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A generalization of Bertrand's theorem to surfaces of revolution

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Abstract. We prove a generalization of Bertrand's theorem to the case of abstract surfaces of revolution that have no 'equators'. We prove a criterion for exactly two central potentials to exist on this type of surface (up to an additive and a multiplicative constant) for which all bounded orbits are closed and there is a bounded nonsingular noncircular orbit. We prove a criterion for the existence of exactly one such potential. We study the geometry and classification of the corresponding surfaces with the aforementioned pair of potentials (gravitational and oscillatory) or unique potential (oscillatory). We show that potentials of the required form do not exist on surfaces that do not belong to any of the classes described.

Bibliography: 33 titles.

Keywords: Bertrand's theorem, inverse problem of dynamics, surface of revolution, motion in a central field, closed orbits.

§ 1. Introduction

1.1. Classical results. In the second half of the 19th century, the following problem on the motion of a point in the Euclidean space $\mathbb{R}^3$ was posed and solved (Bertrand [1]): *find the law of an attractive force if it depends only on distance*, *and its point of application describes a closed curve whatever the initial conditions are*, *provided that the initial speed of the point is less than a certain limiting value.*

Note that, although this problem was stated for the motion of a point in Euclidean space $\mathbb{R}^3$, because the potentials under consideration are central its solution reduces to considering the motion of a point in the Euclidean plane $\mathbb{R}^2$.

A similar problem was also posed and solved (Koenigs [2]): *given that the force inducing the motion of a planet around the Sun depends only on distance*, *and its point of application describes an algebraic curve whatever the initial conditions are*, *and there exist bounded nonsingular noncircular orbits*, *find the law of this force.*

Both problems were solved in Euclidean space $\mathbb{R}^3$ and the answer is the same — the law of attraction can have either Newtonian (that is, gravitational) form with

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the attractive force $-G/r^2$, or Hookean (that is, oscillatory) form with the attractive force $-kr$, where r is the distance from a point to the centre of attraction. In other words, the equations of motion of a point have the form

$$\frac{d^2}{dt^2}\vec{r} = -\frac{G}{r^3}\vec{r} \quad \text{or} \quad \frac{d^2}{dt^2}\vec{r} = -k\vec{r},$$

where $\vec{r} = \vec{r}(t) \in \mathbb{R}^3$ is the radius-vector of the point (planet), $r = |\vec{r}|$, $G = \text{const} > 0$, $k = \text{const} > 0$. The first problem was solved by Bertrand and Darboux [1], [3], see also [4], [5]. The second problem was solved by Koenigs [2].

In [1] Bertrand stated and proved the following theorem (in fact under the additional assumption that the central potential is strongly closing, see Definition 2).

Theorem 1 (Bertrand [1], 1873). *In Euclidean space there exist exactly two laws of attraction with an analytic central potential for which every trajectory of a point P moving around a fixed point O (under the condition that the coordinates of the initial position of the point and the components of its initial velocity are not proportional and the initial speed of the point is less than a certain limiting value depending on the initial position of the point P) is closed, but not necessarily non-self-intersecting. These laws are Newton's law with an attractive force $F_1 = -G/r^2$ and Hooke's law with an attractive force $F_2 = -kr$, where $G > 0$, $k > 0$. For the force law F_β, nonsingular bounded noncircular orbits are defined by periodic functions $r = r(\varphi)$ with minimal positive period $\Phi_\beta = 2\pi/\beta$, $\beta = 1, 2$.*

Furthermore, for both potentials (Newtonian, that is, gravitational, $V_1(r) = -G/r$, and Hookean, that is, oscillatory, $V_2(r) = kr^2/2$), the geometric form of the orbits is the same: these are conic sections, and in the case of closed orbits they are ellipses with focus or centre at the point of attraction. An analogue of Theorem 1 for a sphere and the Lobachevskiĭ plane (with any negative curvature) was proved by Liebmann [6].

The arguments in [1], [6], [7] are based on the following assertion, which follows from [1] and which we shall call Bertrand's technical theorem.

Theorem 2 (Bertrand [1]). *Consider the one-parameter family of differential equations*

$$\frac{d^2 z}{d\varphi^2} + z = \frac{1}{K^2}\Psi(z)$$

on the ray $z > 0$ with parameter $K \in \mathbb{R} \setminus \{0\}$, where $\Psi = \Psi(z)$ is an analytic function such that $\Psi(z) > 0$. (The function $(1/K^2)\Psi(z)$ is called a force function or a function of external forces, and the function $-z$ is called a centrifugal force or an internal force.) We say that the function $\Psi(z)$ is rationally closing if

(i) *for all K all the bounded nonconstant solutions $z = z(\varphi)$ are periodic functions with periods commensurable with 2π,*

(ii) *every point $z > 0$ is a nondegenerate stable equilibrium of the equation for $|K| = \sqrt{\Psi(z)/z}$.*

There exist two and only two rationally closing functions Ψ up to a multiplicative constant: $\Psi(z) = A/z^{\beta^2-1}$, where $\beta = 1, 2$ and $A > 0$ is an arbitrary multiplicative constant. Furthermore, all the bounded nonconstant solutions are periodic functions with minimal positive period $\Phi = 2\pi/\beta$.

In fact, Theorem 2 and certain restrictions in it on the function $\Psi(z)$ and the multiplicative constant A (namely, $\Psi(z) > 0$, (ii), $A > 0$) were not stated explicitly in [1], but this theorem was proved in [1] under these restrictions. Theorem 2 is in a sense a re-formulation of Theorem 1 (and its analogue for the sphere and the Lobachevskiĭ plane, see [6], [7]): here, $z = 1/r$, the function $z(\varphi)$ characterizes the dependence of the distance $r(\varphi)$ on the angle, and its $(2\pi/\beta)$-periodicity (for rational $\beta > 0$) corresponds to the trajectory of motion of a point being closed, and the function $-\Psi(z)$ is the derivative of the potential $V(1/z)$ with respect to the variable z.

In what follows, a one-parameter family of differential equations of the form

$$\frac{d^2 z}{d\varphi^2} + z = \frac{1}{K^2}\Psi(z), \qquad K \neq 0,$$

is called a *family of Bertrand's equations*. It follows from Theorem 2 that if the equations of orbits of the point $z(\varphi)$ form a family of Bertrand equations, where K is the value of the kinetic momentum integral, then the condition that bounded orbits be closed (and that such orbits exist) implies that the potential $V(r)$ has one of the two specific forms (up to an additive and a multiplicative constant).

We call the constant β in Theorems 1, 2 the *Bertrand constant* corresponding to a given force law. This constant is equal to 1 for the gravitational law, and to 2 for the oscillatory law on the following classical surfaces: the Euclidean plane (see Theorem 1), the sphere and the Lobachevskiĭ plane (see [6] or Theorem 5 for $c \neq 0$, $\xi = 1$).

Remark 1. If in condition (i) of Theorem 2 we require that all periods are commensurable with the number $2\pi\xi$ (instead of 2π) for some $\xi > 0$, then we obtain the definition of a $2\pi\xi$*-closing* function. As shown by the proofs of the technical Theorem 2 in [1], [7], its assertion remains valid if in condition (i) we require merely pairwise commensurability of the periods (instead of commensurability with 2π). Hence for irrational ξ there does not exist any $2\pi\xi$-closing function Ψ. This implies a generalization of Theorem 1 to the case of all (not necessarily rational) cones (see Corollary 1 and § 3), as well as a generalization of Liebmann's theorem [6] to the case of all (not necessarily rational) 'real branched coverings' of the sphere and the Lobachevskiĭ plane (see Theorem 5 for $c \neq 0$).

1.2. Statement of the problem and Darboux's description of all Bertrand pairs. Instead of the Euclidean plane we consider some *abstract surface of revolution*, that is, a two-dimensional surface $S \approx (a, b) \times S^1$ with the Riemannian metric

$$ds^2 = dr^2 + f^2(r)\, d\varphi^2, \qquad (r, \varphi \mod 2\pi) \in (a, b) \times S^1, \tag{1}$$

where $f = f(r)$ is an infinitely smooth, positive function on the interval (a, b), $-\infty \leqslant a < b \leqslant \infty$. On this surface we consider the system consisting of a fixed attracting centre ('Sun') and an attracted point ('planet'). A natural question arises: what can be said about the form of central potentials $V(r)$ (laws of attraction) for which all the bounded orbits are closed (and these orbits exist)? In fact, Darboux [4], [5] and Perlick [8] obtained an answer to this question for surfaces without equators (see Theorem 3 and Remark 7), although Darboux posed this question only for genuine (non-abstract) surfaces of revolution in $\mathbb{R}^3$. The question of studying the geometric and dynamical properties of these surfaces also arises, which was partially considered in [9]–[13], and of their classification up to isometry, similarity, and other equivalence relations.

It is also desirable to understand to which abstract surfaces of revolution Theorem 1 can be generalized: on which surfaces there exist exactly two central potentials that ensure the closedness of all bounded trajectories. Do there exist surfaces on which the number of required central potentials (up to an additive and a multiplicative constant) differs from zero and two? A partial answer to this question was recently obtained by Santoprete [12] (see Theorem 4).

To define a closing potential we introduce several definitions.

In what follows, by a *trajectory* we mean a solution $\vec{r}(t)$ of the equation of motion defined on a maximal interval by inclusion $(t_0, t_1) \subset \mathbb{R}^1$, by an *orbit* the image of this map $O = \{\vec{r} = \vec{r}(t) \mid t \in (t_0, t_1)\} \subset S$, by a *phase trajectory* the function $(\vec{r}(t), \dot{\vec{r}}(t))$ with values in the tangent space TS, and by a *phase orbit* its image

$$\widetilde{O} = \{(\vec{r} = \vec{r}(t),\ \dot{\vec{r}} = \dot{\vec{r}}(t)) \mid t \in (t_0, t_1)\} \subset TS.$$

Definition 1. (a) An orbit of a point moving on the surface S according to the force law defined by a central potential $V(r)$ (that is, depending only on r) is said to be *circular* if it coincides with an orbit of the action of the rotation group. A trajectory is said to be *circular* if the corresponding orbit is circular. A circular orbit is called *strongly stable* if the function $V(r) + K_0^2/(2f^2(r))$, which is called the *effective potential*, has a nondegenerate local minimum on this orbit for some $K_0 \neq 0$.

A closed orbit is called *orbitally stable* if the corresponding phase trajectory is orbitally stable for the restriction of the system to the level set of the kinetic momentum containing this phase trajectory.

(b) A trajectory is called *bounded* if it is defined on the entire time axis $t \in \mathbb{R}^1$, and its image is contained in some compact set $[r_1, r_2] \times S^1 \subset (a, b) \times S^1$. An orbit is called *bounded* if the corresponding trajectory is bounded.

(c) A trajectory (and the corresponding orbit) is called *singular* if the value of the kinetic momentum integral K on this trajectory is equal to 0, that is, $\varphi = \text{const}$.

Remark 2. (a) Suppose that we have a two-dimensional surface S with the metric (1), where $f(r)$ is a smooth function, and a central smooth potential $V(r)$ on this surface. We can verify that a circle $\{r_0\} \times S^1$ is a circular orbit if and only if r_0 is a critical point of the effective potential $V(r) + K_0^2/(2f^2(r))$, where K_0 is the value of the kinetic momentum integral on the corresponding trajectory. We note right away that $f > 0$. Now suppose that $f'(r_0)$ and $V'(r_0)$ have the same sign or are simultaneously equal to zero. Then in this case, and only in this case, the circle $\{r_0\} \times S^1$ is a circular orbit, and the value of the kinetic momentum on the corresponding trajectory is equal to

$$K_0 = \pm\sqrt{\frac{V'(r_0)}{f'(r_0)} f^3(r_0)},$$

or to any $K_0 \neq 0$, respectively.

(b) A circle $\{r\}\times S^1$ is a strongly stable circular orbit if and only if it is a circular orbit and the corresponding phase orbit is the Bott critical set of local minima of the restriction of the energy integral to the level surface of the kinetic momentum integral in the phase space. Hence strongly stable circular orbits are orbitally stable.

Note that if a trajectory is unbounded in the sense of Definition 1(b), then the planet 'reaches the edge of the surface'.

We shall study the following five classes of central potentials on abstract surfaces of revolution, and we shall also describe all the surfaces without equators which admit such potentials.

Definition 2. Let $V(r)$ be a central potential on a surface S with the metric (1). This potential is said to be *closing* if

($\exists$) there exists a nonsingular bounded noncircular orbit γ in S,

($\forall$) every nonsingular bounded orbit in S is closed.

The potential $V(r)$ is said to be *locally closing* if

$(\exists)^{\mathrm{loc}}$ there exists a strongly stable circular orbit $\{r_0\}\times S^1$ in S,

$(\forall)^{\mathrm{loc}}$ for every strongly stable circular orbit $\{r_0\}\times S^1$ in S there exists $\varepsilon>0$ such that every nonsingular bounded orbit that is entirely contained in the annulus $[r_0-\varepsilon, r_0+\varepsilon]\times S^1$ and has kinetic momentum level in the interval $(K_0-\varepsilon, K_0+\varepsilon)$ is closed, where K_0 is the value of the kinetic momentum on the corresponding circular trajectory.

The potential $V(r)$ is called *semilocally closing* if conditions ($\exists$), $(\forall)^{\mathrm{loc}}$ hold and the following condition holds:

$(\forall)^{\text{s-loc}}$ any nonsingular bounded orbit in the annulus $U=[a',b']\times S^1$ with kinetic momentum level $\widehat{K}$ is closed, where $a':=\inf r|_\gamma$, $b':=\sup r|_\gamma$, γ is a bounded orbit in ($\exists$), and $\widehat{K}$ is the value of kinetic momentum on this orbit.

The potential $V(r)$ is called *strongly* (*weakly*) *closing* if condition $(\forall)^{\mathrm{loc}}$ holds (its analogue for every orbitally stable circular orbit holds, respectively) and the following condition holds, which is used in [1], [3]–[5], [12] ([8], respectively): any circle $\{r\}\times S^1$ is a strongly stable (orbitally stable, respectively) circular orbit.

Remark 3. (a) Obviously, any closing central potential $V(r)$ is semilocally closing, and any strongly closing potential is locally and weakly closing (see Remark 2). It is easy to show (using the first integrals of total energy and kinetic momentum) that any weakly closing central potential is semilocally closing. We shall show (see Theorem 7(A) or the proof of Theorem 8, step 1) that if $V(r)$ is semilocally closing, then it is locally closing, and there exists a strongly stable circular orbit $\{r=r_0\}$ on which the value of the kinetic momentum integral K coincides with $K|_\gamma$ for a nonsingular bounded noncircular trajectory γ in ($\exists$). Thus, all potentials in Definition 2 are locally closing (without assuming there are no equators on the surface, see below).

(b) We shall show (Theorems 5, 6, 7(B)) that if a surface has no *equators* (that is, circles $\{r\}\times S^1$ such that $f'(r)=0$), then all five notions of a closing potential $V(r)$ (see Definition 2) are equivalent, and the triple $(f(r), V(r), \beta)$ has a specific form (see Theorem 3 or Theorems 5, 6): either the form

$$\big(\xi f_{c,0}(\pm(r-r_0)), V_{c,0,i}(\pm(r-r_0)), i\xi\big)$$

for some $i \in \{1,2\}$, $c \in \mathbb{R}$, and $\xi \in \mathbb{Q} \cap \mathbb{R}_{>0}$, or the form

$$\left(\frac{f_{c,d}(\pm(r-r_0))}{\mu}, V_{c,d,2}(\pm(r-r_0)), \frac{2}{\mu}\right)$$

for some $c \in \mathbb{R}$, $d \in \mathbb{R} \setminus \{0\}$, and $\mu \in \mathbb{Q} \cap \mathbb{R}_{>0}$, where $\{\xi f_{c,0}(r)\}$ and $\{f_{c,d}(r)/\mu\}$ are two-parameter and three-parameter families of surfaces (Bertrand surfaces of the first and second type consisting of one or two connected components). Here, $\{V_{c,0,1}(r)\}$ and $\{V_{c,d,2}(r)\}$ are the corresponding families of closing central potentials up to an additive and a positive multiplicative constant, $\xi, c, d, \mu \in \mathbb{R}$ are the parameters of the families (see the table in § 2.2), and $\beta = 2\pi/\Phi$ is the Bertrand constant.

(c) In contrast to Theorems 1 and 3 (as well as to Perlick's theorem [8]), in our definitions of a closing, locally or semilocally closing potential we do not require that all the circles $\{r\} \times S^1$ be strongly (or orbitally) stable circular orbits. Only in one of the definitions is the existence of at least one such circle required.

The following assertion follows from Darboux's paper [4] of 1877 (see also [5], [8]).

