JOURNAL OF FUNCTIONAL ANALYSIS **157**, 394–412 (1998)
ARTICLE NO. FU983272

Müntz–Szasz Theorems for Nilpotent Lie Groups

Darwyn C. Cook

Department of Mathematics, Daniel Webster College, 20 University Drive, Nashua, New Hampshire 03063

Received August 2, 1996; ; revised February 5 and October 14, 1997; accepted January 4, 1998

The classic Müntz–Szasz theorem says that for $f \in L^2([0,1])$ and $\{n_k\}_{k=1}^{\infty}$, a strictly increasing sequence of positive integers,

$$\left(\int_0^1 x^{n_j} f(x)\,dx = 0\ \forall j \Rightarrow f = 0\right) \Leftrightarrow \sum_{j=1}^{\infty} \frac{1}{n_j} = \infty.$$

We have generalized this theorem to compactly supported functions on $\mathbb{R}^n$ and to an interesting class of nilpotent Lie groups. On $\mathbb{R}^n$ we rephrased the condition above on an integral against a monomial, as a condition on the derivative of the Fourier transform $\hat{f}$. This transform, for compactly supported f, has an entire extension to complex n-space and these derivatives are coefficients in a Taylor series expansion of $\hat{f}$.

For nilpotent Lie groups, we have proven a Müntz–Szasz theorem for the matrix coefficients of the operator valued Fourier transform, on groups that have a fixed polarizer for the representations in general position.

Our work here is inspired by recent work on Paley–Wiener theorems for nilpotent Lie groups by Moss (*J. Funct. Anal.* **114** (1993), 395–411) and Park (*J. Funct. Anal.* **133** (1995), 211–300), who have proven Paley–Wiener theorems on restricted classes of nilpotent Lie groups. Lipsman and Rosenberg (*Trans. Amer. Math. Soc.* **348** (1996), 1031–1050) have extended these results, for matrix coefficients, to any connected, simply connected nilpotent Lie group.

As part of the proof of the Müntz–Szasz theorem for matrix coefficients, We construct a new basis in a nilpotent Lie algebra, which we call an *almost strong Malcev basis*. This new basis has many of the features of a strong Malcev basis, although it can be used to pass through subalgebras that are not ideals. Almost strong Malcev bases are unique up to a fixed strong Malcev basis. We will also show that, using almost strong Malcev bases, we can provide a partial answer to a question posed by Corwin and Greenleaf ("Representations of Nilpotent Lie Groups and Their Applications", Cambridge Univ. Press, Cambridge, UK, 1990) on using additive coordinates for a cross-section.

1. INTRODUCTION

The Müntz–Szasz theorem for the unit interval states that for $f \in L^2[0,1]$ and a strictly increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$,

0022-1236/98 $25.00

$\int_{[0,1]} x^{n_k} f(x)\,dx = 0$ for each k implies that $f = 0$ if and only if $\sum_{k=1}^{\infty} 1/n_k = \infty$. In this paper we explore Müntz–Szasz theorems on noncommutative groups: in particular connected and simply connected nilpotent Lie groups. In this case it is not clear what the equivalent of monomials for nilpotent Lie groups should be. We solve this by stating an equivalent theorem that can be more readily generalized.

For $f \in L^2[0,1]$, and $s \in \mathbb{R}$, the nth derivative of the Euclidean Fourier transform of f, at s, is given by

$$\hat{f}^{(n)}(s) = (2\pi i)^n \int_{[0,1]} x^n (e^{2\pi i s x} f(x))\,dx.$$

This suggests restating the classic Müntz–Szasz theorem in terms of the Fourier transform. For convenience we state this explicitly as a proposition.

Proposition 1.1. *Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of natural numbers and s a fixed real number. Then $\hat{f}^{(n_k)}(s) = 0$ for each k implies $f = 0$ a.e. if and only if $\sum_{k=1}^{\infty} 1/n_k = \infty$.*

In some sense the Müntz–Szasz theorem is a generalization of the Paley–Wiener theorem, which says that $\hat{f}$ has an entire extension to the complex plane for compactly supported f. For our purposes Proposition 1.1 is a step in the right direction, since there are generalizations of the Fourier transform and the Paley–Wiener theorem [3] to nilpotent Lie groups.

A Müntz–Szasz theorem for nilpotent Lie groups can be phrased in terms of the operator valued Fourier transform $\hat{f}_{op}$ of the function f: If sufficiently many of the (partial) derivatives of $\hat{f}_{op}$ vanish then f vanishes. Here, derivatives of the operator valued transform are expressed in terms of derivatives of matrix coefficients for the operator, in some orthonormal basis. We have proven Müntz–Szasz theorems for groups that have a fixed polarizer for all of the *parameterizing functionals.*

In order to compute (partial) derivatives of the matrix coefficients, we make use of the fact that, on nilpotent groups $\mathfrak{G}$, the unitary dual $\hat{\mathfrak{G}}$ has a nice parameterization from the Kirillov orbit model. If $\mathfrak{g}$ is the Lie algebra corresponding to $\mathfrak{G}$, we can consider the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ as a vector space. $\mathfrak{G}$ acts on $\mathfrak{g}$ by the adjoint map $\mathrm{Ad}(x)$ and this can be lifted to the coadjoint action $\mathrm{Ad}^*(x)$ on $\mathfrak{g}^*$ [7]. Kirillov showed that $\hat{\mathfrak{G}}$ is in one-to-one correspondence with $\mathfrak{g}^*/\mathrm{Ad}^*(\mathfrak{G})$, the linear dual of $\mathfrak{g}$ mod the coadjoint orbits of $\mathfrak{G}$.

For the study of harmonic analysis on nilpotent groups, it suffices to consider a collection of orbit representatives that come from a Zariski open set in the full collection of orbit representatives, called the generic representations. In addition, there is a convenient way to choose a representative from each generic coadjoint orbit that gives a nice parameterization of $\hat{\mathfrak{G}}$. We outline the details of the construction in the remaining paragraphs.

Fix a strong Malcev basis $X = \{X_1, ..., X_n\}$ for $\mathfrak{g}$. By definition $\mathfrak{g}_j = \mathbb{R} - \text{span}\{X_1, ..., X_j\}$ is an ideal in $\mathfrak{g}$, for each j. The basis X is a *Jordan–Hölder* basis for the action of $\text{Ad}(x)$. That is, for each j and $X \in \mathfrak{g}$ we have $\text{Ad}(\exp(X))(g_j) \subseteq g_j$. Therefore, $\ell_n, ..., \ell_1$ is a Jordan–Hölder basis for the action of Ad^*, where $\ell_j = X_j^*$. So, for all $X \in \mathfrak{g}$ and $1 \leqslant j \leqslant n$, $\text{Ad}^*(\exp(X))(A_j) \subseteq A_j$—where $A_j = \mathbb{R}\text{-span}\{\ell_n, ..., \ell_j\}$.

