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CONTRIBUTIONS TO STABILITY THEORY

By José L. Massera

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During the preparation of the course on the Theory of Stability which I gave at the International Mathematical Summer Center at Varenna, Italy (September 1954) I found a certain number of new or partially new results belonging to several chapters of this theory. A summary of these results was presented to the Pittsburgh Meeting of the American Mathematical Society (December 27–29, 1954) and published in the Bulletin of the American Mathematical Society, Vol. 61 (1955), pp. 150–153. The present paper contains the precise statements and proofs of these theorems. However, as I worked on the manuscript, I found that it was possible to refine and generalize still further several results, in particular the theorems on asymptotic stability, so that the present exposition is substantially different from the lectures at Varenna.

The majority of the following theorems are valid in general Banach spaces. Those theorems which need the assumption that the dimension of the space is finite are marked with an asterisk.

1. On a generalization of an algebraic theorem of Lyapunov and its application to stability

Lyapunov [14] proved the following

Theorem 1*. *Let $U(x)$ be a positive definite form of degree m in the components of the n-dimensional vector x. If A is a real matrix all whose characteristic roots have negative real parts, there is one and only one form $V(x)$ of degree m such that $(\partial V/\partial x).Ax = U$, where $\partial V/\partial x$ represents the gradient (row) vector and the dot the matrix product (here, inner product), and this form will be negative definite. If at least one characteristic root has positive real part, there is a form V and a positive number a such that $(\partial V/\partial x).Ax = aV + U$, and in this case V assumes positive values for certain values of x.*

As a corollary, Lyapunov's "second method" theorem on asymptotic stability admits a reciprocal in the case of linear systems with constant coefficients:

Theorem 2*. *If the solution $x = 0$ of the linear system with constant coefficients $\dot{x} = Ax$ is asymptotically stable, a positive definite Lyapunov's function $V(x)$ exists whose total derivative $dV/dt = (\partial V/\partial x).Ax$ is negative definite; V may be taken as an algebraic form of any given (even) degree.*

The following theorems correspond to the case of simple (non-asymptotic) stability:

Theorem 3*. *Let A be a real matrix none of whose characteristic roots has a positive real part; assume moreover that the elementary divisors corresponding to roots with vanishing real part are linear. Then a positive definite form V of any given even degree $2m$ exists such that $(\partial V/\partial x).Ax$ is negative semidefinite.*

There is a real non-singular matrix C such that $D = C^{-1}AC$ is the direct sum of three different kinds of matrices: (a) p matrices $L_1, \cdots, L_p$ of order one, $L_h = (0)$, corresponding to the zero characteristic roots of A; (b) q matrices $M_1, \cdots, M_q$ of order two, $M_h = \begin{pmatrix} 0 & \rho_h \\ -\rho_h & 0 \end{pmatrix}$, corresponding to each pair of conjugate purely imaginary characteristic roots $\pm\rho_h$ i (there may be repetitions among the ρ_h); (c) a matrix N of order $r = n - p - 2q$, corresponding to the characteristic roots having negative real parts. We assume that the direct sum is so disposed that the first p rows and columns of D correspond to the matrices L, the following $2q$ to the matrices M and the last r to the matrix N. Let P be the direct sum of the matrices $L_1, \cdots, L_p, M_1, \cdots, M_q$.

If y is the column-vector having the (real) components $y_1, \cdots, y_n$, denote by u the column vector having the components $y_1, \cdots, y_{p+2q}$ and by v the vector having the components $y_{p+2q+1}, \cdots, y_n$. From Theorem 1* follows that a positive definite form $W_2(v)$ of degree $2m$ exists such that $(\partial W_2/\partial v).Nv$ is negative definite. Let $W_1(u) = y_1^{2m} + \cdots + y_p^{2m} + (y_{p+1}^2 + y_{p+2}^2)^m + \cdots + (y_{2q-1}^2 + y_{2q}^2)^m$; it is easy to see that $(\partial W_1/\partial u).Pu = 0$. The positive definite form $W(y) = W_1(u) + W_2(v)$ satisfies then

$$(\partial W/\partial y).Dy = (\partial W_1/\partial u).Pu + (\partial W_2/\partial v).Nv \leqq 0.$$

Finally, the form $V(x) = W(C^{-1}x)$ satisfies the requirements of the theorem because $(\partial V/\partial x).Ax = (\partial W/\partial y).Dy$ (where $x = Cy$).

As a corollary, we have the following reciprocal of Lyapunov's theorem on (simple) stability:

THEOREM 4*. *If the solution $x = 0$ of the linear system with constant coefficients $\dot{x} = Ax$ is stable, there is a positive definite Lyapunov's function $V(x)$ whose total derivative is negative semidefinite; V may be taken as an algebraic form of any given (even) degree.*

In the case of unstability the situation is not so satisfactory. If the unstability is "rough", i.e. if there are characteristic roots with a positive real part, a reciprocal of the corresponding Lyapunov's theorem may be immediately stated on the basis of the second part of Theorem 1*. But if none of the characteristic roots have positive real part and some of the elementary divisors corresponding to roots having vanishing real part are not linear, then the solution $x = 0$ is unstable but, in general, there is no algebraic form V satisfying the corresponding theorem of Lyapunov, i.e. such that V assumes positive values and $dV/dt = aV + U$ where a is a positive constant and U a non-negative function.

This may be easily shown, for instance, on the example $\dot{x} = y$, $\dot{y} = 0$, x, y scalars. We shall see that there is no analytic function $V(x, y)$ with the required properties. In fact, suppose that such a function exists and let $x_n \to 0$, $y_n \to 0$ be sequences such that $V(x_n, y_n) > 0$; we may assume $y_n > 0$, the proof being similar if $y_n < 0$. As V satisfies the equation $dV/dt = yV_x = aV + U$, we have

$$V(x, y_n) = \exp\,[a(x - x_n)/y_n].$$
$$\cdot \left\{ V(x_n, y_n) + \int_{x_n}^{x} \exp\,[-a(t - x_n)/y_n].U(t, y_n)\,dt \right\}$$

when $x \geqq x_n$ whence $V(x, 0) \geqq 0$ when $x > 0$. On the other hand, from the same equation $yV_x = aV + U$ follows $V(x, 0) = -U(x, 0)/a \leqq 0$ so that $V(x, 0) = 0$ when $x \geqq 0$ and finally, V being analytic, $V(x, y) = yV_1(x, y)$, V_1 analytic. Again from the equation we have $U(x, y) = y(V_x - aV_1) = yU_1(x, y)$, U_1 analytic, and we must have $U_1(x, 0) = 0$, otherwise $U(x, y) < 0$ at certain points in the neighborhood of the origin. Then $U_1(x, y) = yU_2(x, y)$, $U_2 \geqq 0$ analytic, and $V(x, y) = y^2(V_{1x} - U_2)/a = y^2V_2(x, y)$, V_2 analytic; we would have then $yV_{2x} = aV_2 + U_2$ and the same argument could be repeated indefinitely, which is absurd.

An interesting remark is that the system $\dot{x} = y$, $\dot{y} = 0$, has a *non-analytic* Lyapunov's function, namely $V(x, y) = \exp[(x - 1)/y]$ when $y > 0$, $= \exp[(x + 1)/y]$ when $y < 0$, $V(x, 0) = 0$ (we assume $|x| < 1$), which satisfies $dV/dt = yV_x = V$ ($U = 0$) and assumes positive values in the neighborhood of the origin. This agrees with recent results of Krasovskiĭ [13] who proved the existence of Lyapunov's functions in the case of unstability under very general conditions.

2. Generalization of certain criteria for unstability due to Četaev

Četaev [6–7] proved in 1934 certain sufficient conditions for unstability which substantially generalize the criteria found by Lyapunov; his proofs were not entirely rigorous but Persidskiĭ [20] gave entirely satisfactory proofs. Persidskiĭ [21] thought at first that his proofs were applicable to the case of infinite dimensional spaces, which is not true as he himself recognized afterwards [22].

These previous results are included in the following general theorem:

THEOREM 5. *Let $\dot{x} = f(x, t)$ be a differential equation defined in the semi-cylinder of (x, t)-space $K: \| x \| \leqq a$, $t \geqq 0$; $f(0, t) = 0$ and we assume f sufficiently regular so that the existence, uniqueness and continuous dependence of the solutions from the initial conditions are granted. Let $G \subset K$ be an open set and assume that a scalar function $V(x, t)$ is defined and Fréchet differentiable* [10] *in the closure of G; assume that V is positive and bounded in G, $V(0, t) = 0$ at the points $(0, t)$ belonging to the closure of G, and that there are two positive continuous functions $b(u)$, $c(t)$ of the positive variables u, t, the function b being non-decreasing, $\int^{+\infty} c(t)\,dt = +\infty$, such that in G $dV/dt = \delta V[(x, t); (f, 1)] \geqq b(V).c(t)$.*[1] *Then, if any one of the following supplementary assumptions is satisfied, the solution $x = 0$ of the differential equation is unstable:*

(a) *G has at least one boundary point $(0, T)$, $T > 0$, and $V(x, t) = 0$ at any boundary point (x, t) of G such that $\| x \| < a$;*

(b) *G has at least one boundary point $(0, T)$, $T > 0$ and, at the boundary points (x, t) of G such that $0 < \| x \| < a$, the vector $g = (f, 1)$ (with the components f relative to the x-space and 1 along the t-axis) enters G in the generalized sense;*

[1] This expression, differential of V relative to the increments $\delta x = f(x, t)$ and $\delta t = 1$, is the "total derivative" of V, i.e. the derivative with respect to t taken along the integral curves of the differential equation.

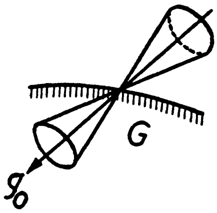

FIG. 1

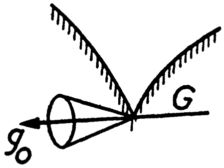

FIG. 2

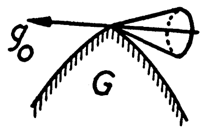

FIG. 3

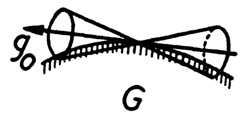

FIG. 4

(c)* *The space is finite dimensional; there is a sequence of numbers* $t_n \to +\infty$ *and a sequence of sets* $G_n \subset G$, G_n *contained in the semi-cylinder* $\| x \| \leqq a$, $t \geqq t_n$, *such that the sets* $G_n \cup \{(0, t): t \geqq 0\} \cup \{(x, t): \| x \| = a, t \geqq 0\}$ *are connected; moreover, at any boundary point* (x, t) *of* G *such that* $0 < \| x \| < a$ *the vector* g *leaves* G *in the generalized sense.*

The phrases *enters* (*leaves*) G *in the generalized sense* mean the following. Let y_0 be a point on the boundary of an open set; we shall say that a vector g_0 attached to the point y_0 *enters* G *in the generalized sense* if, given $\varepsilon > 0$ there is a $\delta > 0$ such that one or other (or both) of the following conditions are satisfied:

(i) For any given vector v which forms with g_0 an angle $< \varepsilon$, the points $y_0 + \lambda v$, $0 < \lambda < \delta$, belong to G (Figures 1 and 2);

(ii) For any vector v which forms with g_0 an angle $< \varepsilon$, the points $y_0 - \lambda v$, $0 < \lambda < \delta$, do not belong to G (Figures 1 and 3).

