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Fast Computation of a General Complete Elliptic Integral of Third Kind by Half and Double Argument Transformations

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Abstract

We developed a novel method to calculate an associate complete elliptic integral of the third kind, $J(n|m) \equiv [\Pi(n|m) - K(m)]/n$. The key idea is the double argument formula of $J(n|m)$ with respect to n. We derived it from the not-so-popular addition theorem of Jacobi's complete elliptic integral of the third kind, $\Pi_1(a|m)$, with respect to a, which is a real or pure imaginary argument connected with n and m as $n = m\ \mathrm{sn}^2(a|m)$. Repeatedly using the half argument transformation (Fukushima 2010a) of a new variable, $y \equiv n/m$, or its complement, $x \equiv (m-n)/m$, we reduce $|y|$ sufficiently small, say less than 0.3 or so. Then, we evaluate the integral for the reduced variable by its Maclaurin series expansion. The coefficients of the series expansion are recursively computed from two other associate complete elliptic integrals, $B(m) \equiv [E(m) - (1-m)K(m)]/m$ and $D(m) \equiv [K(m) - E(m)]/m$. The precise and fast computation of these two integrals is found in our previous work (Fukushima 2011a). Finally, we recover the integral value for the original n by successively applying the double argument formula of $J(n|m)$. The new method is sufficiently precise in the sense the maximum errors are less than around 10 machine epsilons. For the sole computation of $J(n|m)$, the new method runs 1.2-1.5 and 4.7-5.5 times faster than Bulirsch's `cel` and Carlson's R_J, respectively. In the simultaneous computation of three associate complete integrals, the new method runs 1.6-1.7 and 5.3-8.0 times faster than `cel` and Carlson's R_D and R_J, respectively.

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1. Introduction

1.1. Legendre's complete elliptic integral of the third kind

We write Legendre's normal form complete elliptic integral of the third kind [1, Formula 110.08] as

$$\Pi(n|m) \equiv \int_0^{\pi/2} \frac{d\theta}{\left(1 - n\sin^2\theta\right)\sqrt{1 - m\sin^2\theta}}, \tag{1}$$

where n and m are called the characteristic and the parameter, respectively [2, Section 3.2.11]. The modern reference of elliptic integrals is [3, Chapt. 19]. Visit its WEB site, `http://dlmf.nist.gov/`. Other references are [4, 5, 6, 7, 8, 9].

There are significant differences in the notation of the input arguments, especially the sign of n, and other convention in the literature [3, Section 19.1]. In the present article, we follow the sign convention of n given in the above and the notation of [2].

The integral $\Pi(n|m)$ is a complicated bivariate special function [6, Section 22.74]. It is frequently used in mathematical physics and engineering [3, Sections 19.30 through 19.35] as well as two other complete elliptic integrals [1, Formulas 110.06 and 110.07]:

$$K(m) \equiv \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}, \tag{2}$$

$$E(m) \equiv \int_0^{\pi/2} \sqrt{1 - m\sin^2\theta}\, d\theta. \tag{3}$$

See also [1, Introduction].

For example, $\Pi(n|m)$ plays a key role in describing the gravitational or electrostatic potential of a uniform thin disk [10]. Since its partial derivatives are general complete elliptic integrals, and therefore, the acceleration field is expressed as linear combinations of the three normal form complete integrals.

Thus, the computation of $\Pi(n|m)$ is necessary in conducting precise orbital integration of satellites or space probes near Saturn's ring. Although

the total mass of the ring is smaller than that of Mimas, the closest massive satellite of Saturn, and the mean density of the ring is tiny, its gravitational tidal effect becomes non-negligible when a target object is close to Saturn.

Also, $\Pi(n|m)$ is mandatory required in computing Legendre's normal form incomplete elliptic integral of the third kind [1, Formula 110.04]:

$$\Pi(\varphi, n|m) \equiv \int_0^{\varphi} \frac{d\theta}{\left(1 - n\sin^2\theta\right)\sqrt{1 - m\sin^2\theta}}, \tag{4}$$

when the amplitude φ takes an arbitrary value. Indeed, its evaluation reduces to that in the standard domain, $[0, \pi/2)$, as

$$\Pi(\varphi, n|m) = 2j\Pi(n|m) \pm \Pi(\psi, n|m), \quad (0 \le \psi < \pi/2) \tag{5}$$

where

$$\varphi = j\pi \pm \psi, \tag{6}$$

and j is an integer.

A good example appears in the long term computation of free rotation of a general rigid body [11, 12, 13]. The rotation angle is expressed as a linear combination of the argument u and $\Pi(\mathrm{am}(u|m), n|m)$ where $\mathrm{am}(u|m)$ is Jacobian amplitude function [3, Formula 22.16.1]. Since u is in proportion to the physical time, the mean angular velocity of rotation is written as a rational function of $K(m)$ and $\Pi(n|m)$.

1.2. Associate complete elliptic integral of third kind

From a practical viewpoint, however, rather important is not $\Pi(n|m)$ but its modification named the associate complete elliptic integral of the third kind [14]:

$$J(n|m) \equiv \int_0^{\pi/2} \frac{\sin^2\theta\, d\theta}{\left(1 - n\sin^2\theta\right)\sqrt{1 - m\sin^2\theta}}. \tag{7}$$

Jacobi defined his incomplete elliptic integral of the third kind [5, p.420] as

$$\Pi_1(u, a|m) \equiv m\,\mathrm{sn}(a|m)\mathrm{cn}(a|m)\mathrm{dn}(a|m)\int_0^u \frac{\mathrm{sn}^2(v|m)\, dv}{1 - m\,\mathrm{sn}^2(a|m)\mathrm{sn}^2(v|m)}, \tag{8}$$

where $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$ are the three principal Jacobian elliptic functions [3, Section 22.1]. See also [1, p.223] and [7, Section 29, Equation (6)]. His complete integral,

$$\Pi_1(a|m) \equiv \Pi_1(K(m), a|m), \tag{9}$$

is essentially the same as $J(n|m)$:

$$\Pi_1(a|m) = m\,\mathrm{sn}(a|m)\mathrm{cn}(a|m)\mathrm{dn}(a|m)\; J\left(m\,\mathrm{sn}^2(a|m)\middle|\, m\right), \tag{10}$$

where the argument a may be a real or pure imaginary number depending on the sign of $n - m$.

