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ON THE MÜNTZ-SZÁSZ THEOREM FOR $C[0, 1]$

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ABSTRACT. The functions $1, t^{\lambda_1}, t^{\lambda_2}, \cdots$ with complex λ's are shown to be incomplete in $C[0, 1]$ under conditions weaker than those proven by Szász, and a special construction due to P. D. Lax where the functions are complete is given.

In 1916 Szász proved the following classical result:

THEOREM 1. *Suppose* $\operatorname{Re} \lambda_j > 0, j=1, 2, \cdots$, *and, for the sake of simplicity, the* λ*'s are distinct. Then the functions* $1, t^{\lambda_1}, t^{\lambda_2}, \cdots$ *are complete in* $C[0, 1]$ *if*

$$\sum \frac{\operatorname{Re} \lambda_j}{1+|\lambda_j|^2} = \infty \tag{1}$$

and incomplete if

$$\sum \frac{\operatorname{Re} \lambda_j + 1}{1+|\lambda_j|^2} < \infty. \tag{2}$$

When $\liminf_{j\to\infty} \operatorname{Re} \lambda_j > 0$, (1) is the negation of (2) and constitutes a necessary and sufficient condition for completeness. However, under circumstances where neither (1) nor (2) is satisfied, for example if $\lambda_j = 1/j + i\sqrt{j}$ or the λ's are bounded and $\sum \operatorname{Re} \lambda_j < \infty$, the completeness question is not answered by Szász's theorem.

For the L^2 case, it should be noted that there is no such unresolved margin. This is because the L^2 distance between t^λ and the linear span of $\{t^{\lambda_j}\}_{j=1}^\infty$ can be explicitly expressed in terms of a Blaschke product which converges to a nontrivial function if and only if

$$\sum \frac{\operatorname{Re} \lambda_j + \frac{1}{2}}{1+|\lambda_j|^2} < \infty. \tag{3}$$

The purpose of this paper is to reduce the gap in Theorem 1.

THEOREM 2. *Suppose* $\operatorname{Re} \lambda_j > 0$, $j = 1, 2, \cdots$, $\lambda_j \to \infty$ *as* $j \to \infty$ *and the* λ*'s are distinct. If, for some* $\alpha < 1$,

$$\sum \frac{\operatorname{Re} \lambda_j + \exp(-|\lambda_j|^\alpha)}{1+|\lambda_j|^2} < \infty, \tag{4}$$

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then the linear span of $1, t^{\lambda_1}, t^{\lambda_2}, \cdots$ *is not dense in* $C[0, 1]$. *In fact,* t^{λ} *cannot be uniformly approximated by elements of the span unless* λ *equals one of the* λ_j*'s or* 0.

PROOF. We shall construct a bounded linear functional l on $C[0, 1]$, see (14), such that $l(t^{\lambda})=0$ only when $\lambda=0, \lambda_1, \lambda_2, \cdots$. First, a few preliminary constructions. Set

$$e^{-s} = t, \quad F(s) = f(e^{-s}) \tag{5}$$

for $f \in C[0, 1]$. Then $F(s) \in C[0, \infty]$ and has a Laplace transform

$$\hat{F}(w) = \int_0^{\infty} e^{ws} F(s)\, ds \tag{6}$$

for Re $w<0$. Clearly $\hat{F}$ satisfies

$$|\hat{F}(w)| \leqq \frac{\|F\|_{\infty}}{-\mathrm{Re}\, w}. \tag{7}$$

LEMMA. *Choose some* β, $\alpha<\beta<1$, *and, via the Poisson integral, define an analytic function* ψ *in the right half-plane such that*

$$\mathrm{Re}\, \psi(iy) = \exp(-|y|^{\beta}). \tag{8}$$

Then ψ *has the following properties*:

(a) ψ *is bounded.*

(b) *The mapping* $z \to w$ *defined by*

$$w = z - \psi(z) \tag{9}$$

maps the right half-plane 1-1 *onto a region containing* Re $w \geqq 0$.

(c) *Denote by* γ_j *the point whose image under this map is* λ_j:

$$\lambda_j = \gamma_j - \psi(\gamma_j). \tag{10}$$

Then

$$\mathrm{Re}\, \gamma_j = O(\mathrm{Re}\, \lambda_j + \exp(-|\lambda_j|^{\alpha})). \tag{11}$$

Proof later.