Theorem 3 (Darboux [4], [5], the Bertrand surface–potential pairs). *Suppose that on a surface S with the Riemannian metric* (1) *a central potential $V = V(r)$ is defined, where f and V are functions of class C^∞ that have no critical points. The potential V is strongly closing* (*see Definition* 2) *if and only if in the coordinates* $(V, \varphi \mod 2\pi)$ *the Riemannian metric on S has at least one of the following forms*:

$$ds^2 = \frac{A\ dV^2}{\beta^2(AV^2 - BV + C)^2} + \frac{d\varphi^2}{AV^2 - BV + C}$$

or

$$ds^2 = \frac{A\ dV^2}{\beta^2(-V-K)^3(A/(-V-K) - BV + C)^2} + \frac{d\varphi^2}{A/(-V-K) - BV + C},$$

where $A, B, C, K \in \mathbb{R}$, $\beta \in \mathbb{Q} \cap \mathbb{R}_{>0}$ are constants such that $-2AV + B > 0$ or $-A/(V+K)^2 + B > 0$, respectively. Furthermore, nonsingular bounded noncircular orbits are given by periodic functions $r = r(\varphi)$ with minimal positive period $\Phi = 2\pi/\beta$. Surfaces of the same form corresponding to the tuples (β, A, B, C) and $(\beta/\alpha, \alpha^2 A, \alpha^2 B, \alpha^2 C)$ for $\alpha \in \mathbb{Q} \cap \mathbb{R}_{>0}$ are locally isometric to one another.

In fact, neither Theorem 3 and the formulae for the Riemannian metric indicated in it, nor the restrictions on the functions f, V and the constants A, B, K, were stated explicitly in the papers [4], [5] (which were mainly devoted to realizable surfaces of revolution in $\mathbb{R}^3$, rather than to abstract surfaces of revolution). However, it is under these restrictions that this theorem was in fact proved in [4], [5]. (More precisely, the formulae for the Riemannian metric indicated above follow from formula (15.17) in [5] after substituting the solutions (15.10) and (15.11) into it, in view of relation (15.8) and the notation (15.4), while the other assertions of Theorem 3 follow from relations (15.3) and (15.19) in [5].) Perlick [8] generalized Theorem 3 (in different notation) to a wider class of surfaces and potentials: to surfaces of class C^5 without equators and weakly closing central potentials of class C^5, by expressing the Riemannian metric in terms of the coordinates

$(f, \varphi \mod 2\pi)$ (see Remark 7 for the connection between the parameters used by Darboux and Perlick). Although in [8] the result was stated for an even wider class of potentials, it was only proved rigorously for strongly closing potentials, and not quite rigorously for weakly closing potentials.

1.3. The development of the problem, Santoprete's result. The motion of a point in a central force field on various Riemann surfaces has been studied repeatedly by many researchers (see, for example, the collection of papers [14]). Furthermore, they not only studied the 'direct problem of dynamics' — the study of the properties of motion under a given law, but also the 'inverse problem of dynamics' — the search for laws of motion according to which the trajectories of motion would have certain properties (for example, be closed (Bertrand's problem) or algebraic curves (Koenigs' problem)). A detailed history of these problems is given in [15].

An analogue of a Newtonian force as a quantity that is inversely proportional to the area of a sphere of radius r was proposed for the space H^3 by Lobachevskiĭ [16] and Bolyai [17]. In 1860 Serret [18] defined an analogue of gravitational potential on a sphere and solved the Kepler problem on it. In 1870 Schering [19] wrote down an analytic expression for Newton's potential on H^3. In 1873 Lipschitz considered the motion of a body in a central field on the sphere S^2 with the standard metric; however, instead of the potential $-1/\tan r$, he considered the potential $-1/\sin r$. He found a general solution of this problem in terms of elliptic functions [20]. In 1885 Killing [21] generalized Kepler's laws to the sphere S^3 equipped with the standard metric. Similarly to Lobachevskiĭ and Bolyai, he considered a force of attraction as a quantity inversely proportional to the area of a two-dimensional sphere of radius r in S^3. In the following year these results were proved again by Neumann [22]. Killing [21] also proved that the variables in Kepler's problem with two attracting centres on the sphere S^n with the standard metric can be separated, which implies that the problem is integrable. In 1902 Liebmann [7] extended these results to H^3.

In the 1940s this problem was looked at in the framework of relativity theory, namely, the quantum-mechanical one-particle spectral problem for the Newtonian potential was solved on the sphere S^3 by Schrödinger and Stevenson, and on H^3 by Infeld and Schild. In the 1980s central potentials on S^3, H^3, S^n were studied by Kurochkin, Otchik, Bogush, Leemon in the framework of relativity theory. In 1994 Kozlov rediscovered Kepler's laws for spaces of constant sectional curvature [10]. In the same year together with Fëdorov he established the integrability of the classical motion of one particle on the sphere S^n in the field created by Hookean potentials situated at the $2(n+1)$ intersection points of the sphere with the coordinate axes.

As for the inverse problem of dynamics, in 1870 Bertrand [1] solved the problem of finding all the strongly closing (see Definition 2) central potentials on the Euclidean plane. Then in 1877 Darboux described all the Bertrand surface–potential pairs $(f(r), V(r))$, that is, all the abstract surfaces of revolution without equators with strongly closing central potentials on them (see Theorem 3). In particular, he constructed generalized gravitational and oscillatory potentials on 'rational' cones and on 'rational coverings' of the sphere S^2, constructed a generalized oscillatory potential on a certain pear-shaped surface (corresponding to the arc ℓ_4 in our space of parameters, see Fig. 1) and on its 'rational coverings', and proved all its geodesics were closed. In 1902 Liebmann [7] constructed a generalization of the oscillatory potential on S^3 and H^3. Here, in the generalization of the Newtonian potential he required that Kepler's first law hold (that is, that the orbits are ellipses with foci at the centre of the field). The definition of the oscillatory potential was drawn from the requirement that the motion of a particle occurs over ellipses with centre coinciding with the centre of the field. In 1903 Liebmann [6] proved a generalization of Theorem 1 to S^2 and H^2. Later, in 1979, a generalization of Theorem 1 to the space S^n was proved by Higgs [23]. In turn, in 1980, a generalization of Theorem 1 to S^3 was proved by Slavyanovskiĭ [24], and to S^n and H^n by Ikeda and Katayama [25] in 1982. In 1992 Kozlov and Kharin rediscovered the Newtonian and Hookean potentials for the space S^n as a complete solution of the generalized Bertrand problem [9]. In the same year Perlick [8] obtained a description of all weakly closing Bertrand pairs $(f(r), V(r))$ of class C^5 without equators, analogous to Darboux's description (see Theorem 3), by expressing them in the coordinates $(f, \varphi \mod 2\pi)$ using parameters (see Remark 7).

In 2008 Santoprete [12] generalized Theorem 1 to analytic surfaces of revolution without equators which have constant Gaussian curvature and are embedded in $\mathbb{R}^3$. He also proved that on the other surfaces of revolution without equators the number of strongly closing central potentials does not exceed one, and indicated the form of this potential, and gave a necessary condition on the metric (which is also in fact sufficient) in the form of a biquadratic equation on the Bertrand constant for the existence of exactly one potential.

Theorem 4 (Santoprete [12]). *Suppose that $S \subset \mathbb{R}^3$ is a two-dimensional surface of revolution with coordinates $(r, \varphi \mod 2\pi) \in (a, b) \times S^1$ with the analytic Riemannian metric (1) such that the function f has no critical points on (a, b). Then in the class of analytic central potentials on S*

1. *there exist at most two strongly closing potentials (up to an additive and a positive multiplicative constant);*
2. *there are exactly two if and only if $f''f - (f')^2 = -\xi^2$, where ξ is a positive rational constant, and these potentials are the generalized gravitational potential $V_1(r)$ and the generalized oscillatory potential $V_2(r)$;*
3. *if the potential is unique, then $-f''f + (f')^2 =: h$ is not a constant, and the potential has the form of a generalized oscillatory potential.*

Moreover, for any strongly closing central potential, nonsingular bounded noncircular orbits are given by periodic functions $r = r(\varphi)$ with one and the same minimal positive period $\Phi = 2\pi/\beta$, where β is a positive rational constant depending on the potential and satisfying the biquadratic identity

$$\beta^4 - 5\big(-f''f + (f')^2\big)\beta^2 - 5ff''(f')^2 + 4(f'')^2f^2 - 3f'''f'f^2 + 4(f')^4 = 0;$$

different constants correspond to different potentials (up to an additive and a positive multiplicative constant); the constant $\beta_i = i\xi$, $i = 1, 2$, corresponds to the potential $V_i(r)$ in part 2.

Remark 4. For a surface of revolution with the metric (1), the scalar Riemannian curvature R is calculated by the formula $R/2 = c := -f''/f$, where c is the Gaussian curvature of the surface if the surface can be embedded into $\mathbb{R}^3$ (see also Corollary 2(C)). Hence, $h' = -f'''f + f''f' = f^2c'$, where $h := -f''f + (f')^2$. In other words, if condition 2 of Theorem 4 is valid, then the Gaussian curvature of the surface S is constant. A description of all abstract surfaces of revolution of constant scalar curvature and without equators can be obtained from Lemma 2 and Remark 9.

Remark 5. (a) In [12] Santoprete did not formulate the condition that the function f have no critical points, and considered closing potentials instead of strongly closing potentials. Nor did he state part 3 and the last assertion of Theorem 4 as separate assertions. However, these assertions follow from his paper, and these are the necessary conditions for conducting his proof. The part 'if' of part 2 of Theorem 4 follows easily from the theorems of Bertrand and Liebmann [7], [6], while parts 1, 2 (the part 'only if') and part 3 follow easily from the last assertion of the theorem, since $h := -f''f + (f')^2 = (\beta_1^2 + \beta_2^2)/5$, where the β_i are roots of the biquadratic equation given above, and if h is not constant, then at most one constant root β can exist. We point out that the biquadratic equation in Theorem 4 has the form $\beta^4 - 5\beta^2 h + 3ff'h' + 4h^2 = 0$ and turns into Tikochinsky's biquadratic equation [26] in the case of metrics of constant curvature (see Remark 4).

(b) Theorem 4 only looks at surfaces embedded into $\mathbb{R}^3$ as surfaces of revolution, which fact imposes certain conditions on the function $f(r)$. Namely, for a surface with the metric (1) to be embeddable into $\mathbb{R}^3$ as a surface of revolution it is necessary that $|f'(r)| \leqslant 1$ (and sufficient that $|f'(r)| < 1$). By (2), under the hypotheses of part 2 of Theorem 4 this inequality is equivalent to the inequality $|\xi| \leqslant 1$ if $c \geqslant 0$ (see Remark 4) and $\inf f = 0$ (for example, when the interval (a, b) is maximal), and to the inequality

$$|\xi| \leqslant \min\left\{\frac{1}{\cosh(\sqrt{-c}(b - r_0))}, \frac{1}{\cosh(\sqrt{-c}(a - r_0))}\right\}$$

if $c < 0$.

1.4. Some unsolved problems. We list some apparently unsolved problems which are closely connected with the one studied in this paper.

1) Study an analogue of Bertrand's problem on a right cylinder and other abstract surfaces of revolution with equators (see Remarks 3(b), 6), on *pseudo-Riemannian surfaces of revolution* with the pseudo-Riemannian metric $ds^2 = dr^2 - f^2(r)\,d\varphi^2$, as well as in the presence of two central force fields — a potential field and an ideal magnetic field (see the example in [11], § 5).

2) Describe the maximal surfaces of revolution embedded in $\mathbb{R}^3$ (with equators, or without) that have closing central potentials (see Remarks 5(b), 6 and Corollary 2(B)).

3) (Fomenko) Consider a smooth Riemannian manifold M with a distinguished point on it — the 'Sun'. Suppose that another point (call it a 'planet') is moving on M in accordance with Newton's equations under the action of a force of attraction depending only on the distance from the 'Sun'. Suppose for simplicity that the manifold is two-dimensional (although not necessarily a surface of revolution) and

that nonsingular trajectories of the motion are bounded (see Definition 1), although they are not necessarily closed, that is, each one is contained inside some annulus determined by the initial data of the motion of the 'planet'. The planet is moving inside the annulus, and in the case of general position its trajectory may cover this annulus densely. This happens, for example, if the corresponding dynamical system on the cotangent bundle T^*M to M admits an integral of motion whose regular level surfaces are two-dimensional tori. The integral trajectory of the system is moving on these tori. Under the projection of the cotangent bundle T^*M onto M, the tori are projected to annuli. The problem is posed as follows: find the form of the potential ('law of the force of attraction') that ensures just this motion of the 'planet' inside the annuli. To begin with, it would be interesting to examine the case of a triaxial ellipsoid. Of course, it is assumed in the statement of the problem that the initial speeds of the 'planet' are less than a certain quantity and that the 'planet does not fall into the Sun'. Naturally, a multi-dimensional problem can also be stated in a similar fashion.

In solving the problem, one needs to use the information on the topology of the foliation into Liouville tori and their bifurcations ('atoms') that appear for integrable Hamiltonian systems, see [27]–[31].

4) Study the dynamics and the topology of the Liouville foliation in the class of problems about the motion of a particle on an abstract surface of revolution $S \approx (a,b)\times S^1$ with the Riemannian metric (1) in a force field given by an arbitrary (not necessarily closing) central potential. In particular, the problem is integrable, and has an additional first integral — kinetic momentum; hence (for example, in the case of isoenergy nondegeneracy) the closure of almost any orbit of the planet is the annulus in which the effective potential is less than a fixed energy level (see Problem 3). Construct and study the action variables, the Liouville foliation in the phase space (more precisely, in the domain in the phase space in which the common level sets of the integrals of energy and kinetic momentum are compact and connected, and are therefore tori) into Liouville tori, and, on the base of the foliation (which is called the bifurcation complex), study the affine structure which results and construct the integer lattice of the action variables.

The paper is structured as follows. In § 2 we state the main results of the paper (Theorems 5–8, Corollaries 1–3, and Comments 2, 3). In § 3 we discuss the connection between Bertrand's problems on the Euclidean plane and on a cone. In § 4 we prove the generalized technical Bertrand theorem (Theorem 8). In § 5 we prove Theorems 5–7 and Corollaries 2, 3.

The authors are grateful to A. T. Fomenko for posing the problem, to A. V. Bolsinov, A. V. Borisov, M. D. Malykh, S. Yu. Nemirovski, A. A. Oshemkov and A. V. Shchepetilov for useful discussions.

§ 2. The statement of the main results

2.1. A generalization of Bertrand's theorem to abstract surfaces of revolution without equators.

Theorem 5 (C^∞-smooth theorem for the Bertrand surfaces of the first type). *Suppose that S is a smooth two-dimensional surface diffeomorphic to $(a,b)\times S^1$ and equipped with the Riemannian metric* (1) (*that is*, *an abstract surface of revolution*).

Suppose that the function f satisfies the identity $f''f - (f')^2 = -\xi^2$, where $\xi > 0$ is rational, that is, f has one of the following forms:

$$f(r) = \xi f_c(r - r_0) := \begin{cases} \pm\xi(r - r_0), & c = 0, \\ \dfrac{\xi}{\sqrt{c}} \sin(\sqrt{c}\,(r - r_0)), & c > 0, \\ \pm\dfrac{\xi}{\sqrt{-c}} \sinh(\sqrt{-c}\,(r - r_0)), & c < 0, \end{cases} \tag{2}$$

where c is one-half of the scalar Riemannian curvature of this surface; in this case the curvature is constant; $2\pi\xi$ is the total angle at the conical point of the surface (the centre of the field). Suppose further that the function $f'(r)$ has no zeros on the interval (a, b). Then in the class of central potentials on S there exist two and only two (up to an additive and a multiplicative constant) semilocally closing (locally closing, closing, strongly or weakly closing, respectively) potentials $V_1(r)$, $V_2(r)$.

These closing potentials are the generalized gravitational potential $V_1(r)$ and oscillatory potential $V_2(r)$, that is, they have the form

$$V_i(r) = \frac{(-1)^i A|\theta(r)|^{2-i^2}}{i} + B, \qquad i = 1, 2,$$

where $A > 0$, B are some constants and

$$\theta(r) = -\frac{f'(r)}{f(r)} = \pm\sqrt{\frac{\xi^2}{f^2(r)} - c}.$$

The nonsingular noncircular bounded orbits corresponding to the potential $V_i(r)$ are given by periodic functions $r = r(\varphi)$ with minimal positive period $\Phi_i = 2\pi/(i\xi)$, $i = 1, 2$. Furthermore, on the phase trajectory corresponding to a circular orbit $\{r\} \times S^1 \subset (a, b) \times S^1$, the value of the kinetic momentum K is equal to

$$K_i = \pm\xi\sqrt{\frac{A_i}{|\theta(r)|^{i^2}}}, \qquad i = 1, 2.$$

The boundary circle $\{\widehat{r}\} \times S^1$ on which the infimum $\inf f(r)$ (that is, $\sup|\Theta(r)|$) is attained is an attracting centre of the field with closing potential $V_i(r)$ (that is, $\inf V_i(r)$ is attained on it).