Once $J(n|m)$ and two other associate complete elliptic integrals [9, Section 61:13]

$$B(m) \equiv \int_0^{\pi/2} \frac{\cos^2\theta\; d\theta}{\sqrt{1 - m\sin^2\theta}}, \tag{11}$$

$$D(m) \equiv \int_0^{\pi/2} \frac{\sin^2\theta\; d\theta}{\sqrt{1 - m\sin^2\theta}}, \tag{12}$$

are given, all Legendre's normal form complete elliptic integrals are computed without loss of significant figures [15, 16, 17] as

$$K(m) = B(m) + D(m), \tag{13}$$

$$E(m) = B(m) + m_c D(m), \tag{14}$$

$$\Pi(n|m) = B(m) + D(m) + nJ(n|m), \tag{15}$$

where

$$m_c \equiv 1 - m \tag{16}$$

is the complementary parameter [2].

The reverse procedure suffers cancellation problems when $|m|$ and/or $|n|$ are small. Such cases frequently occur in practical applications. For example,

$$m \approx +1 \times 10^{-7}, \;\; n \approx -7 \times 10^{-4}, \tag{17}$$

in case of the Earth rotation [11, 12]. This is the main reason why we prefer the trio of associate integrals to Legendre's three normal form integrals in developing their numerical procedures [17, 18, 14].

1.3. Existing methods

There are four types of method to compute $J(n|m)$: (1) power series expansions with respect to m and/or n [1, Formulas 906.00, 906.05, and 906.06], (2) linear combinations of $K(m)$ and Jacobian Zeta function, $Z(\varphi|m)$, or Heuman's Lambda function, $\Lambda_0(\varphi|m)$, depending on the numerical values of n and m [1, Formulas 410.02, 411.02, 412.02, 413.02, 414.02, 415.02], (3)

Bartky transformation [19] realized by Bulirsch's `cel` [20], and (4) the duplication theorem of Carlson's R_J, the standard symmetric elliptic integral of the third kind [21, 22, 23].

The first method is efficient only when $|m| < |n|$ and $|m|$ is small, say less than 0.1. Otherwise, its speed of convergence significantly slows down. As a result, this method is impractical for general values of n and m.

Recently, we developed a new type of series expansion formulas for the symmetric elliptic integrals [24]. Although its full examination is not completed, our preliminary test indicates that they are slower than the third one. This may be caused by the fact that, in the new series expansions, the complete integrals are first transformed to the incomplete integrals of a special argument, then expanded with respect to some combinations of input arguments.

Next, the second method requires, apart from the computation of $K(m)$, the computation of $Z(\varphi|m)$ or $\Lambda_0(\varphi|m)$. These functions are defined [1, Formulas 140.01 and 150.03] as

$$Z(\varphi|m) \equiv E(\varphi|m) - \frac{E(m)}{K(m)} F(\varphi|m), \tag{18}$$

$$\Lambda_0(\varphi|m) \equiv \frac{2}{\pi} \left[E(m)F(\varphi|m_c) + K(m)E(\varphi|m_c) - K(m)F(\varphi|m_c) \right], \tag{19}$$

where $F(\varphi|m)$ and $E(\varphi|m)$ are Legendre's normal form incomplete elliptic integrals of the first and the second kind, respectively. They are defined [1, Formulas 110.02 and 110.03] as

$$F(\varphi|m) \equiv \int_0^\varphi \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}, \tag{20}$$

$$E(\varphi|m) \equiv \int_0^\varphi \sqrt{1 - m\sin^2\theta}\, d\theta. \tag{21}$$

Therefore, the core part of this method finally reduces to the simultaneous evaluation of $B(m)$ and $D(m)$ [17] as well as their incomplete versions [18]:

$$B(\varphi|m) \equiv \int_0^\varphi \frac{\cos^2\theta\, d\theta}{\sqrt{1 - m\sin^2\theta}}, \tag{22}$$

$$D(\varphi|m) \equiv \int_0^\varphi \frac{\sin^2\theta\, d\theta}{\sqrt{1 - m\sin^2\theta}}, \tag{23}$$

which are similarly related to $F(\varphi|m)$ and $E(\varphi|m)$ as

$$F(\varphi|m) = B(\varphi|m) + D(\varphi|m), \tag{24}$$

$$E(\varphi|m) = B(\varphi|m) + m_c D(\varphi|m). \tag{25}$$

Experimentally, we find that this method is much slower than the third method even using the fastest procedures [17, 18]. This is mainly because it requires the evaluation of many different complete and incomplete elliptic integrals, $K(m)$, $E(m)$, $F(\varphi|m)$, $E(\varphi|m)$, $F(\varphi|m_c)$, and $E(\varphi|m_c)$.

In the third method, $J(n|m)$ is expressed as

$$J(n|m) = \texttt{cel}\left(k_c, n_c, 0, 1\right), \tag{26}$$

where

$$k_c \equiv \sqrt{m_c} = \sqrt{1-m}, \tag{27}$$

and

$$n_c \equiv 1 - n, \tag{28}$$

are the complementary modulus and the complementary characteristic, respectively [2], and `cel` is defined [20, p307] as

$$\texttt{cel}\left(k_c, p, a, b\right) \equiv \int_0^{\pi/2} \frac{\left(a\cos^2\theta + b\sin^2\theta\right) d\theta}{\left(\cos^2\theta + p\sin^2\theta\right)\sqrt{\cos^2\theta + k_c^2\sin^2\theta}}. \tag{29}$$

Finally, in the last method, $J(n|m)$ is written [3, Formula 19.25.2] as

$$J(n|m) = \frac{1}{3} R_J\left(0, m_c, 1, n_c\right), \tag{30}$$

where R_J is defined [3, Formula 19.16.2] as

$$R_J\left(x, y, z, p\right) \equiv \frac{3}{2}\int_0^{\infty} \frac{dt}{(t+p)\sqrt{(t+x)(t+y)(t+z)}}. \tag{31}$$

1.4. Comparison of existing methods

Fig. 1 shows the relative errors of $J(n|m)$ computed by the last two methods described in the previous subsection. The errors are measured as the difference from the quadruple precision computation by `qcel`, the quadruple

Figure 1: Relative errors of $J(n|m)$. Shown are the m-dependence of the relative errors of $J(n|m)$ computed by three methods in the double precision environment: Bulirsch's `cel`, Carlson's R_J, and the new method. Overlapped are the results for various values of n in the range $-\sqrt{m} < n < m/\left(1+\sqrt{1-m}\right)$. The manner of error distribution is unchanged in the single precision environment.