We shall say that g_0 leaves G if $-g_0$ enters G. According to these definitions a vector can simultaneously enter and leave G (Figures 2 and 3); it is also clear that a vector may neither enter nor leave G (Figure 4).

To prove the theorem in the cases (a) and (b) let us choose a point $(x_0, t_0) \in G$, $x_0 \neq 0$, close to $(0, T)$. If the solution $x = 0$ were stable it would be possible to choose $\| x_0 \|$ so small that the solution $x(t)$ starting at (x_0, t_0) satisfy $0 < \| x(t) \| < a$ when $t \geqq t_0$. At any point of this solution which belongs to G we have $dV/dt > 0$, whence $V[x(t), t] \geqq V(x_0, t_0) = V_0 > 0$ provided that in the interval (t_0, t) the solution has not abandoned G. In the case of assumption (a) the solution would never abandon G and

$$V[x(t), t] = V_0 + \int_{t_0}^{t} (dV/dt)\, dt \geqq V_0 + b(V_0) \int_{t_0}^{t} c(t)\, dt;$$

it would follow that, as $t \to +\infty$, $V \to +\infty$, against the assumption that V is bounded. This contradiction shows that $x = 0$ is unstable.

In the case of assumption (b) the proof finishes as before once we have proved that the solution $x(t)$ cannot abandon G although it may have isolated points at the boundary of G. This is in fact true. Let $y_1 = (x_1, t_1)$, $x_1 = x(t_1)$, be the first point of the solution on the boundary of G; $g_1 = (f(x_1, t_1), 1)$ enters G so that there are numbers $\varepsilon > 0$, $\delta > 0$ such that for any vector v forming with g_1 an angle $< \varepsilon$, either the points $y_1 - \lambda v$ do not belong to G, or the points $y_1 + \lambda v$ belong to G for any λ, $0 < \lambda < \delta$. The first alternative is impossible because, if $\delta_1 > 0$ is such that in the interval $(t_1 - \delta_1, t_1)$ the vectors $g(t) = (f[x(t), t], 1)$ form with g_1 an angle $< \varepsilon$, the point $y(t) = y_1 - (t_1 - t)g(\tau)$, $t < \tau < t_1$, would not belong to G for $t_1 - t$ small enough, against the assumption that y_1 was the first point of the integral curve on the boundary of G. Therefore the second alternative takes place and there is an arc $(t_1, t_1 + \delta_2)$, $\delta_2 > 0$, of the integral curve contained in G. The argument can then be repeated again and our assertion is proved.

In the case of assumption (c), if the solution $x = 0$ were stable, there would exist a $\delta > 0$ such that all the solutions $x(t)$ satisfying $x(0) = x_0$, $\| x_0 \| = \delta$, satisfy $0 < \| x(t) \| < a$ for all $t \geqq 0$. If each of these solutions leaves G for a certain $t = t(x_0)$, it cannot enter G again, which is proved by a similar argument as in case (b). By the continuity of the dependence of the solutions from the initial conditions and Heine-Borel's theorem applied to the compact set $\| x_0 \| = \delta$, $t = 0$, a $T > 0$ would exist such that for $t \geqq T$ all the solutions considered are outside G; these solutions generate then a tube which separates the axis $x = 0$ from the cylinder $\| x \| = a$, against the connectedness assumption made on the G_n. Then at least one solution should remain entirely in G, but this leads to the same contradiction as in cases (a) and (b).

REMARK 1. The assumption that V is differentiable may be replaced by the weaker one that V satisfies a local Lipschitz condition; in this case dV/dt should be interpreted as a generalized total derivative (see Section 3).

REMARK 2. In the theorems relative to assumptions (b) and (c) the crucial part of the argument consists in the proof that the solutions which start at points in G do not leave G, respectively, those which start from points outside G do not enter G. This proof is based on the fact that the vector g enters or leaves G at the boundary points in the sense already explained. The same argument leads then to the proof of other theorems with other assumptions relative to the behaviour of g at the boundary of G, applicable to cases where g neither enters nor leaves G in the generalized sense. We have, for instance, the following result which is interesting in itself and may be applied to other problems:

THEOREM 6. *Let G be an open set in a real Hilbert space Y whose boundary is smooth in the following sense: given any point y_0 in the boundary of G, there is a local coordinate system such that the boundary points of G in the neighborhood of y_0 may be represented as the graph of a function $u = \varphi(z)$, where u represents the component of y along a certain one-dimensional subspace and z the projection of y on the orthogonal subspace, the function φ having a bounded differential in the neighborhood of y_0; it is assumed that the points of G correspond locally to the set $u < \varphi(z)$. Let $g(y)$ be a vector field in Y which satisfies a local Lipschitz condition; let $p(z, u)$ be the projection of g on the z-space and $q(z, u)$ the* (*scalar*) *component along the u-axis. If at any boundary point of G the inequality $q - \delta\varphi[z; p] \leqq 0$ is satisfied* (i.e. *if g forms an acute or straight angle with the inner normal to the boundary*), *any solution of the differential equation $\dot{y} = g(y)$ starting from any point of G will remain in the future in the closure of G.*

Let $y(t)$ be a solution and assume that $y_0 = y(t_0)$ belongs to the boundary of G;

we assume that y_0 is the origin of the local coordinate system. We shall prove that there is an interval $t_0 < t < t_1$ such that in this interval $y(t)$ belongs to the closure of G, which proves the theorem.

Let $z(t)$ and $u(t)$ be the components of $y(t)$ and let $v(t) = u(t) - \varphi[z(t)]$. $v(t)$ satisfies the equation

$$\dot{v} = q[z(t), v + \varphi(z(t))] - \delta\varphi[z(t); p(z(t), v + \varphi(z(t)))] = Q(v, t).$$

By virtue of the assumptions made on g and φ, Q satisfies a Lipschitz condition with respect to v, so that

$$Q(v, t) \leqq Q(0, t) + K \mid v \mid \leqq K \mid v \mid,$$

because, again from the assumptions $Q(0, t) \leqq 0$. From the classical theorems it follows then that, if $t \geqq t_0$, $v(t)$ is $\leqq$ the solution of the equation $\dot{w} = K \mid w \mid$ with the initial condition $w(t_0) = 0$. This solution being nothing else but $w(t) = 0$, we have $v(t) \leqq 0$, $u(t) \leqq \varphi[z(t)]$ which proves our assertion.

3. On asymptotic stability

The purpose of this section is to study necessary and sufficient conditions, of the kind considered in Lyapunov's "second method", for asymptotic stability. We first state a few definitions and notations relative to the regularity of functions in real Banach spaces.

We will say that a function $y = f(x)$ belongs to the class C_0 in a region R if for any bounded set $R' \subset R$ a Lipschitz condition is satisfied, i.e. there is a constant $M(R')$ such that, for any pair of points x', x'' in R', we have $\| f(x') - f(x'') \| \leqq M \| x' - x'' \|$; we shall say that it belongs to the class C_s in R, s positive integer, if it has Fréchet differentials [10] $\delta^{(p)}f[x; h_1, h_2, \cdots, h_p]$, $1 \leqq p \leqq s$, continuous, also with respect to x, in R and if these differentials are uniformly bounded (as multilinear operators) in any bounded set $R' \subset R$; we shall say that it belongs to the class C_∞ if it has differentials of any given order in R, each one of which is uniformly bounded in any bounded set $R' \subset R$. If the space is finite dimensional (locally compact), $f \in C_s$, $0 \leqq s \leqq \infty$, in the neighborhood of each point of a closed region R implies $f \in C_s$ in R. We shall say that $f \in \bar{C}_s$ in a region R if there is a unique Lipschitz constant (if $s = 0$) or a unique bound for the norm of each one of the successive differentials (if $s > 0$) valid for all R.

We give now a few definitions concerning the stability of the solution $x = 0$ of an equation $\dot{x} = f(x, t)$, $f(0, t) = 0$. We assume that f is continuous in the semi-cylinder $\| x \| \leqq a$, $t \geqq 0$, and satisfies supplementary conditions which guarantee the uniqueness of the solutions and continuity of their dependence on the initial conditions.

(i) *Stability in the sense of Lyapunov*: given $\varepsilon > 0$ and $t_0 \geqq 0$, there is a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\| x_0 \| < \delta$ implies $\| F(t, x_0, t_0) \| < \varepsilon$ when $t \geqq t_0$ ($x = F(t, x_0, t_0)$ being the general solution of the equation, $x_0 = F(t_0, x_0, t_0)$).

(ii) *Uniform stability* [20]: given $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that $\| x_0 \| < \delta$, $t_0 \geqq 0$, imply $\| F(t, x_0, t_0) \| < \varepsilon$, for $t \geqq t_0$.

(iii) *Asymptotic stability in the sense of Lyapunov*: (i) holds and, for each $t_0 \geqq 0$ there is a $\delta_0(t_0) > 0$ such that $\| x_0 \| < \delta_0$ implies $F(t, x_0, t_0) \to 0$ when $t \to +\infty$.

(iv) *Equiasymptotic stability* [18]: there is a $\delta_0 > 0$ and, for each $\varepsilon > 0$, a $T(\varepsilon)$ such that $\| x_0 \| < \delta_0$ implies $\| F(t, x_0, 0) \| < \varepsilon$ when $t \geqq T(\varepsilon)$.

(v) There is a $\delta_0 > 0$ and, for each $\varepsilon > 0$, a $T(\varepsilon)$ such that $\| x_0 \| < \delta_0$, $t_0 \geqq 0$, imply $\| F(t, x_0, t_0) \| < \varepsilon$ when $t \geqq t_0 + T(\varepsilon)$.

(vi) *Uniform-asymptotic stability* [17]: both (ii) and (v) hold.

(vii) *Exponential-asymptotic stability* [15]: there is a $\nu > 0$ and, for each $\varepsilon > 0$, a $\delta = \delta(\varepsilon) > 0$ such that $\| x_0 \| < \delta$, $t_0 \geqq 0$, imply $\| F(t, x_0, t_0) \| < \varepsilon . \exp[-\nu(t - t_0)]$ when $t \geqq t_0$.[2]

THEOREM 7. *The following implications hold*:

(a) (vii) $\to$ (vi) $\to$ (ii) $\to$ (i).

(b) (vii) $\to$ (vi) $\to$ (v) $\to$ (iv) $\to$ (iii) $\to$ (i).

(c) If $f \in \bar{C}_0$ in the semi-cylinder $\| x \| \leqq a$, $t \geqq 0$, (v) $\to$ (vi).

(d) If f is periodic in t or independent of t, (i) $\to$ (ii).

(e)* If f is periodic in t or independent of t and the space is finite dimensional, (iii) $\to$ (vi) (Cf. [18]).

(f) If f is linear (vi) $\to$ (vii); more precisely, in this case there is a constant N such that it is possible to choose $\delta(\varepsilon) = \varepsilon N$, i.e. $\| F(t, x_0, t_0) \| < N \|x_0\| . \exp[-\nu(t - t_0)]$.