Let $\xi = p/q$. Then a q-sheeted covering $\widetilde{S}$ of every such surface S can be represented as a branched p-sheeted covering (with one or two branching points) of one of the three 'basic' surfaces: Euclidean plane, sphere, Lobachevskiĭ plane. Here the covering space $\widetilde{S}$ is determined by the following property: this is a branched covering of minimal degree that locally isometrically covers a punctured plane (sphere, Lobachevskiĭ plane, respectively).

As an illustration we consider a special case of Theorem 5: the case of a cone (that is, $c = 0$) which cannot necessarily be embedded into three-dimensional ambient space as a surface of revolution (that is, it is not necessarily true that $\xi \leqslant 1$).

Corollary 1 (C^∞-smooth closing central potentials on cones). *Suppose that $S \approx (0,+\infty) \times S^1$ is the standard cone with the Riemannian metric*

$$ds^2 = dr^2 + \xi^2 r^2\, d\varphi^2, \qquad (r, \varphi \mod 2\pi) \in (0,+\infty) \times S^1, \tag{3}$$

where $\xi > 0$, that is, the angle at the apex of the cone is equal to $2\pi\xi$. If ξ is not a rational number, then there do not exist closing (locally, semilocally, strongly or weakly closing) central potentials on the cone under consideration. If ξ is rational, then there exist two and only two (up to an additive and a multiplicative constant) closing (locally, semilocally, strongly or weakly closing) central potentials on the cone: the gravitational and oscillatory potentials

$$V_i(r) = \frac{(-1)^i A r^{i^2-2}}{i} + B, \qquad i = 1, 2,$$

where $A > 0$, B are arbitrary constants. The corresponding nonsingular noncircular bounded orbits are given by periodic functions $r = r(\varphi)$ with minimal positive period $\Phi_i = 2\pi/(i\xi)$. The apex of the cone $\{0\} \times S^1$ is an attracting centre of the field for each potential V_1, V_2.

In the special case of strongly closing analytic potentials, Theorem 5 and Corollary 1 follow from the strengthened version of the technical Theorem 2 (see Remark 1), and under the additional condition $\xi \leqslant 1$, from Theorem 4.

Theorem 6 (C^∞-smooth theorem for the Bertrand surfaces of the second type). *Suppose that S is a smooth two-dimensional surface diffeomorphic to $(a,b) \times S^1$ and equipped with the Riemannian metric (1). Suppose that the function f does not satisfy the identity $f''f - (f')^2 = -\xi^2$ for any rational $\xi > 0$, and suppose that the function $f(r)$ has no critical points on (a,b). Then there exists at most one semilocally closing (locally closing, closing, strongly or weakly closing) central potential (up to an additive and a multiplicative constant). Furthermore, there is exactly one potential (up to an additive and a multiplicative constant) if and only if there exists a smooth function $\theta = \theta(r)$ without zeros on (a,b) such that $\theta'(r) > 0$ and the Riemannian metric in the coordinates $(\theta, \varphi \mod 2\pi)$ has the form*

$$ds^2 = \frac{d\theta^2}{(\theta^2 + c - d\theta^{-2})^2} + \frac{d\varphi^2}{\mu^2(\theta^2 + c - d\theta^{-2})}, \tag{4}$$

where μ is a positive rational constant, d is a nonzero constant, and c is an arbitrary real constant.

Furthermore, the function $\theta(r)$ and the triple of numbers (μ, c, d) are unique (if they exist), and the closing potential is a generalized oscillatory potential, that is, it has the form

$$V_2(r) = \frac{A}{2\theta^2(r)} + B, \qquad A, B \in \mathbb{R}, \quad A(\theta^4(r) + d) > 0.$$

The corresponding nonsingular noncircular bounded orbits are given by periodic functions $r = r(\varphi)$ with minimal positive period $\Phi = \pi\mu$. On the phase trajectory corresponding to a circular orbit $\{r\} \times S^1 \subset (a,b) \times S^1$, the value of the kinetic momentum K is equal to

$$K = \pm\frac{1}{\mu}\sqrt{\frac{A}{\theta^4(r) + d}};$$

the boundary circle $\{r_0\}\times S^1$ *on which the infimum* $\inf f(r)$ (*that is*, $\sup A|\theta(r)|$) *is attained, is an attracting centre of the field with the closing potential* V_2 (*that is*, $\inf V_2(r)$ *is attained on it*).

Remark 6. We point out that each of Theorems 5–7 supposes merely C^∞-smoothness (and not necessarily analyticity) of the functions $f(r), V(r)$ and includes five assertions:

1) a description of closing central potentials;
2) a description of semilocally closing central potentials;
3) a description of locally closing central potentials;
4) a description of strongly closing central potentials;
5) a description of weakly closing central potentials.

Under the hypotheses of Theorems 5–7, all the five classes of potentials coincide for abstract surfaces of revolution without equators (see Remark 3(b)). In Theorems 5–7 of this paper (as in [1], [3]–[5], [8], [12], and other papers known to us) surfaces with equators, for example, the cylinder, are not considered. The assumption that there are no equators is essential, since all surfaces of revolution with closed geodesics (that is, the Tannery surfaces, see [32], Theorem 4.13) possess a closing central potential equal to a constant. It can be shown that in the absence of equators, condition (4) for $c,d\in\mathbb{R}$, as a condition on the function f, is equivalent to the biquadratic equation in Theorem 4 for the Bertrand constant $\beta:=2\pi/\Phi=i/\mu$ depending on the type of the strongly closing potential V_i, $i=1,2$.

Remark 7. If we set $d=0$ and $\mu=1/\xi$ in Theorem 6, then the corresponding surfaces coincide with the surfaces in Theorem 5, and the potential with the oscillatory potential $V_2(r)$. The surface–potential pairs in Theorems 5 and 6 obviously coincide with the pairs in Theorem 3. The three-parameter family of pairs $(f_{\beta,K}(r),V_{K,G}(r))$ in Perlick's paper [8] coincides with the family for the gravitational potential described in Theorem 5 with the multiplicative constant $A_1=\xi/2$, where $\beta:=\xi$, $K:=-c\mu^2$, $G:=-2B$ are the parameters of the family in [8]. The four-parameter family of pairs $(f_{\beta,D,K}(r),V_{D,K,G}(r))$ in Perlick's paper [8] coincides with the family described in Theorem 6 (for any $d\in\mathbb{R}$) for the multiplicative constant $A_2=\pm1/(2\mu^2)$, where $\beta:=2/\mu$, $D:=\mu^2c$, $K:=-4\mu^4d$, $G:=-2B$ are the parameters of the family in [8].

2.2. Geometry and classification of Bertrand surfaces. Below we construct a unified family of maximal analytic (generally speaking, non-connected) surfaces containing any of the surfaces described in Theorems 5 and 6 and study the geometry and classification of these surfaces.

We consider the family of functions $Q_{c,d}(\theta):=\theta^2+c-d\theta^{-2}$, $c,d\in\mathbb{R}$. We introduce the notation

$$I_{c,d}:=\bigl\{\theta\in\mathbb{R}\mid\theta<0,\ Q_{c,d}(\theta)>0,\ Q'_{c,d}(\theta)\neq0\bigr\}=:\bigcup_{k=1}^{k_{c,d}}I_{c,d,k},$$

where $I_{c,d,k}=(\theta_{c,d,k,\min},\theta_{c,d,k,\max})\subset I_{c,d}$ is the maximal interval by inclusion, k is the index of the interval, $k_{c,d}$ is the number of intervals, that is, $k_{c,d}:=1$ for $d\geqslant0$,

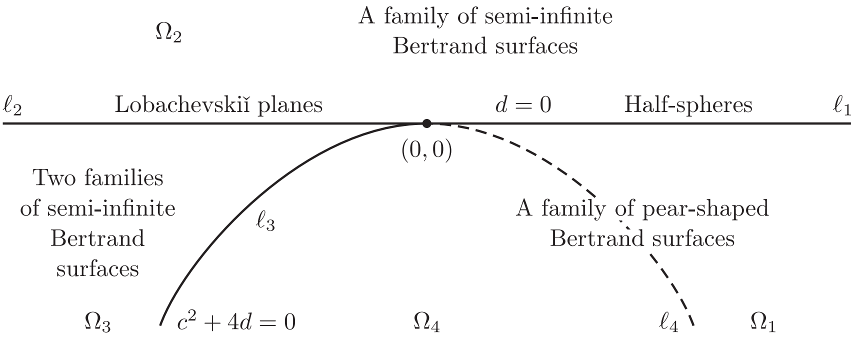

Figure 1. The plane of parameters $\mathbb{R}^2(c, d)$ divided into subsets: domains, curves, and a point (on the arc ℓ_4 there are no bifurcations of the family).

and $k_{c,d} := 2$ for $d < 0$. Let

$$r_{c,d}(\theta) = \int \frac{d\theta}{Q_{c,d}(\theta)}$$

be some antiderivative in the domain $I_{c,d}$.

Remark 8. The functions $r_{c,d}(\theta)$ and maximal intervals $I_{c,d,k}$ can be given by explicit formulae depending on $(c, d) \in \mathbb{R}^2$. Namely, we set $\Delta = \Delta(c, d) := c^2 + 4d$. The straight line $\{d = 0\}$ and the parabola $\{\Delta = 0\}$, which are tangent to each other, divide the plane $\mathbb{R}^2$ into the following subsets: the domains

$$\Omega_1 = \{\Delta > 0,\ d < 0 < c\}, \qquad \Omega_2 = \{d > 0\},$$
$$\Omega_3 = \{c < 0,\ d < 0 < \Delta\}, \qquad \Omega_4 = \{\Delta < 0\},$$

the curves

$$\ell_1 = \{d = 0 < c\}, \qquad \ell_2 = \{c < 0 = d\},$$
$$\ell_3 = \{c < 0 = \Delta\}, \qquad \ell_4 = \{\Delta = 0 < c\},$$

and the point $\{(0, 0)\}$ (see Fig. 1). In the domain $\overline{\Omega}_i$, let $x_i = x_i(c, d)$ and $y_i = y_i(c, d)$, where $(c, d) \in \overline{\Omega}_i$, $i \in \{1, 2, 3, 4\}$, denote the following functions:

$$x_1 := \sqrt{\frac{c - \sqrt{\Delta}}{2}}, \qquad x_2 = x_3 := \sqrt{\frac{-c + \sqrt{\Delta}}{2}}, \qquad x_4 := \sqrt{\frac{\sqrt{-d}}{2} - \frac{c}{4}},$$
$$y_1 = y_2 := \sqrt{\frac{c + \sqrt{\Delta}}{2}}, \qquad y_3 := \sqrt{\frac{-c - \sqrt{\Delta}}{2}}, \qquad y_4 := \sqrt{\frac{\sqrt{-d}}{2} + \frac{c}{4}}.$$

Then the function $r_{c,d}(\theta)$ can be given by the following formulae:

$$r_{c,d}(\theta) = \begin{cases} -\dfrac{1}{\theta}, & (c,d) = (0,0), \\ \dfrac{1}{y_1^2 - x_1^2}\left(-x_1 \tan^{-1}\dfrac{\theta}{x_1} + y_1 \tan^{-1}\dfrac{\theta}{y_1}\right), & (c,d) \in \Omega_1, \\ \dfrac{1}{y_1}\tan^{-1}\dfrac{\theta}{y_1}, & (c,d) \in \ell_1, \\ \dfrac{1}{x_2^2 + y_2^2}\left(\dfrac{x_2}{2}\ln\left|\dfrac{\theta - x_2}{\theta + x_2}\right| + y_2 \tan^{-1}\dfrac{\theta}{y_2}\right), & (c,d) \in \Omega_2, \\ \dfrac{1}{2x_2}\ln\left|\dfrac{\theta - x_2}{\theta + x_2}\right|, & (c,d) \in \ell_2, \\ \dfrac{1}{x_3^2 - y_3^2}\left(\dfrac{x_3}{2}\ln\left|\dfrac{\theta - x_3}{\theta + x_3}\right| - \dfrac{y_3}{2}\ln\left|\dfrac{\theta - y_3}{\theta + y_3}\right|\right), & (c,d) \in \Omega_3, \\ -\dfrac{1}{2}\dfrac{\theta}{\theta^2 - x_3^2} - \dfrac{1}{4x_3}\ln\left|\dfrac{\theta + x_3}{\theta - x_3}\right|, & (c,d) \in \ell_3, \\ \dfrac{1}{4y_4}\Bigg(\tan^{-1}\dfrac{\theta + x_4}{y_4} + \tan^{-1}\dfrac{\theta - x_4}{y_4} & \\ \qquad\qquad -\dfrac{1}{2}\ln\dfrac{(\theta + x_4)^2 + y_4^2}{(\theta - x_4)^2 + y_4^2}\Bigg), & (c,d) \in \Omega_4, \\ \dfrac{1}{2y_4}\tan^{-1}\dfrac{\theta}{y_4} - \dfrac{1}{2}\dfrac{\theta}{\theta^2 + y_4^2}, & (c,d) \in \ell_4. \end{cases}$$

The intervals $I_{c,d,k}$, on which the function $r_{c,d}(\theta)$ is defined, have the form

$$I_{c,d} = \bigcup_{k=1}^{k_{c,d}} I_{c,d,k} := \begin{cases} (-\infty, 0), & (c,d) \in \{(0,0)\} \cup \ell_1, \\ (-\infty, -x_3), & (c,d) \in \Omega_2 \cup \ell_2, \\ (-\infty, -x_3) \cup (-y_3, 0), & (c,d) \in \Omega_3 \cup \ell_3, \\ (-\infty, -\sqrt[4]{-d}) \cup (-\sqrt[4]{-d}, 0), & (c,d) \in \ell_3 \cup \Omega_4 \cup \ell_4 \cup \Omega_1. \end{cases}$$

The way these intervals depend on $(c,d) \in \mathbb{R}^2$ and $k \in \{1, k_{c,d}\}$ is shown in Fig. 2 and in the table below, together with the set of values of the function $r_{c,d}|_{I_{c,d,k}}$.

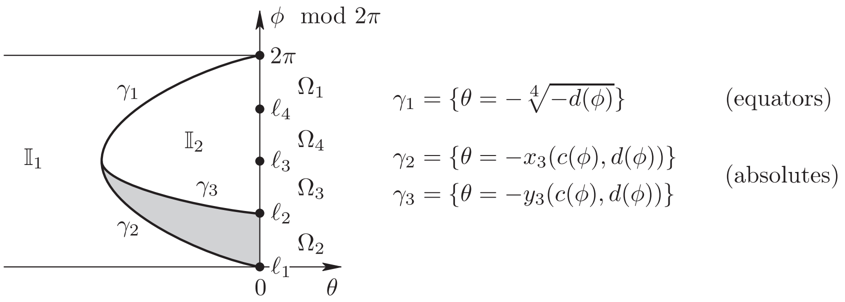

Figure 2. The unions of the intervals $\mathbb{I}_k := \bigcup_\phi I_{c(\phi),d(\phi),k} \times \{\phi\}$ for the family of Bertrand surfaces $S_{c(\phi),d(\phi),k}$ $(k = 1,2)$, where $(c(\phi), d(\phi)) := 8(\sqrt{2}\cos(\phi + \pi/4), \sqrt{2}\sin(\phi + \pi/4) - 1)$.

