Function Signatures
Radar Cross Sections

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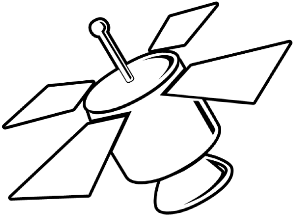
Models of Radar Cross Sections for Satellites



Spherical Harmonics Expansion

$$f(r,\theta,\phi) \approx a_{0,0}Y_0^0 + a_{1,-1}Y_1^{-1} + a_{1,0}Y_1^0 + a_{1,1}Y_1^1 + \dots$$

where

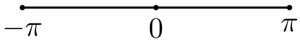
$$Y_n^m(\theta,\phi) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}}P_n^m(\cos\theta)e^{im\phi}$$

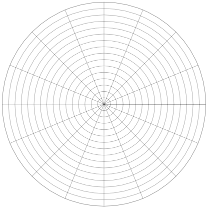
Overview

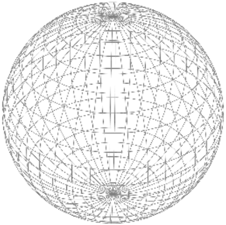
Goal: RCS Models in 3D

RCS models have been built in 2D.
Extend to 3D using the same MoM code.

Fourier Domains in 1-, 2-, and 3D







$\theta \in [-\pi, \pi]$

$\theta \in [-\pi, \pi]$
$r \in [0, 1]$

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$r \in [0, 1]$
$\phi \in [0, \pi]$

Why We Love Fourier for Smooth Functions I

Smooth Functions, Beautiful Representations

- **Weierstrass Approximation Theorem:** Any continuous function on $[a, b]$ can be uniformly approximated by polynomials. Fourier provides a similar approximation, using trigonometric bases instead of polynomials.
- **Riesz-Fischer Theorem:** Hunting license - Fourier coefficients $(a_n, b_n) \in l^2$, guaranteeing convergence in the L^2 sense.
- **Uniform Convergence for Smooth Periodic Functions:** Smooth (C^∞ or C^k) functions, Fourier series converge uniformly, ensuring no oscillatory artifacts (Gibbs phenomenon disappears).

Why We Love Fourier for Smooth Functions II

- **Orthogonality of Basis:** Sines and cosines form an orthogonal basis in L^2, enabling direct computation of coefficients via projection (Parseval's theorem quantifies energy distribution).
- **Spectral Insights:** Decomposes a function into frequency components, making smoothness and structure explicit. High smoothness = rapid Fourier coefficient decay.
- **Compact Representation:** Smooth functions require fewer terms for accurate approximation. High efficiency for practical computation and storage.

Why We Love Fourier for Smooth Functions III

- **Universality:** Fourier's reach extends to PDEs, signal processing, and quantum mechanics. Smooth functions unlock the full power of these tools.

Takeaway: Fourier connects smoothness, convergence, and representation, offering unmatched clarity and utility for periodic and localized phenomena.

Fourier and Extensions to 2- and 3D

1D: $$f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

2D: $$f(r,\theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n,2} a_n^m R_n^m(r) e^{in\theta}$$

3D: $$f(r,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m \sqrt{\frac{(2m+1)(m-n)!}{4\pi(m+n)!}} P_l^m(\cos\theta) e^{in\phi}$$

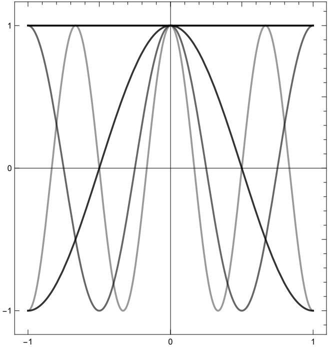
Fourier and Extensions to 2- and 3D

- **1D Fourier Series:** Decomposes a periodic function $f(\theta)$ into a sum of complex exponentials with coefficients a_n capturing the amplitudes of each frequency component.
- **2D Zernike Polynomials:** Extends Fourier analysis to two dimensions using radial functions $R_n^m(r)$, often employed in circular domains or optical applications.
- **3D Spherical Harmonics:** Represents functions on a sphere using harmonics $Y_l^m(\theta, \phi)$ and radial components $R_l(r)$, crucial in fields like quantum mechanics and gravitational modeling.

