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Toshio Fukushima
National Astronomical Observatory of Japan

EXPANSIONS OF ELLIPTIC MOTION BASED ON ELLIPTIC FUNCTION THEORY

EUGENE BRUMBERG
National Astronomical Observatory, Mitaka, Tokyo 181, Japan
(on leave from Institute of Applied Astronomy, St.–Petersburg 197042, Russia)
iparan@sovamsu.sovusa.com

and

TOSHIO FUKUSHIMA
National Astronomical Observatory, Mitaka, Tokyo 181, Japan
toshio@spacetime.mtk.nao.ac.jp

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Abstract. New expansions of elliptic motion based on considering the eccentricity e as the modulus k of elliptic functions and introducing the new anomaly w (a sort of elliptic anomaly) defined by $w = \frac{\pi u}{2K} - \frac{\pi}{2}$, $g = \operatorname{am} u - \frac{\pi}{2}$ (g being the eccentric anomaly) are compared with the classic (e, M), (e, v) and (e, g) expansions in multiples of mean, true and eccentric anomalies, respectively. These (q, w) expansions turn out to be in general more compact than the classical ones. The coefficients of the (e, v) and (e, g) expansions are expressed as the hypergeometric series, which may be reduced to the hypergeometric polynomials. The coefficients of the (q, w) expansions may be presented in closed (rational function) form with respect to q, k, $k' = (1 - k^2)^{1/2}$, K and E, q being the Jacobi nome related k while K and E are the complete elliptic integrals of the first and second kind respectively. Recurrence relations to compute these coefficients have been derived.

Key words: Elliptic two–body problem, elliptic anomaly, hypergeometric polynomials, elliptic functions

1. Introduction

In Brumberg (E.V., 1993) it is shown that taking the eccentricity e as the modulus k of elliptic functions and using well–known trigonometric expansions of elliptic functions with rational coefficients with respect to the Jacobi nome $q = q(k)$ one may replace the classic (e, M), (e, v) and (e, g) trigonometric expansions in multiples of the mean anomaly M, true anomaly v and eccentric anomaly g, respectively, by the (q, w) trigonometric expansions; w being the new anomaly related with the argument of ellipitic functions. Even for large eccentricities q remains rather small. The aim of the present paper is to make a general comparison of the (q, w) expansions with the traditional ones for the typical elliptic motion functions $(r/a)^n \exp imv$ and to give an efficient recurrence technique to compute the coefficients of (q, w) expansions, which will be called below the elliptic Hansen–like coefficients. These recurrence relations are only a little more complicated as compared with the recurrences for the classic Hansen coefficients (Vinh, 1969; Giacaglia, 1976; Hughes, 1981). The derived relations may be used to compute the elliptic Hansen–like coefficients either analytically using some computer algebra system (*Mathematica, Maple,* etc.) or numerically. So far, we have applied these

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recurrences only for numerical calculations using the software package elaborated earlier (Fukushima, 1991).

In contrast to the coefficients of the (e, M) expansions, the elliptic Hansen–like coefficients always have a closed form, being rational functions of q, k, $k' = (1-k^2)^{1/2}$, K and E (K and E are the complete elliptic integrals of the first and second kind respectively). This represents one more advantage of (q, w) expansions to stimulate their application in constructing a perturbation theory for highly–eccentric orbits.

In Section 2 we give for the sake of convenience the basic relations of presenting the solution of the two–body problem in terms of q and w. Section 3 is devoted to numerical comparison of (q, w) expansions with the traditional ones and to the derivation of the analytical expressions for the coefficients of the (e, v) and (e, g) expansions. Sections 4 and 5 deal with the recurrence relations for the elliptic Hansen–like coefficients and with the initial values for them. In Section 6 we briefly describe the results of computation by means of these recurrences.

2. Basic Relations

Defining the modulus k and the amplitude φ of elliptic functions by

$$k = e, \quad \varphi = g + \frac{\pi}{2} \tag{2.1}$$

and introducing the new anomaly u as

$$\varphi = \mathrm{am}\, u\,, \tag{2.2}$$

or

$$u = F(\varphi, k)\,, \tag{2.3}$$

$F(\varphi, k)$ being the elliptic integral of the first kind,

$$F(\varphi, k) = \int_0^{\varphi} (1 - k^2 \sin^2\varphi)^{-1/2} d\varphi\,,$$

one obtains the closed form solution for the orbital coordinates X, Y and the radius vector r in the two–body problem in terms of elliptic functions:

$$X = r\cos v = a\,(\mathrm{sn}\, u - k)\,, \tag{2.4}$$

$$Y = r\sin v = -a\, k'\, \mathrm{cn}\, u \tag{2.5}$$

and

$$r = a\,(1 - k\,\mathrm{sn}\, u)\,, \tag{2.6}$$

where a is a semi–major axis.

The relationship of u with time t is given by the Kepler equation

$$\operatorname{am} u + k \operatorname{cn} u = M + \frac{\pi}{2}\,. \tag{2.7}$$

The transformation given by (2.1) and (2.2) may be regarded as another attempt to realize Gyldén's idea of considering the classic (mean, true or eccentric) anomaly as the elliptic amplitude of a new independent variable (Nacozy, 1977).

The solution (2.4)–(2.7) is not of much interest by itself but it enables one to use the well–known trigonometric expansions of the Jacobi elliptic functions to be applied in the perturbed two–body problem. Changing u by

$$w = \frac{\pi u}{2K} - \frac{\pi}{2}\,, \tag{2.8}$$

it turns out rather easy to construct the trigonometric expansions in multiples of w for the typical functions of elliptic motion such as $\sin g$, $\cos g$, $\sin v$, $\cos v$, r/a and a/r. The Hansen expansion

$$\left(\frac{r}{a}\right)^n \exp imv = \sum_{s=-\infty}^{\infty} X_s^{n,m}(e) \exp isM\,, \quad i = \sqrt{-1} \tag{2.9}$$

is replaced by

$$\left(\frac{r}{a}\right)^n \exp imv = \sum_{s=-\infty}^{\infty} B_s^{n,m}(q) \exp isw\,, \tag{2.10}$$

$q = q(k)$ being the Jacobi nome which serves as a small parameter in the trigonometric expansions of elliptic functions.

All these formulae are given here for convenience so they may be used later. The detailed derivation of the elliptic motion expansions as well as the solution technique of the Kepler equation (2.7) and the elementary perturbation techniques based on (2.1) may be found in Brumberg (E.V., 1993).

3. Comparison of (e, v), (e, g), (e, M) and (q, w) Expansions

Along with expansions (2.9) and (2.10) in multiples of the mean anomaly M and the anomaly w, respectively, we shall consider the expansions in multiples of the true anomaly v and the eccentric anomaly g. These expansions, which frequently occur in celestial mechanics perturbation theory, have the form

$$\left(\frac{r}{a}\right)^n \exp imv = \sum_{s=-\infty}^{\infty} Y_s^{n,m}(e) \exp isv \tag{3.1}$$

and

$$\left(\frac{r}{a}\right)^n \exp imv = \sum_{s=-\infty}^{\infty} Z_s^{n,m}(e) \exp isg\,. \tag{3.2}$$