TABLE. Intervals $I_{c,d,k}$ and their images (see Comments 1–3)

<table>
<tr><td colspan="3">$(c,d)\in\ell_2$
Lobachevskiĭ plane</td><td colspan="3">$(c,d)\in\Omega_2$
Family of semi-infinite
Bertrand surfaces</td><td colspan="3">$(c,d)\in\ell_1$
Half-sphere</td></tr>
<tr><td>$I_{c,d}$
$r_{c,d}(I_{c,d})$
$F_{c,d}(I_{c,d})$
$R_{c,d}(I_{c,d})$
sgn A_2</td><td colspan="2">$(-\infty,-x_2)$
$(0,\infty)$
$(0,\infty)$
$2c$
$+$</td><td>$I_{c,d}$
$r_{c,d}(I_{c,d})$
$F_{c,d}(I_{c,d})$
$R_{c,d}(I_{c,d})$
sgn A</td><td colspan="2">$(-\infty,-x_2)$
$(-\frac{\pi y_2}{2\sqrt{\Delta}},\infty)$
$(0,\infty)$
$(2c,-\frac{2\Delta}{x_2^2})$
$+$</td><td>$I_{c,d}$
$r_{c,d}(I_{c,d})$
$F_{c,d}(I_{c,d})$
$R_{c,d}(I_{c,d})$
sgn A_2</td><td colspan="2">$(-\infty,0)$
$(-\frac{\pi}{2y_1},0)$
$(0,\frac{1}{y_1})$
$2c$
$+$</td></tr>
<tr><td colspan="3">$y_2=y_3=0,\ x_2=x_3=\sqrt{-c}$</td><td colspan="3">$x_2>0,\ y_2>0$</td><td colspan="3">$x_1=x_2=0,\ y_1=y_2=\sqrt{c}$</td></tr>
<tr><td colspan="3">$(c,d)\in\Omega_3$
Two families of semi-infinite
Bertrand surfaces</td><td colspan="3">$(c,d)=(0,0)$
Euclidean plane</td><td colspan="3">$(c,d)\in\Omega_1$
Family of pear-shaped
Bertrand surfaces</td></tr>
<tr><td>$I_{c,d,k}$
$r_{c,d}(I_{c,d,k})$
$F_{c,d}(I_{c,d,k})$
$R_{c,d}(I_{c,d,k})$
k, sgn A</td><td>$(-\infty,-x_3)$
$(0,\infty)$
$(0,\infty)$
$(2c,-\frac{2\Delta}{x_3^2})$
$1,\ +$</td><td>$(-y_3,0)$
$(-\infty,0)$
$(\infty,0)$
$(-\frac{2\Delta}{y_3^2},\infty)$
$2,\ -$</td><td>$I_{c,d}$
$r_{c,d}(I_{c,d})$
$F_{c,d}(I_{c,d})$
$R_{c,d}(I_{c,d})$
sgn A_2</td><td colspan="2">$(-\infty,0)$
$(0,\infty)$
$(0,\infty)$
0
$+$</td><td>$I_{c,d,k}$
$r_{c,d}(I_{c,d,k})$
$F_{c,d}(I_{c,d,k})$
$R_{c,d}(I_{c,d,k})$
k, sgn A</td><td>$(-\infty,-\sqrt[4]{-d})$
$(\frac{-\pi}{2(x_1+y_1)},-r_1)$
$(0,\frac{1}{x_1+y_1})$
$(2c,32y_4^2)$
$1,\ +$</td><td>$(-\sqrt[4]{-d},0)$
$(-r_1,0)$
$(\frac{1}{x_1+y_1},0)$
$(32y_4^2,\infty)$
$2,\ -$</td></tr>
<tr><td colspan="3">$0<y_3<x_3$</td><td colspan="3"></td><td colspan="3">$0<x_1<y_1$</td></tr>
<tr><td colspan="3">$(c,d)\in\ell_3$
Two semi-infinite
Bertrand surfaces</td><td colspan="3">$(c,d)\in\Omega_4$
Family of pear-shaped
Bertrand surfaces</td><td colspan="3">$(c,d)\in\ell_4$
Pear-shaped
Bertrand surface</td></tr>
<tr><td>$I_{c,d,k}$
$r_{c,d}(I_{c,d,k})$
$F_{c,d}(I_{c,d,k})$
$R_{c,d}(I_{c,d,k})$
k, sgn A</td><td>$(-\infty,-y_3)$
$(0,\infty)$
$(0,\infty)$
$(2c,0)$
$1,\ +$</td><td>$(-y_3,0)$
$(-\infty,0)$
$(\infty,0)$
$(0,\infty)$
$2,\ -$</td><td>$I_{c,d,k}$
$r_{c,d}(I_{c,d,k})$
$F_{c,d}(I_{c,d,k})$
$R_{c,d}(I_{c,d,k})$
k, sgn A</td><td>$(-\infty,-\sqrt[4]{-d})$
$(-\frac{\pi}{4y_4},-r_4)$
$(0,\frac{1}{2y_4})$
$(2c,32y_4^2)$
$1,\ +$</td><td>$(-\sqrt[4]{-d},0)$
$(-r_4,0)$
$(\frac{1}{2y_4},0)$
$(32y_4^2,\infty)$
$2,\ -$</td><td>$I_{c,d,k}$
$r_{c,d}(I_{c,d,k})$
$F_{c,d}(I_{c,d,k})$
$R_{c,d}(I_{c,d,k})$
k, sgn A</td><td>$(-\infty,-y_4)$
$(\frac{-\pi}{4y_4},-r_{4,1})$
$(0,\frac{1}{2y_4})$
$(2c,16c)$
$1,\ +$</td><td>$(-y_4,0)$
$(-r_{4,1},0)$
$(\frac{1}{2y_4},0)$
$(16c,\infty)$
$2,\ -$</td></tr>
<tr><td colspan="3">$x_3=y_3=x_4=\sqrt{-c/2},\ y_4=0$</td><td colspan="3">$x_4>0,\ y_4>0$</td><td colspan="3">$x_4=0,\ y_4=y_1=x_1=\sqrt{c/2}$</td></tr>
</table>

By construction each of the functions $r_{c,d}|_{I_{c,d,k}}\colon I_{c,d,k}\to r_{c,d}(I_{c,d,k})$ is strictly monotonic and therefore has an inverse function, which is denoted by $\theta_{c,d,k}=\theta_{c,d,k}(r)$. Thus, the function $\theta_{c,d,k}\colon r_{c,d}(I_{c,d,k})\to I_{c,d,k}$ is such that $\theta_{c,d,k}(r_{c,d}(\theta))\equiv\theta$. We consider the three-parameter family of abstract surfaces of revolution $S_{c,d}=\bigcup_{k=1}^{k_{c,d}}S_{c,d,k}$ with the Riemannian metrics $ds^2_{\mu,c,d}$, where

$$\begin{gathered} S_{c,d,k}=r_{c,d}(I_{c,d,k})\times S^1, \qquad ds^2_{\mu,c,d}|_{S_{c,d,k}}=dr^2+\frac{1}{\mu^2}f^2_{c,d,k}(r)\,d\varphi^2, \\ S_{c,d}\approx I_{c,d}\times S^1, \quad ds^2_{\mu,c,d}=\frac{d\theta^2}{(\theta^2+c-d\theta^{-2})^2}+\frac{d\varphi^2}{\mu^2(\theta^2+c-d\theta^{-2})}, \end{gathered} \tag{5}$$

where $c,d\in\mathbb{R}$, $\mu>0$ are parameters, $k\in\{1,k_{c,d}\}$, and

$$f_{c,d,k}(r):=\frac{1}{\sqrt{Q_{c,d}(\theta_{c,d,k}(r))}}.$$

This surface consists of $k_{c,d}$ connected components. The first connected component ($k=1$) is called the *main* one, the second ($k=2$ for $d<0$) the *additional* one, and the family (5) is called the *family of* (*maximal*) *Bertrand surfaces*.

Let $\gamma=\gamma(\phi)=(c_\phi,d_\phi)$, $0\leqslant\phi\leqslant2\pi$, be some curve in $\mathbb{R}^2$, not necessarily continuous, intersecting every set of the form $\{(\lambda c,\lambda^2 d)\mid\lambda>0\}\subset\mathbb{R}^2$ at a unique point. For example, the image of the curve $\gamma=\gamma(\phi)$ may be the union of the two straight lines $\mathbb{R}\times\{\pm1\}$ and the three points $\{(-1,0),(0,0),(1,0)\}$.

Theorem 7. *Suppose that a two-dimensional surface S is diffeomorphic to $(a,b)\times S^1$ and is equipped with the C^∞-smooth Riemannian metric* (1).

(A) *If a C^∞-smooth central potential is closing it is semilocally closing*; *if it is strongly closing it is weakly closing*; *if it is weakly closing it is semilocally closing*; *and if it is semilocally closing it is locally closing.*

(B) *Suppose that $f(r)$ has no critical points on (a,b). Then any C^∞-smooth locally closing central potential is closing and strongly closing, and the following assertions are equivalent*:

(a) *on the surface S there is a closing central potential $V(r)$*;

(b) *there exists a triple $(\mu,c,d)\in\mathbb{R}^3$ and a function $\theta=\theta(r)$ with no zeros on the interval (a,b) such that $\mu\in\mathbb{Q}\cap\mathbb{R}_{>0}$ and* (4) *holds, that is,*

$$\begin{gathered} f(r)=\frac{1}{\mu}f_{c,d,k}(\eta(r-r_0)), \quad \theta(r)=\eta\theta_{c,d,k}(\eta(r-r_0)), \qquad r\in(a,b), \\ \bigl(\eta(a-r_0),\eta(b-r_0)\bigr)\subseteq r_{c,d}(I_{c,d,k}) \end{gathered} \tag{6}$$

for some $r_0\in\mathbb{R}$, $\eta\in\{+,-\}$, and $k\in\{1,k_{c,d}\}$, that is, the surface (S,ds^2) can be isometrically and S^1-equivariantly embedded into the Bertrand surface (5) *by the map $(r,\varphi)\mapsto(\eta(r-r_0),\varphi)$*;

(c) *there exists a tuple (μ,λ,ϕ,r_0,k) such that $\mu\in\mathbb{Q}\cap\mathbb{R}_{>0}$, $\lambda\in\mathbb{R}\setminus\{0\}$, $0\leqslant\phi\leqslant2\pi$, $r_0\in\mathbb{R}$, $k\in\{1,k_{\gamma(\phi)}\}$, $|\lambda|=1$ for $\gamma(\phi)=(0,0)$, and the function $f(r)$ has the form*

$$f(r)=\frac{1}{\mu|\lambda|}f_{\gamma(\phi),k}(\lambda(r-r_0)), \qquad \bigl(\lambda(a-r_0),\lambda(b-r_0)\bigr)\subseteq r_{\gamma(\phi)}(I_{\gamma(\phi),k}),$$

that is, the surface $(S, \lambda^2 ds^2)$, which is similar to the original one with similarity coefficient λ, can be isometrically and S^1-equivariantly embedded into the surface $(S_{\gamma(\phi),k}, ds^2_{\mu,\gamma(\phi),k})$ by the map $(r, \varphi) \mapsto (\lambda(r - r_0), \varphi)$.

The tuples of numbers in (b) *and* (c) *are unique. The tuples (μ, η, r_0, k) in* (b) *and $(\mu, \frac{\lambda}{|\lambda|}, r_0, k)$ in* (c) *coincide. The parameters c, d and the function $\theta(r)$ in* (b) *are connected with the parameters λ, ϕ in* (c) *by the relations $c = \lambda^2 c_\phi$, $d = \lambda^4 d_\phi$, $\theta(r) = \lambda\theta_{\gamma(\phi),k}(\lambda(r - r_0))$.*

The potential $V = V(r)$ in (a) *has the form*

$$V_i(r) = \frac{(-1)^i A_i |\theta(r)|^{2-i^2}}{i} + B_i,$$

where $i \in \{1, 2\}$ for $d = 0$, $i = 2$ for $d \neq 0$, $A_i, B_i \in \mathbb{R}$ are any multiplicative and additive constants such that $A_1 > 0$, $A_2(\theta^4(r) + d) > 0$.

(C) *The nonsingular noncircular bounded orbits corresponding to the closing central potential V_i are given by periodic functions $r = r(\varphi)$ with minimal positive period $\Phi_i = 2\pi\mu/i$, where μ is the same as in* (b) *and* (c), *$i = 1, 2$. Furthermore, on the phase trajectory corresponding to a circular orbit $\{r\} \times S^1 \subset (a, b) \times S^1$, the value of the kinetic momentum K is equal to*

$$K_i = \pm\frac{1}{\mu}\sqrt{\frac{A_i}{|\theta(r)|^{i^2} + d}};$$

the boundary circle $\{\widehat{r}\} \times S^1$ on which the infimum $\inf f(r)$ (*that is,* $\sup A_i|\theta(r)|$) *is attained is an attracting centre of the field with the potential V_i* (*that is,* $\inf V_i(r)$ *is attained on it*).

Comment 1. The table indicates the maximal intervals $I_{c,d,k}$ in which the coordinate θ takes values on the Bertrand surface (5), and the ranges of the monotonic functions $r_{c,d}(\theta)$, $F_{c,d}(\theta) := f_{c,d,k}(r_{c,d}(\theta))$, $R_{c,d}(\theta)$ (the natural parameter on a meridian, the radius of the parallel for $\mu = 1$, scalar curvature). The table uses the notation $r_{4,1} := (\pi/2 - 1)/(4y_4)$ and

$$r_1 := \frac{\pi}{2}\,\frac{y_1}{y_1^2 - x_1^2} - \frac{\tan^{-1}\sqrt{y_1/x_1}}{y_1 - x_1}, \qquad r_4 := \frac{\pi}{8y_4} - \frac{1}{4x_4}\ln\frac{\sqrt{x_4^2 + y_4^2} - x_4}{y_4}.$$

Corollary 2 (geometry of the Bertrand surfaces). (A) *On any Bertrand surface $(S, ds^2) = (S_{c,d,k}, ds^2_{\mu,c,d})$* (*see* (5)) *there exists a unique boundary circle $\{\widehat{r}\} \times S^1$ determined by the condition* $\lim_{r\to\widehat{r}} f(r) = 0$ (*that is, contractible to a point called a pole of the surface*), *for which* $\lim_{r\to\widehat{r}}\theta(r) = -\infty$ *in the case $k = 1$, and* $\lim_{r\to\widehat{r}}\theta(r) = 0$ *in the case $k = 2$. At the pole, the surface has a conical singularity with total angle* $2\pi\lim_{r\to\widehat{r}}|f'(r)|$ *equal to $2\pi/\mu$ for $k = 1$, or ∞ for $k = 2$. The pole is an attracting centre for any closing central potential.*

(B) *The surface $(S, ds^2) = ((a, b) \times S^1, ds^2) \subset (S_{c,d,k}, ds^2_{\mu,c,d})$ can be realized in $\mathbb{R}^3$ as a surface of revolution if and only if $|f'(r)| \leqslant 1$ everywhere on S, where*

$$f'(r) = -\frac{\theta + d\theta^{-3}}{\mu\sqrt{\theta^2 + c - d\theta^{-2}}}.$$

In particular, the additional Bertrand surface ($k = 2$) *is always non-realizable; for* $d \leqslant 0 \leqslant c$ *the main Bertrand surface* ($k = 1$) *is realizable if and only if* $\mu \geqslant 1$ (*including the standard half-sphere for* $(\mu, c, d) = (1, 1, 0)$, *and the Euclidean plane for* $(\mu, c, d) = (1, 0, 0)$); *for* $(c, d) \in \ell_3$ *the main Bertrand surface* ($k = 1$) *is realizable if and only if* $\mu \geqslant 2$; *for* $(c, d) \in \Omega_2 \cup \ell_2 \cup \Omega_3$ *the main Bertrand surface* ($k = 1$) *is non-realizable for any* μ (*including the standard Lobachevskiĭ plane for* $(\mu, c, d) = (1, -1, 0)$).

(C) *The scalar Riemannian curvature* $R = -2f''(r)/f(r) =: R_{c,d}(\theta)$ *on the Bertrand surface* (5) *satisfies the relations*

$$\frac{R_{c,d}(\theta)}{2} = c - \frac{6d}{\theta^2} - \frac{3cd}{\theta^4} + \frac{2d^2}{\theta^6}, \qquad \frac{R'_{c,d}(\theta)}{4!} = \frac{d}{\theta^5}\left(\theta^2 + c - \frac{d}{\theta^2}\right) = \frac{d}{\mu^2 f^2 \theta^5},$$

that is, $R = R(\theta)$ *is constant for* $d = 0$, *is increasing for* $d < 0$, *is decreasing for* $d > 0$, *and has the following limit values: at the main pole,* $R \to 2c$; *at the additional pole,* $R \to +\infty$; *on the equator,* $R \to 32y_4^2(c, d) > 0$ *for* $d \neq 0$; *on the main absolute,* $R \to -2\Delta(c, d)/x_3^2(c, d) \leqslant 0$ *for* $(c, d) \neq (0, 0)$; *on the additional absolute,* $R \to -2\Delta(c, d)/y_3^2(c, d) \leqslant 0$. *In particular,* $R > 0$ *for* $d \leqslant 0 \leqslant c$ *and* $(c, d) \neq (0, 0)$, *and also for* $(c, d) \in \overline{\Omega}_4$ *and* $k = 2$; $R < 0$ *for* $c \leqslant 0 \leqslant d$ *and* $(c, d) \neq (0, 0)$, *and also for* $(c, d) \in \overline{\Omega}_3$ *and* $k = 1$; R *has nonconstant sign for the remaining* (c, d, k) *such that* $(c, d) \neq (0, 0)$.

(D) *On any surface of revolution or, more generally, on its* N*-dimensional analogue, that is, on a Riemannian manifold* $(a, b) \times S^{N-1}$ *with the Riemannian metric* $ds^2 = dr^2 + f^2(r)\,d\varphi^2$, *where* $N \in \mathbb{N}$ *and* $d\varphi^2$ *denotes the standard Riemannian metric on the* $(N-1)$*-dimensional unit sphere* S^{N-1}, *the Laplace-Beltrami operator acts on central functions* $h = h(r)$ *by the formula*

$$\Delta h(r) = h''(r) + \frac{(N-1)h'(r)f'(r)}{f(r)} = \frac{(f^{N-1}(r)h'(r))'}{f^{N-1}(r)}.$$

For $N = 3$ *the generalized gravitational-Coulombic potential* $\Theta = \Theta(r)$ *defined by the condition* $\Theta'(r) = 1/f^2(r)$ *is a harmonic function.*

Comment 2. We now describe the bifurcations of the Bertrand surfaces (5) as the pair of parameters $(c, d) \in \mathbb{R}^2$ moves around the origin.

