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Possible potentials responsible for stable circular relativistic orbits

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Abstract

Bertrand's theorem in classical mechanics of the central force fields attracts us because of its predictive power. It categorically proves that there can only be two types of forces which can produce stable, circular orbits. In this paper an attempt has been made to generalize Bertrand's theorem to the central force problem of relativistic systems. The stability criterion for potentials which can produce stable, circular orbits in the relativistic central force problem has been deduced and a general solution of it is presented. It is seen that the inverse square law passes the relativistic test but the kind of force required for simple harmonic motion does not. Special relativistic effects do not allow stable, circular orbits in the presence of a force which is proportional to the negative of the displacement of the particle from the potential centre.

1. Introduction

The central force problem in non-relativistic classical mechanics is one of the most useful topics in physics. Closely linked with the central force problem is the Keplerian orbit theory which is a cornerstone for understanding planetary motions in the solar system or the motion of electrons near the nucleus. In classical mechanics, there is an important theorem called Bertrand's theorem which proposes that there can only be two types of central potentials, the Coulomb type and the simple harmonic type, which can produce stable, circular orbits for particles moving around the potential source. A good presentation of Bertrand's theorem can be found in [1]. In this paper, we try to generalize the results of Bertrand's theorem when the orbiting particle can have relativistic velocities.

We first set up the relativistic orbit equation for a particle in the central potential presumed to be dependent on the radial coordinate only. The relativistic central force orbits were previously studied in [2–5]. A brief description of the central force problem in a relativistic setting in a Coulomb potential was presented in the book on classical theory of fields by Landau and Lifshitz [6]. Before one starts the main analysis of the stability of orbits of relativistic

particles in a central force potential it is better to specify the assumptions one makes in arriving at definite results. In this paper, we use the same assumptions and approximations as utilized by Boyer in [2] and Landau in [6]. None of the specific references cited above present a Lorentz covariant treatment of the relativistic central force problem. The main reason is that all of them assume a central potential $V(r)$ where $r = |\mathbf{r}|$ is the distance between the source and the orbiting particle. The form of the potential only depends on the position coordinates of the orbiting particle. The form of $V(r)$ is not Lorentz covariant. In such cases the results of the whole analysis are valid in a particular frame where the origin of the coordinate system coincides with the potential centre.

The references cited above assume the particle that produces the potential $V(r)$ to be static in the specific coordinate system utilized by the observer. If the source of the potential does not have any velocity, then the retarded nature of the interactions, owing to the finite velocity of light, does not complicate the calculation of the orbit of the relativistic particle. A specific example will make the point clear. In classical electrodynamics if the source of the Coulomb potential $V(r)$ moves with a velocity \mathbf{v}_s and the orbiting particle has a velocity \mathbf{v} , then the potential $V(r)$ gets a relativistic correction. The magnitude of the lowest order relativistic correction to the Coulomb potential was calculated by Darwin in 1920 and it looks like

$$\frac{V(r)}{2c^2} \left[\mathbf{v}_s \cdot \mathbf{v} + \frac{(\mathbf{v}_s \cdot \mathbf{r})(\mathbf{v} \cdot \mathbf{r})}{r^2} \right].$$

For a better understanding of the Darwin correction, one can look at [7]. In our case $\mathbf{v}_s = 0$, and consequently there will be no relativistic modification of $V(r)$. Moreover we do not consider any general relativistic effects due to $V(r)$. We briefly comment on the general relativistic generalization of the central force problem in section 4. In this paper, the background spacetime is assumed to be flat.

It is shown that the stability condition of the perturbed orbits around a stable circular orbit gives rise to a nonlinear differential equation for the central potential. The Newtonian or the Coulomb potential satisfies the resulting differential equation with some restrictions on the possible value of the angular momentum of the orbiting particle. Except the Newtonian potential solution we present a more general solution of the differential equation for the potential which can give rise to stable, circular orbit for relativistic particles. This solution gives rise to a force which is not common in physics except in its Newtonian inverse square law limit. The equation of the orbit of a relativistic particle in such a non-trivial force shows that the orbit will precess and the precession angle can be calculated.