The proofs for (a) and (b) are trivial. To prove (c) it is enough to prove that (v) $\to$ (ii). Let M be a Lipschitz constant of f in the semi-cylinder $\| x \| \leqq a$, $t \geqq 0$, and define $\delta(\varepsilon) = \varepsilon . e^{-MT(\varepsilon)}$. Then, if $\| x_0 \| < \delta(\varepsilon)$, we have $\| F(t, x_0, t_0) \| \leqq \| x_0 \| . e^{M(t-t_0)} < \varepsilon$ if $t_0 \leqq t \leqq t_0 + T(\varepsilon)$ and, from (v), also when $t \geqq t_0 + T(\varepsilon)$.

Part (d) may be proved thus: if $x = 0$ were not uniformly stable, there would be an $\varepsilon > 0$ and sequences $x_{0n} \to 0$, t_{0n}, t_n, $t_{0n} \leqq t_n$, such that $\| F(t_n, x_{0n}, t_{0n}) \| \geqq \varepsilon$. Assuming that the period is 1, let $\tau_{0n} = t_{0n} - [t_{0n}] \leqq 1$ ($[a]$ denotes the largest integer in a), $\tau_n = t_n - [t_{0n}] \geqq \tau_{0n} \geqq 0$ and we have $\| F(\tau_n, x_{0n}, \tau_{0n}) \| = \| F(t_n, x_{0n}, t_{0n}) \| \geqq \varepsilon$. But $x_{1n} = F(0, x_{0n}, \tau_{0n})$ exists for n sufficiently large and tends to zero (as a consequence of the continuous dependence of the solutions from the initial conditions) and we would have $\| F(\tau_n, x_{1n}, 0) \| = \| F(\tau_n, x_{0n}, \tau_{0n}) \| \geqq \varepsilon$ against the stability assumption.

Part (e)* is proved thus: from the assumption, there is a $\delta_1 > 0$ such that $\| x_0 \| < \delta_1$ implies $F(t, x_0, 0) \to 0$ as $t \to +\infty$. From the continuity theorem, a $\delta_0 > 0$ exists such that $\| x_0 \| \leqq \delta_0$, $0 \leqq t_0 \leqq 1$, implies that $F(t, x_0, t_0)$ exists for $0 \leqq t \leqq t_0$ and $\| F(0, x_0, t_0) \| < \delta_1$. If the solution $x = 0$ were not uniform-asymptotically stable, for each $\delta_0 > 0$ (in particular for the one just defined) an

[2] In a personal communication, J. Kurzweil has indicated to me the usefulness of introducing still other definitions of asymptotic stability which make it possible to prove interesting necessary and sufficient conditions. The work of J. Kurzweil on this subject is not yet published.

$\varepsilon > 0$ and sequences x_{0n}, t_{0n}, T_n would exist such that $\| x_{0n} \| \leqq \delta_0$, $T_n \to +\infty$, $\| F(t_{0n} + T_n, x_{0n}, t_{0n}) \| \geqq \varepsilon$. The points x_{0n} have a point of accumulation x_0, $\| x_0 \| \leqq \delta_0$, and the numbers $\tau_{0n} = t_{0n} - [t_{0n}]$ a point of accumulation τ_0, $0 \leqq \tau_0 \leqq 1$. By virtue of the definition of δ_0, $x_1 = F(0, x_0, \tau_0)$ exists and $\| x_1 \| < \delta_1$, and, because of asymptotic stability, there is an integer N such that $\| F(N, x_0, \tau_0) \| = \| F(N, x_1, 0) \| < \frac{1}{2}\delta(\varepsilon)$, where $\delta(\varepsilon)$ is the function characterizing the uniform stability already proved in (d). The continuous dependence on initial conditions shows that if n is large enough we have $\|F(N, x_{0n}, \tau_{0n}) \| < \delta(\varepsilon)$ and, from the uniform stability,

$$\| F(\tau, x_{0n}, \tau_{0n}) \| = \| F(\tau + [t_{0n}], x_{0n}, t_{0n}) \| < \varepsilon,$$

for $\tau \geqq N$, which contradicts $\| F(t_{0n} + T_n, x_{0n}, t_{0n}) \| \geqq \varepsilon$ with $T_n \to +\infty$.

In the proof of (f) we have $F(t, x_0, t_0) = R(t)R^{-1}(t_0)x_0$, where $R(t)$ is the solution of the equation $\dot{X} = f(X, t)$ in the space of linear operators, with $R(0) = I$. From the uniform stability follows that $\| R(t)R^{-1}(t_0) \| \leqq \varepsilon_0/\delta(\varepsilon_0) = N \geqq 1$ for a certain fixed $\varepsilon_0 > 0$ and every $t \geqq t_0 \geqq 0$. From assumption (v) follows $\| R(t)R^{-1}(t_0) \| < \frac{1}{2}$ when $t \geqq t_0 + T(\frac{1}{2}\delta_0)$, whence $\| R(t)R^{-1}(t_0) \| < 2^{-n}N$ when $t \geqq t_0 + nT(\frac{1}{2}\delta_0)$, $n = 0, 1, \cdots$. If we take $\nu = \log 2/T(\frac{1}{2}\delta_0)$ we have $\| R(t)R^{-1}(t_0) \| < 2N.\exp[-\nu(t - t_0)]$ for all $t \geqq t_0 \geqq 0$.

The following examples clarify still further the relations between the different types of stability.

Example 1. (ii) is not implied by (iv), not even for linear bounded ($\epsilon\bar{C}_0$) equations of the first order. Consider

$$\dot{x} = -[13 + 12 \sin \log (t + 1) + 12t(t + 1)^{-1} \cos \log (t + 1)].x.$$

The general integral is $x = x(0).\exp\{-[13 + 12 \sin \log (t + 1)].t\}$, and (iv) is satisfied because $| x | \leqq | x(0) |.e^{-t}$. But, if $t_n = \exp[(4n + 1)\pi/2] - 1$ and $t'_n = \exp[(4n + 3)\pi/2] - 1$, we have

$$x'_n/x_n = \exp(-t'_n + 25t_n) = \exp\{[(25 - e^{\pi}).\exp[(4n + 1)\pi/2] - 24\} \to \infty$$

as $n \to \infty$, because $25 > e^{\pi}$, so that (ii) is not satisfied.

Example 2. (vi) is not implied by (v), not even for linear equations of the first order. Consider $\dot{x} = (6t \sin t - 2t).x$ whose general integral is $x = x(0).\exp(6 \sin t - 6t \cos t - t^2)$. If $T > 6$, $t \geqq t_0 + T$, $t_0 \geqq 0$,

$$\begin{aligned} x/x_0 &= \exp(6 \sin t - 6t \cos t - t^2 - 6 \sin t_0 + 6t_0 \cos t_0 + t_0^2) \\ &\leqq \exp[12 + (t + t_0)(6 - t + t_0)] \leqq \exp[12 + T(6 - T)], \end{aligned}$$

which is $< \varepsilon$ if $T = T(\varepsilon)$ is large enough, so that (v) is satisfied. But, if $t_n = n\pi$, we have

$$\begin{aligned} x_{2n+1}/x_{2n} &= \exp(12n\pi + 4n^2\pi^2 + 6(2n + 1)\pi - (2n + 1)^2\pi^2) \\ &= \exp[(4n + 1)\pi(6 - \pi)] \to \infty \end{aligned}$$

when $n \to \infty$, so that (ii) is not satisfied.

EXAMPLE 3. (viii) is not implied by (vi), not even for analytic autonomous equations of the first order. Consider simply $\dot{x} = -x^3$, whose general integral is

$$x = x_0.[1 + 2x_0^2(t - t_0)]^{-\frac{1}{2}}.$$

EXAMPLE 4. In infinite dimensional spaces (vi) is not implied by (iii), not even for linear autonomous systems. The example, in Hilbert space of sequences, is $\dot{x}_n = -x_n/n$.

THEOREM 8. *Given a differential equation* $\dot{x} = f(x, t)$, *the sets of solutions satisfying one of the properties* (iv), (v), (vi), (vii), (ii) + (iii) or (ii) + (iv) *are open* (*in the uniform topology*) *in the space of all solutions.*

In this sense, the properties mentioned are "rough".

The statement is obvious in the cases of properties (iv), (v) and (vii). Let $x_0(t)$ be a solution having properties (ii) and (iii), let $\delta(\varepsilon)$ be the function characterizing its uniform stability and let $x_1(t)$ be a solution so near to x_0 that $x_1(t) - x_0(t) \to 0$ as $t \to +\infty$. Given $\varepsilon > 0$, define $T(\varepsilon)$ so that $\| x_1(t) - x_0(t) \| \leqq \frac{1}{2}\delta(\frac{1}{2}\varepsilon)$ when $t \geqq T(\varepsilon)$, and let $\delta_1(\varepsilon)$ be the supremum of all positive numbers δ such that for any solution $x_2(t)$ and any values t, t_0 satisfying the conditions $0 \leqq t_0 \leqq t \leqq T(\varepsilon)$, $\| x_2(t_0) - x_1(t_0) \| \leqq \delta$, we have $\| x_2(t) - x_1(t) \| \leqq \frac{1}{2}\delta(\frac{1}{2}\varepsilon)$. Then, if $t_0 \geqq 0$, and $\| x_2(t_0) - x_1(t_0) \| \leqq \delta_1$, we may consider the following alternatives: (a) $t_0 \leqq T(\varepsilon)$, $t_0 \leqq t \leqq T(\varepsilon)$, imply, by the definition of δ_1 that $\| x_2(t) - x_1(t) \| \leqq \frac{1}{2}\delta(\frac{1}{2}\varepsilon) < \varepsilon$; (b) if $t_0 \leqq T(\varepsilon)$, $t \geqq T(\varepsilon)$, by the definition of δ_1 we shall have $\| x_2[T(\varepsilon)] - x_1[T(\varepsilon)] \| \leqq \frac{1}{2}\delta(\frac{1}{2}\varepsilon)$ and as $\| x_1[T(\varepsilon)] - x_0[T(\varepsilon)] \| \leqq \frac{1}{2}\delta(\frac{1}{2}\varepsilon)$, we have $\| x_2[T(\varepsilon)] - x_0[T(\varepsilon)] \| \leqq \delta(\frac{1}{2}\varepsilon)$, whence, because of the uniform stability of x_0, $\| x_1(t) - x_0(t) \| < \frac{1}{2}\varepsilon$ and $\| x_2(t) - x_0(t) \| < \frac{1}{2}\varepsilon$ and finally $\| x_2(t) - x_1(t) \| < \varepsilon$; (c) if $t \geqq t_0 \geqq T(\varepsilon)$, from the definition of $T(\varepsilon)$ follows $\| x_1(t_0) - x_0(t_0) \| \leqq \frac{1}{2}\delta(\frac{1}{2}\varepsilon)$ and on the other hand $\| x_2(t_0) - x_1(t_0) \| \leqq \delta_1(\varepsilon) \leqq \frac{1}{2}\delta(\frac{1}{2}\varepsilon)$ and then, as in case (b), follows $\| x_2(t) - x_1(t) \| \leqq \varepsilon$. In every case we have then $\| x_2(t) - x_1(t) \| \leqq \varepsilon$ when $t \geqq t_0$ so that x_1 has property (ii) and it is obvious that it has property (iii).

