Bertrand's Theorem: Closed, Perturbatively Stable Orbits

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Overview

Proof Types

1. Naïve expansion
2. Dedicated Web Sites
3. Federation of American Scientists

Bertrand's Paper En français

COMPTES RENDUS

DES SÉANCES

DE L'ACADÉMIE DES SCIENCES.

SÉANCE DU LUNDI 20 OCTOBRE 1873.

PRÉSIDENCE DE M. DE QUATREFAGES.

MÉMOIRES ET COMMUNICATIONS

DES MEMBRES ET DES CORRESPONDANTS DE L'ACADÉMIE.

MÉCANIQUE ANALYTIQUE. — *Théorème relatif au mouvement d'un point attiré vers un centre fixe;* par M. J. BERTRAND.

Bertrand 1873

Bertrand's Paper in English

EDVCATIO PHYSICORVM
QVO NON ASCENDAM?

An English Translation of Bertrand's Theorem

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Abstract
A beautiful theorem due to J. L. F. Bertrand concerning the laws of attraction that admit bounded closed orbits for arbitrarily chosen initial conditions is translated from French into English.

Keywords: Classical Mechanics, Orbits.

Resumen
Un hermoso teorema debido a la J. L. F. Bertrand sobre las leyes de la atracción que admiten limitadas órbitas cerradas arbitrariamente escogidas para las condiciones iniciales es traducido del francés al inglés.

Palabras clave: Mecánica Clásica, Órbitas.

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Santos, Soares, and Tort 2011

Bertrand's Paper in English

An English translation of Bertrand's theorem

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October 30, 2018

Abstract

A beautiful theorem due to J. L. F. Bertrand concerning the laws of attraction that admit bounded closed orbits for arbitrarily chosen initial conditions is translated from French into English.

Santos, Soares, and Tort 2011

Proofs

1. Théorème relatif au mouvement d'un point attiré vers un centre fixe (Bertrand 1873)
2. An English translation of Bertrand's theorem (Bertrand 1873)
3. A simplified proof of Bertrand's theorem (Tikochinsky 1988)
4. A pedagogical relook at Bertrand's theorem (TU and Das 2019)
5. Alternative proof of Bertrand's theorem using a phase space approach (Quilantán, Río-Correa, and Medina 1995)
6. A proof of bertrand's theorem using the theory of isochronous potentials (Ortega and Rojas 2019)
7. A partial proof of Bertrand's theorem (Musgrove 2020)

Proof Types

1. Lyapunov stability
2. PDE linearization
3. Phase portraits
4. Dimensional reduction

Lyapunov

Lyapunov's second method for stability [edit source]

Lyapunov, in his original 1892 work, proposed two methods for demonstrating stability.[1] The first method developed the solution in a series which was then proved convergent within limits. The second method, which is now referred to as the Lyapunov stability criterion or the Direct Method, makes use of a *Lyapunov function V(x)* which has an analogy to the potential function of classical dynamics. It is introduced as follows for a system $\dot{x} = f(x)$ having a point of equilibrium at $x = 0$. Consider a function $V : \mathbb{R}^n \to \mathbb{R}$ such that

- $V(x) = 0$ if and only if $x = 0$
- $V(x) > 0$ if and only if $x \neq 0$
- $\dot{V}(x) = \frac{d}{dt}V(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \nabla V \cdot f(x) \leq 0$ for all values of $x \neq 0$. Note: for asymptotic stability, $\dot{V}(x) < 0$ for $x \neq 0$ is required.

Then *V(x)* is called a Lyapunov function and the system is stable in the sense of Lyapunov. (Note that $V(0) = 0$ is required; otherwise for example $V(x) = 1/(1+|x|)$ would "prove" that $\dot{x}(t) = x$ is locally stable.) An additional condition called "properness" or "radial unboundedness" is required in order to conclude global stability. Global asymptotic stability (GAS) follows similarly.

It is easier to visualize this method of analysis by thinking of a physical system (e.g. vibrating spring and mass) and considering the energy of such a system. If the system loses energy over time and the energy is never restored then eventually the system must grind to a stop and reach some final resting state. This final state is called the attractor. However, finding a function that gives the precise energy of a physical system can be difficult, and for abstract mathematical systems, economic systems or biological systems, the concept of energy may not be applicable.

Lyapunov's realization was that stability can be proven without requiring knowledge of the true physical energy, provided a Lyapunov function can be found to satisfy the above constraints.

Papier de Bertrand en français

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Literature: Texts

1. **Arnol'd 2013,** §2.8D, p. 37
2. **Goldstein 1980,** App. A, p. 601
3. **José and Saletan 1998,** §2.3.3, p. 88

Bertrand's Paper in French

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Bertrand 1873

Conservative Force

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