Table 1: Averaged CPU times to compute $J(n|m)$. Listed are the averaged CPU times to compute $J(n|m)$ in the single and double precision environments. They are the means of the CPU times for $2^{15} \times 2^{15}$ pairs of n and m evenly distributed in the domain, $0 < m < 1$ and $-\sqrt{m} < n < m/\left(1+\sqrt{1-m}\right)$. The unit of CPU time is ns at a PC with an Intel Core i7-2675QM run at 2.20 GHz clock. Our routine `celj` assumes that $B(m)$ and $D(m)$ are externally provided.

method	procedure	single	double
duplication	`rj`	276	790
Bartky transf.	`cel`	88	172
new	`celj`	59	144

precision extension of `cel`. In the figure, the errors are plotted as functions of m in the standard domain,

$$0 < m < 1, \tag{32}$$

for various values of n in the domain,

$$-k < n < m^* \equiv \frac{m}{1+k_c}, \tag{33}$$

where

$$k \equiv \sqrt{m}, \tag{34}$$

is the modulus and k_c is the complementary modulus defined in Eq.(27). We limit the domain of m and n as above since other cases can be reduced to this by the transformation formulas summarized in Appendix A. Both of Bulirsch's and Carlson's methods are sufficiently precise, say with the maximum error being less than around a dozen machine epsilons. Then, the main question of concern is the computational speed.

Table 1 compares the averaged CPU times of these procedures to compute $J(n|m)$ in the single and double precision environments. Also, Table 2 is

Table 2: Averaged CPU times to compute $B(m)$, $D(m)$, and $J(n|m)$ simultaneously. Same as Table 1 but for the simultaneous computation of $B(m)$, $D(m)$, and $J(n|m)$.

method	procedure	single	double
duplication	`rj`, `rd`	547	1675
Bartky transf.	`cel`	174	340
new	`celbd`, `celj`	103	208

for the simultaneous computation of $B(m)$, $D(m)$, and $J(n|m)$. The listed CPU times are those uniformly averaged with respect to n and m in the same domain described in Eqs (32) and (33). The tables clearly show that Bulirsch's approach is significantly faster than Carlson's method. This is due to the quadratic convergence of Bartky transformation.

1.5. Outline of this work

Triggered by practical needs in the research of orbital and rotational motions of celestial bodies [11, 12, 13, 10], we have conducted a series of studies to seek for precise and fast procedures to compute Jacobian elliptic functions and the complete and the incomplete elliptic integrals as well as their inversion with respect to the input arguments. They are (1) the computation of $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$ in [25, 26, 27], (2) that of $K(m)$ or $E(m)$ in [26], (3) that of $F(\varphi|m)$ in [28], (4) that of $B(m)$ and $D(m)$ in [17], (5) that of $B(\varphi|m)$ and $D(\varphi|m)$ in [18], (6) that of $J(\varphi, n|m)$ in [14], (7) the inversion of a general incomplete integral including $B(\varphi|m)$, $D(\varphi|m)$, and $J(\varphi, n|m)$ with respect to φ or

$$u \equiv F(\varphi|m), \tag{35}$$

in [29], and (8) the inversion of $K(m)$ or $E(m)$ with respect to m in [30]. This time, the computation of $J(n|m)$ is our target in this article.

During the investigation of the numerical computation of $J(\varphi, n|m)$ [14], we noticed the existence of the addition theorem of $\Pi_1(u, a|m)$ with respect to the second argument a [6, Section 22.74, Example 3]. In case of the computation of the incomplete integral, $J(\varphi, n|m)$, the theorem itself is not so efficient as the addition theorem with respect to u, which we adopted in [14]. Nevertheless, it becomes useful in the case of the complete integral, $J(n|m)$. The addition theorem of $\Pi_1(a|m)$ with respect to a is derived from

that of $\Pi_1(u, a|m)$ by setting $u = K(m)$. These addition theorems are not mentioned in other references [4, 5, 1, 8, 7, 9, 3].

At any rate, we translate the addition theorem of $\Pi_1(a|m)$ into the double argument transformation of $J(n|m)$ with respect to the new variables defined as

$$y \equiv \frac{n}{m}. \tag{36}$$

$$x \equiv 1 - y = \frac{m - n}{m}, \tag{37}$$

If $y < 1$, we can employ the same approach we took in [28, 18, 14]. This fact has driven us to develop a novel method to compute $J(n|m)$ by the half argument transformation of x or y and the double argument transformation of $J(n|m)$. The resulting method is as precise as the existing methods and runs faster than them. See Fig. 1 and Tables 1 and 2.

In Section 2, we summarize the necessary formulas required to develop the new method. In Section 3, we describe the details of the new method. In Section 4, we illustrate the results of numerical experiments.

2. Formulas

Before describing our method of computation, we first derive the necessary formulas of the associate complete elliptic integral of the third kind, $J(n|m)$, which are not found in the literature.

2.1. Addition theorem of $\Pi_1(a|m)$

The addition theorem of $\Pi_1(u, a|m)$ with respect to a [5, p.428] is written as

$$\begin{aligned} \Pi_1(u, a+b|m) &= \Pi_1(u, a|m) + \Pi_1(u, b|m) \\ &\quad - m\, u\, \mathrm{sn}(a|m)\, \mathrm{sn}(b|m)\, \mathrm{sn}(a+b|m) \\ &\quad - \frac{1}{2} \log \left(\frac{1 - m\, \mathrm{sn}(a|m)\, \mathrm{sn}(b|m)\, \mathrm{sn}(u|m)\, \mathrm{sn}(a+b-u|m)}{1 + m\, \mathrm{sn}(a|m)\, \mathrm{sn}(b|m)\, \mathrm{sn}(u|m)\, \mathrm{sn}(a+b+u|m)} \right). \end{aligned} \tag{38}$$

See also [6, Section 22.74, Example 3]. Putting $u = K(m)$, we obtain its complete case:

$$\Pi_1(a+b|m) = \Pi_1(a|m) + \Pi_1(b|m) - mK(m)\, \mathrm{sn}(a|m)\, \mathrm{sn}(b|m)\, \mathrm{sn}(a+b|m). \tag{39}$$

Note that the logarithm term vanishes in the complete case since

$$-\mathrm{sn}(a+b-K|m) = \mathrm{sn}(K-a-b|m) = \mathrm{sn}(a+b+K|m), \tag{40}$$

where we used the abbreviation

$$K \equiv K(m). \tag{41}$$

Refer to [1, Formulas 122.00 and 122.03].

