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Remarks on Theorems of Müntz-Szász Type

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1. In this rather expository note we will describe a few simple observation concerning extensions of the classical Müntz-Szász theorem.

Let X be a normed linear space and X^* the dual space of X. A subset Γ of X^* is called total over X if $\gamma(x)=0$ for all $\gamma\in\Gamma$ implies $x=0$. A subset E of X is called fundamental if E spans a dense linear subspace of X.

PROPOSITION 1. *Let $\phi(t)$ be a strictly increasing real-valued continuous function on the interval $[0, 1]$ with $\phi(0)=0$ and let $\{n(k): k=1, 2,\dots\}$ be a strictly increasing sequence of positive real numbers tending to infinity. Then we have the following*:

(a) *Let $C_0[0, 1]$ be the space of (real- or complex-valued) continuous functions on $[0, 1]$ vanishing at 0, equipped with the usual uniform norm. Then, the system $\{\phi^{n(k)}: k=1, 2,\dots\}$ is fundamental in $C_0[0, 1]$ if and only if $\sum_{k=1}^{\infty} n(k)^{-1}=+\infty$.*

(b) *If $\sum_{k=1}^{\infty} n(k)^{-1}=+\infty$, then $\{\phi^{n(k)}: k=1, 2,\dots\}$ is total over $L^1[0, 1]$.*

(c) *If $\sum_{k=1}^{\infty} n(k)^{-1}<+\infty$ and if $\phi(t)$ is absolutely continuous, then $\{\phi^{n(k)}: k=1, 2,\dots\}$ is not total over $L^1[0, 1]$.*

PROPOSITION 2. *If $\phi(t)$ is an absolutely continuous function on $[0, 1]$ which is not one-to-one, then the system $\{\phi^n: n=0, 1,\dots\}$ is not total over $L^1[0, 1]$.*

PROPOSITION 3. *There exists a non-negative continuous function $\phi(t)$ on $[0, 1]$ with $\phi(0)=\phi(1)=0$ such that the system $\{\phi^{n(k)}: k=1, 2,\dots\}$ is total over $L^1[0, 1]$ provided $\sum_{k=1}^{\infty} n(k)^{-1}=+\infty$.*

2. The facts stated in the preceding section follow at once from the classical Müntz-Szász theorem but we will give a sketchy proof in order to make the description self-contained.

Proposition 1 follows from the classical one by a mere change of variable: $t\to\phi(t)$. Proposition 2 is also simple but we want to derive it from the Cauchy theorem.

PROPOSITION 2′. *Let $\psi(t)$ be a complex-valued continuous function defined*

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on an interval $[a, b]$, $a<b$. *Suppose that* $\psi(t)$ *is absolutely continuous and that* $\psi(a)=\psi(b)$. *Then, there exists a non-trivial integrable function* $f(t)$ *on* $[a, b]$ *such that*

$$\int_a^b f(t)\psi(t)^n dt=0$$

for $n=0, 1, \ldots$. *If* ψ *is real-valued, then* f *can be made real-valued.*

PROOF. We denote by L the curve defined by ψ: $[a, b]\to\mathbf{C}$, which may be degenerate. Our assumption says that L is closed and rectifiable. So the Cauchy theorem shows that

$$\int_L z^n dz=0$$

for $n=0, 1, \ldots$. Using the parametrization $t\to\psi(t)$ of the curve L, we have

$$\int_a^b \psi(t)^n d\psi(t)=0.$$

Since ψ is absolutely continuous, $d\psi(t)=\psi'(t)dt$ with $\psi'\in L^1[a, b]$ and therefore

$$\int_a^b \psi(t)^n\psi'(t)dt=0.$$

If $\psi'(t)\neq 0$ on a set of positive Lebesgue measure, then $f=\psi'$ satisfies the required condition. On the other hand, if $\psi'(t)=0$ a.e., then ψ is constant on $[a, b]$ and we have only to take as f any nontrivial function in $L^1[a, b]$ with $\int_a^b f(t)dt=0$.

