in L_2(0,+\infty)$, because $\varphi(t)\in L_2(0,+\infty)$.

In view of Theorem B, (3.9) gives that $\Phi(w)\in H_2^+$, and almost everywhere on $(-\infty,+\infty)$

$$\frac{1}{\sqrt{2\pi}}\frac{d}{dt}\int_{-\infty}^{+\infty}\frac{e^{ity}-1}{iy}\Phi\left(e^{i\frac{\pi}{2}\operatorname{sign}y}|y|\right)dy=\begin{cases}t^{\mu-1}\varphi(t^{\frac{1}{\rho}}) & \text{for } t\in(0,+\infty),\\ 0 & \text{for } t\in(-\infty,0).\end{cases}\tag{3.10}$$

Since $\Phi(w)$ belongs to H_2^+ we conclude by Theorem A that it belongs to $H_2[1]$.

Returning to the variable ζ, we obtain on the basis of (3.8) and (3.10) that $F(\zeta)\in H_2[\rho]$ and (3.6) holds.

Moreover, in view of (3.9) and Theorem B we obtain

$$\|\Phi\|_{2,1}=\|\Phi\|_2=\left\{\int_0^{+\infty}|\tau^{\mu-1}\varphi(\tau^{\frac{1}{\rho}})|^2d\tau\right\}^{\frac{1}{2}}=\sqrt{\rho}\,\|\varphi\|_{L_2(0,+\infty)}. \tag{3.11}$$

In view of (3.8) we also get

$$\|\Phi\|_{2,1}=\sqrt{\rho}\,\|F\|_{2,\rho},$$

which together with (3.11) implies (3.7).

If $F(\zeta)\in H_2[\rho]$ it is easy to see that the function

$$\Phi(w)=F(w^{\frac{1}{\rho}})\,w^{\mu-1},\quad w\in\Delta(1),$$

belongs to $H_2[1]$. In view of Theorem A this function belongs to H_2^+. Hence by Theorem B we have the representation

$$w^{\mu-1}F(w^{\frac{1}{\rho}})=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}e^{-w\tau}\tilde{\varphi}(\tau)\,d\tau,\quad w\in\Delta(1),$$

where $\tilde{\varphi}(\tau)\in L_2(0,+\infty)$. Substituting $w^{1/\rho}=\zeta$ and $\tau^{1/\rho}=t$ and setting

$$\varphi(t)=\tilde{\varphi}(t^\rho)\,t^{\frac{\rho-1}{2}},\quad t\in(0,+\infty),$$

we obtain

$$F(\zeta)=\frac{\rho}{\sqrt{2\pi}}\zeta^{\mu\rho-1}\int_0^{+\infty}e^{-\zeta^\rho t^\rho}t^{\mu\rho-1}\varphi(t)\,dt,\quad \zeta\in\Delta(\rho),$$

where obviously $\varphi(t)\in L_2(0,+\infty)$. The lemma is established.

Now we shall prove another criterion for closedness of systems of simple rational fractions in $H_2[\rho]$.

THEOREM 2. *Let* $\frac{1}{2}<\alpha<+\infty$ *and* $1/\rho=2-1/\alpha$, *and let* $\{\lambda_k\}_1^\infty$ *be a sequence of complex numbers from the angular domain* $\Delta(\alpha)$. *Let* $s_k\geqslant 1$ *be the multiplicity of the appearance of* λ_k *on the segment* $\{\lambda_1,\ldots,\lambda_k\}$ *of this sequence.*

The system of rational fractions

$$r(\zeta;\lambda_k)=\frac{(s_k-1)!}{(\zeta+\lambda_k)^{s_k}}\quad(1\leqslant k<+\infty) \tag{3.12}$$

is closed in the Hilbert space $H_2[\rho]$ *if and only if*

$$\sum_{k=1}^{\infty}\frac{\operatorname{Re}\lambda_k^\alpha}{1+|\lambda_k|^{2\alpha}}=+\infty. \tag{3.13}$$