2.2. Addition theorem of $J(n|m)$

Using the expression of $\Pi_1(a|m)$ in terms of $J(n|m)$, Eq.(10), we rewrite the addition theorem of $\Pi_1(a|m)$ as

$$J_3 = \frac{s_1 c_1 d_1 J_1 + s_2 c_2 d_2 J_2 - s_1 s_2 s_3 K}{s_3 c_3 d_3}, \tag{42}$$

where, for the sake of brevity, we posed

$$a_3 = a_1 + a_2, \tag{43}$$

and used the notation

$$s_j \equiv \mathrm{sn}(a_j|m), \quad (j=1,2,3) \tag{44}$$

$$c_j \equiv \mathrm{cn}(a_j|m), \quad (j=1,2,3) \tag{45}$$

$$d_j \equiv \mathrm{dn}(a_j|m), \quad (j=1,2,3) \tag{46}$$

$$J_j \equiv J(n_j|m), \quad (j=1,2,3) \tag{47}$$

while

$$n_j \equiv m s_j^2. \quad (j=1,2,3) \tag{48}$$

Once s_j, c_j, and d_j for $j=1$ and 2 are known, those for $j=3$ are obtained by the addition theorems of Jacobian elliptic functions [1, Formula 123.01] as

$$s_3 = (s_1 c_2 d_2 + s_2 c_1 d_1)\,\nu_3, \tag{49}$$

$$c_3 = (c_1 c_2 - s_1 s_2 d_1 d_2)\,\nu_3, \tag{50}$$

$$d_3 = (d_1 d_2 - m s_1 s_2 c_1 c_2)\,\nu_3, \tag{51}$$

where ν_3 is the common factor defined as

$$\nu_3 \equiv \frac{1}{1 - m s_1^2 s_2^2}. \tag{52}$$

Using these, we rewrite the above expression of J_3 as

$$J_3 = \frac{r_1 J_1 + r_2 J_2 - s_3 \sqrt{n_1 n_2} K}{r_3}, \tag{53}$$

where

$$r_j \equiv \sqrt{n_j \left(1 - n_j\right) \left(m - n_j\right)}, \quad (j = 1, 2, 3) \tag{54}$$

and s_3 is directly expressed in terms of n_1, n_2, and m as

$$s_3 = \frac{\sqrt{n_1 \left(1 - n_2\right) \left(m - n_2\right)} + \sqrt{n_2 \left(1 - n_1\right) \left(m - n_1\right)}}{m - n_1 n_2}. \tag{55}$$

2.3. Double argument formula of $J(n|m)$

As a corollary, we obtain a rewriting of the addition theorem of $J(n|m)$ for the case $a_1 = a_2 = a$:

$$J_D = \frac{2scdJ - s^2 s_D K}{s_D c_D d_D}, \tag{56}$$

where we used another abbreviation:

$$s \equiv \mathrm{sn}(a|m), \tag{57}$$

$$c \equiv \mathrm{cn}(a|m), \tag{58}$$

$$d \equiv \mathrm{dn}(a|m), \tag{59}$$

$$J \equiv J\left(m s^2 \middle| m\right), \tag{60}$$

$$s_D \equiv \mathrm{sn}(2a|m), \tag{61}$$

$$c_D \equiv \mathrm{cn}(2a|m), \tag{62}$$

$$d_D \equiv \mathrm{dn}(2a|m), \tag{63}$$

$$J_D \equiv J\left(m s_D^2 \middle| m\right). \tag{64}$$

This form is inconvenient when $|s|$ is small or a is pure imaginary. Then, using the half argument formulas of Jacobian elliptic functions [1, Formula 124.02]:

$$s = \frac{s_D}{\sqrt{(1+c_D)(1+d_D)}}, \tag{65}$$

$$c = \sqrt{\frac{c_D + d_D}{1+d_D}}, \tag{66}$$

$$d = \sqrt{\frac{c_D + d_D}{1+c_D}}, \tag{67}$$

we finally obtain the double argument formula of $J(n|m)$:

$$J_D = \frac{2(c_D + d_D)J - y_D K}{c_D d_D (1+c_D)(1+d_D)}, \tag{68}$$

where we introduced a new variable in place of s_D defined in Eq.(61) as

$$y_D \equiv s_D^2. \tag{69}$$

Then, c_D and d_D are computed from y_D as

$$c_D = \sqrt{1 - y_D}, \tag{70}$$

$$d_D = \sqrt{1 - m y_D}. \tag{71}$$

Neither s nor s_D remains. As a result, this expression is effective as long as $y_D < 1$, especially even when $y_D < 0$.

2.4. Special addition theorem of $J(n|m)$

As another corollary, by putting

$$a_1 = a, \tag{72}$$

$$a_2 = a_A \equiv K - a, \tag{73}$$

and therefore

$$a_3 = a_1 + a_2 = K, \tag{74}$$

we rewrite the addition theorem, Eq.(53), as

$$J = \frac{s s_A K - s_A c_A d_A J_A}{scd}, \tag{75}$$

where the term with the suffix 3 disappears since

$$s_3 = \mathrm{sn}(K|m) = 1, \tag{76}$$

$$c_3 = \mathrm{cn}(K|m) = 0. \tag{77}$$

Refer to [1, Formula 122.02].

The special addition theorems of Jacobian elliptic functions [1, Formulas 122.00 and 122.03] are expressed as

$$s_A = \frac{c}{d}, \tag{78}$$

$$c_A = \frac{k_c s}{d}, \tag{79}$$

$$d_A = \frac{k_c}{d}, \tag{80}$$

where k_c is the complementary modulus defined in Eq.(27). Using these, we further rewrite the above expression of J as

$$J(n|m) = \frac{1}{n_c}\left[K(m) - \left(\frac{m_c}{n_c}\right) J\left(n_A|\, m\right)\right]. \tag{81}$$

where n_c and m_c are the complementary characteristic and the complimentary parameter defined in Eqs (28) and (16), respectively, and

$$n_A \equiv \frac{m-n}{n_c}. \tag{82}$$

This is the special addition theorem of $J(n|m)$.