As (vi) implies (ii) + (iv) which in turn implies (ii) + (iii), the cases of properties (vi) and (ii) + (iv) are now a consequence of what has been already proved.

For equations of the first order, the set of solutions having property (iii) is also open because in this case (iii) implies (iv) (Theorem 7 of [18]). But already for systems of order 2 the set (iii) need not be open, as Example 4 in [18] shows.

The sets of solutions having properties (i) or (ii) are not open, in general, as the following example shows:

EXAMPLE 5. Consider the autonomous equation of the first order, $f \in \bar{C}_\infty$, $\dot{x} = \exp\{-[\sin(\pi/x)]^{-2}\}$. It has the solutions $x = \pm 1/n$, n integer, and the other solutions are contained in the strips $1/(n + 1) < | x | < 1/n$. Therefore $x = 0$ has property (ii) with $\delta(\varepsilon) = 2\varepsilon$. However, the solutions $x = \pm 1/n$ are unstable.

We shall now see which is the relation between the different types of stability and the existence of Lyapunov's functions satisfying certain assumptions. We shall assume always that a function of Lyapunov $V(x, t)$ belongs to the class C_0 and that $V(0, t) = 0$. We shall call *generalized total derivative* (right) the expression

$$V'(x, t) = \limsup_{h \to 0^+} [V(x + hf(x, t), t + h) - V(x, t)]/h.$$

If $V \in C_1$, we have $V'(x, t) = \delta V[(x, t); (f(x, t), 1)]$.[3]

LEMMA 1. *Let $f(x, t)$ be continuous and $x = F(t, x_0, t_0)$ be any of the integrals of the equation $\dot{x} = f(x, t)$ passing through the point (x_0, t_0). The upper right derivative of the function $V[F(t, x_0, t_0), t]$ for $t = t_0$ is equal to $V'(x_0, t_0)$.*

The proof of the Lemma is immediate, using the fact that $V \in C_0$.

We shall consider the following supplementary assumptions relative to V:

(viii) $V(x, t)$ is positive definite ($V(x, t) \geqq a(\| x \|)$, $a(r)$ being continuous and increasing when $r > 0$, $a(0) = 0$) and $V'(x, t) \leqq -c[V(x, t)]$, where $c(v)$ is a function having the same properties as $a(r)$.

(ix) V is bounded and satisfies (viii).

(x) V is positive definite and has an infinitely small upper bound, and V' is negative definite: $a(\| x \|) \leqq V(x, t) \leqq b(\| x \|)$, $V'(x, t) \leqq -c(\| x \|)$, a, b, c being functions having the same properties of $a(r)$ in (viii).

Lyapunov proved in 1892 [14] the following

THEOREM 9. *If $V \in C_1$ exists satisfying assumption* (x), *the solution $x = 0$ is asymptotically stable* ((x) *implies* (iii)).

In 1935–37 Malkin [15] and Persidskiĭ [19] proved theorems which, taking into account Theorem 7 (f) may be stated thus:

THEOREM 10*. *If the system is linear, a necessary and sufficient condition for exponential-asymptotic stability* (vii) *is that a function V exists satisfying assumption* (x) (*for linear systems,* (x) *is equivalent to* (vi)).

I proved in 1949 [18]:

THEOREM 11*. *If a function $V \in C_1$ exists which satisfies assumption* (x), *the solution $x = 0$ is equiasymptotically stable* ((x) *implies* (iv)); *conversely, if the differential equation is periodic and $f \in C_1$ (in this case C_1 and $\bar{C}_1$ coincide), and if the solution $x = 0$ is asymptotically stable, there is a $V \in \bar{C}_1$ satisfying assumption* (x) (*for periodic systems* (iii) *implies* (x)).

In 1951, by an entirely different method, Barbašin [1] proved the converse, in the case of autonomous equations, showing that if $f \in C_s$, $s \geqq 1$, it is possible to find $V \in C_s$. In 1954, Malkin [17], using my method of 1949, proved:

THEOREM 12*. *If $f \in \bar{C}_1$, a necessary and sufficient condition for the uniform-asymptotic stability of the solution $x = 0$ is that a function $V \in \bar{C}_1$ exist satisfying assumption* (x) (*for equations with $f \in \bar{C}_1$,* (x) *is equivalent to* (vi)).

In my course at Varenna, I proved that this last theorem could be completed in the sense that, if $f \in \bar{C}_1 \cap C_s$, $s > 1$, it is possible to find $V \in \bar{C}_1 \cap C_s$.

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[3] This generalization of the idea of total derivative is essentially due to T. Yoshizawa [23] who applied it to the investigation of periodic solutions.

The main purpose of this section is to prove the following theorems:

THEOREM 13. *If a function* $V \in C_0$ *exists satisfying assumption* (x), *the solution* $x = 0$ *is uniform-asymptotically stable* ((x) *implies* (vi)).

THEOREM 14*. *If the space is finite dimensional, if* $f \in C_0$ *and if the solution* $x = 0$ *is uniform-asymptotically stable, there is a function* $V \in C_\infty$ *satisfying assumption* (x) ((vi) + (C_0) *imply* (x)). *Moreover, if* $f \in \bar{C}_0$, *it is possible to find* $V \in \bar{C}_\infty$. *If f does not depend on t or is periodic in t, it is possible to find V with the same properties.*

THEOREM 15. *If there is a function* $V \in C_0$ *satisfying assumption* (viii), *the solution* $x = 0$ *is asymptotically stable* ((viii) *implies* (iii)).

THEOREM 16. *If there is a function* $V \in C_0$ *satisfying assumption* (ix) *and if the solution* $x = 0$ *is uniformly stable, it is uniform-asymptotically stable* ((ix) + (ii) *imply* (vi)).

THEOREM 17. *If there is a function* $V \in C_0$ *satisfying assumption* (ix) *and if* $f \in \bar{C}_0$, *the solution* $x = 0$ *is uniform-asymptotically stable* ((ix) + ($\bar{C}_0$) *imply* (vi)).

THEOREM 18. *If the space is a real Hilbert space, if there is a function* $V \in C_0$ *satisfying assumption* (ix) *and if f is such that* $(x, f) \leqq \| x \| . g(\| x \|)$, *where* (x, y) *denotes the inner product and* $g(r)$ *is a positive continuous function defined for* $r > 0$ *such that* $\int_0 dr/g(r) = +\infty$, *the solution* $x = 0$ *is uniform-asymptotically stable.*[4]

Theorems 13 and 14*, together with Theorem 7 (b) and (e)*, are more general and stronger than Theorems 9, 11*, 12* and the already mentioned result of Varenna; these same theorems, together with Theorem 7 (f) imply Theorem 10*. Theorem 15 generalizes Theorem 9. The three last theorems give sufficient criteria for uniform-asymptotic stability which may be easier to apply than Theorem 13. The reciprocal of Theorem 16 is obviously true, in finite dimensional spaces and if $f \in C_0$, as a consequence of Theorem 14* and the obvious fact that (x) implies (ix).

These results (apart from the complements relative to the regularity properties of V) may be visualized in Figure 5:[5]

It seems clear from this diagram that the most important concept of asymptotic stability is uniform-asymptotic stability. To this it must be added that the set of uniform-asymptotically stable solutions of a given equation is open (Theorem 8).

PROOF OF THEOREM 13. Taking into account Lemma 1, the proof is not different from that of Theorem 9. Given $\varepsilon > 0$, let $\delta(\varepsilon) > 0$ be such that $a(\varepsilon) > b[\delta(\varepsilon)]$; then, if $\| x_0 \| < \delta(\varepsilon)$, $t_0 \geqq 0, t \geqq t_0$, $x = F(t, x_0, t_0)$, from the assumption on V' and Lemma 1, we have $a(\| x \|) \leqq V(x, t) \leqq V(x_0, t_0) \leqq b[\delta(\varepsilon)] < a(\varepsilon)$

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[4] It should be noted that only the *positive* values of (x, f) are restricted.

[5] The relations between the different assumptions indicated in the diagram are valid in infinite dimensional spaces, except the implication (iii) $\rightarrow$ (P) $\rightarrow$ (vi). The symbols (P), (L) indicate that f is, respectively, periodic in t or linear in x.

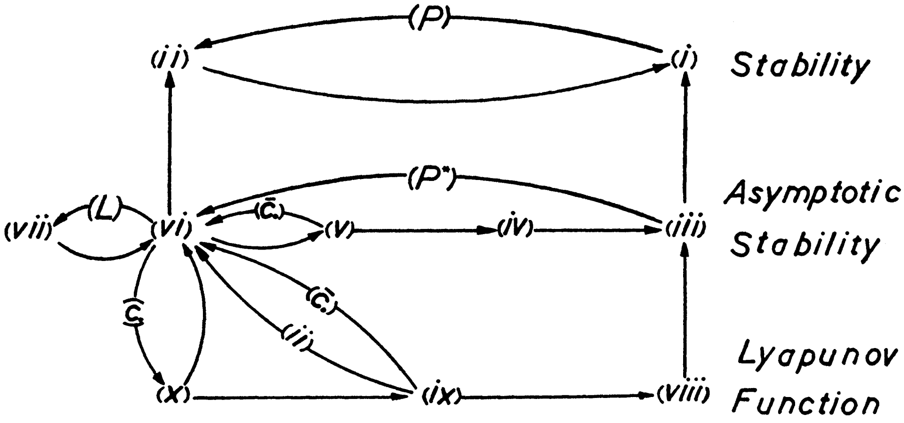

FIG. 5

whence $\| x \| < \varepsilon$, which proves (ii). On the other hand, if $\| x_0 \| < \delta_0 = \delta(a)$ (a being the radius of the semicylinder where the assumptions of the theorem are satisfied) and if $T(\varepsilon) = b(\delta_0)/c[\delta(\varepsilon)]$, and if we had $\| x \| \geqq \delta(\varepsilon)$ in the interval $t_0 \leqq t \leqq t_0 + T(\varepsilon)$, we would have $V'(x, t) \leqq -c[\delta(\varepsilon)]$ in this interval and $V(x, t) \leqq V(x_0, t_0) - c[\delta(\varepsilon)].(t - t_0) \leqq b(\delta_0) - c[\delta(\varepsilon)].(t - t_0)$ which leads to absurdity when $t = t_0 + T(\varepsilon)$. There is therefore a t_1 in this interval such that $\| x_1 \| < \delta(\varepsilon)$ whence $\| F(t, x_0, t_0) \| = \| F(t, x_1, t_1) \| < \varepsilon$ when $t \geqq t_1$ and a fortiori when $t \geqq t_0 + T(\varepsilon)$.

PROOF OF THEOREM 15. Lyapunov's theorem on (simple) stability shows that the solution $x = 0$ is stable (i). Along the trajectory through (x_0, t_0) we have $\int_{V(x_0,t_0)}^{V(x,t)} dV/c(V) \leqq -(t - t_0)$ and as $t \to +\infty$ the integral should tend to $-\infty$ which is only possible if $V(x, t) \to 0$ and from $V(x, t) \geqq a(\| x \|)$ it follows $x \to 0$ so that (iii) is satisfied.

