ELLIPTIC ANOMALY IN CONSTRUCTING LONG-TERM AND SHORT-TERM DYNAMICAL THEORIES

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(Received: 28 June 2001; accepted: 23 February 2001)

Abstract. The techniques of Brumberg and Brumberg (1999) based on the use of elliptic anomaly are specified in this paper in two aspects. The iteration technique (Broucke, 1969) to construct short-term semi-analytical theories of motion in rectangular coordinates in lines of Encke and Hill is re-elaborated in terms of elliptic anomaly resulting in extending this technique for high-eccentricity orbits. In constructing long-term semi-analytical theories the key point is to integrate trigonometric functions of several angular arguments related to time by different differential expressions. This problem is reduced in the paper to linear algebraic recurrence relations admitting efficient solution by iterations.

Key words: elliptic anomaly, iteration techniques, trigonometric series integration

1. Introduction

By using the elliptic anomaly instead of any of three traditional ones (mean, true and eccentric anomalies) one may reformulate many classical techniques of the perturbation theory of celestial mechanics in attempting to make them more effective in dealing with highly eccentric orbits. On the other hand, the recent monograph (Brumberg and Brumberg, 1999) is focused on the long-term analytical and semi-analytical theories of motion when all angular (trigonometric) arguments (mean longitudes, longitudes of pericentres and nodes) preserve their literal form whereas metric variables (semi-major axes, eccentricities and inclinations) may be treated in the numerical form. The key point in constructing both the short-term and long-term semi-analytical theories of motion resides in integrating functions dependent on several angular arguments involving one or more elliptic anomalies. Such integration techniques are considered below with the aim of actual construction of short-term (Section 2) or long-term (Section 3) theories of motion.

2. Iteration Procedure

This section deals with a short-term theory constructed by iterations in rectangular coordinates as developed by Broucke (1969). More specifically, we consider here method (III) by Broucke actually representing an iteration version of the classical

Celestial Mechanics and Dynamical Astronomy **80:** 159–166, 2001.

techniques by Encke and Hill and modify it by introducing the elliptic anomaly. The final series of classical short-term theories represent trigonometric series in multiples of the fast angular variables (e.g., mean longitudes) with the polynomial with respect to time coefficients. In using the elliptic anomaly w the final series of the perturbed two-body problem represent polynomial-trigonometric series in w.

In the method (III) by Broucke (1969) (see also Brumberg, 1995) the equations of the perturbed two-body problem

$$\ddot{\mathbf{r}} = -Gm\frac{\mathbf{r}}{r^3} + \mathbf{X}, \tag{2.1}$$

are compared with the equations of the unperturbed two-body problem

$$\ddot{\boldsymbol{\rho}} = -Gm\frac{\boldsymbol{\rho}}{\rho^3}\,. \tag{2.2}$$

The correction vector $\mathbf{s}$

$$\mathbf{r} = \boldsymbol{\rho} + \mathbf{s} \tag{2.3}$$

is determined by the equation

$$\ddot{\mathbf{s}} + S\mathbf{s} = -Gm\left(\frac{\mathbf{r}}{r^3} - \frac{\boldsymbol{\rho}}{\rho^3}\right) + S(\mathbf{r} - \boldsymbol{\rho}) + \mathbf{X}\,, \tag{2.4}$$

S being an arbitrary 3×3 matrix. Method (III) by Broucke corresponds to the option

$$S = \frac{Gm}{\rho^3}E\,, \tag{2.5}$$

E being a unit matrix. In solving (2.4) by iterations

$$\mathbf{r}^{(m+1)} = \boldsymbol{\rho} + \mathbf{s}^{(m)}, \tag{2.6}$$

the right-hand side of (2.4) is regarded at each step of iterations as a known function of time $\mathbf{Q} = \mathbf{Q}(t)$. If $\mathbf{Q}^{(m)}$ results from substituting $\mathbf{r}^{(m)}$ into the right-hand side of (2.4) then a more accurate solution is obtained by (2.6) with

$$\mathbf{s}^{(m)} = \mathbf{C}_1^{(m)}q_1 + \mathbf{C}_2^{(m)}q_2 - q_1\int_0^t q_2\mathbf{Q}^{(m)}\,\mathrm{d}t + q_2\int_0^t q_1\mathbf{Q}^{(m)}\,\mathrm{d}t\,, \tag{2.7}$$

with arbitrary constants $\mathbf{C}_1^{(m)}, \mathbf{C}_2^{(m)}$ and q_1, q_2 representing two linearly independent solutions of the homogeneous Equation (2.4) without the right-hand member. One has therewith

$$q_1\dot{q}_2 - q_2\dot{q}_1 = 1\,. \tag{2.8}$$

In accordance with the choice (2.5)

