

Lecture 11

Tue. 9.27.2016

Poincare, Lagrange, Hamiltonian, and Jacobi mechanics

(Unit 1 Ch. 12, Unit 2 Ch. 2-7, Unit 3 Ch. 1-3, Unit 7 Ch. 1-2)

Parabolic and 2D-IHO orbital envelopes (Review of Lecture 9 p.56-81 and a generalization.)

Clues for future assignments (Web Simulation: CouIt)

Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum, JerkIt (Vert Driven Pendulum))

1D-HO phase-space control (Old Mac OS & Web Simulations of “Catcher in the Eye”)

Exploring phase space and Lagrangian mechanics more deeply

A weird “derivation” of Lagrange’s equations

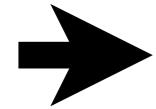
Poincare identity and Action, Jacobi-Hamilton equations

How Classicists might have “derived” quantum equations

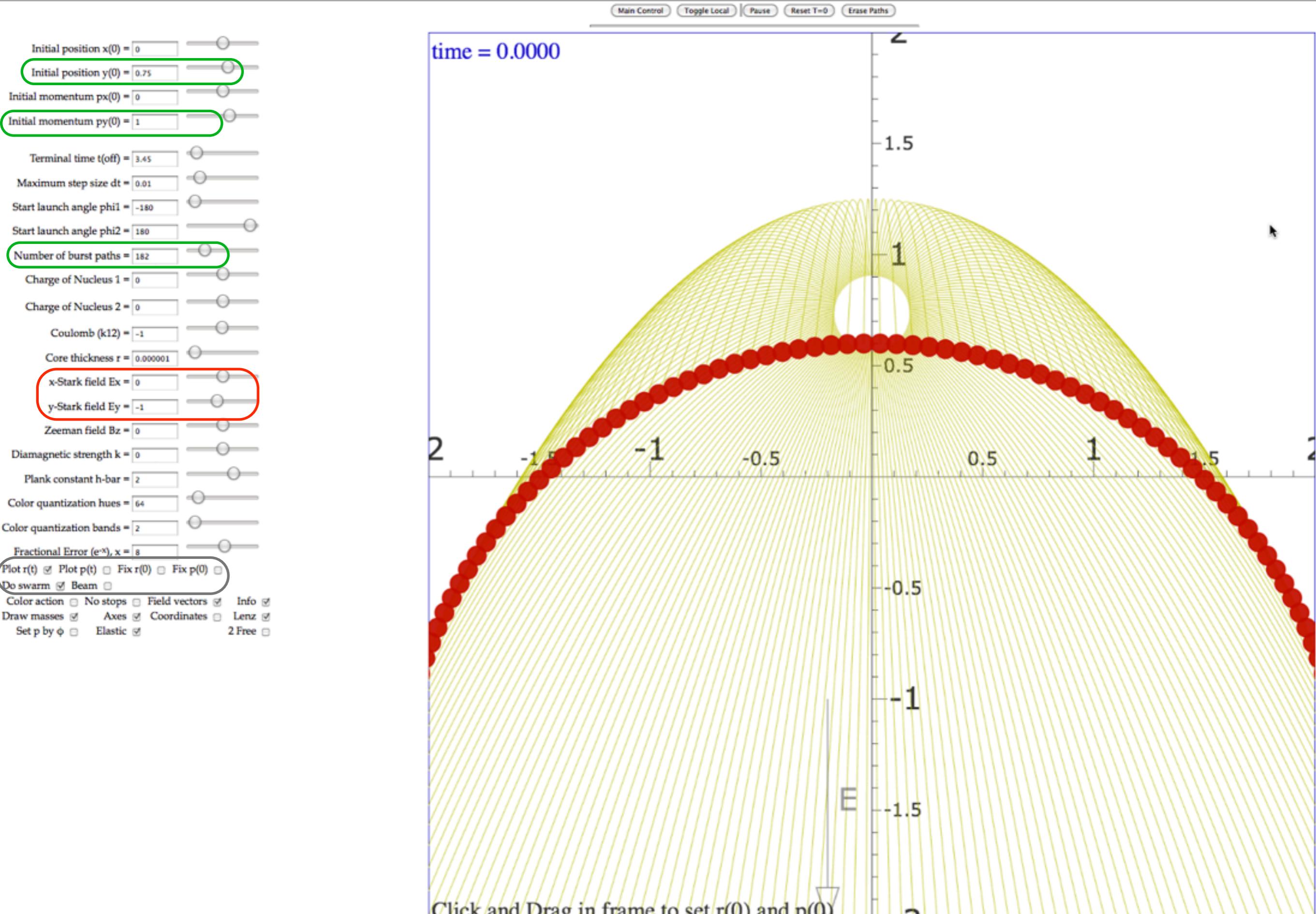
Huygen’s contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

(“Color-Quantization” simulations: Davis-Heller “Color-Quantization” or “Classical Chromodynamics”)



*Parabolic ~~1,2D~~ HHO orbital envelopes (Review of Lecture 9 p.56-81 and a generalization.)
Some clues for future assignments ([Web Simulation: CouIIt](#))*



[Link ⇒ CoulIt - Simulation of the Volcanoes of Io](#)

Parabolic orbital trajectory envelopes

then drops. When at $y=1.0\ldots$

Q1. ...where is its focus?

Q2. ...where is the blast wave? center falls as far as 90° ball rises

Q3. How high can $\alpha=45^\circ$ path rise ? $1/2$ as high

Q4. Where on x -axis does $\alpha=45^\circ$ path hit ? $x=2$

Q5. Where is blast wave then? centered on 45° normal

Q6 Where is $\alpha=45^\circ$ path focus? $(x=1, y=0)$

Q7 Guess for $\alpha=30^\circ$ path focus? $(x=1.73, y=0)$

and its focus? directrix?

Q7 Where is $\alpha=45^\circ$ "kite" geometry?

Q8 Where is $\alpha=0^\circ$ path focus?

directrix?

Focus of envelope

envelope
orbit trajectory

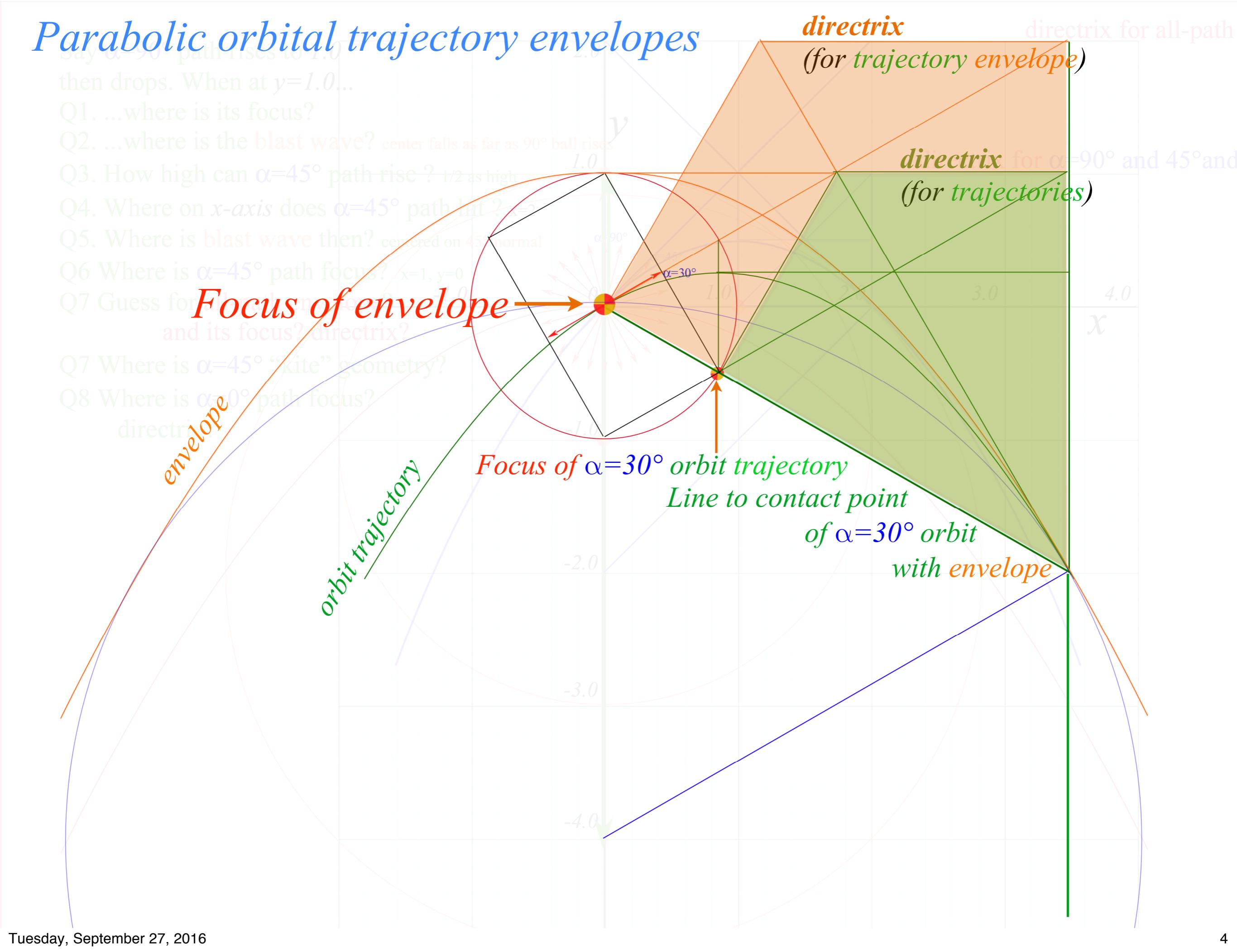
Focus of $\alpha=30^\circ$ orbit trajectory

Line to contact point

**of $\alpha=30^\circ$ orbit
with envelope**

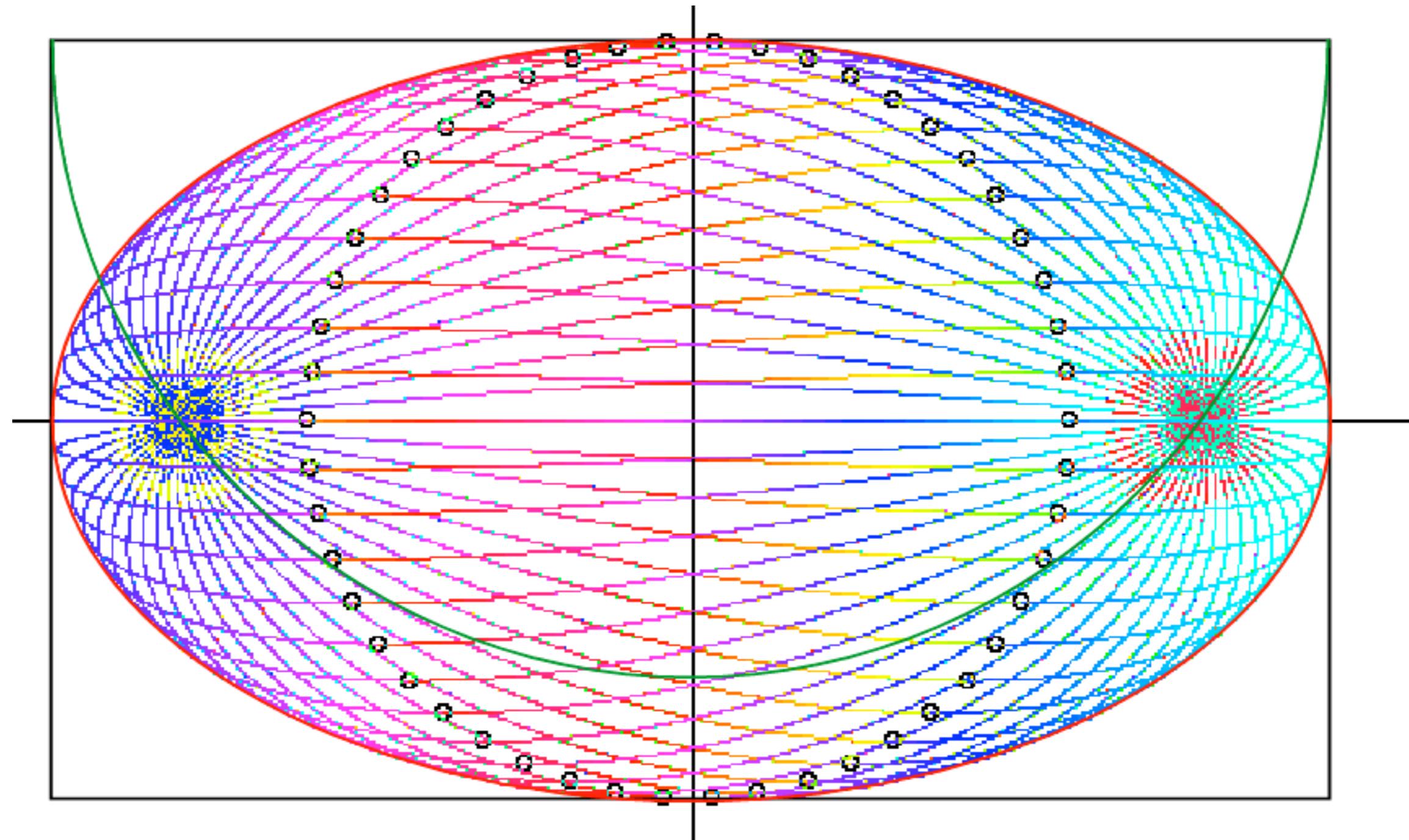
**directrix
(for trajectory envelope)**

**directrix
(for $\alpha=90^\circ$ and 45° and
trajectories)**



Parallel to 2D-IHO orbital envelopes (~~Review of Lecture 9, 15, 21 and a generalization.~~)
Some clues for future assignments ([Web Simulation: CouIIt](#))

Exploding-starlet elliptical envelope and contacting elliptical trajectories



*(Web Simulation: CouIIt - Exploding*Starlet {IHO Potential})*

Initial position $x(0) = 1$

Initial position $y(0) = 0$

Initial momentum $p_x(0) = 0$

Initial momentum $p_y(0) = 1$

Terminal time $t(\text{off}) = 3.45$

Maximum step size $dt = 0.01$

Start launch angle $\phi_1 = -180$

Start launch angle $\phi_2 = 180$

Number of burst paths = 51

Charge of Nucleus 1 = 0

Charge of Nucleus 2 = 0

Coulomb (k_{12}) = 0

Core thickness $r = 0.000001$

x-Stark field $E_x = 0$

y-Stark field $E_y = 0$

Zeeman field $B_z = 0$

Diamagnetic strength $k = -0.638$

Plank constant $h\bar{}$ = 2

Color quantization hues = 64

Color quantization bands = 2

Fractional Error (e^{-x}), $x = 8$

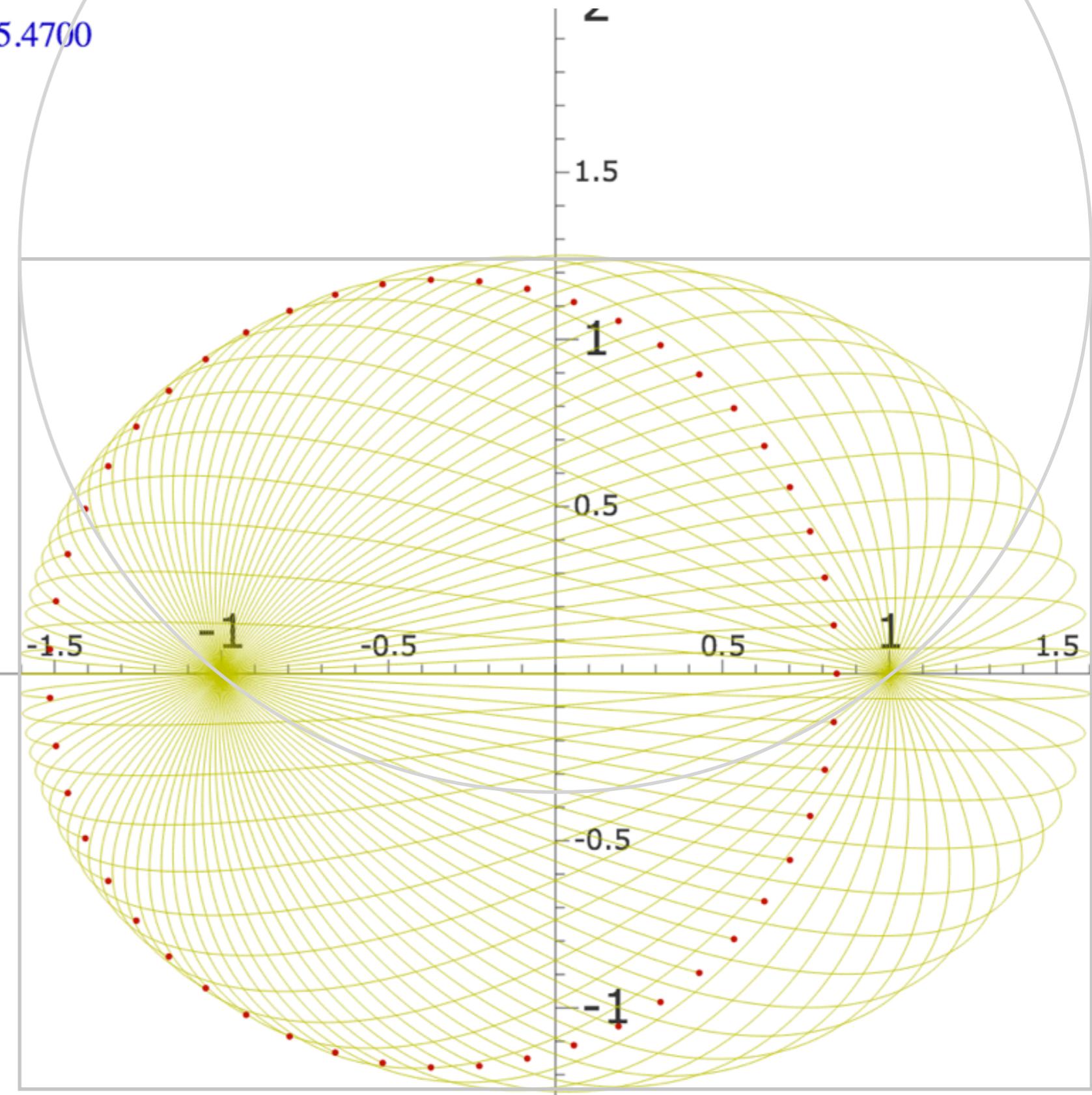
Particle Size = 2

Fix $r(0)$ Fix $p(0)$ Do swarm Beam

Plot $r(t)$ Plot $p(t)$

Color action No stops Field vectors Info
 Draw masses Axes Coordinates Lenz
 Set p by ϕ Elastic 2 Free
 Save to GIF

time = 5.4700



(Web Simulation: CouIIt - Exploding*Starlet {IHO Potential})

Examples of Hamiltonian mechanics in phase plots



1D Pendulum and phase plot (Web Simulations: [Pendulum](#), [Cycloidulum](#), [JerkIt](#) (Vert Driven Pendulum))

Circular pendulum dynamics and elliptic functions

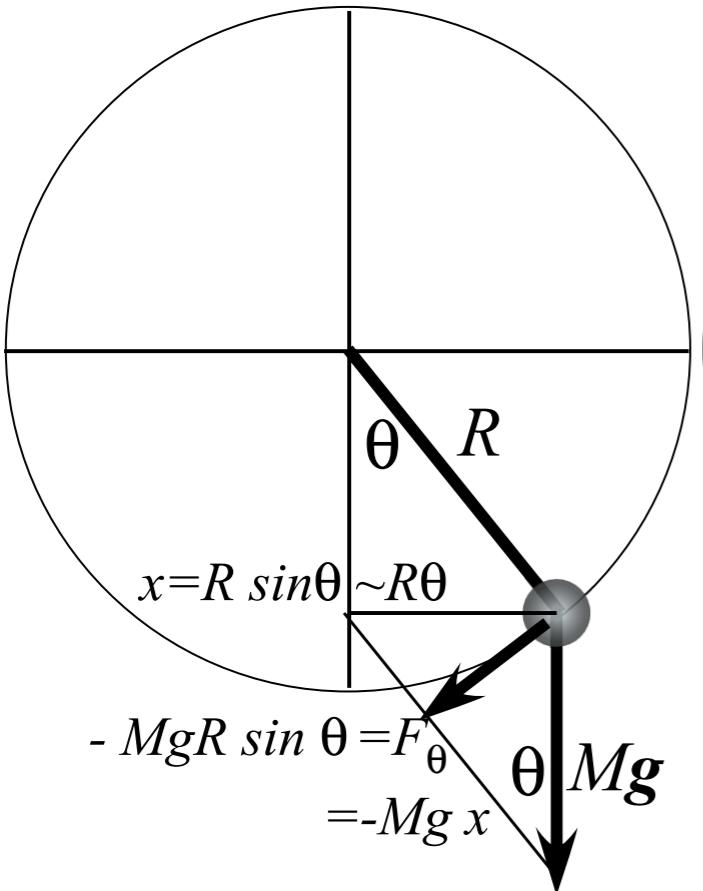
Cycloid pendulum dynamics and “sawtooth” functions

1D-HO phase-space control (Old Mac OS & [Web Simulations](#) of “Catcher in the Eye”)

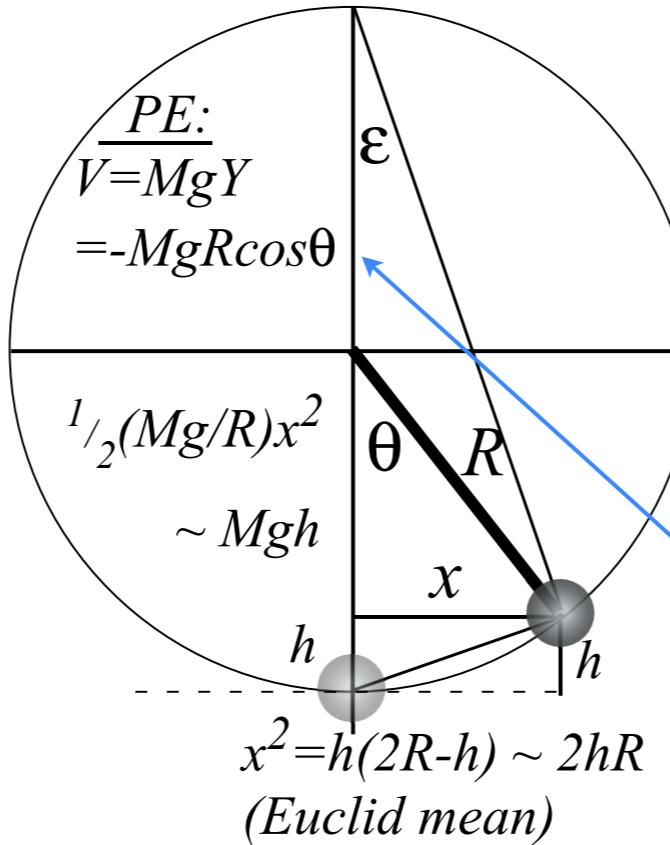
1D Pendulum and phase plot

(Unit 2 Chapter 7 Fig. 2)