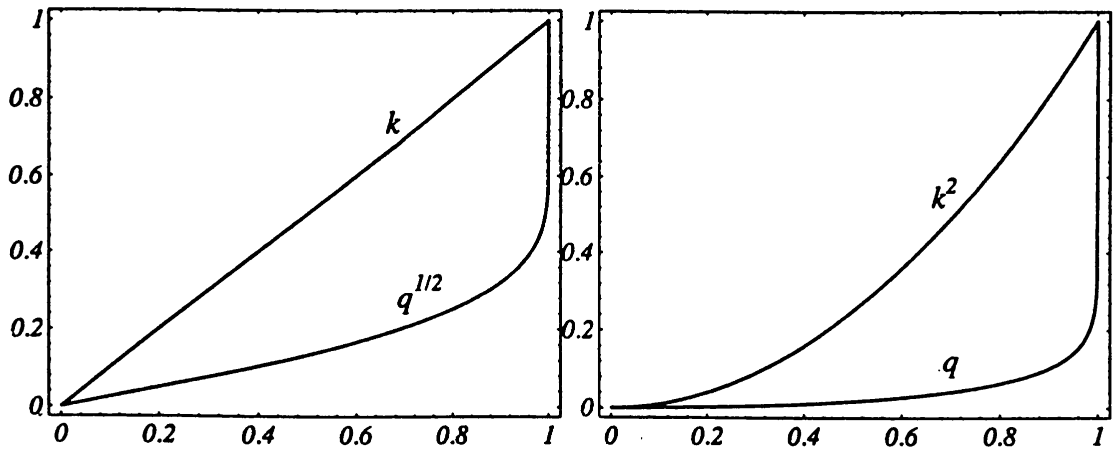

Fig. 1. Graphs of k, $\sqrt{q}$, k^2 and q

We compare the (q, w) expansion (2.10) with others in three respects:

– applicability for large eccentricities;
– compactness of the corresponding coefficients;
– compactness of the corresponding trigonometric series themselves.

First of all, let us note that the order of the coefficients $X_s^{n,m}(e)$, $Y_s^{n,m}(e)$ and $Z_s^{n,m}(e)$ is $e^{|m-s|}$ and these coefficients may be represented by series in powers of e^2. The coefficients $B_s^{n,m}(q)$ are of the order $(\sqrt{q})^{|m-s|}$ (which is the same as $e^{|m-s|}$ only for small eccentricities) and may be represented in the series in powers of q. Even for large eccentricities the Jacobi nome q remains rather small (for example, $q = 0.2622\ldots$ corresponds to $k^2 = 0.99$). Graphs of k, q, k^2 and $\sqrt{q}$ are shown on Fig. 1. This makes it evident that (q, w) series involving q as a small parameter are valid even for large values of eccentricities when all other series under consideration using e as a small parameter become inefficient. Moreover, when dealing with elliptic function expansions it is always possible to make a Landen transformation to replace q by $q_1 = q^2$ as is done in general planetary theory (Brumberg V.A., 1993). In principle, this permits one to reduce the eccentricity parameter to any arbitrary small value desired.

Consider now the coefficients $X_s^{n,m}(e)$, $Y_s^{n,m}(e)$, $Z_s^{n,m}(e)$ and $B_s^{n,m}(q)$ with respect to presenting them in compact form. As is well known, the Hansen coefficients are represented by infinite series in powers of e^2 or β^2 with

$$\beta = \frac{k}{1+k'}, \quad k' = \left(1-k^2\right)^{\frac{1}{2}}, \quad k = e, \tag{3.3}$$

and only for $s = 0$ are these series reduced to finite hypergeometric polynomials in e^2 or β^2. The elliptic Hansen–like coefficients $B_s^{n,m}(q)$ may be represented by infinite series in powers of q (Brumberg E.V., 1993). Therefore, since $q \ll k^2$, in actual analytical calculations with some definite accuracy it is possible to retain here fewer terms as compared with the Hansen coefficient expansions. Moreover, as will be shown in Sections 4 and 5, the coefficients $B_s^{n,m}(q)$ may be represented for all values of n, m and s in closed form as rational functions of q, k, k', K and E.

As for the coefficients $Y_s^{n,m}(e)$ and $Z_s^{n,m}(e)$, they may be expressed as hypergeometric series with respect to β^2. Indeed, introducing

$$\sigma = \exp iv\,, \tag{3.4}$$

we may present r/a as follows:

$$\begin{aligned}\frac{r}{a} &= \frac{1-e^2}{1+e\cos v} = \frac{2k'^2}{k}\,\sigma\left(\sigma^2+\frac{2}{k}\sigma+1\right)^{-1}\\ &= \frac{\left(1-\beta^2\right)^2}{1+\beta^2}\left(1+\beta\sigma^{-1}\right)^{-1}(1+\beta\sigma)^{-1}\,.\end{aligned} \tag{3.5}$$

Hence, we conclude that

$$\left(\frac{r}{a}\right)^n \exp imv = \frac{\left(1-\beta^2\right)^{2n}}{\left(1+\beta^2\right)^n}\left(1+\beta\sigma^{-1}\right)^{-n}(1+\beta\sigma)^{-n}\,\sigma^m. \tag{3.6}$$

The expression on the right–hand side of (3.6) differs only by a factor from the function $\gamma\,(n,x,y,\nu,\alpha,z)$ introduced in Brumberg (1980). Using the expansion of this function in the exponential series we obtain directly

$$\begin{aligned}Y_s^{n,m}(e) &= (-1)^{|m-s|}\frac{(n)_{|m-s|}}{(1)_{|m-s|}}\beta^{|m-s|}\frac{\left(1-\beta^2\right)^{2n}}{\left(1+\beta^2\right)^n}\,\times\\ &\quad\times F\left(n,n+|m-s|,1+|m-s|,\beta^2\right)\,,\end{aligned} \tag{3.7}$$

where

$$F\,(a,b,c,z) = \sum_{s=0}^{\infty}\frac{(a)_s(b)_s}{(c)_s(1)_s}\,z^s \tag{3.8}$$

is the hypergeometric function and $(a)_s$ is the Pochhammer symbol

$$(a)_0 = 1\,,\quad (a)_s = a\,(a+1)\ldots(a+s-1)\,.$$

In a similar way, introducing

$$\zeta = \exp ig\,, \tag{3.9}$$

we derive

$$\frac{r}{a} = \left(1+\beta^2\right)^{-1}\left(1-\beta\zeta^{-1}\right)(1-\beta\zeta)\,, \tag{3.10}$$

$$\sigma = \zeta\left(1-\beta\zeta^{-1}\right)(1-\beta\zeta)^{-1} \tag{3.11}$$

and, finally,

$$Z_s^{n,m}(e) = \frac{(-n-m)_{(M+)}\ (-n+m)_{(M-)}}{(1)_{|m-s|}}\beta^{|m-s|}\left(1+\beta^2\right)^{-n} \times$$
$$\times F(-n-m+(M+)\,,\ -n+m+(M-)\,,\ 1+|m-s|,\beta^2)\,. \tag{3.12}$$

where $M+ = \max(0, m-s)$ and $M- = \max(0, s-m)$ From the definition (3.8) it is evident that the hypergeometric series $F(a,b,c,z)$ reduces to a polynomial if a or b is a negative integer. It is known (Erdélyi, 1953) that if k, j, l are integers, then $F(a+k, b+j, c+l, z)$ can be reduced, by repeated applications of the Gauss relations between hypergeometric contiguous functions, to $F(a,b,c,z)$ and one of its contiguous functions. It follows from this that the coefficients $Y_s^{n,m}(e)$ and $Z_s^{n,m}(e)$ determined by (3.7) and (3.12) may be eventually expressed in closed form as hypergeometric polynomials. Expressions analogous to (3.7) and (3.12) were obtained by Brown and Shook (1933) although they did not mention the possibility of presenting them in closed form.