$$q_1 = \frac{1}{\sqrt{n}}(\cos g - \mathrm{e}), \qquad q_2 = \frac{1}{\sqrt{n}}\sin g \tag{2.9}$$

with the mean motion n, eccentric anomaly g and the relationship to time by means of the Kepler equation

$$g - \mathrm{e} \sin g = M\,, \tag{2.10}$$

$M = nt + M_0$ being the mean anomaly. The Keplerian solution $\boldsymbol{\rho}$ of (2.2) is also expressed in terms of (2.9)

$$\boldsymbol{\rho} = \sqrt{n}a\left(\mathbf{A}_1 q_1 + \sqrt{1-\mathrm{e}^2}\mathbf{A}_2 q_2\right) \tag{2.11}$$

with the semi-major axis a, eccentricity e and unit orthogonal vectors $\mathbf{A}_1$ and $\mathbf{A}_2$.

Method (III) is very simple and sufficiently efficient for small and moderate eccentricities. Its efficiency for high eccentricities may be increased by using the elliptic anomaly w (Brumberg, 1992) defined by the relations

$$\sin g = -\operatorname{cn} u\,, \qquad \cos g = \operatorname{sn} u\,, \qquad w = \frac{\pi u}{2K} - \frac{\pi}{2}\,, \tag{2.12}$$

with modulus $k = \mathrm{e}$ of Jacobi elliptic functions and $K = K(k)$ being the complete elliptic integral of the first kind. Then the particular solutions (2.9) are presented by fast converging trigonometric series in multiples of w

$$q_1 = \frac{1}{\sqrt{n}}\left[-k + \frac{2\pi}{kK}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{q^{m-1/2}}{1-q^{2m-1}}\cos(2m-1)w\right] \tag{2.13}$$

and

$$q_2 = \frac{1}{\sqrt{n}}\frac{2\pi}{kK}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{q^{m-1/2}}{1+q^{2m-1}}\sin(2m-1)w\,, \tag{2.14}$$

$q = q(k)$ being the Jacobi nome. The Kepler Equation (2.10) takes the form

$$w + \sum_{m=1}^{\infty} d_m \sin mw = M \tag{2.15}$$

with coefficients d_m determined by

$$d_m = (-1)^{\mathrm{E}(\frac{m+1}{2})}\frac{2q^{m/2}}{1+q^m}\Delta_m\,, \qquad \Delta_m = \begin{cases} \frac{2}{m}\,, & m \text{ is even} \\ \frac{\pi}{K}\,, & m \text{ is odd} \end{cases}\,. \tag{2.16}$$

At the first step of iterations one should substitute into the right–hand member of (2.4) the unperturbed solution (2.11) with (2.13), (2.14) and some functions of time due to the disturbing factors. These functions may be expanded usually into trigonometric series in multiples of some linear functions of time. The latter may be expressed as linear functions of the mean anomaly M. Therefore, at the first step of iterations one has to deal with integrals of the type

$$T_{pr} = \int_0^t \exp \mathrm{i}(pM + rw + \varphi)\,\mathrm{d}M \tag{2.17}$$

with real (non-integer) p, integer r and some constant phase angle φ. Such integrals were expressed in (Brumberg and Brumberg, 1999) in the form

$$T_{pr} = \frac{\mathrm{i}}{p}\Bigg\{ - \exp \mathrm{i}(pM + rw + \varphi) + \\ + r \sum_{s=-\infty}^{\infty} \frac{F_s^{(p)}}{p+r+s} \exp \mathrm{i}\big[(p+r+s)w + \varphi\big]\Bigg\}\Bigg|_0^t \tag{2.18}$$

for $p \neq 0$. Functions $F_s^{(p)}$ depending on q represent the coefficients of the expansion

$$\exp \mathrm{i}pM = \sum_{s=-\infty}^{\infty} F_s^{(p)} \exp \mathrm{i}(p+s)w \,. \tag{2.19}$$

Different techniques to compute these coefficients are exposed in (Brumberg and Brumberg, 1999). Functions $F_s^{(p)}$ decrease very quickly with $|s| \to \infty$ making series (2.18) rather compact as contrasted to the series in the mean anomaly. By substituting (2.19) into (2.18) one may express T_{pr} only in terms of w

$$T_{pr} = -\frac{\mathrm{i}}{p} \sum_{s=-\infty}^{\infty} \frac{p+s}{p+r+s} F_s^{(p)} \exp \mathrm{i}\big[(p+r+s)w + \varphi\big]\Bigg|_{w_0}^{w} \,. \tag{2.20}$$

To evaluate T_{pr} for $p = 0$ one needs the expansion

$$\frac{\mathrm{d}M}{\mathrm{d}w} = \sum_{s=-\infty}^{\infty} \Phi_s \exp \mathrm{i}sw \,. \tag{2.21}$$