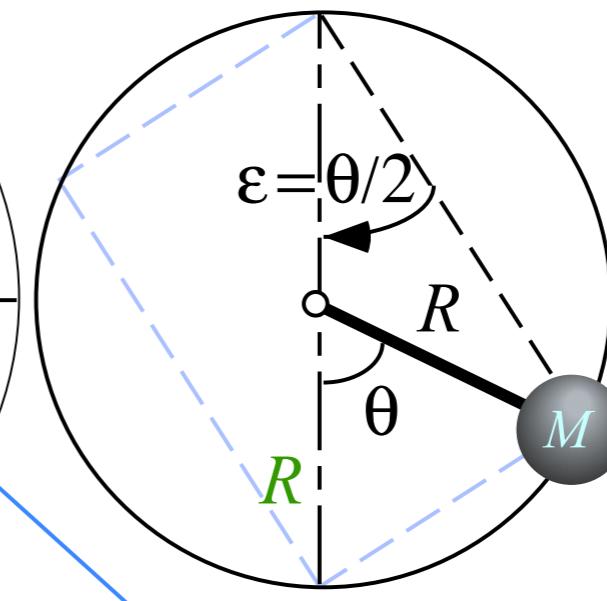
(a) Force geometry



(b) Energy geometry



(c) Time geometry



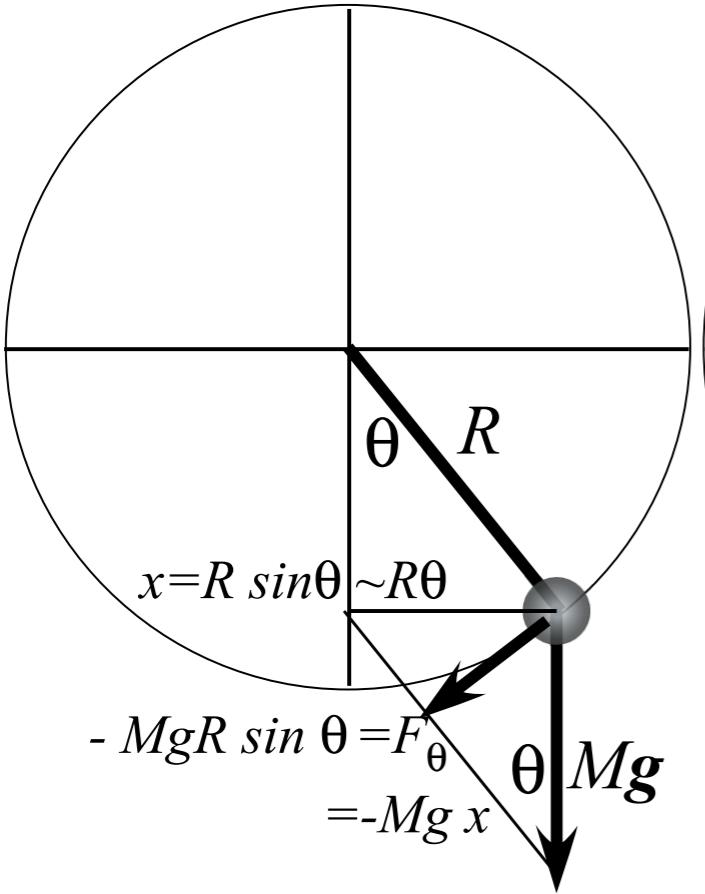
NOTE: Very common loci of ± sign blunders

Lagrangian function $L = KE - PE = T - U$ where potential energy is $U(\theta) = -MgR \cos \theta$

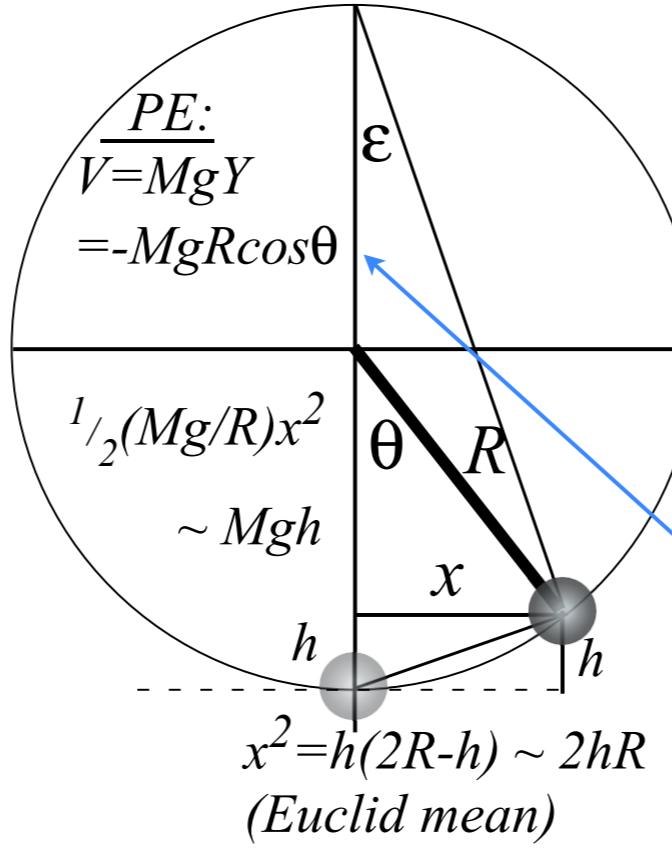
$$L(\dot{\theta}, \theta) = \frac{1}{2} I \dot{\theta}^2 - U(\theta) = \frac{1}{2} I \dot{\theta}^2 + MgR \cos \theta$$

1D Pendulum and phase plot

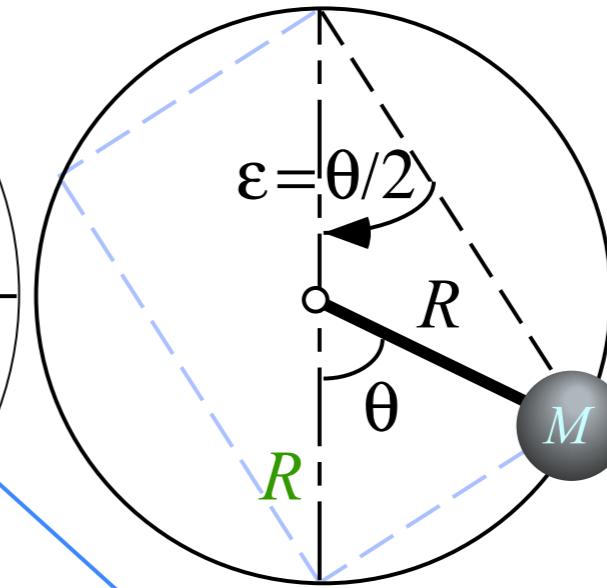
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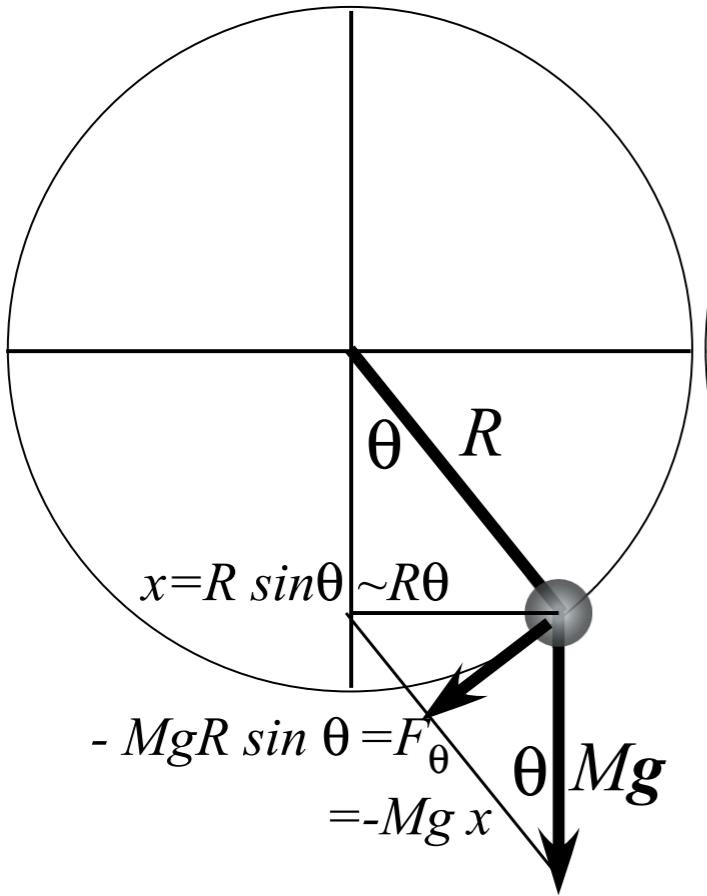
$$L(\dot{\theta}, \theta) = \frac{1}{2} I \dot{\theta}^2 - U(\theta) = \frac{1}{2} I \dot{\theta}^2 + MgR \cos \theta$$

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

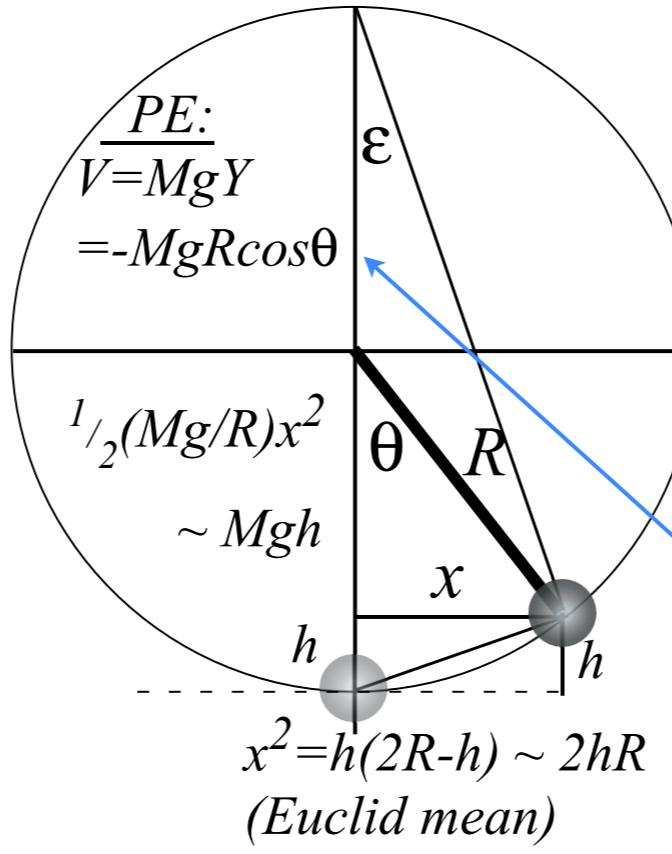
$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.}$$

1D Pendulum and phase plot

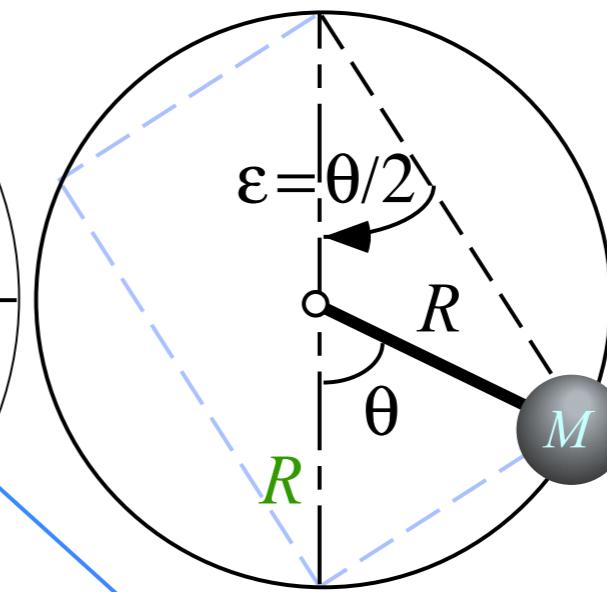
(a) Force geometry



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(c) Time geometry



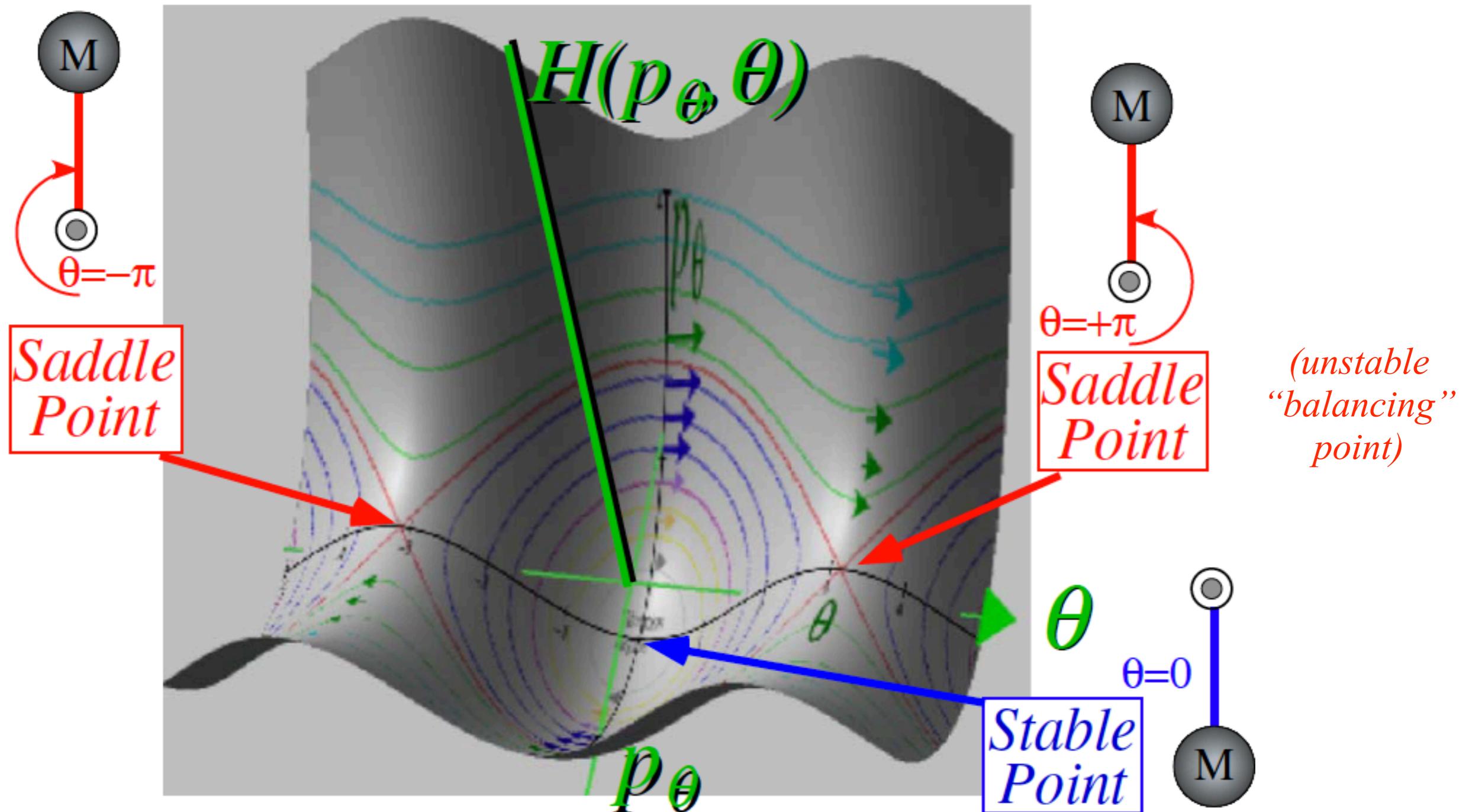
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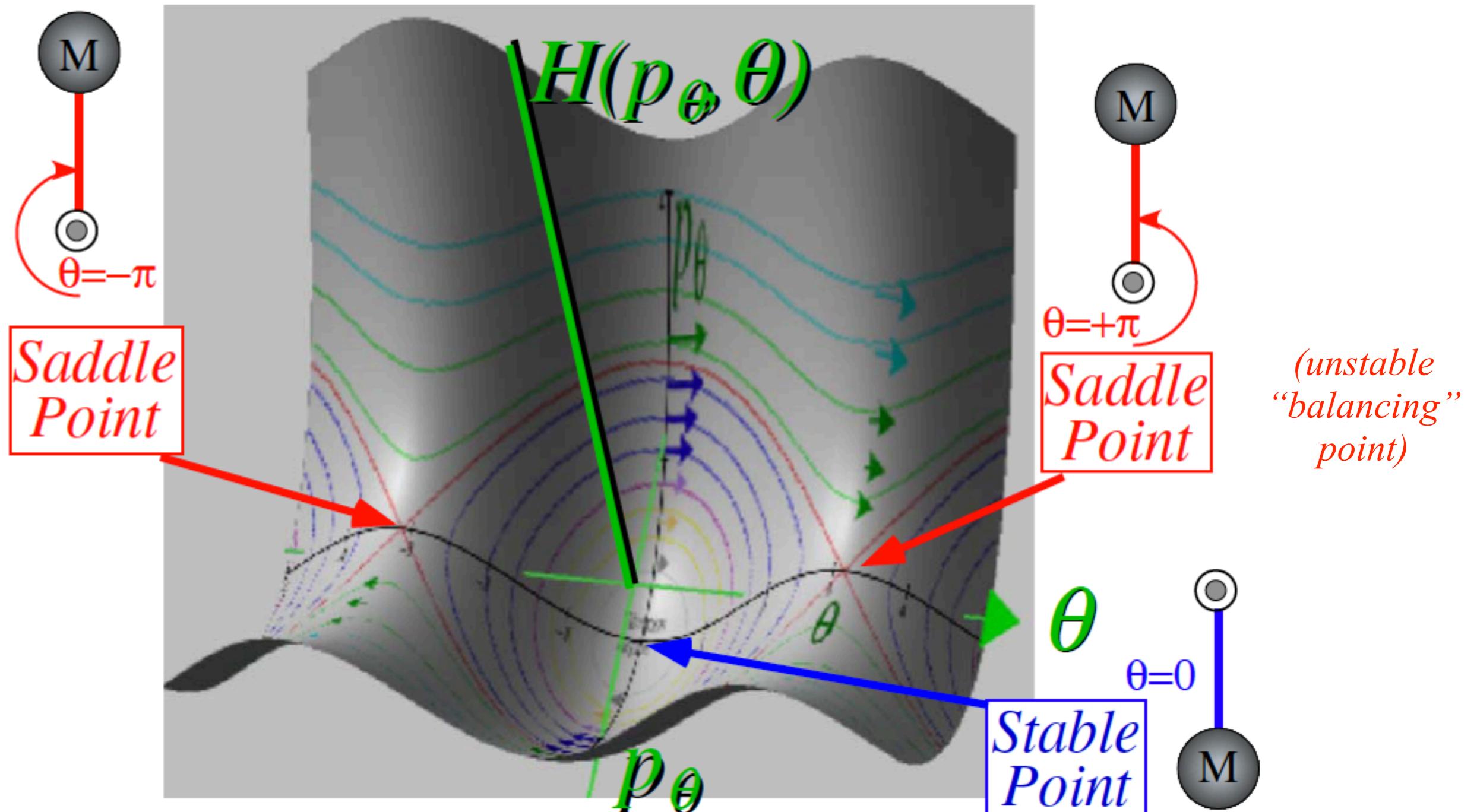
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implies: $p_\theta = \sqrt{2I(E + MgR \cos \theta)}$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ, p_θ)

$$H(p_\theta, \theta) = E = \frac{1}{2I} p_\theta^2 - MgR \cos \theta, \quad \text{or: } p_\theta = \sqrt{2I(E + MgR \cos \theta)}$$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ, p_θ)

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Funny way to look at Hamilton's equations:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix} = \mathbf{e}_H \times (-\nabla H) = (\overrightarrow{\text{H-axis}}) \times (\overrightarrow{\text{fall line}}), \text{ where: } \begin{cases} (\overrightarrow{\text{H-axis}}) = \mathbf{e}_H = \mathbf{e}_q \times \mathbf{e}_p \\ (\overrightarrow{\text{fall line}}) = -\nabla H \end{cases}$$