So far we have discussed the comparative characteristics of the coefficients $X_s^{n,m}(e)$, $Y_s^{n,m}(e)$, $Z_s^{n,m}(e)$ and $B_s^{n,m}(q)$. Now we want to compare the four representations (2.9), (3.1), (3.2) and (2.10) of the same functions $(r/a)^n \exp imv$. We restrict ourselves to the consideration of the four sets of values $(n,m) = (2,0)$, $(2,2)$, $(-3,0)$ and $(-3,2)$ because these correspond to the initial terms in the expansion of the Earth satellite perturbation function caused by the luni–solar attraction and the non–sphericity influence. We evaluate the practical convergence of the expansions under consideration by computing their coefficients for the three values of the eccentricity $e = 0.1$, $e = 0.5$ and $e = 0.9$ by taking the Fourier quadratures; thus

$$X_s^{n,m}(e) = \frac{1}{2\pi}\int_0^{2\pi}(1-e\cos g)^n \cos(mv - sM)\,dM\,. \tag{3.13}$$

Here $g = g(M)$ is determined by the Kepler equation and $v = v(g)$ is given by Broucke and Cefola (1973):

$$v = g + 2\arctan\left(\frac{e\sin g}{1+k'-e\cos g}\right). \tag{3.14}$$

Similarly

$$Y_s^{n,m}(e) = \frac{k'^{2n}}{2\pi}\int_0^{2\pi}\frac{1}{(1+e\cos v)^n}\cos(mv - sv)\,dv\,, \tag{3.15}$$

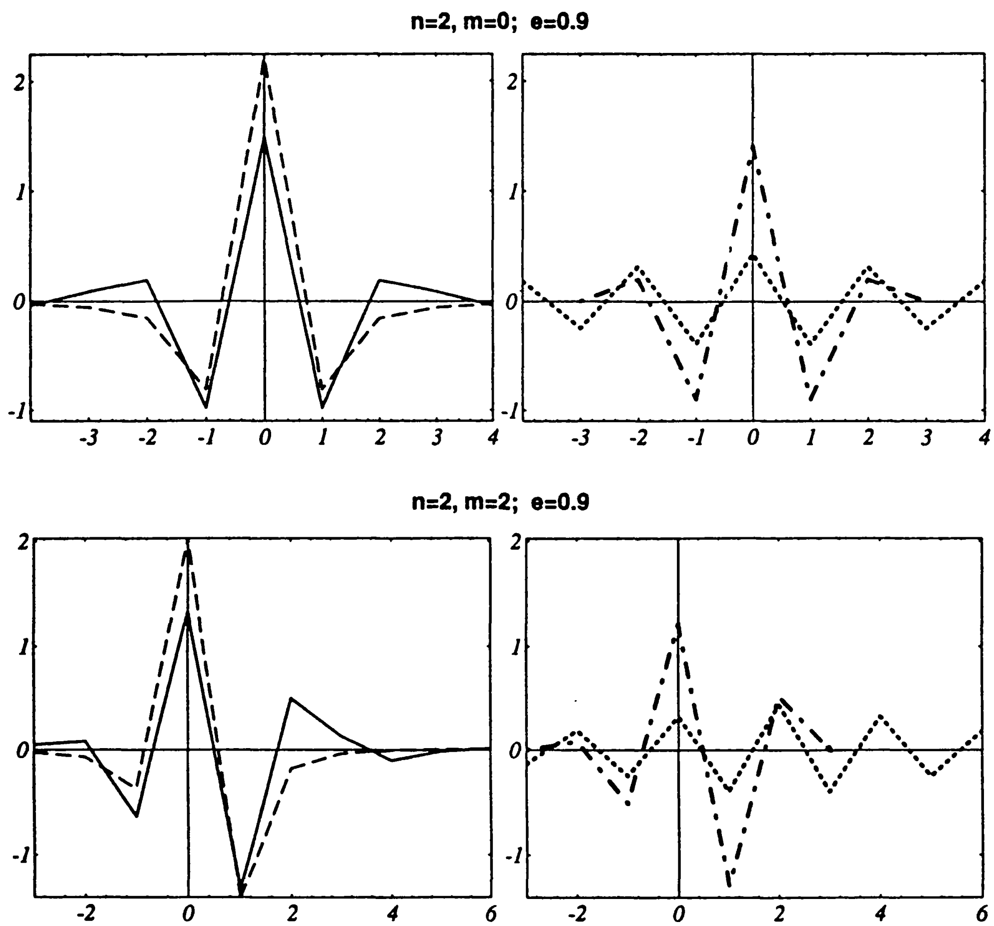

Fig. 2. Graphs of $X_s^{n,m}$ [— —], $Y_s^{n,m}$ [· · · · ·], $Z_s^{n,m}$ [— · —] and $B_s^{n,m}$ [——] as functions of s (abscissa axis) for $(n, m) = (2, 0), (2, 2)$ and $e = 0.9$

and

$$Z_s^{n,m}(e) = \frac{1}{2\pi} \int_0^{2\pi} (1 - e\cos g)^n \cos(mv - sg)\, dg\,, \tag{3.16}$$

with $v = v(g)$ determined by (3.14). Finally

$$B_s^{n,m}(q) = \frac{1}{2\pi} \int_0^{2\pi} (1 - k\,\mathrm{sn}\, u)^n \cos(mv - sw)\, dw\,, \tag{3.17}$$

where the relationship between v and u is determined by (2.4), (2.5) and (2.6) and $u = u(w)$ is given by (2.8).

The calculations have been performed using *Mathematica 2.1* (Wolfram, 1991) on a Macintosh Quadra 700. Some results of our calculations are presented in Figs. 2–4. We do not reproduce the results for $e = 0.1$ because they are practically

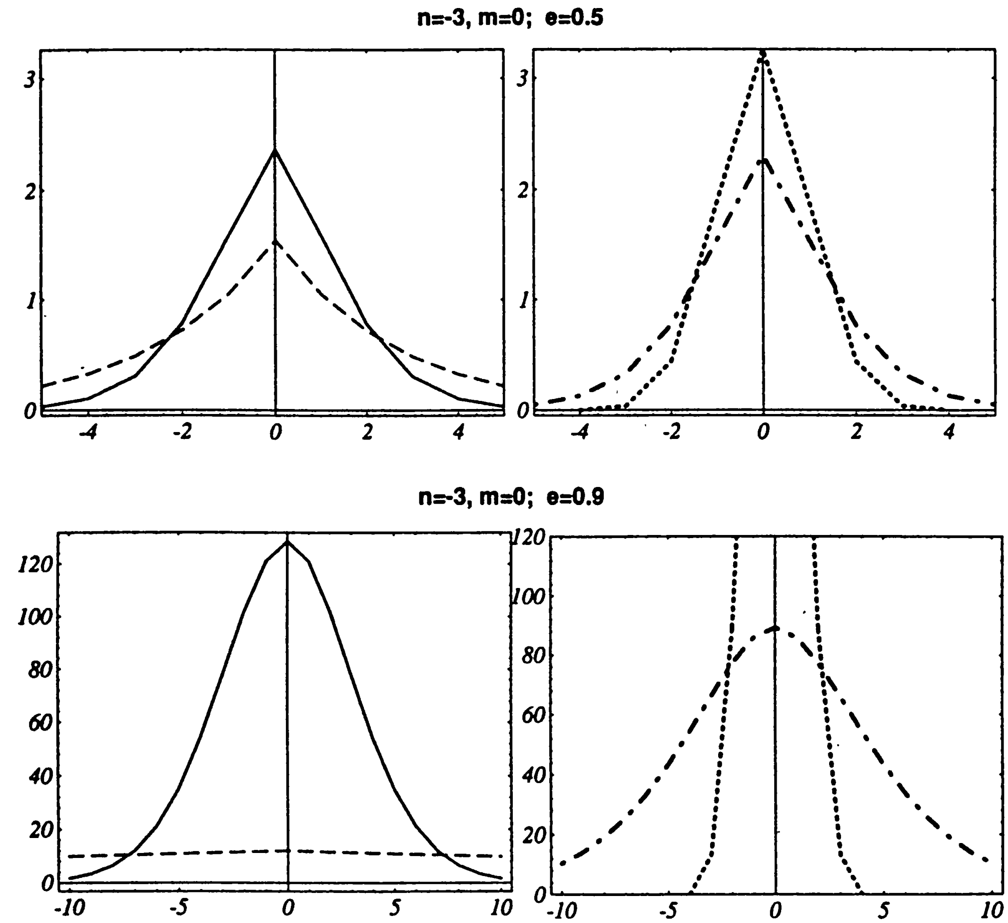

Fig. 3. Graphs of $X_s^{n,m}$ [– –], $Y_s^{n,m}$ [· · · · ·], $Z_s^{n,m}$ [– · –] and $B_s^{n,m}$ [——] as functions of s (abscissa axis) for $(n,m) = (-3,0)$, $e = 0.5$ and $(n,m) = (-3,0)$, $e = 0.9$; $Y_0^{-3,0}(0.9) = 322.9$

