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Quaternion Regularization in Celestial Mechanics, Astrodynamics, and Trajectory Motion Control. III

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Abstract—The present paper[1] analyzes the basic problems arising in the solution of problems of the optimum control of spacecraft (SC) trajectory motion (including the Lyapunov instability of solutions of conjugate equations) using the principle of the maximum. The use of quaternion models of astrodynamics is shown to allow: (1) the elimination of singular points in the differential phase and conjugate equations and in their partial analytical solutions; (2) construction of the first integrals of the new quaternion; (3) a considerable decrease of the dimensions of systems of differential equations of boundary value optimization problems with their simultaneous simplification by using the new quaternion variables related with quaternion constants of motion by rotation transformations; (4) construction of general solutions of differential equations for phase and conjugate variables on the sections of SC passive motion in the simplest and most convenient form, which is important for the solution of optimum pulse SC transfers; (5) the extension of the possibilities of the analytical investigation of differential equations of boundary value problems with the purpose of identifying the basic laws of optimum control and motion of SC; (6) improvement of the computational stability of the solution of boundary value problems; (7) a decrease in the required volume of computation.

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1. THE FORMULATION OF THE PROBLEM OF OPTIMUM CONTROL OF SC CENTER OF MASS

The motion of the center of mass of SC (the material point B of variable masses) will be considered in the coordinate system $OX_1X_2X_3(X)$ with origin at the center of attraction O. The coordinate axes of this coordinate system are parallel to the axes of the inertial coordinate system. The controlled motion of the center of mass of the spacecraft (SC) in the central Newtonian field of gravity is described by a differential vector equation of the second order [4, 5]

$$d^2\mathbf{r}/dt^2 + fMr^{-3}\mathbf{r} = \mathbf{p} \tag{1.1}$$

or by a system of two vector differential equations of the first order

$$d\mathbf{r}/dt = \mathbf{v},\ \ d\mathbf{v}/dt = -fMr^{-3}\mathbf{r} + \mathbf{p}, \quad \mathbf{p} = (T/m)\mathrm{e} = \mathbf{T}/m, \tag{1.2}$$

where $\mathbf{r}$, $\mathbf{v}$ are the radius vector and the vector of velocity of SC center of mass in the coordinate system X, $r = |\mathbf{r}|$, f is the gravitational constant, M is the mass of an attracting body, $m = m(t)$ is the SC mass, $\mathbf{p}$ is the vector of acceleration of SC center of mass due to engine thrust, accepted as a control, $\mathbf{T} = T\mathrm{e}$ is the thrust vector, T and $\mathbf{e}$ are the magnitude and unit vector of the thrust direction.

In space flight mechanics, the following problem of optimum control of the motion of SC center of mass in a Newtonian gravitational field is of fundamental importance: it is required to determine the restricted-in-magnitude control $\mathbf{p}$:

$$0 \le p \le p_{\max} < \infty,\ \ p = |\mathbf{p}|, \tag{1.3}$$

which transfers the SC, the motion of the center of mass of which is described by equations (1.1) or (1.2), from the given initial state

$$\mathbf{r}(t_0) = \mathbf{r}(0) = \mathbf{r}^0,\ \ \mathbf{v}(t_0) = \mathbf{v}(0) = \mathbf{v}^0, \tag{1.4}$$

to the final state

$$\mathbf{r}(t_1) = \mathbf{r}^*,\ \ \mathbf{v}(t_1) = \mathbf{v}^* \tag{1.5}$$

or to a state that belongs to some movable diversity, in the general case, and minimizes the functional

$$J = \int_0^{t_1} (\alpha_1 + \alpha_2 p^2(t))dt,\ \ \alpha_1,\ \alpha_2 = \mathrm{const} \ge 0 \tag{1.6}$$

[1] The article is based on [1]. The article is a continuation of [2, 3].

or the functional

$$J = \int_0^{t_1} (\alpha_1 + \alpha_2 p(t))dt = \alpha_1 t_1 + \alpha_2 Q(t_1), \tag{1.7}$$
$$\alpha_1, \alpha_2 = \text{const} \geq 0,$$

where α_1, α_2 are weighting coefficients of the functional. The time t_1 of controlled motion is supposed to be nonspecified.