Ad^* can be defined as a quotient action on $\mathfrak{g}^*/A_j$. Let $d_j(\ell)$ be the orbit dimension for the equivalence class of ℓ under this action. The *generic* ℓ are those ℓ whose orbits have the maximal possible dimension $d_j(\ell)$ for each j. A *generic orbit* is an $\text{Ad}^*(\mathfrak{G})$ orbit that contains a generic ℓ: Generic orbits are comprised of only generic ℓ. The Chevalley–Rosenlicht theorem shows that the collection of generic orbits U is a Zariski-open set in $\mathfrak{g}^*$, and in particular its complement has Euclidean measure 0.

For generic ℓ let $d_j = d_j(\ell)$ (d_j is well defined by the definition of the generic ℓ). A *jump index* s_j is an index such that the orbit dimension increases, that is $d_{s_j} = d_{s_j - 1} + 1$. The remaining indices, those indices for which the orbits do not increase in dimension, are called the *non-jump indices*, which we typically denote as t_j.

The generic orbits, jump indices, and non-jump indices all depend on the original strong Malcev basis. The following proposition, which we will make use of in Section 3, gives an alternate method for detecting non-jump indices [7].

PROPOSITION 1.2. *An index i is a non-jump index if and only if for every $\ell \in U$ there is a vector $Y \in \mathbb{R}\text{-span}\{X_1, ..., X_{i-1}\}$ such that $X_i + Y \in r_\ell$. Here r_ℓ is the radical of ℓ.*

Let S, T be the collection of jump and non-jump indices, respectively, and let $V_S = \mathbb{R}\text{-span}\{\ell_{s_i} \mid s_i \in S\}$, $V_T = \mathbb{R}\text{-}span\{\ell_{t_i} \mid t_i \in T\}$. Then V_S has the same dimension as the generic orbits, and, for a generic orbit O_ℓ, orthogonal projection maps O_ℓ diffeomorphically onto V_S. From this it can be seen that a convenient way to choose orbit representatives is to take members of the Zariski-open set $U \cap V_T$.

The preceding remarks form the basis for the construction of the parameterizing map $\psi(\ell_T, \ell_S)$: $(U \cap V_T) \times V_S \to U$. ψ is rational in ℓ_T, and, for fixed ℓ_T, $\psi(\ell_T, \cdot)$ is a polynomial diffeomorphism onto the orbit O_{ℓ_T} of ℓ_T. If p is projection onto V_S, the point $\psi(\ell_T, \ell_S)$ is the unique point in the orbit O_{ℓ_T} such that $p(\psi(\ell_T, \ell_S)) = \ell_S$. The functionals from $U \cap V_T$ are called the parameterizing functionals.

2. EUCLIDEAN MÜNTZ–SZASZ THEOREMS

We will proceed using Proposition 1.1 as a template for a Müntz–Szasz Theorem on nilpotent groups. In this case the matrix coefficients of some fixed φ on the group will play the role of the Euclidean Fourier transform. To make use of the classic Müntz–Szasz theorem we need to address two problems that will arise.

The smallest non-Euclidean nilpotent Lie group is the Heisenberg group of dimension three, while the classic Müntz–Szasz theorem is stated in one dimension. More importantly Müntz–Szasz theorems are always stated on a subset of $[0, \infty)$, since a Müntz–Szasz theorem on $[-1, 1]$, for example, would in general be false (consider the sequence of all even powers against the function $f(x)=x$). On the other hand, our matrix coefficients are given as an integral over a nilpotent Lie group. This integral introduces a polynomial q inside of our function ψ, destroying any *a priori* knowledge of where the support of φ lies. The following definition allows us to reformulate the Euclidean Müntz–Szasz theorem to include functions of arbitrary compact support.

DEFINITION 2.1. A Müntz–Szasz sequence is a sequence of integers $1 < n_1 < n_2 < \cdots$ such that $\sum_{k=1}^{\infty} 1/n_{e_k} = \sum_{k=1}^{\infty} 1/n_{o_k} = \infty$. Here $\{n_{e_k}\}_{k=1}^{\infty}$, and $\{n_{o_k}\}_{k=1}^{\infty}$ are the subsequences of even and odd terms of $\{n_k\}_{k=1}^{\infty}$, respectively.

Using Müntz–Szasz sequences, we can extend the classic Müntz–Szasz theorem to $[-b, b]$ for arbitrary real b. The extension from $[-b, b]$ to an n-dimensional box, with arbitrary sides, follows from standard real analysis arguments. As in Proposition 1.1, we can translate the Müntz–Szasz theorem on n-space to the n-dimensional Euclidean Fourier transform. We are able to pass to a function f of arbitrary compact support because the support of f is contained in some n-dimensional box. These arguments are the content of the following theorem.

PROPOSITION 2.2. *Let $\mu_1, ..., \mu_n$ be n Müntz–Szasz sequences. Suppose that $f \in L_c^2(\mathbb{R}^n)$ and s is a fixed element of $\mathbb{C}^n$. Then*

$$\frac{\partial^{\mu_1(k_1)+\cdots+\mu_n(k_n)}}{\partial s_1^{\mu_1(k_1)} \cdots \partial s_n^{\mu_n(k_n)}} \hat{f}(s_1, ..., s_n) = 0,$$

for every $k_1, ..., k_n \in \mathbb{N}^n$, implies $f = 0$ if and only if each μ_i is a Müntz–Szasz sequence.

We have lost some generality in the sequences that we may consider from the classic Müntz–Szasz theorem; however, the gain in the class of functions that the theorem will apply to is necessary for the generalization to nilpotent groups.

3. ALMOST STRONG MALCEV BASIS

In order to talk about derivatives of operator valued Fourier transforms, we will make use of group coordinates arising from a weak Malcev basis passed through a fixed polarizer m. In this section we construct a new type of weak Malcev basis to make the coordinate system as simple as possible. Since this new basis will have many of the properties of a strong Malcev basis, we call it an *almost strong Malcev basis*. Notice that the construction of this basis does not depend on m being a polarizer, only that m is a subalgebra.

One of the goals of this section will be to derive a formula for the action of the generic representations on a Hilbert space, using almost strong Malcev bases. As expected, the action of π_ℓ has the form

$$(\pi_\ell(x)(f))(y) = e^{2\pi i(\sum_{j=1}^{a} \alpha_{i_j}(x,\, y)\, \ell(M_j))} f(\beta_{e_1}(x, y), ..., \beta_{e_c}(x, y)),$$

where a is the dimension of a polarizer for generic ℓ, c is the dimension of a crosssection for the polarizer, and α_i, β_i, are polynomials. By making use of an almost strong Malcev basis, we can ensure that the polynomials α and β resulting from factoring the right translation by x have nice triangularity conditions:

$$\alpha_{i_k}(x, y) = x_{i_k} + A_{i_k}(x_{i_k+1}, ..., x_n, y_{e_s} \text{ with } e_s \geqslant i_k)$$

$$\beta_{e_k}(x, y) = x_{e_k} + y_{e_k} + B_{e_k}(x_{e_k+1}, ..., x_n, y_{e_{k+1}}, ..., y_{e_c}),$$

where A and B are polynomials in x and y. The indices i_k and e_k will be explained in the next section. Throughout this chapter $\mathfrak{G}$ will be a connected, simply connected nilpotent Lie group with corresponding Lie algebra $\mathfrak{g}$.

