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SUPERINTEGRABILITY IN CLASSICAL MECHANICS: A CONTEMPORARY APPROACH TO BERTRAND'S THEOREM

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Superintegrable Hamiltonians in three degrees of freedom posses more than three functionally independent globally defined and single-valued constants of motion. In this contribution and under the assumption of the existence of only periodic and plane bounded orbits in a classical system we are able to establish the superintegrability of the Hamiltonian. Then, using basic algebraic ideas, we obtain a contemporary proof of Bertrand's theorem. That is, we are able to show that the harmonic oscillator and the Newtonian gravitational potentials are the only 3D potentials whose bounded orbits are all plane and periodic.

1. Introduction

Nowadays classical mechanics is in a state of chaos, in the sense that the overwhelming majority of mechanical systems exhibit strong sensitivity to any change in the initial conditions. However, the typical classical mechanics problems, namely the harmonic oscillator and the Kepler problem, posses a very different behavior; all of its bounded orbits are periodic and, furthermore, they all are plane curves. These extraordinary properties have been known for a long time and inspired the classical theorem of Bertrand.[1] He was able to clarify the special role played by such problems by showing that they are the only ones, in three degrees of freedom, allowing only periodic and plane bounded orbits.

What separates the harmonic oscillator and the Kepler problem from the rest of 3D Hamiltonian systems is that they are paradigmatic examples of *superintegrable* systems,[2–4] in which the Hamilton-Jacobi equation is separable in more than one

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coordinate system and in which there exist more than three functionally independent globally defined single-valued constants of motion. Not all the constants in a superintegrable systems have to be in involution but they need to be functionally independent, otherwise some may be trivial. A set of n constants of motion I_i, where $3 \leq n < 6$, is said to be functionally independent if the rank of the $n \times 6$ Jacobian $\partial(I_1, \ldots, I_n)/\partial(x_i, p_i)$ is precisely n. If a 3D Hamiltonian posseses five such isolating constants then it is called a maximally superintegrable system.

In this contribution we address the problem of finding all 3D Hamiltonians in which all finite orbits are plane and periodic. Using the Hamiltonian formalism of classical mechanics and its relationship with the associated symmetry algebras, trough the Poisson brackets, we are able to obtain a new proof of Bertrand theorem. In contrast to the proofs we are aware of,[1,5,6] this one does not require expanding the Hamiltonian in a power series.

2. Hamiltonians with Plane and Periodic Bounded Orbits

Let us first ask for the kind of 3D Hamiltonians admitting bounded plane orbits *only*. If we work in the framework of Hamiltonian classical mechanics, assume that all orbits are planar, *i. e.* that they are curves with vanishing torsion, it can be shown[7] that the Hamiltonian has to be SO(3) invariant, that is, that it must have the form

$$H = \frac{|\mathbf{p}|^2}{2m} + V(|\mathbf{r}|) \tag{1}$$

where $\mathbf{p}$ is the linear momentum, $\mathbf{r}$ the position vector, $|\mathbf{a}|$ stands for the magnitude of $\mathbf{a}$, and $V(|\mathbf{r}|)$ is the potential energy function. As both the energy E (*i. e.* the value of the Hamiltonian) and the angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, are conserved quantities, we must conclude thus that any Hamiltonian system allowing only plane orbits is integrable.

Many properties of Hamiltonian systems can be expressed using Poisson brackets (PB); given any two dynamical quantities, $A(q,p)$ and $B(q,p)$, its Poisson bracket is defined as

$$\{A, B\} = \frac{\partial A}{\partial q_s}\frac{\partial B}{\partial p_s} - \frac{\partial A}{\partial p_s}\frac{\partial B}{\partial q_s} \tag{2}$$

where q and p stand for any set of canonically conjugated variables and we are using the summation convention (summing from 1 to 3) for repeated indices. The constancy of $\mathbf{L}$ may be written as the vanishing of its PB with the Hamiltonian, $\{\mathbf{L}, H\} = 0$. The angular momentum characterizes the orbital plane and, besides, the SO(3) symmetry properties of Hamiltonian systems are summarized in the relationships

$$\{L_a, L_b\} = \epsilon_{abc} L_c \quad a, b, c = 1, 2, 3 \tag{3}$$

between the infinitesimal generators of rotations, L_a. The ϵ_{abc} (the Levi-Civita antisymmetric symbol) are the structure constants of SO(3).

