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Review Article

A Review of Fundamentals of Lyapunov Theory

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1. Introduction

Stability theory plays a significant role in engineering systems. For any given control system, it is required to have a stable system since an unstable control system is useless. Lyapunov stability criterion is a general and useful approach to analyze the stability of nonlinear systems. Lyapunov stability concepts include two approaches: Lyapunov indirect method and Lyapunov direct method. For Lyapunov indirect method the idea of system linearization around a given point is used and one can achieve local stability with small stability regions. On the other hand the Lyapunov direct method is the most important tool for design and analysis of nonlinear systems. This method can be applied directly to a nonlinear system without the need to linearization and achieves global stability. The fundamental concept of the Lyapunov direct method is that if the total energy of a system is continuously dissipating, then the system will eventually reach an equilibrium point and remain at that point. Hence, the Lyapunov direct method consists of two steps. Firstly, a suitable scalar function is chosen and this function is referred as Lyapunov function [1]. Secondly, we have to evaluate its first-order time derivative along the trajectory of the system. If the derivative of a Lyapunov function is decreasing along the system trajectory as time increase, then the system energy is dissipating and the system will finally settle down.

In this paper some of important theorems based on Lyapunov theory are reviewed. We summarize the tools of Lyapunov stability theory. A survey of the results is presented with no proofs and we give examples to illustrate the procedure for studying the stability of nonlinear system. The interested reader should consult a standard text, such as Vidyasagar (1992) [2], Khalil (2002) [3], or Sastry (1999) [4], for details.

2. Equilibrium point and asymptotic stability

Consider a dynamical system which satisfies

$$\begin{aligned} \dot{x} &= f(x,t), \qquad t \geq 0 \\ x(t_0) &= x_0, \qquad t_0 \geq 0 \end{aligned} \tag{1}$$

where $x \in \Re^n$, f is a given nonlinear continuous function in t where $t \in \Re^+$. We assume that $f(x,t)$ satisfies the standard conditions for the existence and uniqueness of solutions. The nonlinear system (1) is said to be autonomous if $f(t,x)$ does not depend explicitly on time, i.e., if the system can be written as

$$\dot{x} = f(x), \tag{2}$$

The system is called non-autonomous. A state x^* is an equilibrium point of the system if $f(x^*) = 0$. Intuitively and somewhat crudely speaking, we say an equilibrium point is locally stable if all solutions which start near x^* (meaning that the initial conditions are in a neighborhood of x^*) remain near x^* for all time. The equilibrium point x^* is said to be locally asymptotically stable if x^* is locally stable and, furthermore, all solutions starting near x^* tend towards x^* as $t \to \infty$.

We first consider the Lyapunov direct method. For the case of autonomous systems, based on the Lyapunov direct method the stability theorem can be stated as follows:

Theorem 1 Let $x = 0$ be an equilibrium point for (2) where $f : U \to \Re^n$ is a locally Lipchitz and $U \subset \Re^n$ a domain that contains the origin. Let $V : U \to \Re$ be a continuously differentiable, positive definite function in U.

1. If $\dot{V}(x)$ is negative semidefinite, then $x = 0$ is a stable equilibrium point.
2. If $\dot{V}(x)$ is negative definite, then $x = 0$ is an asymptotically stable equilibrium point.

In both cases above V is called a Lyapunov function. Moreover, if the conditions hold for all $x \in \Re^n$ and $\|x\| \to \infty$ implies that $V(x) \to \infty$, then $x = 0$ is globally stable in case 1 and globally asymptotically stable in case 2.

Proof. The proof of this Theorem can be found in Khalil [3].

Remark 1. Please note that the definitions of negative (or positive) definite and negative (or positive) semidefinite are given in Khalil [3]. In both cases above V is called a Lyapunov function. Moreover, if the conditions hold for all $x \in \Re^n$ and $\|x\| \to \infty$, then $x = 0$ is globally stable in case 1 and globally asymptotically stable in case 2.

To ensure that the equilibrium is asymptotically stable we have to seek a scalar function of the states and this function is positive definite in a region U around the equilibrium point: $V(x) > 0$, except $V(0) = 0$. The existence of a Lyapunov function is sufficient to prove stability (in the sense of Lyapunov) in the region U. If $\dot{V}(x)$ is negative definite, the equilibrium is asymptotically stable [5].

Example 1 Consider the scalar system

$$\dot{x} = -x^3, \qquad x \in \Re \tag{3}$$

We want to investigate the stability of the origin $x = 0$.

Consider a candidate Lyapunov function:

$$V(x) = \frac{1}{2}x^2 \tag{4}$$

The derivative of $V(x)$ is

$$\begin{aligned} \dot{V} &= x \cdot (-x^3) \\ &= -x^4. \end{aligned} \tag{5}$$

Obviously, from (5) $\dot{V}(x)$ is negative definite for all $x \in \Re$. Therefore, by Theorem 1, the origin is globally asymptotically stable.