3.1. *Defining Almost Strong Malcev Basis*

In this section we will construct a basis for $\mathfrak{g}$, passing through a fixed subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, such that the basis acts like a strong Malcev basis in $\mathfrak{h}$ and also as a basis for the cross section $\mathfrak{h}\backslash\mathfrak{g}$.

We start with a fixed, but arbitrary, strong Malcev basis $\{X_1, ..., X_n\}$ of $\mathfrak{g}$. Choose a linear (vector space) basis $M = \{M_1, ..., M_a\}$ of $\mathfrak{h}$. Since we are considering M as a linear basis, we can assume that the vectors M_k have the form

$$M_k = X_{i_k} + \sum_{j=1}^{i_k - 1} \omega_j^k X_j,$$

for a distinct indices $i_i < \cdots < i_a$.

Denote by I the ordered set of indices $\{i_i, ..., i_a\}$ and by $E = \sim I$ the (ordered) set $\{e_1, ..., e_c\}$ of indices not in I. We may further triangularize this basis by assuming that

$$M_k = X_{i_k} + \sum_{\substack{j=1 \\ j \notin I}}^{i_k - 1} \omega_j^{i_k} X_j = X_{i_k} + \sum_{e_p < i_k} \omega_{e_p}^{i_k} X_{e_p},$$

again considering our basis as a linear basis. We are now in a position to extend our basis to a weak Malcev basis of $\mathfrak{g}$. We will use the following lemma to show that $\{M_1, ..., M_a\}$ is a strong Malcev basis for $\mathfrak{h}$ that extends in a nice way to a weak Malcev basis for $\mathfrak{g}$. Weak Malcev is the best we can do, since we are not assuming that $\mathfrak{h}$ is an ideal.

Lemma 3.1.1. *Let $V_p = \sum_{s=1}^{p} v_s X_s$ be an arbitrary vector in $\mathfrak{g}$, with p chosen such that $v_p \neq 0$. Then*

$$[X_j, V_p] \in \mathbb{R}\text{-span}\{M_s \textit{ with } i_s < A, X_{e_s} \textit{ with } e_s < A\}$$

where $A = \min\{j, p\}$.

Proof. Notice that

$$\mathbb{R}\text{-span}\{M_s \text{ with } i_s < A, X_{e_s} \text{ with } e_s < A\} = \mathbb{R}\text{-span}\{X_1, ..., X_{A-1}\}.$$

The lemma follows since $\{X_1, ..., X_n\}$ is a strong Malcev basis. ∎

Theorem 3.1.2. *$B = \{M_1, ..., M_a, X_{e_1}, ..., X_{e_c}\}$ is a weak Malcev basis for $\mathfrak{g}$ passing through $\mathfrak{h}$.*

Proof. The fact that the i_j and e_k are all distinct shows that B is a vector space basis for $\mathfrak{g}$. It remains to show that the basis is weak Malcev.

Claim. $\{M_1, ..., M_a\}$ is a strong Malcev basis of $\mathfrak{h}$.

Proof. By construction $\{M_1, ..., M_a\}$ is a linear basis of $\mathfrak{h}$. Since $\mathfrak{h}$ is an algebra, we also know that $[M_j, M_k]$ is in the span of the M_i. Lemma 3.1.1 now shows that this basis is strong Malcev. ∎

To complete the proof of the theorem it suffices to check that:

(1) $[X_{e_j}, X_{e_k}] \in \mathbb{R}\text{-span}\{M_1, ..., M_a, X_{e_1}, ..., X_{e_d}\}$ where $d = \min\{j, k\}$.

(2) $[X_{e_j}, M_k] \in \mathbb{R}\text{-span}\{M_1, ..., M_a, X_{e_1}, ..., X_{e_j}\}$.

This has been shown in Lemma 3.1.1. ▮

We have proven the following theorem:

THEOREM 3.1.3. *Let* $\{X_1, ..., X_n\}$ *be a strong Malcev basis for the Lie algebra* $\mathfrak{g}$. *Let* $\mathfrak{h}$ *be an a-dimensional subalgebra of* $\mathfrak{g}$. *Then there exist two disjoint collections of indices* $I = \{i_1, ..., i_a\}$ *and* $E = \{e_1, ..., e_c\}$, *along with vectors* $\{M_1, ..., M_a\}$, *such that*:

(1) $M_s = X_{i_s} + \sum_{e_p < i_s} \omega^{i_s}_{e_p} X_{e_p}$.

(2) $\{M_1, ..., M_a\}$ *is a strong Malcev basis for* $\mathfrak{h}$.

(3) $\{M_1, ..., M_a, X_{e_1}, ..., X_{e_c}\}$ *is a weak Malcev basis for* $\mathfrak{g}$ *through* $\mathfrak{h}$.

DEFINITION 3.1.4. An almost strong Malcev basis for $\mathfrak{g}$, passing through $\mathfrak{h}$, will be a basis satisfying the conditions of Theorem 3.1.3.

THEOREM 3.1.5. *Let* $\mathfrak{h}$ *be an a-dimensional subalgebra of* $\mathfrak{g}$. *The almost strong Malcev basis in Definition* 3.1.4 *is unique up to the choice of strong Malcev basis* $\{X_1, ..., X_n\}$.

Proof. Suppose that there are two bases $P = \{M^P_1, ..., M^P_a, X^P_{e_1}, ..., X^P_{e_c}\}$ and $Q = \{M^Q_1, ..., M^Q_a, X^Q_{e_1}, ..., X^Q_{e_c}\}$ satisfying the hypothesis of Theorem 3.1.3.

Claim. To show that $P = Q$, it suffices to show that the internal indices, $I^P = \{i^P_1, ..., i^P_a\}$ and $I^Q = \{i^Q_1, ..., i^Q_a\}$, for these two bases are the same.

Proof of Claim. Suppose that $I^P = I^Q$, which implies that $E^P = E^Q$. Therefore $X^P_{e_j} = X^Q_{e_j}$ for $j = 1, ..., c$, so we may drop the superscript on these vectors. Since the internal indices are the same, by property 2 of almost strong bases $M^P_k - M^Q_k \in \mathbb{R}\text{-span}\{X_{e_j} \mid e_j \in E\}$. Of course we also have $M^P_k - M^Q_k \in \mathfrak{h}$, and, by the linear independence of the vectors $\{M^P_1, ..., M^P_a, X^P_{e_1}, ..., X^P_{e_c}\}$, we must have $M^P_k - M^Q_k = 0$. ▮

The proof that $i^P_k = i^Q_k$ proceeds by induction on k. So consider i^P_1, i^Q_1. We may as well assume that $i^P_1 < i^Q_1$. By the definition of almost strong Malcev basis, $i^P_1 < i^Q_1 < i^Q_2 < \cdots < i^Q_a$. If this were true the vectors $\{M^P_1, M^Q_1, ..., M^Q_a\}$ would be linearly independent, which contradicts $\dim(\mathfrak{h}) = a$. The induction step is handled similarly. ▮

The following corollary is useful for identifying members of an almost strong Malcev basis.