It follows from (4) and (11) that the Blaschke product

$$B(z) = \prod_{j=0}^{\infty} \frac{\gamma_j - z}{\bar{\gamma}_j + z} \frac{\bar{\gamma}_j}{\gamma_j} = \prod_{j=0}^{\infty} \left(1 - \frac{2z\, \mathrm{Re}\, \gamma}{|\gamma_j|^2 + \gamma_j z}\right) \tag{12}$$

converges. From (7) and (8), we have

$$|\hat{F}(iy - \psi(iy))| \leqq \exp(|y|^{\beta})\, \|F\|_{\infty}. \tag{13}$$

Choosing a δ, $\alpha<\beta<\delta<1$, we define the linear functional $l(f)$ as follows:

$$l(f) = \int_{-i\infty}^{i\infty} \hat{F}(z - \psi(z))\exp(-z^{\delta})B(z)\, dz. \tag{14}$$

Since $|B(iy)|=1$ and $|\exp(|y|^{\beta})\exp(-(iy)^{\delta})|$ is integrable, it follows from (13) that l is bounded. Now set $f=t^{\lambda}$; then $F(s)=e^{-\lambda s}$, $\hat{F}(w)=1/(\lambda-w)$, and $\hat{F}(z-\psi(z))=-1/(z-\psi(z)-\lambda)$ is meromorphic with one pole, γ, in the right half-plane. In this case, the contour in (14) can be shifted to the right, with the total contribution resulting from the residue at $z=\gamma$ since the integrand vanishes in a dominated way at infinity:

$$l(t^{\lambda}) = 2\pi i \frac{\exp(-\gamma^{\delta})B(\gamma)}{1 - \psi'(\gamma)}. \tag{15}$$

Therefore $l(t^{\lambda})$ vanishes if and only if B vanishes at γ; this happens only when $\gamma=\gamma_0, \gamma_1, \cdots$, i.e., when $\lambda=0$, $\lambda_1, \lambda_2, \cdots$. This completes the proof except for the lemma.

Setting

$$\psi(x + iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-|t|^{\beta}}}{x + i(y - t)}\, dt$$

gives property (8) and the boundedness of Re ψ. But

$$\begin{aligned}
|\mathrm{Im}\, \psi(x + iy)| &= \left|\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-|t|^{\beta}}(t - y)}{x^2 + (y - t)^2}\, dt\right| \\
&= \left|\frac{1}{\pi}\int_{0}^{\infty} \frac{t(e^{-|t+y|^{\beta}} - e^{-|t-y|^{\beta}})}{x^2 + t^2}\, dt\right| \\
&< \sup_y \int_0^1 \left|\frac{e^{-|t+y|^{\beta}} - e^{-|t-y|^{\beta}}}{t}\right| dt + \int_{-\infty}^{\infty} e^{-|t|^{\beta}}\, dt \\
&< 2\beta \sup_y \int_0^1 \frac{e^{-|t-y|^{\beta}}}{|t - y|^{1-\beta}}\, dt + \int_{-\infty}^{\infty} e^{-|t|^{\beta}}\, dt < \infty.
\end{aligned}$$

So $|\psi(z)|<k$ for some k, which is property (a). Applying the argument principle to $z-\psi(z)$ in the right half-plane gives (b). Now

$$\begin{aligned}
\mathrm{Re}\, \psi(z) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{xe^{-|t|^{\beta}}}{x^2 + (y - t)^2}\, dt \\
&\leqq \frac{1}{\pi}\int_{-|y/2|}^{|y/2|} \frac{e^{-|t|^{\beta}}x}{x^2 + (y - t)^2}\, dt + \frac{e^{-|y/2|^{\beta}}}{\pi}\int_{-\infty}^{\infty} \frac{x}{x^2 + (y - t)^2}\, dt \\
&< \frac{4x|y|}{\pi(x^2 + y^2)} + e^{-|y/2|^{\beta}},
\end{aligned}$$

$$\mathrm{Re}\, \psi(z) < \mathrm{Re}(z/2) + \exp(-|\mathrm{Im}(z/2)|^{\beta}), \qquad |z| \text{ large.} \tag{16}$$