Unlike the non-relativistic case, in the relativistic case there exists no radial effective potential minimizing which we can obtain the radius of a circular orbit. In the relativistic case, a first-order perturbation from a circular orbit is enough to determine the stability criterion of the orbit. In the non-relativistic case, one uses higher order perturbations from a circular orbit to specify the form of the potential. In the relativistic case, the general solution of the form of the potential from a first-order perturbation from a circular orbit is such that all higher order corrections become irrelevant. As a consequence of this fact the general form of the potential which can produce stable, circular orbits for relativistic particles contains more parameters than the corresponding expressions of non-relativistic potentials.

This paper is organized as follows. Section 2 sets the conventions and derives the orbit equation of a relativistic particle in a central orbit. Section 3 generalizes Bertrand's theorem for the relativistic case. In this section, the stability condition for the circular orbits is interpreted as a nonlinear differential equation for the potential. The solutions of the nonlinear stability equation are also derived in section 3. In section 4, the connection of this work with some related works which exist in the literature are discussed. This section gives a wider view to the

readers who really want to understand the stability of orbits in special relativity and general relativity. Section 5 summarizes the important points presented in this paper.

2. The orbit equation

In this section, we derive the orbit equation of the relativistic particle in the presence of a potential $V(r)$ which is purely a function of the radial coordinate. The Lagrangian of a relativistic particle of mass m in the presence of a continuous radial potential $V(r)$ is

$$\mathcal{L} = -mc^2\sqrt{1 - v^2/c^2} - V(r), \quad (1)$$

where the velocity of the particle \mathbf{v} in plane polar coordinates is given as

$$\mathbf{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta,$$

where $\hat{e}_r, \hat{e}_\theta$ are the mutually orthogonal unit vectors along the radial and the angular directions. The form of the Lagrangian in equation (1) immediately shows that the angular momentum

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\gamma\dot{\theta} \quad (2)$$

is a constant, where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

The total energy E of the particle in the presence of the potential $V(r)$ is

$$E = mc^2\gamma + V(r). \quad (3)$$

Although from the definition of γ it looks like a function of r, \dot{r} and $\dot{\theta}$, it can be shown that in a central force field γ is only a function of the radial coordinate r . The reason for such behaviour of γ can be understood from the following. As energy and angular momentum are constant functions of r, \dot{r} and $\dot{\theta}$ we can use the conservation conditions of E and L to re-express \dot{r} and $\dot{\theta}$ as functions of r, E and L . As E and L are constants so in a central force field \dot{r} and $\dot{\theta}$ are functions of r alone. Consequently γ is only a function of r . In special relativity the energy of the particle in a central potential can also be written as

$$E = \sqrt{p^2c^2 + m^2c^4} + V(r), \quad (4)$$

where $p = |\mathbf{p}|$,

$$\begin{aligned} \mathbf{p} &= m\gamma\mathbf{v} = m\gamma(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta) \\ &= p_r\hat{e}_r + p_\theta\hat{e}_\theta, \end{aligned} \quad (5)$$

and

$$p_r = m\gamma\dot{r}, \quad p_\theta = m\gamma r\dot{\theta} = \frac{L}{r}.$$

Because $(p_r/p_\theta) = (\dot{r}/r\dot{\theta})$, we have

$$p_r = \frac{L}{r^2} \frac{dr}{d\theta}.$$

With the above information on the various momentum components we can now rewrite equation (4) as

$$(E - V)^2 = \left(\frac{L}{r^2} \frac{dr}{d\theta} \right)^2 c^2 + \frac{L^2 c^2}{r^2} + m^2 c^4. \quad (6)$$

Instead of r we use the variable $u = \frac{1}{r}$ in terms of which equation (6) becomes

$$(E - V)^2 = L^2 c^2 \left(\frac{du}{d\theta} \right)^2 + u^2 L^2 c^2 + m^2 c^4.$$