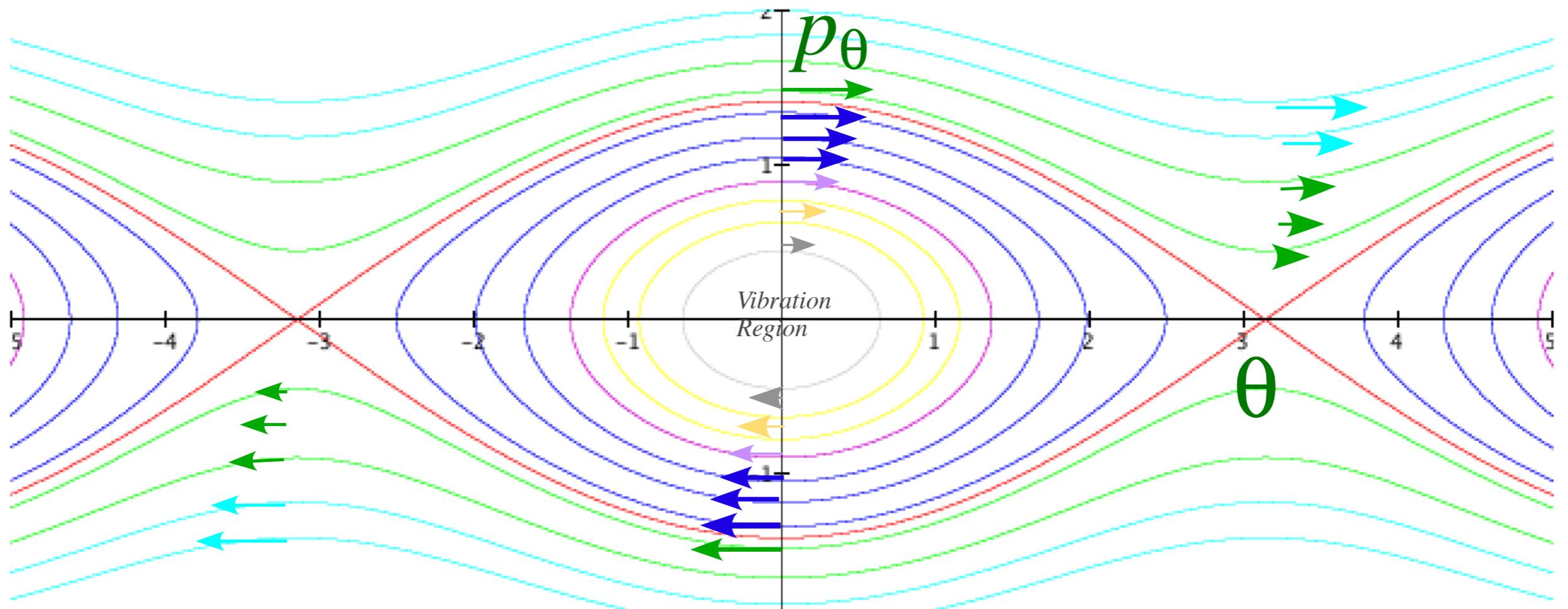


Fig. 2.7.2 Phase portrait or topography map for simple pendulum

(Unit 2 Chapter 7 Fig. 2)

Examples of Hamiltonian mechanics in phase plots

- *1D Pendulum and phase plot (Web Simulations: [Pendulum](#), [Cycloidulum](#), [JerkIt](#) (Vert Driven Pendulum))*
- *Circular pendulum dynamics and elliptic functions*
- *Cycloid pendulum dynamics and “sawtooth” functions*
- *1D-HO phase-space control (Old Mac OS & [Web Simulations](#) of “Catcher in the Eye”)*

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.} \quad \text{implies: } p_\theta = \sqrt{2I(E + MgR \cos \theta)}$$

Let $E = MgY = -MgR \cos \theta_0$ be potential energy where $KE = 0$ or $p_\theta = 0$

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$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E + MgR \cos \theta)} / I \quad \text{where: } I = MR^2$$

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Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time 0 to } \theta_0) = \tau_{1/4}$$

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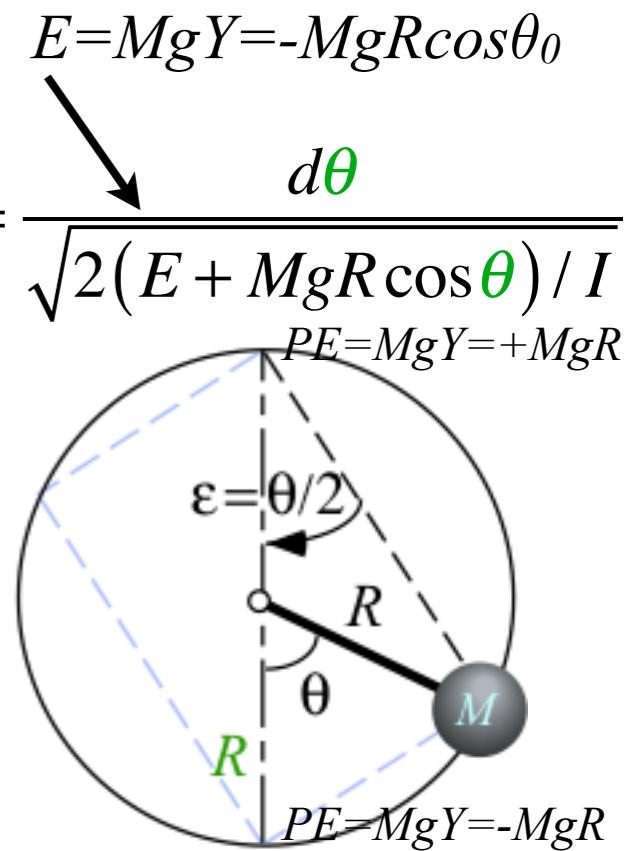
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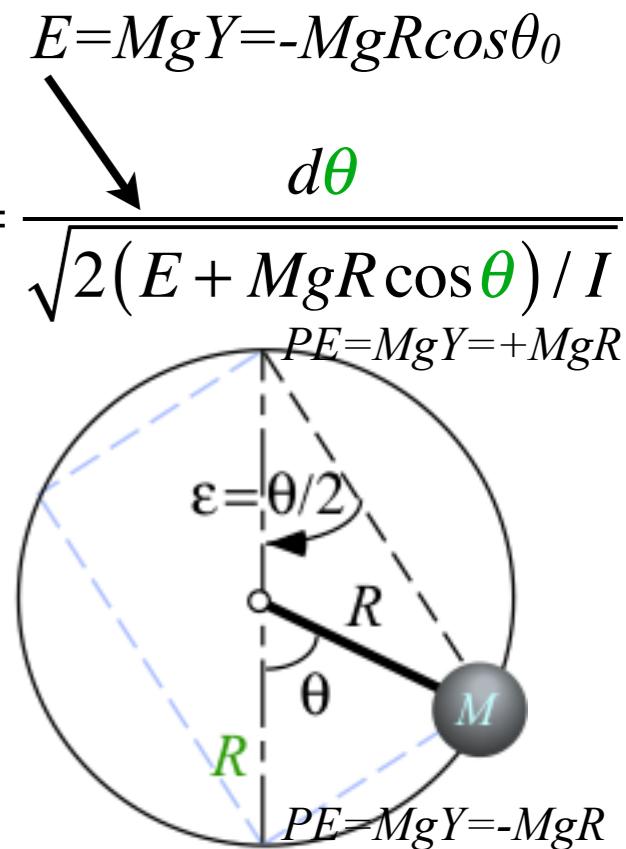
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Circular pendulum dynamics and elliptic functions

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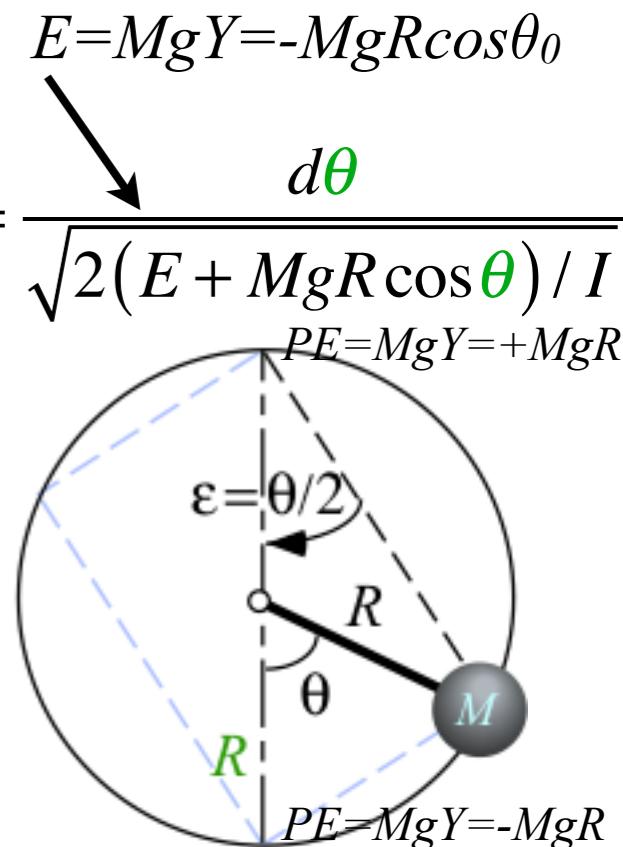
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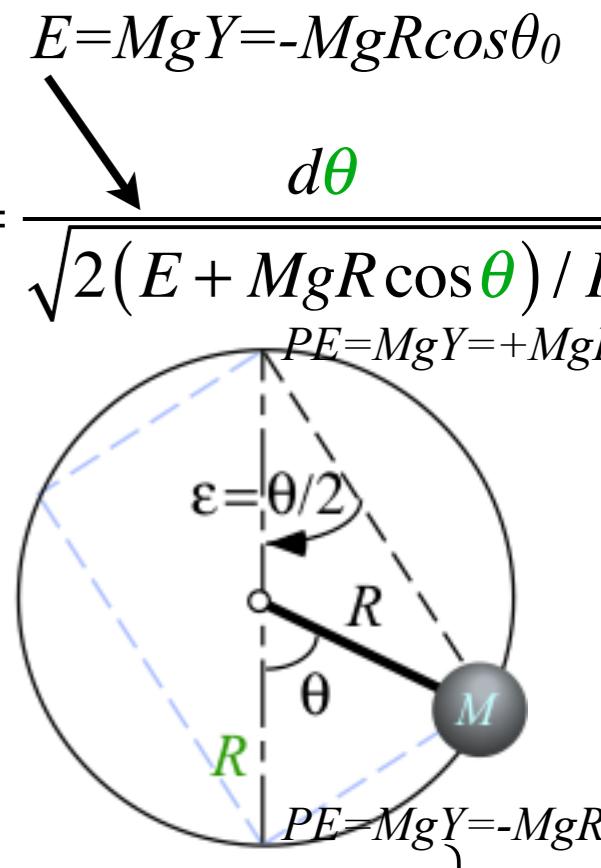
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Circular pendulum dynamics and elliptic functions

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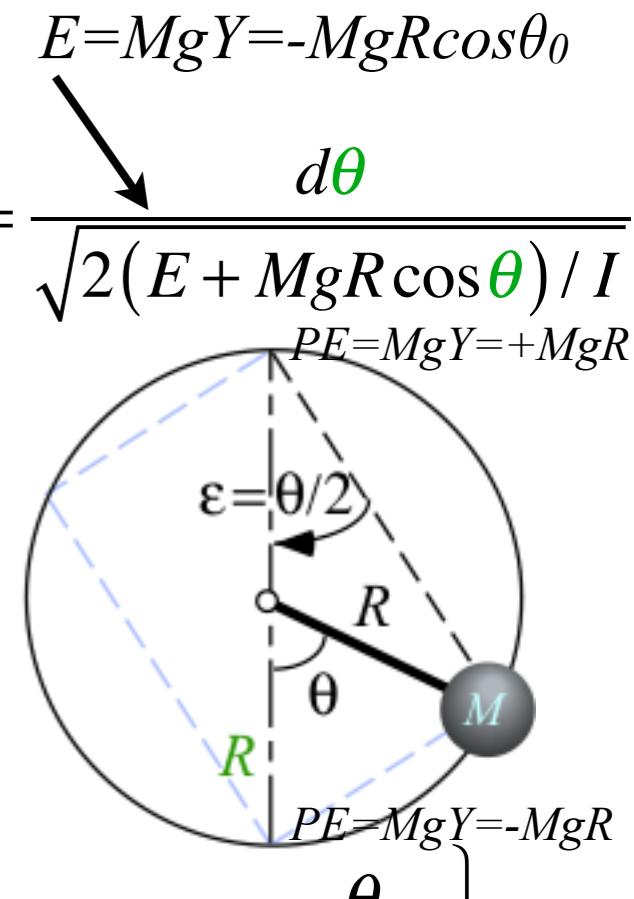
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The integral is an *elliptic integral of the first kind*: $F(k, \varepsilon_0) = am^{-1}$ or the "inverse amu" function.

$$F(k, \varepsilon_0) \equiv \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{1 - k^2 \sin^2 \varepsilon}} \equiv am^{-1}(k, \varepsilon_0)$$



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Uses a half-angle coordinate $\varepsilon = \theta/2$

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - 2 \sin^2 \varepsilon, \quad \cos \theta - \cos \theta_0 = 2 \sin^2 \varepsilon_0 - 2 \sin^2 \varepsilon$$

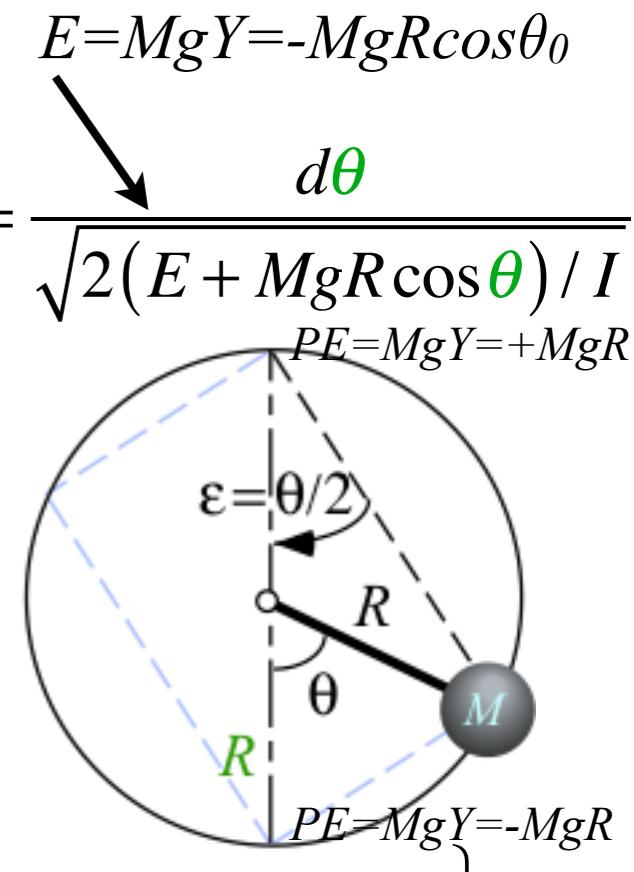
$$\tau_{1/4} = \sqrt{\frac{I}{MgR}} \int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\sin^2 \varepsilon_0 - \sin^2 \varepsilon}} = \sqrt{\frac{R}{g}} \int_0^{\varepsilon_0} \frac{k d\varepsilon}{\sqrt{1 - k^2 \sin^2 \varepsilon}}, \text{ where: } \left\{ \begin{array}{l} 1/k = \sin \varepsilon_0 = \sin \frac{\theta_0}{2} \\ I = MR^2 \end{array} \right.$$

The integral is an *elliptic integral of the first kind*: $F(k, \varepsilon_0) = am^{-1}$ or the "inverse amu" function.

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For low amplitude $\varepsilon \ll 1$: $\sin \varepsilon_0 \approx \varepsilon_0$ reduces $\tau_{1/4}$ to $\tau \frac{2\pi}{4}$



Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR \cos \theta$

$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.} \quad \text{implies: } p_\theta = \sqrt{2I(E + MgR \cos \theta)}$$

Let $E = MgY = -MgR \cos \theta_0$ be potential energy where $KE = 0$ or $p_\theta = 0$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E + MgR \cos \theta)} / I \quad \text{where: } I = MR^2 \quad \text{or: } dt = \frac{d\theta}{\sqrt{2(E + MgR \cos \theta)} / I}$$

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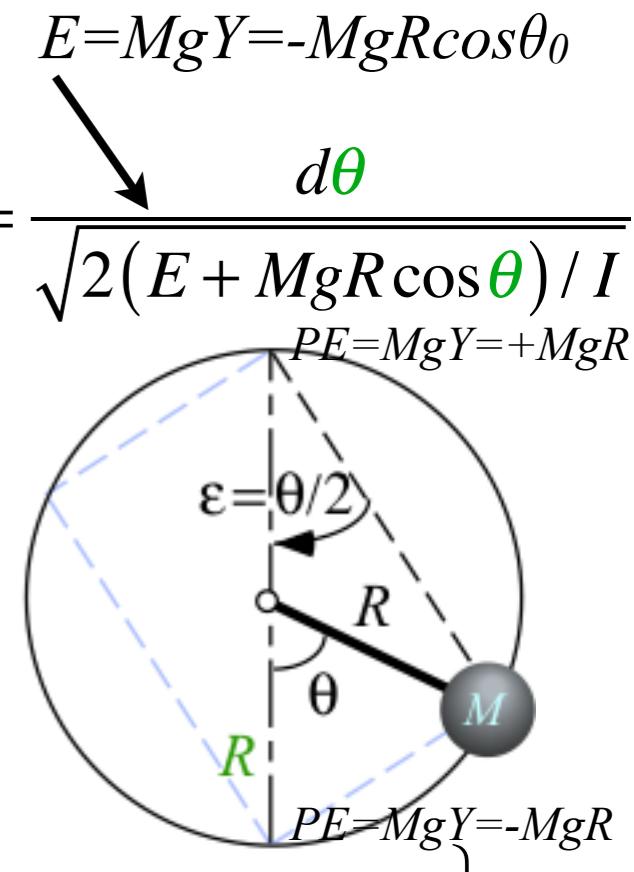
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$$\text{low } \varepsilon \ll 1: t = \sqrt{\frac{R}{g}} \int_0^{\varepsilon(t)} \frac{d\varepsilon}{\sqrt{\varepsilon_0^2 - \varepsilon^2}} = \sqrt{\frac{R}{g}} \sin^{-1} \frac{\varepsilon}{\varepsilon_0} \Big|_0^{\varepsilon(t)} = \sqrt{\frac{R}{g}} \sin^{-1} \frac{\varepsilon(t)}{\varepsilon_0} \quad \text{For low amplitude } \varepsilon \ll 1: \sin \varepsilon_0 \approx \varepsilon_0 \text{ reduces } \tau_{1/4} \text{ to } \tau \frac{2\pi}{4}$$



$$\left. \begin{array}{l} 1/k = \sin \varepsilon_0 = \sin \frac{\theta_0}{2} \\ I = MR^2 \end{array} \right\}$$

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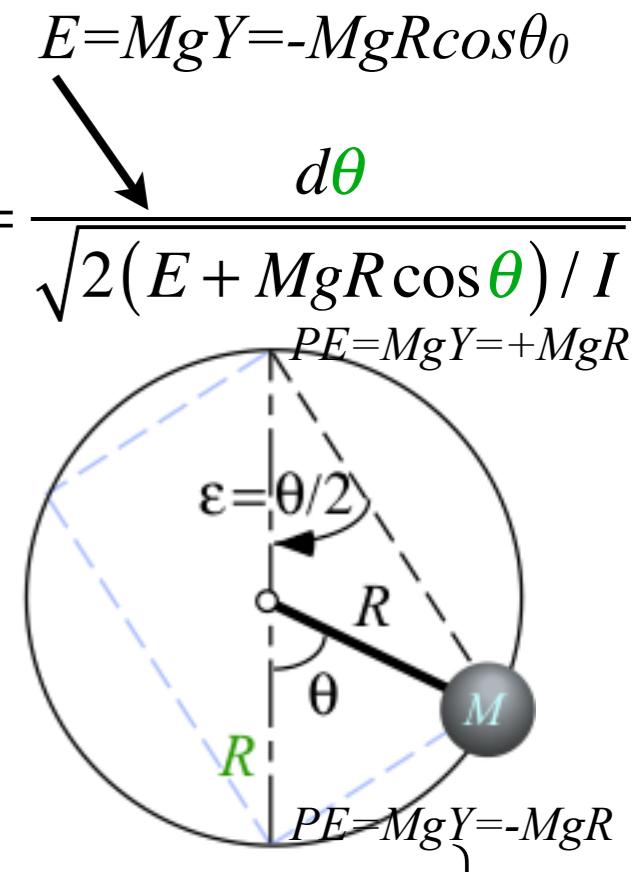
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..reduces to sine...

$$\varepsilon(t) = \varepsilon_0 \sin \sqrt{\frac{g}{R}} t = \varepsilon_0 \sin \omega t, \text{ where: } \omega = \sqrt{\frac{g}{R}}$$