PROOF. First we shall prove the following formula (cf. [2], Lemma 3.9, and also [7], Lemma 2):

$$\frac{n!}{(\zeta+z)^{n+1}} = \rho\zeta^{\mu\rho-1}\int_0^{+\infty} e^{-\zeta^\rho t^\rho} t^{\mu\rho-1} E_\rho^{(n)}(-zt;\ \mu)\, t^n\, dt \quad (n=0, 1, 2, \ldots),$$
$$\zeta \in \Delta(\rho), \quad z \in \Delta(\alpha), \quad \mu = \frac{1+\rho}{\rho}. \tag{3.14}$$

Let $L_\rho(\nu) = \{\zeta;\ \mathrm{Re}\ \zeta^\rho = \nu,\ |\arg \zeta| \leqslant \pi/2\}$ be a contour in the complex plane and let $D_\rho^*(\nu)$ and $D_\rho(\nu)$, respectively, be regions lying to the left and to the right of the contour $L_\rho(\nu)$. It is known (cf. [2], p. 149) that

$$\int_0^{+\infty} e^{-\zeta^\rho t} E_\rho(zt^{\frac{1}{\rho}};\ \mu)\, t^{\mu-1}\, dt = \frac{\zeta^{1-\mu\rho}}{\zeta - z} \quad (\mu > 0), \tag{3.15}$$

when $z \in D_\rho^*(\nu)$ and $\zeta \in D_\rho(\nu)$.

If z lies in the angular domain $|\arg(-z)| < \pi/2\alpha$, then $\zeta^{1-\mu\rho}(\zeta - z)^{-1}$ is analytic in $|\arg \zeta| < \pi/2\rho$.

On the other hand, for the considered values of z, the integral on the left-hand side of (3.15) converges absolutely and uniformly in the region $|\arg \zeta| < \pi/2\rho$, and represents there an analytic function.

Hence, by the principle of analytic continuation, (3.15) holds provided ζ and z vary in the regions

$$|\arg \zeta| < \frac{\pi}{2\rho}, \quad |\arg(-z)| < \frac{\pi}{2\alpha},$$

or, equivalently, the equality

$$\frac{1}{\zeta + z} = \rho\zeta^{\mu\rho-1}\int_0^{+\infty} e^{-\zeta^\rho t^\rho} t^{\mu\rho-1} E_\rho(-zt;\ \mu)\, dt \tag{3.15$^{\mathrm{I}}$}$$

holds for $\zeta \in \Delta(\rho)$ and $z \in \Delta(\alpha)$.

Differentiating (3.15$^{\mathrm{I}}$) n times with respect to z, we obtain (3.14); the differentiation under the integral sign is possible in view of the absolute and uniform convergence of the resulting integrals.

Substituting $n = s_k - 1$ and $z = \lambda_k$ into (3.14), we obtain

$$\frac{(s_k - 1)!}{(\zeta + \lambda_k)^{s_k}} = \rho\zeta^{\mu\rho-1}\int_0^{+\infty} e^{-\zeta^\rho t^\rho} t^{\mu\rho-1} E_\rho^{(s_k-1)}(-\lambda_k t;\ \mu)\, t^{s_k-1}\, dt,$$
$$k = 1, 2, \ldots; \quad \mu = \frac{1+\rho}{2\rho}, \quad \zeta \in \Delta(\rho). \tag{3.16}$$

Since for $\lambda_k \in \Delta(\alpha)$, $1/\rho = 2 - 1/\alpha$ and $\mu = (1+\rho)/2\rho$ the function $E_\rho^{(s_k-1)}(-\lambda_k t;\ \mu)t^{s_k-1}$ belongs to $L_2(0, +\infty)$(see [6]), it follows from (3.16) by Lemma 3.1 that

$$r(\zeta;\ \lambda_k) \in \mathcal{H}_2[\rho] \quad (1 \leqslant k < +\infty). \tag{3.17}$$