3. Method

3.1. Strategy of computation

Below we describe a new method to compute $J(n|m)$, the associate complete elliptic integral of the third kind when n and m are subject to the conditions, Eqs (32) and (33). Refer to Appendix A for the procedure to reduce the domain of the input arguments to this assumed case. See Fig. 2 for the diagram showing the reduced domain.

For n and m satisfying this condition, we (1) use repeatedly the half argument formulas of the new variables, y and x, which are defined in Eqs

Figure 2: n-m Diagram. Illustrated are the curves of boundary of the parameter domain, $-\sqrt{m} < n < m/\left(1+\sqrt{1-m}\right)$. Also shown are the two lines, $n/m = \pm 0.325$, indicating the region where the Maclaurin series expansion is effective to compute $J(n|m)$ in the double precision environment.

(36) and (37), respectively, in order to reduce $|n|$ while keeping m unchanged, (2) calculate an approximate value of the integral for the reduced n by the Maclaurin series expansion of $J(n|m)$ with respect to y, and (3) recover the integral value corresponding to the original n by successive applications of the double argument formula of $J(n|m)$.

These three steps are explained in Sections 3.2, 3.3, and 3.4, respectively. In order to complete this algorithm, we must specify the condition to terminate the half argument transformation and to shift to the Maclaurin series evaluation. We discuss it in Section 3.5.

3.2. Half argument transformation

When $|n|$ is not small, say when $|n/m| > 0.3$ or so, we reduce it by the following sequence:

$$y_0 = \frac{n}{m}, \tag{83}$$

$$y_{\ell+1} = \frac{y_\ell}{(1+c_\ell)\,(1+d_\ell)}, \quad (\ell = 0, 1, \cdots) \tag{84}$$

where

$$c_\ell \equiv \sqrt{1-y_\ell}, \tag{85}$$

$$d_\ell \equiv \sqrt{1-m y_\ell}. \tag{86}$$

This is the half argument transformation of the sine amplitude function [1, Formula 124.02] if we understand that

$$y = \mathrm{sn}^2(a|m), \tag{87}$$

$$c = \mathrm{cn}(a|m), \tag{88}$$

$$d = \mathrm{dn}(a|m), \tag{89}$$

where a is the argument of Jacobi's normal form complete elliptic integral of the third kind, $\Pi_1(a|m)$, given in Eq.(10), and $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$ are the three principal Jacobian elliptic functions [3, Section 22.2].

If $y_0 > 1/2$, the above sequence of y faces loss of one significant bit or more in the process to calculate c_ℓ and/or d_ℓ. In such case, we introduce another variable,

$$x \equiv 1 - y, \tag{90}$$

and use its sequence:

$$x_0 = \frac{m-n}{m}, \tag{91}$$

$$x_{\ell+1} = \frac{c_\ell + d_\ell}{1 + d_\ell}. \ (\ell = 0, 1, \cdots) \tag{92}$$

This time, c_ℓ and d_ℓ are computed differently as

$$c_\ell = \sqrt{x_\ell}, \tag{93}$$

$$d_\ell = \sqrt{m_c + m x_\ell}, \tag{94}$$

where m_c is the complementary parameter defined in Eq.(16). This is the half argument formula of the cosine amplitude function [1, Formula 124.02]. When x_ℓ becomes large, say when $x_\ell > 1/2$, the square root operation to compute c_ℓ and/or d_ℓ in Eqs (93) and (94) faces a loss of significant bits. Then, we terminate this sequence when $x_\ell > 1/2$ and shift to that of y described in the above by translating

$$y_\ell = 1 - x_\ell. \tag{95}$$

Originally, we invented the half argument transformation when $y > 0$ [28]. Unexpectedly, we learn that it does work even when $y < 0$. The transformation converges fairly fast. In average, one application of it is enough to reduce $|y|$ sufficiently small, say less than 0.3 or so. This will be shown later in Section 4.

3.3. Maclaurin series expansion

Assume that $|y|$ is sufficiently small, say less than 0.3 or so. Then, noting that

$$n = my, \tag{96}$$

which is derived from the definition of y given in Eq.(36), we compute $J(n|m)$ by its Maclaurin series expansion with respect to y while fixing m:

$$J(n|m) = \int_0^{K(m)} \frac{\mathrm{sn}^2(v|m)\, dv}{1 - my\, \mathrm{sn}^2(v|m)} = \sum_{j=0}^{\infty} D_j(m) y^j, \tag{97}$$

where the coefficients are defined as

$$D_j(m) \equiv m^j \int_0^{K(m)} \mathrm{sn}^{2j+2}(v|m)\, dv. \tag{98}$$

This series expansion is not found in the existing literature [1, 9, 8, 3].

The coefficients of the series expansion are recursively computed [4, p.60] once $B(m)$ and $D(m)$ are given:

$$D_0(m) = D(m), \tag{99}$$

$$D_1(m) = \frac{1}{3}\left[(1+2m)D(m) - B(m)\right], \tag{100}$$

$$D_j(m) = (1-I_j)\,(1+m)D_{j-1}(m) - (1-2I_j)\,mD_{j-2}(m), \quad (j=2,3,\cdots) \tag{101}$$

where

$$I_j \equiv \frac{1}{2j+1}, \tag{102}$$

is a numerical constant, which can be pre-computed and stored. See also the formulas 310.02, 310.04, and 310.05 of [1] after substituting $u = K(m)$. There appear no small divisor m nor n in the series expansions as well as the recurrence formulas of coefficients. Therefore, this formulation suffers no cancellation problems.

Also the following properties of the coefficient functions hold when $0 < m < 1$:

$$0 < \frac{D_j(m)}{D(m)} < 1, \tag{103}$$

$$\frac{dD_j(m)}{dm} > 0, \tag{104}$$

$$D_j(m) > D_{j+1}(m), \tag{105}$$

which are easily proved from the definition of $D_j(m)$ given in Eq. (98). These relations provide us a rough estimate of the relative magnitude of the truncated terms as

$$\frac{1}{J(n|m)}\left|\sum_{j=p}^{\infty} D_j(m)y^j\right| \approx \frac{D_p(m)\,|y^p|}{D(m)} < |y|^p\,. \tag{106}$$

This assures that the series quickly converges when $|y|$ is sufficiently small, say less than 0.3 or so.