Functional (1.6) for $\alpha_1 = 0, \alpha_2 = 1$ is used in problems of the mechanics of space flight with low-thrust engines [6], and functional (1.7) is used, for the same values of constants α_1 and α_2, in problems with high-thrust engines [5]. Functional (1.6) characterizes energy expenses for SC transfer from an initial to final state and the time spent for this transfer. Functional (1.7) characterizes the total (in some proportion) SC time and characteristic velocity for performing controlled motion. For $\alpha_1 = 0$, $\alpha_2 = 1$ the minimum of functional (1.7) implies the minimum of characteristic velocity Q. For $\alpha_2 = 0$ functionals (1.6), (1.7) transfer to the functional $J = t_1$, and the formulated problem represents a high-speed response problem in this case. Note that the solution of the problem of optimum control of motion of SC center of mass for functional (1.7) is much more difficult than the solution of a similar problem for functional (1.6), because of the nonanalytical character of the integrand function in equation (1.7), which results in laborious calculations in determining optimum control.

The formulated problem will be considered using Pontryagin's maximum principle. We introduce the additional variable g, which satisfies, when minimizing functional (1.6), the differential equation $\dot{g} = \alpha_1 + \alpha_2 p^2(t)$ and the initial condition $g(0) = 0$, or, when minimizing functional (1.7), the differential equation $\dot{g} = \alpha_1 + \alpha_2 p(t)$ and the initial condition $g(0) = 0$. We introduce conjugate vector variables ψ_r and ψ_v, corresponding to vector phase variables $\mathbf{r}$ and $\mathbf{v}$, and the scalar conjugate variable ψ_0 corresponding to the scalar phase variable g.

The Hamilton–Pontryagin function will be as follows:

$$H = \psi_0 \sigma + \psi_r \cdot \mathbf{v} - fMr^{-3} \psi_v \cdot \mathbf{r} + \psi_v \cdot \mathbf{p}, \tag{1.8}$$

where for functional (1.6)

$$\sigma = \alpha_1 + \alpha_2 p^2, \quad \alpha_1, \alpha_2 \geq 0, \tag{1.9}$$

and for functional (1.7)

$$\sigma = \alpha_1 + \alpha_2 |\mathbf{p}|, \quad \alpha_1, \alpha_2 \geq 0. \tag{1.10}$$

The system of equations for the conjugate variables has a well-known form:

$$\begin{aligned} d\psi_v/dt &= -\psi_r, \quad d\psi_r/dt \\ &= fMr^{-3}\psi_v - 3fMr^{-5}(\psi_v \cdot \mathbf{r})\mathbf{r}, \end{aligned} \tag{1.11}$$

$$d\psi_0/dt = 0. \tag{1.12}$$

In accordance with the principle of the maximum, $\psi_0(t_1) \leq 0$, therefore, in virtue of equation (1.12) and the homogeneity of function H in conjugate variables, one can choose any $\psi_0(t) = \text{const} < 0$ by appropriate redetermination of the other variables. Further on, in the expression (1.8) for the function H, the multiplier ψ_0 is supposed to be equal to –1.

The optimum control[2] $\mathbf{p}^0$, found from the condition of maximum of function H defined by relations (1.8)–(1.10), with respect to variable $\mathbf{p}$ with regard to constraint (1.3), has the form:

$$\mathbf{p}^0 = p^0 \psi_v / |\psi_v|. \tag{1.13}$$

Here, in for the minimization of the functional (1.6) in the case of $\alpha_2 > 0$,

$$p^0 = \begin{cases} (2\alpha_2)^{-1}|\psi_v|, & \text{if } (2\alpha_2)^{-1}|\psi_v| \leq p_{\max}, \\ p_{\max}, & \text{if } (2\alpha_2)^{-1}|\psi_v| > p_{\max}, \end{cases} \tag{1.14}$$

and in the case of $\alpha_2 = 0$

$$p^0 = p_{\max}. \tag{1.15}$$

In the minimization of the functional (1.7),

$$p^0 = \begin{cases} p_{\max}, & \text{if } |\psi_v| - \alpha_2 \geq 0, \\ 0, & \text{if } |\psi_v| - \alpha_2 < 0, \\ \forall p \in [0, p_{\max}], & \text{if } |\psi_v| - \alpha_2 \equiv 0. \end{cases} \tag{1.16}$$

Note that among the control modes there exist: (1) a mode corresponding to the maximum value of acceleration $\mathbf{p}$; (2) a mode corresponding to the minimum (zero) value of acceleration, as well as (3) a particular mode that can arise when meeting the third of the conditions in (1.16).