(A) For $(c, d) \in \Omega_1 \cup \ell_4 \cup \Omega_4$ the Bertrand surface consists of two connected components on which $\theta \in (-\infty, -(-d)^{1/4})$ and $\theta \in (-(-d)^{1/4}, 0)$, respectively. These components are two halves of the 'pear-shaped' analytic Riemannian surface $(-\infty, 0) \times S^1$ cut along the unique equator (where $f' = 0$). Being *pear-shaped* for a surface of revolution means that there exists a unique equator, the functions r and f are monotonic on each half of the surface outside the equator, take values in finite intervals (that is, the surface is bounded), and in each half there is a pole (that is, $\inf f = 0$, so that $\sup f$ is attained on the equator), and at the poles the surface has conical singularities with different total angles (equal to $2\pi/\mu$ and ∞ in the main and additional poles, respectively). For $\mu \in \mathbb{Q}$ the corresponding closing (oscillatory) potential on different halves is proportional to the same analytic function, but the signs of the coefficients of proportionality are different on different halves. For each sign of the potential, all the nonsingular bounded orbits are closed (that is, on the entire surface without the poles the potential is closing, semilocally and locally closing, but neither weakly nor strongly closing). The length of every closed geodesic formed by two meridians is equal to $\mu L/2$, while for $\mu \in \mathbb{Q}$ all the other geodesics are closed and cross the equator, and all the geodesics, except for the equator and meridians, have exactly the same length qL, where L is the length of the equator and $\mu/2 =: q/p$ is an irreducible fraction. These pear-shaped surfaces are examples of *Tannery surfaces* ([32], Theorem 4.13 and Appendix A), that is, surfaces of revolution all of whose geodesics are closed. For $\mu = 2$ we obtain examples of *Zoll surfaces* ([32], Corollary 4.16), that is, surfaces all of whose geodesics have the same length.

As the point $(c, d) \in \Omega_1$ approaches the ray ℓ_1, the additional half 'vanishes' (degenerates into the equator), while the main half turns into a punctured half-sphere (for $\mu = 1$) or into a surface locally isometric to it (for an arbitrary $\mu > 0$). The maximal analytic continuation of a half-sphere, a sphere, is similar to pear-shaped surfaces, with one distinction — it is symmetric with respect to the equator. Since the oscillatory potential is equal to θ^{-2}, it has a singularity on the equator of the sphere (that is, for $\theta = 0$), and therefore it cannot be extended to the entire sphere. The gravitational potential on the 'main' half-sphere is equal to θ, and its analytic continuation to the sphere (as for the oscillatory potential on a pear-shaped surface) is closing, semilocally and locally closing, but neither weakly nor strongly closing.

As the point $(c, d) \in \Omega_4$ approaches the arc ℓ_3, the equator lengthens infinitely and recedes from the poles ('turns into the absolute'). As a result, a pear-shaped surface decomposes into two semi-infinite surfaces (see (C) below).

(B) For $(c, d) \in \Omega_2$ the surface is connected (that is, consists of only one half — the main one) and *semi-infinite*, that is, r and f are increasing (as functions of each other), and there is a pole ($\inf r > -\infty$, $\inf f = 0$) and an 'absolute' ($\sup r = \sup f = +\infty$). Thus, every point is at finite distance from the pole and can be at any distance from it. At the pole the surface has a conical singularity with total angle $2\pi/\mu$. All geodesics with $K \neq 0$ have infinite length (in both directions).

As the point $(c, d) \in \Omega_2$ approaches the ray ℓ_1 or ℓ_2, the surface turns into a punctured half-sphere or a Lobachevskiĭ plane (in the case $\mu = 1$).

(C) For $(c, d) \in \Omega_3 \cup \ell_3$ the surface consists of two connected components — the main and additional ones, each component is semi-infinite (see above), and the components are not isometric (and not similar) to each other, even locally. At the poles the surface has conical singularities with different total angles (equal to $2\pi/\mu$ and ∞ at the main and additional poles, respectively). All the geodesics with $K \neq 0$ have infinite length.

As the point $(c, d) \in \Omega_3$ approaches the ray ℓ_2, the additional component 'vanishes', while the main one turns into a punctured Lobachevskiĭ plane (for $\mu = 1$) or into a surface locally isometric to it (for any $\mu > 0$).

Corollary 3 (Classification of the Bertrand surfaces). *Consider the quadruples of numbers (μ, c, d, θ_0) such that $\mu, c, d \in \mathbb{R}$, $\mu > 0$, $\theta_0 \in I_{c,d}$, and pairs of numbers $(\lambda, \phi) \in \mathbb{R}_{>0} \times [0, 2\pi]$ such that $\lambda = 1$ in the case $\gamma(\phi) = (0, 0)$. On the Bertrand surface $(S_{c,d}, ds^2_{\mu,c,d})$, consider the coordinates $\theta, \varphi \mod 2\pi$ (see (5)), the meridians $I_{c,d} \times \{\varphi_0\} \subset S_{c,d}$, and the parallels $\{\theta_0\} \times S^1 \subset S_{c,d}$.*

(A) *For different quadruples* $(\mu_i, c_i, d_i, \theta_i)$, $i = 1, 2$, *the Bertrand surfaces* $(S_{c_i,d_i}, ds^2_{\mu_i,c_i,d_i})$ *are not isometric to each other* (*with the family of meridians being preserved*) *in any neighbourhoods of the parallels* $\{\theta_i\} \times S^1 \subset S_{c_i,d_i}$.

The surface $(S_{c,d}, ds^2_{\mu,c,d})$ *is locally isometric to the surface* $(S_{c,d}, ds^2_{1,c,d})$. *More precisely, a neighbourhood of any meridian of the surface* $(S_{c,d}, ds^2_{\mu,c,d})$ *can be isometrically embedded into the surface* $(S_{c,d}, ds^2_{1,c,d})$ *by the association* $(\theta, \varphi) \mapsto (\theta, \varphi/\mu)$. *For different triples* (c_i, d_i, θ_i), $i = 1, 2$, *the surfaces* $(S_{c_i,d_i}, ds^2_{1,c_i,d_i})$ *are not locally isometric to each other* (*with the family of meridians being preserved*) *in any neighbourhoods of the points* $(\theta_i, 0) \in S_{c_i,d_i}$.

(B) *The surface* $(S_{c,d}, ds^2_{\mu,c,d})$ *is similar to the surface* $(S_{\gamma(\phi)}, ds^2_{\mu,\gamma(\phi)})$ *and is locally similar to the surface* $(S_{\gamma(\phi)}, ds^2_{1,\gamma(\phi)})$ *with similarity coefficient* $1/\lambda$, *where the pair* (λ, ϕ) *is uniquely determined by the relations* $c = \lambda^2 c_\phi$ *and* $d = \lambda^4 d_\phi$. *More precisely, the surface* $(S_{c,d}, \lambda^2 ds^2_{\mu,c,d})$, *which is similar to the surface* $(S_{c,d}, ds^2_{\mu,c,d})$, *is isometrically mapped onto* $(S_{\gamma(\phi)}, ds^2_{\mu,\gamma(\phi)})$ *by the association* $(\theta, \varphi) \mapsto (\theta/\lambda, \varphi)$. *For different triples* $(\mu_i, \phi_i, \theta_i)$, $i = 1, 2$, *the surfaces* $(S_{\gamma(\phi_i)}, ds^2_{\mu_i,\gamma(\phi_i)})$ *are not similar to each other* (*with the family of meridians being preserved*) *in any neighbourhoods of the parallels* $\{\theta_i\} \times S^1 \subset S_{\gamma(\phi_i)}$. *For different pairs* (ϕ_i, θ_i), $i = 1, 2$, *the surfaces* $(S_{\gamma(\phi_i)}, ds^2_{1,\gamma(\phi_i)})$ *are not locally similar to each other* (*with the family of meridians being preserved*) *in any neighbourhoods of the points* $(\theta_i, 0) \in S_{\gamma(\phi_i)}$.

(C) *The surface* $(S_{c,d}, ds^2_{\mu,c,d})$ *is projectively diffeomorphic to a domain on one of the three surfaces* $(S_{c_0,d_0}, ds^2_{\mu,c_0,d_0})$ *for* $d_0 := \operatorname{sgn} d \in \{-1, 0, 1\}$ *and for any* $c_0 \geqslant c_0(c, d)$, *where*

$$c_0(c,d) := \begin{cases} 0 & \text{for } d = 0, \\ -2 & \text{for } d < 0, \\ \dfrac{c}{\sqrt{d}} & \text{for } d > 0. \end{cases}$$

More precisely, there is an embedding $(S_{c,d}, ds^2_{\mu,c,d}) \hookrightarrow (S_{c_0,d_0}, ds^2_{\mu,c_0,d_0})$ *that takes* (*nonparametrized*) *geodesics to geodesics and is defined by the following association. For* $d = 0$ *it is defined by the association* $(\theta, \varphi) \mapsto (\theta, \varphi)$, *which is bijective for* $c \geqslant 0$, *while for* $c < 0$ *it takes the surface to the punctured disc* $(-\infty, -\sqrt{-c}) \times S^1 \subset S_{c_0,0}$. *For* $d \neq 0$ *it is defined by the association* $(\theta, \varphi) \mapsto (\theta/|d|^{1/4}, \varphi)$, *which is bijective for* $(c, d) \in \Omega_1 \cup \ell_4 \cup \Omega_4 \cup \ell_3$, *for* $(c, d) \in \Omega_3$ *it maps the surface onto the union of the two punctured discs* $(-\infty, -x_3(c,d)/(-d)^{1/4}) \times S^1$ *and* $(-y_3(c,d)/(-d)^{1/4}, 0) \times S^1$, *and for* $d > 0$ *it maps the surface onto the punctured disc* $(-\infty, -x_3(c,d)/d^{1/4}) \times S^1$.

Comment 3. The Bertrand surfaces (5) are analytic, pairwise non-isometric surfaces (even locally, with the action of the rotation group being preserved) and form a smooth three-parameter family with the parameters (μ, c, d) or (μ, λ, ϕ). The two parameters $\mu, \lambda > 0$ have a simple geometric meaning — when they vary (with a suitable linear scaling of the coordinate axes), the components of the Riemannian metric are multiplied by constants, and therefore the resulting surface is locally similar to the original one with the family of meridians being preserved. In the classification of the metrics (5) up to isometry (local isometry, similarity, local similarity) preserving the family of meridians, a complete invariant is the tuple of parameters (μ, c, d) ((c, d), (μ, ϕ), ϕ, respectively), while the other parameters are not essential in the classification. The class of projective (locally projective) diffeomorphicity of the surface (5) is completely determined by the parameter $\mu > 0$ and the sign $\operatorname{sgn} d$ (by the sign $\operatorname{sgn} d$, respectively). The properties of projectively equivalent metrics were studied in [33], for example.

2.3. Equations of the orbits of the motion of a point on the Bertrand surface in the field of a closing potential. We write explicitly the equations of nonsingular orbits of the motion of a point on the Bertrand surfaces (5) (that is, (2) and (4)).

A surface of the first type, $f(r) = \xi f_c(r)$, has a nonsingular orbit $\theta = \theta_{c,0}(r(\varphi))$ for the gravitational potential $V(r) = V_{c,0,1}(r) = -A|\theta| + B$ for $A > 0$:

$$\theta = \pm\frac{\xi^2 A}{K^2}\left(1 + \sqrt{1 + 2\frac{E_1 K^2}{\xi^2 A^2}}\sin(\xi(\varphi - \varphi_0))\right), \qquad \Phi = \frac{2\pi}{\xi}.$$

A surface of the first type, $f(r) = \xi f_c(r)$, has a nonsingular orbit $\theta = \theta_{c,0}(r(\varphi))$ for the oscillatory potential $V(r) = V_{c,0,2}(r) = A/(2\theta^2) + B$ for $A > 0$:

$$\theta^2 = \frac{\xi^2 E_1}{K^2}\left(1 + \sqrt{1 - \frac{AK^2}{\xi^2 E_1^2}}\sin(2\xi(\varphi - \varphi_0))\right), \qquad \Phi = \frac{\pi}{\xi}.$$

A surface of the second type, $f(r) = f_{c,d}(r)/\mu$, has a nonsingular orbit $\theta = \theta_{c,d}(r(\varphi))$ for the oscillatory potential $V(r) = V_{c,d,2}(r) = A/(2\theta^2) + B$ for $(\theta^4 + d)A > 0$:

$$\theta^2 = \frac{E_1}{\mu^2 K^2}\left(1 + \sqrt{1 - \frac{\mu^2 K^2}{E_1^2}\left(A - \mu^2 K^2 d\right)}\sin\left(2\frac{\varphi - \varphi_0}{\mu}\right)\right), \qquad \Phi = \pi\mu.$$

Here,

$$E = \frac{\dot r^2 + f^2(r)\dot\varphi^2}{2} + V(r) = K^2\frac{\mu^4\left(\frac{d\theta}{d\varphi}\right)^2 + \frac{1}{f^2(r)}}{2} + V(r)$$

is the constant of total energy, $K \neq 0$ is the kinetic momentum integral, $E_1 = E - B - \mu^2 K^2 c/2$, $c, d \in \mathbb{R}$, $\xi = 1/\mu$, μ is a positive rational number determined by the metric (see (1), (2), (4)), Φ is the minimal positive period of the periodic functions $r(\varphi)$ that are not constants.

2.4. Generalization of the technical Bertrand theorem (Theorem 2). We define a *generalized family of Bertrand's equations* to be the one-parameter family of differential equations

$$\frac{d^2 z}{d\varphi^2} + \rho(z) = \frac{1}{K^2}\Psi(z)$$

on an interval $(a, b) \subset \mathbb{R}^1$ with parameter $K \in \mathbb{R} \setminus \{0\}$, where $\Psi(z)$ and $\rho(z)$ are functions of class C^∞ defined on the interval (a, b). The following definition is similar to Definition 2.

Definition 3. The function $\Psi = \Psi(z)$ on the interval (a, b) is said to be *closing for the function* $\rho = \rho(z)$ (or *ρ-closing*) if

(∃) there exists a value of the parameter $K = \widehat{K} \in \mathbb{R}\setminus\{0\}$ for which the equation has a bounded nonconstant solution $\widehat{z} = \widehat{z}(\varphi)$,

($\forall$) all the bounded nonconstant solutions $z = z(\varphi)$ of the equation with all possible values of the parameter K are periodic functions with pairwise commensurable periods.

The function $\Psi(z)$ is said to be *locally closing for the function* $\rho(z)$ (or *locally ρ-closing*) if

$(\exists)^{\mathrm{loc}}$ there exists a value of the parameter $K = K_0$ for which the equation has a nondegenerate stable equilibrium $z_0 \in (a, b)$,

$(\forall)^{\mathrm{loc}}$ for any pair $(K_0, z_0) \in (\mathbb{R} \setminus \{0\}) \times (a, b)$ satisfying condition $(\exists)^{\mathrm{loc}}$ there exist $\varepsilon, \delta > 0$ such that all the bounded nonconstant solutions $z = z(\varphi)$ of the equation, with values of the parameter $K \in (K_0 - \delta, K_0 + \delta)$, such that $z(\mathbb{R}^1) \subseteq [z_0 - \varepsilon, z_0 + \varepsilon]$ are periodic functions with pairwise commensurable periods.

The function $\Psi(z)$ is said to be *semilocally closing for the function* $\rho(z)$ (or *semilocally ρ-closing*) if conditions ($\exists$), $(\forall)^{\mathrm{loc}}$ hold and the following condition holds:

$(\forall)^{\text{s-loc}}$ all the bounded nonconstant solutions $z = z(\varphi)$ of the equation for $K = \widehat{K}$ such that $z(\mathbb{R}^1) \subseteq \widehat{z}(\mathbb{R}^1)$ are periodic functions with pairwise commensurable periods, where $\widehat{K}$ and $\widehat{z} = \widehat{z}(\varphi)$ are a value of the parameter and a solution in ($\exists$).

The function $\Psi(z)$ is called *strongly* (*weakly*) *ρ-closing* if any point $z_0 \in (a, b)$ is a nondegenerate stable (stable, respectively) equilibrium of the equation for some $K = K_0$ depending on z_0, and condition $(\forall)^{\mathrm{loc}}$ holds (its analogue holds for every pair $(K_0, z_0) \in (\mathbb{R} \setminus \{0\}) \times (a, b)$ such that z_0 is a stable equilibrium of the equation for $K = K_0$, respectively).