Formulas (3.16) also show that if

$$P(t)=\sum_{k=1}^{n} a_k \sqrt{2\pi} E_{\rho}^{(s_k-1)}(-\lambda_k t;\mu)\, t^{s_k-1}, \quad \mu=\frac{1+\rho}{2\rho}, \tag{3.18}$$

and

$$\hat{P}(\zeta)=\sum_{k=1}^{n} a_k r(\zeta;\lambda_k) \tag{3.19}$$

are linear combinations of functions of the systems (3.3) (for $\omega = 0$) and (3.12), respectively, then

$$\hat{P}(\zeta)=\frac{\rho}{\sqrt{2\pi}}\zeta^{\mu\rho-1}\int_0^{+\infty} e^{-\zeta^\rho t^\rho} t^{\mu\rho-1} P(t)\,dt, \quad \mu=\frac{1+\rho}{2\rho}, \quad \zeta\in\Delta(\rho). \tag{3.20}$$

Furthermore, in view of Lemma 3.1 the class $H_2[\rho]$ coincides with the class of functions $F(\zeta)$ which can be represented in the form

$$F(\zeta)=\frac{\rho}{\sqrt{2\pi}}\zeta^{\mu\rho-1}\int_0^{+\infty} e^{-\zeta^\rho t^\rho} t^{\mu\rho-1}\varphi(t)\,dt, \quad \zeta\in\Delta(\rho) \quad \left(\mu=\frac{1+\rho}{2\rho}\right), \tag{3.21}$$

where $\varphi(t)$ is an arbitrary function from $L_2(0, +\infty)$.

Subtracting (3.20) from (3.21), we conclude on the basis of Lemma 3.1 that

$$\|F-\hat{P}\|_{2,\rho}=\|\varphi-P\|_{L_2(0,+\infty)}. \tag{3.22}$$

From all of what was said above it follows that the system (3.12) is closed in $H_2[\rho]$ if and only if the system (3.3) (for $\omega = 0$) is closed in $L_2(0, +\infty)$. This together with Theorem D (for $\omega = 0$) implies the statement of the theorem.

Now we shall prove a theorem which represents a substantial generalization of the well-known theorem of Szász [3]:

THEOREM 3. *Let $1 < \alpha < \infty$ and $1/\gamma = 1 - 1/\alpha$, let $\{\lambda_k\}_1^\infty$ be an arbitrary sequence of complex numbers from the angular region $\Delta(\alpha)$ and let $s_k \geqslant 1$ be the multiplicity of the appearance of λ_k on the segment $\{\lambda_1, \ldots, \lambda_k\}$ of this sequence. The system of functions*

$$\omega(z;\lambda_k)=e^{-\lambda_k z} z^{s_k-1} \quad (1\leqslant k<+\infty) \tag{3.23}$$

is closed in the Hilbert space $H_2[\gamma]$ if and only if

$$\sum_{k=1}^{\infty}\frac{\operatorname{Re}\lambda_k^\alpha}{1+|\lambda_k|^{2\alpha}}=+\infty. \tag{3.24}$$

PROOF. Together with the system $\{\omega(z;\lambda_k)\}_1^\infty$ we shall consider the system of rational fractions $\{r(\zeta;\lambda_k)\}_1^\infty$, setting

$$r(\zeta;\lambda_k)=\frac{(s_k-1)!}{[\zeta+\lambda_k)^{s_k}}\qquad (1\leqslant k<+\infty), \tag{3.25}$$

where

$$\frac{1}{\rho}=2-\frac{1}{\alpha}. \tag{3.26}$$

By a direct verification it is easy to see that the functions of system (3.23) belong to $H_2[\gamma]$ and also that for all k $(1\leqslant k<+\infty)$ we have

$$r(\zeta;\lambda_k)=\frac{1}{\sqrt{2\pi}}e^{-i\varphi}\int_0^{+\infty}e^{-\zeta te^{-i\varphi}}\omega(te^{-i\varphi};\lambda_k)\,dt,\qquad \zeta\in D(\varphi), \tag{3.27}$$

for any $\varphi\in(-\pi/2\gamma,\pi/2\gamma)$. It follows from (3.27) that if

$$P(z)=\sum_{k=1}^{n}a_k\omega(z;\lambda_k) \tag{3.28}$$

and

$$\hat{P}(\zeta)=\sum_{k=1}^{n}a_kr(\zeta;\lambda_k) \tag{3.29}$$

are linear combinations of functions from the systems (3.23) and (3.25) respectively, then

$$\hat{P}(\zeta)=\frac{1}{\sqrt{2\pi}}e^{-i\varphi}\int_0^{+\infty}e^{-\zeta te^{-i\varphi}}P(te^{-i\varphi})\,dt,\qquad \zeta\in D(\varphi), \tag{3.30}$$

for any $\varphi\in(-\pi/2\gamma,\pi/2\gamma)$.