PROOF OF THEOREM 16. Let B be an upper bound of V and

$$T(\varepsilon) = \int_{a(\varepsilon)}^{B} dV/c(V).$$

If $\| x_0 \| < \delta_0 = \delta(a)$, $t_0 \geqq 0$, $t \geqq t_0 + T(\varepsilon)$, we have

$$\int_{V(x_0,t_0)}^{V(x,t)} dV/c(V) \leqq -T(\varepsilon) = \int_{B}^{a(\varepsilon)} dV/c(V)$$ and, because $V(x_0, t_0) \leqq B$,

this implies $V(x, t) \leqq a(\varepsilon)$ whence $\| x \| \leqq \varepsilon$.

PROOF OF THEOREM 17. After Theorem 16 it is enough to prove (ii). Let $T(\varepsilon)$ be defined as in the proof of Theorem 16 and, if M is a Lipschitz constant of f, define $\delta(\varepsilon) = \varepsilon.\exp[-M.T(\varepsilon)]$. If $\| x_0 \| < \delta(\varepsilon)$, $t_0 \geqq 0$, and if $t_0 \leqq t \leqq t_0 + T(\varepsilon)$, we have $\| F(t, x_0, t_0) \| \leqq \| x_0 \|.\exp[M.T(\varepsilon)] < \varepsilon$. If there was a point (x_1, t_1) on the solution such that $\| x_1 \| = \varepsilon$, we should then have $t_1 \geqq t_0 + T(\varepsilon)$ and this leads to the same contradiction as in the proof of Theorem 16.

Proof of Theorem 18. Let $\delta(\varepsilon) > 0$ be such that $\int_{\delta(\varepsilon)}^{\varepsilon} dr/g(r) > T(\varepsilon)$, where $T(\varepsilon)$ is the same as in the proof of Theorem 16. As $d \| x \|^2/dt = 2(x,f) \leqq 2 \| x \|.g(\| x \|)$, if $\| x_0 \| < \delta(\varepsilon)$, $t_0 \geqq 0$, and if $t_0 \leqq t \leqq t_0 + T(\varepsilon)$ we will have $\int_{\|x_0\|}^{\|x\|} d \| x \| /g(\| x \|) \leqq T(\varepsilon)$ whence $\| x \| < \varepsilon$; the argument proceeds then as before.

For the proof of Theorem 14* we need the following auxiliary result which will be also used in the proof of theorems of Section 4.

Lemma 2. *Given any real function of real variables $A(r, t)$, defined, continuous and positive in the quadrant $Q: r > 0$, $t \geqq 0$, there are two functions $g(r)$, $h(t) \in C_\infty$ in $[0, +\infty)$, $h(t) > 0$, $g(r) > 0$ when $r > 0$, $g(0) = 0$, such that $g(r)h(t) \leqq A(r, t)$, $g(r) \mid h'(t) \mid \leqq A(r, t)$, $\mid g'(r) \mid h(t) \leqq A(r, t)$ in Q.*

Let us first define in $[0, +\infty)$ a continuous function $a(r)$ by

$$a(r) = a(1/r)$$
$$= \inf \{\exp(-r^2), A(\rho, \tau), \rho A(\rho, \tau): (2r)^{-1} \leqq \rho \leqq 2r, 0 \leqq \tau \leqq r, r \geqq 1\}.$$

$a(r)$ is positive for $r > 0$, $a(0) = 0$, increasing for $r < 1$, decreasing for $r > 1$.

Consider how any kernel function $K(r) \in C_\infty$, defined in $(-\infty, +\infty)$, $K \geqq 0$, $K(r) = 0$ for $\mid r \mid \geqq 1$, $\int_{-\infty}^{+\infty} K(r)dr = 1$, and define

$$g(r) = 2r^{-1} \int_{-\infty}^{+\infty} a(\rho)K[2(\rho - r)r^{-1}]\, d\rho = 2r^{-1} \int_{r/2}^{3r/2} a(\rho)K[2(\rho - r)r^{-1}]\, d\rho, \quad r \neq 0,$$

$$g(0) = 0$$

or

$$g(r) = \int_{-\infty}^{+\infty} a[\tfrac{1}{2}(2 + \rho)r]K(\rho)\, d\rho.$$

Obviously $g(r) \in C_\infty$ for $r \neq 0$ and, owing to the bound $a(r) \leqq \exp(-r^{-2})$, also for $r = 0$. An easy computation gives

$$g'(r) = -r^{-1} \int_{-\infty}^{+\infty} a[\tfrac{1}{2}(2 + \rho)r].[(2 + \rho)K'(\rho) + K(\rho)]\, d\rho.$$

I say that, for each fixed t, $\inf \{A(r, t)/ \mid g'(r) \mid , A(r, t)/g(r): r > 0\} > 0$. The infimum could be zero only if $A(r, t)/ \mid g'(r) \mid$ or $A(r, t)/g(r) \to 0$ either when $r \to 0$ or when $r \to +\infty$. But this is impossible; in fact, if $r \geqq 2(t + 1) \geqq 2$, we have $g(r) \leqq a(\tfrac{1}{2}r) \leqq A(r, t)$ and, on the other hand, if

$$M = \int_{-\infty}^{+\infty} (2 + \rho) \mid K'(\rho) \mid d\rho, \mid g'(r) \mid \leqq r^{-1}.a(\tfrac{1}{2}r).(M + 1) \leqq A(r, t).(M + 1);$$

and a similar argument applies if $r \leqq [2(t + 1)]^{-1} \leqq \tfrac{1}{2}$.

Let then $b(t) = \inf\{A(\rho, \tau)/|g'(\rho)|, A(\rho, \tau)/g(\rho): \rho > 0, 0 \leqq \tau \leqq t\}$,

$$c(t) = \int_{-\infty}^{+\infty} b(\tau)K(\tau - t - 1)\, d\tau = \int_{-1}^{1} b(\tau + t + 1)K(\tau)\, d\tau$$

$$h(t) = \int_{t}^{+\infty} c(\tau).\, e^{-\tau}\, d\tau.$$

b is a decreasing function and the same is then true of c; $c \in C_\infty$ whence $h \in C_\infty$.

$$|h'(t)| \leqq c(t).e^{-t} \leqq c(t) \leqq b(t)$$

and, from the second mean value theorem,

$$h(t) = c(t) \int_{t}^{t'} e^{-\tau}\, d\tau \leqq c(t) \leqq b(t).$$

This completes the proof of the Lemma.

Proof of Theorem 14*. Let $\delta(\varepsilon)$, $T(\varepsilon)$ be the functions characterizing the uniform-asymptotic stability of the solution $x = 0$; we may assume that they are continuous strictly monotonic functions. Let $L(r, T)$ be a Lipschitz constant of the function $\| F(t + \tau, x, t) \|$ in the region $0 < r \leqq \| x \| \leqq b = \inf\{1, \delta(a)\}$, $0 \leqq t \leqq T$, $0 \leqq \tau \leqq T(\frac{1}{2}r)$; we may define $L(r, T) = L(b, T)$ for $r \geqq b$. By Lemma 2 there are functions $g(r), M(T) \in C_\infty$, defined for $r > 0$, $T \geqq 0$, $M(T) > 1$ for $0 \leqq T < +\infty$, $0 < g(r) < \delta(r)$ for $0 < r < 1$, $g(0) = 0$, such that $g(r)L[\frac{1}{2}\delta(r), T] \leqq M(T)$ for $0 < r < 1$, $T \geqq 0$; it is enough to apply the Lemma to the function $A(r, t) = \{L[\frac{1}{2}\delta(r), t]\}^{-1}$ and take $M(T) = [h(T)]^{-1}$. Let $G(r) = \int_0^r dr \int_0^r g(r) dr$, whence $G'(r)$ is an increasing function $< g(r)$. We also have

$$2G(\tfrac{1}{2}r) = 2\int_0^{r/2} dr \int_0^r g(r)\, dr = \int_0^r dr \int_0^{r/2} g(r)\, dr < G(r).$$

For $\| x \| \leqq b$, $t \geqq 0$, define

$$U(x, t) = \sup\{G(\| F(t + \tau, x, t) \|).(1 + 2\tau)(1 + \tau)^{-1}: \tau \geqq 0\}.$$

As $1 \leqq (1 + 2\tau)(1 + \tau)^{-1} \leqq 2$ for $\tau \geqq 0$, we have

$$G(\| x \|) \leqq U(x, t) \leqq 2G[\varepsilon(\| x \|)] \leqq 2\| x \|,$$

where $\varepsilon = \varepsilon(\delta)$ is the inverse function of $\delta = \delta(\varepsilon)$; this proves that U is positive definite and has an infinitely small upper bound.

If $x \neq 0$, $\tau \geqq T(\frac{1}{2}\| x \|)$, we have

$$G(\| F(t + \tau, x, t) \|).(1 + 2\tau)(1 + \tau)^{-1} \leqq 2G(\tfrac{1}{2}\| x \|) \leqq G(\| x \|) \leqq U(x, t),$$

so that the supremum defining U is reached in the interval $0 \leqq \tau \leqq T(\frac{1}{2}\| x \|)$. The function $G(\| F(t + \tau, x, t) \|)(1 + 2\tau)(1 + \tau)^{-1}$ has, in the region $0 < r \leqq \| x \| \leqq 2r$, $0 \leqq t \leqq T$, $0 \leqq \tau \leqq T(\frac{1}{2}r)$, a Lipschitz constant

$$\leqq 4G'[\epsilon(2r)].\mathrm{L}(r, T) + G[\varepsilon(2r)] \leqq 4g[\varepsilon(2r)].L(r, T) + 1 \leqq 4M(T) + 1,$$

so that the function U has in the region $\| x \| \leqq b$, $0 \leqq t \leqq T$, a Lipschitz constant $\leqq 4M(T) + 1$ (for $x = 0$ this follows from $U(x, t) \leqq 2 \| x \|$) and $U \in C_0$.

If $f \in \bar{C}_0$, we may take $L(r, T)$ independent of T, $M(T)$ constant, whence $U \in \bar{C}_0$. If f is periodic in t, U is also obviously periodic in t.

Finally, if $h > 0$, $x' = F(t + h, x, t)$, if $0 \leqq \tau' \leqq T(\frac{1}{2}\| x' \|)$ is such that $U(x', t + h) = G(\| F(t + h + \tau', x', t + h) \|).(1 + 2\tau')(1 + \tau')^{-1}$ and if $\tau = \tau' + h$, we have

$$\begin{aligned} U(x', t + h) &= G(\| F(t + \tau, x, t) \|).(1 + 2\tau)(1 + \tau)^{-1} \\ &\qquad \cdot [1 - h(1 + \tau')^{-1}(1 + 2\tau)^{-1}] \\ &\leqq U(x, t).[1 - h(1 + \tau')^{-1}(1 + 2\tau)^{-1}], \end{aligned}$$

whence

$$\begin{aligned} &[U(x', t + h) - U(x, t)].h^{-1} \\ &\qquad \leqq - U(x, t).[1 + T(\tfrac{1}{2} \| x' \|)]^{-1}[1 + 2h + 2T(\tfrac{1}{2} \| x' \|)]^{-1} \end{aligned}$$

and by letting $h \to 0$, we find

$$U'(x, t) \leqq -2 \| x \|.[1 + 2T(\tfrac{1}{2} \| x \|)]^{-2} = -c(\| x \|)$$

and U' is negative definite. A remark which will be used later is that $c(r)$ is an increasing function.