COROLLARY 3.1.6. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with strong Malcev basis $\{X_1, ..., X_n\}$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ and $\{M_1, ..., M_a, X_{e_1}, ..., X_{e_c}\}$ the almost strong Malcev basis for $\mathfrak{g}$ through $\mathfrak{h}$. If a vector $X \in \mathfrak{h}$ has the form $X = X_j + \sum_{e_p < j} \alpha_{e_p} X_{e_p}$, then $X = M_k$ for some k.*

Proof. For each $1 \leqslant k \leqslant a$ express M_k in terms of the original strong Malcev basis: $M_k = X_{i_k} + \sum_{e_p < i_k} \omega^{i_k}_{e_p} X_{e_p}$. We must have $j = i_k$ for some k, for otherwise X would be linearly independent from all of the M_k. Therefore $X - M_k \in \mathfrak{h} \cap (\mathbb{R}\text{-span}\{X_{e_p} \mid e_p \in E\}) = \{0\}$. ∎

Strong Malcev coordinates are generally preferable to weak Malcev coordinates because of the resulting triangular dependencies. In general, when using weak Malcev coordinates, the most that can be said is that the resulting coordinate system is polynomial. In the next theorem we exploit the fact that, when using almost strong Malcev coordinates, we get triangular dependencies similar to those from strong Malcev coordinates. As a result, in Corollary 3.1.9, we give a partial answer to a question proposed by Corwin and Greenleaf [7, p. 23].

THEOREM 3.1.7. *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, and let $M = \{M_1, ..., M_a, X_{e_1}, ..., X_{e_c}\}$ be the almost strong Malcev basis for $\mathfrak{g}$ through $\mathfrak{h}$. Define $\varphi\colon \mathbb{R}^c \to \mathfrak{G}$ by*

$$\varphi(s_e) = \varphi(s_{e_1}, ..., s_{e_c}) = \exp\left(\sum_{j=1}^{s} s_{e_j} X_{e_j}\right).$$

Then the map $\phi(s) = \mathfrak{H} \cdot \varphi(s)$ is an onto diffeomorphism.

Proof. Label the internal indices for the almost strong Malcev basis as $\{i_1, ..., i_a\}$ and the external indices as $\{e_1, ..., e_c\}$.

Define the map $\Phi\colon \mathbb{R}^n \to \mathfrak{G}$ by the equation

$$\Phi(s) = \exp\left(\sum_{k=1}^{a} s_{i_k} M_k\right) \exp\left(\sum_{j=1}^{c} s_{e_j} X_{e_j}\right) = \tilde{\varphi}(s_i) \cdot \varphi(s_e).$$

Since the internal and external indices form two disjoint sets, and the internal indices are in one-to-one correspondence with the M_k, Φ is well defined.

LEMMA 3.1.8. *Φ is a polynomial diffeomorphism onto $\mathfrak{G}$, with polynomial inverse, such that*

$$\Phi(s) = \exp\left(\sum_{j=1}^{n} P_j(s)\, X_j\right),$$

where the polynomials P_j *satisfy* $P_j(s) = s_j + \tilde{P}_j(s_{j+1}, ..., s_n)$ *for some polynomial* $\tilde{P}_j$.

Proof. The proof is very similar to the proof that strong Malcev coordinates are polynomial with polynomial inverse. Here we will write the basis vector M_k in terms of the original strong Malcev basis:

$$\begin{aligned}\Phi(s_i, s_e) &= \exp\left(\sum_{k=1}^{a} s_{i_k}\left(X_{i_k} + \sum_{e_p < i_k} \omega_{e_p}^{i_k} X_{e_p}\right)\right) \exp\left(\sum_{j=1}^{c} s_{e_j} X_{e_j}\right)\\ &= \exp\left(\sum_{k=1}^{n} P_k(s)\, X_k\right).\end{aligned}$$

By virtue of the Campbell–Baker–Hausdorff formula, and the fact that in our setup vectors bracket down, the polynomials P have the form

$$P_k(s) = \begin{cases} s_{i_j} + \tilde{P}_{i_j}(s_{i_{j+1}}, ..., s_{i_a}, s_{e_l} \text{ with } e_l > i_j) & \text{if } \ k = i_j \quad \text{for some } j,\\ s_{e_j} + \tilde{P}_{e_j}(s_{e_{j+1}}, ..., s_{e_c}, s_{i_l} \text{ with } i_l > e_j) & \text{if } \ k = e_j \quad \text{for some } j. \end{cases}$$

Notice that $\{i_{i+1}, ..., i_a\} \cup \{e_l \mid e_l > i_j\} = \{i_j + 1, ..., n\}$. So the polynomials P may be rewritten as:

$$\begin{aligned}P_k(s) &= \begin{cases} s_{i_j} + \tilde{P}_{i_j}(s_{i_j+1}, ..., s_n) & \text{if } \ k = i_i \quad \text{for some } j\\ s_{e_j} + \tilde{P}_{e_j}(s_{e_j+1}, ..., s_n) & \text{if } \ k = e_j \quad \text{for some } j \end{cases}\\ &= s_k + \tilde{P}_k(s_{k+1}, ..., s_n).\end{aligned}$$

The proof of the lemma now follows from the triangularity of the polynomials P_j. The equations $t_j = P_j(s)$ can be solved recursively, starting with $t_n = P_n(s)$, for polynomials $\tilde{Q}$ such that

$$s_j = t_j + \tilde{Q}(t_{j+1}, ..., t_n).$$

This shows that Φ has a polynomial inverse. ∎

Define $\pi: \mathbb{G} \to \mathfrak{H}\backslash\mathfrak{G}$ as the canonical quotient map. Then:

$$\phi(s_{e_1}, ..., s_{e_c}) = \mathfrak{H} \cdot \exp\left(\sum_{j=1}^{c} s_{e_j} X_{e_j}\right) = \pi(\varphi(s_e)).$$

This shows that ϕ is analytic. Since Φ and π are onto, and $\pi(\Phi(s_i, s_e)) = \pi(\Phi(s_e))$ for any $s_i = (s_{i_1}, ..., s_{i_a})$, we can also see that ϕ is onto.

To show that ϕ is one-to-one, suppose that $\phi(s_e) = \phi(t_e)$. This is equivalent to saying that $\varphi(s_e) = h \cdot \varphi(t_e)$ for some $h \in \mathfrak{H}$. Because $\tilde{\varphi}$ is onto $\mathfrak{H}$, we can choose t_i such that $h = \tilde{\varphi}(t_i)$. Then $\varphi(s_e) = \tilde{\varphi}(t_i) \cdot \varphi(t_e)$, and, since Φ

is one-to-one, we must have $s_e = t_e$ and $t_i = 0$, which is what we wanted to show.