Example 2 Consider the pendulum system without friction:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 \end{aligned} \tag{6}$$

We want to study the stability of the equilibrium point around the origin. A natural candidate Lyapunov function for this system is the energy function (Khalil [2], p. 106)

$$V(x) = a(1-\cos x_1) + \frac{1}{2}x_2^2 \tag{7}$$

Notice that $V(0) = 0$ and $V(x)$ is positive definite over the domain $-\pi < x_1 < \pi.$ The derivative of $V(x)$ along the trajectories of the system is:

$$\begin{aligned} \dot{V} &= a \sin x_1 \dot{x}_1 + x_2 \dot{x}_2 \\ &= a x_2 \sin x_1 - a x_2 \sin x_1 = 0. \end{aligned} \tag{8}$$

Since $V(x)$ is positive definite and $\dot{V}(x) = 0,$ then by Theorem 1, the origin $x = 0$ is globally asymptotically stable.

Comments 1 The Lyapunov approach is a useful method since it allows to assess the stability of equilibrium points of a system without solving the differential equations that describe the system. However, there is no generally applicable method for finding Lyapunov functions. Trial and error and mathematical/physical insight are often used. Alternative procedures include as the variable gradient method (Khalil [3]) and the sum of squares decomposition ([6] and [7]).

3. Uniform stability and uniform asymptotic stability

Uniform stability is a concept which guarantees that the equilibrium point is not losing stability. Next we discuss on the sufficient conditions for uniform stability and uniform asymptotic stability of non-autonomous systems.

Theorem 2 Let $x = 0$ be an equilibrium point for (1) and $U \subset \Re^n$ a domain containing it. Let $V : U \times [0, \infty] \to \Re$ be a continuously differentiable function that satisfies:

$$W_1(x) \le V(x,t) \le W_2(x) \tag{9}$$

$$\dot{V}(x,t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \le -W_3(x) \tag{10}$$

for all $t \ge t_0$, and $x \in U$, where $W_1(x), W_2(x)$ and $W_3(x)$ are continuous positive definite functions on U. Then, $x = 0$ is uniformly asymptotically stable and V is called a Lyapunov function. Furthermore, if $W_3(x) = 0,$ then $x = 0$ is uniformly stable.

Proof: The proof of Theorem 2 can be found in Khalil [3].

The following corollary gives sufficient conditions for global uniform asymptotic stability for non-autonomous systems.

Corollary 1 Suppose that the assumptions of Theorem 2 hold for all $x \in \Re^n$ and $W_1(x) \to \infty$ for $\|x\| \to \infty$, then x = 0 is globally uniformly asymptotically stable.

Proof: The proof of Corollary 1 can be found in Khalil [3].

Example 3 Consider the scalar system described by

$$\dot{x} = -x^3 + \frac{x^3}{2}\sin(t), \qquad x(t_0) = x_0 \tag{11}$$

Consider a candidate Lyapunov function:

$$V(x) = \frac{1}{2}x^2 \tag{12}$$

We obtain the derivative of $V(x)$ as

$$\begin{aligned} \dot{V} &= x \cdot (-x^3 + \frac{x^3}{2}\sin(t)) \\ &= -x^4\left[1 - \frac{1}{2}\sin(t)\right] \end{aligned} \tag{13}$$

Obviously, if we choose $W_1(x) = W_2(x) = V(x)$ and $W_3(x) = ax^4$, where $a < 0.5$, then the assumptions of Theorem 2 are satisfied globally. Therefore, the origin is globally uniformly asymptotically stable.

The following corollary gives sufficient conditions for global exponential stability for non-autonomous systems.

Corollary 2 Suppose that the assumptions of Theorem 2 are replaced by

$$c_1\|x\|^q \le V(x,t) \le c_2\|x\|^q \tag{14}$$

$$\dot{V}(x,t) \le -c_3\|x\|^q \tag{15}$$

for some positive constants c_1, c_2, c_3 and q. Then $x = 0$ is exponentially stable. Furthermore, if the assumptions are satisfied for all $x \in \Re^n$, then $x = 0$ is globally exponentially stable.

Proof. The proof of this Corollary can be found in Khalil [3].

Comment 2 Form Theorem 2, the function V is required to be dominated by a time-invariant function $W_2(x)$. This means that V is decrescent. Therefore, sufficient conditions for uniform stability and uniform asymptotic stability are achieved if there is a continuously differentiable, positive definite and decrescent function V that satisfies the condition (10). One can also conclude that the origin is uniformly stable if $\dot{V}$ is negative semi-definite. If $\dot{V}$ is negative definite then the equilibrium point is uniformly asymptotically stability.

4. Lyapunov indirect method.

The following theorem gives conditions under which we can draw conclusions about the local stability of an equilibrium point of a nonlinear system by investigating the stability of a linearized system.

Theorem 3 Let $x = 0$ be an equilibrium point for the nonlinear system $\dot{x} = f(x)$, where $f : D \to \Re^n$ is continuously differentiable and D is a neighborhood of the origin. Let the Jabobian matrix A at $x = 0$ be:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} \quad . \tag{16}$$

Let λ_i, $i = 1, \ldots, n$ be the eigenvalues of A. Then,
1. The origin is asymptotically stable if $\mathrm{Re}(\lambda_i) < 0$ for all eigenvalues of A.
2. The origin is unstable if $\mathrm{Re}(\lambda_i) > 0$ for any of the eigenvalues of A.
Proof: The proof of this theorem can be found in Khalil [3].