Furthermore, since $1<\alpha<+\infty$, $1/\gamma=1-1/\alpha$ and $1/\rho=2-1/\alpha$, we have $\frac{1}{2}<\rho<1$ and $1/\gamma+1=1/\rho$. Consequently we can use Theorem 1, where we set $\alpha=\rho$ and $\beta=\gamma$.

By Theorem 1, $H_2[\rho]$ coincides with the class of those functions $F(\zeta)$ which can be represented in the form

$$F(\zeta)=\frac{1}{\sqrt{2\pi}}e^{-i\varphi}\int_0^{+\infty}e^{-\zeta te^{-i\varphi}}f(te^{-i\varphi})\,dt,\qquad \zeta\in D(\varphi), \tag{3.31}$$

for any $\varphi\in(-\pi/2\gamma,\pi/2\gamma)$, where $f(z)$ is an arbitrary function from $H_2[\gamma]$. Subtracting (3.30) from (3.31) and using Theorem 1, we arrive at

$$\|F-\hat{P}\|_{2,\rho}\leqslant 2\|f-P\|_{2,\gamma}\leqslant 2\sqrt{2}\,\|F-\hat{P}\|_{2,\rho}. \tag{3.32}$$

From what was said above it follows that the system (3.23) is closed in $H_2[\gamma]$ if and only if (3.25) is closed in $H_2[\rho]$. Hence, on the basis of Theorem 2, we obtain the statement of our theorem.

Observe that in the limiting case when $\gamma = +\infty$ ($\alpha = 1$) Theorem 4 goes into the Szász theorem in its classical formulation [4], provided we identify $H_2[+\infty]$ with $L_2(0, +\infty)$.

The system of functions

$$\{e^{-\lambda_k z} z^{s_k - 1}\}_1^\infty \quad (\operatorname{Re}\lambda_k > 0)$$

is closed in $L_2(0, +\infty)$ *if and only if*

$$\sum_{k=1}^{\infty} \frac{\operatorname{Re}\lambda_k}{1+|\lambda_k|^2} = +\infty.$$

3.2. (a) If (3.13) is not satisfied, then the system of rational fractions (3.12) is clearly not closed in $H_2[\rho]$. Therefore it is natural to ask about a complete intrinsic description of this nonclosed system.

Let $\frac{1}{2} < \alpha < +\infty$ and $1/\rho = 2 - 1/\alpha$, and let $\{\lambda_k\}_1^\infty$ be an arbitrary sequence of complex numbers from the angular region $\Delta(\alpha)$. Let L_ρ be the boundary of $\Delta(\rho)$. Assuming that

$$\sum_{k=1}^{\infty} \frac{\operatorname{Re}\lambda_k^\alpha}{1+|\lambda_k|^{2\alpha}} < +\infty, \tag{3.33}$$

we denote by $\lambda_2\{\Delta(\rho); \lambda_k\}$ the class of functions $F(\zeta) \in H_2[\rho]$ whose boundary values coincide almost everywhere on L_ρ with the boundary values of a function $F^{(-)}(z)$ which is meromorphic in the region $|\arg(-z)| < \pi/2\alpha$, has poles only among the points $z = -\lambda_k$, and is such that

$$F^{(-)}(-z) B_\alpha(z) \in \mathscr{H}_2[\alpha],$$

where $B_\alpha(z)$ is a convergent Blaschke product for the angular domain $\Delta(\alpha)$ with zeros at the points $z = \lambda_k$:

$$B_\alpha(z) = \prod_{k=1}^{\infty} \frac{z^\alpha - \lambda_k^\alpha}{z^\alpha + \bar{\lambda}_k^\alpha} \varkappa_k, \quad \varkappa_k = \frac{|1-\lambda_k^{2\alpha}|}{1-\lambda_k^{2\alpha}}, \quad z \in \Delta(\alpha). \tag{3.34}$$