the same for all four types of series. In all cases the coefficients of the (q, w) expansions decrease faster than the coefficients of the (e, M) expansions. Moreover, for $e = 0.9$ the (e, M) expansions become practically invalid. The (e, v) expansions behave as good as the (q, w) ones for $n = -3$ but are not so good for $n = 2$. On the contrary, the (e, g) expansions are quite concurrent with the (q, w) series for $n = 2$ but are worse for $n = -3$. We may conclude that in all cases the (q, w) expansions contain the merit of the (e, v) expansion for the case $n < 0$ and that of the (e, g) expansions for the case $n > 0$, and have evident advantages as compared with the (e, M) expansions.

4. Recurrence Relations for the Elliptic Hansen Coefficients

In this section we derive recurrence relations which permit one to compute the table of elliptic Hansen-like coefficients. The most simple recurrence relation is

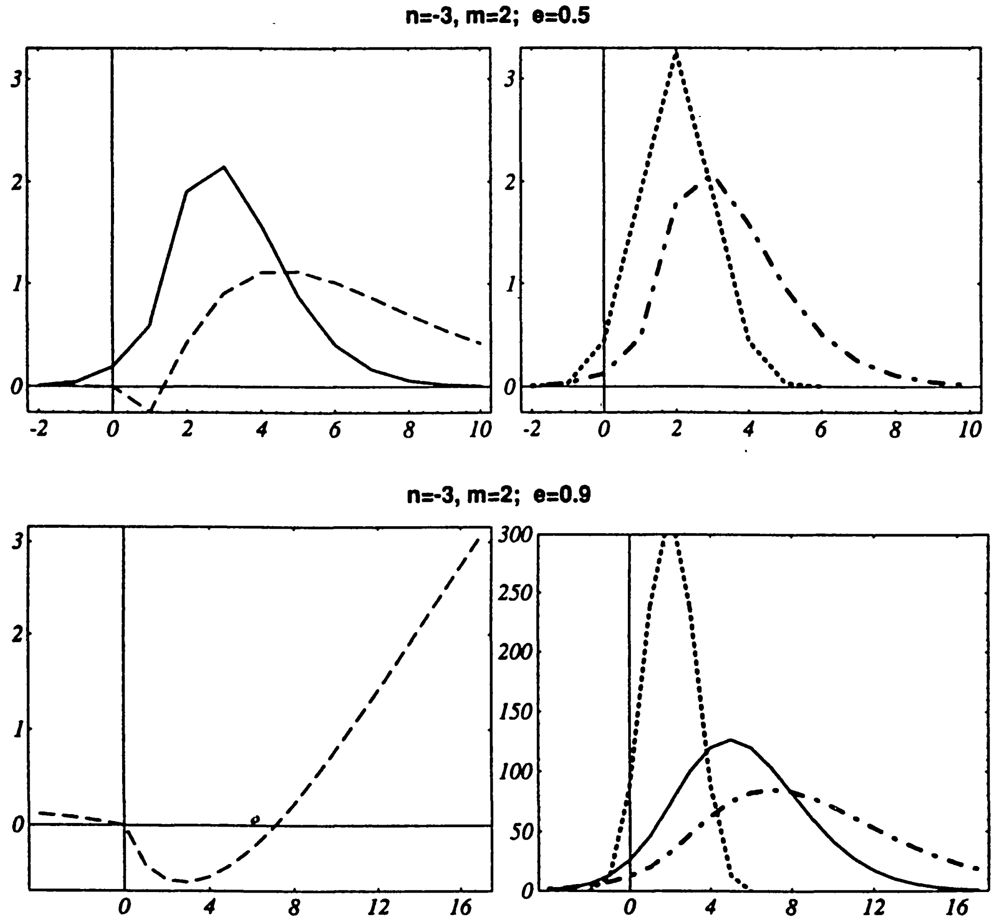

Fig. 4. Graphs of $X_s^{n,m}$ [– –], $Y_s^{n,m}$ [$\cdots\cdot$], $Z_s^{n,m}$ [– · –] and $B_s^{n,m}$ [——] as functions of s (abscissa axis) for $(n, m) = (-3, 2)$, $e = 0.5$ and $(n, m) = (-3, 2)$, $e = 0.9$; $Y_2^{-3,2}(0.9) = 322.9$

the same as for the classic Hansen coefficients. Indeed, writing the equation for the radius vector in the form

$$\left(1 - e^2\right) = \frac{r}{a}\left[1 + \frac{1}{2}\,e\left(\exp iv + \exp(-iv)\right)\right], \tag{4.1}$$

multiplying the both sides by $(r/a)^n \exp imv$ and applying the defining relation for the elliptic Hansen–like coefficients (2.10) one has

$$k'^2 B_s^{n,m} = B_s^{n+1,m} + \frac{k}{2}\left(B_s^{n+1,m+1} + B_s^{n+1,m-1}\right). \tag{4.2}$$

Now let $\mathbf{B} = (B_{nm})$ be a $(2N+1)\times(2M+1)$ matrix $(n = -N, -N+1, \ldots, N-1, N;$

$m = -M, -M+1, \ldots, M-1, M$), each element B_{nm} representing a set of $B_s^{n,m}(q)$ with fixed n and m while s runs over all necessary values:

$$\mathbf{B} = \begin{pmatrix} B_s^{-N,-M} & \ldots & B_s^{-N,0} & B_s^{-N,1} & \ldots & B_s^{-N,M} \\ \vdots & & \vdots & \vdots & & \vdots \\ B_s^{\;0,-M} & \ldots & B_s^{\;0,0} & B_s^{\;0,1} & \ldots & B_s^{\;0,M} \\ \vdots & & \vdots & \vdots & & \vdots \\ B_s^{\;N,-M} & \ldots & B_s^{\;N,0} & B_s^{\;N,1} & \ldots & B_s^{\;N,M} \end{pmatrix}, \quad s = 0, \pm 1, \pm 2, \ldots$$

If all elements of two neighbouring columns are known (for $m = 0$ and $m = 1$, for instance), then all other elements may be found from (4.2). Replacing in (4.2) n by $n-1$ and m by $m \pm 1$ we obtain two formulae equivalent to (4.2) but containing in the left–hand side the quantities to be determined by recurrence:

$$B_s^{n,m} = -B_s^{n,m+2} + \frac{2}{k}\left(k'^2 B_s^{n-1,m+1} - B_s^{n,m+1}\right), \quad m = -1, -2, \ldots \tag{4.3}$$

and

$$B_s^{n,m} = -B_s^{n,m-2} + \frac{2}{k}\left(k'^2 B_s^{n-1,m-1} - B_s^{n,m-1}\right), \quad m = 2, 3, \ldots . \tag{4.4}$$

The recurrence (4.3) enables one to move "to the left" along the columns of (B_{nm}) and the relation (4.4) provides moving "to the right".