The function U satisfies then all the conditions stated in Theorem 14* except that $U \in C_0$ instead of $U \in C_\infty$. We are now going to smooth out U so that this last condition is also satisfied.

Let $N(t) \geqq 1$ be a Lipschitz constant of $f(\xi, \tau)$ in the region $\| \xi \| \leqq b$, $0 \leqq \tau \leqq t$. It is possible to find real functions of real variables $\theta(t)$, $\rho(r)$, $\varphi(u) \in C_\infty$, θ always positive, $\rho(0) = \varphi(0) = 0$, ρ and φ positive, increasing and with increasing first derivatives for $r, u > 0$, such that

$$\begin{aligned} \theta(t).[4M(t + 1) + 1].N(t + 1) &\leqq 1 \\ | \theta'(t) | .[4M(t + 1) + 1].N(t + 1) &\leqq 1 \\ \rho(r) \leqq 1, \ \rho(r) &\leqq \tfrac{1}{2}r \\ 6\rho(r) + 4r | \rho'(r) | &\leqq c(\tfrac{1}{2}r) \\ \varphi(3r).| \rho^{(s)}(r) |.[\rho(r)]^{-p} &\to 0 \end{aligned}$$

when $r \to 0$ for any fixed integers p, s. To find θ, ρ satisfying the first conditions presents no difficulties. Once this is done, for each integer m define

$$\varphi_m = \inf \{[\rho(r)]^p. | \rho^{(s)}(r) |^{-1}: 0 \leqq p \leqq m, 0 \leqq s \leqq m, (3m)^{-1} \leqq r \leqq 1\} > 0,$$

so that φ_m tends to zero monotonically when $m \to \infty$. Define $\varphi(m^{-1}) = \varphi_{m+1}^2$, $\varphi(u)$ linear between m^{-1} and $(m - 1)^{-1}$. Then, if p, s are any fixed integers, if

$m \geqq \sup(p, s)$ and if $(3m)^{-1} \leqq r \leqq [3(m-1)]^{-1}$, we will have

$$[\rho(r)]^p . \mid \rho^{(s)}(r) \mid^{-1} \geqq \varphi_m, \qquad \varphi(3r) \leqq \varphi[(m-1)^{-1}] = \varphi_m^2,$$

whence $\varphi(3r). \mid \rho^{(s)}(r) \mid .[\rho(r)]^{-p} \leqq \varphi_m \to 0$ when $r \to 0$. It only remains to smooth out φ conveniently so that $\varphi \in C_\infty$.

Let $K(x, t) \in \bar{C}_\infty$ be a nonnegative kernel function, vanishing outside the set $\| x \| \leqq 1$, $0 \leqq t \leqq 1$, such that $\int K(x, t)\, dx\, dt = 1$, extended over the whole space. If the vectors x are n-dimensional, define

$$V(x, t) = \int \varphi[U(\xi, \tau)].K\{(\xi - x).[\rho(\| x \|).\theta(t)]^{-1}, (\tau - t).[\rho(\| x \|).\theta(t)]^{-1}\}. [\rho(\| x \|).\theta(t)]^{-n-1}\, d\xi\, d\tau \qquad \text{when } x \neq 0,$$

$$V(0, t) = 0.$$

By means of the change of variables

$$\xi = x + \rho(\| x \|).\theta(t).X, \qquad \tau = t + \rho(\| x \|).\theta(t).T,$$

we find

$$V(x, t) = \int \varphi[U(x + \rho(\| x \|).\theta(t).X, t + \rho(\| x \|).\theta(t).T)].K(X, T)\, dX\, dT,$$

which is also valid when $x = 0$. As it is enough to extend this last integral to the region $\| X \| \leqq 1$, $0 \leqq T \leqq 1$, the variables ξ, τ satisfy

$$\| \xi - x \| \leqq \rho(\| x \|).\theta(t), \qquad t \leqq \tau \leqq t + \rho(\| x \|).\theta(t)$$

and, in particular, $\| x \|/2 \leqq \| \xi \| \leqq 3 \| x \|/2$, $t \leqq \tau \leqq t + 1$. It follows that V is defined in the region $\| x \| \leqq 2b/3$, $t \geqq 0$, and we have

$$\varphi[G(\tfrac{1}{2} \| x \|)] \leqq \varphi[U(\xi, \tau)] \leqq \varphi(3 \| x \|),$$

whence

$$\varphi[G(\tfrac{1}{2} \| x \|)] \leqq V(x, t) \leqq \varphi(3 \| x \|)$$

which shows that V is positive definite and has an infinitely small upper bound. V obviously belongs to C_∞ for any $x \neq 0$ and, the derivatives being bounded by integrals of the form

$$\int \varphi(3 \| x \|).\mid D_1 K \mid.\mid D_2 \rho \mid.\mid D_3 \theta \mid.\rho^{-p}\theta^{-q}\, d\xi\, d\tau,$$

where the D_i represent certain derivatives of the functions indicated and p, q integers, for each fixed $t \geqq 0$ these derivatives tend to zero as $x \to 0$, which proves that $V \in C_\infty$ also when $x = 0$. If $f \in \bar{C}_0$ we have $U \in \bar{C}_0$, M, N and therefore θ may be taken constants, the bounds for the derivatives of V are independent of t and $V \in \bar{C}_\infty$. If f is periodic in t, U has the same property and moreover θ can again be taken constant so that V is periodic.

We must only prove then that V' is negative definite. Let (x, t), (x', t') be any two points on the same integral curve $t < t' < t + 1$. We have

$$
\begin{aligned}
&V(x', t') - V(x, t) \\
&= \int \{\varphi[U(x' + \rho(\| x' \|)\theta(t')X, t' + \rho(\| x' \|)\theta(t')T)] \\
&\qquad\qquad -\varphi[U(x + \rho(\| x \|)\theta(t)X, t + \rho(\| x \|)\theta(t)T)]\}.K(X, T)\, dX\, dT \\
&= \int \{\varphi[U(x' + \rho(\| x' \|)\theta(t')X, t' + \rho(\| x' \|)\theta(t')T)] \\
&\qquad\qquad - \varphi[U(\xi'', t' + \rho(\| x \|)\theta(t)T)]\}.K(X, T)\, dX\, dT \\
&\qquad + \int \{\varphi[U(\xi'', t' + \rho(\| x \|)\theta(t)T)] \\
&\qquad\qquad - \varphi[U(x + \rho(\| x \|)\theta(t)X, t + \rho(\| x \|)\theta(t)T)]\}.K(X, T)\, dX\, dT,
\end{aligned}
$$

where $\xi'' = F(t' + \rho(\| x \|)\theta(t)T, x + \rho(\| x \|)\theta(t)X, t + \rho(\| x \|)\theta(t)T)$. If we divide by $t' - t$, the absolute value of the integrand of the first integral is bounded by

$$
\begin{aligned}
&\varphi'[U(\xi_0, \tau_0)].[4M(t + 1) + 1].\{ \| x' + \rho(\| x' \|)\theta(t')X - \xi'' \| \\
&\qquad\qquad + | \rho(\| x' \|)\theta(t') - \rho(\| x \|)\theta(t) | T\}.K(X, T).(t' - t)^{-1} \\
&\qquad \leqq \varphi'[U(\xi_0, \tau_0)].[4M(t + 1) + 1].\{ \| x' + \rho(\| x \|)\theta(t)X - \xi'' \| \\
&\qquad\qquad +2 | \rho(\| x' \|)\theta(t') - \rho(\| x \|)\theta(t) | \}.K(X, T).(t' - t)^{-1},
\end{aligned}
$$

where (ξ_0, τ_0) tends to $(\xi, \tau) = (x + \rho(\| x \|)\theta(t)X, t + \rho(\| x \|)\theta(t)T)$ as $t' \to t$.

We have
$$
\begin{aligned}
\xi'' &= x + \rho(\| x \|)\theta(t)X + (t' - t).f(\xi_1, \tau_1) \\
x' &= F(t', x, t) = x + (t' - t).f(\xi_2, \tau_2),
\end{aligned}
$$

where (ξ_1, τ_1), (ξ_2, τ_2) tend, when $t' \to t$, respectively to (ξ, τ), (x, t). We then have

$$x' + \rho(\| x \|)\theta(t)X - \xi'' = (t' - t)[f(\xi_1, \tau_1) - f(\xi_2, \tau_2)]$$

and the lim sup of the integrand considered is

$$
\begin{aligned}
&\leqq \varphi'[U(\xi, \tau)].[4M(t + 1) + 1].\{N(t + 1)\rho(\| x \|)\theta(t) + 2\rho(\| x \|) | \theta'(t) | \\
&\qquad\qquad + 2 | \rho'(\| x \|) | . \| x \|.N(t + 1)\theta(t)\}K(X, T) \\
&\leqq \varphi'[U(\xi, \tau)].c(\tfrac{1}{2} \| x \|).K(X, T)/2,
\end{aligned}
$$

where the expression $\| x \|.N(t + 1)$ is an upper bound of $d \| x \|/dt$.

As to the integrand of the second integral, after division by $t' - t$ its lim sup when $t' \to t$ is the generalized total derivative of $\varphi(U)$ at the point (ξ, τ), so that it is bounded by

$$-\varphi'[U(\xi, \tau)].c(\| \xi \|).K(X, T) \leqq -\varphi'[U(\xi, \tau)].c(\tfrac{1}{2} \| x \|).K(X, T).$$

Finally,

$$V'(x, t) \leqq -\frac{1}{2} \int \varphi'[U(\xi, \tau)].c(\tfrac{1}{2} \| x \|).K(X, T)\, dX\, dT$$
$$\leqq -\tfrac{1}{2}\varphi'[G(\tfrac{1}{2} \| x \|)].c(\tfrac{1}{2} \| x \|).$$

REMARK. The first part of the proof of Theorem 14*, i.e. the construction of the function U, may be carried out without changes in infinite dimensional spaces; the second part, on the contrary, requires the existence of an integral with such strong invariance properties that the assumption that the dimension is finite is unavoidable. I have not been able therefore to extend Theorem 14* to infinite dimensional spaces, which would require an entirely different method of proof, but the following results, not so general and strong in other respects, are true in general spaces:

THEOREM 19. *If $f \in C_0$ and the solution $x = 0$ is uniform-asymptotically stable, there is a function $V \in C_0$ satisfying assumption* (x) ((vi) + (C_0) *imply* (x)). *If $f \in \bar{C}_0$ it is possible to find $V \in \bar{C}_0$. If f does not depend on t or is periodic in t it is possible to find V with the same properties.*

THEOREM 20. *If $f \in \bar{C}_0 \cap C_s$ and the solution $x = 0$ is uniform-asymptotically stable, there is a function $V \in \bar{C}_0 \cap C_s$ which satisfies assumption* (x). *If $f \in \bar{C}_m$ it is possible to find $V \in \bar{C}_m$. If f does not depend on t or is periodic in t it is possible to find V with the same properties.*

The proof of Theorem 19 is the first part of the proof of Theorem 14*. The proof of Theorem 20 may be carried out by essentially the same method used in [18] and will be published elsewhere.