In [7] one can find a complete intrinsic description, under assumption (3.33), of the closure of system (3.12) in the metric of $L_{2,\omega}(L_\rho)$ $(-1 < \omega < 1)$; however, this result is not explicitly formulated there.

With our notation this result (for $\omega = 0$) can be formulated as follows:

THEOREM E. *Under hypothesis* (3.33) *the closure of system* (3.12) *in the metric of* $H_2[\rho]$ *coincides with the class* $\lambda_2\{\Delta(\rho); \lambda_k\}$.

3.2. (b) If condition (3.24) is not satisfied, i.e. if

$$\sum_{k=1}^{\infty} \frac{\operatorname{Re}\lambda_k^{\alpha}}{1+|\lambda_k|^{2\alpha}} < +\infty, \tag{3.35}$$

then system (3.23) is not closed in $H_2[\gamma]$.

Now we shall establish a theorem which gives a complete intrinsic description of the closure in $H_2[\gamma]$ of a nonclosed system (3.23).

THEOREM 4. *Under the hypothesis* (3.35) *the closure of system* (3.23) *in the metric of* $H_2[\gamma]$ *coincides with the class of functions which can be represented in the form*

$$f(re^{i\varphi}) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi} \frac{d}{dr} \int_{-\infty}^{+\infty} \frac{e^{iry}-1}{iy} F(iye^{-i\varphi})\, dy, \quad z = re^{i\varphi} \in \Delta(\gamma), \tag{3.36}$$

where $F(\zeta)$ *is any function of the class* $\lambda_2\{\Delta(\rho); \lambda_k\}$ *and* $1/\rho = 2 - 1/\alpha$.

PROOF. Together with the nonclosed system (3.23) we shall consider the system of rational fractions

$$r(\zeta; \lambda_k) = \frac{(s_k-1)!}{(\zeta+\lambda_k)^{s_k}} \quad (1 \leqslant k < +\infty). \tag{3.37}$$

Observe that if $1 \leqslant k < +\infty$, then

$$r(\zeta; \lambda_k) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi} \int_0^{+\infty} e^{-\zeta te^{-i\varphi}} \omega(te^{-i\varphi}; \lambda_k)\, dt, \quad \zeta \in D(\varphi), \tag{3.38}$$

for any $\varphi \in (-\pi/2\gamma, \pi/2\gamma)$.

Assume now that $f(z)$ belongs to the closure of the incomplete system (3.23) and let the sequence

$$P_n(z) = \sum_{k=1}^{m_n} a_k^{(n)} \omega(z; \lambda_k) \quad (1 \leqslant n < +\infty) \tag{3.39}$$

converge to $f(z)$ in the metric of $H_2[\gamma]$:

$$\lim_{n\to+\infty} \| f - P_n \|_{2,\gamma} = 0. \tag{3.40}$$

Since $\frac{1}{2} < \rho < 1$ and $1/\gamma = 1/\rho - 1$, we can use Theorem 1, setting $\alpha = \rho$ and $\beta = \gamma$. According to Theorem 1, the formula

$$F(\zeta) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi} \int_0^{+\infty} e^{-\zeta te^{-i\varphi}} f(te^{-i\varphi})\, dt, \quad \zeta \in D(\varphi), \quad \varphi \in \left(-\frac{\pi}{2\gamma}, \frac{\pi}{2\gamma}\right), \tag{3.41}$$

defines a function $F(\zeta) \in H_2[\rho]$.