We know that ϕ is a homeomorphism. It remains to show that ϕ is a diffeomorphism. Since the map Φ is a diffeomorphism, the vectors

$$\left\{ \frac{\partial \Phi}{\partial s_{i_k}}, \frac{\partial \Phi}{\partial s_{e_j}} \,\middle|\, 1 \leqslant k \leqslant a, 1 \leqslant j \leqslant c \right\}$$

are linearly independent at every point $s \in \mathbb{R}^n$. The map $d\pi$ carries these vectors onto a spanning set for the tangent space of $\mathfrak{H}\backslash\mathfrak{G}$. However $\pi(\tilde{\varphi}(s_i)) = \pi(\Phi(s_i, 0)) = \mathfrak{H}$ for every s_i. It follows that

$$d\pi \left(\frac{\partial \tilde{\varphi}}{\partial s_{i_k}} \right) = 0$$

for $1 \leqslant k \leqslant a$.

Therefore the vectors $\{ d\pi(\partial \phi / \partial s_{e_1}), ..., d\pi(\partial \phi / \partial s_{e_s}) \}$ form a spanning set for $\mathfrak{H}\backslash\mathfrak{G}$. Since the dimension of $\mathfrak{H}\backslash\mathfrak{G}$ is c, these vectors actually form a basis. But $\phi = \pi(\varphi)$.

This shows that the vectors $\partial \phi / \partial s_{e_k}$ are c linearly independent vectors tangent to $\mathfrak{H}\backslash\mathfrak{G}$, which is what we needed to show. ∎

COROLLARY 3.1.9. *In the almost strong Malcev basis* $M = \{ M_1, ..., M_a, X_{e_1}, ..., X_{e_c} \}$, $S = \{ \sum_{j=1}^{c} t_j X_{e_j} \mid t_j \in \mathbb{R} \}$ *is a cross-section for* $\mathfrak{H}\backslash\mathfrak{G}$.

3.2. *Calculating the Action of Unitary Representations*

We will use an almost strong Malcev basis, passing through a polarizer, to calculate the action of an irreducible unitary representation on the group $\mathfrak{G}$. In this section we will show that the action of such a representation has the form outlined in the beginning of this section.

Fix $\ell = \sum_{j=1}^{n} \ell_j X_j^* \in \mathfrak{g}^*$, a polarizer m for ℓ, and let $\mathfrak{M} = \exp(m)$—the Lie subgroup of $\mathfrak{G}$ corresponding to m. Then π_ℓ acts on the Hilbert space H_{π_ℓ} by right translation:

$$\pi_\ell(x) f(y) = f(yx)$$

for $x, y \in \mathfrak{G}$. Here H_{π_ℓ} is the collection of functions f with $|f| \in L^2(\mathfrak{M}\backslash\mathfrak{G})$ such that

$$f(ab) = \chi_\ell(a) f(b) = e^{2\pi i \ell(\log(a))} f(b),$$

for every $a \in \mathfrak{M}$ and $b \in \mathfrak{G}$. Notice that it is sufficient to calculate $\pi_\ell(x) f(y)$ for y coming from a cross-section of $\mathfrak{M}\backslash\mathfrak{G}$.

The calculation of π_ℓ amounts to factoring an arbitrary product yx, $x \in \mathfrak{G}$, y in a cross section for $\mathfrak{M}\backslash\mathfrak{G}$, into a product $\alpha\beta$ where $\alpha \in \mathfrak{M}$ and β comes from the cross-section for $\mathfrak{M}\backslash\mathfrak{G}$.

Let $\{M_1, ..., M_a, X_{e_1}, ..., X_{e_c}\}$ be the almost strong Malcev basis for $\mathfrak{g}$ passing through m, with respect to the fixed strong Malcev basis $\{X_1, ..., X_n\}$, as in Theorem 3.1.3. We will use $S = \{\exp(\sum_{k=1}^{c} t_{e_k} X_{e_k}) \mid t \in \mathbb{R}^c\}$ for the cross-section of $\mathfrak{M}\backslash\mathfrak{G}$.

Let $x = \exp(\sum_{k=1}^{n} x_k X_k) \in \mathfrak{G}$ and $y = \exp(\sum_{k=1}^{c} y_{e_k} X_{e_k}) \in S$. Choose the unique $\alpha = \exp(\sum_{k=1}^{a} \alpha_{i_k} M_k) \in M$ and $\beta = \exp(\sum_{k=1}^{a} \beta_{e_k} X_{e_k}) \in S$ such that $yx = \alpha\beta$.

We need to calculate α_{i_j} and β_{e_k} in terms of the coordinates x_s and y_t. First we calculate yx. By the Campbell–Baker–Hausdorff formula we have

$$yx = \exp\left(\sum_{k=1}^{n} p_k(x, y)\, X_k\right),$$

where

$$p_k(x, y) = \begin{cases} x_{i_j} + \tilde{p}_{i_j}(x_{i_j+1}, ..., x_n, y_{e_s} \text{ with } e_s \geqslant i_j) \\ \qquad \text{if} \quad k = i_j \quad \text{for some } j \\ x_{e_j} + y_{e_j} + \tilde{p}_k(x_{e_j+1}, ..., x_n, y_{e_j+1}, ..., y_{e_c}) \\ \qquad \text{if} \quad k = e_j \text{ for some } j. \end{cases}$$

Using Lemma 3.1.8 we can express the product $\alpha\beta$ in polynomial coordinates:

$$\alpha\beta = \Phi(\alpha_i, \beta_e) = \exp\left(\sum_{j=1}^{n} P_j(\alpha_i, \beta_e)\, X_j\right)$$

(We have abused notation here, since the coordinates α_i and β_e are out of sequence with the coordinates of Φ.) Lemma 3.1.8 shows that the polynomials P_j have the form

$$P_j(\alpha, \beta) = \begin{cases} a_{i_k} + \tilde{P}_{i_k}(\alpha_{i_{k+1}}, ..., \alpha_{i_a}, \beta_{e_p} \text{ with } e_p > i_k) \\ \qquad \text{if} \quad j = i_k \text{ for some } k; \\ \beta_{e_k} + \tilde{P}_{e_k}(\alpha_{i_p} \text{ with } i_p > e_k, \beta_{e_k+1}, ..., \beta_{e_c}) \\ \qquad \text{if} \quad j = e_k \text{ for some } k. \end{cases}$$

Since α and β were chosen so that $yx = \alpha\beta$, we must have $p_k(x, y) = P_k(\alpha, \beta)$ for $k = 1, ..., n$. Use this to solve for α and β. By the triangular dependencies of the polynomials p_k and P_k, there are polynomials A and B such that

$$\alpha_{i_k}(x, y) = x_{i_k} + A_{i_k}(x_{i_k+1}, ..., x_n, y_{e_s} \text{ with } e_s \geqslant i_k),$$

$$\beta_{e_k}(x, y) = x_{e_k} + y_{e_k} + B_{e_k}(x_{e_k+1}, ..., x_n, y_{e_{k+1}}, ..., y_{e_c}).$$

We have proven the following:

THEOREM 3.2.1. *Let $\mathfrak{g}$ be a connected, simply connected nilpotent Lie algebra with strong Malcev basis $\{X_1, ..., X_n\}$. Let $\ell \in \mathfrak{g}^*$ and let m be a polarizer for ℓ. Finally let $\{M_1, ..., M_a, X_{e_1}, ..., X_{e_c}\}$ be the almost strong Malcev basis for $\mathfrak{g}$ through m. If we use this basis to calculate the action of π_ℓ on $L^2(\mathbb{R}^c)$, then that action is given by the formula*

$$\pi_\ell(x)\, f(y) = e^{2\pi i(\sum_{j=1}^{a} \alpha_{i_j}(x, y)\, \ell(M_j))} f(\beta_{e_1}(x, y), ..., \beta_{e_c}(x, y)).$$

In addition there exist polynomials A, B such that

$$\alpha_{i_k}(x, y) = x_{i_k} + A_{i_k}(x_{i_k+1}, ..., x_n y_{e_s} \textit{ with } e_s > i_k),$$

$$\beta_{e_k}(x, y) = x_{e_k} + y_{e_k} + B_{e_k}(x_{e_k+1}, ..., x_n, y_{e_{k+1}}, ..., y_{e_c}).$$

In the calculations leading up to the proof of Theorem 3.2.1 we treated the coefficents ω as constants (the constants that give the change of basis from the original strong Malcev basis to the almost strong Malcev basis). However, in this section we are dealing with subalgebras that may depend on the choice of dual vector ℓ. In this case the polarizers are said to rotate with ℓ. In terms of the almost strong Malcev basis through the polarizer, this means that the coefficients ω are dependent on ℓ. It is known that the coefficients ω can be chosen to vary rationally with ℓ [1]. It is an interesting exercise to go back and repeat our calculations, this time keeping track of the ω. The polynomials α and β of Theorem 3.2.1 will also be polynomial in ω, and hence rational in ℓ. This fact can be used to give an alternate proof for the Paley–Wiener theorem for nilpotent Lie groups.

In the next section we will make use of Theorem 3.2.1 when $\ell \in U \cap V_T$, the set of parameterizing functionals, and there is one fixed polarizer m for all such ℓ. As a consequence we will not need to keep track of the coefficients ω.

3.3. *Calculating the Action of Parameterizing Representations*

We are going to prove a Müntz–Szasz theorem, for the matrix coefficients of the operator valued Fourier transform, as functions of the parameterizing functionals. By making this limitation we can make further refinements to Theorem 3.2.1. As in Section 3.1, we will start with a fixed strong Malcev basis $\{X_1, ..., X_n\}$ of the Lie algebra $\mathfrak{g}$. With respect to this basis, we will let U denote the collection of generic orbits and $S = \{s_1, ..., s_0\}$, $T = \{t_1, ..., t_r\}$ will denote the collection of jump and non-jump indices respectively. Then 0 is the dimension of the generic orbits, and r is the dimension of the radical r_ℓ for generic $\ell \in \mathfrak{g}^*$. If we let $V_T = \{X^*_{t_i} \mid t_i \in T\}$, then $U \cap V_T$ is the collection of parameterizing functionals.

In the previous section we showed that

$$\pi_{\ell_T}(x)\, f(y) = e^{2\pi i(\sum_{j=1}^{a} \alpha_{i_j}(x,\, y)\, \ell_T(M_j))} f(\beta_{e_1}(x, y), ..., \beta_{e_c}(x, y)).$$

Here $M = \{M_1, ..., M_a, X_{e_1}, ..., X_{e_c}\}$ is an almost strong Malcev basis for $\mathfrak{g}$ through a polarizer m for ℓ_T. *We will assume that there is one fixed polarizer m for all $\ell_T \in U \cap V_T$.* In this case the polynomials α, β will be independent of ℓ_T, as we have been writing them.

When an almost strong Malcev basis is passed through a polarizer for a generic representation, the set T of non-jump indices must be contained in the set of internal indices I, which is what the next lemma shows.

LEMMA 3.3.1. *For each $t_i \in T$ there is a vector $Y_{t_i} \in \mathbb{R}$-span$\{X_{e_k} \mid e_k < t_i\}$ such that $X_{t_i} + Y_{t_i}$ is an element of our almost strong Malcev basis M through m.*

Proof. Fix a parameterizing functional $\ell \in U \cap V_T$. By Proposition 1.2, for each $t_j \in T$ we may choose a vector $Y_{t_j} \in \mathbb{R}$-span$\{X_1, ..., X_{t_j - 1}\}$ such that $X_{t_j} + Y_{t_j} \in r_\ell$.

Define the projection map P: $\mathfrak{g} \to \mathbb{R}$-span$\{X_{e_1}, ..., X_{e_c}\}$ with respect to the almost strong Malcev basis M. Let $\tilde{Y}_{t_j} = P(Y_{t_j})$, so that $Y_{t_j} = \tilde{Y}_{t_j} + M_{t_j}$ for some vector $M_{t_j} \in m$. That is, we still have $X_{t_j} + \tilde{Y}_{t_j} \in m$ and this vector has the correct form to be part of a strong Malcev basis. By Corollary 3.1.6, $X_{t_j} + \tilde{Y}_{t_j} = M_k$ for some k. ∎

Remark. Lemma 3.3.1 shows that the non-jump indices are contained in the internal indices for the almost strong Malcev basis. By taking set theoretic compliments, we can see that the external indices for the basis are always jump indices. Notice that Lemma 3.3.1 does not depend on the polarizer being fixed. In terms of the sets we have defined, $E \subseteq S$ and $T \subseteq I$.

These comments allow us to calculate $\ell_T(M_j)$ in the action of π_{ℓ_T}.

THEOREM 3.3.2. *Suppose that there is one fixed polarizer for the parameterizing functionals $\ell_T = \sum_{j=1}^{r} \ell_{t_j} X^*_{t_j} \in U \cap V_T$. Then π_{ℓ_T} can be modeled in* *a fixed modeling space* $L^2(\mathbb{R}^c) \cong L^2(\sum_i exp(\mathbb{R}X_{e_i}))$. *The action of* π_{ℓ_T} *is given by*

$$\pi_{\ell_T}(x)\, f(y) = e^{2\pi i(\sum_{j=1}^{r} \ell_{t_j}(x_{t_j} + A_{t_j}(x_{t_j+1}, ..., x_n, y_k \text{ with } e_k > t_j)))} \cdot f(\beta_{e_1}(x, y), ..., \beta_{e_c}(x, y)),$$

where $\beta_{e_k}(x, y) = x_{e_k} + y_{e_k} + B_{e_k}(x_{e_k+1}, ..., x_n;\, y_{e_{k+1}}, ..., y_{e_c})$, A_{t_j}, *and* B_{e_k} *are polynomial in* x, y.

Proof. By the remarks preceding the theorem

$$\ell_T(M_j) = \ell_T\left(X_{i_j} + \sum_{e_p < i_j} \omega^{i_j}_{e_p} X_{e_p}\right) = \ell_T\left(X_{i_j}\right) = \begin{cases} 0 & \text{if} \quad i_j \in T \\ \ell_{t_k} & \text{if} \quad i_j = t_k \quad \text{for some } k. \end{cases}$$

On the other hand, Lemma 3.3.1 shows that each ℓ_{t_k}, is represented by $\ell_T(M_j)$ for some j. The theorem now results from Theorem 3.2.1. ▮

4. A MÜNTZ–SZASZ THEOREM FOR MATRIX COEFFICIENTS

In Section 2 we proved a Müntz–Szasz theorem for compactly supported L^2 functions in $\mathbb{R}^n$. In this section we will prove a similar theorem for compactly supported L^2 functions, on a certain class of connected, simply connected nilpotent Lie groups. The class of groups we will deal with throughout this section is the class mentioned in the previous section: *the groups that have one fixed polarizer for all of the parameterizing functionals* $\ell_T \in U \cap V_T$. For example, this includes all of the n-step chain groups.