3.4. Double argument transformation

Assume that (1) the variable y_L, obtained after the L times applications of the half argument transformations, becomes sufficiently small such that (2) the corresponding integral value, J_L, is successfully obtained by the series expansion formula described in the previous subsection. Then, the original integral,

$$J_0 \equiv J(n|m), \tag{107}$$

is computed by the successive application of the double argument formula of $J(n|m)$ given in Section 2.3:

$$J_\ell = \frac{2\,(c_\ell + d_\ell)\,J_{\ell+1} - y_\ell K}{c_\ell d_\ell\,(1+c_\ell)\,(1+d_\ell)}, \quad (\ell = L-1, \cdots, 1, 0) \tag{108}$$

where $K \equiv K(m)$ is computed from $B(m)$ and $D(m)$ as given in Eq.(13) and we assumed that the intermediate values of y_ℓ, c_ℓ, and d_ℓ are all saved during the process of the half argument transformation.

3.5. Terminating condition of half argument transformations

We terminate the half argument transformations and move to the evaluation of the Maclaurin series expansion when $|y|$ is less than a critical value, y_C. If we set y_C too large, the convergence of the Maclaurin series slows down or diverges. Then, the total computational amount increases. On the other hand, if we set y_C too small, both the numbers of the half and the double argument transformations become large. Thus, the total computational amount increases again. Therefore, there exists an optimal value of y_C to minimize the total CPU time.

This value may depend on the computing precision desired and the computer architecture adopted. In general, it seems difficult to predict it theoretically. Then, we experimentally determine it. By adopting a PC with a recent computer chip, the Intel Core i7-2675QM, we learn that

$$y_C = 0.325 \pm 0.025 \tag{109}$$

in the double precision environment. A fairly large uncertainty shows that the valley of the minimum is shallow. The lines showing this critical value are already depicted in the n-m diagram, Fig. 2, where we omitted the uncertainty for simplicity. In the single precision environment, this slightly changes as

$$y_C = 0.250 \pm 0.025. \tag{110}$$

Table 3: Convergence of the half argument transformation. Listed are the sequences of y_ℓ and/or $x_\ell \equiv 1 - y_\ell$ starting from $y_0 = n/m$ until $|y_\ell| < y_C = 0.325$. Shown are two typical and two extreme cases under the condition $0 < m < 1$ and $-\sqrt{m} < n < m/\left(1 + \sqrt{1-m}\right)$: (1) $m = 1/3$ and $n = -1/2$, (2) $m = 1/2$ and $n = 1/3$, (3) $m = \epsilon$ and $n = -1/\sqrt{\epsilon}$, and (4) $m = 1 - \epsilon$ and $n = 1 - \sqrt{\epsilon}$. Here $\epsilon \approx 1.11 \times 10^{-16}$ is the machine epsilon in the double precision environment.

	Case 1	Case 2		Case 3	Case 4	
ℓ	y_ℓ	x_ℓ	y_ℓ	y_ℓ	x_ℓ	y_ℓ
0	−1.50E+00	3.33E−01		−8.55E+23	1.05E−08	
1	−2.61E−01	7.67E−01	2.33E−01	−9.49E+07	2.05E−04	
2				−4.87E+03	2.83E−02	
3				−3.44E+01	2.88E−01	
4				−2.47E+00	6.98E−01	3.02E−01
5				−4.32E−01		
6				−9.83E−02		

4. Numerical experiments

The computational errors and the CPU time of the new method are already shown in Fig. 1 and Tables 1 and 2. In short, the new method is as precise as the existing methods and outperforms them significantly in terms of the computational speed. Below, we shall investigate the reason.

First of all, we examine the manner of convergence of the half argument transformation. Let us begin with the sequence of y. If $y_\ell < 1$, then both c_ℓ and d_ℓ are real and positive since $0 < m < 1$, and therefore $|y_{\ell+1}| < |y_\ell|$. Initially, the assumption is satisfied as $y_0 = n/m < 1$. Thus, the sequence decreases $|y|$. Since $x \equiv 1 - y$, this means that the same sequence increases x. Typically, at most one application is enough to make $|y|$ sufficiently small, say less than 0.3 or so. See Cases 1 and 2 in Table 3.

Next, let us examine extreme cases. First, assume that y_0 is negative and its magnitude is large, say $y_0 < -10$. Then, the sequence rapidly decreases $|y|$. This is because

$$|y_{\ell+1}| < \left| \frac{y_\ell}{2c_\ell} \right| \approx \frac{\sqrt{|y_\ell|}}{2}, \tag{111}$$

since $d_\ell > 1$ in that case. Consider an extreme case when

$$m = \epsilon, \;\; n = -k = \frac{-1}{\sqrt{\epsilon}}, \tag{112}$$

where

$$\epsilon \equiv 2^{-53} \approx 1.11 \times 10^{-16}, \tag{113}$$

is the double precision machine epsilon. Then, starting from a negative huge number, $y_0 = n/m = -1/(\epsilon\sqrt{\epsilon}) \approx -10^{24}$, we find that 6 iterations of the transformation deflates $|y|$ sufficiently as $y_6 \approx -0.1$. See Case 3 in Table 3.

On the other hand, if x_0 is positive and small, say when $0 < x_0 < 0.3$, then the sequence rapidly increases x. This is because

$$x_{\ell+1} > c_\ell = \sqrt{x_\ell}, \tag{114}$$

since $c_\ell < 1$ in that case. Consider another extreme case when

$$m = 1 - \epsilon, \;\; n = \frac{m}{1 + \sqrt{1-m}} \approx 1 - \sqrt{\epsilon}. \tag{115}$$

Then, starting from a positive tiny number,

$$x_0 = \frac{m-n}{m} \approx \sqrt{\epsilon} \approx 10^{-8}, \tag{116}$$

we find that 4 iterations of the transformation inflates x greater than $1/2$ as $x_4 \approx 0.70$. Thus, $y_4 \equiv 1 - x_4 \approx 0.30$ becomes less than y_C. See Case 4 in Table 3.

Table 4 shows the averaged number of transformations for m and n distributed uniformly in the domain, $0 < m < 1$ and $-\sqrt{m} < n < m/(1 + \sqrt{1-m})$. In the double precision case, the expected number of transformations is as small as 0.8 despite that the extreme case requires 6 iterations. This situation is only slightly changed in the single precision computation. Meanwhile, the necessary number of terms of Maclaurin series depends on the precision. The averaged number in the double precision environment is 15. This number becomes 5 in the single precision computation.