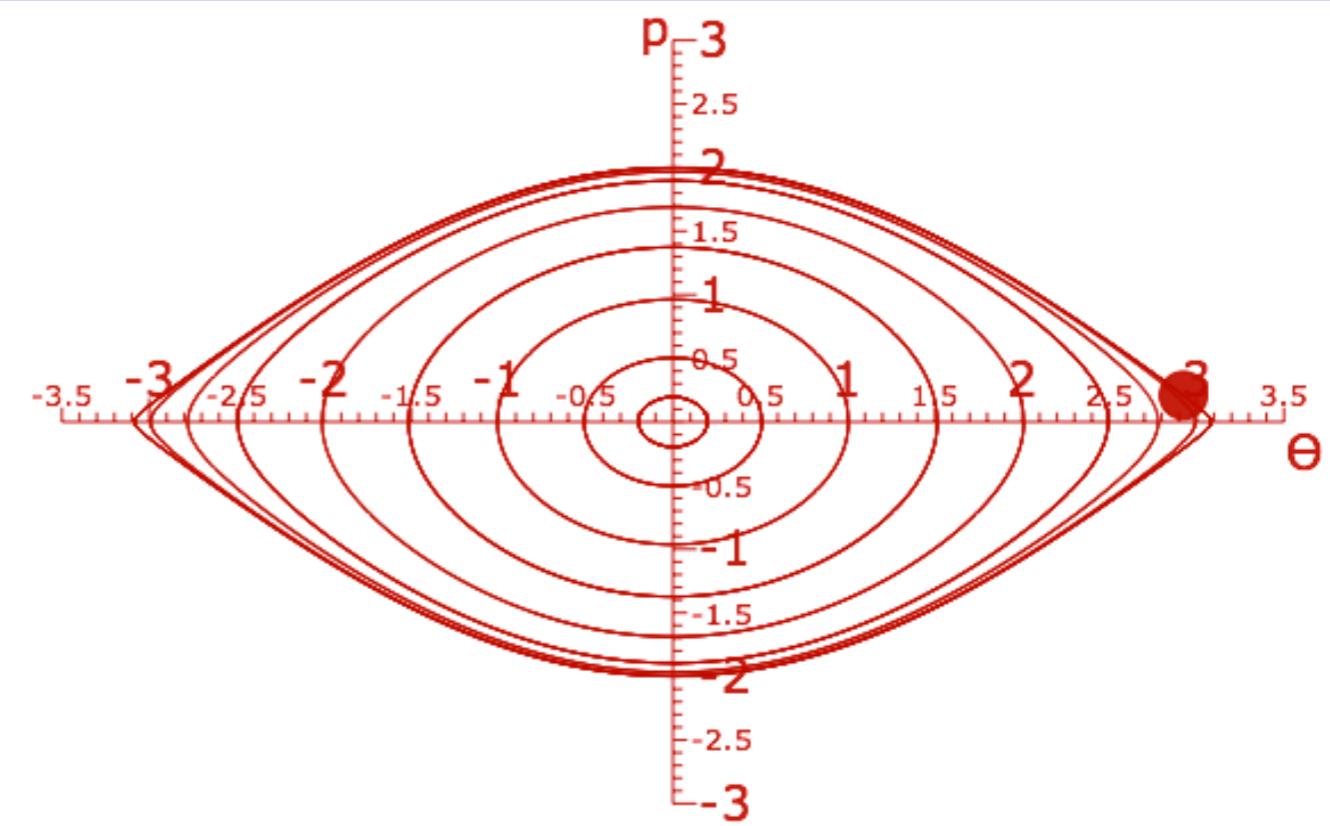
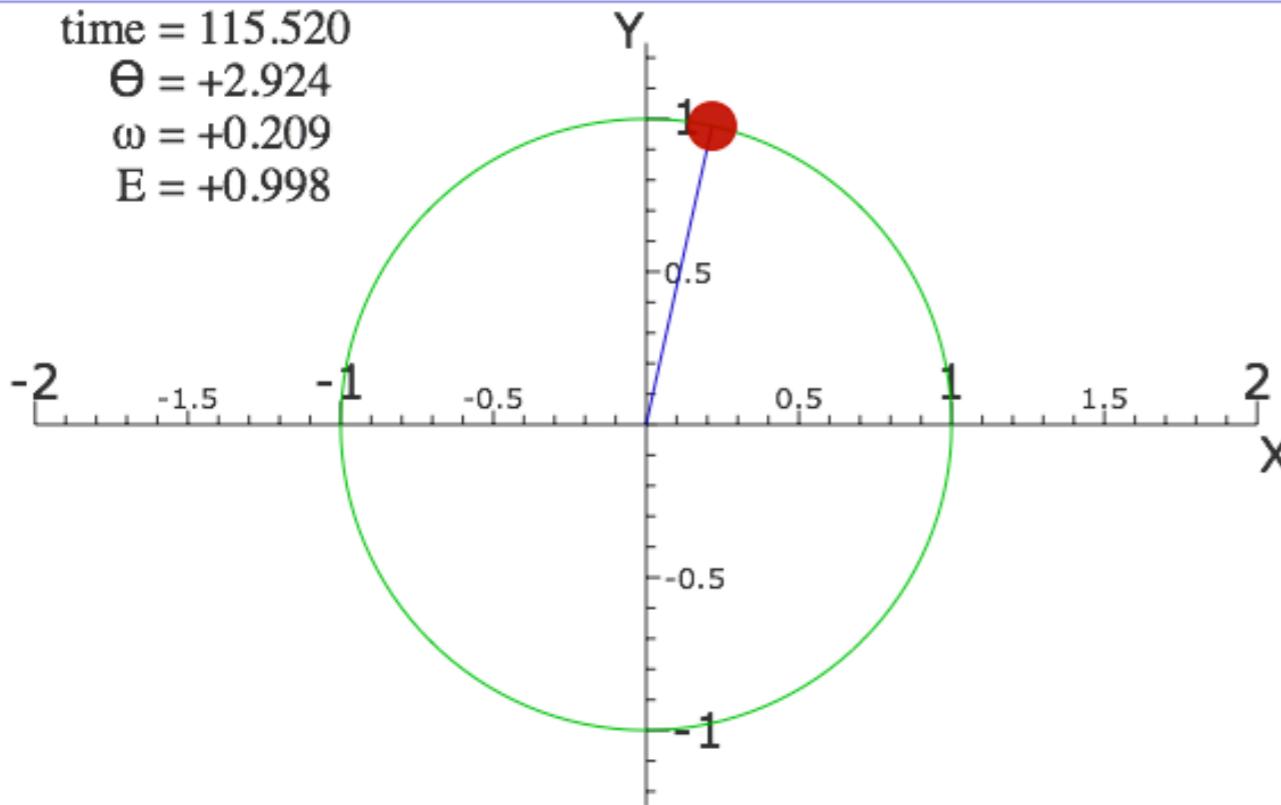
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Circular pendulum dynamics and elliptic functions

time = 115.520
 $\Theta = +2.924$
 $\omega = +0.209$
 $E = +0.998$

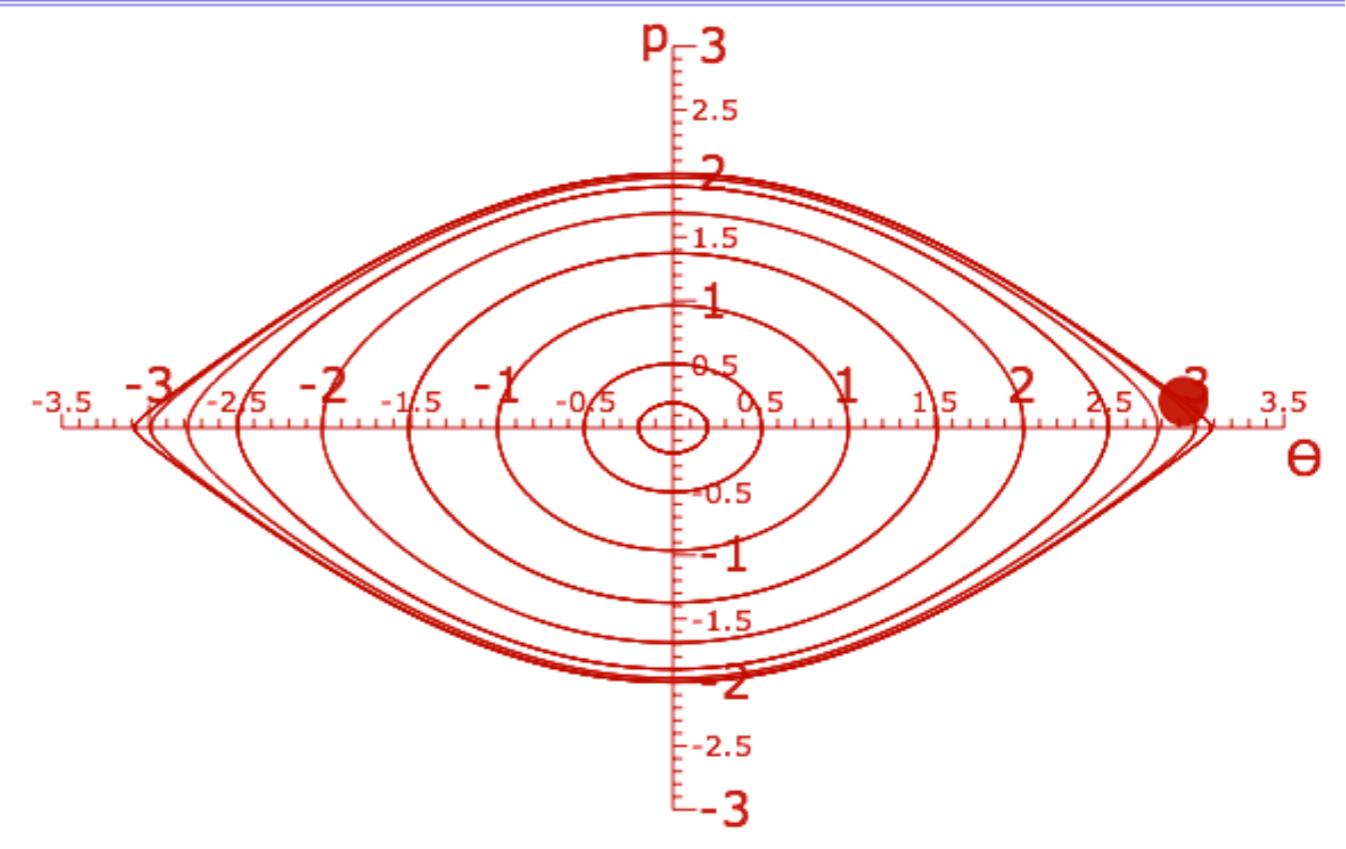
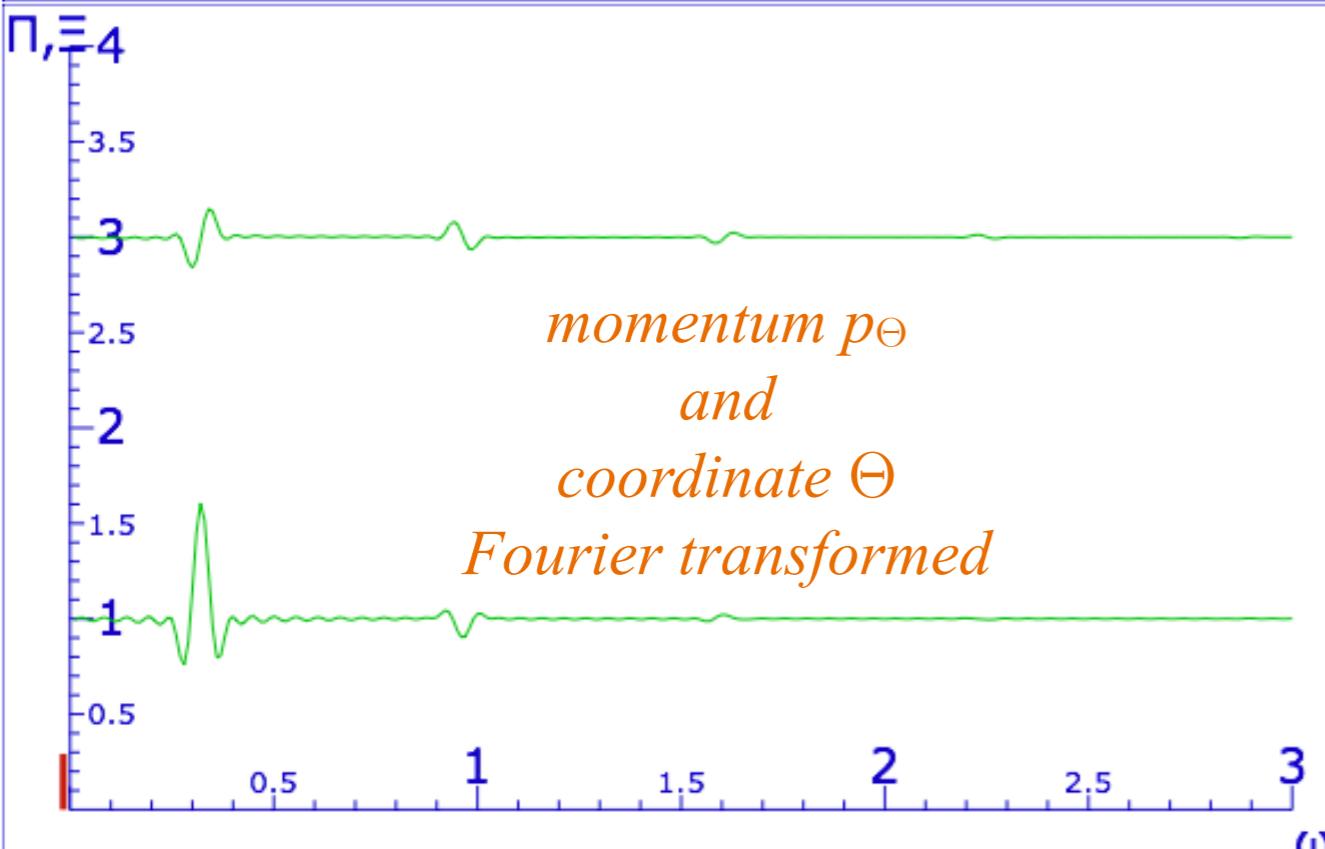
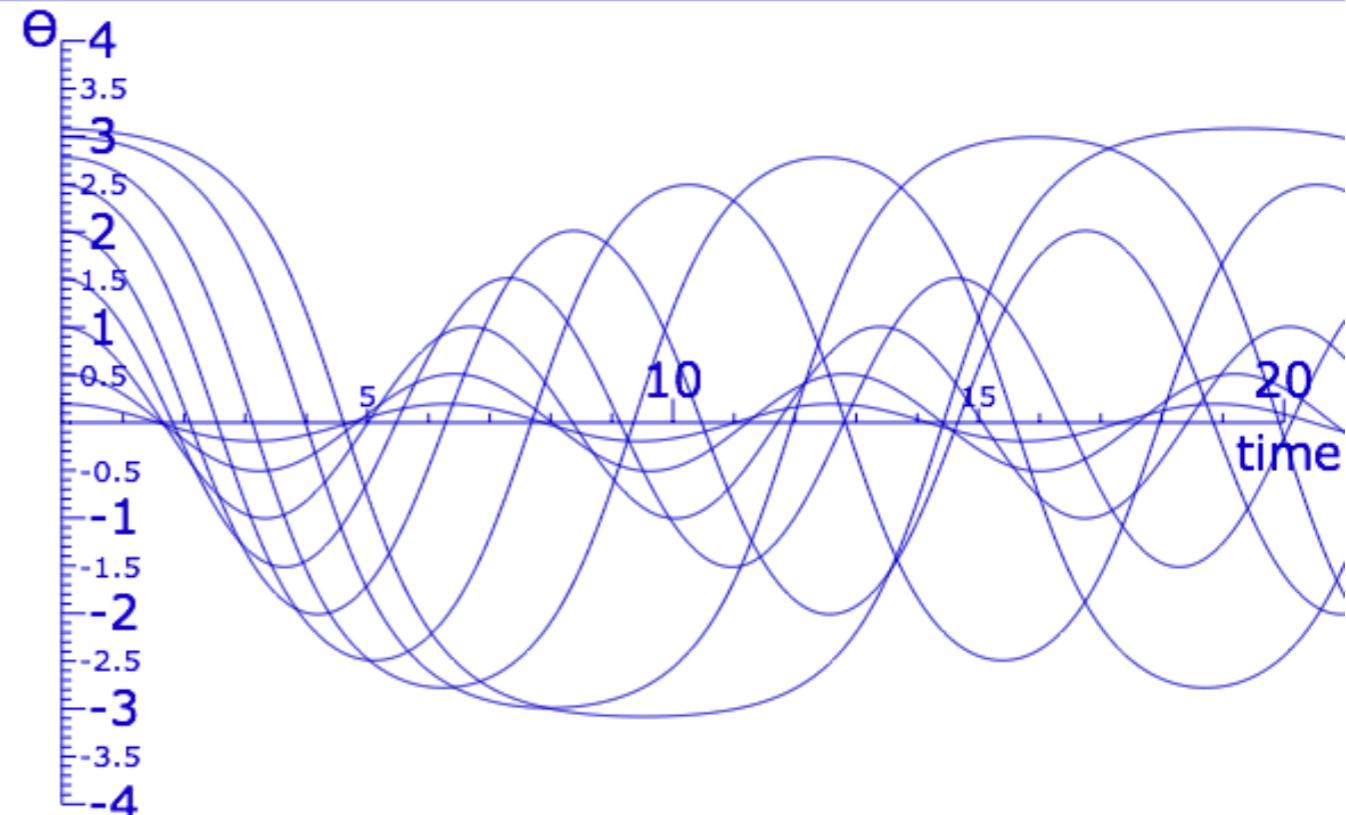
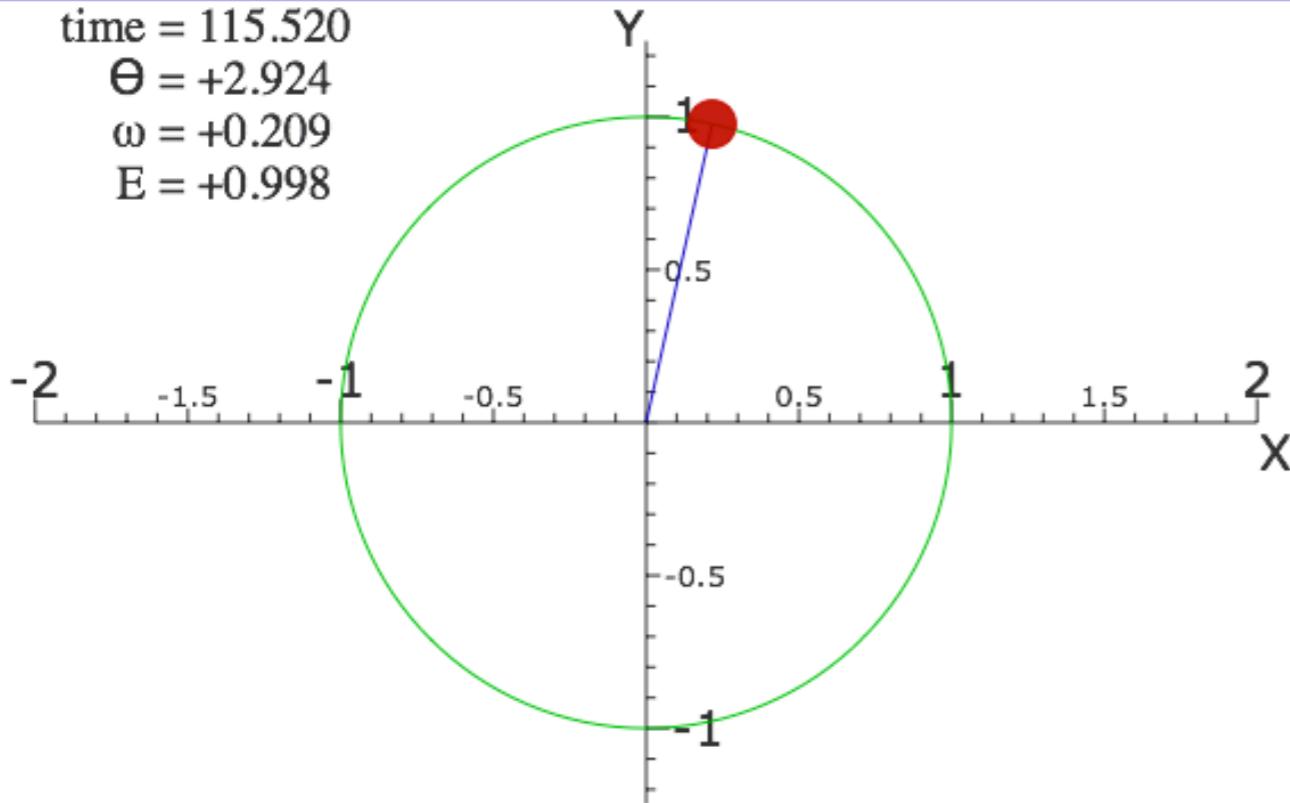


(Simulations of pendulum)

(See also: Simulation of cycloidally constrained pendulum)

Circular pendulum dynamics and elliptic functions

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Examples of Hamiltonian mechanics in phase plots