For $\varphi \in L^2(\mathfrak{G}) \cap L^1(\mathfrak{G})$, and $f, g \in L^2(\mathbb{R}^c)$, we define the matrix coefficient of the operator valued transform $\tilde{\varphi}_{op}(\ell_T) = \pi_{\ell_T}(\varphi)$ by

$$\langle \pi_{\ell_T}(\varphi)\, f, g\rangle = \int_{\mathfrak{G}} \varphi(x) \langle \pi_{\ell_T}(x)\, f, g\rangle\, dx.$$

In fact this relation also defines $\pi_{\ell_T}(\varphi)$. The next theorem shows that, if we use almost strong Malcev coordinates to model the parameterizing representations π_{ℓ_T} in the fixed modeling space $L^2(\mathbb{R}^c)$, these matrix coefficients are actually Euclidean Fourier transforms of certain $L^2 \cap L^1$ functions. Theorem 4.3 is a Müntz–Szasz theorem for the matrix coefficients of compactly supported φ, with some restrictions on the choice of basis vectors. Theorem 4.4 is a Müntz–Szasz theorem for the operator valued transform of compactly supported φ.

Recall that $T=\{t_1, ..., t_r\}$ and $S=\{s_1, ..., s_o\}$ are the non-jump and jump indices, respectively, r is the dimension of the polarizer for generic ℓ, and o is the dimension of the generic orbits.

THEOREM 4.1. *Assume $\varphi \circ \exp \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $f, g \in L^2(\mathbb{R}^c)$. In addition assume $g \in L^\infty(\mathbb{R}^c)$. Then there, exist polynomials q and β such that*

$$\begin{aligned}\langle \pi_{\ell_T}(\varphi) f, g\rangle &= \int_{\mathbb{R}^{n+c}} e^{2\pi i \sum_{j=1}^r l_{t_j} x_{t_j}} \varphi(q(x, y))\, f(\beta(x, y))\, g(y)\, dy\, dx \\ &= \hat{\Phi}_{f, g}(\ell_T)\end{aligned}$$

where:

$$(1) \quad q_j(x, y) = \begin{cases} x_{t_i} - A_{t_i}(x_{t_i+1}, ..., x_n; y_{e_k} \text{ with } e_k > t_i) \\ \qquad \text{if } \quad j = t_i \quad \text{for some } i; \\ x_{s_i} \qquad \text{if } \quad j = s_i \quad \text{for some } i. \end{cases}$$

$$(2) \quad \beta_{e_j}(x, y) = x_{e_j} + y_{e_j} + \tilde{\beta}_{e_j}(x_{e_j+1}, ..., x_n; y_{e_{j+1}}, ..., y_{e_c}).$$

$$(3) \quad \Phi_{f, g}(x_T) = \int_{\mathbb{R}^{3c}} \varphi(q(x, y))\, f(\beta(x, y))\, g(y)\, dy\, dx_S \in L^1(\mathbb{R}^r) \cap L^2(\mathbb{R}^r).$$

(4) *If, in addition to the conditions of the theorem, φ and g are compactly supported, then so is $\Phi_{f, g}$.*

Remarks. Here c is the codimension of the polarizer and $x_T = (x_{t_1}, ..., x_{t_r})$, $x_S = (x_{s_1}, ..., x_{s_o})$. In (5), $3c$ is the correct dimension since $n + c - r = (n - r) + c = 2c + c = 3c$. The polynomials A_{t_i} are the polynomials from Theorem 3.3.2.

Proof. In Theorem 3.3.2 we calculated the action of π_{ℓ_T} using an almost strong Malcev basis. We use that calculation here to calculate the matrix coefficient $\hat{\Phi}_{f, g}$:

$$\langle \pi_{\ell_T}(\varphi) f, g\rangle = \int_{\mathbb{R}^{n+c}} e^{2\pi i (\sum_{j=1}^r \ell_{t_j}(z_{t_j} + A_{t_j}(z, y)))} \varphi(z)\, f(\beta(z, y))\, g(y)\, dy\, dz. \tag{1}$$

For $z \in \mathbb{R}^c$ define f_z as $f_z(y) = f(\beta(z, y))$. Notice that $\|f_z\|_2 = \|f\|_2$ for almost every z. If we define $H(x, y)$ by the formula

$$H(x, y) = \varphi(q(x, y))\, f(\beta(x, y))\, g(y),$$

we see that $\|H\|_1 \leqslant \|\varphi\|_1\, \|f\|_2^2\, \|g\|_2^2$, where we have made use of the Schwarz inequality and the above remarks on f_z.

Fubini's theorem now shows that we may change the order of integration in (1), which allows us to define $\Phi_{f,g}$ by making a change of variables:

$$\langle \pi_{\ell_T}(\varphi) f, g\rangle = \int_{\mathbb{R}^{n+c}} e^{2\pi i \sum_{j=1}^{r} l_{t_j} x_{t_j}} \varphi(q(x, y)) \cdot f(\beta(x, y))\, g(y)\, dx_T\, dx_S\, dy = \hat{\Phi}_{f,g}(\ell_T).$$

Fubini's theorem implies that $\Phi_{f,g}$ is L^1. To prove (3), it remains to show that $\Phi_{f,g} \in L^2(\mathbb{R}^r)$. By undoing the change of variables that resulted in the polynomials q inside of φ, we can show that $H \in L^2(\Re^{n+e})$:

$$\begin{aligned}\|H\|_2^2 &= \int_{\mathbb{R}^{n+c}} |\varphi(q(x, y))\, f(\beta(x, y))\, g(y)|^2\, dy\, dz \\ &= \int_{\mathbb{R}^{n+c}} |\varphi(z)\, f(\beta(z, y))\, g(y)|^2\, dy\, dz \\ &\leqslant \|g\|_\infty^2\, \|\phi\|_2^2\, \|f\|_2^2.\end{aligned}$$

We use the bound in the last line to justify any necessary applications of Fubini's theorem.

It follows that $\hat{H} \in L^2(\mathbb{R}^{n+c})$. But $\hat{\Phi}_{f,g}(\ell_T) = \hat{H}(\ell_T, 0)$ (where 0 is an $n+c-r$ dimensional 0), so, by Fubini's theorem, $\Phi_{f,g} \in L^2(\mathbb{R}^r)$. (Fubini's theorem would guarantee that $\hat{H}(\ell_T, s) \in L^2(\mathbb{R}^r)$ for almost every s. Here $\hat{H}$ is the transform of an L^1 function, and hence continuous. So we may assume $s = 0$.) It follows that $\Phi_{f,g} \in L^2(\mathbb{R}^r)$. Note that the behavior of the matrix coefficient $\hat{\Phi}_{f,g}$ we have described is separate from the Plancherel theorem for nilpotent Lie groups. In our case we use Lebesgue measure on the function *and* transform side.