The proofs of Theorems 5–7 are based on the following generalization of Theorem 2 using the changes of variables $z(r) - \zeta = -\Theta(r) = -\mu^2\theta(r)$, where

$$\Theta'(r) = \frac{1}{f^2(r)}, \qquad \rho(z(r)) = \frac{f'(r)}{f(r)},$$
$$\Psi(z) = f^2(r)V'(r) = -\frac{dV(r(z))}{dz}, \qquad D = \mu^8 d.$$

Theorem 8 (the generalized technical Bertrand theorem). *Consider the one-parameter family of differential equations*

$$\frac{d^2 z}{d\varphi^2} + \rho(z) = \frac{1}{K^2}\Psi(z)$$

on an interval $(a, b) \subset \mathbb{R}^1$ with parameter $K \in \mathbb{R} \setminus \{0\}$, where $\Psi = \Psi(z)$ and $\rho = \rho(z)$ are functions of class C^∞ defined on the interval (a, b). If Ψ is semilocally ρ-closing (*or ρ-closing, or strongly or weakly ρ-closing*), *then this function is locally ρ-closing.*

Suppose that the function ρ has no zeros on the interval (a, b). Then in the interval (a, b) the classes of closing, semilocally closing, locally closing, strongly closing and weakly closing for ρ functions Ψ coincide, and there exists at most two closing functions $\Psi(z)$ up to a positive multiplicative constant, and these functions are determined by the following conditions:

(a) *if* $\rho'|_{(a,b)} = \mathrm{const} > 0$, *then there exist exactly two ρ-closing functions Ψ* (*up to a positive multiplicative constant*) *on (a, b), namely,* $\Psi_i(z) = A_i/\rho^{i^2-1}(z)$,

$i = 1, 2$ (*that is, the functions corresponding to the generalized gravitational and oscillatory force laws on the corresponding surface of revolution*), *where* $A_i \neq 0$ *is an arbitrary multiplicative constant such that* $A_i \rho^i(z) > 0$, *and the minimal positive period of any bounded nonconstant solution is equal to* $\Phi_i = 2\pi/(i\sqrt{\rho'})$;

(b) *if* $\rho|_{(a,b)}$ *is a rational function of the form*

$$\rho(z) = \frac{(z-\zeta)^4 + D}{\mu^2 (z-\zeta)^3}, \qquad D = \mathrm{const} \neq 0, \quad \mu = \mathrm{const} > 0, \quad \zeta = \mathrm{const} \notin (a,b),$$

then there exists a unique ρ*-closing function* (*up to a positive multiplicative constant*) *on* (a,b): $\Psi(z) = \Psi_2(z) = A/(z-\zeta)^3$ (*that is, corresponding to the oscillatory force law on the corresponding surface of revolution*), *where* $A \neq 0$ *is an arbitrary multiplicative constant such that* $A((z-\zeta)^4 + D) > 0$, *and the minimal positive period of any bounded nonconstant solution is equal to* $\Phi = \pi\mu$;

(c) *if* $\rho(z)$ *has none of the above-mentioned forms, then there do not exist* ρ*-closing functions on* (a,b).

In cases (a) *and* (b), *every point* $z \in (a,b)$ *is a nondegenerate stable equilibrium of the equation for* $K = K_i := \pm\sqrt{A_i/\rho^{i^2}(z)}$, $i = 1, 2$ (*in case* (a)) *and* $K = \pm\mu\sqrt{A/((z-\zeta)^4 + D)}$ (*in case* (b)), *while for the other values of the parameter* K *the point is not an equilibrium.*

We point out that in Theorem 8 (in contrast to Theorem 2) we do not require the functions $\Psi_i(z)$ and $\rho(z)$ to be analytic, and the constant $\mu > 0$ does not have to be rational (since here we do not require that all periods be commensurable with 2π, but merely that the periods be pairwise commensurable, as in Remark 1).

§ 3. Special case: a cone

By Corollary 1, Theorem 1 is generalized to the family of 'rational' cones. Other cones ('irrational' ones) do not admit a generalization of Bertrand's theorem. We now explain what role is played by the condition that the constant ξ be rational in Corollary 1. We consider the construction described after Theorem 5: we cut the cone S along a directrix and develop it. We obtain some sector in the Euclidean plane. We observe that the angle at the apex (equal to $2\pi\xi$) can be even greater than 2π; in this case the cone does not embed into $\mathbb{R}^3$ as a surface of revolution, but Corollary 1 remains valid. Consider further the surface $\widetilde{S}$ that is simultaneously a branched covering of the cone S and a branched covering of the Euclidean plane, where the number of sheets of each of the coverings is the least possible. The surface $\widetilde{S}$ can be constructed as follows. We superimpose the sectors obtained by developing the cone cut along a directrix onto the plane, rotating each sector in turn so that the edge of each succeeding sector coincides with the opposite edge of the preceding one. In this way we obtain a branched covering $\widetilde{S} \to \mathbb{R}^2$ over the plane. Here, if the cone is 'rational', that is, the angle at the apex is commensurable with 2π (in other words, $\xi \in \mathbb{Q}$, so that $\xi = p/q$), then after q steps of constructing $\widetilde{S}$ the edge of the next sector will coincide with the edge of the first sector; in this case we stop the process of constructing $\widetilde{S}$, and the covering $\widetilde{S} \to \mathbb{R}^2$ is p-sheeted. If not, edges of different sectors never coincide and the covering will be infinitely sheeted. The tiled plane and the image of a trajectory of motion of a point on the surface $\widetilde{S}$ are depicted in Fig. 3.

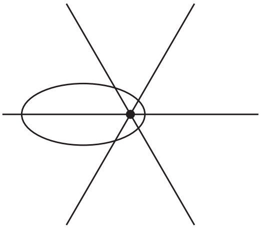

Figure 3. The image of a trajectory of motion on a cone on the tiled plane; $\xi = 1/6$.

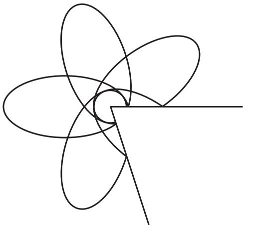

Figure 4. The image of a trajectory of motion on the cone on the surface $\widetilde{S}$ represented by the union of sectors situated 'one on top of another' on the plane (each sector has its own arc of the trajectory on it); $\xi = 5/6$.

Figure 5. A closed trajectory on a rational cone; $\xi = 1/8$.

Thus, the surface $\widetilde{S}$ is a q-sheeted cover of the original cone if ξ is a rational equal to p/q, and is an infinitely sheeted cover otherwise. The covering is constructed naturally: its sheets are the sectors of which the surface $\widetilde{S}$ is composed. Every such sector is a development of the cone; hence we can define the map $\widetilde{S} \to S$ taking to a point with coordinates (r, φ) on the cone all points of the surface $\widetilde{S}$ that have the same coordinates as it on the sector to which they belong. Graphically this can be represented as follows. We take the sectors obtained by developing the cone cut along a directrix, and place them over the development of the cone 'one on top of the other'—q copies if ξ is rational, and a denumerable number of them otherwise; the opposite edges of sectors situated one above another are regarded as glued together. Points of these sectors are points of the surface $\widetilde{S}$, and the map $\widetilde{S} \to S$ is given by the natural projection. The image of a trajectory of motion on the cone on the surface $\widetilde{S}$ represented as explained above is shown in Fig. 4. In Fig. 5 we give an example of a closed trajectory on a 'rational' cone, seen from the top.

§ 4. Proof of the generalized technical Theorem 8

A generalized Bertrand equation has the form $z''_{\varphi\varphi} = -U'(z)$, where

$$U(z) := \int \frac{\rho(z) - \Psi(z)}{K^2}\,dz.$$

Hence its solutions satisfy the energy conservation law

$$\frac{(z')^2}{2} + U(z) =: E,$$

and therefore this equation can be reduced to the first order equation $(z')^2 = 2E - 2U$, where E is a constant (called the energy level in the generalized Bertrand problem) depending on the solution, that is, to the equation

$$(z')^2 = R(z) + \varkappa W(z) + 2E = 2E - 2U_\varkappa(z),$$

where

$$R(z) := -2\int \rho(z)\,dz, \qquad \varkappa := \frac{2}{K^2},$$
$$W(z) := \int \Psi(z)\,dz, \qquad U_\varkappa(z) := -\frac{R(z) + \varkappa W(z)}{2}.$$

The quantity K^2E coincides with the *energy level* (see § 2.3), and the function $K^2U_\varkappa(z)$ with the *effective potential* (see Definition 1(a)) in the problem of the motion of a point on the surface S in a central field with potential $V = V(r)$.

First of all we prove three auxiliary results on the properties of solutions of the differential equation $z''_{\varphi\varphi} = -U'(z)$, where $U = U(z)$ is a smooth function on the interval (a, b).

Proposition 1. *If a function $z = z(\varphi)$ is a solution of the equation $z''_{\varphi\varphi} = -U'(z)$ and $z'(\varphi_0) = 0$, then the graph of the function $z = z(\varphi)$ is symmetric with respect to the straight line $\{\varphi = \varphi_0\}$; in other words, $z(\varphi) = \widetilde{z}(\varphi) := z(\varphi_0 - (\varphi - \varphi_0))$.*

Proof. The fact that the function $\widetilde{z}(\varphi)$ satisfies the equation $z''_{\varphi\varphi} = -U'(z)$ is proved by direct substitution. And since $z(\varphi_0) = \widetilde{z}(\varphi_0)$ and $z'(\varphi_0) = \widetilde{z}'(\varphi_0)$, the solutions $z(\varphi)$ and $\widetilde{z}(\varphi)$ coincide. The proposition is proved.

Proposition 2. *Let $a < a' < b' < b$ and let $E' \in \mathbb{R}$. Then the following conditions are equivalent*:

(a) *there exists a bounded solution $z(\varphi)$ of the equation $z''_{\varphi\varphi} = -U'(z)$ with energy level E such that $a' = \inf z(\mathbb{R}^1)$ and $b' = \sup z(\mathbb{R}^1)$*;

(b) *$U(a') = U(b') = E'$ and $U|_{(a',b')} < E'$.*

If $\widehat{z} \in \{a', b'\}$ and condition (a) *holds, then the relations $U'(\widehat{z}) \neq 0$ and $\widehat{z} \in z(\mathbb{R}^1)$ are equivalent.*

Proposition 3. *Suppose that $a < a' < b' < b$, $U(a') = U(b') = E'$, $U|_{(a',b')} < E'$, and let $E_0 := \min U|_{[a',b']}$. Then the following conditions are equivalent*:

(a) *for any $E \in (E_0, E']$ any bounded solution $z_E(\varphi)$ of $z''_{\varphi\varphi} = -U'(z)$ with energy level E such that $z_E(0) \in (a', b')$ is periodic*;

(b) *there exists a closed interval $[c_1, c_2] \subset (a', b')$ such that $U'|_{[a',c_1)} < 0$, $U'|_{[c_1,c_2]} = 0$ and $U'|_{(c_2,b']} > 0$.*

When these conditions hold, the least positive period of a solution $z_E(\varphi)$ is equal to

$$\Phi(E) = 2 \int_{z_1(E)}^{z_2(E)} \frac{dz}{\sqrt{2E - 2U(z)}}, \tag{7}$$

where the values $z_1 = z_1(E) \in [a', c_1)$ and $z_2 = z_2(E) \in (c_2, b']$ are determined by the conditions $U(z_1) = U(z_2) = E$. The function $\Phi = \Phi(E)$ is continuous on the half-interval $(E_0, E']$. If $U''(c_1) = 0$, then

$$\lim_{E \to E_0} \Phi(E) = \infty. \tag{8}$$

Proof of Propositions 2 *and* 3. *Step* 1. Suppose that either of conditions (a) and (b) of Proposition 2 holds. Then there exists a point $z_0 \in (a', b')$ such that $U(z_0) < E'$. We fix any number $E \in (U(z_0), E']$. Let $z(\varphi)$ be a local solution of the differential equation $z''_{\varphi\varphi} = -U'(z)$ such that $z(0) = z_0$ and $z'(0) = \sqrt{2E - 2U(z_0)}$. Then the energy level on the solution $z(\varphi)$ is equal to E. Let $(z_1, z_2) \subseteq (a, b)$ be the maximal interval by inclusion containing the point z_0 on which $U(z) < E$. Then the solution $z(\varphi)$ can be extended to the interval (φ_1, φ_2) (using the energy conservation law), where

$$\varphi_i := \int_{z_0}^{z_i} \frac{dz}{\sqrt{2E - 2U(z)}}, \qquad i = 1, 2.$$

Hence, when either (a) or (b) of Proposition 2 holds, the interval (a', b') contains the maximal interval by inclusion containing the point z_0 on which $U(z) < E'$. Therefore, $(z_1, z_2) \subseteq (a', b')$ (as $E \leqslant E'$).

We now show that the solution $z(\varphi)$ is bounded and $z(\mathbb{R}^1) \subseteq [z_1, z_2]$ (that is, $z_1 = \inf z(\mathbb{R}^1)$ and $z_2 = \sup z(\mathbb{R}^1)$). (This implies conditions (a) and (b) of Proposition 2 are equivalent.) If $\varphi_1 = -\infty$ and $\varphi_2 = \infty$, then the required result is obvious, since the solution we have constructed is defined on the whole $\mathbb{R}^1$ and $z(\mathbb{R}^1) = (z_1, z_2)$ (and therefore is aperiodic). It remains to consider the case where $|\varphi_i| < \infty$ holds for some $i \in \{1, 2\}$. In this case, since $(z_1, z_2) \subseteq [a', b'] \subset (a, b)$

the solution can be extended to the point φ_i, hence $z(\varphi_i) = z_i$, $U(z_i) = E$, and $z'(\varphi_i) = 0$. If $\varphi_1 = -\infty$ and $\varphi_2 < +\infty$, then by Proposition 1 the solution is bounded and $z(\mathbb{R}^1) = z((-\infty, \varphi_2]) = (z_1, z_2]$ (and therefore the solution is aperiodic). It is shown in similar fashion that if $\varphi_1 > -\infty$ and $\varphi_2 = +\infty$, then the solution is bounded and $z(\mathbb{R}^1) = z([\varphi_1, \infty)) = [z_1, z_2)$ (and therefore the solution is aperiodic), and when $\varphi_1 > -\infty$ and $\varphi_2 < +\infty$ the solution is periodic, $z(\mathbb{R}^1) = z([\varphi_1, \varphi_2]) = [z_1, z_2]$, and the minimal positive period is equal to

$$\Phi = 2 \int_{z_1}^{z_2} \frac{dz}{\sqrt{2E - 2U(z)}}.$$

Thus, the solution is always bounded and $z(\mathbb{R}^1) \subset [z_1, z_2]$, as required.

We will now prove that for any $i \in \{1, 2\}$ the relations $U'(z_i) \neq 0$ and $z_i \in z(\mathbb{R}^1)$ are equivalent. Suppose for definiteness that $i = 2$. Suppose that $U'(z_2) = 0$, that is, the point z_2 is an equilibrium. If $z_2 \in z(\mathbb{R}^1)$ (that is, $\varphi_2 < \infty$, see above), then both solutions $z(\varphi)$ and $z_2(\varphi) \equiv z_2$ satisfy the same system of initial conditions $z(\varphi_2) = z_2$, $z'(\varphi_2) = 0$ and therefore they coincide, which contradicts the condition $z_1 < z_2$. Now suppose that $U'(z_2) \neq 0$. Then $U|_{[z_0,z_2]}(z) \leqslant E - c(z_2 - z)$ for some $c > 0$, hence $\varphi_2 \leqslant (z_2 - z_0)\sqrt{2/c} < \infty$, and therefore, $z_2 \in z(\mathbb{R}^1)$ (see above). The proof of Proposition 2 is complete.

Step 2. If condition (b) of Proposition 3 holds, then $U'(z_1) \neq 0$ and $U'(z_2) \neq 0$, and therefore, by Proposition 2, the image $z(\mathbb{R}^1)$ of the solution $z(\varphi)$ is a closed interval, so the solution is periodic (see step 1). This fact and Proposition 2 imply that condition (b) of Proposition 3 implies condition (a) of Proposition 3.