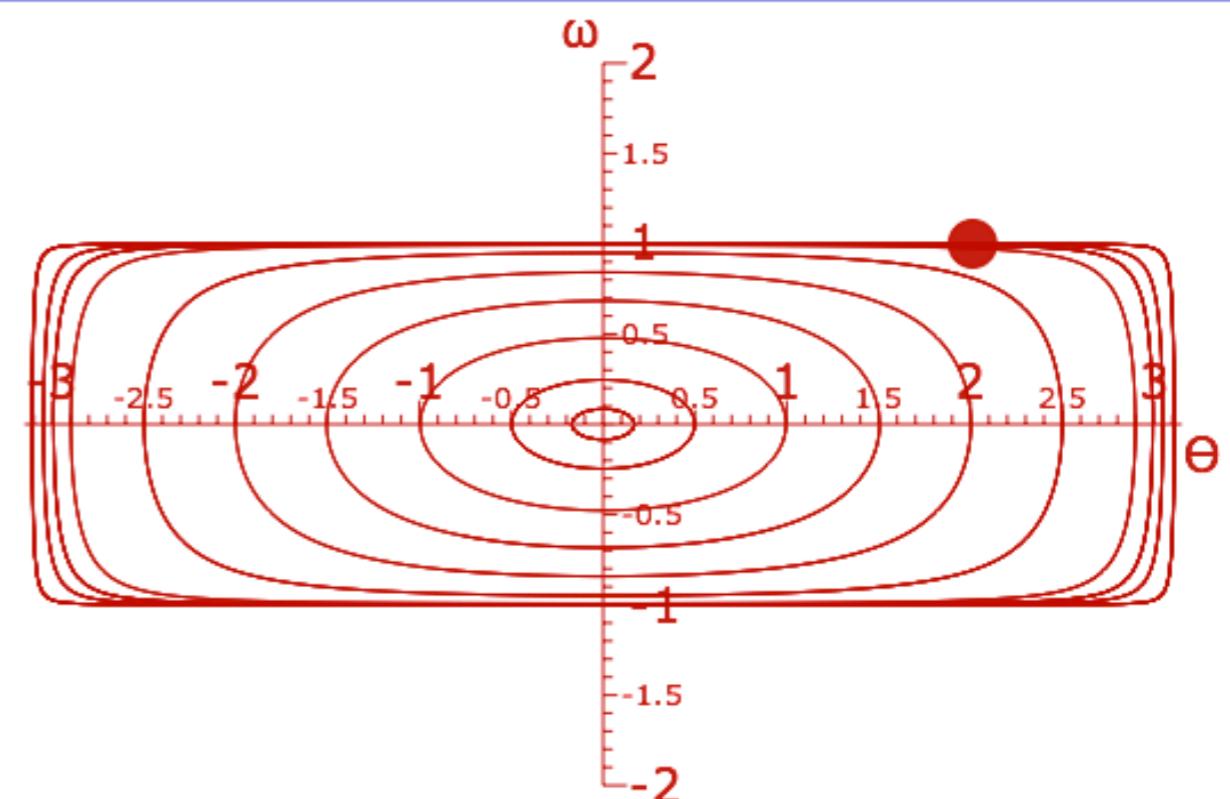
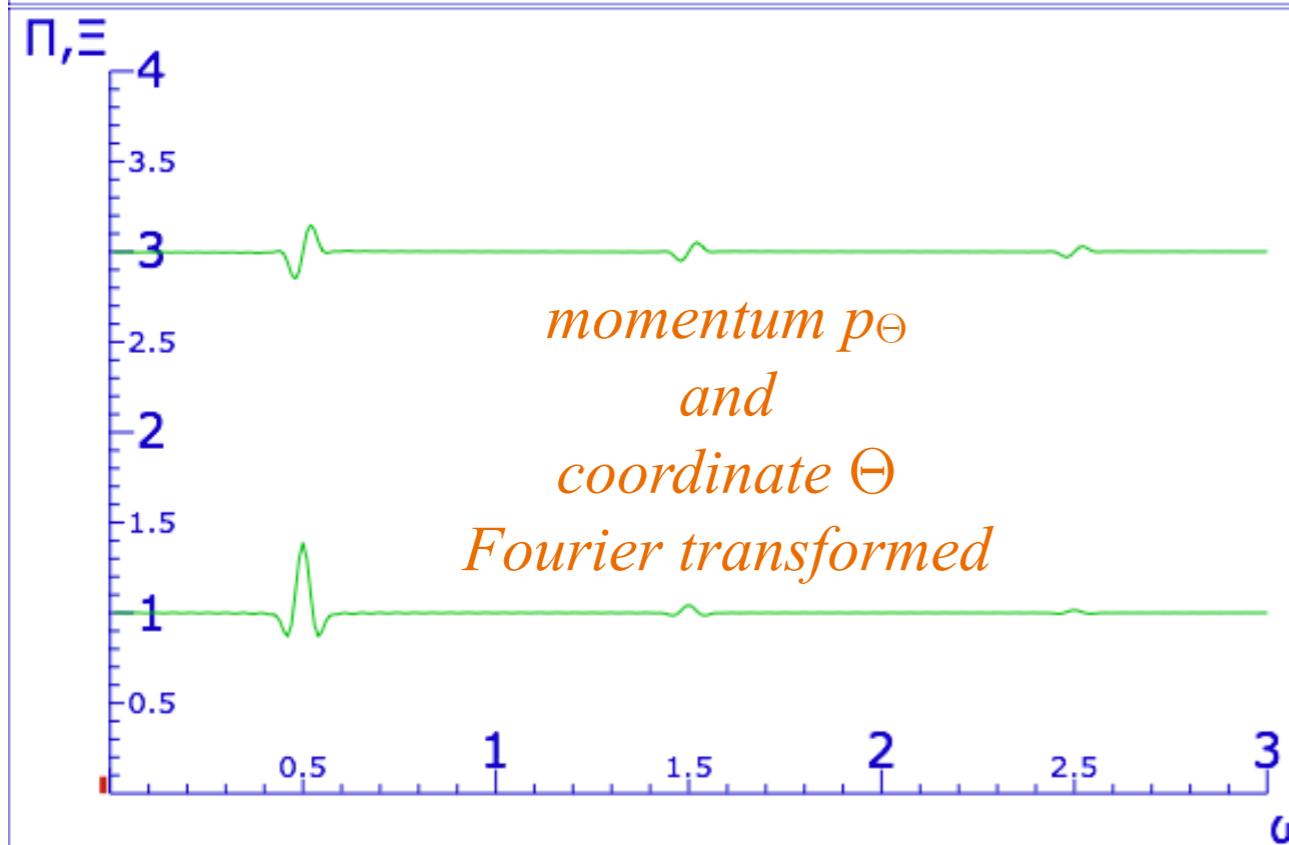
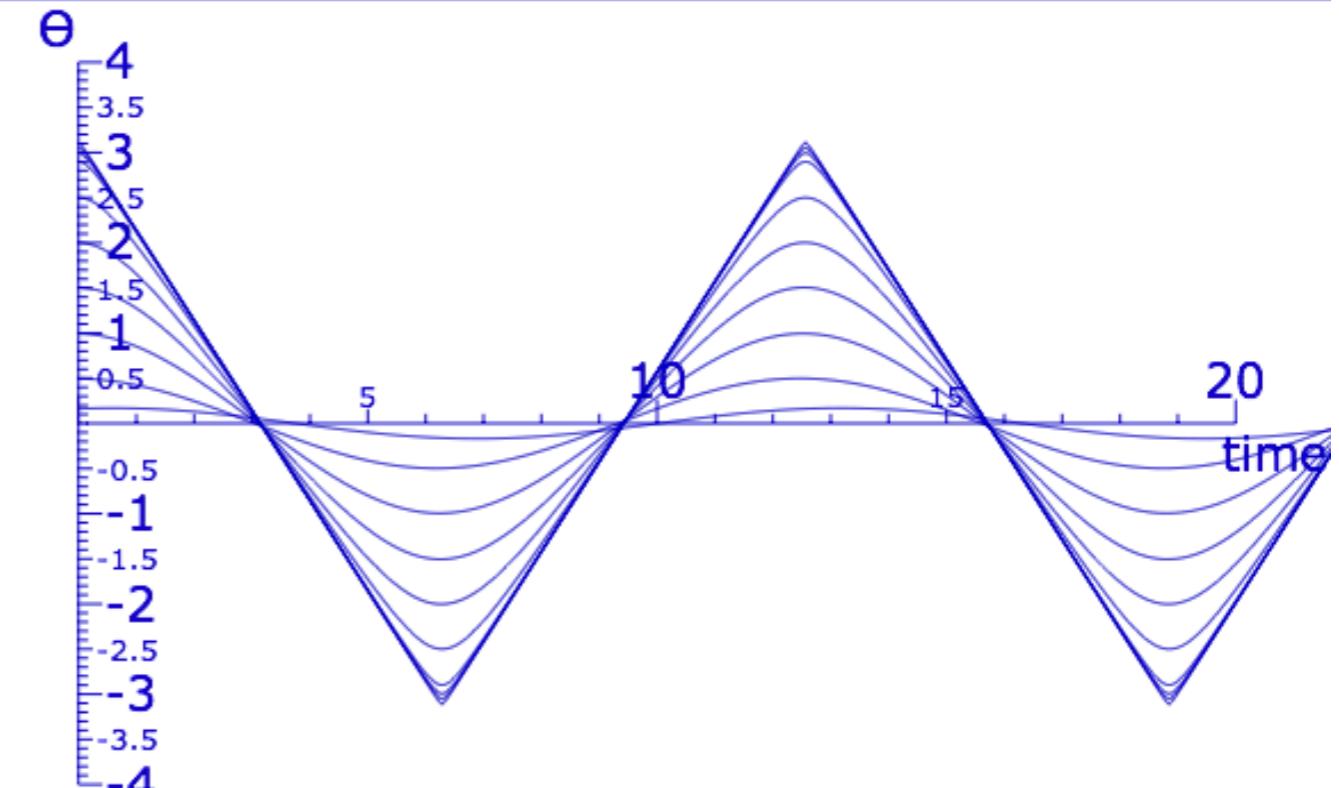
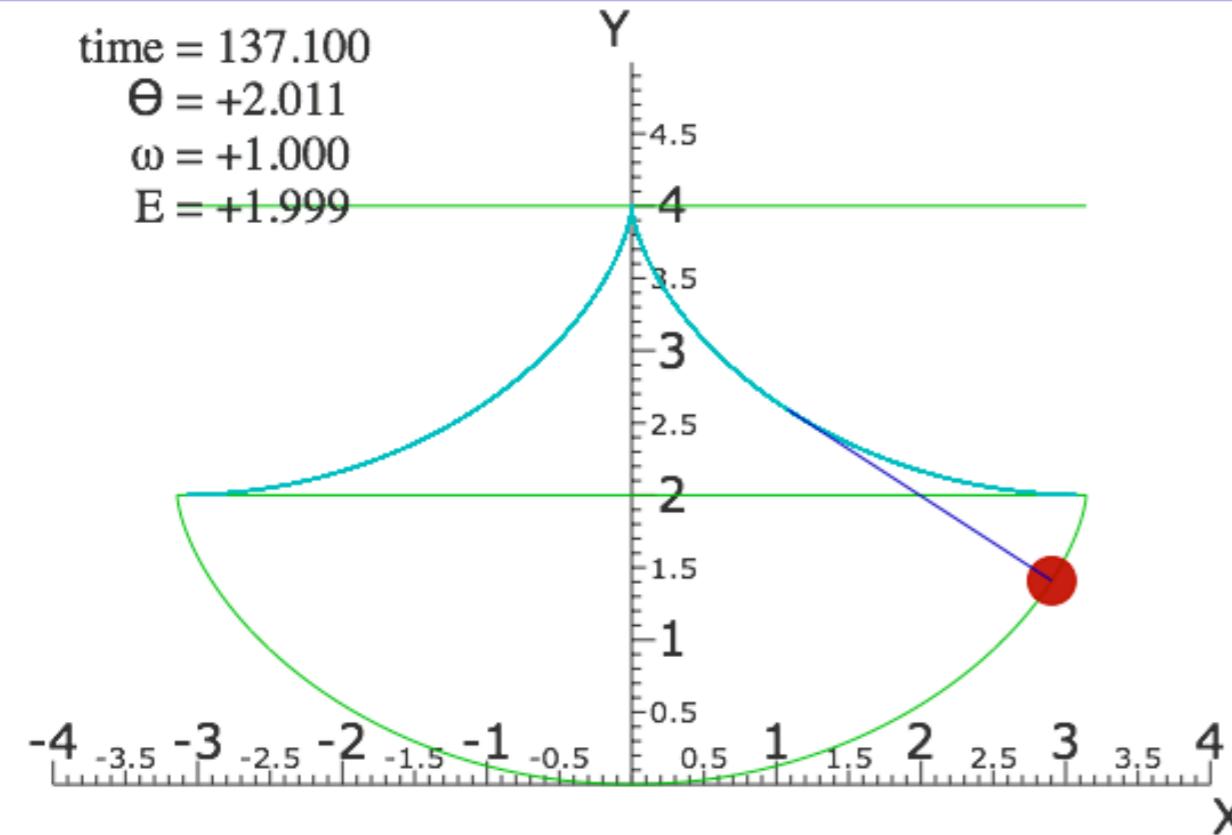
1D Pendulum and phase plot (Web Simulations: [Pendulum](#), [Cycloidulum](#), [JerkIt](#) (Vert Driven Pendulum))

Circular pendulum dynamics and elliptic functions

 *Cycloid pendulum dynamics and “sawtooth” functions*

1D-HO phase-space control (Old Mac OS & [Web Simulations](#) of “Catcher in the Eye”)

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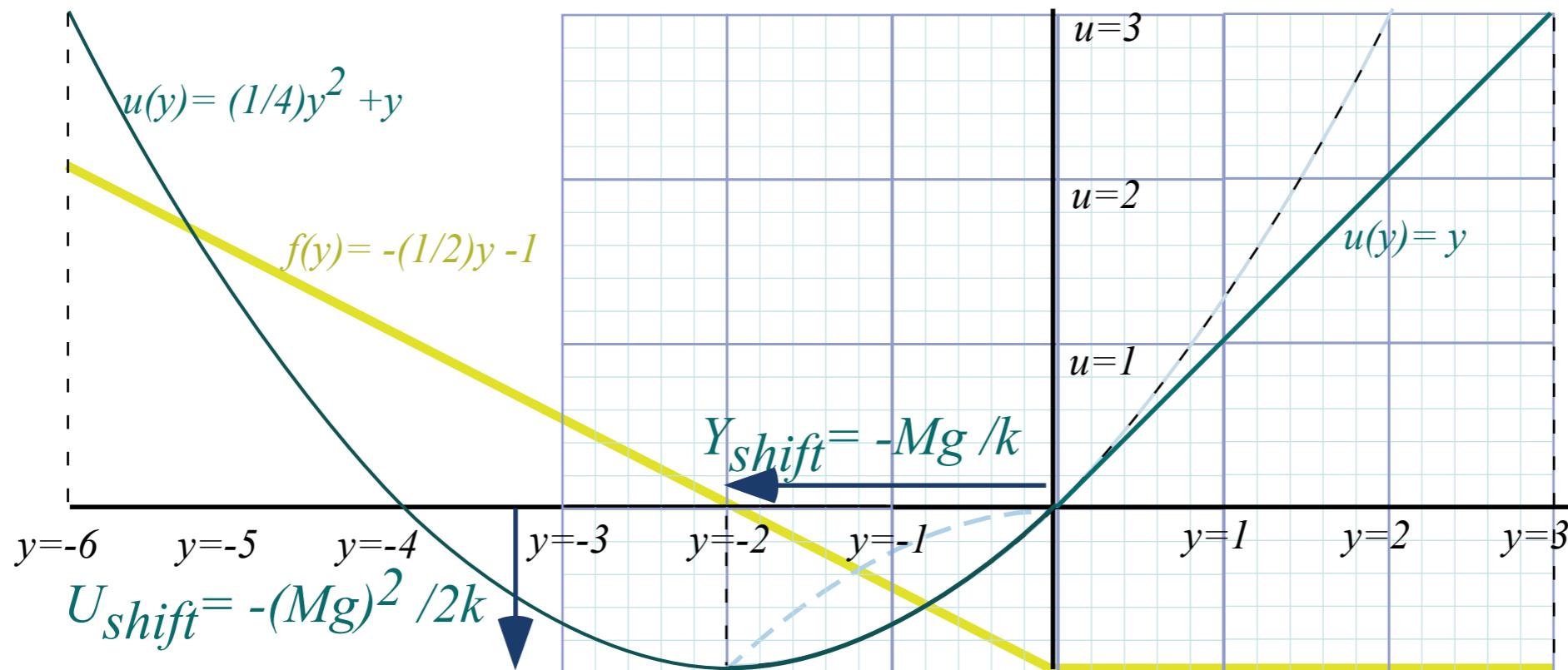
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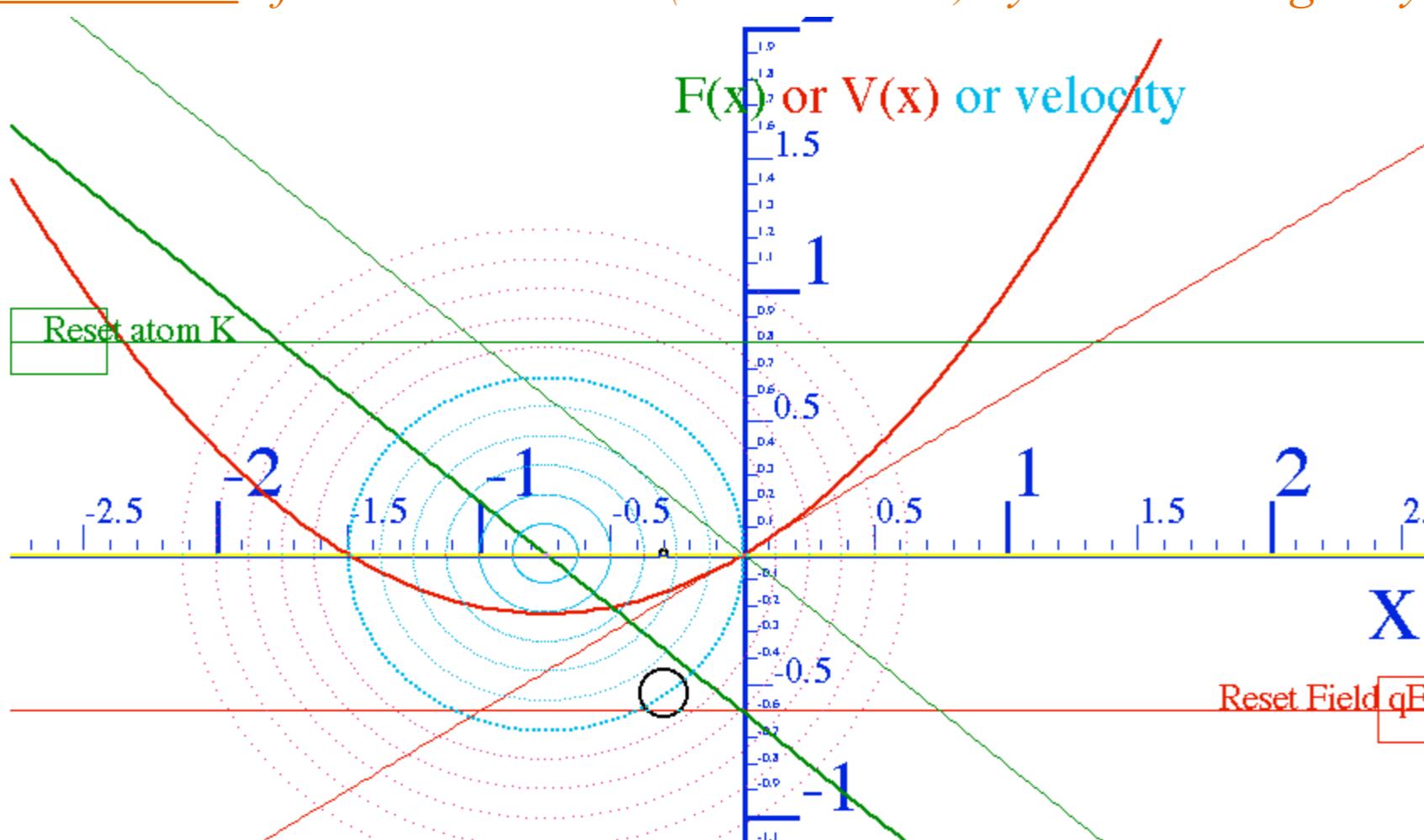
$$F(Y) = -kY - Mg$$

$$U(Y) = (1/2)kY^2 + Mg Y$$



Unit 1
Fig. 7.4

Web Simulation of atomic classical (or semi-classical) dynamics using varying phase control



Exploring phase space and Lagrangian mechanics more deeply



A weird “derivation” of Lagrange’s equations

Poincare identity and Action, Jacobi-Hamilton equations

How Classicists might have “derived” quantum equations

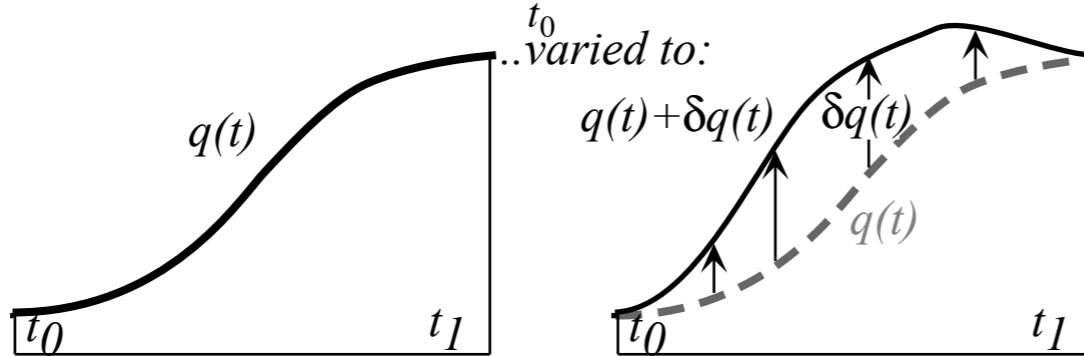
Huygen’s contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

A strange “derivation” of Lagrange’s equations by Calculus of Variation

Variational calculus finds extreme (minimum or maximum) values to entire integrals

$$\text{Minimize (or maximize): } S(q) = \int_{t_0}^{t_1} dt L(q(t), \dot{q}(t), t).$$



An arbitrary but small variation function $\delta q(t)$ is allowed at every point t in the figure along the curve except at the end points t_0 and t_1 . There we demand it not vary at all. (1)

$$\delta q(t_0) = 0 = \delta q(t_1) \quad (1)$$

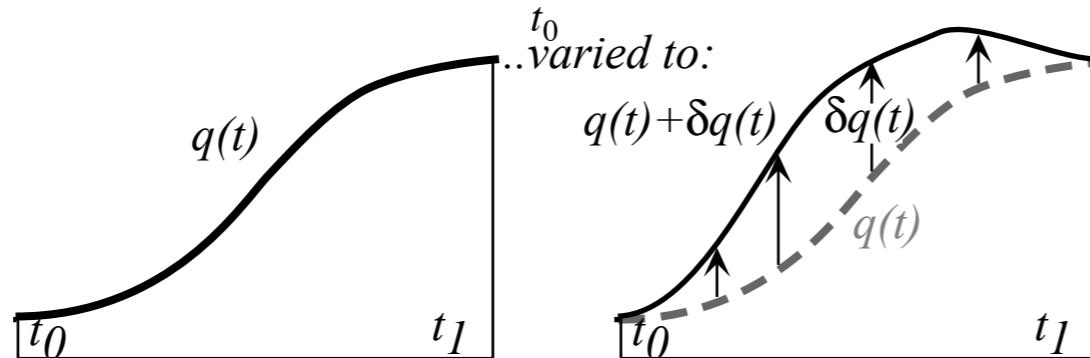
1st order $L(q+\delta q)$ approximate:

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A weird “derivation” of Lagrange’s equations

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(1) ←

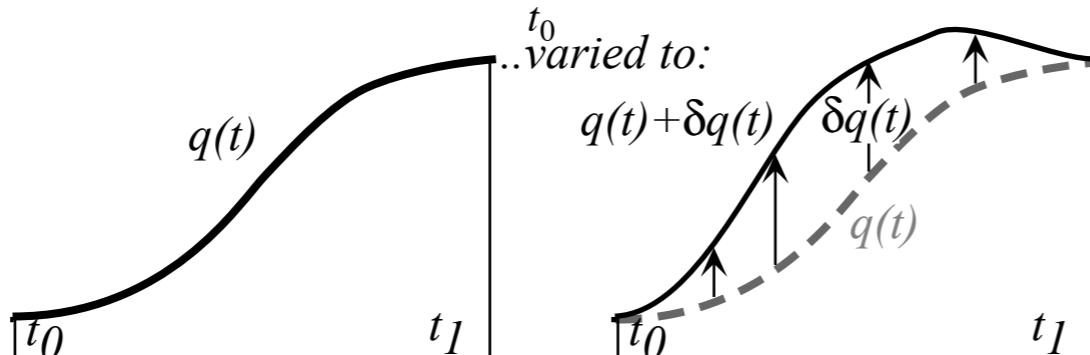
Replace $\frac{\partial L}{\partial \dot{q}} \delta \dot{q}$ with $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q$

$$u \cdot \frac{dv}{dt} = \frac{d}{dt} (uv) - \frac{du}{dt} v$$

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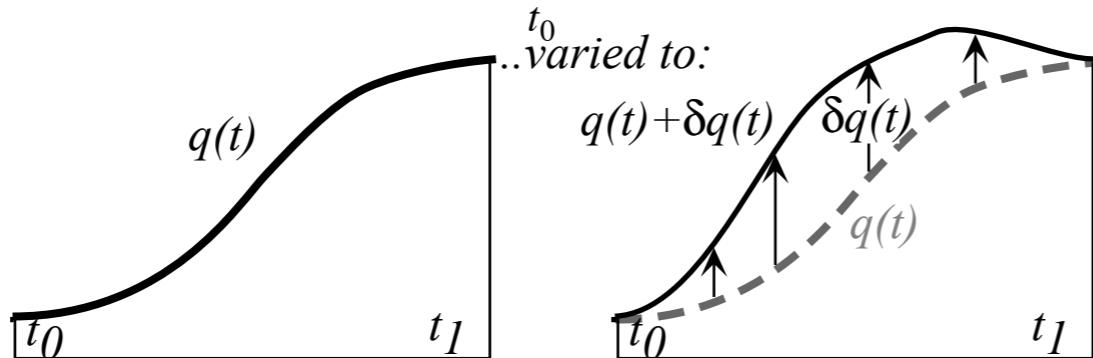
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$$= \int_{t_0}^{t_1} dt L(q, \dot{q}, t) + \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q + \left. \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \right|_{t_0}^{t_1}$$

(1)

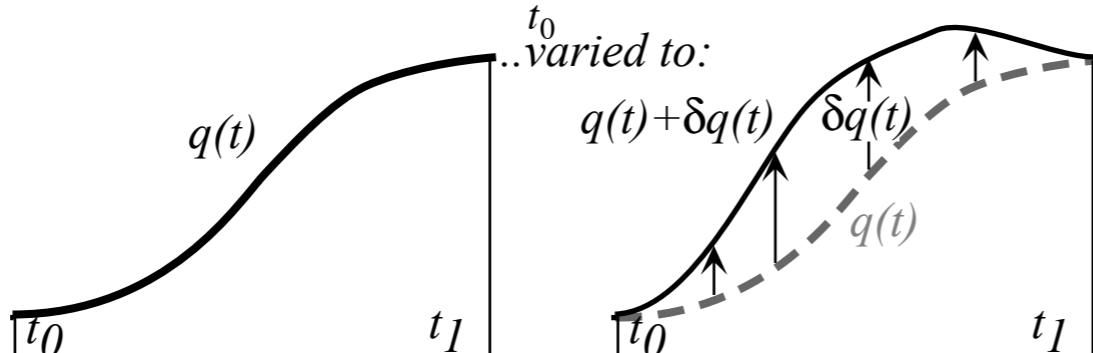
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Third term vanishes by (1). This leaves first order variation: $\delta S = S(q + \delta q) - S(q) = \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q$

Extreme value (actually *minimum* value) of $S(q)$ occurs if and only if Lagrange equation is satisfied!