Q. E. D.

In order to prove Proposition 3, let $\phi_0(t)$ be a strictly increasing continuous function on $[0, 1]$ with $\phi_0(0)=0$ and $\phi_0(1)=1$ such that $\phi_0'(t)=0$ a.e. on $[0, 1]$. For existence of such a function, see Gelbaum and Olmsted [1; p. 105]. We then define a function $\phi(t)$ by setting $\phi(t)=2t$ for $0\leqq t\leqq 1/2$ and $=\phi_0(2(1-t))$ for $1/2\leqq t\leqq 1$. Now, take any strictly increasing sequence $\{n(k)\}$ of positive real numbers tending to infinity such that $\sum_{k=1}^{\infty} n(k)^{-1}=+\infty$. Suppose that a function $g\in L^1[0, 1]$ satisfies

$$I_k=\int_0^1 \phi(t)^{n(k)}g(t)dt=0$$

for $k=1, 2, \ldots$. Then we have

$$\begin{aligned} I_k&=\int_0^{1/2}(2t)^{n(k)}g(t)dt+\int_{1/2}^1 \phi_0(2(1-t))^{n(k)}g(t)dt \\ &=\frac{1}{2}\int_0^1 t^{n(k)}[g(t/2)dt-g((2-\phi_0^{-1}(t))/2)d(\phi_0^{-1}(t))]. \end{aligned}$$

Since $\{t^{n(k)}: k=1, 2, \ldots\}$ is fundamental in the space $C_0[0, 1]$, it follows that

$$g(t/2)dt - g((2-\phi_0^{-1}(t))/2)d(\phi_0^{-1}(t)) = 0$$

on (0, 1]. Since $\phi_0(t)$ is singular on [0, 1], so is its inverse $\phi_0^{-1}(t)$ and therefore the measures dt and $d(\phi_0^{-1}(t))$ are mutually singular. Thus we conclude that

$$g(t/2)dt = g((2-\phi_0^{-1}(t))/2)d(\phi_0^{-1}(t)) = 0$$

on (0, 1]. Hence we see that $g(t)=0$ a.e. on [0, 1]. Namely, the system $\{\phi^{n(k)}: k=1, 2,\ldots\}$ is total over $L^1[0, 1]$, as desired.

3. Finally, we note that in Proposition 3 the function ϕ is allowed to possess an arbitrary number of ups and downs. In the following we just indicate how this is possible by dealing with the simplest case. Namely, we set

$$\phi(t) = \begin{cases} 3t & \text{for } 0 \leqq t \leqq 1/3 \\ \phi_1(2-3t) & \text{for } 1/3 \leqq t \leqq 2/3 \\ \phi_2(3t-2) & \text{for } 2/3 \leqq t \leqq 1, \end{cases}$$

where ϕ_1 and ϕ_2 are strictly increasing continuous functions on [0,1] with $\phi_1(0) = \phi_2(0) = 0$ and $\phi_1(1) = \phi_2(1) = 1$. We wish to choose functions ϕ_1 and ϕ_2 in such a way that ϕ satisfies the conclusion of Proposition 3. Thus we proceed just as in the proof of Proposition 3. Namely, let $g \in L^1[0, 1]$ and suppose

$$I_k = \int_0^1 \phi(t)^{n(k)} g(t)dt = 0$$

for $k=1, 2,\ldots$. Then the definition of $\phi(t)$ shows that

$$I_k = \frac{1}{3}\int_0^1 t^{n(k)}\{g(t/3)dt - g([2-\phi_1^{-1}(t)]/3)d\phi_1^{-1}(t) + g([2+\phi_2^{-1}(t)]/3)d\phi_2^{-1}(t)\}.$$

Assuming that $\sum_{k=1}^{\infty} n(k)^{-1} = +\infty$, we infer that the measure in the braces vanishes identically on (0, 1]. So, if we can make the three measures dt, $d\phi_1^{-1}(t)$ and $d\phi_2^{-1}(t)$ mutually singular, then the desired conclusion will be drawn as in the proof of Proposition 3. Thus we have only to show the following

LEMMA. *There exists on the interval* [0, 1] *a countable set of probability measures* $d\mu_n$ *which satisfies the conditions*:

(i) *each* $d\mu_n$ *is continuous*;

(ii) *the closed support of* $d\mu_n$ *is the whole interval* [0, 1];

(iii) $d\mu_n$ *are mutually singular and also singular with respect to the Lebesgue measure.*

Although we suspect that the fact may be known to many people, we cannot find out any explicit reference so far. The proof we furnish below is based on the idea appearing in Wiener [5].