Lowest Order Fourier Basic Functions

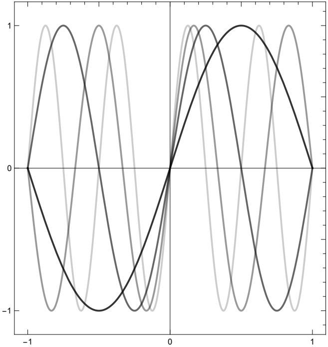
$$f(-\theta) = f(\theta)$$

Even Parity

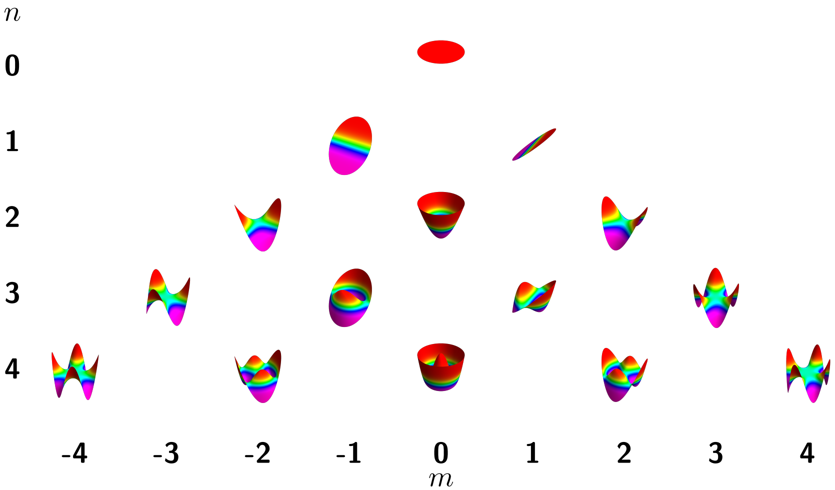


$$f(-\theta) = -f(\theta)$$

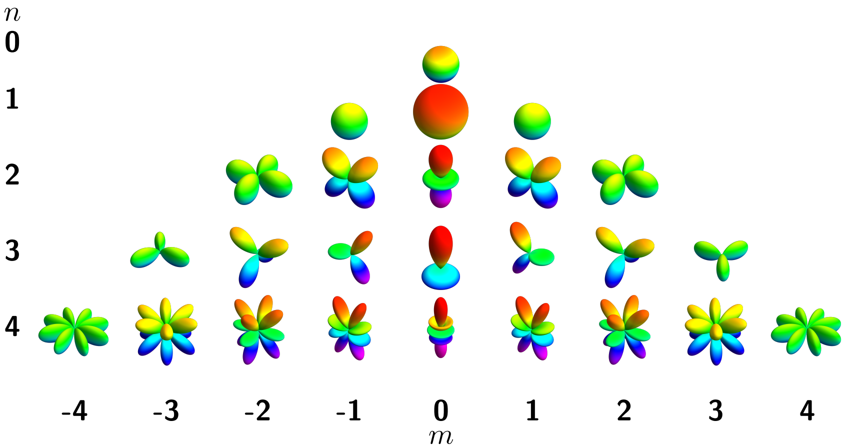
Odd Parity



Lowest Order Zernike Disk Polynomials



Lowest Order Spherical Harmonics

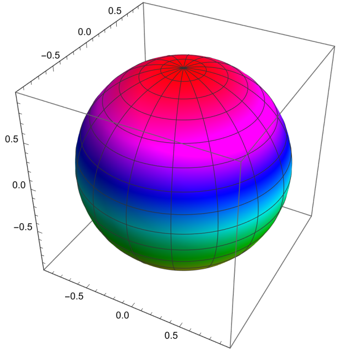
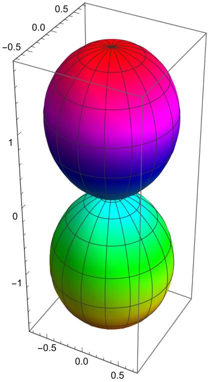


Lowest Order Spherical Harmonics

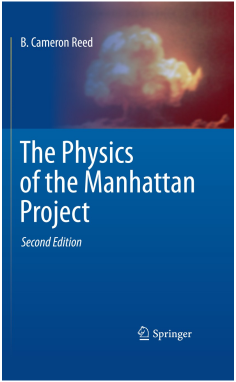
Elegant, powerful theorems

1. **Bohr-Wheeler Spontaneous Fission Limit**
2. **Shape of Earth**

Comparison of Liquid Drop Models

Constant Radius	Variable Radius
	

Historical Survey



Appendix E: Formal Derivation of the Bohr–Wheeler Spontaneous Fission Limit.

Summary of Liquid Drop Model

1. Nucleus is liquid drop of variable radius
2. Fluid is incompressible
3. Volume is conserved

1939 Paper on Nuclear Fission

SEPTEMBER 1, 1939 PHYSICAL REVIEW VOLUME 56

The Mechanism of Nuclear Fission

NIELS BOHR
University of Copenhagen, Copenhagen, Denmark, and The Institute for Advanced Study, Princeton, *y*

AND

JOHN ARCHIBALD WHEELER
Princeton University, Princeton, New Jersey

(Received June 28, 1939)

On the basis of the liquid drop model of atomic nuclei, an account is given of the mechanism of nuclear fission. In particular, conclusions are drawn regarding the variation from nucleus to nucleus of the critical energy required for fission, and regarding the dependence of fission cross section for a given nucleus on energy of the exciting agency. A detailed discussion of the observations is presented on the basis of the theoretical considerations. Theory and experiment fit together in a reasonable way to give a satisfactory picture of nuclear fission.