Step 3. Suppose that condition (a) of Proposition 3 holds. Since $a' < b'$ and $U(a') = U(b')$, the set $A := \{z \in (a', b') \mid U'(z) = 0\}$ is non-empty. We set

$$E^* := \sup U|_A, \qquad B := \{z \in [a', b'] \mid U(z) \leqslant E^*\}, \qquad c_1 := \inf B, \quad c_2 := \sup B.$$

Then

$$U'|_{[a',c_1)} < 0, \quad U'|_{(c_2,b']} > 0, \quad E_0 \leqslant U|_{[c_1,c_2]} \leqslant U(c_1) = U(c_2) = E^* < E'. \qquad (9)$$

Suppose that $U([c_1, c_2]) \neq E^*$. Then $c_1 < c_2$, and by (9) there exists an interval $(z_1^*, z_2^*) \subseteq (c_1, c_2)$ such that $U|_{(z_1^*,z_2^*)} < E^*$, $U(z_1^*) = U(z_2^*) = E^*$, and one of the endpoints z_i^* ($i \in \{1, 2\}$) of this interval belongs to A (that is, $U'(z_i^*) = 0$). Now let $z^*(\varphi)$ be a bounded solution with energy level E^* corresponding to the interval (z_1^*, z_2^*) by Proposition 2. Since this solution is bounded, it is periodic by condition (a) of Proposition 3, and hence $z^*(\mathbb{R}^1) = [z_1^*, z_2^*]$ (see step 1). Hence, $U'(z_i^*) \neq 0$ by Proposition 2. The contradiction thus obtained shows that $U|_{[c_1,c_2]} = E^* = E_0$, that is, condition (b) of Proposition 3 holds.

Step 4. Suppose that either of conditions (a) and (b) of Proposition 3 holds. Formula (7) was proved at step 1. We now prove (8). Suppose that $U''(c_1) = 0$. Then for any $z \in [c_1 - \varepsilon, c_1]$ we have $E_0 \leqslant U(z) \leqslant E_0 + c(c_1 - z)^3$ for some $c > 0$. Hence,

$$\Phi(E) \geqslant \int_{c_1-((E-E_0)/c)^{1/3}}^{c_1} \frac{dz}{\sqrt{2E - 2E_0}} \to +\infty$$

as $E \to E_0$ for $E_0 < E \leqslant E'$. This completes the proof of Proposition 3.

Proposition 4 (the local generalized technical Bertrand theorem). *Let $\Psi = \Psi(z)$ and $\rho = \rho(z)$ be C^∞-smooth functions on (a, b).*

(A) *Suppose that $\rho(z_0) \neq 0$ for some $z_0 \in (a, b)$. Suppose that the point z_0 is a nondegenerate stable equilibrium of the generalized Bertrand equation for $K = K_0 > 0$, that is,*

$$K_0^2 = \frac{\Psi(z_0)}{\rho(z_0)} > 0, \qquad \rho'(z_0) - \frac{\rho(z_0)}{\Psi(z_0)}\Psi'(z_0) > 0.$$

Suppose that the pair (z_0, K_0) satisfies condition $(\forall)^{\mathrm{loc}}$ in Definition 3. Then there exist $\varepsilon_0, \beta > 0$ such that the functions Ψ and ρ on $[z_0 - \varepsilon_0, z_0 + \varepsilon_0]$ satisfy the relations

$$3\Psi''\Psi = 4(\Psi')^2, \qquad \Psi\rho' - \Psi'\rho = \beta^2\Psi. \tag{10}$$

(B) *Suppose that relations (10) hold on an interval $I \subset (a, b)$. Suppose that some point $z_0 \in I$ is an equilibrium of the generalized Bertrand equation on (a, b) for some $K = K_0 \neq 0$, and $\{z_0\} \subsetneq I$. Then the identities*

$$\Psi(z) = A_i(z - \zeta)^{1-i^2}, \quad \rho(z) = \beta^2\frac{(z - \zeta)^4 + D}{i^2(z - \zeta)^3}, \qquad i \in \{1, 2\},$$

hold in the interval I, where A_i, D, ζ are some constants such that $D = 0$ for $i = 1$, $z - \zeta \neq 0$, $(z - \zeta)A_1 > 0$, $((z - \zeta)^4 + D)A_2 > 0$ for any $z \in I$. Furthermore, any point $z \in I$ is a nondegenerate stable equilibrium of the generalized Bertrand equation for

$$K = \pm\sqrt{\frac{\Psi(z)}{\rho(z)}};$$

all bounded nonconstant solutions $z = z(\varphi)$ of the generalized Bertrand equation on I for all possible values of the parameter K are periodic with minimum positive period $\Phi = 2\pi/\beta$.

Proof. Step 1. We set $\varkappa := 2/K^2$, $\varkappa_0 := 2/K_0^2$, $E_0 := U_{\varkappa_0}(z_0)$. A point $z_0 \in (a, b)$ is a nondegenerate stable equilibrium of $z''_{\varphi\varphi}(\varphi) = -U'_\varkappa(z)$ for $\varkappa = \varkappa_0$ if and only if $U'_{\varkappa_0}(z_0) = 0$ and $U''_{\varkappa_0}(z_0) > 0$. Hence,

$$-2U'_{\varkappa_0}(z_0) = R'(z_0) + \varkappa_0 W'(z_0) = -2\rho(z_0) + \frac{2}{K_0^2}\Psi(z_0) = 0$$

(and so $\Psi(z_0) \neq 0$ in view of $\rho(z_0) \neq 0$) and

$$-2U''_{\varkappa_0}(z_0) = R''(z_0) + \varkappa_0 W''(z_0) = -2\rho'(z_0) + 2\frac{\rho(z_0)}{\Psi(z_0)}\Psi'(z_0) < 0.$$

We choose $\varepsilon, \delta > 0$ satisfying condition $(\forall)^{\mathrm{loc}}$ in Definition 3 for the pair (z_0, K_0) to be so small that

$$U''_\varkappa|_{[z_0-\varepsilon, z_0+\varepsilon]} > 0 \quad \text{and} \quad E_{0,\varepsilon,\varkappa} < E'_{\varepsilon,\varkappa}$$

for $K = \sqrt{2/\varkappa} \in [K_0 - \delta, K_0 + \delta]$, where we have put

$$E'_{\varepsilon,\varkappa} := \min\{U_\varkappa(z_0 - \varepsilon), U_\varkappa(z_0 + \varepsilon)\} \quad \text{and} \quad E_{0,\varepsilon,\varkappa} := \min U_\varkappa|_{[z_0-\varepsilon, z_0+\varepsilon]}.$$

For any pair $(E, \varkappa)$ such that

$$K = \sqrt{\frac{2}{\varkappa}} \in [K_0 - \delta, K_0 + \delta], \qquad E \in (E_{0,\varepsilon,\varkappa}, E'_{\varepsilon,\varkappa}] \tag{11}$$

we consider a bounded solution $z_{E,\varkappa}(\varphi)$ of $z''_{\varphi\varphi} = -U'_{\varkappa}(z)$ with energy level E such that $z_{E,\varkappa}(\mathbb{R}^1) \subset [z_0 - \varepsilon, z_0 + \varepsilon]$. Such a solution exists by Proposition 2 and is periodic by condition $(\forall)^{\mathrm{loc}}$ in Definition 3. We write down the value $\Phi(E, \varkappa)$ of the minimal positive period of this solution according to Proposition 3:

$$\Phi(E, \varkappa) = 2 \int_{z_1(E,\varkappa)}^{z_2(E,\varkappa)} \frac{dz}{\sqrt{R(z) + \varkappa W(z) + 2E}}, \tag{12}$$

where integration is taken over the interval between the two zeros z_1 and z_2 of the denominator (and therefore, by Proposition 2, z_1 and z_2 are the minimum and maximum of the periodic solution $z_{E,\varkappa} = z_{E,\varkappa}(\varphi)$ of $z''_{\varphi\varphi}(\varphi) = -U'_{\varkappa}(z)$).

Using the fact that $z_1 = z_1(E, \varkappa)$ and $z_2 = z_2(E, \varkappa)$ make the denominator on the right-hand side of equation (12) vanish, we express the constants $\varkappa$ and E in terms of them and consider the resulting expressions on the set of all pairs $(z_1, z_2) \in (a, b)$ such that $z_0 - \varepsilon \leqslant z_1 < z_2 \leqslant z_0 + \varepsilon$:

$$\varkappa(z_1, z_2) := \frac{R(z_2) - R(z_1)}{W(z_1) - W(z_2)}, \qquad 2E(z_1, z_2) := \frac{R(z_1)W(z_2) - R(z_2)W(z_1)}{W(z_1) - W(z_2)}. \tag{13}$$

We extend the definitions of these expressions to the set of pairs (z_1, z_1) of coinciding numbers by the relations

$$\varkappa(z_1, z_1) := -\frac{R'(z_1)}{W'(z_1)} = 2\frac{\rho(z_1)}{\Psi(z_1)}, \qquad 2E(z_1, z_1) := -R(z_1) + \frac{R'(z_1)}{W'(z_1)} W(z_1).$$

Since $W'(z_0) = \Psi(z_0) \neq 0$, for sufficiently small $\varepsilon_0 > 0$ the functions (13) are well defined and continuous in (z_1, z_2) for $z_0 - \varepsilon_0 \leqslant z_1 \leqslant z_2 \leqslant z_0 + \varepsilon_0$. Consequently, for sufficiently small $\varepsilon_0 > 0$ the corresponding values of the parameter $K = \sqrt{2/\varkappa(z_1, z_2)}$ of the generalized Bertrand equation (for all possible pairs of points $z_1, z_2 \in [z_0 - \varepsilon_0, z_0 + \varepsilon_0]$ such that $z_1 < z_2$) belong to the δ-neighbourhood of the number

$$\sqrt{\frac{2}{\varkappa(z_0, z_0)}} = \sqrt{\frac{2}{\varkappa_0}} = K_0.$$

Hence for any $z_1, z_2 \in [z_0 - \varepsilon_0, z_0 + \varepsilon_0]$ such that $z_1 < z_2$ the pair $(E, \varkappa) := (E(z_1, z_2), \varkappa(z_1, z_2))$ satisfies (11) (that is, it belongs to the domain of the function $\Phi = \Phi(E, \varkappa)$ in (12)), and for any such pair, formula (12) has the form

$$\Phi(E(z_1, z_2), \varkappa(z_1, z_2)) = 2 \int_{z_1}^{z_2} \frac{dz}{\sqrt{R(z) + \varkappa(z_1, z_2)W(z) + 2E(z_1, z_2)}}. \tag{14}$$

Step 2. We now set $z_1 = c - h$, $z_2 = c + h$, $z = c + ht$, where $c \in (z_0 - \varepsilon_0, z_0 + \varepsilon_0)$, $0 < h \ll 1$. Next, by expanding W and R at these points into a Taylor series with respect to powers of h and substituting them into (14) taking account of (13), in the limit as $h \to 0$ we obtain the following:

$$\lim_{h \to 0} \frac{\Phi(E(c-h, c+h), \varkappa(c-h, c+h))}{2\pi} = \sqrt{\frac{2W'(c)}{R'(c)W''(c) - W'(c)R''(c)}} =: \frac{1}{\beta(c)}.$$

Hence we obtain the following relation in the interval $(z_0-\varepsilon, z_0+\varepsilon)$:

$$R''W' - R'W'' = -2\beta^2 W', \tag{15}$$

where we have set $\beta := \beta(c)$, $R' := R'(c)$, $W' := W'(c)$, $R'' := R''(c)$, $W'' := W''(c)$. Substituting $W' = \Psi$ and $R' = -2\rho$ into this relation we obtain the relation $\Psi\rho' - \Psi'\rho = \beta^2\Psi$ in the interval $(z_0-\varepsilon, z_0+\varepsilon)$, that is, the second relation in (10).

Since the function (14) is continuous on its domain (that is, for $z_0-\varepsilon_0 \leqslant z_1 < z_2 \leqslant z_0+\varepsilon_0$) and all its values are pairwise commensurable in view of condition $(\forall)^{\mathrm{loc}}$ in Definition 3, it follows that this function is constant and is equal to its limit value $2\pi/\beta(z_0)$. In particular, the function $\beta = \beta(c)$ is constant on the interval $(z_0-\varepsilon, z_0+\varepsilon)$.

Step 3. We now expand the integral (14) in powers of h and consider the coefficient at h^2:

$$\frac{\pi\beta^{-3}}{4!\,W'}\biggl(\frac{3}{4}(R^{\mathrm{IV}}W' - R'W^{\mathrm{IV}}) + \frac{R'W''' - R'''W'}{W'}\biggl(\frac{R'W''' - R'''W'}{8\beta^2} + W''\biggr)\biggr). \tag{16}$$

From equations (15) with $\beta = \mathrm{const}$ we obtain

$$R'W''' - R'''W' = 2\beta^2 W'', \qquad R^{\mathrm{IV}}W' - R'W^{\mathrm{IV}} = \frac{2\beta^2}{W'}((W'')^2 - 2W'W'''). \tag{17}$$

Using the expressions (17) we see that under the condition $\beta = \mathrm{const}$ the coefficient (16) of h^2 in the expansion of the integral (14) depends only on the values of the function W and its derivatives at the point c and is equal to

$$\frac{\pi}{4!}\,\frac{1}{\beta(W')^2}[4(W'')^2 - 3W'W'''].$$

Equating this to zero and setting $W' = \Psi$ we obtain the following relation in the interval $(z_0-\varepsilon, z_0+\varepsilon)$:

$$3\Psi''\Psi = 4(\Psi')^2, \tag{18}$$

that is, the first relation in (10). Part (A) of Proposition 4 is proved.

Step 4. We now prove part (B) of Proposition 4 concerning an explicit solution of the system of differential equations (10). Since the point $z_0 \in I$ is an equilibrium we have $\Psi(z_0) \neq 0$. We now find all the smooth solutions of equation (18) in the interval I such that $\Psi(z_0) \neq 0$: these are the solutions $\Psi_i(z) = A_i(z-\zeta)^{1-i^2}$, $i = 1, 2$, where $A_i \neq 0$ and $\zeta \notin I$ are constants. There are no other smooth solutions defined on the interval I and satisfying the condition $\Psi(z_0) \neq 0$, since the initial conditions $(\Psi(z_0), \Psi'(z_0))$ of the above-mentioned solutions form the whole set $\mathbb{R}^2 \setminus (\{0\} \times \mathbb{R})$, and the indicated solutions have no zeros (even on the whole $\mathbb{R}$) and therefore are solutions of the second order differential equation

$$\Psi'' = \frac{4}{3}\,\frac{(\Psi')^2}{\Psi}$$

solved for the higher derivative. Finally, using $\beta = \mathrm{const} > 0$ and the second equation in (10), knowing $\Psi = \Psi_i(z)$ we find ρ on the interval I:

$$\rho_i(z) = \beta^2\,\frac{z-\zeta + D(z-\zeta)^{-3}}{i^2}, \qquad i = 1, 2,$$

respectively, where D is some constant such that $D=0$ for $i=1$ and $D\neq-(z-\zeta)^4$ for any $z\in I$. From the equations $U'_{\varkappa_0}(z_0)=0$ and $\varkappa_0=2/K_0^2>0$ we have

$$\frac{2}{\varkappa_0}=\frac{\Psi(z_0)}{\rho(z_0)}>0,$$

and so the sign of the constant A_i is determined uniquely by the sign of the function ρ on I, $i=1,2$. We verify directly that for any pair of functions (Ψ_i,ρ_i), $i=1,2$, above, the function Ψ_i is ρ_i-closing and strongly ρ_i-closing on I and that $\Phi=2\pi/\beta$ is conducted. Proposition 4 is proved.

Proof of Theorem 8. *Step* 1. First suppose that the function Ψ is semilocally ρ-closing on (a,b) (see Definition 3). Let $z=z(\varphi)$ be a bounded solution of the generalized Bertrand equation with the parameter $K=K_0$ and energy level E'. Such a solution exists in view of condition $(\exists)$ in Definition 3. Set $\varkappa_0:=2/K_0^2$, $a':=\inf z(\mathbb{R}^1)$, $b':=\sup z(\mathbb{R}^1)$, $E_0:=\min U_{\varkappa_0}|_{[a',b']}$. Then $[a',b']\subset(a,b)$. If $z_{E,\varkappa_0}(\varphi)$ is a bounded solution of the equation for $K=K_0$ with energy level $E\in(E_0,E']$ such that $z_{E,\varkappa_0}(0)\in(a',b')$, then by Proposition 2 we have $z_{E,\varkappa_0}(\mathbb{R}^1)\subset[a',b']$, and by condition $(\forall)^{\text{s-loc}}$ in Definition 3 the solution $z_{E,\varkappa_0}(\varphi)$ is periodic. Therefore condition (a) in Proposition 3 holds for $U=U_{\varkappa_0}$, and hence by Proposition 3 condition (b) of this proposition holds for $U=U_{\varkappa_0}$. We claim that the corresponding closed interval $[c_1,c_2]\subset(a',b')$ is a nondegenerate local minimum point of the function $U_{\varkappa_0}$. Indeed, by (8) and condition $(\forall)^{\text{s-loc}}$ in Definition 3 we have $U''_{\varkappa_0}(c_1)>0$ (since if $U''_{\varkappa_0}(c_1)=0$ the set of values of the period function contains an interval in view of (8) and therefore does not consist of pairwise commensurable numbers, which fact contradicts condition $(\forall)^{\text{s-loc}}$ in Definition 3). Therefore the closed interval $[c_1,c_2]\subset(a',b')$ is a nondegenerate local minimum point of the function $U_{\varkappa_0}$, that is, condition $(\exists)^{\text{loc}}$ in Definition 3 holds. Hence the function Ψ is locally ρ-closing on (a,b).