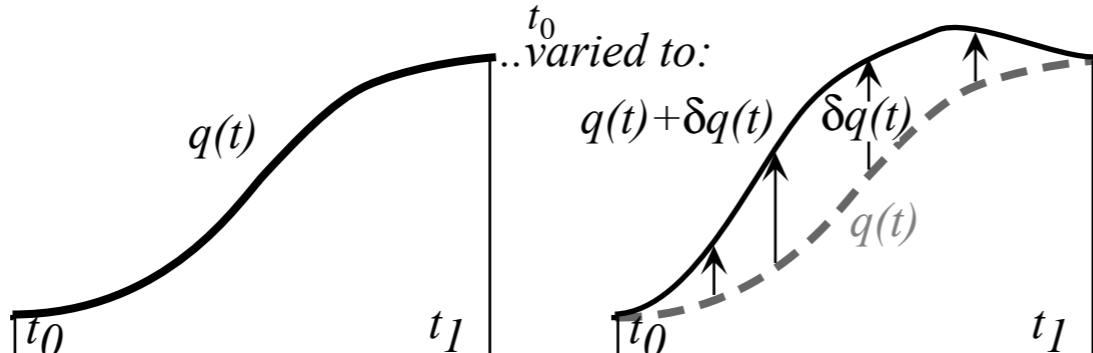
$$\delta S = 0 \Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$$

Euler-Lagrange equation(s)

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$$\delta S = 0 \Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$$

Euler-Lagrange equation(s)

But, WHY is nature so inclined to fly JUST SO as to minimize the Lagrangian $L = T - U$???

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Legendre-Poincare identity and Action

Legendre transform $L(\mathbf{v}) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p})$ becomes *Poincare's invariant differential* if dt is cleared.

$$L \cdot dt = \mathbf{p} \cdot \mathbf{v} \cdot dt - H \cdot dt = \mathbf{p} \cdot d\mathbf{r} - H \cdot dt \quad \left(\mathbf{v} = \frac{d\mathbf{r}}{dt} \text{ implies: } \mathbf{v} \cdot dt = d\mathbf{r} \right)$$

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This is the time differential dS of *action* $S = \int L \cdot dt$ whose time derivative is rate L of *quantum phase*.

$$dS = L \cdot dt = \mathbf{p} \cdot d\mathbf{r} - H \cdot dt \quad \text{where: } L = \frac{dS}{dt}$$

Legendre-Poincare identity and Action

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$$L \cdot dt = \mathbf{p} \cdot \mathbf{v} \cdot dt - H \cdot dt = \mathbf{p} \cdot d\mathbf{r} - H \cdot dt \quad \mathbf{v} = \frac{d\mathbf{r}}{dt}$$

This is the time differential dS of *action* $S = \int L \cdot dt$ whose time derivative is rate L of *quantum phase*.

$$dS = L \cdot dt = \mathbf{p} \cdot d\mathbf{r} - H \cdot dt \quad \text{where: } L = \frac{dS}{dt}$$

Unit 8 shows DeBroglie law $\mathbf{p} = \hbar \mathbf{k}$ and Planck law $H = \hbar \omega$ make *quantum plane wave phase* Φ :

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Q: When is the *Action*-differential dS integrable?

A: A differential $dW = f_x(x, y)dx + f_y(x, y)dy$ is *integrable* to a $W(x, y)$ if: $f_x = \frac{\partial W}{\partial x}$ and: $f_y = \frac{\partial W}{\partial y}$

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Similar to conditions for integrating work differential $dW = \mathbf{f} \cdot d\mathbf{r}$ to get potential $W(\mathbf{r})$. That condition is no curl allowed: $\nabla \times \mathbf{f} = 0$ or ∂ -symmetry of W :

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(Given “quantum wave”)

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or \mathbf{p} -op in \mathbf{r} -basis
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Schrodinger time
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 $i\hbar \dot{\psi}(\mathbf{r}, t) = H \psi(\mathbf{r}, t)$

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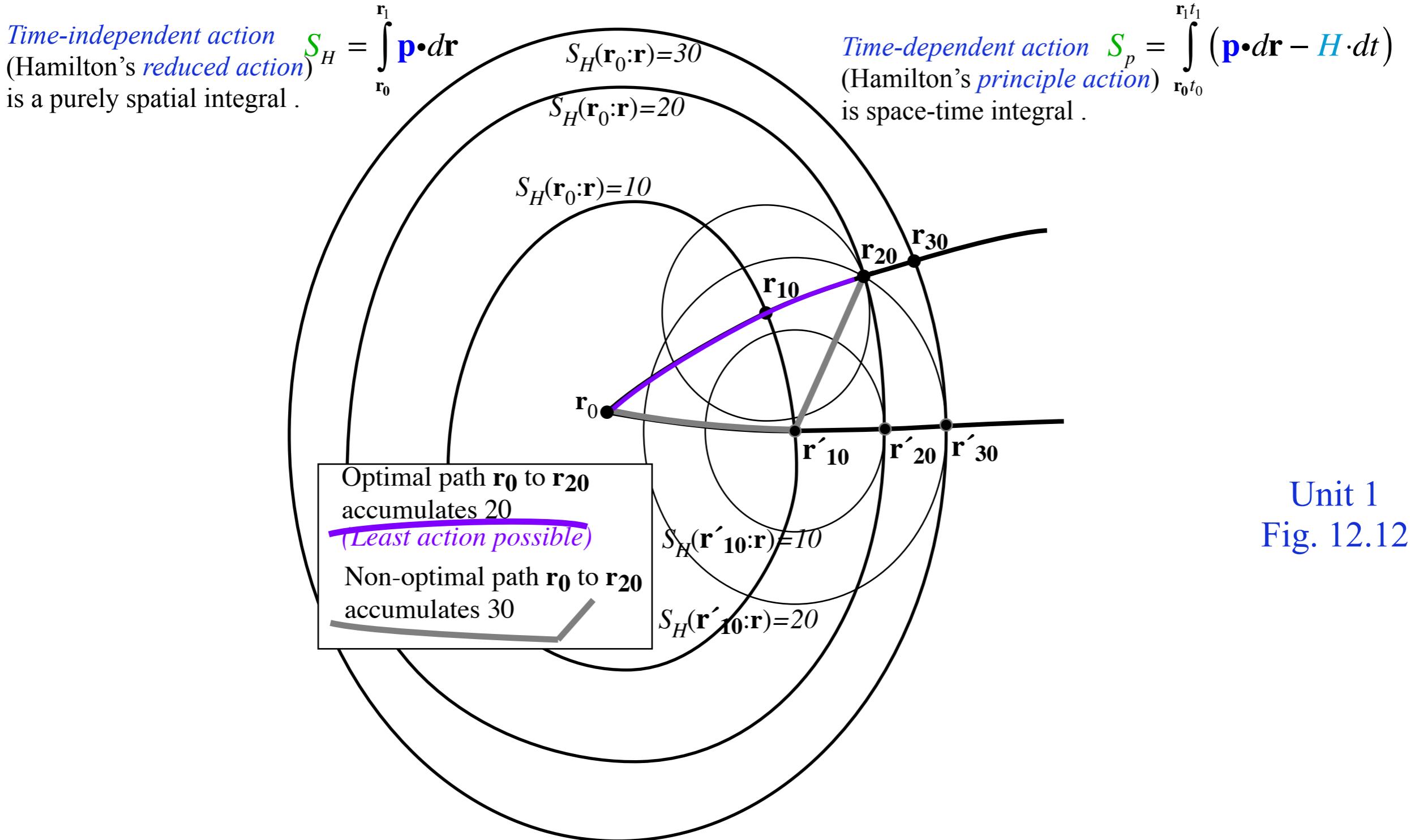
How to do quantum mechanics if you only know classical mechanics



Huygen's contact transformations enforce minimum action

Each point \mathbf{r}_k on a wavefront “broadcasts” in all directions.

Only **minimum action** path interferes constructively



Unit 1
Fig. 12.12

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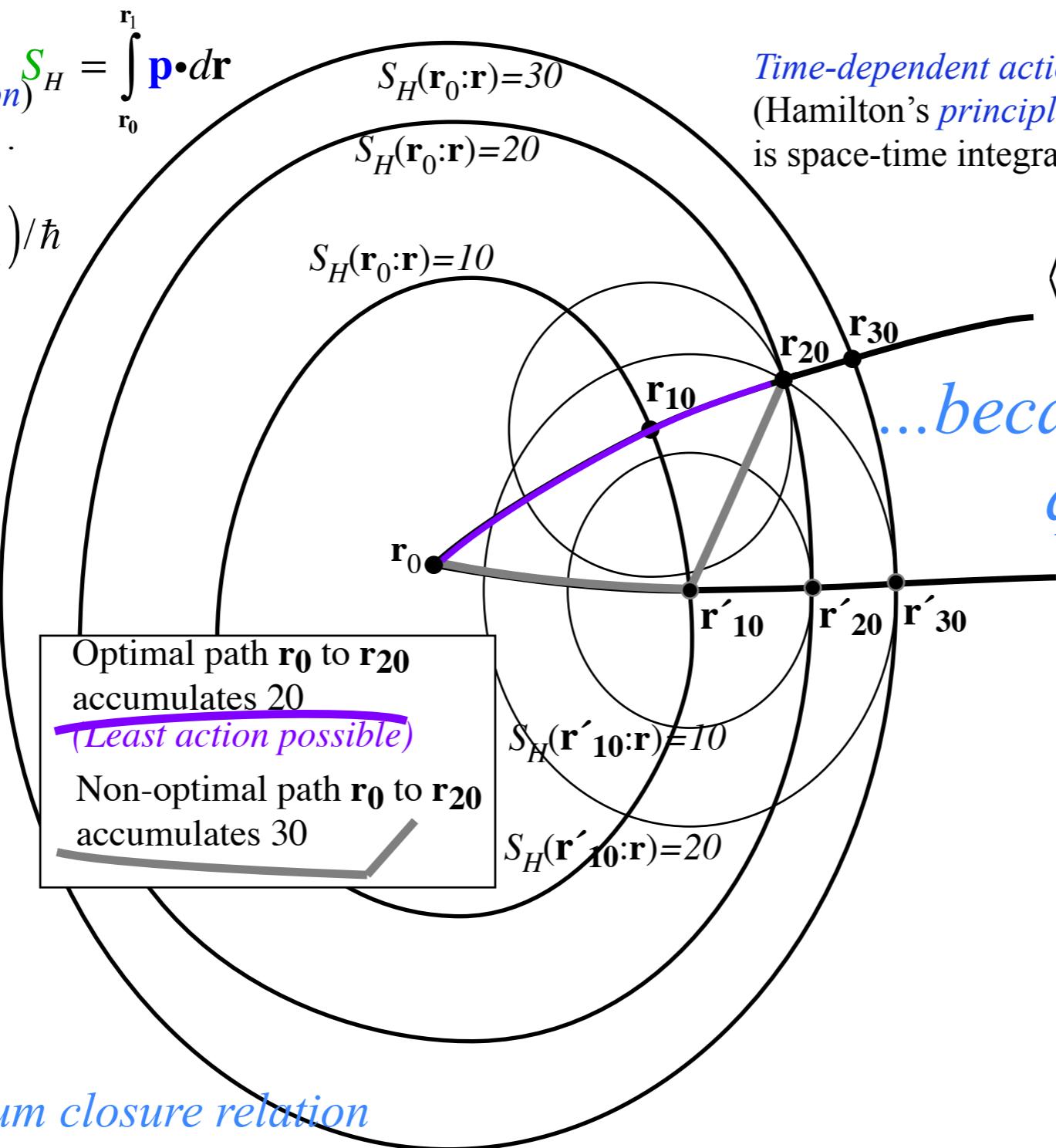
Only **minimum action** path interferes constructively

Time-independent action
(Hamilton's *reduced action*)
 $S_H = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{p} \cdot d\mathbf{r}$
is a purely spatial integral .

$$\langle \mathbf{r}_1 | \mathbf{r}_0 \rangle = e^{i S_H(\mathbf{r}_0 : \mathbf{r}_1) / \hbar}$$

Time-dependent action
(Hamilton's *principle action*)
 $S_p = \int_{\mathbf{r}_0 t_0}^{\mathbf{r}_1 t_1} (\mathbf{p} \cdot d\mathbf{r} - \mathbf{H} \cdot dt)$
is space-time integral .

$$\langle \mathbf{r}_1, t_1 | \mathbf{r}_0, t_0 \rangle = e^{i S(\mathbf{r}_0, t_0 : \mathbf{r}_1, t_1) / \hbar}$$



Unit 1
Fig. 12.12

Feynman's path-sum closure relation

$$\sum_{\mathbf{r}'} \langle \mathbf{r}_1 | \mathbf{r}' \rangle \langle \mathbf{r}' | \mathbf{r}_0 \rangle \equiv \sum_{\mathbf{r}'} e^{i (S_H(\mathbf{r}_0 : \mathbf{r}') + S_H(\mathbf{r}' : \mathbf{r}_1)) / \hbar} = e^{i S_H(\mathbf{r}_0 : \mathbf{r}_1) / \hbar} = \langle \mathbf{r}_1 | \mathbf{r}_0 \rangle$$

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How to do quantum mechanics if you only know classical mechanics

Davis-Heller “Color-Quantization” or “Classical Chromodynamics”



How to do quantum mechanics if you only know classical mechanics

Bohr quantization requires quantum phase S_H/\hbar in amplitude to be an integral multiple n of 2π after a closed loop integral $S_H(\mathbf{r}_0 : \mathbf{r}_0) = \int_{r_0}^{r_0} \mathbf{p} \cdot d\mathbf{r}$. The integer n ($n = 0, 1, 2, \dots$) is a *quantum number*.

$$I = \langle \mathbf{r}_0 | \mathbf{r}_0 \rangle = e^{i S_H(\mathbf{r}_0 : \mathbf{r}_0)/\hbar} = e^{i \Sigma_H/\hbar} = 1 \text{ for: } \Sigma_H = 2\pi \hbar n = hn$$

Numerically integrate Hamilton's equations and Lagrangian L . Color the trajectory according to the current accumulated value of action $S_H(\mathbf{0} : \mathbf{r})/\hbar$. Adjust energy to quantized pattern (if closed system*)

$$S_H(\mathbf{0} : \mathbf{r}) = S_p(\mathbf{0}, 0 : \mathbf{r}, t) + Ht = \int_0^t L dt + Ht.$$

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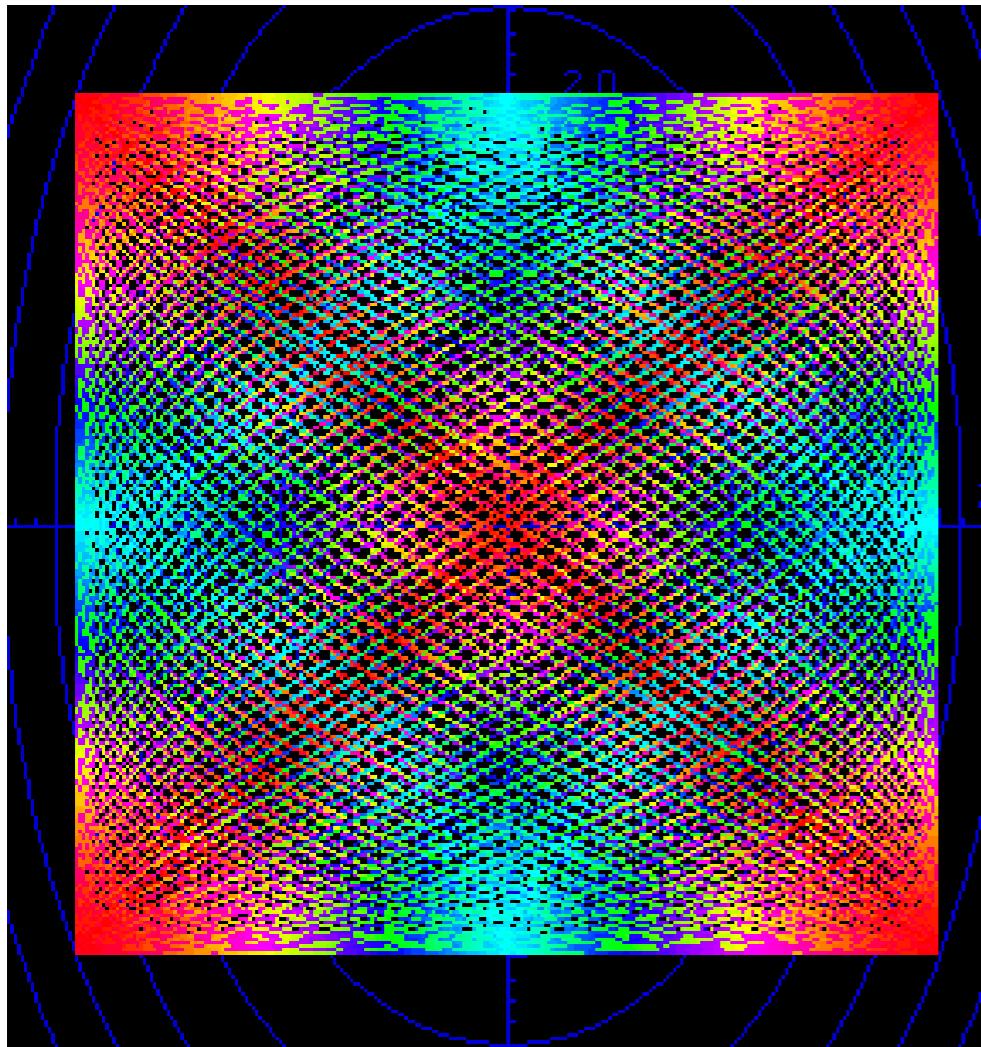
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The hue should represent the phase angle $S_H(\mathbf{0} : \mathbf{r})/\hbar$ modulo 2π as, for example,

$0 = \text{red}$, $\pi/4 = \text{orange}$, $\pi/2 = \text{yellow}$, $3\pi/4 = \text{green}$, $\pi = \text{cyan}$ (opposite of red), $5\pi/4 = \text{indigo}$, $3\pi/2 = \text{blue}$, $7\pi/4 = \text{purple}$, and $2\pi = \text{red}$ (full color circle).