Having discussed the consequences of the planar nature of the orbits, we have to ask what are the consequences of their periodicity. It is easy to see that every periodic orbit is necessarily closed. But, if it is closed, then every point on the orbit must be fixed in 3-space and any quantity specifying its position must be a constant of motion different from E and from $\mathbf{L}$. Thence, as a consequence of the periodicity of the bounded orbits of a Hamiltonian system, extra time-independent globally-valid conserved quantities should exist. Therefore, any Hamiltonian system with only plane *and* periodic orbits has to be superintegrable. The components of the extra constants, together with $\mathbf{L}$ and H, play the role of the infinitesimal generators of the dynamical symmetry group of the system.[8]

3. The Role of the Extra Constants of Motion

As we have just mentioned, the extra conserved quantity, $\mathcal{A}$, fixes the position of at least one of the points in the periodic orbit. Then it follows that, irrespective of its precise nature, $\mathcal{A}$ must lay on the orbital plane and it has to be orthogonal to $\mathbf{L}$: $\mathcal{A} \cdot \mathbf{L} = 0$. Furthermore, being a constant, the PB of $\mathcal{A}$ with the Hamiltonian must vanish

$$\{\mathcal{A}, H\} = 0. \tag{4}$$

What is the nature of the extra conserved quantity? To investigate this, we should be aware that the only property that matters for determining it, is its ability for fixing the orbit in place. Note that $\mathcal{A}$ has to be defined on the 2D orbital plane; so, any geometrical quantity defined there and able to fix the position of at least one point in a orbit should be enough for our purposes. Let us analyse the general form such quantities might have. To begin with, we may choose a vector, which is enough for fixing a single point. However, there are more possibilities, we may choose a rank-2 tensor that is enough for fixing two points, positioned orthogonal to each other along its principal directions, on the orbit; note that this calls for diagonalizable tensor. It appears that we could select tensors of any higher rank for fixing the orbit. However, any tensor of rank greater than two is bound to be an useless complication. This is so because the maximum number of independent directions on a plane is precisely two, hence any rank-higher-than-2 quantity gives redundant information. Thence, we are lead to conclude that any useful extra conserved quantity can only be either i) a vector or ii) a symmetric rank-2 tensor.

Can we say something about the bounded periodic orbits from the previous discussion? The orbits of a Hamiltonian having an extra vectorial constant of motion must have a single dynamical symmetry axis, whereas the orbits of a Hamiltonian having a rank-2 tensor conserved quantity have two dynamical symmetry axes. In other words, in the former case only one of the orbit's symmetry axes can pass through the origin whereas in the latter two of the these axes pass through the origin. Thus the center of force coincides with the geometrical center of the orbit, this is the main difference between the two cases. Let us now address the two cases separately.

4. An Extra Vectorial Constant: the Kepler Problem

If we assume that the additional conserved quantity is a vector, this inmediately implies the value of its PB with $\mathbf{L}$ has to be[9]

$$\{\mathcal{A}_a, L_b\} = \epsilon_{abc}\mathcal{A}_c. \tag{5}$$

The most general expression $\mathcal{A}$ could have and still be laying on the orbital plane is

$$\mathcal{A} = \alpha(r,p)\mathbf{r} + \beta(r,p)(\mathbf{L}\times\mathbf{p}), \tag{6}$$

where $r \equiv |\mathbf{r}|$, $p \equiv |\mathbf{p}|$, and α and β stand for two at yet unknown functions in phase space. At this point we might wonder that, though expression (6) is rather general, why not try other expressions which could come as easily to the mind as it? We address this point at the end of the section.

If we now use expression (6) to evaluate the PBs (4) and (5), it is possible to obtain expressions for α, β and the associated potential energy function $V(r)$.[3,4] From the aforementioned equations, it follows that β must be a constant and the two relationships

$$\alpha(r,p) - \beta m r \frac{dV(r)}{dr} = 0 \tag{7}$$

and

$$p\frac{\partial\gamma}{\partial r} - m\frac{\partial\gamma}{\partial p}\frac{dV(r)}{dr} - \beta m p \frac{dV(r)}{dr} = 0, \tag{8}$$

where, for the sake of compactness, we have introduced $\gamma \equiv \alpha(r,p) - \beta p^2$, from these equations we conclude that α is of the form

$$\alpha(r) = \frac{c}{r}, \tag{9}$$

with c an integration constant. We can also find the explicit form of the 3D potential allowing only plane and periodic bounded orbits as solutions

$$V(r) = -\frac{k}{r}, \tag{10}$$

where $k \equiv c/\beta m$ is assumed to be positive. This is the Newtonian gravitational potential, which is thus the only system with periodic plane bounded orbits and an extra vector constant of motion. Notice that, as we have anticipated, the bounded orbits of the system have one dynamical symmetry axis only.