If we differentiate the last equation with respect to θ and then divide the resulting equation by $du/d\theta$, we obtain the desired equation of the orbit of a particle of mass m possessing momentum \mathbf{p} moving in the presence of a general central potential $V(r)$ as

$$\frac{d^2 u}{d\theta^2} + u = \frac{(V - E) dV}{L^2 c^2 du}. \quad (7)$$

Using equation (3) we can rewrite the above equation in the form

$$\frac{d^2 u}{d\theta^2} + u = -\frac{m\gamma}{L^2} \frac{dV}{du}. \quad (8)$$

Writing $L = \gamma \ell$, where $\ell = mr^2 \dot{\theta}$ is the non-relativistic angular momentum, the above equation in the non-relativistic limit ($\gamma \rightarrow 1$) transforms exactly to the form we get in a conventional non-relativistic treatment of the problem as given in [1].

3. Circular, stable closed orbits

Let us define

$$J(u) \equiv \frac{(V - E) dV}{L^2 c^2 du}. \quad (9)$$

Suppose equation (7) admits a circular orbit of radius $r_0 = 1/u_0$. For small perturbations around this circular orbit, we can Taylor expand $J(u)$ around u_0 . Keeping up to first-order terms in the perturbation of u we obtain

$$J(u) = J(u_0) + (u - u_0) \left(\frac{dJ}{du} \right)_{u_0}. \quad (10)$$

Noting that $J(u_0) = u_0$ for the circular orbit, we can now write equation (7) as

$$\frac{d^2 u}{d\theta^2} + (u - u_0) = (u - u_0) \left(\frac{dJ}{du} \right)_{u_0}.$$

If we define $x \equiv u - u_0$, then the above equation can be written as

$$\frac{d^2 x}{d\theta^2} + \zeta^2 x = 0, \quad (11)$$

where ζ^2 is defined as

$$\zeta^2 \equiv 1 - \left(\frac{dJ}{du} \right)_{u_0}. \quad (12)$$

From equation (11) it is clear that if the orbit of the relativistic particle in a general central potential has to be stable then $\zeta^2 > 0$ and if the orbit has to be closed then ζ must be a rational number.

3.1. A differential equation for the potential $V(r)$ producing stable and closed circular orbits

The rational number ζ as predicted, in equation (12), from the stability criterion of closed circular orbits in the central force problem is an interesting input in the theory. The interesting property about this rational number is that it is a constant and so it does not depend on the details of the orbit which one tries to perturb. The reason for the constancy of ζ is the following. For any circular orbit of radius r_0 , a specific ζ specifies the number of undulations of the perturbed orbit. If ζ is a rational number, then the number of undulations of the perturbed orbit will be such that they form a closed geometrical structure. Now suppose one takes another circular orbit of radius $r_0 + \delta r$ where $\delta r \ll r_0$. If ζ has a different value on this orbit, then the number of undulations due to a perturbation will be different. In the limit $\delta r \rightarrow 0$ in a continuous manner, the two unperturbed circular orbits tend to each other but the number of undulations on the circular orbits will not match as ζ is not a continuous variable but can only have discrete rational values. Consequently the number of cycles of the perturbations will change discontinuously with radius and the perturbed orbits cannot be closed at this discontinuity. As we are only interested in stable, closed orbits we can conclude that ζ must be a constant and does not change discretely with r . The discussion on the constancy of ζ as given above closely follows the analysis given in [1] where the author gives a nice discussion on the role of ζ in the case of non-relativistic orbits.

As ζ is a constant and must not depend upon the choice of u_0 or x one can interpret equation (12) as an independent differential equation by itself,

$$1 - \left(\frac{dJ}{du} \right) = \zeta^2, \quad (13)$$

whose solutions would give us information about the general form of the central potential $V(r)$. Using equation (9) we can write the last equation as

$$(V - E) \frac{d^2 V}{du^2} + \left(\frac{dV}{du} \right)^2 = L^2 c^2 (1 - \zeta^2), \quad (14)$$

which is a nonlinear second-order differential equation. The right-hand side of the above equation is a constant which can be written as

$$d = L^2 c^2 (1 - \zeta^2). \quad (15)$$

Equation (14) admits multiple solutions for V . The constant E is the total energy of the particle.