The problem is to find all elements with $m = 0$ and $m = 1$. We have chosen the two basic columns with these values of m because it is rather easy to determine initial values for the corresponding recurrence relations. The relevant recurrence formulae are based on the differential relations of the second order. We shall see below that in contrast to the theory of the classic Hansen coefficients the first–order relations cannot be expressed in rational form with respect to the functions of r and v.

To derive recurrences based on differential relations we need first of all the well–known expressions (Gradshteyn and Ryzhik, 1965) for the derivatives of the Jacobi elliptic functions:

$$\begin{aligned} \frac{d}{du}\,\mathrm{sn}\,u &= \mathrm{cn}\,u\,\mathrm{dn}\,u\,, \\ \frac{d}{du}\,\mathrm{cn}\,u &= -\,\mathrm{sn}\,u\,\mathrm{dn}\,u \end{aligned}$$

and

$$\frac{d}{du}\,\mathrm{dn}\,u = -k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,.$$

From the basic relations of the Section 2 one may express the Jacobi elliptic functions in terms of r and trigonometric functions of v as follows:

$$k\,\mathrm{sn}\,u = 1 - \frac{r}{a}, \tag{4.5}$$

$$\mathrm{dn}^2 u = 2\frac{r}{a} - \left(\frac{r}{a}\right)^2 \tag{4.6}$$

and

$$ikk'\mathrm{cn}\,u = k'^2 - \frac{r}{a} - k\frac{r}{a}\exp iv\,. \tag{4.7}$$

From (4.6) it is seen that the first–order derivatives of the Jacobi elliptic functions cannot be expressed in rational form with respect to r. In view of (2.8), differentiating the function (r/a) with respect to w, we have

$$\frac{d}{dw}\left(\frac{r}{a}\right) = -\frac{2K}{\pi}\,k\,\mathrm{cn}\,u\,\mathrm{dn}\,u \tag{4.8}$$

and then

$$\frac{d^2}{dw^2}\left(\frac{r}{a}\right) = \frac{4K^2}{\pi^2}\,k\,\mathrm{sn}\,u\left(1 + k^2 - 2k^2\,\mathrm{sn}^2 u\right). \tag{4.9}$$

With the aid of (4.5) we put the last relation into the form

$$\frac{d^2}{dw^2}\left(\frac{r}{a}\right) = \frac{4K^2}{\pi^2}\left[-k'^2 + \left(5 - k^2\right)\frac{r}{a} - 6\left(\frac{r}{a}\right)^2 + 2\left(\frac{r}{a}\right)^3\right]. \tag{4.10}$$

Taking the square of (4.8), and expressing $\mathrm{cn}\,u$ and $\mathrm{dn}\,u$ in terms of $\mathrm{sn}\,u$, we have

$$\left[\frac{d}{dw}\left(\frac{r}{a}\right)\right]^2 = \frac{4K^2}{\pi^2}\left(1 - k^2\,\mathrm{sn}^2 u\right)\left(k^2 - k^2\,\mathrm{sn}^2 u\right), \tag{4.11}$$

or, using again (4.5),

$$\left[\frac{d}{dw}\left(\frac{r}{a}\right)\right]^2 = \frac{4K^2}{\pi^2}\left[-2k'^2\frac{r}{a} + \left(5 - k^2\right)\left(\frac{r}{a}\right)^2 - 4\left(\frac{r}{a}\right)^3 + \left(\frac{r}{a}\right)^4\right]. \tag{4.12}$$

Now we are able to find the second derivatives of $(r/a)^n$: thus

$$\frac{d^2}{dw^2}\left(\frac{r}{a}\right)^n = n\left(\frac{r}{a}\right)^{n-1}\frac{d^2}{dw^2}\left(\frac{r}{a}\right) + n\,(n-1)\left(\frac{r}{a}\right)^{n-2}\left[\frac{d}{dw}\left(\frac{r}{a}\right)\right]^2.$$

Using here (4.10) and (4.12) one has finally:

$$\begin{aligned}\frac{d^2}{dw^2}\left(\frac{r}{a}\right)^n = \frac{4K^2}{\pi^2}n\Bigg[&(1-2n)k'^2\left(\frac{r}{a}\right)^{n-1} + n\left(5 - k^2\right)\left(\frac{r}{a}\right)^n - \\ &- 2\,(1+2n)\left(\frac{r}{a}\right)^{n+1} + (n+1)\left(\frac{r}{a}\right)^{n+2}\Bigg].\end{aligned} \tag{4.13}$$

This relation represents one of our two basic differential relations of the second order needed to derive the recurrence formulae. Substituting into both sides of (4.13) the defining relation (2.10) for the elliptic Hansen–like coefficients with $m = 0$, we get

$$(1-2n)\,k'^2 B_s^{n-1,0} + \left[n\left(5-k^2\right) + \frac{s^2}{n}\frac{\pi^2}{4K^2}\right] B_s^{n,0} - \\ -2\,(1+2n)B_s^{n+1,0} + (n+1)B_s^{n+2,0} = 0\,. \tag{4.14}$$

This formula gives the recurrence relation for the column $m = 0$. Indeed, replacing n by $-n+1$ we obtain the formula to move "up" along the column (B_{n0}):

$$(2n-1)\,k'^2 B_s^{-n,0} = \left[(n-1)\left(5-k^2\right) + \frac{s^2}{n-1}\frac{\pi^2}{4K^2}\right] B_s^{-n+1,0} - \\ -2\,(2n-3)B_s^{-n+2,0} + (n-2)B_s^{-n+3,0}, \quad n = 2, 3, \ldots . \tag{4.15}$$

Replacing now in (4.14) n by $n-2$ we get the formula to move "down" along the column (B_{n0}):

$$(n-1)B_s^{n,0} = 2\,(2n-3)B_s^{n-1,0} + \left[(n-2)\left(k^2-5\right) - \frac{s^2}{n-2}\frac{\pi^2}{4K^2}\right] B_s^{n-2,0} \\ + (2n-5)\,k'^2 B_s^{n-3,0}, \quad n = 3, 4, \ldots . \tag{4.16}$$

Hence, all the elements of the column $m = 0$ may be determined from (4.15) and (4.16) provided that the initial values $B_s^{-1,0}$, $B_s^{0,0}$, $B_s^{1,0}$ and $B_s^{2,0}$ are known. These values will be given in the next Section.

We shall now deal with the general case $m \neq 0$. Differentiating v with respect to w we have, from (2.4)-(2.6), (2.8), or alternatives,

$$\frac{dv}{dw} = \frac{2K}{\pi}k'\frac{1+k\,\mathrm{sn}\,u}{\mathrm{dn}\,u} = \frac{2K}{\pi}\,k'\,\frac{a}{r}\,\mathrm{dn}\,u\,. \tag{4.17}$$

Taking the square of this expression and using (4.6) there results

$$\left(\frac{dv}{dw}\right)^2 = \frac{4K^2}{\pi^2}k'^2\left(2\,\frac{a}{r} - 1\right). \tag{4.18}$$

Multiplying (4.8) and (4.17) we get

$$\frac{dv}{dw}\frac{d}{dw}\left(\frac{r}{a}\right) = -\frac{4K^2}{\pi^2}\,\frac{a}{r}\,kk'\;\mathrm{cn}\,u\,\mathrm{dn}^2 u\,,$$

or, after substituting (4.6) and (4.7)

$$\frac{dv}{dw}\frac{d}{dw}\left(\frac{r}{a}\right) = i\,\frac{4K^2}{\pi^2}\left[\,2k'^2 + \left(k^2-3\right)\frac{r}{a} - 2k\,\frac{r}{a}\exp iv + \right. \\ \left. + \left(\frac{r}{a}\right)^2 + k\left(\frac{r}{a}\right)^2\exp iv\right]. \tag{4.19}$$