4. On asymptotic stability in the large

Barbašin and Krasovskiĭ [2] defined the following concept:

The solution $x = 0$ of the differential equation $\dot{x} = f(x, t)$, where f is defined for all x and $t \geqq 0$ and $f(0, t) = 0$, is said to be *asymptotically stable for arbitrary perturbations* (shortly, *asymptotically stable in the large*) if it is stable and, moreover, for any (x_0, t_0), $t_0 \geqq 0$, we have $\lim_{t \to +\infty} F(t, x_0, t_0) = 0$.

In the same paper, Barbašin and Krasovskiĭ proved conditions which are essentially necessary and sufficient for such type of stability, assuming that the equation is autonomous (f independent of t). At Varenna, I was able to generalize these results to periodic equations. In a new paper [3], Barbašin and Krasovskiĭ considerably generalize the previous results; to this end, they introduce the following definition:

The solution $x = 0$ is *uniform-asymptotically stable in the large* if two positive continuous functions exist $\varepsilon(\delta)$ and $T(\delta, \varepsilon)$, defined for any positive values of the arguments, $\lim_{\delta\to 0}\varepsilon(\delta) = 0$, such that (a) $\| x_0 \| < \delta$, $t_0 \geqq 0$, $t \geqq t_0$, imply $\| F(t, x_0, t_0) \| < \varepsilon(\delta)$; (b) $\| x_0 \| < \delta$, $t_0 \geqq 0$, $t \geqq t_0 + T(\delta, \varepsilon)$, imply $\| F(t, x_0, t_0) \| < \varepsilon$. It is always possible to assume that $\varepsilon(\delta)$ is an increasing function.

However, the theorems of [3] still require certain unnatural restrictions on the function f, namely, the existence of bounds not depending on t for f and its first partials. The following theorems eliminate these restrictions and moreover its contentions are much stronger than those of the preceding results.

THEOREM 21*. *If the space is finite dimensional and the differential equation is autonomous or periodic, asymptotic stability in the large implies uniform-asymptotic stability in the large.*

The proof is entirely similar to that of Theorem 7 (d) and (e)*.

THEOREM 22. *If a Lyapunov's function $V(x, t) \in C_0$ exists which is positive definite, infinitely large*[6] *and has an infinitely small upper bound, and such that its generalized total derivative is negative definite, the solution $x = 0$ is uniform-asymptotically stable in the large.*

The proof is immediate, by the usual methods.

THEOREM 23*. *If the space is finite dimensional, $f \in C_0$, and the solution $x = 0$ is uniform-asymptotically stable in the large, there is a function $V(x, t) \in C_\infty$, positive definite, infinitely large, having an infinitely small upper bound, such that V' is negative definite. If $f \in \bar{C}_0$,*[7] *it is possible to find $V \in \bar{C}_\infty$. If f is periodic in t or independent of t, it is possible to find V having the same properties.*

The proof of Theorem 23* is almost identical to that of Theorem 14*. We first construct the function U as in Theorem 14*; the supremum defining U is now reached for $\tau \leqq T(\| x \|, \frac{1}{2}\| x \|)$. The functions g, G are defined as in Theorem 14*, $L(r, T)$ being a Lipschitz constant of $\| F(t + \tau, x, t) \|$ in the region $0 < r \leqq \| x \| \leqq 1/r$, $0 \leqq t \leqq T$, $0 \leqq \tau \leqq T(r, \frac{1}{2}r)$, when $r \leqq 1$; if $r > 1$, the first inequality should be replaced by $0 < 1/r \leqq \| x \| \leqq r$. As $G \to \infty$ when $r \to +\infty$, U is infinitely large, the other properties of U being proved as in Theorem 14*.

Let now $M(r, t) \geqq 1$, $N(r, t) \geqq 1$, be Lipschitz constants of $U(\xi, \tau)$, $f(\xi, \tau)$ in the region $\| \xi \| \leqq r$, $0 \leqq \tau \leqq t$. From Lemma 2 follows the existence of real functions of real variables $\theta(t)$, $\rho(r)$, $\varphi(u) \in C_\infty$, θ always positive, $\rho(0) = \varphi(0) = 0$, ρ and φ positive, increasing and with increasing first derivatives for r, $u > 0$, $\varphi \to +\infty$ when $u \to +\infty$, such that

$$\rho(r) \leqq 1, \qquad \rho(r) \leqq \tfrac{1}{2}r, \qquad \theta(t) \leqq 1$$

$$2M(r, t + 1)[N(r, t + 1)\rho(r)\theta(t) + 2\rho(r). \mid \theta'(t) \mid + rN(r, t + 1). \mid \rho'(r) \mid \theta(t)] \leqq c(\tfrac{1}{2}r)$$

[6] I.e. [2], the function $a(r)$ in assumption (viii) of Section 3 tends to infinity with r.

[7] In the present case, we shall say that $f \in \bar{C}_s$ if the differentials up to the order s are bounded uniformly with respect to t, i.e. $f \in \bar{C}_s$ in any semi-cylinder which is the cartesian product of any bounded set in x-space by the interval $[0, +\infty)$.

$\varphi(3r).\, |\, \rho^{(s)}(r)\, |\, .[\rho(r)]^{-p} \to 0$ when $r \to 0$, p, s being any fixed integers. It suffices to apply the Lemma with $A(r, t) = c(\frac{1}{2}r).[8(r + 1)M(r, t + 1)N(r, t + 1)]^{-1}$. With these definitions the proof proceeds in an entirely similar way as for Theorem 14*.

REMARK. With respect to the validity of Theorem 23* in infinite dimensional spaces the same remarks apply as for Theorem 14*; in particular, it is possible to prove theorems similar to Theorems 19 and 20.

5. On stability in the first approximation

The following theorem is stronger than one proved by Malkin [16]:

THEOREM 24*. *In a finite dimensional space, let $f(x) \epsilon C_0$ be a positively homogeneous vectorial function of degree m ($f(kx) = k^m f(x)$ for any vector x and any positive number k). If the solution $x = 0$ of the differential equation $\dot{x} = f(x)$ (we assume $f(0) = 0$) is asymptotically stable, there is a number $M > 0$ such that, for any function $g(x, t)$, $g(0, t) = 0$, satisfying the inequality $\| g(x, t) \| \leqq M. \| x \|^m$, uniformly in t in a neighborhood of $x = 0$, the solution $x = 0$ of the equation $\dot{x} = f(x) + g(x, t)$ is uniform-asymptotically stable.*

By Theorems 7 (e)* and 14*, there is a function $V(x) \epsilon C_\infty$ which is positive definite and such that $\delta V[x; f]$ is negative definite. Let us call $S(1)$ the surface $V = V_0 =$ const., V_0 being so small that the surface is entirely contained in the neighborhood where g is defined. The vector f on the surface $S(1)$ forms with the inner normal an angle whose cosine is $> 3\alpha > 0$.

Consider the surface $S(\theta)$ homothetical to $S(1)$ with respect to the origin and with a ratio θ, $0 < \theta \leqq 1$ (these surfaces may intersect each other). At corresponding points of $S(1)$ and $S(\theta)$ the normals are parallel and also the field vectors $f(x)$ and $\theta^m f(x)$; therefore the latter also forms with the inner normal an angle whose cosine is $> 3\alpha$. It is now possible to choose the constant M so small that the vectors $f(x)$ and $f(x) + g(x, t)$ form such a small angle that the cosine of the angle formed by the latter with the inner normal is $> 2\alpha$ (this is possible because, owing to the assumption of asymptotic stability, f vanishes only at $x = 0$). As a consequence, if an integral curve of $\dot{x} = f + g$ intersects $S(\theta)$ at the point (x_0, t_0) it cannot intersect it again for $t > t_0$.

Consider the inner parallel surface $S'(1)$ to $S(1)$ taken so close to $S(1)$ that any ray passing through any point x of $S(1)$ and forming with the inner normal an angle whose cosine is $> \alpha$ intersects $S'(1)$ at a first point x' whose distance from x is $< \beta$; this is possible because $S(1)$ has continuous curvature (V belongs to C_∞). Let $C(x, 1)$ be the portion of cone formed by all the segments xx' belonging to the rays considered. We may assume that the distance between $S(1)$ and $S'(1)$ is so small that for each $t \geqq 0$ and each $y \epsilon C(x, 1)$, the vector $f(y) + g(y, t)$ forms with the inner normal at x an angle whose cosine is $> \alpha$. Let $G(1)$ be the open set bounded by $S(1)$ and $S'(1)$, r, r' the radii of two spheres centered at the origin such that the first one contains $S(1)$ and the second one is interior to $S'(1)$, and let $\| f(x) \| \geqq 2N > 0$ for $\| x \| \geqq r'$.

By means of a homothetical contraction of ratio θ we obtain, besides the surface $S(\theta)$ already considered, the surface $S'(\theta)$, the open set $G(\theta)$ and the cones $C(x, \theta)$ such that: (a) a ray through any point x of $S(\theta)$ forming an angle with the inner normal whose cosine is $>\alpha$ intersects $S'(\theta)$ at a point x' whose distance from x is $<\beta\theta$; (b) the sphere of radius θr contains $S(\theta)$ and the sphere of radius $\theta r'$ is interior to $S'(\theta)$; $\| f(x) \| \geqq 2\theta^m N$ in $G(\theta)$. By choosing the constant M still smaller if it were necessary so that $\| f(x) + g(x, t) \| \geqq \frac{1}{2} \| f(x) \|$, it easily follows that: if (x_0, t_0) is any point, $x_0 \epsilon G(\theta)$, $t_0 \geqq 0$, and if y_0 is the foot of the normal to $S(\theta)$ passing through x_0, the solution of the equation $\dot{x} = f + g$ starting at (x_0, t_0) stays in $C(y_0, \theta)$ until its intersection with $S'(\theta)$ which takes place at a point (x_0', t_0') with $t_0' \leqq t_0 + \theta^{1-m}\beta/N$.

After these preliminaries the proof of the theorem proceeds as follows. In the first place, given $\varepsilon > 0$ define $\delta(\varepsilon) = r'\varepsilon/r$; then, if $\| x_0 \| < \delta(\varepsilon)$, x_0 is interior to $S'(\varepsilon/r)$ and therefore the solution $x = F(t, x_0, t_0)$ of $\dot{x} = f + g$ stays in $S(\varepsilon/r)$ for $t \geqq t_0$ so that $\| F(t, x_0, t_0) \| < (\varepsilon/r).r = \varepsilon$, which proves condition (ii) of Section 3 (uniform stability).