PROOF. For the sake of brevity of exposition, we are going to construct two probability measures $d\mu_1$ and $d\mu_2$ on $I=[0, 1]$ such that they are mutually singular and singular with respect to the Lebesgue measure. We define a map π of the square $I\times I$ to the segment I as follows: for any $(x, y)\in I\times I$ we expand x and y into binary fractions as

$$x=0.x_1x_2...x_n... \quad \text{and} \quad y=0.y_1y_2...y_n...$$

with x_i, $y_i=0$, 1 and then set

$$t=0.x_1y_1x_2y_2...x_ny_n....$$

The map $(x, y)\Leftrightarrow t$ is one-to-one unless x or y or t admits a terminating binary representation other than 0 or 1. Let Q be the set of numbers in (0, 1) admitting terminating binary representations. Then Q is a countable subset of I, so that $(Q\times I)\cup(I\times Q)$ has the zero plane (Lebesgue) measure. Let N be the image of $(Q\times I)\cup(I\times Q)$ under the correspondence $(x, y)\rightarrow t$. A generic element in N has one of the following binary representations: (i) $0.*...*0*0*0*...$ and (ii) $0.*...*1*1*1*....$ So N has the zero linear (Lebesgue) measure.

Let π denote the restriction to $(I\backslash Q)\times(I\backslash Q)$ of the map $(x, y)\rightarrow t$. So π maps $(I\backslash Q)\times(I\backslash Q)$ bijectively onto $I\backslash N$. It therefore induces an isomorphism, say π^*, between the space of (Borel) measures on $(I\backslash Q)\times(I\backslash Q)$ onto that on $I\backslash N$. As shown by Wiener in [5], we have $\pi^*(dx\times dy)=dt$. In order to get measures $d\mu_1$, $d\mu_2$ with the required properties, we use the singular function ϕ_0 in the preceding section and set

$$d\nu_1=d\phi_0(x)\times dy \quad \text{and} \quad d\nu_2=dx\times d\phi_0(y)$$

on $I\times I$. Since $d\phi_0$ is a continuous measure and Q is countable, we have $\nu_i((Q\times I)\cup(I\times Q))=0$ for $i=1, 2$, so that both $d\nu_1$ and $d\nu_2$ can be supposed to have supports in $(I\backslash Q)\times(I\backslash Q)$. Thus,

$$d\mu_i=\pi^*(d\nu_i)$$

for $i=1, 2$ are well-defined as measures on I supported by $I\backslash N$. Now it is easy to verify the conditions in the lemma for these measures. Namely, since $d\nu$, $i=1, 2$, are continuous measures and since π is one-to-one, the measures $d\mu_i$ are also continuous. Since $dx\times dy$, $d\nu_1$ and $d\nu_2$ are mutually singular, so are measures dt, $d\mu_1$ and $d\mu_2$. Finally, we have to show that the support of each $d\mu_i$ is dense in I. For this purpose, take any interval J of the form $(j/2^{2k}, (j+1)/2^{2k})$. Then there exists a square S of the form $(l/2^k, (l+1)/2^k)\times(m/2^k, (m+1)/2^k)$ such that $\pi(S\cap[(I\backslash Q)\times(I\backslash Q)])=J\cap(I\backslash N)$. Since ϕ_0 is strictly increasing, it is clear that

$$\nu_i(S\cap[(I\backslash Q)\times(I\backslash Q)])>0$$

for $i=1, 2$. Hence, we have $\mu_i(J)>0$ for $i=1, 2$, which implies the desired result.

The argument for getting more measures is the same, if we use higher dimensional cubes. Q. E. D.

Discussion of similar sort has appeared in papers by Hwang, Pan and Wu [2], Takahasi [3], and Takahasi and Takeuchi [4]. A remaining question is the following: Can one relax the absolute continuity assumption on ϕ in Proposition 1, (c)?

References

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Added in Proof. K. Yabuta kindly pointed out that, in view of recent work of Brown and others, Riesz products can be used to produce an uncountably many collection of measures with the properties mentioned in Lemma.