Equations (1.2), (1.11) of the considered boundary value problem have, for the optimum control and optimum trajectory (more precisely, for any $\mathbf{p} \| \psi_v$,, in particular, when meeting the necessary optimality conditions (1.13)–(1.16), which take into account the restriction imposed on the control magnitude), the vector (1.17) and scalar (1.18) first integrals [7–9]:

$$((dr/dt) \times \psi_v + (d\psi_v/dt) \times \mathbf{r})\big|_{p=p^0} = \text{const}, \tag{1.17}$$

$$H(\mathbf{r}, \mathbf{v}, \psi_r, \psi_v, \mathbf{p}^0(\psi_v)) = 0. \tag{1.18}$$

[2] Here and hereafter the control satisfying the necessary optimality conditions (Pontryagin's maximum principle) is called the optimum control, and the optimum trajectory is the trajectory that meets this control.

The problem of optimum control of motion of SC center of mass is thus reduced, in the case of a fixed right end of the trajectory (for the specified final values $\mathbf{r}(t_1)$ and $\mathbf{v}(t_1)$ of the radius vector and velocity vector of SC center of mass) to the two-point boundary value problem that is described (when the acceleration vector of SC center of mass $\mathbf{p}$ is used as a control) by twelve scalar nonlinear differential equations of the first order (1.2), (1.11), (1.13)–(1.15) in the case of the minimization of functional (1.6), or by equations (1.2), (1.11), (1.13), (1.16), in the case of the minimization of functional (1.7).

Upon integrating the equations, there will appear twelve arbitrary integration constants; the thirteenth unknown quantity will be the time of controlled motion t_1. To determine unknown constants and time t_1, we have thirteen conditions: twelve boundary conditions (1.4), (1.5) and the equality

$$H^0\Big|_{t_1} = H(\mathbf{r}, \mathbf{v}, \psi_r, \psi_v, \mathbf{p}^0)\Big|_{t_1} = 0, \qquad (1.19)$$

which takes place in accordance with (1.18) for the optimum control $\mathbf{p}^0$ and the optimum trajectory of the SC.

In the general case, when the left and (or) the right end of the trajectory appertains to some diversities, the dimension of the boundary value problem may increase (due to the appearance of additional differential equations describing these diversities). Besides, in this case, when solving the problem, there arises the necessity of constructing and accounting for the transversality conditions. The dimension of the two-point boundary value problem also increases [5] by two units (i.e., it has the fourteenth order), when the thrust vector $\mathbf{T}$ is used as a control. The dimension of the problem can also increase when the other models of motion of the SC's center of mass are used.

2. PROBLEMS OF OPTIMUM CONTROL OF MOTION OF SC CENTER OF MASS AND HOW TO INCREASE THE EFFICIENCY OF THEIR SOLUTION

The principal difficulty in the problems of optimum control of motion of SC center of mass lies in the solution of boundary value problems for the system of differential equations constructed by means of Pontryagin's maximum principle. We indicate the basic factors that make the solution of boundary value problems difficult to optimize: (1) the nonlinearity and the high dimension of systems of differential equations for boundary value problems and, in some cases, their nonstationary character; (2) the openness of the problem of existence and uniqueness of the solution; (3) the Lyapunov instability of solutions of conjugate equations, i.e., instability with respect to unknown initial conditions for conjugate variables, which should be found in the course of solving the problem; (4) the existence of singular points in differential equations and in their partial analytical solutions, where the equations and their solutions are degenerated; (5) the nonanalytical character of one of the basic quality functionals (characteristic velocity) that complicates the numerical solution of problems; (6) the absence of analytical solutions of equations in the majority of cases; (7) the computational instability of solutions of systems of differential equations; (8) and the absence, as a consequence of the factors listed above, of regular algorithms for numerical solution of such boundary value problems as would guarantee their solution without the intervention of the researcher.