As for the time-consuming operations, namely the operations other than the addition/subtraction and the multiplication, each half argument transformation requires two calls of the square root and one division operation while each double argument transformation of $J(n|m)$ needs one division operation only. Therefore, the major computational labor is, in average, 2 calls

Table 4: Numbers of the half argument transformations. Listed are the maximum and the mean values of the half argument transformations until $|y| < y_C$ in the single and double precision environments. These values are obtained from the statistics for $2^{15} \times 2^{15}$ pairs of n and m evenly distributed in the domains, $0 < m < 1$ and $-\sqrt{m} < n < m/\left(1 + \sqrt{1-m}\right)$. Also, illustrated are the number of terms of the Maclaurin series expansions needed.

number of	single		double	
	max.	mean	max.	mean
half argument transformations in x	4	0.12	5	0.06
total half argument transformations	5	0.89	6	0.79
terms of truncated Maclaurin series	9	5.20	34	15.13

of the square root, 2 divisions, and a single evaluation of a 5-th or 15-th order polynomial by Horner's method together with their coefficients determination by the linear three-term recurrence formulas. This computational amount is fairly small. As a result, although the method is of the linear convergence, it runs significantly faster than Bulirsch's `cel`, which converges quadratically, as already shown in Table 1.

5. Conclusion

We developed a fast method to compute the associate complete elliptic integral of the third kind, $J(n|m)$. Using the special value formulas and transformations, we first reduce the domain of input arguments as $0 < m < 1$ and $-\sqrt{m} < n < m/(1 + \sqrt{1-m})$. Then, by repeatedly applying the half argument transformations with respect to the variables $y \equiv n/m$ and/or $x \equiv (m-n)/m$, we reduce the magnitude of y such that $|y| < y_C$. The critical value, y_C, is determined to minimize the total computational time while keeping the computational error being less than 10 machine epsilons. The determined value of y_C is 0.250 and 0.325 in the single and double precision environments, respectively. When $|y| < y_C$, we compute the integral value by truncated Maclaurin series expansion. The coefficients of the series expansion are calculated recursively from two other associate complete elliptic integrals, $B(m)$ and $D(m)$, which we compute by our fast procedure [17]. The averaged degree of the resulting polynomial is 5 and 15 in the single and the double precision environments, respectively. Finally, we recover the original integral

value, $J(n|m)$, by repeatedly applying the double argument formula to the value of J computed by the truncated series expansion.

By choosing rather large value of y_C, the averaged number of the half and the double argument transformations are as small as less than 1. This leads to small numbers of the time-consuming operations, the square root and the division. As a result, the new method runs fairly fast. For the sole computation of $J(n|m)$, the new method runs 1.2-1.5 and 4.7-5.5 times faster than Bulirsch's `cel` and Carlson's R_J, respectively. In the simultaneous computation of three associate complete integrals, the new method runs 1.6-1.7 and 5.3-8.0 times faster than `cel` and Carlson's R_D and R_J, respectively.

The present formulation keeps the value of m unchanged during the computation. Then, it would be beneficial to develop a procedure to invert numerically $J(n|m)$ with respect to n while fixing m.

The double precision Fortran function of the developed procedure, `celj`, and its companion subroutine, `celbd`, are available from the author's personal WEB page at ResearchGate:

`https://www.researchgate.net/profile/Toshio_Fukushima/`

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APPENDIX

Appendix A. Transformation of Input Arguments

The associate complete elliptic integral of the third kind, $J(n|m)$, is real-valued if $m < 1$ and $n \neq 1$. When $n > 1$, the integrand has a pole at

$$\theta = \pm \sin^{-1} \sqrt{\frac{1}{n}}. \tag{A.1}$$

In this case, we understand that the integral is defined by its principal value.

Under these conditions and understanding, we explain below a series of procedures to reduce m and n such that

$$0 < m < 1, \tag{A.2}$$

$$-k < n < m^*. \tag{A.3}$$

Here we introduced an auxiliary parameter

$$m^* \equiv \frac{m}{1+k_c}, \tag{A.4}$$

while $k = \sqrt{m}$ and $k_c = \sqrt{1-m}$ are the modulus and the complementary modulus defined in Eqs(34) and (27), respectively.

First, in Appendix A.1, we reduce the domain of m such that $0 < m < 1$ under the condition $n \neq 1$. Then, in Appendix A.2, we reduce the domain of n such that $-k < n < m^*$ under the condition $0 < m < 1$. All the following transformation formulas are numerically confirmed by using Mathematica [2]. For the case of principal values, we confirmed them by comparing the real parts. Also refer to [14, Appendix], which describes a similar procedure for the incomplete integral, $J(\varphi, n|m)$.

Appendix A.1. Transformation of parameter

Under the conditions $m < 1$ and $n \neq 1$, we reduce the domain of m such that $0 < m < 1$. First, if $m = 0$, the integral is real-valued if $n < 1$ and is expressed as

$$J(n|0) = \frac{\pi}{2(n_c + \sqrt{n_c})}. \tag{A.5}$$

where $n_c = 1-n$ is the complementary characteristic defined in Eq.(28). This expression is derived from those of $K(m)$ and $\mathit{\Pi}(n|m)$ [1, Formula 111.01] written as

$$K(0) = \frac{\pi}{2}, \tag{A.6}$$

$$\mathit{\Pi}(n|0) = \frac{\pi}{2\sqrt{n_c}}, \tag{A.7}$$

by using the connection formulas, Eqs (13) and (15). See also [3, Formulas 19.6.2 and 19.6.3].

Next, if $m < 0$, we transform m by the negative parameter transformation:

$$J(n|m) = \frac{J\left(n_N|\, m_N\right)}{m_c k_c}, \tag{A.8}$$

where $m_C = 1 - m$ is the complementary parameter defined in Eq.(16) and n_N and m_N are defined as

$$n_N \equiv \frac{n-m}{m_c}, \tag{A.9}$$

$$m_N \equiv \frac{-m}{m_c}, \tag{A.10}$$

such that

$$n_N \neq 1, \tag{A.11}$$

$$0 < m_N < 1. \tag{A.12}$$

This is obtained from the corresponding transformation formulas of $K(m)$ and $\Pi(n|m)$ [1, Formula 160.02]. See also [3, Formula 19.7.5]. Thus, the domain of m is reduced as $0 < m < 1$ while the condition $n \neq 1$ is kept.