Differentiating (4.17) one obtains

$$\frac{d^2v}{dw^2} = \frac{4K^2}{\pi^2}\,kk'\,\mathrm{cn}\,u\,\frac{a}{r}\left(\frac{a}{r}\,\mathrm{dn}^2u - k\,\mathrm{sn}\,u\right)$$

and then, by virtue of (4.5), (4.6) and (4.7),

$$\frac{d^2v}{dw^2} = i\frac{4K^2}{\pi^2}\left(1 + k\exp iv - k'^2\,\frac{a}{r}\right). \tag{4.20}$$

Now we may take the second derivative of our basic function $(r/a)^n\exp imv$:

$$\frac{d^2}{dw^2}\left[\left(\frac{r}{a}\right)^n\exp imv\right] = \left\{\frac{d^2}{dw^2}\left(\frac{r}{a}\right)^n + 2inm\left(\frac{r}{a}\right)^{n-1}\frac{dv}{dw}\frac{d}{dw}\left(\frac{r}{a}\right) + \right. \\ \left. + \left[-m^2\left(\frac{r}{a}\right)^n\left(\frac{dv}{dw}\right)^2 + im\left(\frac{r}{a}\right)^n\frac{d^2v}{dw^2}\right]\right\}\exp imv$$

and after using (4.13), (4.18), (4.19) and (4.20) we have finally the most general differential relation of the second order valid for any n and m:

$$\frac{d^2}{dw^2}\left[\left(\frac{r}{a}\right)^n\exp imv\right] = \frac{4K^2}{\pi^2}\left\{(n+m)(1-2n-2m)\,k'^2\left(\frac{r}{a}\right)^{n-1}\exp imv\right. \\ + \left[5n^2 + 6nm + m^2 - m - k^2\,(n+m)^2\right]\left(\frac{r}{a}\right)^n\exp imv \\ + km\,(4n-1)\left(\frac{r}{a}\right)^n\exp i(m+1)v - 2n(1+m+2n)\left(\frac{r}{a}\right)^{n+1}\exp imv \\ \left. -2knm\left(\frac{r}{a}\right)^{n+1}\exp i(m+1)v + n(n+1)\left(\frac{r}{a}\right)^{n+2}\exp imv\right\}. \tag{4.21}$$

As to be expected, putting $m = 0$ in (4.21) yields (4.13). Again, substituting into (4.21) the basic relation (2.10) we obtain the recurrence formula as follows:

$$(n+m)\,(1-2n-2m)\,k'^2B_s^{n-1,m} + \\ + \left[5n^2 + 6nm + m^2 - m - k^2(n+m)^2 + \frac{\pi^2}{4K^2}s^2\right]B_s^{n,m} + \\ + km(4n-1)B_s^{n,m+1} - 2n(1+m+2n)B_s^{n+1,m} - \\ - 2knmB_s^{n+1,m+1} + n(n+1)B_s^{n+2,m} = 0\,. \tag{4.22}$$

This formula may be used to determine either $B_s^{n+2,m}$, $n = 1, 2, \ldots$ or $B_s^{n-1,m}$, $n = 0, -1, -2, \ldots$ provided that we know all the values for the column $m + 1$ and the initial values $B_s^{0,m}$, $B_s^{1,m}$ and $B_s^{2,m}$. Changing the signs of m and s we may deduce from (4.22) the conjugate recurrence formula

$$
\begin{aligned}
&(n-m)(1-2n+2m)\,k'^2 B_s^{n-1,m} + \\
&\qquad + \left[5n^2 - 6nm + m^2 + m - k^2(n-m)^2 + \frac{\pi^2}{4K^2}s^2\right] B_s^{n,m} - \\
&\qquad - km(4n-1)B_s^{n,m-1} - 2n(1-m+2n)B_s^{n+1,m} + \\
&\qquad + 2knmB_s^{n+1,m-1} + n(n+1)B_s^{n+2,m} = 0\,.
\end{aligned}
\tag{4.23}
$$

This formula should be used to determine either $B_s^{n+2,m}$, $n = 1, 2, \ldots$ or $B_s^{n-1,m}$, $n = 0, -1, -2, \ldots$ provided that we know the same initial values and all the values for the column $m - 1$. In particular, the formula (4.23) is used for the column $m = 1$. Initial values needed to apply (4.22) and (4.23) are given in the next Section.

5. Initial Values for the Recurrences

In Brumberg (E.V., 1993) the (q, w) expansions for a/r, r/a and $\exp iv$ have been obtained. In exponential form these expansions are as follows:

$$
\frac{a}{r} = \frac{E}{k'^2 K} + \frac{\pi^2}{2k'^2 K^2} \sum_{m=-\infty}^{\infty} |m| \frac{q^{\frac{|m|}{2}}}{1 - q^{|m|}} \exp imw\,,
\tag{5.1}
$$

$$
\frac{r}{a} = 1 + \frac{\pi}{K} \sum_{m=-\infty}^{\infty} (-1)^{\frac{|2m-1|+1}{2}} \frac{q^{\frac{|2m-1|}{2}}}{1 - q^{|2m-1|}} \exp i(2m-1)w
\tag{5.2}
$$

and

$$
\exp iv = \frac{E - K}{kK} + \frac{\pi^2}{kK^2} \sum_{m=-\infty}^{\infty} |m| \frac{q^{(1-\frac{1}{2}\,\mathrm{sgn}\,m)|m|}}{1 - q^{2|m|}} \exp imw\,.
\tag{5.3}
$$

For simplicity we adopted an apparently complex form in the above expressions although the imaginary parts actually cancel out each other. These expansions correspond to the values $(n, m) = (-1, 0)$, $(1, 0)$ and $(0, 1)$, respectively. For $(n, m) = (0, 0)$ we have an evident expansion $1 = 1$. Besides this, we need the expansions for $(r/a)^2$, $(r/a)\exp iv$ and $(r/a)^2 \exp iv$ corresponding to the values $(n, m) = (2, 0)$, $(1, 1)$ and $(2, 1)$, respectively.

Considering that

$$
\left(\frac{r}{a}\right)^2 = 1 - 2\,k\,\mathrm{sn}\,u + k^2 \mathrm{sn}^2 u\,,
$$

it is necessary to have an expansion for $\mathrm{sn}^2 u$. Using the standard trigonometric expansion for the Jacobi zeta–function $Z(u)$, we have

$$\begin{aligned}\mathrm{sn}^2 u &= \frac{1}{k^2}\left(1 - \frac{E}{K} - \frac{dZ}{du}\right)\\ &= \frac{K-E}{k^2 K} - \frac{2\pi^2}{k^2 K^2}\sum_{m=1}^{\infty} m\,\frac{q^m}{1-q^{2m}}\cos m\frac{\pi u}{K}\,.\end{aligned} \tag{5.4}$$

Combining (5.4) with the standard expansion for sn u and introducing w instead of u we have

$$\begin{aligned}\left(\frac{r}{a}\right)^2 &= 2 - \frac{E}{K} + \frac{2\pi}{K}\sum_{m=1}^{\infty}(-1)^m\left[\frac{2q^{m-\frac{1}{2}}}{1-q^{2m-1}}\cos(2m-1)w - \right.\\ &\left. - \frac{\pi}{K}m\frac{q^m}{1-q^{2m}}\cos 2mw\right].\end{aligned}$$

Transforming to exponential form there results

$$\begin{aligned}\left(\frac{r}{a}\right)^2 &= 2 - \frac{E}{K} + \frac{\pi}{K}\sum_{m=-\infty}^{\infty}\left[(-1)^{\frac{|2m-1|+1}{2}}\frac{2q^{\frac{|2m-1|}{2}}}{1-q^{|2m-1|}}\exp i(2m-1)w - \right.\\ &\left. - (-1)^{|m|}\frac{\pi}{K}|m|\frac{q^{|m|}}{1-q^{2|m|}}\exp i2mw\right].\end{aligned} \tag{5.5}$$