But $\gamma_j=\lambda_j+\psi(\gamma_j)$, and hence $\operatorname{Re}\gamma_j=\operatorname{Re}\lambda_j+\operatorname{Re}\psi(\gamma_j)$. Applying (16) yields

$$\operatorname{Re}\gamma_j = O(\operatorname{Re}\lambda_j + \exp(-|\operatorname{Im}(\gamma_j/2)|^\beta)). \tag{17}$$

Since $|\lambda_j-\gamma_j|<k$, it follows that

$$\exp(-|\operatorname{Im}(\gamma_j/2)|^\beta) = O(\operatorname{Re}\lambda_j + \exp(-|\gamma_j|^\alpha)), \qquad \alpha < \beta. \tag{18}$$

Combining (17) and (18) gives (11), the final part of the lemma.

COROLLARY. *Suppose* $\operatorname{Re}\lambda_j>0$, $j=1, 2, \cdots$, $\lambda_j\to i\gamma$ *as* $j\to\infty$ *and the* λ*'s are distinct. If, for some* $\alpha<1$,

$$\sum \operatorname{Re}\lambda_j + \exp\left(-\left|\frac{1}{\lambda_j - i\gamma}\right|^\alpha\right) < \infty, \tag{19}$$

then $1, t^{\lambda_1}, t^{\lambda_2}, \cdots$ *is not complete in* $C[0, 1]$.

This follows from setting

$$l(f) = \int_{-i\infty}^{i\infty} \hat{F}\left(i\gamma + \frac{1}{z-\psi(z)}\right) e^{-z^\delta} B(z)\, dz. \tag{20}$$

COROLLARY. *Suppose* $\lambda_{j,k}\to i\gamma_k$ *and* $\lambda_{j,0}\to\infty$ *as* $j\to\infty$, $k=1, 2, \cdots, l$, *the* λ*'s are distinct and with positive real parts. If, for some* $\alpha<1$,

(21)
$$\sum_{j=1}^{\infty} \operatorname{Re}\left[\frac{\lambda_{j,0} + \exp(-|\lambda_{j,0}|^\alpha)}{1 + |\lambda_{j,0}|^2} + \sum_{k=1}^{l} \lambda_{j,k} + \exp\left(-\left|\frac{1}{\lambda_{j,k} - i\gamma_k}\right|^\alpha\right)\right] < \infty,$$

then the functions $1, t^{\lambda_{j,k}}, j=1, 2, \cdots, k=0, 1, \cdots, l$, *are not complete in* $C[0, 1]$.

The result is a direct consequence of setting

$$\hat{F}(w_0, w_1, \cdots, w_l) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \exp(w_0 s_0 + w_1 s_1 + \cdots + w_l s_l) \times F(s_0 + s_1 + \cdots + s_l)\, ds_0\, ds_1 \cdots ds_l, \tag{22}$$

$$l(f) = \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \hat{F}\left(z_0 - \psi(z_0), i\gamma_1 + \frac{1}{z_1 - \psi(z_1)}, \cdots, i\gamma_l + \frac{1}{z_l - \psi(z_l)}\right) \times \prod_{k=0}^{l} e^{-z_k^\delta} B_k(z_k)\, dz_k. \tag{23}$$

COROLLARY. *Suppose the* λ_j*'s are positive and distinct for* $j=1, 2, \cdots$. *Then* $1, t^{\lambda_1}, t^{\lambda_2}, \cdots$ *are complete in* $C[0, 1]$ *if and only if*

$$\sum [\lambda_j/(1 + \lambda_j^2)] = \infty. \tag{24}$$

This generalization of Müntz's theorem is stated, though not proven, in [**2**, p. 29]. Its proof may be simplified by letting $\psi\equiv 1$ in the preceding calculations.

For complex λ's, however, Lax has shown by example that (1) is not necessary for completeness.