We can write down the explicit expression for the extra vectorial conserved quantity, *i. e.* the rank-1 Laplace tensor:

$$\mathcal{A} = \frac{1}{mk}(\mathbf{p}\times\mathbf{L}) - \frac{\mathbf{r}}{r}, \tag{11}$$

where we arbitrarily choose $c = -1$. This constant quantity is also known as the Laplace-Runge-Lenz vector though the original proposal seems to come from

Bernoulli.[13] As is well known, the vector $\mathcal{A}$ points from the origin to the pericenter of the orbit and its magnitude, $|\mathcal{A}| = \sqrt{1+2EL^2/mk^2}$, coincides with its eccentricity.

Since we only guessed the form of $\mathcal{A}$, what prevents us from using alternative expressions for it? Let us try as a different *ansatz* the expression

$$\mathcal{A}' = \alpha(r,p)\mathbf{p} + \beta(r,p)(\mathbf{L}\times\mathbf{r}) \tag{12}$$

which is as general and consistent with the required conditions as expression (6). If we repeat the analysis done, but now using expression (12) istead of (6), we obtain the same conclusion regarding the potential energy function but, for the extra constant, we get the different expression

$$\mathcal{A}' = \mathbf{p} + \frac{km}{rL^2}(\mathbf{L}\times\mathbf{r}) \tag{13}$$

which is known as the Hamilton vector[4,10,11] a quantity closely related with the hodograph of the problem.[12] The Hamilton vector is related to the Laplace vector trhough $\mathcal{A}' = mk(\mathbf{L}\times\mathcal{A})/L^2$.

5. An Extra Rank-2 Tensor Constant: the Harmonic Oscillator

If now we assume a rank-2 tensor as the extra constant of motion, the most general expression for A, diagonalizable and laying on the orbital plane, is

$$\mathcal{A}_{ij} = \alpha(r,p)p_ip_j + \beta(r,p)(x_ip_j + p_ix_j) + \delta(r,p)x_ix_j \tag{14}$$

where, i and j run from 1 to 3, α, and β and δ are functions to be determined, as in the previous section. This tensor must transform appropriately under rotations, therefore[9]

$$\{\mathcal{A}_{ij}, L_s\} = \epsilon_{isn}\mathcal{A}_{nj} + \epsilon_{jsn}\mathcal{A}_{in}. \tag{15}$$

If we carry on a similar analysis as the one we did in the previous section but now using equations (14) and (15), we obtain that α and δ must be constants and that β must vanish. We also get the equation

$$\frac{1}{r}\frac{dV(r)}{dr} = \frac{\delta}{m\alpha}, \tag{16}$$

which gives us inmmediately the harmonic oscillator potential

$$V(r) = \frac{1}{2}kr^2 \tag{17}$$

where $k \equiv \delta/m\alpha$. Therefore, the requirement of periodic and plane orbits together with the extra rank-2 tensor constant has led us directly to the harmonic oscillator, which is thus the only system with periodic plane bounded orbits and an extra rank-2 tensor constant of motion. The explicit form of the rank-2 Laplace tensor is

$$\mathcal{A} = \frac{1}{2m}\mathbf{pp} + \frac{k}{2}\mathbf{rr}. \tag{18}$$

As we have anticipated, the center of the orbits of this system coincide with the center of force.

6. Conclusions

The results of the last two sections taken together establish Bernard's result. Our goal here is to offer a modernized proof of Bertrand's theorem. We do this making the conection between Bertrand's classical last century result, with some XX century classical mechanics concepts, like the notion of superintegrability, and symmetries and their relation with constants of motion. We think the proof is not only interesting but also that the technique may be generalized to investigate the existence of superintegrable systems with a different set of conditions from the one used here.

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