To prove (4), assume ϕ and g are compactly supported. It suffices to check that $H(x, y)$ is compactly supported. From the compact support of g, we can see that y is compactly supported. By the compact support of φ, and by property (3) from Theorem 4.1, we can see that x_S is compactly supported. So it remains to see that x_T is compactly supported. We will work inductively starting with x_{t_r},

Notice by the triangularity conditions on the q_j that

$$q_{t_r}(x, y) = x_{t_r} - A_{t_r}(y_{e_k} \text{ with } e_k > t_r).$$

Since the y are compactly supported, and so is φ, we must have x_{t_r} compactly supported. Now assume that $x_{t_r}, ..., x_{t_{i+1}}$ are compactly supported, and consider x_{t_i}. Notice that $q_{t_i}(x, y) = x_{t_i} + \hat{q}_{t_i}(x_{t_i+1}, ..., x_n;\, y_{e_k}$ with $e_k > t_i)$. $\tilde{q}_{t_i}$ is a continuous function on a compact set, and hence bounded. Therefore, by the compact support of φ, x_{t_i} must be compactly supported. ∎

COROLLARY 4.2. *If* $\varphi \in L^1(\mathfrak{G})$ *and* $f, g \in L^2(\mathbb{R}^c)$, *then*

$$\lim_{|\ell_T| \to \infty} \langle \pi_{\ell_T}(\varphi) f, g \rangle = 0.$$

Proof. By Theorem 4.1 we may use the Riemann–Lebesgue lemma for Euclidean L^1 functions. For this we need $\Phi_{f,g} \in L^1(\mathbb{R}^r)$, and the conditions we supplied here are the ones necessary to show this. ▮

We are now in a position to prove a Müntz–Szasz theorem for these matrix coefficients. For a differentiable function f of k variables, let

$$f^{(n)}(s) = f^{(n_1, \ldots, n_k)}(s_1, \ldots, s_k) = \frac{\partial^{n_1 + \cdots + n_k}}{\partial s_1^{n_1} \cdots \partial s_k^{n_k}} f(s_1, \ldots, s_k).$$

THEOREM 4.3. *Let* $\mu = (\mu_1, \ldots, \mu_r)$ *be an r-tuple of sequences and* ℓ_T *a fixed element of the complexified dual. Fix* $f, g \in L^2(\mathbb{R}^c)$. *Assume that g has compact support and is essentially bounded. Then, for compactly supported* φ, *if each of the sequences* μ_j *is a Müntz–Szasz sequence and*

$$\hat{\Phi}_{f,g}^{(\mu_1(k_1), \ldots, \mu_r(k_r))}(\ell_T) = 0,$$

for every $k_1, \ldots, k_r$, *then* $\hat{\Phi}_{f,g}(\ell_T) = 0$ *for all* ℓ_T.

Proof. Theorem 4.1 shows that $\Phi_{f,g}$ is compactly supported. The result follows from the Euclidean Müntz–Szasz theorem. ▮

Remark. As remarked earlier, although we may have *a priori* knowledge of where the support of φ lies, we lose that knowledge when we pass to $\varphi(q)$. Hence the need for a more general Müntz–Szasz theorem.

For $\mathfrak{G}$, as specified at the beginning of this section, we are ready to state a Müntz–Szasz theorem for the operator valued Fourier transform. We make use of what we know about the matrix coefficients of compactly supported functions from our previous work. We would like to write a partial derivative of the operator valued transform in terms of sums of derivatives of matrix coefficients; however, there is no guarantee that these sums will converge. We discuss this further in the remarks following the theorem.

THEOREM 4.4 (Müntz–Szasz). *Let* $\{\xi_i\}_{i=1}^{\infty} \subset L_c^2(\mathbb{R}^c) \cap L^\infty(\mathbb{R}^c)$ *be a fixed orthonormal basis of* $L^2(\mathbb{R}^c)$ *and let* $\varphi \in L_c^2(\mathfrak{G})$. *Then*:

(1) *The operator valued Fourier transform* $\hat{\varphi}_{op}(\ell_T)$: $L^2(\mathbb{R}^c) \to L^2(\mathbb{R}^c)$ *of* φ *has an entire extension to* $\mathbb{C}^r$, *in the sense that*

$$\hat{\varphi}_{op}(\ell_T)(f) = \sum_{j=1}^{\infty} \hat{\Phi}_{f, \xi_j}(\ell_T)\, \xi_j$$

and each matrix coefficient $\hat{\Phi}_{f, \xi_j}(\ell_T)$ *has such an extension.*

(2) *if, for each pair of basis vectors* $\{\xi_i, \xi_j\}$, *there exists a parameterizing functional* ℓ_T *and an r-tuple of Müntz–Szasz sequences* $\mu_{i,j}$ *such that* $\hat{\Phi}^{(\mu_{i,j})}_{\xi_i, \xi_j}(\ell_T) = 0$, *then* $\varphi = 0$.

Proof. Notice that $L^2(\mathbb{R}^c) \subset L^1(\mathbb{R}^c)$, so that (1) follows from Theorem 4.1 and the Paley–Wiener theorem. In (2), the existence of the Müntz–Szasz sequence $\mu_{i,j}$ ensures that $\hat{\Phi}_{\xi_i, \xi_j}(\ell_T) = 0$ for all ℓ_T and i, j, by Theorem 4.4. Finally,

$$\|\hat{\varphi}_{op}(\ell_T)\|^2_{H-S} = \sum_{i,j=1}^{\infty} |\hat{\Phi}_{\xi_i, \xi_j}(\ell_T)|^2 = 0,$$

for all ℓ_T. Hence

$$\|\phi\|^2_2 = \int_{V_T} \|\hat{\varphi}_{op}(\ell_T)\|^2_{H-S}\, |\mathrm{Pf}(\ell_T)|\, d\ell_T = 0,$$

by the nilpotent Plancherel theorem. ∎

Remark. Let $\mu = (\mu_1, ..., \mu_r)$ be an r-tuple of Müntz–Szasz sequences. Formally define a partial derivative of $\hat{\varphi}_{op}(\ell_T)$ by the formula

$$\hat{\varphi}^{(\mu)}_{op}(\ell_T)(f) = \sum_{j=1}^{\infty} \hat{\Phi}^{(\mu)}_{f, \xi_j}(\ell_T)\, \xi_j.$$

As remarked before the theorem, these sums do not necessarily converge; however, for this formal definition Theorem 4.4 shows that if enough of these formal derivatives vanish, for all f coming from the orthonormal basis and a fixed ℓ_T, then φ itself must vanish. In this sense Theorem 4.4 is the equivalent of a Müntz–Szasz theorem for the operator valued transform on a nilpotent Lie group.

ACKNOWLEDGEMENT

The author thanks his dissertation advisor Leonard Richardson for his suggestions on the line of thought for this paper and for his continuing advice.

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