It is interesting to note that the differential equation for the potential stemming from the stability of closed, circular orbits in the relativistic case does not have a non-relativistic analogue. Although the orbit equation (8) has a proper non-relativistic limit the same cannot be said about equation (14). The reason for such behaviour can be clearly seen if we rewrite equation (14) in a slightly different way. From the expression of the energy of the particle in the central force field γ can always be written as $(E - V)/mc^2$. As the total energy is a constant in this case we must have $d\gamma/du = -(1/mc^2) dV/du$. Consequently equation (14) can also be written as

$$\gamma \frac{d^2 \gamma}{du^2} + \left(\frac{d\gamma}{du} \right)^2 = \frac{L^2 (1 - \zeta^2)}{m^2 c^2}, \quad (16)$$

which gives a differential equation of γ . The above equation obviously does not have a well-defined non-relativistic limit. The relativistic stability condition produces an ill-defined non-relativistic limit due to the fact that in the relativistic case $J(u)$ as given in equation (9) depends upon the velocity of the orbiting particle¹. In the non-relativistic case, $J(u)$ was

¹ The $V - E$ in $J(u)$ is proportional to γ which depends upon the velocity of the particle.

purely a function of the radial coordinate of the orbiting particle. A perturbation from the circular orbit in the relativistic case consists of two kinds of perturbations. One is related to the change in the position of the particle from its previous orbit and the other is the change in velocity from the velocity it had previously on the circular orbit. In the non-relativistic case only a radial perturbation from the circular orbit fixes the shape of the stability condition. As the stability condition of the orbit depends upon the velocity of the relativistic particle and the corresponding non-relativistic stability condition does not depend upon the velocity of the particle, the non-relativistic limit of equation (14) or equation (16) is not well defined.

In the case of non-relativistic motion we know that the inverse square law potential and the simple harmonic potential have the capability of producing stable, closed circular orbits. In the present case to obtain the forms of the potentials which can produce stable, closed orbits we have to solve equation (14). As it is a non-trivial equation we first try to see whether the potentials which produced stable, closed orbits in the non-relativistic regime still satisfy equation (14). Let us try to see whether any power-law solution of the form

$$V(u) = -\alpha u^\tau, \quad (17)$$

where $\alpha > 0$ satisfies equation (14). In the above equation α and τ are constants. If we substitute the above form of the potential in equation (14) we obtain

$$u^{2(\tau-1)}\{\alpha^2\tau(\tau-1) + \alpha^2\tau^2\} - u^{\tau-2}\{E\alpha\tau(\tau-1)\} = d.$$

This directly shows that the above relation can be valid for any u only if $\tau = 1$, when $\alpha^2 = d$ or

$$L = \frac{\alpha}{c\sqrt{1-\zeta^2}}, \quad (18)$$

on using equation (15). In this case we see that by choosing $\tau = 1$ in equation (17) we obtain the Coulomb or Newtonian potential. The last equation shows that for stable, circular orbits the particle's angular momentum must satisfy some condition. Equation (18) implies that $\zeta^2 < 1$, and as $\zeta^2 > 0$ for a stable orbit, we have

$$0 < \zeta^2 < 1. \quad (19)$$

The above equation gives

$$L > \frac{\alpha}{c}, \quad (20)$$

giving a lower bound on the angular momentum of the orbiting particle. This lower bound of the angular momentum of the orbiting particle was previously obtained in a different way by Boyer in [2]. It must be noted here that except $\tau = 1$ no other values of τ are allowed in the potential which can produce stable circular orbits of relativistic particles. In non-relativistic mechanics we also have the harmonic oscillator potential corresponding to $\tau = -2$ and $\alpha < 0$ in equation (17), but interestingly relativistic effects forbid this value of τ .