If Ψ is ρ-closing or strongly or weakly ρ-closing, then it is automatically locally ρ-closing (see Remark 3(a)).

Step 2. Now suppose that the function Ψ is locally ρ-closing on (a,b) and the function ρ has no zeros on (a,b). Let $z_0\in(a,b)$ be a nondegenerate stable equilibrium of the equation for some $K=K_0>0$ (which exists by condition $(\exists)^{\text{loc}}$ in Definition 3). Since condition $(\forall)^{\text{loc}}$ holds, by part (A) of the local generalized technical Bertrand theorem (see Proposition 4) for $\varkappa=2/K^2$ and $\varkappa_0=2/K_0^2$ the functions Ψ,ρ satisfy the system of differential equations (10) in some neighbourhood $(z_0-\varepsilon,z_0+\varepsilon)\subset(a,b)$ of the point z_0. Let $(a_0,b_0)\subseteq(a,b)$ be the maximal interval by inclusion containing the point z_0 on which the differential relations (10) hold. As the functions Ψ, ρ are smooth, relations (10) hold on the interval $I:=[a_0,b_0]\cap(a,b)$. This fact and the local generalized technical Bertrand theorem (see Proposition 4) imply that in some neighbourhood of the interval I in (a,b) one of the formulae for the pair (Ψ,ρ) indicated in Proposition 4(B) is true. Hence, $I=(a_0,b_0)=(a,b)$, since otherwise the interval (a_0,b_0) is not maximal.

Thus, one of the formulae for the pair (Ψ,ρ) indicated in Proposition 4(B) is true on the whole interval (a,b). By Proposition 4(B) Ψ is ρ-closing and strongly ρ-closing. Hence Ψ is also semilocally and weakly ρ-closing (see Remark 3(a)), that is, it satisfies any of the five definitions of a ρ-closing function (see Definition 3).

Step 3. If $\rho(z)$ is a linear function increasing on (a,b), then both possible ρ-closing force functions

$$\Psi_i(z)=A_i(z-\zeta)^{1-i^2},\qquad i=1,2,$$

occur, and the corresponding angular periods are equal to $\Phi_i=2\pi/\beta_i=2\pi/(i\sqrt{\rho'})$. If $\rho(z)$ has the form $\rho_2(z)$ for $D\neq 0$, then there is exactly one possible ρ-closing force function — this is $\Psi_2(z)=A(z-\zeta)^{-3}$, and $\Phi=2\pi/\beta$. In both cases we deduce from the energy conservation law that solutions $z=z(\varphi)$ of the corresponding generalized Bertrand equations have the form indicated in § 2.3, where

$$z(r)-\zeta=-\Theta(r)=-\mu^2\theta(r),\qquad \Psi(z)=f^2(r)V'(r)=-\frac{dV(r(z))}{dz},\qquad D=\mu^8 d.$$

Finally, if $\rho=\rho(z)$ does not belong to either of the two indicated forms, then there does not exist any ρ-closing force function Ψ.

Theorem 8 is proved.

§ 5. The general case of motion in a central force field

Proof of Theorems 5 and 6. *Step* 1. We describe the system under consideration. Suppose that on a surface $S\approx(a,b)\times S^1$ with coordinates $(r,\varphi \mod 2\pi)$ we are given the Riemannian metric (1) and a potential $V=V(r)$ which depends only on the coordinate r. We will find a condition on the metric that is a criterion for it to be possible to generalize Bertrand's result to the corresponding system. We let $(\,\cdot\,)^{\cdot}$ denote the derivative with respect to t, $(\,\cdot\,)'_r$ the derivative with respect to r, and $(\,\cdot\,)'_\varphi$ the derivative with respect to φ.

Step 2. The Lagrangian of the motion has the form

$$L=\frac{1}{2}(\dot r^2+f^2(r)\dot\varphi^2)-V(r),$$

and the Euler-Lagrange equations are as follows:

$$ff'_r\dot\varphi^2-V'_r-\ddot r=0,\qquad (\dot\varphi f^2)^{\cdot}=0,\qquad K:=\dot\varphi f^2=\text{const}. \tag{19}$$

For $K=0$ the motion occurs along the straight line $\{\varphi=\text{const}\}$. Now suppose that $K\neq 0$. Since $\dot\varphi=K/f^2\neq 0$, we can introduce the parameter φ instead of t on a trajectory of motion $(r(t),\varphi(t))$.

Lemma 1. *For $K\neq 0$ the function $r=r(\varphi)$ defining an orbit of motion of a point on the surface $S\approx(a,b)\times S^1$ with the metric (1) in the central field with potential $V(r)$ satisfies the following identity*:

$$K^2\left(-(\Theta\circ r)''_{\varphi\varphi}+\frac{f'_r(r)}{f(r)}\right)=f^2(r)V'_r(r), \tag{20}$$

where $\Theta=\Theta(r)$ is an arbitrary function such that

$$d\Theta(r)=\frac{dr}{f^2(r)}.$$

In other words, the function $z(\varphi):=-\Theta\circ r(\varphi)$ is a solution of the generalized Bertrand equation for

$$\rho(z)=\frac{f'_r(r(z))}{f(r(z))},\qquad \Psi(z)=f^2(r(z))V'_r(r(z)).$$

Proof. The derivatives with respect to t and φ are connected by the relations

$$\frac{dr}{dt} = \frac{K}{f^2(r)}\,\frac{dr}{d\varphi} = K\frac{d(\Theta\circ r)}{d\varphi}, \qquad \frac{d^2r}{dt^2} = \frac{K^2}{f^2(r)}\,\frac{d^2(\Theta\circ r)}{d\varphi^2},$$

where $d\Theta(r) = dr/f^2(r)$. The first equation in (19) takes the form

$$-(\Theta\circ r)''_{\varphi\varphi}\frac{K^2}{f^2(r)} + f'_r(r)\frac{K^2}{f^3(r)} = V'_r(r),$$

or, equivalently, (20), which is what we required to show. The lemma is proved.

By Lemma 1 the functions $r = r(\varphi)$ defining the motion of a point on the surface under consideration in a central force field coincide with the solutions of the equations that form the family of generalized Bertrand's equations with parameter K equal to the value of kinetic momentum on the corresponding trajectories of motion, where

$$z(r) = -\Theta(r), \qquad d\Theta(r) = \frac{dr}{f^2(r)}, \qquad \rho(z) = \frac{f'_r(r(z))}{f(r(z))},$$

$$\Psi(z) = f^2(r)V'_r(r) = -\frac{d}{dz}V(r(z)), \qquad \frac{\Psi(z)}{\rho(z)} = \frac{f^3(r(z))}{f'_r(r(z))}V'_r(r(z)),$$

and the effective potential satisfies

$$K^2U_{2/K^2}(z) = \frac{K^2}{2f^2(r(z))} + V(r(z)).$$

Hence each of conditions $(\exists)$ and $(\exists)^{\mathrm{loc}}$ on the potential $V = V(r)$ (see Definition 2) is equivalent to the condition of the same name on the function $\Psi = \Psi(z)$ (see Definition 3), and each of conditions $(\forall)$, $(\forall)^{\mathrm{loc}}$, and $(\forall)^{\text{s-loc}}$ on the potential V is equivalent to the 'rational analogue' of the condition of the same name on the function Ψ. Here, the rational analogue of the condition on the function Ψ is obtained from this condition by replacing the requirement of pairwise commensurability of the indicated periods with the (stronger) requirement of commensurability of these periods with 2π. We call such ρ-closing functions Ψ *rationally ρ-closing.* By Definition 1(a), a point $z_0 = -\Theta(r_0)$ is a stable equilibrium of the generalized Bertrand equation for $K = K_0$ if and only if the circle $\{r_0\}\times S^1$ is a strongly stable circular orbit such that the value of the kinetic momentum integral on the corresponding trajectory is equal to K_0. Therefore, the potential V is closing (locally, semilocally, strongly or weakly closing) for the metric (1) if and only if the function Ψ is rationally ρ-closing (locally, semilocally, strongly or weakly rationally ρ-closing, respectively). It follows from Theorem 8 that, for any surface S with the Riemannian metric (1), any semilocally closing potential V is locally closing, and if f has no critical points on (a,b), then all five classes of closing potentials (see Definition 2) coincide and have the following properties.

First, there exist at most two (up to an additive and a multiplicative constant) closing central potentials $V(r)$.

Second, the existence of exactly two closing central potentials V_1 and V_2 (up to an additive and a multiplicative constant) is equivalent to the condition

$$\rho'(z) := \frac{d}{dz}\left(\frac{f'_r(r(z))}{f(r(z))}\right) = \text{const} := \xi^2 > 0 \tag{21}$$

(which is equivalent to the condition $f''_{rr}(r)f(r)-(f'_r(r))^2=-\xi^2$), where $\xi=\sqrt{\rho'(z)}$ is a positive rational constant corresponding to the Bertrand constant $\beta_i=i\xi$ (where $i=1,2$, depending on the form of the potential) in Theorem 8 (see Proposition 4(B)). In case (21) we have

$$z(r)=\frac{f'_r(r)}{\xi^2 f(r)}+\zeta.$$

The function $z(r)$ was defined up to an additive constant; for definiteness we set

$$z(r):=\frac{f'_r(r)}{\xi^2 f(r)},$$

hence

$$\rho(z)=\xi^2 z, \qquad \Theta(r)=-\frac{f'(r)}{\xi^2 f(r)}.$$

By Theorem 8 rationally ρ-closing functions have the form $\Psi_i(z)=A_i z^{1-i^2}$, $i=1,2$, where $A_1 z>0$, $A_2>0$. Hence by Lemma 1 the closing potentials are gravitational and oscillatory, that is, they have the form

$$V_i(r)=\left(-\int\Psi_i(z)\,dz\right)\bigg|_{z=-\Theta(r)}=\frac{(-1)^i|A_i|\,|\Theta(r)|^{2-i^2}}{i}+B, \qquad i=1,2,$$

where $B=\mathrm{const}\in\mathbb{R}$.

Third, the existence of exactly one closing central potential is equivalent to the condition

$$\rho=-\frac{\Theta}{\mu^2}-D\frac{\Theta^{-3}}{\mu^2},$$

where $D\neq 0$, $\mu\in\mathbb{Q}_{>0}$, and $\rho=f'_r/f$. Integrating the equation

$$\frac{f'_r}{f}=-\frac{\Theta+D\Theta^{-3}}{\mu^2}$$

with respect to Θ gives

$$\frac{1}{f^2}=\frac{\Theta^2-D\Theta^{-2}+C}{\mu^2},$$

where C is an arbitrary constant, so that

$$f(r)=\frac{\mu}{\sqrt{\Theta^2(r)+C-D\Theta^{-2}(r)}}.$$

Hence the rationally ρ-closing function is given by

$$\Psi_2(z)=\frac{A_2}{z^3},$$

where $A_2(z^4+D)>0$. Consequently, the closing central potential satisfies

$$V_2(r)=\left(-\int\Psi_2(z)\,dz\right)\bigg|_{z=-\Theta(r)}=\frac{A_2|\Theta(r)|^{-2}}{2}+B,$$

where $B=\mathrm{const}\in\mathbb{R}$.

The formulae for the periods Φ_i of the functions $r = r(\varphi)$ defining noncircular nonsingular bounded orbits, and for the kinetic momenta K_i, $i = 1, 2$, for circular orbits, are repetitions of the formulae in Theorem 8. The value of the kinetic momentum K on a circular orbit $\{r\} \times S^1$ is found from the relations

$$K^2 = \frac{\Psi(z)}{\rho(z)} = \frac{f^3(r(z))}{f_r'(r(z))} V_r'(r(z))$$

(see Theorem 8). The attracting centre is situated where the potential is smallest. The sign of the force is

$$\begin{aligned} -\operatorname{sgn} V_i'(r) &= -\operatorname{sgn} \Psi_i(-\Theta(r)) = -\operatorname{sgn} \rho(-\Theta(r)) \\ &= -\operatorname{sgn} f'(r) = \operatorname{sgn}\big((\Theta^4(r) + D)\Theta(r)\big), \end{aligned}$$

and therefore $\inf f(r)$ and $\sup(A_2|\Theta(r)|)$ are attained at the attracting centre.

Step 3. Thus, we have proved that the identity $f''f - (f')^2 = -\xi^2$ holds, where $\xi \in \mathbb{Q} \cap \mathbb{R}_{>0}$, if and only if there are exactly two closing potentials (up to an additive and a positive multiplicative constant). We now produce an explicit form of the functions f for which this identity holds.

Lemma 2. *The solutions of the equation $f''f - (f')^2 = -\xi^2$ for $\xi > 0$ that have no zeros are the following functions $f = f(r)$ and only these*:

$$\frac{\xi}{\alpha}\sin(\alpha r + \beta), \qquad \pm\frac{\xi}{\alpha}\sinh(\alpha r + \beta), \qquad \pm\xi r + \beta, \tag{22}$$

where $\alpha \neq 0$, β are arbitrary real constants and r belongs to an interval in which $f(r) \neq 0$.

Proof. Suppose that $f'(r) \neq 0$ in a neighbourhood of some point. We set $f' = p(f)$. Then $f'' = p'p$. Let $w = p^2$. Then the original differential equation takes the form $w'f = 2w - 2\xi^2$. Its solution is the function $w = w(f) = C_1 f^2 + \xi^2$. Hence we obtain the following:

$$\pm dr = \frac{df}{\sqrt{C_1 f^2 + \xi^2}}.$$

Depending on the sign of the constant C_1 we obtain the solutions (22) for $C_1 < 0$, $C_1 > 0$, and $C_1 = 0$, respectively. The initial conditions $(f(0), f'(0))$ of these solutions form the set

$$\left\{\left(\frac{\xi}{\alpha}\sin\beta, \xi\cos\beta\right), \pm\left(\frac{\xi}{\alpha}\sinh\beta, \xi\cosh\beta\right), (\beta, \pm\xi)\right\} \supset (\mathbb{R} \setminus \{0\}) \times \mathbb{R},$$
$$\alpha \in \mathbb{R} \setminus \{0\}, \qquad \beta \in \mathbb{R}.$$

Since f is a solution of a second order differential equation solved for the higher derivative and with a smooth right-hand side (as $f \neq 0$), there are no other solutions without zeros.

Remark 9. It can be proved in a similar fashion that the solutions of the equation $-f''f + (f')^2 = h = \text{const} \leqslant 0$ without zeros are the functions

$$f(r) = \frac{\sqrt{-h}}{\alpha}\cosh(\alpha r + \beta) \quad \text{for } h < 0, \qquad f(r) = \alpha e^{\beta r} \quad \text{for } h = 0,$$

where $\alpha \neq 0$, $\beta \in \mathbb{R}$.

Taking account of the changes of variables $\Theta(r) = \mu^2\theta(r)$, $C = \mu^4 c$, $D = \mu^8 d$, Theorems 5 and 6, as well as the orbit formulae in § 2.3, are proved.

We have also proved parts (A), (C) of Theorem 7, the equivalence of conditions (a) and (b) and the uniqueness of the tuple in condition (b) of Theorem 7(B). The equivalence of conditions (b) and (c) of Theorem 7(B) follows from the relations

$$Q_{c_1,d_1}(\theta) = \lambda^{-2}Q_{c,d}(\lambda\theta), \qquad r_{c_1,d_1}(\theta) = \lambda(r_{c,d}(\lambda\theta) - r_0),$$
$$\lambda^{-1}f_{c_1,d_1,k}(\lambda(r - r_0)) = f_{c,d,k}(r)$$

for $c := \lambda^2 c_1$, $d := \lambda^4 d_1$, $\lambda > 0$. Parts (A), (B), (C) of Corollary 2 are derived from the formulae given in it and the table in § 2.2. Corollary 2(D) is a well-known fact concerning arbitrary surfaces of revolution and their multi-dimensional analogues. The first part of Corollary 3(A) follows from the uniqueness of the triple (c, d, k) and the uniqueness of the isometric embedding of the surface S into the Bertrand surface $(S_{c,d,k}, ds^2_{\mu,c,d})$, see condition (b) of Theorem 7(B), in view of the uniqueness of the numbers r_0, η (for $\mu > 0$, but not necessarily $\mu \in \mathbb{Q} \cap \mathbb{R}_{>0}$). The first part of Corollary 3(B) follows from the equivalence of conditions (b) and (c) of Theorem 7(B). The second parts of the above-mentioned corollaries (about being pairwise non-isometric, locally non-isometric, non-similar, locally non-similar) follow from the first parts and the fact that the factors $\mu, \lambda > 0$ are arbitrary. The associations in Corollary 3(C) take geodesics to geodesics by the explicit formulae for geodesics; see the analogue of § 2.3 for $A = 0$.

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