Rewriting (4.7) in the form

$$\frac{r}{a}\exp iv = \mathrm{sn}\,u - k - ik'\mathrm{cn}\,u \tag{5.6}$$

and substituting the expansions for sn u and cn u we get

$$\begin{aligned}\frac{r}{a}\exp iv &= -k + \frac{2\pi}{kK}\sum_{m=1}^{\infty}(-1)^{m+1}\left[\frac{q^{m-\frac{1}{2}}}{1-q^{2m-1}}\cos(2m-1)w + \right.\\ &\left. + ik'\frac{q^{m-\frac{1}{2}}}{1+q^{|2m-1|}}\sin(2m-1)w\right]\end{aligned}$$

and then

$$\begin{aligned}\frac{r}{a}\exp iv &= -k + \frac{\pi}{kK}\sum_{m=-\infty}^{\infty}(-1)^{\frac{|2m-1|-1}{2}}\left[\frac{q^{\frac{|2m-1|}{2}}}{1-q^{|2m-1|}} + \right.\\ &\left. + (-1)^{\frac{1-\mathrm{sgn}(2m-1)}{2}}k'\frac{q^{\frac{|2m-1|}{2}}}{1+q^{|2m-1|}}\right]\exp i(2m-1)w\,.\end{aligned} \tag{5.7}$$

It remains to get the last function needed for the initial values. Multiplying both sides of (5.6) by r/a we have

$$\left(\frac{r}{a}\right)^2 \exp iv = -k + (1+k^2)\,\mathrm{sn}\,u - k\,\mathrm{sn}^2 u - ik'\mathrm{cn}\,u + ikk'\mathrm{sn}\,u\,\mathrm{cn}\,u\,.$$

The expansions for all functions entering into the right–hand side are known with the exception of $\mathrm{sn}\,u\,\mathrm{cn}\,u$. Using the standard expansion for $\mathrm{dn}\,u$ there results

$$\mathrm{sn}\,u\,\mathrm{cn}\,u = -\frac{1}{k^2}\frac{d}{du}\mathrm{dn}\,u = \frac{2\pi^2}{k^2K^2}\sum_{m=1}^{\infty} m\,\frac{q^m}{1+q^{2m}}\sin m\frac{\pi u}{K}\,. \tag{5.8}$$

Combining this expansion with those for $\mathrm{sn}\,u$, $\mathrm{sn}^2 u$ and $\mathrm{cn}\,u$ one gets

$$\begin{aligned}\left(\frac{r}{a}\right)^2 \exp iv &= -k + \frac{E-K}{kK} + \\ &+ \frac{2\pi}{kK}\sum_{m=1}^{\infty}(-1)^{m+1}\Bigg[\left(1+k^2\right)\frac{q^{m-\frac{1}{2}}}{1-q^{2m-1}}\cos(2m-1)w + \\ &+ ik'\frac{q^{m-\frac{1}{2}}}{1+q^{2m-1}}\sin(2m-1)w - \frac{\pi}{K}\,m\,\frac{q^m}{1-q^{2m}}\cos 2mw - \\ &- ik'\frac{\pi}{K}\,m\,\frac{q^m}{1+q^{2m}}\sin 2mw\Bigg]\end{aligned}$$

or, finally,

$$\begin{aligned}\left(\frac{r}{a}\right)^2 \exp iv &= -k + \frac{E-K}{kK} + \\ &+ \frac{\pi}{kK}\sum_{m=-\infty}^{\infty}\Bigg\{(-1)^{\frac{|2m-1|-1}{2}}\Bigg[\left(1+k^2\right)\frac{q^{\frac{|2m-1|}{2}}}{1-q^{|2m-1|}} + \\ &+ (-1)^{\frac{1-\mathrm{sgn}(2m-1)}{2}}k'\frac{q^{\frac{|2m-1|}{2}}}{1+q^{|2m-1|}}\Bigg]\exp i(2m-1)w + \\ &+ \frac{\pi}{K}(-1)^{|m|}|m|\times \\ &\times\left[\frac{q^{|m|}}{1-q^{2|m|}} + k'\,\mathrm{sgn}(m)\frac{q^{|m|}}{1+q^{2|m|}}\right]\exp i2mw\Bigg\}.\end{aligned} \tag{5.9}$$

Now we may list all initial values necessary for our recurrence formulae. For the column $m=0$ we have from (5.1), (5.2) and (5.5):

$$B_s^{-1,0} = \begin{cases}\dfrac{E}{k'^2K}\,, & s=0\\[2ex] \dfrac{\pi^2}{2k'^2K^2}|s|\dfrac{q^{\frac{|s|}{2}}}{1-q^{|s|}}\,, & s\neq 0\end{cases} \tag{5.10}$$

$$B_s^{0,0} = \begin{cases} 1\,, & s=0 \\ 0\,, & s\neq 0 \end{cases} \tag{5.11}$$

$$B_s^{1,0} = \begin{cases} 1\,, & s=0 \\ 0\,, & s=\text{ even} \\ \dfrac{\pi}{K}(-1)^{\frac{|s|+1}{2}}\dfrac{q^{\frac{|s|}{2}}}{1-q^{|s|}}\,, & s=\text{ odd} \end{cases} \tag{5.12}$$

$$B_s^{2,0} = \begin{cases} 2-\dfrac{E}{K}\,, & s=0 \\ -\dfrac{\pi^2}{K^2}(-1)^{\frac{|s|}{2}}\dfrac{|s|}{2}\dfrac{q^{\frac{|s|}{2}}}{1-q^{|s|}}\,, & s=\text{ even} \\ \dfrac{2\pi}{K}(-1)^{\frac{|s|+1}{2}}\dfrac{q^{\frac{|s|}{2}}}{1-q^{|s|}}\,, & s=\text{ odd} \end{cases} \tag{5.13}$$

The three initial values for the column $m = 1$ result from the expansion (5.3), (5.7) and (5.9):

$$B_s^{0,1} = \begin{cases} \dfrac{E-K}{kK}\,, & s=0 \\ \dfrac{\pi^2}{kK^2}|s|\dfrac{q^{(1-\frac{1}{2}\,\mathrm{sgn}\,s)|s|}}{1-q^{2|s|}}\,, & s\neq 0 \end{cases} \tag{5.14}$$

$$B_s^{1,1} = \begin{cases} -k\,, & s=0 \\ 0\,, & s=\text{ even} \\ \dfrac{\pi}{kK}(-1)^{\frac{|s|-1}{2}}q^{\frac{|s|}{2}}\left[\dfrac{1}{1-q^{|s|}}+(-1)^{\frac{1-\mathrm{sgn}\,s}{2}}\dfrac{k'}{1+q^{|s|}}\right], & s=\text{ odd} \end{cases} \tag{5.15}$$

$$B_s^{2,1} = \begin{cases} -k+\dfrac{E-K}{kK}\,, & s=0 \\ \dfrac{\pi^2}{kK^2}(-1)^{\frac{|s|}{2}}\dfrac{|s|}{2}q^{\frac{|s|}{2}}\left[\dfrac{1}{1-q^{|s|}}+\mathrm{sgn}(s)\dfrac{k'}{1+q^{|s|}}\right], & s=\text{ even} \\ \dfrac{\pi}{kK}(-1)^{\frac{|s|-1}{2}}q^{\frac{|s|}{2}}\left[\dfrac{1+k^2}{1-q^{|s|}}+(-1)^{\frac{1-\mathrm{sgn}\,s}{2}}\dfrac{k'}{1+q^{|s|}}\right], & s=\text{ odd} \end{cases} \tag{5.16}$$