3.2. The general solution of the differential equation for the potential and the nature of orbits

We can find the general form of the force which can produce stable, circular relativistic orbits. Noticing that the left-hand side of equation (14) can also be written as

$$\frac{d^2}{du^2} \left[\frac{(V-E)^2}{2} \right],$$

it can be easily shown that

$$V(r) - E = -\sqrt{d \left(b + \frac{1}{r} \right)^2 + a} \quad (21)$$

satisfies equation (14) where d is as given in equation (15) and b and a are two other dimensional, integration constants. For an attractive force, $b > 0$ and $d > 0$ but a can have any sign. If we assume that as $r \rightarrow \infty$, $V(r) \rightarrow 0$, then we get a relation between the constants d , b and a as

$$E = \sqrt{db^2 + a}. \quad (22)$$

From equation (21) we obtain the force acting on the particle,

$$\mathbf{F} = -\nabla V(r),$$

as

$$\mathbf{F} = -\frac{d(b + \frac{1}{r})}{r^2 \sqrt{d(b + \frac{1}{r})^2 + a}} \hat{r}, \quad (23)$$

From the form of the force and equation (22) we immediately see that if $a = 0$ we have $b = E/\sqrt{d}$ and we get back the Newtonian or the Coulombic potential. From the form of the potential as written in equation (17) we can furthermore identify $\alpha = \sqrt{d}$ and consequently when $a = 0$ we have $b = E/\alpha$. If $a \neq 0$, then the form of the force is non-trivial. The form of the force as given in equation (23) cannot be reduced to the harmonic oscillator force in any limits of the constants. This shows that special relativistic effects do not allow stable circular orbits in the presence of a force which is proportional to the negative of the displacement vector. Although the force expression in equation (23) is mathematically interesting, in physics we do not encounter such a force, except the $a = 0$ limit.

From the expression of $V-E$ as given in equation (21) we obtain

$$J(u) \equiv \frac{(V - E) dV}{L^2 c^2 du} = \frac{d}{L^2 c^2} (u + b), \quad (24)$$

yielding

$$\frac{d^2 u}{d\theta^2} + u = \frac{d}{L^2 c^2} (u + b), \quad (25)$$

which gives the orbit equation of the relativistic particle which is acted on by a force given by equation (23). As in general $d = L^2 c^2 (1 - \zeta^2)$ for a stable closed orbit where ζ must be a rational number, we obtain

$$\frac{1}{r} = \frac{1}{R} \cos(\zeta\theta) + \frac{b(1 - \zeta^2)}{\zeta^2}, \quad (26)$$

where

$$R = Lc\zeta \left[\frac{b^2 L^2 c^2 (1 - \zeta^2)}{\zeta^2} + a - m^2 c^4 \right]^{-1/2}. \quad (27)$$

The equation of the orbit (26) shows that in the most general case we have precession of the orbits dictated by the condition $(2\pi + \delta\theta)\zeta = 2\pi$, which predicts that the orbit precesses by an angle

$$\delta\theta = \frac{2\pi(1 - \zeta)}{\zeta}, \quad (28)$$

per orbit.

3.3. The case of large perturbations

Till now we have utilized a first-order perturbation from a circular orbit as described in equation (10) in the beginning of this section. To include higher order perturbations from a circular orbit we require more terms in the Taylor series expansion of $J(u)$ in equation (10). The second-order effects will come from terms proportional to $(d^2 J/du^2)_{u_0}$. If the general form of the potential $V(r)$ satisfies equation (21), then it is immediately clear from equation (24) that all derivatives of $J(u)$, except the first, vanish. Consequently in the relativistic case it is impossible to restrict the constants ζ , a and b by higher order perturbation terms to the circular orbit. For higher order perturbations from circular orbits, the form of the potential as given in equation (21) remains the same.