Let us prove condition (v). Define $\delta_0 = r'^2/r$ and let ε be any number, $0 < \varepsilon < \delta_0$. The open sets $G(\theta)$ for $0 < \theta \leqq 1$ cover the compact set $r'\varepsilon/r \leqq \| x \| \leqq r'$; let then $G(\theta_1), \cdots, G(\theta_k)$ be a finite covering family and $T(\varepsilon) = \beta N^{-1}. \sum_{i=1}^{i=k} \theta_i^{1-m}$. If (x_0, t_0) is any point, $\| x_0 \| < \delta_0$, $t_0 \geqq 0$, we will have $x_0 \epsilon G(\theta_{i_0})$; the integral curve of $\dot{x} = f + g$ through (x_0, t_0) intersects $S'(\theta_{i_0})$ at a point (x_1, t_1) with $t_1 - t_0 < \theta_{i_0}^{1-m}\beta/N$. If $r'\varepsilon/r \leqq \| x_1 \| \leqq r'$, we will have $x_1 \epsilon G(\theta_{i_1})$, $i_1 \neq i_0$, and the integral curve intersects $S'(\theta_{i_1})$ at a point (x_2, t_2) with $t_2 - t_1 < \theta_{i_1}^{1-m}\beta/N$, and so on. The successive indices $i_0, i_1, \cdots$ are different from each other because once the curve intersects a certain $S'(\theta)$ it can never again enter $G(\theta)$. Therefore, after $\leqq k$ steps which, in the variable t represent at most a total interval of length $T(\varepsilon)$, the condition $r'\varepsilon/r \leqq \| x \| \leqq r'$ will not be satisfied and as we always have $\| x \| \leqq r'$, because of the uniform stability already proved, a certain $\tau \leqq t_0 + T(\varepsilon)$ will exist such that $\| F(\tau, x_0, t_0) \| < r'\varepsilon/r$, and, again because of the uniform stability, we will have $\| F(t, x_0, t_0) \| < \varepsilon$ for $t \geqq \tau$ and a fortiori for $t \geqq t_0 + T(\varepsilon)$ which proves condition (v).

THEOREM 25*. *If the linear system* $\dot{x} = A(t)x$ *is reducible* [14] *and its solution* $x = 0$ *is asymptotically stable, there is a number* $M > 0$ *such that, for any function* $g(x, t)$, $g(0, t) = 0$, *satisfying the inequality* $\| g(x, t) \| \leqq M \| x \|$ (*uniformly in* t) *in a certain neighborhood of the origin, the solution* $x = 0$ *of the system* $\dot{x} = A(t)x + g(x, t)$ *is uniform-asymptotically stable.*

There is a change of variables $y = B(t)x$ with B, $\dot{B}$ and B^{-1} bounded such that $C = BAB^{-1} + \dot{B}B^{-1}$ is a constant matrix. The equation $\dot{x} = Ax + g$ is transformed into $\dot{y} = Cy + B(t).g[B^{-1}(t)y, t] = Cy + h(y, t)$. Because $\| h(y, t) \| \leqq M. \| B(t) \|. \| B^{-1}(t) \|. \| y \|$, Theorem 24* applies and yields the result immediately.

The following theorem is a generalization of results of Lyapunov [14] and Četaev [8–9]:

THEOREM 26*. *Let* $\dot{x} = A(t)x$ *be a linear system, where* A *is a bounded matrix; let* $\chi_1, \cdots, \chi_n$ *be its order numbers (opposite values of Lyapunov's "characteristic* *numbers") and*

$$\sigma = \sum \chi_i - \liminf_{t \to +\infty} (1/t) . \int_{t_0}^{t} \operatorname{trace} A(\tau)\, d\tau \geqq 0$$

its "degree of irregularity". Then, if $\chi_i < -\sigma/(m - 1)$, $i = 1, \cdots, n$, $m > 1$, *for any function* $g(x, t)$ *such that* $\| g(x, t) \| \leqq \| x \|^m$ *(uniformly in t) in a neighborhood of the origin, the solution* $x = 0$ *of the system* $\dot{x} = A(t)x + g(x, t)$ *is asymptotically stable.*

Let $\varepsilon > 0$ be a number such that $\chi_i < -\varepsilon < -\sigma/(m - 1)$, D the diagonal matrix $\{\chi_1 + \varepsilon, \cdots, \chi_n + \varepsilon\}$ and $X(t)$ the matrix formed by a normal system of solutions in such a way that the i^{th} column has the order number χ_i. Let us perform the change of variables $x = X(t).e^{-Dt}.z$; the equation in z becomes

$$\dot{z} = Dz + e^{Dt}.X^{-1}(t).g[X(t).e^{-Dt}.z, t].$$

The i^{th} row in X^{-1} is formed by the elements Δ_{ij}/Δ, where Δ is the determinant of X and Δ_{ij} the minor corresponding to x_{ij}. The order number of this row is therefore

$$\leqq \sup_j \chi(\Delta_{ij}) + \chi(1/\Delta) \leqq \chi_1 + \cdots + \chi_{i-1} + \chi_{i+1} + \cdots + \chi_n + \chi\left\{\exp\left[-\int_{t_0}^{t} \operatorname{trace} A(\tau)\, d\tau\right]\right\} = \sigma - \chi_i .$$

Then, the i^{th} column of $X(t).e^{-Dt}$ has the order number $\chi_i - \chi_i - \varepsilon = -\varepsilon$ and the i^{th} row of $e^{Dt}.X^{-1}(t)$ an order number $\leqq \chi_i + \varepsilon + \sigma - \chi_i = \sigma + \varepsilon$. Therefore, if t is large enough, $X(t).e^{-Dt}.z$ belongs to the neighborhood of the origin where the assumption on g holds and

$$\| e^{Dt}.X^{-1}(t).g[X(t).e^{-Dt}.z, t] \| \leqq \| e^{Dt}.X^{-1}(t) \|.\| X(t).e^{-Dt} \|^m.\| z \|^m$$

and, because $\chi\{\| e^{Dt}.X^{-1}(t) \| . \| X(t).e^{-Dt} \|^m\} \leqq \sigma + \varepsilon - m\varepsilon < 0$, we will have

$$\| e^{Dt}.X^{-1}(t).g[X(t).e^{-Dt}.z, t] \| \leqq K. \| z \|^m$$

for sufficiently small $\| z \|$, uniformly in t.

Because $\chi_i + \varepsilon < 0$, by the classical theorems on stability in the first approximation when the linear part has constant coefficients, the solution $z = 0$ will be asymptotically stable; a fortiori, the same thing holds true for the solution $x = 0$ of the system $\dot{x} = Ax + g$.

REMARK. We have actually proved a more precise result, namely that the order number of any solution of $\dot{x} = Ax + g$ is $\leqq \sup_i \{\chi_i\}$, because $\chi(x) \leqq \chi[X(t).e^{-Dt}] + \chi(z) \leqq -\varepsilon + \chi(z) \leqq -\varepsilon$, and $-\varepsilon$ is as near to sup χ_i as we please.

6. On the stability of linear systems

The following theorem, which is true in infinite dimensional spaces, generalizes results of Dini-Hukuwara [12] and Caligo [5]:

THEOREM 27. *If a fundamental operator solution $R(t)$ of the linear differential equation $\dot{x} = A(t).x$ satisfies the condition $\| R(t)R^{-1}(t_0) \| \leqq N.\exp[-\nu(t - t_0)]$, $N, \nu > 0, t \geqq t_0$, and if $B(t)$ is a linear operator such that*

$$\int_0^{+\infty} [\, \| B(t) \| - \nu N^{-1}] \, dt < +\infty,$$

the equation $\dot{x} = [A(t) + B(t)].x$ is stable; if the preceding integral diverges to $-\infty$, the equation is asymptotically stable.

It may be noticed that the assumption on R is equivalent, by Theorem 7 (f), to the assumption that the solution $x = 0$ of the equation $\dot{x} = Ax$ is uniform-asymptotically stable.

The general solution of the equation $\dot{y} = (A + B)y$ satisfies the integral equation

$$y(t) = R(t)R^{-1}(t_0)y_0 + \int_{t_0}^{t} R(t)R^{-1}(\tau)B(\tau)y(\tau) \, d\tau.$$

If we write $r(t) = \| y(t) \| .e^{\nu t}$, we have

$$r(t) \leqq Nr(t_0) + N \int_{t_0}^{t} \| B(\tau) \| .r(\tau) \, d\tau$$

and, by a known lemma [4], $r(t) \leqq Nr(t_0).\exp\left[N \int_{t_0}^{t} \| B(\tau) \| \, d\tau \right]$, whence

$$\| y(t) \| \leqq N.\| y_0 \| . \exp\left\{ N \int_{t_0}^{t} [\, \| B(\tau) \| - \nu N^{-1}] \, d\tau \right\},$$

which proves the theorem.

7. On total and asymptotic stability

We shall say that the solution $x = 0$ of a differential equation $\dot{x} = f(x, t)$ is *totally stable* (in the Soviet terminology: stable under constantly acting perturbations) if, given $\varepsilon > 0$, there is a $\delta > 0$ such that for any equation $\dot{y} = f(y, t) + g(y, t)$ such that $\| g(y, t) \| < \delta$, the general solution $y = G(t, y_0, t_0)$ satisfies the condition $\| G(t, y_0, t_0) \| < \varepsilon$ for any values such that $\| y_0 \| < \delta$, $t \geqq t_0 \geqq 0$.

Goršin [11] and Malkin [17] proved that, under fairly general assumptions, asymptotic stability implies total stability. The following are partial reciprocals of these results.

THEOREM 28. *If the solution $x = 0$ of the linear equation $\dot{x} = A(t)x$ is totally stable, it is uniform-asymptotically stable.*

If $\dot{x} = Ax$ is totally stable, $\dot{y} = Ay + \varepsilon y$ will be stable for sufficiently small $\varepsilon > 0$. But the solutions of both equations are related by $y(t) = e^{\varepsilon t}.x(t)$, which proves the theorem.

THEOREM 29*. *If the space is finite dimensional and if the solution $x = 0$ of the autonomous or periodic differential equation $\dot{x} = f(x, t)$, $f \in C_1$, is totally stable, it is uniform-asymptotically stable.*

Because $f \in C_1$ and independent of or periodic in t, we will have

$$\| f(x, t) - A(t)x \| / \| x \| \to 0$$

when $x \to 0$, uniformly in t, where $A(t)$ is the Jacobian matrix of f with respect to x for $x = 0$; A is, respectively, a constant or periodic matrix. Let $g(x, t) = f(x, t) - A\ (t)x$.

$x = 0$ being totally stable, the solution $y = 0$ of the equation $\dot{y} = A(t)y + \varepsilon y + g(y, t)$ is stable for sufficiently small $\varepsilon > 0$. Because the matrix $A(t) + \varepsilon I$ of the linear part of the new equation is constant or periodic and $\| g(y, t) \| / \| y \| \to 0$ when $y \to 0$, the characteristic exponents of the equation $\dot{z} = A(t)z + \varepsilon z$ must be $\leqq 0$; otherwise the nonlinear system would be unstable [14]. Then, by the same argument as for Theorem 28, the characteristic exponents of the equation $\dot{z} = A(t)z$ will be $\leqq -\varepsilon$ and, by Theorem 24*, the solution $x = 0$ of $\dot{x} = f(x, t)$ will be uniform-asymptotically stable.

Theorem 29* together with the above mentioned results of Goršin and Malkin shows that, at least for sufficiently regular autonomous or periodic systems, the concepts of total and asymptotic stability are equivalent.

INSTITUTO DE MATEMATICA Y ESTADISTICA
MONTEVIDEO, URUGUAY

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