By differentiating (2.15) one easily finds

$$\Phi_0 = 1 \,, \qquad \Phi_s = \tfrac{1}{2} s d_s \,, \qquad \Phi_{-s} = \Phi_s \qquad (s > 0) \,. \tag{2.22}$$

In (Brumberg and Brumberg, 1999) T_{0r} has been evaluated without its M-secular part. The complete expression for T_{0r} is as follows:

$$T_{0r} = \Bigg\{ \Phi_{-r} w \, \exp \mathrm{i}\varphi - \mathrm{i} \sum_{s=-\infty}^{\infty}{}^{(-r)} \frac{\Phi_s}{r+s} \exp \mathrm{i}\big[(r+s)w + \varphi\big]\Bigg\}\Bigg|_{w_0}^{w} \,. \tag{2.23}$$

Evidently, in the sequence of iterations one meets w outside the sign of the trigonometric functions (this is characteristic for short-term theories of motion). Changing with the aid of (2.21) the integration variable from M to w one has to deal with integrals of the type

$$I_{kpr} = \int_{w_0}^{w} w^k \exp \mathrm{i}(pM + rw + \varphi) \, \mathrm{d}w \tag{2.24}$$

with integer $k,\ r$ and real (non-integer) p. Substitution of (2.19) leads to the series of elementary integrals resulting finally to

$$I_{kpr} = \sum_{s=-\infty}^{\infty} F_s^{(p)} J(k,\ p+r+s,\ w) \exp \mathrm{i}\big[(p+r+s)w + \varphi\big]\Big|_{w_0}^{w}, \tag{2.25}$$

$J(k, p, w)$ being a polynomial of w

$$J(k, p, w) = \sum_{n=0}^{k} \frac{(-k)_n w^{k-n}}{(\mathrm{i}p)^{n+1}}. \tag{2.26}$$

Hence, from the technical point of view the iterations themselves present no difficulties.

3. Integration Technique for Several Angular Variables

Let us consider a classical problem of integrating of a trigonometric series in several angular variables related to one and the same single argument (time t) by means of some differential expressions. Such a problem arises, in particular, in GPT, a general planetary theory exposed in (Brumberg, 1995), when one abandons the series in the mean longitudes of the planets in favour of more compact series in some other angular variables.

Let us say we have to integrate a series

$$S = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} S_{kl} \exp \mathrm{i}(kx + ly), \qquad S_{00} = 0 \tag{3.1}$$

with respect to time t. The result is

$$\int S\,\mathrm{d}t = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} S_{kl} I_{kl} \tag{3.2}$$

with integral

$$I_{kl} = \int \exp \mathrm{i}(kx + ly)\,\mathrm{d}t\,. \tag{3.3}$$

Hence, we have to deal with integral (3.3) with k, l being non-zero integers and x, y representing some angular variables (anomalies) relating to one or two bodies. It is assumed that these variables are related to time t by the differential expressions

$$n(1 + f)\,\mathrm{d}t = \mathrm{d}x \tag{3.4}$$

and

$$m(1 + \varphi)\,\mathrm{d}t = \mathrm{d}y, \tag{3.5}$$

where n and m are real constants while f and φ are some functions small by their magnitudes as compared with 1. The case $n = m$ is possible provided that x and y are related to one and the same body. By multiplying (3.4) and (3.5) by k and l, respectively, and adding the results one has

$$(nk + ml)\,\mathrm{d}t = k\mathrm{d}x + l\mathrm{d}y - (knf + lm\varphi)\,\mathrm{d}t. \tag{3.6}$$

Substituting (3.6) into (3.3) and integrating the first term one gets

$$(nk + ml)I_{kl} = -\mathrm{i}\exp\mathrm{i}(kx + ly) - \int (nkf + ml\varphi)\exp\mathrm{i}(kx + ly)\,\mathrm{d}t. \tag{3.7}$$

The integral remaining in the right-hand member is often of the same type as the original integral (3.3) but due to the smallness of f and φ it has smaller magnitude than integral (3.3). By repeating this procedure one may hope that at some step of the integration process one may neglect the remaining integral and express the integral (3.3) in the ‘closed’ form. This procedure may be applied to any number of variables. In classical celestial mechanics a typical example is given with classical anomalies, for example, $x = v$ (true anomaly) and $y = g$ (eccentric anomaly). Then