Deformations from Sphericity

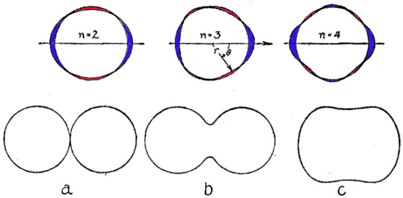


FIG. 2. Small deformations of a liquid drop of the type $\delta r(\theta)=\alpha_n P_n(\cos\ \theta)$ (upper portion of the figure) lead to characteristic oscillations of the fluid about the spherical form of stable equilibrium, even when the fluid has a uniform electrical charge. If the charge reaches the critical value $(10\times\text{surface tension}\times\text{volume})^{\frac{1}{2}}$, however, the spherical form becomes unstable with respect to even infinitesimal deformations of the type $n=2$. For a slightly smaller charge, on the other hand, a finite deformation (c) will be required to lead to a configuration of *unstable equilibrium*, and with smaller and smaller charge densities the critical form gradually goes over (c, b, a) into that of two uncharged spheres an infinitesimal distance from each other (a).

Bohr-Wheeler Derivation

to the value

$$\boxed{r(\theta)=R[1+\alpha_0+\alpha_2 P_2(\cos\theta) + \alpha_3 P_3(\cos\theta)+\cdots],} \quad (8)$$

where the α_n are small quantities. Then a straightforward calculation shows that the surface energy plus the electrostatic energy of the comparison drop has increased to the value

$$E_{S+E}=4\pi(r_0 A^{\frac{1}{3}})^2 O[1+2\alpha_2^2/5+5\alpha_3^2/7+\cdots + (n-1)(n+2)\alpha_n^2/2(2n+1)+\cdots] + 3(Ze)^2/5r_0 A^{\frac{1}{3}}[1-\alpha_2^2/5-10\alpha_3^2/49-\cdots - 5(n-1)\alpha_n^2/(2n+1)^2-\cdots], \quad (9)$$

where we have assumed that the drop is composed of an incompressible fluid of volume $(4\pi/3)R^3=(4\pi/3)r_0^3 A$, uniformly electrified to a charge Ze, and possessing a surface tension O.

Express Radius as a Function of Polar Angle

$$\xcancel{r = R_0} \tag{2.1}$$

$$\Downarrow$$

$$r(\theta) = R_0 \left(1 + \alpha_0 + \alpha_2 P_2(\cos\theta)\right) \tag{2.2}$$

Express Radius as a Function of Polar Angle

Adjust α_0, α_2 so that volume is conserved:

$$r(\theta) = R_0 \left(1 + \alpha_0 + \alpha_2 P_2(\cos\theta)\right) \tag{2.2}$$

Lowest Order Spherical Harmonics

Elegant, powerful theorems

1. Riesz–Fischer Theorem
2. Weierstrass Approximation Theorem: Interval
3. Müntz–Szász theorem: Unit disk (Moragues n.d.; Siegel 1972)
4. Trent's Theorem: (Trent 1981)
5. absent: Unit sphere

Riesz–Fischer Theorem: Hunting License

Theorem (Riesz-Fischer)

Let $\{\phi_n\}$ be an orthonormal sequence of functions on Ω and suppose $\sum |a_n|^2$ converges. Denote the partial sum as

$$s_\tau = a_0\phi_0 + a_1\phi_1 + \cdots + a_\tau\phi_\tau.$$

There exists a function $F \in L^2(\Omega)$ such that $\{s_\tau\}$ converges to F in $L^2(\Omega)$, and such that

$$F = \sum_{k=0}^{\infty} a_k\phi_k,$$

almost everywhere.

Weierstrass Approximation Theorem: Foundation

Theorem (Weierstrass Approximation Theorem)

Polynomials are dense in the space of continuous functions $C^\infty([0,1])$ with respect to the uniform norm.

Proof.

See, for example, Jackson 1934; Białas and Nakamura 1996; Young 2006; Aktan and Vural 2024. □

Müntz–Szász Theorem: Culled Subsets

Theorem (Spans of $C\,[0,1]$)

Suppose $(\lambda_j)_{j=0}^{\infty}$ is a sequence with $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$. The the space $\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$ is dense in $C\,[0,1]$ if an only if

$$\sum_{j=1}^{\infty} \frac{1}{j} = \infty \tag{3.1}$$

Let λ_k be a sequence of numbers which grows without bound. The set of functions like (??) is a span of $C\,[0,1]$, that vanish at $x = 0$ iff

$$\sum_{k \in K} \frac{1}{\lambda_k} = \infty$$

Trent Theorem: Extension to Closed Unit Disk

Theorem (Trent's Theorem)

Let the set S be the following set of continuous functions in the complex plane

$$S = \{z^n \bar{z}^m,\ n, m = 0, 1, 2, \ldots\}.$$

The closed span of S is $\overline{D}_2$ iff the sum

$$\sum_{m \in M_j'} \frac{1}{m}$$

diverges for every integer j.

Proof.

See A Müntz–Szász theorem for $C\left(\bar{D}\right)$, Trent 1981, p.

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