Appendix A.2. Transformation of characteristic

Let us reduce the domain of n such that $-k < n < m^*$ under the conditions $0 < m < 1$ and $n \neq 1$. Noting the inequalities

$$-\infty < -k < 0 < m^* < m < k < 1 < 1 + k_c < +\infty, \tag{A.13}$$

when $0 < m < 1$, we describe the reduction procedures case by case.

Appendix A.2.1. Case 1: $-\infty < n < -k$

We transform n by the first characteristic transformation:

$$n_1 \equiv \frac{m}{n}, \tag{A.14}$$

such that

$$-k < n_1 < 0. \tag{A.15}$$

Note that

$$n_1 = n \leftrightarrow n = \pm k. \tag{A.16}$$

The corresponding transformation of $J(n|m)$ is

$$J(n|m) = -\left(\frac{m\, J\left(n_1|\, m\right)}{n^2} + \frac{K(m)}{n} + \frac{\pi}{2\sqrt{h}}\right), \tag{A.17}$$

where

$$h \equiv n(1-n)(n-m), \tag{A.18}$$

is another auxiliary parameter [14]. This expression is obtained from that of $\Pi(\varphi, n|m)$ [3, Formula 19.7.8] by substituting $\varphi = \pi/2$.

Appendix A.2.2. Case 2: $m^* < n < k$

We transform n by the second characteristic transformation:

$$n_2 \equiv \frac{m-n}{n_c}, \tag{A.19}$$

such that

$$-k < n_2 < m^*. \tag{A.20}$$

Note that

$$n_2 = n \leftrightarrow n = m^* \text{ or } n = 1 + k_c. \tag{A.21}$$

This is a dual of the first characteristic transformation in the sense that the complementary quantities follow it as

$$n_{2c} \equiv 1 - n_2 = \frac{m_c}{n_c}. \tag{A.22}$$

The corresponding transformation of $J(n|m)$ is

$$J(n|m) = \frac{1}{n_c}\left(K(m) - \frac{m_c\, J\left(n_2|\, m\right)}{n_c}\right), \tag{A.23}$$

which is nothing but the special addition theorem of $J(n|m)$ given in Eq. (81). This can be derived from that of $\Pi(\varphi, n|m)$ [3, Formula 19.7.9] by substituting $\varphi = \pi/2$.

Appendix A.2.3. Case 3: $k < n < 1$

We transform n by the third characteristic transformation:

$$n_3 \equiv \frac{mn_c}{m-n}, \tag{A.24}$$

such that

$$-k < n_3 < 0. \tag{A.25}$$

This transformation is the synthesis of the first and the second characteristic transformations because

$$n_3 = \frac{m}{n_2} = \frac{m-n_1}{1-n_1}. \tag{A.26}$$

The corresponding transformation of $J(n|m)$ is

$$J(n|m) = \frac{mm_c\, J\left(n_3|\, m\right)}{(n-m)^2} - \frac{K(m)}{n-m} + \frac{\pi}{2\sqrt{h}}, \tag{A.27}$$

which is obtained by the synthesis of Eqs (A.17) and (A.23). It is also derived from that of $\Pi(\varphi, n|m)$ [3, Formula 19.7.10] by substituting $\varphi = \pi/2$.

Appendix A.2.4. Case 4: $1 < n < 1 + k_c$

We transform n by the third characteristic transformation, Eq. (A.24), such that

$$0 < n_3 < m^*. \tag{A.28}$$

This time, the corresponding transformation of $J(n|m)$ becomes a little different as

$$J(n|m) = \frac{1}{n-m}\left(\frac{mm_c\, J\left(n_3|\, m\right)}{n-m} - K(m)\right). \tag{A.29}$$

Note that two different forms of $R_C(0, y)$ depending on the sign of y [3, Formula 19.6.15] lead to the change from Eq. (A.27).

Appendix A.2.5. Case 5: $1 + k_c < n < +\infty$

We transform n by the first characteristic transformation, Eq. (A.14), such that

$$0 < n_1 < m^*. \tag{A.30}$$

Again, the corresponding transformation of $J(n|m)$ becomes a little different as

$$J(n|m) = \frac{-1}{n}\left(\frac{m\, J\left(n_1|\, m\right)}{n} + K(m)\right). \tag{A.31}$$

Appendix A.2.6. Case 6: special values

Finally, we present the special values when $n = 0$, $n = m$, $n = \pm k$, $n = m^*$, or $n = 1 + k_c$. Their values can be computed by one of the procedures described in the previous cases:

$$J(0|m) = D(m), \tag{A.32}$$

$$J(m|m) = \frac{B(m)}{m_c}, \tag{A.33}$$

$$J(-k|m) = \frac{1}{2k}\left(K(m) - \frac{\pi}{2(1+k)}\right), \tag{A.34}$$

$$J\left(m^*|\, m\right) = \frac{K(m)}{2k_c}, \tag{A.35}$$

$$J(k|m) = \frac{1}{2k}\left(\frac{\pi}{2(1-k)} - K(m)\right), \tag{A.36}$$

$$J\left(1 + k_c|\, m\right) = \frac{-K(m)}{2k_c}. \tag{A.37}$$

These special value formulas are obtained as follows: (1) $J(0|m)$ and $J(m|m)$ are calculated from those of $\Pi(0|m)$ and $\Pi(m|m)$ [1, Formula 111.06], (2) $J(-k|m)$ is obtained from Eq. (A.17) by substituting $n = n_1 = -k$ and solving it with respect to J (see also [3, Formula 19.6.2]), (3) $J(m^*|m)$ is derived from Eq. (A.23) by substituting $n = n_2 = m^*$ and solving it with respect to J, (4) $J(k|m)$ is computed from Eq. (A.23) by noting $n_2 = -k$ in this case and using the expression of $J(-k|m)$ in the above, and (5) $J(1 + k_c|m)$ is computed from Eq. (A.27) by noting $n_3 = m^*$ in this case and using the expression of $J(m^*|m)$ in Eq. (A.35). The last three special value formulas, Eqs (A.35) through (A.37), as well as their translation in terms of $\Pi(n|m)$ are missing in the literature [4, 5, 1, 7, 8, 9, 3].

In conclusion, by using the above various transformations and the special value formulas, we reduce the domain of m and n such that $0 < m < 1$ and $-k < n < m^*$.

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