$$\begin{aligned} f \;&=\; \frac{1}{\eta^3}(1 + \mathrm{e}\cos v)^2 - 1 = \frac{1}{\eta^3}\Big\{1 - \eta^3 + \mathrm{e}\big[\exp\mathrm{i}v + \exp(-\mathrm{i}v)\big] + \\ &\quad + \tfrac{1}{2}\mathrm{e}^2 + \tfrac{1}{4}\mathrm{e}^2\big[\exp 2\mathrm{i}v + \exp(-2\mathrm{i}v)\big]\Big\}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \varphi \;&=\; \frac{\mathrm{e}\cos g}{1 - \mathrm{e}\cos g} = \frac{\mathrm{e}}{2\eta^2}\Big\{\exp\mathrm{i}g + \exp(-\mathrm{i}g) + \tfrac{1}{2}\mathrm{e}\big[\exp\mathrm{i}(v - g) + \\ &\quad + \exp\mathrm{i}(-v + g) + \exp\mathrm{i}(v + g) + \exp\mathrm{i}(-v - g)\big]\Big\} \end{aligned} \tag{3.9}$$

with $\eta = \sqrt{1 - \mathrm{e}^2}$. For small eccentricities one may neglect remaining integrals after several steps. Similar technique has been applied recently to treat the problem of motion of the major planets (Gerasimov et al., 2000). This particular case with v and g may also be treated rigorously by the algorithm of Jefferys (1971).

Extending this classical technique for more sophisticated anomalies let f and φ be represented by the series

$$f = \sum_{r=-\infty}^{\infty} f_r \exp\mathrm{i}rx \tag{3.10}$$

and

$$\varphi = \sum_{r=-\infty}^{\infty} \varphi_r \exp\mathrm{i}ry\,, \tag{3.11}$$

coefficients f_r and φ_r being, at least, of the order $|r|$ with respect to some small parameter (with vanishing f_0 and φ_0 for zero value of this parameter). By substituting these series into (3.7) one gets a recurrent relation

$$(nk + ml)I_{kl} = -\mathrm{i}\exp\mathrm{i}(kx + ly) - \sum_{r=-\infty}^{\infty} (nkf_r I_{k+r,l} + ml\varphi_r I_{k,l+r}). \qquad (3.12)$$

Let us assume that coefficients S_{kl} are of the order $|k| + |l|$ with respect to the small parameter and the maximal order of the terms to be taken into account is equal N. It means that we should know coefficients I_{kl} only for $|k| + |l| \leqslant N$ up to the order $N - |k| - |l|$, inclusively. These coefficients may be found by iterations. At first step one gets their initial approximate value from (3.12) by putting $f_r = \varphi_r = 0$. Then the same formula permits one to improve their value until the prescribed accuracy will be achieved. Eventually, the result of the integration (3.2) is presented in the same form as the original series (3.1) itself.

This technique may be applied, for instance, in dealing with two-argument series of the general planetary theory with elliptic functions (Brumberg, 1995). The basic relation (3.4) takes therewith the form

$$\frac{\pi}{2K(k)}(n' - n)(1 + f)\,\mathrm{d}t = \mathrm{d}w \qquad (3.13)$$

with n, n' being the mean motions of the disturbed and disturbing planets, respectively, modulus k of elliptic functions being expressed in terms of semi-major axes a and a'

$$k^2 = \frac{4aa'}{(a + a')^2} \qquad (3.14)$$

and $K(k)$ being the complete elliptic integral of the first kind. Then,

$$f = \frac{1}{\mathrm{dn}\,u} - 1\,, \qquad u = \frac{K}{\pi}\,w\,. \qquad (3.15)$$

The standard trigonometric expansion of $1/\,\mathrm{dn}\,u$ results in (3.10) with $x = w$ and coefficients

$$f_0 = \frac{\pi}{2k'K} - 1\,, \qquad f_{-r} = f_r = \frac{(-1)^{|r|}\pi}{k'K}\frac{q^{|r|}}{1 + q^{2|r|}}\,, \qquad (3.16)$$

k' and q being the complementary modulus and Jacobi nome, respectively.

In the second-order general planetary theory one deals with the triplet of planets i (disturbed planet), and j and k (disturbing planets). Introducing anomalies w_{ij} and w_{ik} by analogy with (3.13)–(3.15) and putting $x = w_{ij}$, $y = w_{ik}$ one may apply the technique described above to integrate the two-argument series of the right-hand members. This technique might be more effective than the Hansen device to express one of the anomalies in terms of another one (Brumberg, 1995).

4. Conclusion

Complementary to (Brumberg and Brumberg, 1999) two integration techniqies to compress short-term (Section 2) and long-term (Section 3) semi-analytical theories of motion with using the elliptic anomalies have been considered. Application of these techniques is not restricted to only this kind of anomalies. For example, the integration technique of Section 3 may be extended to any angular variables satisfying relations (3.4) and (3.10) including the KS regularizing arguments for many-body problem.

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