THEOREM 3 (LAX). *Let l_j be any sequence of positive numbers going to infinity, and α_j any positive sequence tending to zero. Define $\lambda_{j,k}$ by*

$$\lambda_{j,k} = \alpha_j + i2\pi k/l_j, \qquad j = 1, 2, \cdots, k = 0, \pm 1, \pm 2, \cdots. \tag{25}$$

Then $\{1, t^{\lambda_{j,k}}\}_{j=1}^{\infty}{}_{k=-\infty}^{\infty}$ is complete in $C[0, 1]$.

If we choose α_j and l_j so that

$$\sum \alpha_j l_j < \infty, \tag{26}$$

then $\sum_{j,k} [\operatorname{Re} \lambda_{j,k}/(1+|\lambda_{j,k}|^2)]$ converges since

$$\sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} \frac{\operatorname{Re} \lambda_{j,k}}{1 + |\lambda_{j,k}|^2} < \sum_{j=1}^{\infty} \alpha_j \left(1 + 2\int_0^{\infty} \frac{dx}{1 + (x/l_j)^2}\right) = O\left(\sum \alpha_j l_j\right).$$

Unfortunately, this example is somewhat pathological since the accumulation points of $\lambda_{j,k}$ are precisely the imaginary axis.

PROOF OF THEOREM 3. Let X be the closure in $C_0[0, \infty]$ of the linear space spanned by $\{e^{-\lambda_{j,k}s}\}_{j=1}^{\infty}{}_{k=-\infty}^{\infty}$, where

$$C_0[0, \infty] = \left\{F \in C[0, \infty]: \lim_{s\to\infty} F(s) = 0\right\}.$$

To prove that $X = C_0[0, \infty]$, an equivalent formulation of the theorem, it suffices to show that for any $F \in C_0[0, \infty]$, there is a $G \in X$ such that

$$\|F - G\|_\infty \leqq \tfrac{2}{3}\|F\|_\infty. \tag{27}$$

This is because successive approximations give a sequence in X converging to F.

Let F be an element of $C_0[0, \infty]$. Since $\lim_{s\to\infty} F(s) = 0$, we may choose J so large that

$$|F(s)| \leqq \tfrac{1}{3}\|F\|_\infty \quad \text{for all } s > l_J - 1. \tag{28}$$

For convenience, denote l_J by l, and α_J by α. Define $H(s)$ on $[0, l]$ by

$$\begin{aligned} H(s) &= e^{\alpha s}F(s), \qquad 0 \leqq s < l - 1, \\ &= e^{\alpha s}F(s)(l - s) + F(0)(s - l + 1), \qquad l - 1 \leqq s \leqq l. \end{aligned} \tag{29}$$

H is continuous, and $H(0) = H(l)$. Define L to be the periodic extension of H. Then L can be uniformly approximated by linear combinations of

$\exp(i2\pi ks/l)$, $k=0, \pm 1, \pm 2, \cdots$. Therefore $G(s)=\frac{1}{3}e^{-\alpha s}L(s)$ can be uniformly approximated by linear combinations of $\exp(-\lambda_{J,k}s)$, $k=0, \pm 1, \pm 2, \cdots$, and hence belongs to X. Clearly

$$G(s) = \tfrac{1}{3}F(s) \quad \text{for } s \in [0, l-1]. \tag{30}$$

For $s \in [l-1, l]$,

$$\begin{aligned} |G(s)| &\leqq \tfrac{1}{3}(|F(s)|\,(l-s) + |F(0)|\,(s-l+1)) \\ &\leqq \tfrac{1}{3}(\|F\|_\infty(l-s) + \|F\|_\infty(s-l+1)) = \tfrac{1}{3}\|F\|_\infty. \end{aligned} \tag{31}$$

Since $|G(s)| \leqq |G(s-l)|$ for $s>l$, it follows that

$$\|G\|_\infty = \tfrac{1}{3}\|F\|_\infty. \tag{32}$$

From (30) we have

$$|F(s)-G(s)| \leqq \tfrac{2}{3}\|F\|_\infty, \qquad s \in [0, l-1]. \tag{33}$$

Combining (32) and (28) gives

$$|F(s)-G(s)| \leqq |F(s)| + |G(s)| \leqq \tfrac{2}{3}\|F\|_\infty, \qquad s > l-1. \tag{34}$$

Combining (33) and (34) we see that $\|F-G\|_\infty \leqq \frac{2}{3}\|F\|_\infty$, as asserted in (27).

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