Interpolating action on a palette of 32 colors is enough precision for low quanta.



simulation
by
“Color $U(2)$ ”

Unit 1
Fig.
12.13

*closed system
has quantized E.
Standing wave has
only two phases (\pm)
cyan and *red*

Wavepacket and Color-quantization:
M. J. Davis and E. J. Heller, J. Chem. Phys. 75, 246 (1981)

[The Semiclassical Way to Molecular Spectroscopy:](#)
[Eric J. Heller, Acc. Chem. Res. 1981, 14, 368-375](#)

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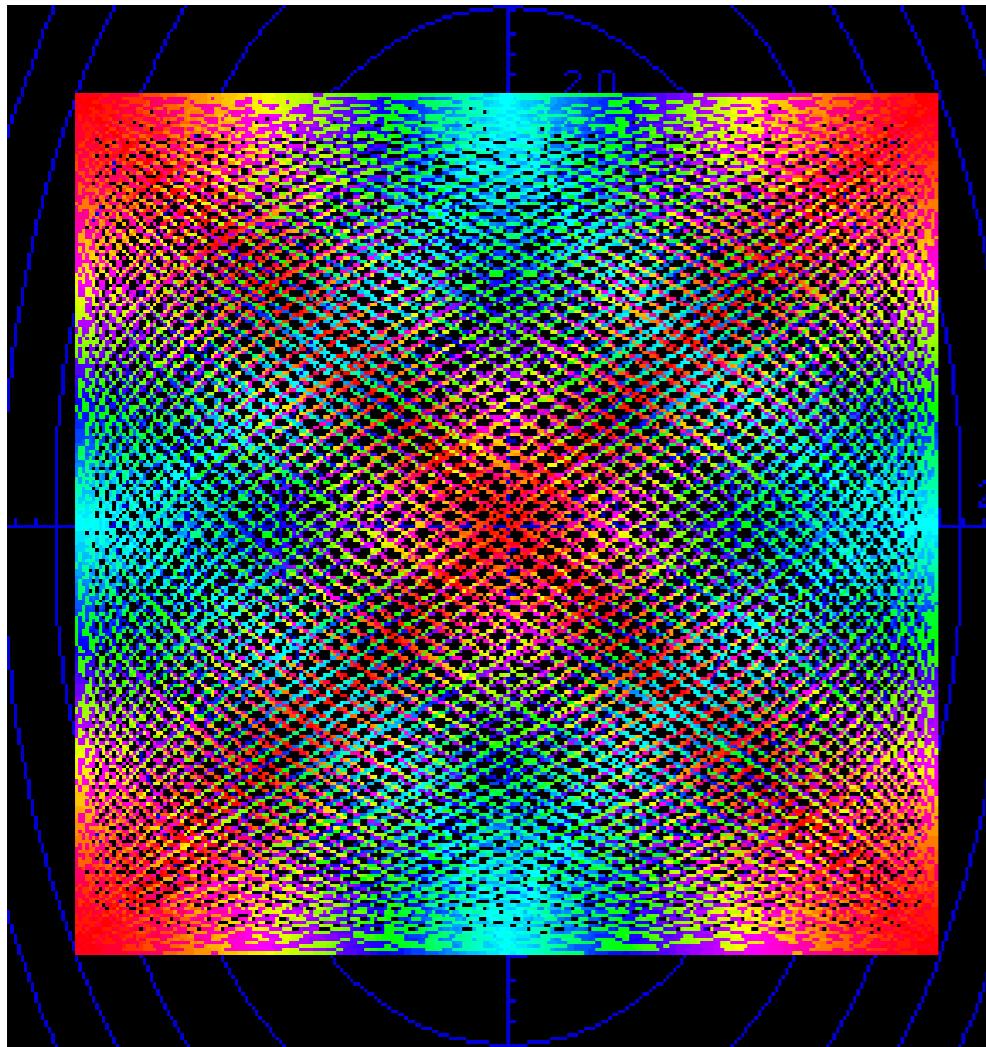
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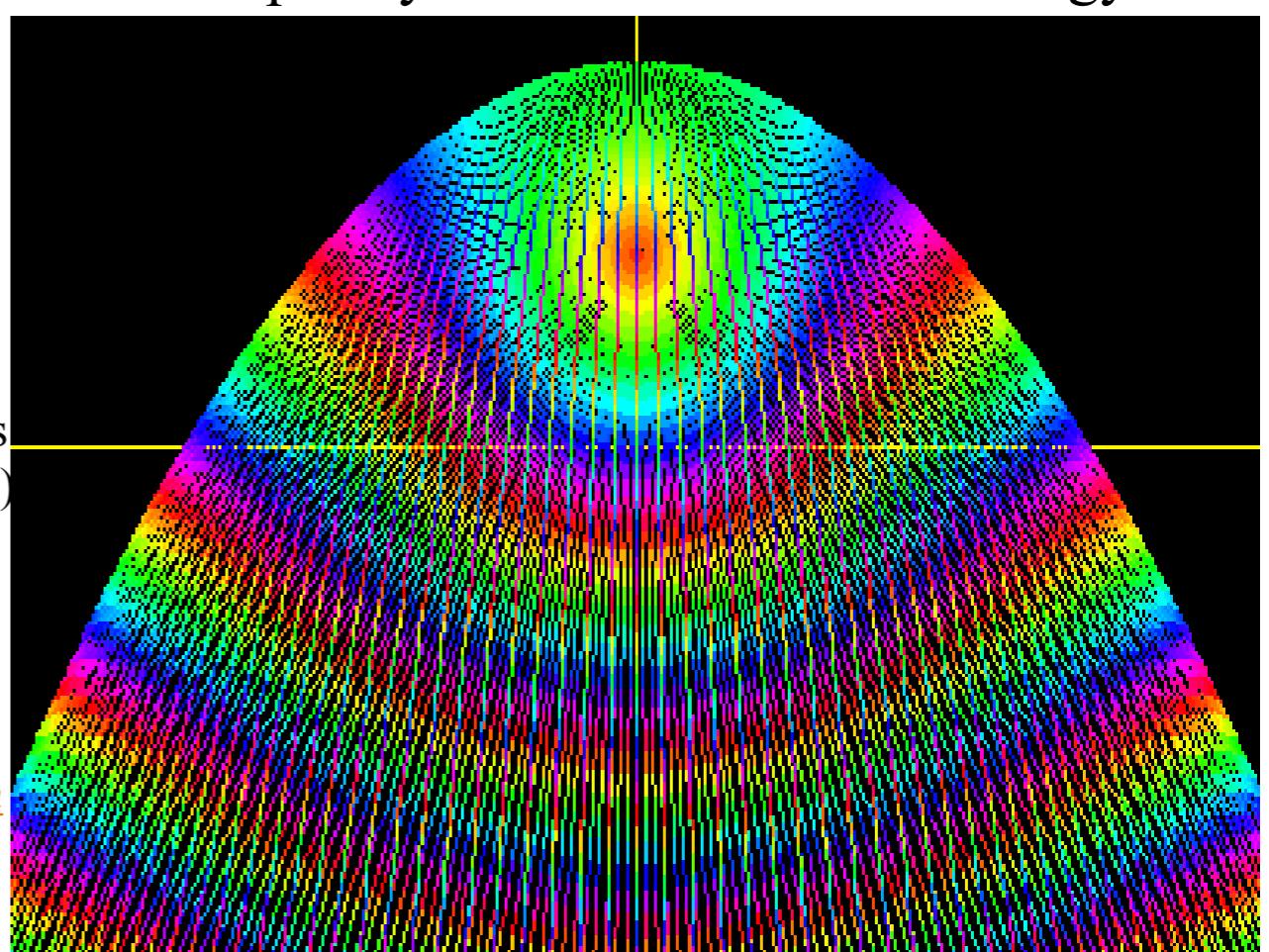


Simulation
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Unit 1
Fig.
12.13

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Standing wave has
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Unit 1
Fig.
12.14
Web Simulation
by
“CoulIt”



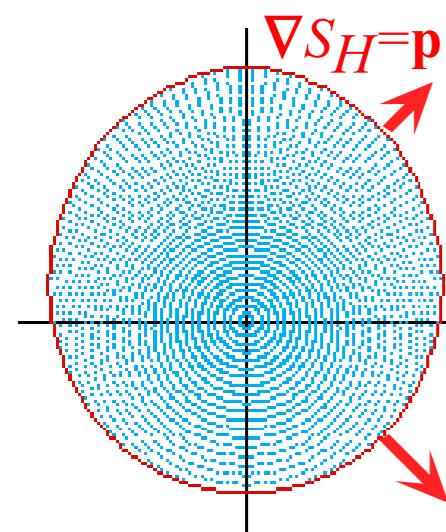
*open system has continuous energy

A moving wave has a *quantum phase velocity* found by setting $S=const.$ or $dS(0,0:r,t)=0=\mathbf{p} \cdot d\mathbf{r} - Hdt$.

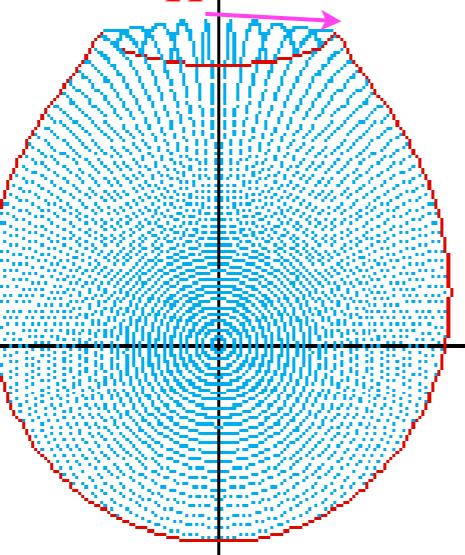
$$\mathbf{v}_{phase} = \frac{d\mathbf{r}}{dt} = \frac{\mathbf{H}}{\mathbf{p}} = \frac{\omega}{\mathbf{k}}$$

Quantum “phase wavefronts”

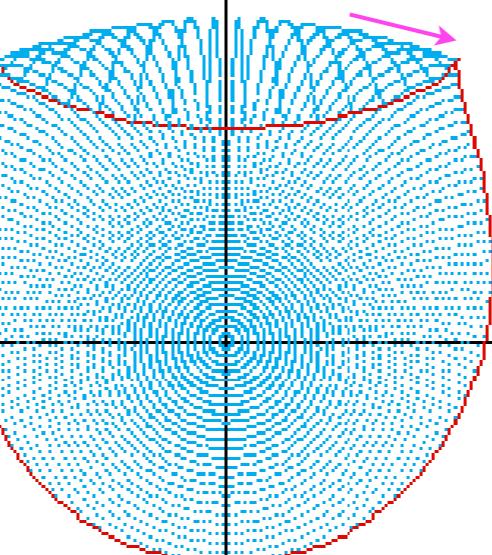
(a) $S_H=0.3$



(b) $S_H=0.35$

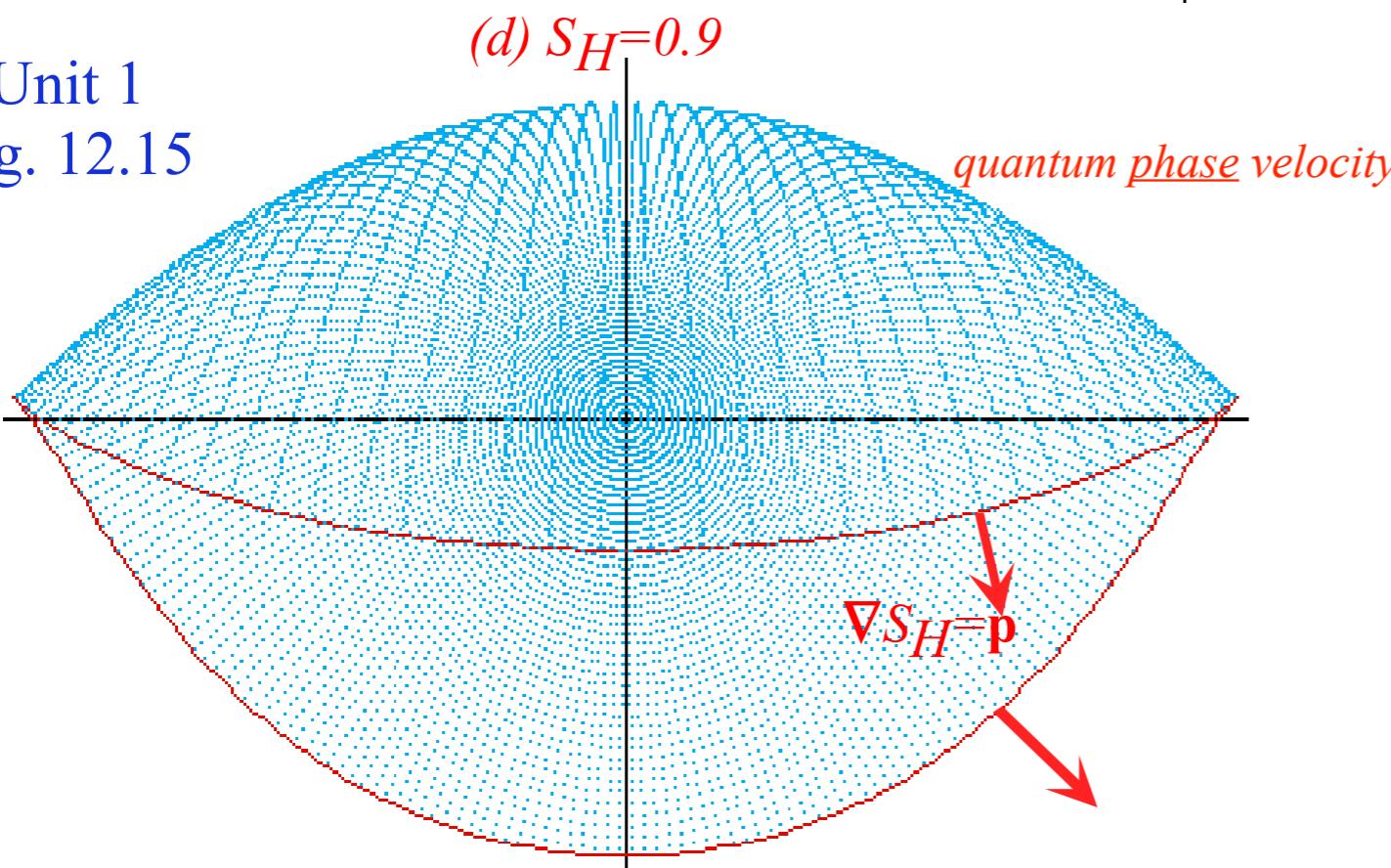


(c) $S_H=0.4$



wavefront
“cat ears”
scoot outward..

(d) $S_H=0.9$

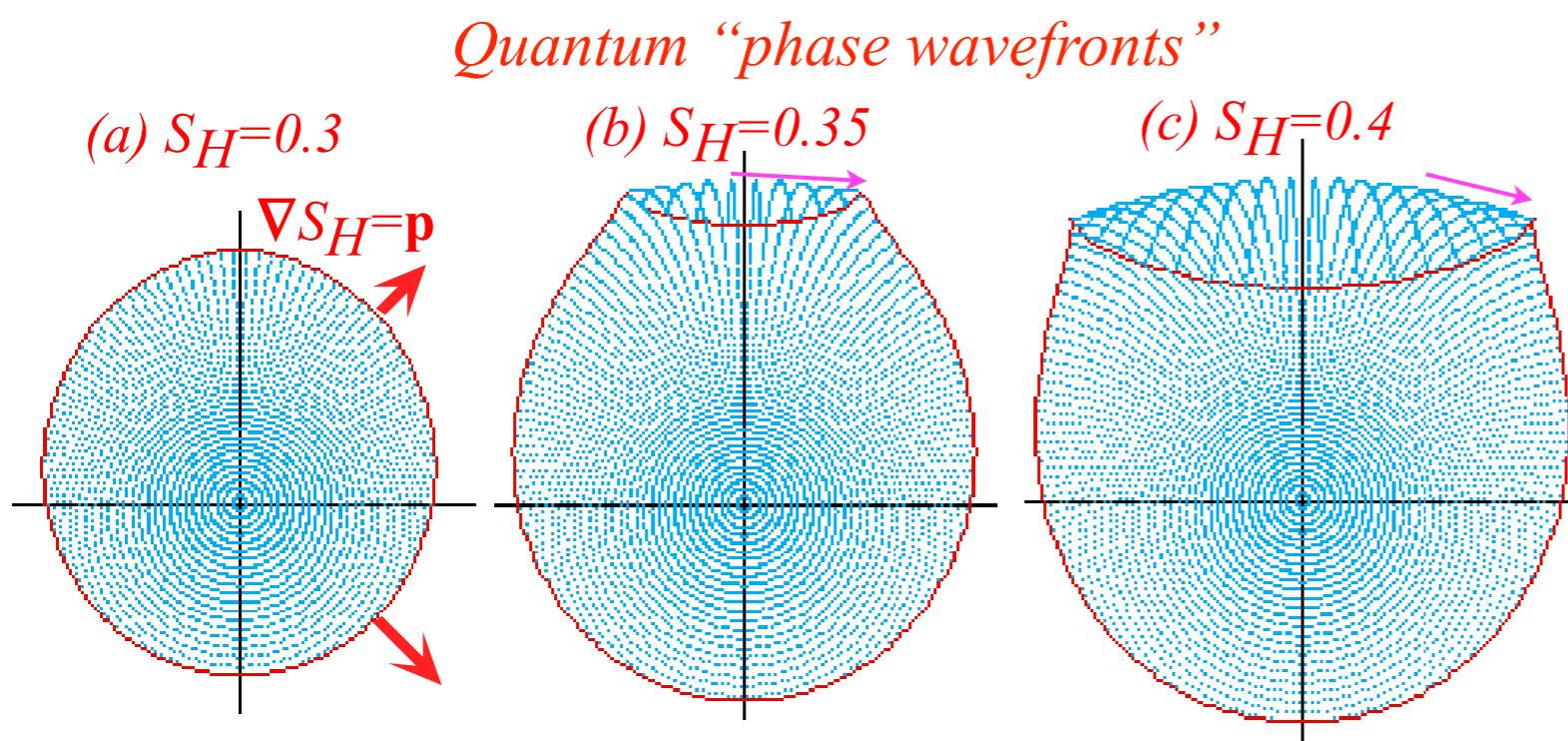


Unit 1
Fig. 12.15

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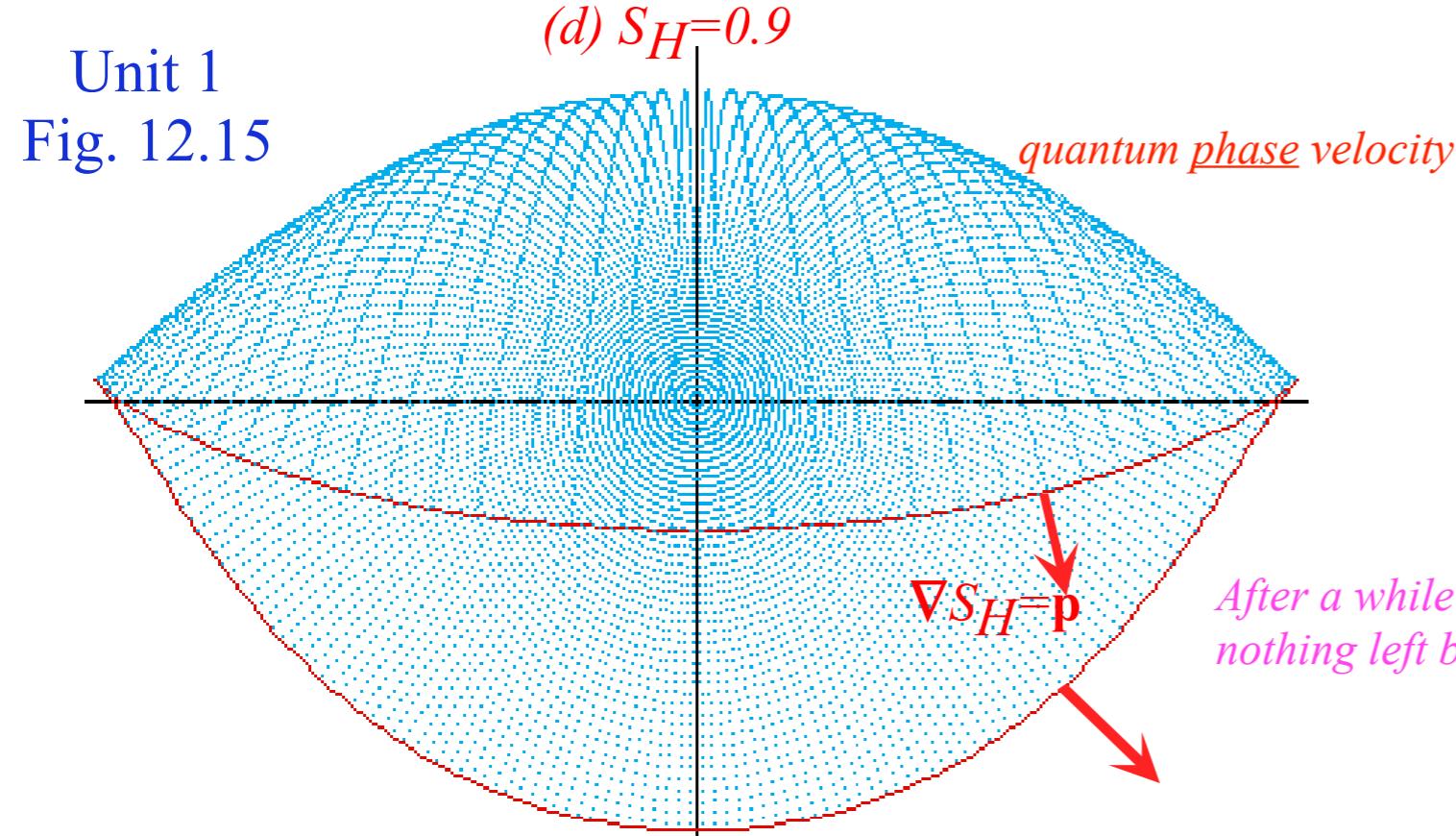
This is quite the opposite of classical particle velocity which is *quantum group velocity*.



$$\mathbf{V}_{group} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \frac{\partial \mathbf{H}}{\partial \mathbf{p}} = \frac{\partial \omega}{\partial \mathbf{k}}$$

Note: This is Hamilton's 1st Equation

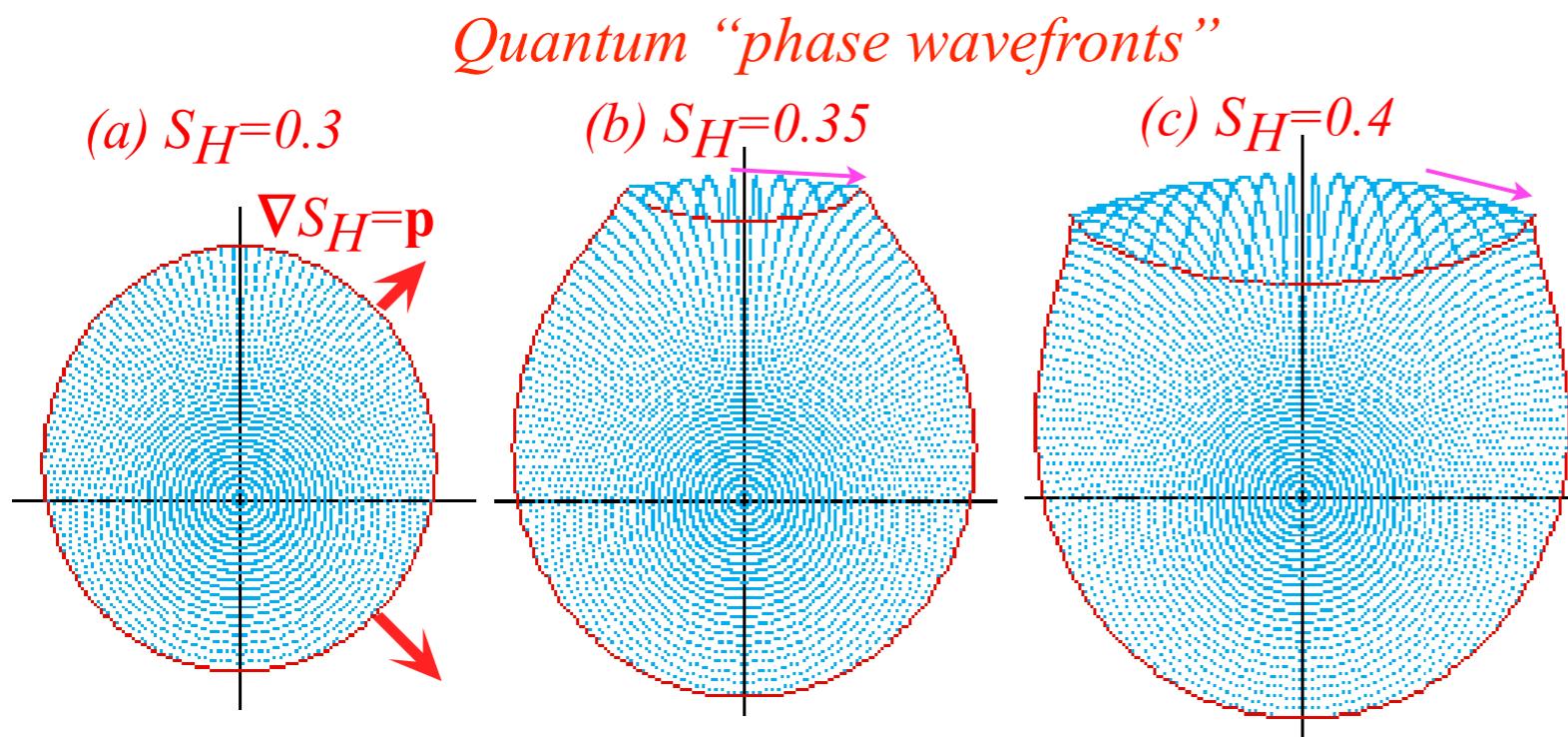
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Unit 1
Fig. 12.15