On the other hand, if we set

$$\hat{P}_n(\zeta) = \sum_{k=1}^{m_n} a_k^{(n)} r(\zeta; \lambda_k) \quad (1 \leqslant n < +\infty), \tag{3.42}$$

then by (3.38) and (3.39) we obtain

$$\hat{P}_n(\zeta) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi} \int_0^{+\infty} e^{-\zeta te^{-i\zeta}} P_n(te^{-i\varphi})\, dt, \quad \zeta \in D(\varphi), \quad \varphi \in \left(-\frac{\pi}{2\gamma}, \frac{\pi}{2\gamma}\right), \tag{3.43}$$

for all n ($1 \leqslant n < +\infty$). Subtracting (3.43) from (3.41), by Theorem 1 we get

$$\|F-\hat{P}_n\|_{2,\rho} \leqslant 2\|f-P_n\|_{2,\gamma} \quad (1 \leqslant n < +\infty). \tag{3.44}$$

It follows from (3.44), (3.40) and (3.42) that $F(\zeta)$ belongs to the closure of system (3.37) in the metric of $H_2[\rho]$. This together with Theorem E shows that $F(\zeta) \in \lambda_2\{\Delta(\rho); \lambda_k\}$. Inverting (3.41) on the basis of Theorem 1 (where we set $\alpha = \rho$ and $\beta = \gamma$), we obtain a representation of the form (3.36) for the function $f(z)$.

Conversely, let $f(z)$ admit a representation of the form (3.36), where $F(\zeta) \in \lambda_2\{\Delta(\rho); \lambda_k\}$.

Since $\frac{1}{2} < \rho < 1$ and $1/\gamma = 1/\rho - 1$, we can use Lemma 2.2, setting $\alpha = \rho$ and $\beta = \gamma$.

Since $F(\zeta)$ belongs to $\lambda_2\{\Delta(\rho), \lambda_k\}$, it follows from the definition of this class that we also have $F(\zeta) \in H_2[\rho]$. This together with (3.36) and Lemma 2.2 shows that $f(z) \in H_2[\gamma]$.

On the other hand, since $F(\zeta) \in \lambda_2\{\Delta(\rho); \lambda_k\}$, by Theorem E there exists a sequence

$$\hat{P}_n(\zeta) = \sum_{k=1}^{m_n} a_k^{(n)} r(\zeta; \lambda_k) \quad (1 \leqslant n < +\infty) \tag{3.45}$$

of linear combinations of functions of system (3.37), satisfying the condition

$$\lim_{n\to+\infty} \|F-\hat{P}_n\|_{2,\rho} = 0. \tag{3.46}$$

Set

$$P_n(z) = \sum_{k=1}^{m_n} a_k^{(n)} \omega(z; \lambda_k) \quad (1 \leqslant n < +\infty). \tag{3.47}$$

It is not difficult to see that for $1 \leqslant n < +\infty$

$$P_n(re^{i\varphi}) = \frac{1}{\sqrt{2\pi}} e^{-i\varphi} \frac{d}{dr} \int_{-\infty}^{+\infty} \frac{e^{iry}-1}{iy} \hat{P}_n(iye^{-i\varphi})\, dy, \qquad z = re^{i\varphi} \in \Delta(\gamma). \tag{3.48}$$

Subtracting (3.48) from (3.36) and using Lemma 2.2, we obtain

$$\|f-P_n\|_{2,\gamma}^{*} \leqslant \sqrt{2}\, \|F-\hat{P}_n\|_{2,\rho}^{*} \quad (1 \leqslant n < +\infty).$$

Hence by Lemma 1.1 we also have

$$\|f-P_n\|_{2,\gamma} \leqslant 2\sqrt{2}\, \|F-\hat{P}_n\|_{2,\rho} \quad (1 \leqslant n < +\infty).$$

This together with (3.46) yields

$$\lim_{n\to+\infty} \|f-P_n\|_{2,\gamma} = 0.$$

Therefore, by (3.47) $f(z)$ belongs to the closure of system (3.23) in the metric of $H_2[\gamma]$. The theorem is completely established. In the extreme case when $\alpha = 1$ ($\gamma = +\infty$) this theorem goes into the main Theorem 3 of [4].

The basic results of this paper were published without proofs in the note [9].

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