4. Connection of this work with some related works

One of the findings of this paper is related to the absence of stable, circular orbits for relativistic particles in the presence of a harmonic oscillator potential. The trajectories of relativistic particles in a three-dimensional harmonic oscillator potential have been studied previously by Homorodean in [8]. The method followed in the referred work is completely equivalent to the one followed in this work. It is interesting to note that in Homorodean's analysis the general shape of the orbit in the relativistic case is not an ellipse, or a circle, but a rosette-shaped curve. In the presence of the oscillator potential the angular momentum of the orbiting particle with a specific energy has an upper bound. The trajectory of the relativistic particle can be a circle only when it has the highest angular momentum for a fixed energy. In [8], the author does not give any information about the stability of the orbits. In the non-relativistic limit the orbit of the particle can be circular. In the light of the findings in [8] of Homorodean, the prediction of the absence of a stable, circular orbit in the oscillator potential is a sensible result.

Bertrand's theorem in non-relativistic classical mechanics has inspired some authors to propose a spacetime (a metric to be precise) where any bounded trajectory of a particle is periodic in nature. This kind of spacetime is named Bertrand spacetime. The works of Perlick, Ballesteros, Enciso, Herranz and Ragnisco, in [9, 10], try to generalize the results of the classical Bertrand's theorem on a flat 3-space to a curved 3-manifold. In [9], the author found that a specific form of a spacetime which is asymptotically flat can support Keplerian orbits. The asymptotically flat Bertrand space cannot support closed trajectories expected in an oscillator type of potential. One of the findings of this paper predicts that even in flat space relativistic effects forbid closed, stable trajectories of particles in the presence of an oscillator potential.

5. Conclusion

The outline of the paper is based on the well-known Bertrand's theorem on central potentials and orbits of particles as described in most classical mechanics books. Like the non-relativistic case the relativistic particle's orbit around a potential source takes place in a plane where the angular momentum and presumably the energy of the orbiting particle remain constant. The main difference between the non-relativistic orbits and relativistic orbits crops up in the orbit equation itself. Unlike the non-relativistic case, in the relativistic case the orbit equation depends upon the total energy of the particle. The main aim of the paper was to find the possible forms of central potentials which can produce stable circular orbits for relativistic particles. The stability condition for the orbits can be transformed to a nonlinear differential equation for the central potential. It is seen that one of the solutions of the nonlinear differential

equation for the central potential is just the normal Coulomb potential. But relativity affects the properties of the orbits by curtailing the angular momentum values beyond a certain limit. Apart from the Coulomb potential solution we find that the stability equation has another general mathematically interesting solution which is unlike any potential which we use in conventional physics. In a specific limit, the general solution reproduces the Newtonian or Coulomb form. In the relativistic version of the central force problem, we lack some restrictions on the potential which can produce stable circular orbits. In the non-relativistic version minimizing the effective potential one can figure out the radius of the circular orbits and higher order perturbation corrections to the stability condition of the orbits could be used for unravelling the exact nature of the potential. In the relativistic version none of those restrictions remain and consequently the general solution of the potential contains some constants whose values cannot be analytically calculated.

An important fact which emerges from the paper is about the non-existence of the harmonic oscillator potential as a solution of the stability equation. In non-relativistic treatment of Bertrand's theorem it is well known that only two kinds of potentials can produce stable circular orbits, one is of the Coulomb type and the other is of the harmonic oscillator type. The Coulomb form of the potential passes the stability test for circular orbits but the harmonic type does not.

The paper focussed on some mathematical properties of relativistic particle orbits in a central potential. Before we finally conclude it is pertinent to say something on the practical side of the relativistic central force problem. Atomic physics always remains a store house of exciting phenomena and one of the places where one may like to apply the tools of relativistic central force problems lies inside the atom. This fact was discussed in [2]. People have studied the Schrödinger equation and the Dirac equation in the presence of the Coulomb potential. It could be quite interesting to study the analogous problems using the potential presented in this paper instead of the Coulomb potential. This attempt may seriously shed some light on the physics of the atoms.

Moreover as there exists some work on the general relativistic generalization of Bertrand's theorem one may expect that in the simplest of the situations, where spacetime remains flat, the results of this work can be applied for the orbits of very fast moving bodies interacting via Newtonian gravity with a massive source.

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