It is seen that all initial values are represented in closed form. More precisely, they are rational functions with respect to q, k, k', K and E. Taking into account that the recurrence relations have closed form coefficients, we may conclude that all

TABLE I
Examples of elliptic Hansen–like coefficients $B_s^{n,0}(q)$ for $e = 0.1$

s	$n=-4$	$n=-3$	$n=-2$	$n=-1$
−4	0.0001942809	0.0000771023	0.0000223352	0.0000031724
−3	0.0024591012	0.0011868098	0.0004457435	0.0000949348
−2	0.0258955936	0.0153818864	0.0076267317	0.0025252495
−1	0.2078135894	0.1539156017	0.1015836606	0.0504100162
0	1.0514042728	1.0306100228	1.0152088751	1.0050441601
s	$n=+1$	$n=+2$	$n=+3$	$n=+4$
0	1.0000000000	1.0050062815	1.0150188444	1.0300752516
1	−0.0500313875	−0.1000627751	−0.1504696337	−0.2016274346
2	0.0000000000	0.0024999970	0.0074999911	0.0150249979
3	0.0000314073	0.0000628145	−0.0000305422	−0.0003734270
4	0.0000000000	−0.0000031407	−0.0000094222	−0.0000126258

TABLE II
Examples of elliptic Hansen–like coefficients $B_s^{n,1}(q)$ for $e = 0.1$

s	$n=-4$	$n=-3$	$n=-2$	$n=-1$
−4	0.0000051974	0.0000012899	0.0000001602	0.0000000000
−3	0.0000875839	0.0000273554	0.0000047947	0.0000000000
−2	0.0012964966	0.0005140951	0.0001275977	0.0000000397
−1	0.0162090113	0.0083430946	0.0031804687	0.0006297326
0	0.1560857065	0.1028020726	0.0509504745	0.0001262623
s	$n=+1$	$n=+2$	$n=+3$	$n=+4$
0	−0.1000000000	−0.1500628145	−0.2006262572	−0.2520659564
1	0.9974948993	1.0024980380	1.0125011827	1.0275418804
2	0.0000000000	−0.0498746073	−0.0999992142	−0.1507485071
3	−0.0006265709	−0.0006297116	0.0018577363	0.0068482493
4	0.0000000000	0.0000626570	0.0001256281	0.0000646977

elliptic Hansen–like coefficients $B_s^{n,m}(q)$ may be presented in such closed form. As mentioned above the Hansen coefficients $X_s^{n,m}(e)$ are known to be represented in closed form only for $s = 0$. The possibility of expressing $Z_s^{n,m}(e)$ in closed form was shown (implicitly) in Vinh (1969) based on the recurrence relations for them. As stated in Section 3, both $Y_s^{n,m}(e)$ and $Z_s^{n,m}(e)$ may be represented in closed form in virtue of their hypergeometric series representation (3.7) and (3.12).

6. Numerical Results

Using the recurrence relations of Section 4 we have computed the three–dimensional tables of the elliptic Hansen–like coefficients $B_s^{n,m}(q)$ for n, m and s taking all integer values from the interval $(-5, 5)$. Such tables were computed for

TABLE III
Examples of elliptic Hansen–like coefficients $B_s^{n,0}(q)$ for $e = 0.9$

s	$n = -4$	$n = -3$	$n = -2$	$n = -1$
−4	582.1841451706	54.1145076854	4.3255050178	0.2092864915
−3	762.0964364304	77.5726588914	7.2887793378	0.4911019412
−2	929.3622452481	101.8227861881	11.0335530235	1.0330921993
−1	1049.9163770227	120.8720761358	14.5647509179	1.7798380321
0	1094.1723255466	128.3634164933	16.3687146791	2.7040972236
s	$n = +1$	$n = +2$	$n = +3$	$n = +4$
0	1.0000000000	1.4862215275	2.4586645825	4.2340364749
1	−0.4909685133	−0.9819370265	−1.8007699674	−3.2753317635
2	0.0000000000	0.1962875179	0.5888625536	1.3524971967
3	0.0451568412	0.0903136823	0.0799330431	−0.0415225569
4	0.0000000000	−0.0397644334	−0.1192933001	−0.2362624803

TABLE IV
Examples of elliptic Hansen–like coefficients $B_s^{n,1}(q)$ for $e = 0.9$

s	$n = -4$	$n = -3$	$n = -2$	$n = -1$
−4	198.6847338447	12.8644085296	0.4970222099	0.0001493759
−3	314.1529688613	23.4945849001	1.1783771864	0.0021273569
−2	466.6834132883	39.9233960633	2.5875788727	0.0244966420
−1	647.1420453947	62.3762251062	5.1659904034	0.2037448826
0	832.4438962189	88.3659170673	8.9114827274	0.4510650726
s	$n = +1$	$n = +2$	$n = +3$	$n = +4$
0	−0.9000000000	−1.4402461417	−2.4180916581	−4.1854335603
1	0.7391508770	1.1810225389	2.0380723401	3.6053782654
2	0.0000000000	−0.3111924416	−0.7990436494	−1.7029321449
3	−0.0719978735	−0.1126390306	−0.0558929072	0.1482242288
4	0.0000000000	0.0634372711	0.1626625322	0.2861699524

the values of the eccentricities $e = 0.0001, 0.001, 0.01, 0.1, 0.5, 0.9$. Part of them for the values $e = 0.1$ and $e = 0.9$ are shown in Tables. The process of calculation is straightforward and involves no difficulties. The results of calculation by the recurrences were checked by the numerical evaluation of the quadratures (3.13)–(3.17) with the aid of *Mathematica 2.1* and of the routine `qromb` of *Numerical Recipes in FORTRAN (Ver.2)* separately. When $e < 0.01$, the values derived from recurrence relations are degraded heavily by the accumulation of errors. This is caused by a small divisor $2/k$ in the main recurrence formulas (4.3, 4.4). This situation is the same as we face in calculating the Bessel functions $J_{n+2}(x)$ from $J_n(x)$ and $J_{n+1}(x)$ for small arguments. While, for the cases $e \geq 0.1$, all values derived from recurrence relations coincide with those evaluated by a numerical quadrature at the level of round-off error; 10^{-13} or so in double precision arithmetics. As for the computing time at a workstation HP 9000/715/50, the case $e = 0.9$ and $-5 \leq n, m, s \leq 5$ requires 1.2s for recurrence relations and 25.8s for numerical quadratures. This demonstrates the possibility of using efficiently the (q, w) expansions in the theory of perturbations for highly eccentric orbits; say larger than 0.1.

7. Conclusion

Comparison with the classic (e, M) (e, v) and (e, g) expansions in multiples of mean, true and eccentric anomalies, respectively, shows the efficiency of the new (q, w) expansions based on application of elliptic function theory to the two–body problem. The new expansions are valid for large eccentricities, admit closed form representation of their coefficients (elliptic Hansen–like coefficients) and are more compact than the traditional series of elliptic motion. Calculation of the elliptic Hansen–like coefficients $B_s^{n,m}(q)$ may be performed without difficulties with the aid of the recurrence relations.

The recurrence technique is well adapted to compute the table of the elliptic Hansen–like coefficients except the case the eccentricity is small. On the other hand, computation of a specific coefficient with fixed values of n, m, and s by recurrences demands the calculation of at least two columns of the matrix $\mathbf{B}$ and might be done more easily by using its integral representation or the analytic expansion formula derived earlier. The possibility of representing the elliptic Hansen–like coefficients in closed form is of theoretical importance, but it is not clear so far whether this way is more advantageous in practice as compared with expanding the elliptic Hansen–like coefficients in powers of q.

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