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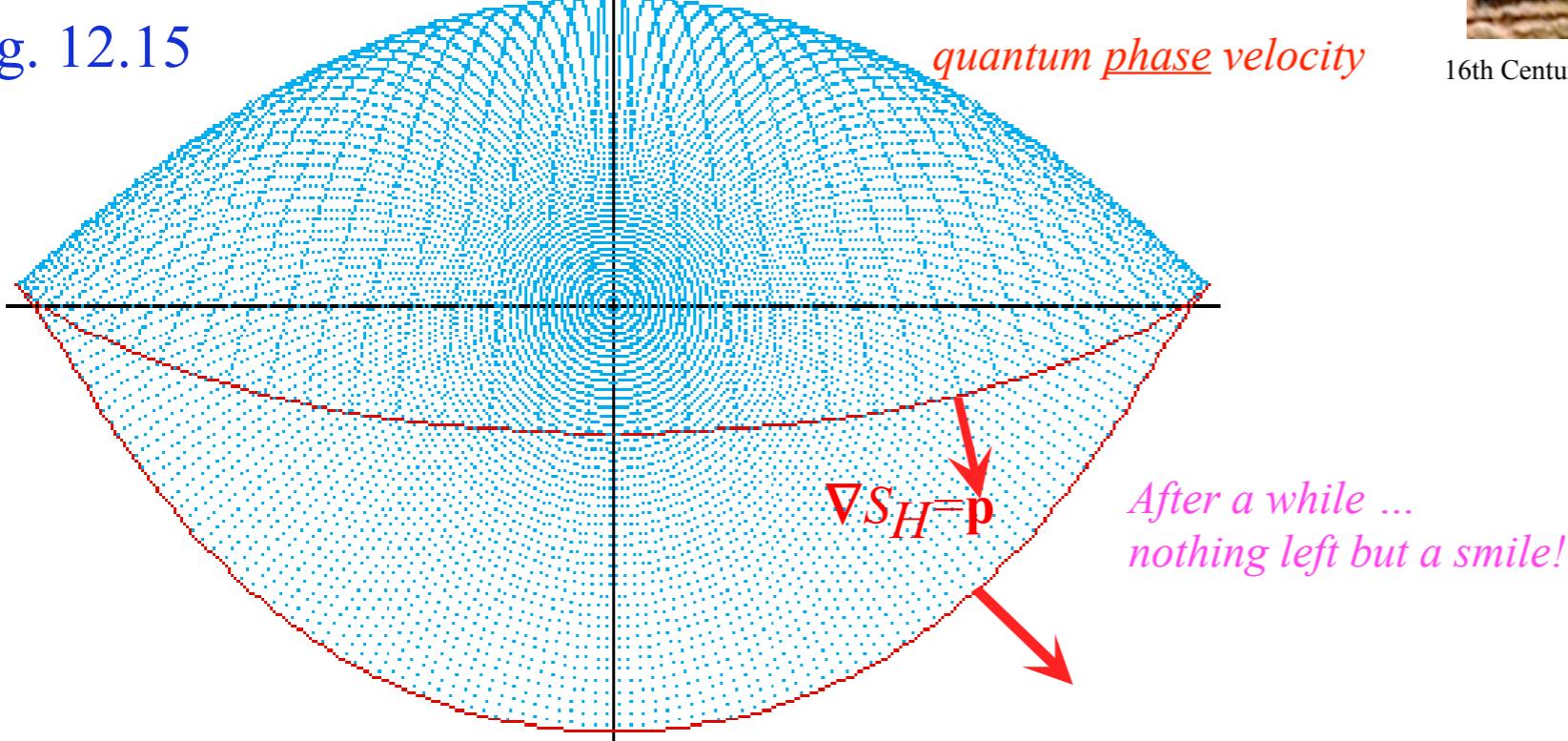
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16th Century carving on St. Wifred's in Grappenhall

...on St. Nicolas

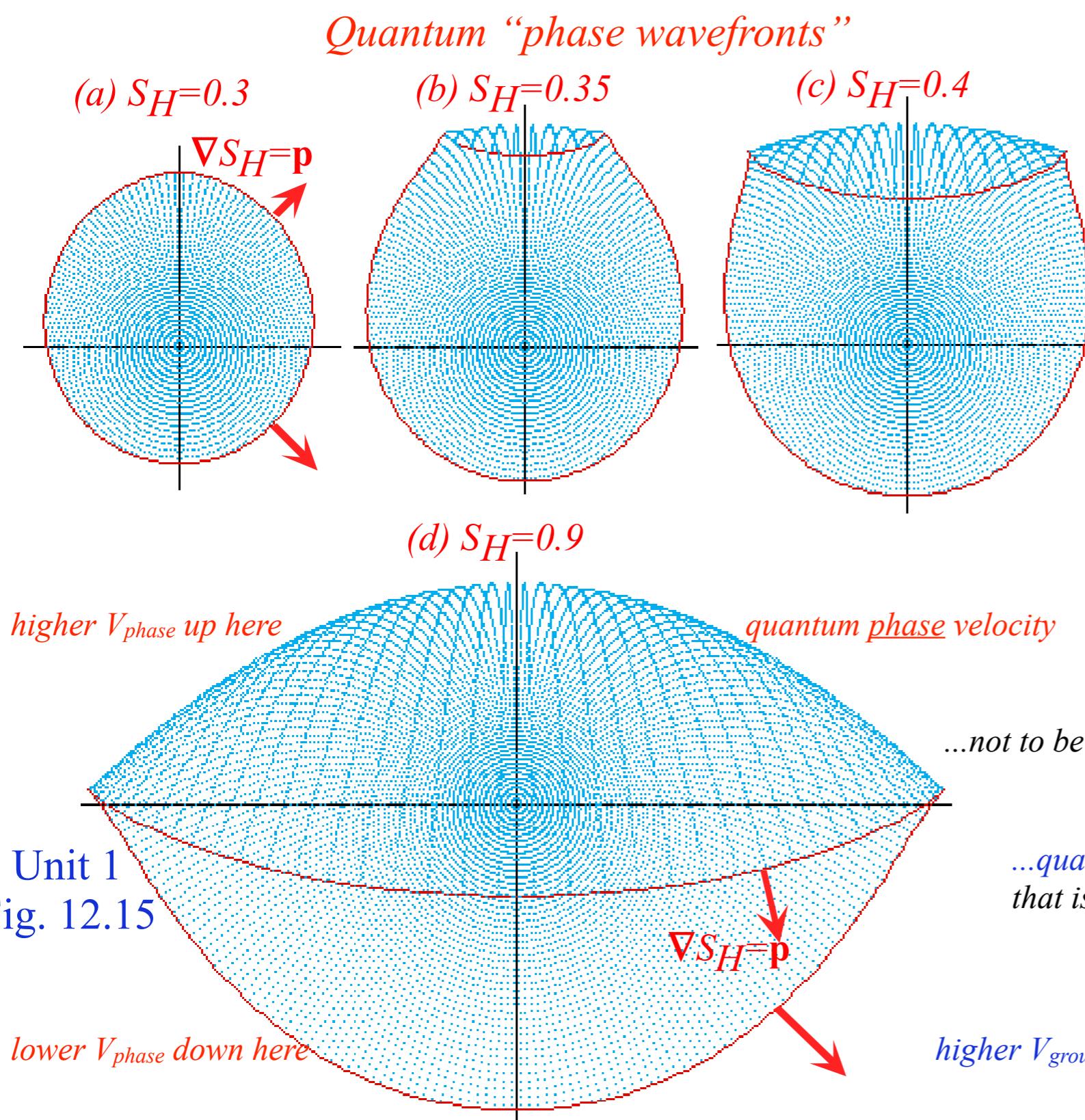


From *Alice's Adventures in Wonderland* by Lewis Carroll (1865)

A moving wave has a *quantum phase velocity* found by setting $S=const.$ or $dS(0,0:r,t)=0=\mathbf{p}\cdot d\mathbf{r}-Hdt$.

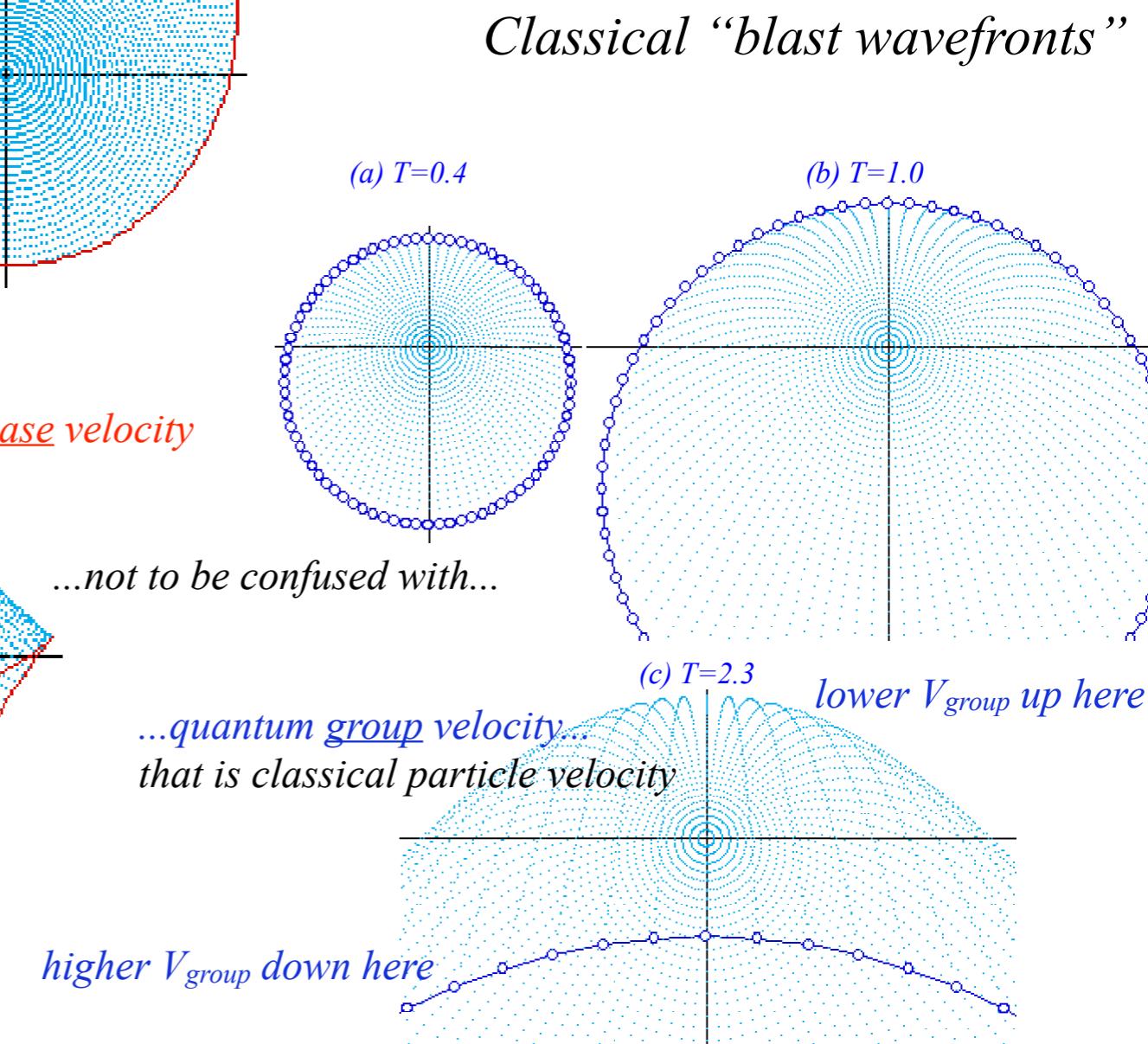
$$V_{phase} = \frac{dr}{dt} = \frac{H}{p} = \frac{\omega}{k}$$

This is quite the opposite of classical particle velocity which is *quantum group velocity*.



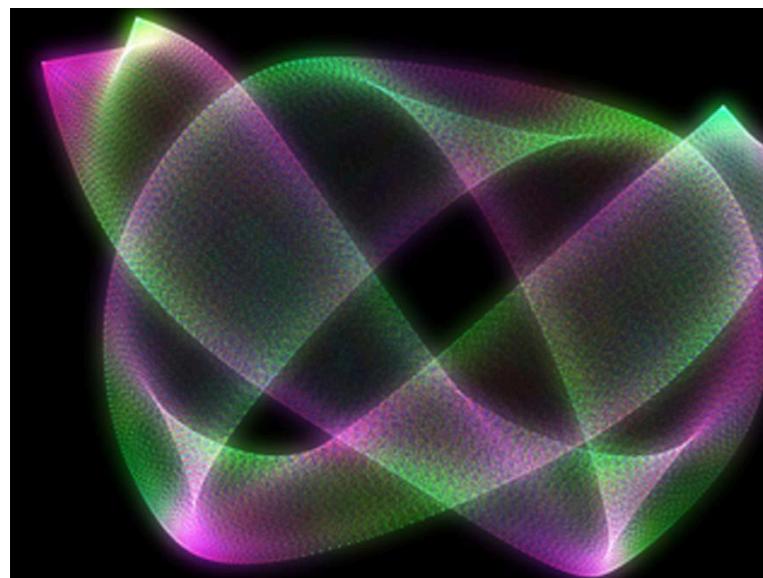
$$V_{group} = \frac{dr}{dt} = \dot{r} = \frac{\partial H}{\partial p} = \frac{\partial \omega}{\partial k}$$

Note: This is Hamilton's 1st Equation



Check out the Heller Galleries

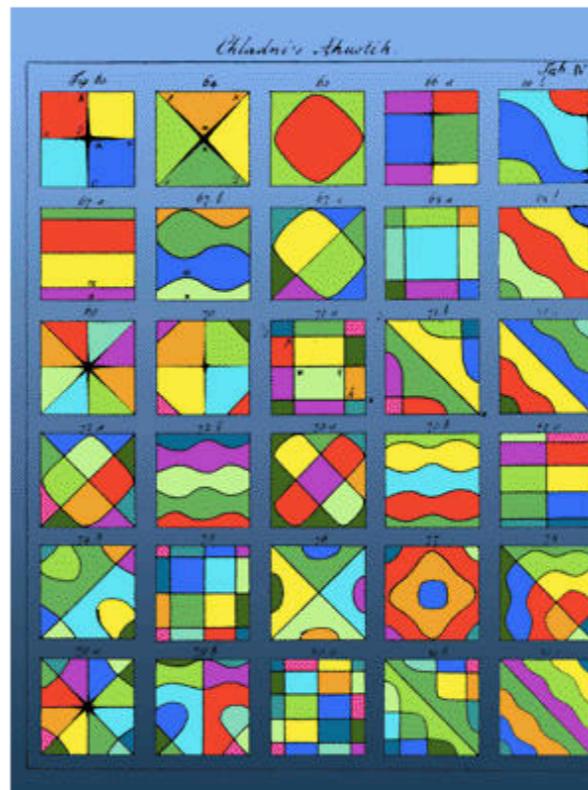
<http://jalbum.net/en/browse/user/album/1696720>



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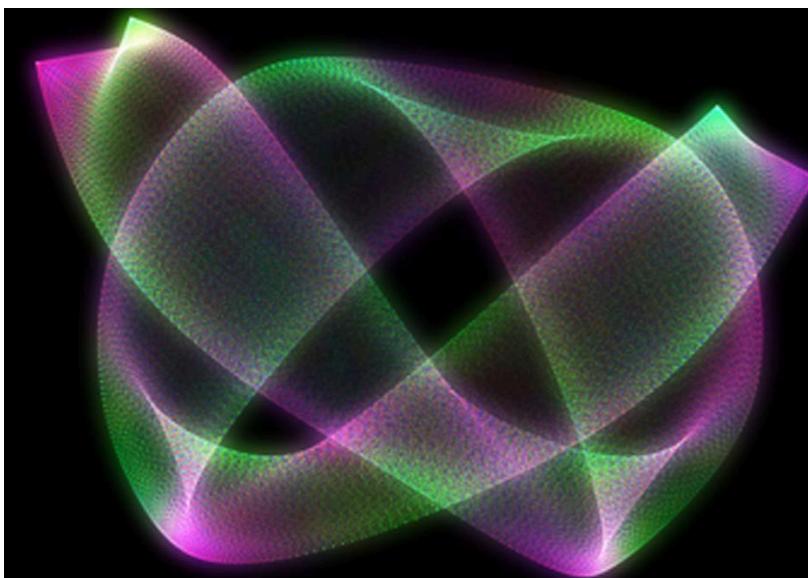
[Chladni](#)



The diagrams of Ernst Chladni (1756-1827) are the scientific, artistic, and even the sociological birthplace of the modern field of wave physics and quantum chaos. Educated in Law at the University of Leipzig, and an amateur musician, Chladni soon followed his love of science and wrote one of the first treatises on acoustics, "Discovery of the Theory of Pitch". Chladni had an inspired idea: to make waves in a solid material visible. This he did by getting metal plates to vibrate, stroking them with a violin bow. Sand or a similar substance spread on the surface of the plate naturally settles to the places where the metal vibrates the least, making such places visible. These places are the so-called nodes, which are wavy lines on the surface. The plates vibrate at pure, audible pitches, and each pitch has a unique nodal pattern. Chladni took the trouble to carefully diagram the patterns, which helped to popularize his work. Then he hit the lecture circuit, fascinating audiences in Europe with live demonstrations. This culminated with a command performance for Napoleon, who was so impressed that he offered a prize to anyone who could explain the patterns. More than that, according to Chladni himself, Napoleon remarked that irregularly shaped plate would be much harder to understand! While this was surely also known to Chladni, it is remarkable that Napoleon had this insight. Chladni received a sum of 6000 francs from Napoleon, who also offered 3000 francs to anyone who could explain the patterns. The mathematician Sophie Germain took the prize in 1816, although her solutions were not completed until the work of Kirchoff thirty years later. Even so, the patterns for irregular shapes remained (and to some extent remains) unexplained. Government funding of waves research goes back a long way! (Chladni was also the first to maintain that meteorites were extraterrestrial; before that, the popular theory was that they were of volcanic origin.) One of his diagrams is the basis for image, which is a playfully colored version of Chladni's original line drawing. Chladni's original work on waves confined to a region was followed by equally remarkable progress a few years later.

Check out the Heller Galleries

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National Science Foundation (NSF)
Arlington, VA

September-November 2002

Selected images.

http://search.nsf.gov/search?ie=&site=nsf&output=xml_no_dtd&proxyreload=1&client=nsf&lr=&proxystylesheet=http%3A%2F%2Fwww.nsf.gov%2Fsearch%2Fnsf_new.xslt&oe=&btnG.x=0&btnG.y=0&q=eric+heller

University Museum, University of Arkansas, Fayetteville, AK

October 2002 - December 2002

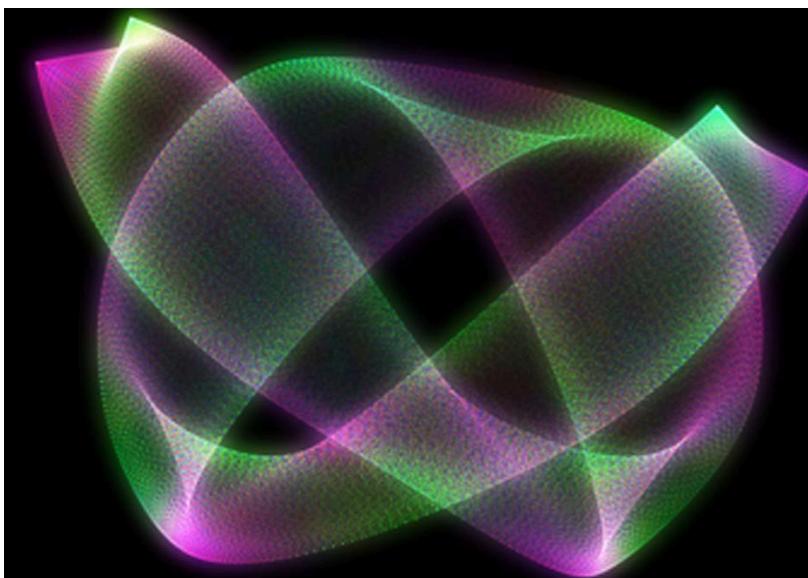
"Approaching Chaos: Visions from the Quantum Frontier"

Approaching Chaos is supported by a grant from the National Science Foundation and by MIT Museum and the Center for Theoretical Physics at the Massachusetts Institute of Technology.

[Bessel 21](#)

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Approaching Chaos is supported by a grant from the National Science Foundation and by MIT Museum and the Center for Theoretical Physics at the Massachusetts Institute of Technology.

*UAF Museum closed after this exhibit



Lecture 11 ends here
Thur. 9/27/2016

Bessel 21