5. Estimate the region of attraction by using Lyapunov function.

Note that a classical problem of Lyapunov theory is the determination of a domain of attraction [8]. The function $V(x) = x^T Px$ can be used to estimate the region of attraction. Suppose $\dot{V}(x) < 0,$ $0 < \|x\| < r.$ Letting $c = \min\limits_{\|x\|=r} x^T Px = \lambda_{\min}(P)r^2$ we obtain $\{x^T Px < c\} \subset \{\|x\| < r\}$. This means that all trajectories beginning in the set $\{x^T Px < c\}$ approach the origin as $t \to \infty$. Hence, the set $\{x^T Px < c\}$ is a subset of the region of attraction.

Example 4 Consider the autonomous pendulum with friction

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2 \end{aligned} \tag{21}$$

where $a,\ b > 0$. The Jacobian matrix A at the equilibrium point $x = 0$ is given by:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \tag{22}$$

Then the eigenvalues of A are $(-1 \pm j\sqrt{3})/2$. Hence, the origin is asymptotically stable. Taking $Q = I$ the Lyapunov equation becomes

$$PA + A^T P = -I \tag{23}$$

Solving (23) we obtain

$$P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix} \tag{24}$$

and $\lambda_{\min}(P) = 0.691.$ Thus, the system (17) has a candidate Lyapunov function

$$\begin{aligned} V(x) &= x^T Px \\ &= 1.5x_1^2 - x_1x_2 + x_2^2 \end{aligned} \tag{25}$$

We obtain the derivative of $V(x)$ as

$$\begin{aligned} \dot{V}(x) &= (3x_1 - x_2)(-x_2) + (-x_1 + 2x_2)[(x_1 + (x_1^2 - 1)x_2)] \\ &= -(x_1^2 + x_2^2) - (x_1^3x_2 - 2x_1^2x_2^2) \\ &\le -\|x\|^2 + |x_1||x_1x_2||x_1 - 2x_2| \\ &\le -\|x\|^2 + \frac{\sqrt{5}}{2}\|x\|^4 \end{aligned} \tag{26}$$

where $|x_1| \le \|x_1\|,$ $|x_1x_2| \le \frac{1}{2}\|x\|^2, |x_1 - 2x_2| \le \sqrt{5}\|x\|.$ Thus, we obtain

$$\dot{V}(x) < 0, \quad 0 < \|x\|^2 < \frac{2}{\sqrt{5}}$$

Obviously, taking $c = \lambda_{\min}(P)r^2 = 0.619 \times \frac{2}{\sqrt{5}} = 0.618$, $\{V(x) < c\}$ is an estimate of the region of attraction.

Comments 3 If $A(x)$ has one eigenvalue in the right hand side of the complex plane, thereby implying that the equilibrium point is unstable. Furthermore, if $A(x)$ has all of its eigenvalues in the left hand side of the complex plane with one or more eigenvalues on the $j\omega$-axis, we cannot use Lyapunov indirect method to say anything about stability. Therefore, this tool has some limitation when applied to linear systems.

6. Discussion on advantages/disadvantages of Lyapunov theory

The Lyapunov theory has been one of the most effective tools in the control of dynamical systems. This claim is evidenced by the fact that, although classical and well established in the literature, this concept is even today extensively exploited in many different practical and theoretical problems. We believe that Lyapunov approach is one of the most powerful tools to deal with the control problem of uncertain systems. The main advantages of the approach can be listed as follows. Firstly, this theory is practically necessary when dealing with uncertain (especially nonlinear) systems with time–varying parameters. Secondly, the theory proposes techniques that are effective and insightful. Lastly, for important classes of problems and special classes of functions the theory is supported by efficient numerical tools such as those based on linear matrix inequalities (LMIs).

However, the Lyapunov theory presents several drawbacks that appear evident when it is applied to uncertain systems control. We can summarize the principal ones as follows. Firstly, it is not always clear how to choose a candidate Lyapunov function. Secondly, the theory basically works for state feedback controls but the output feedback is still a very complicate problem to be faced by using this theory. Thirdly, the theory is conservative when applied to constant uncertain parameter or slowly time–varying parameters.

8. Conclusions

In this paper a review of studying of stability of nonlinear control systems using the Lyapunov methods has been proposed. For practical implementation we have to find the Lyapunov function specifically for each nonlinear system. Our suggestion is that the Lyapunov function usually represents the total energy of a system. Its derivative is continuously decreasing so that the system eventually reaches an equilibrium point and remains at that point. Asymptotic stability properties have been restated and the usefulness of both Lyapunov indirect and direct methods has been presented. Furthermore we have referred to conditions for uniform stability and uniform asymptotic stability, and the use of Lyapunov function for determination of a domain of attraction. Examples have been proposed to illustrate the procedure for studying the stability of nonlinear systems.

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