The efficiency of the analytical investigation and numerical solution of some problems of optimum control of SC trajectory motion can be increased due to the use of quaternion models of astrodynamics, the review and analysis of which are given in [2, 3, 10, 11]: the quaternion equations of motion in the regular Kustaanheimo–Stiefel variables; the equations of motion in quaternion osculating elements; the equations of motion written in rotating coordinate systems and using Euler (Rodrigues–Hamilton) parameters and Hamilton's quaternions for describing the angular (rotational) motion of these coordinate systems; the quaternion equations of SC orbit orientation, SC orbital plane, and the orbital and other rotating coordinate systems.

We have drawn this conclusion on the basis of the analytical investigation and numerical solution of some space flight mechanics problems: the problems of optimum soft or hard rendezvous of a controlled space vehicle with a noncontrolled vehicle moving over a Keplerian orbit, this rendezvous occurring in near-earth space and in a Martian orbit [12–31]; the problems of optimum SC insertion into the target orbit [29, 32]; the problems of optimum SC orbit reorientation accomplished either by means of jet thrust orthogonal to SC orbital plane [33–43], or by means of jet thrust whose direction in space is determined from the solution of the optimization problem [44].

These problems were considered in continuous and pulse formulations using various quaternion models of astrodynamics. Numerical solutions of problems were constructed by the author of this paper jointly with Ya.G. Sapunkov, L.A. Chelnokova, Yu.V. Afanasyeva and I.A. Pankratov. In so doing, the combined functionals (1.6), (1.7) were used as quality functionals. Now we will analyze the aforementioned factors and consider the benefits of using quaternion astrodynamics models in the problems of the optimum control of motion of the SC center of mass.

3. NUMERICAL SOLUTION OF THE BOUNDARY VALUE OPTIMIZATION PROBLEM AND THE STABILITY OF DIFFERENTIAL EQUATIONS OF THE BOUNDARY VALUE PROBLEM

One of the principal difficulties in the considered problem of the optimum control of motion of the SC's center of mass, as well as of other optimum control problems, lies in the solution of the boundary value problem [45], which consists in our case in the integration of differential equations (1.2), (1.11) with regard to the boundary conditions specified in the form of (1.4), (1.5) or belonging to some diversities, to which the optimum motion of the SC's the center of mass should satisfy. The necessary signs of optimality, expressed by differential equations (1.2) and (1.11) for the phase $\mathbf{r}$, $\mathbf{v}$, and the conjugate variables ψ_r, ψ_v and by relations (1.13)–(1.16), determine, as is known, the inner properties of optimum motions by describing their local behavior in the neighborhood of each point on the given trajectory [45]. In virtue of these properties, each optimum motion evolves in time in some particular manner, starting from the initial conditions $\mathbf{r}(t_0)$, $\mathbf{v}(t_0)$ and $\psi_r(t_0)$, $\psi_v(t_0)$.

The initial conditions of motion of the SC's center of mass $\mathbf{r}(t_0)$, $\mathbf{v}(t_0)$ are specified, as a rule. The initial conditions $\psi_r(t_0)$, $\psi_v(t_0)$ of differential conjugate equations determine (according to the conditions of the principle of maximum) the direction in the phase space, along which the imaging point $(\mathbf{r}, \mathbf{v})$ moves away from the initial state $(\mathbf{r}(t_0), \mathbf{v}(t_0))$. The principal difficulty of the solution of the boundary value problem consists in determining the initial conditions $\psi_r(t_0)$, $\psi_v(t_0)$ of the integration of the conjugate equations providing the "aim" of optimum motion to the specified final state (or to the specified diversity of final states), and in determining the time t_1 of a controlled motion.

Effective methods of overcoming this difficulty do not currently exist because of the impossibility of obtaining an explicit dependence between the quantities $\mathbf{r}(t_1)$, $\mathbf{v}(t_1)$ and $\psi_r(t_0)$, $\psi_v(t_0)$ caused by the nonintegrability of differential equations of the problem in the closed form and the absence of regular algorithms for the numerical solution of the problem (regular in the sense of a guaranteed solution without researcher intervention).

In order to find values $\psi_r(t_0)$, $\psi_v(t_0)$ we use, as a rule, an iterative computational algorithm based on the application of Newton's method and on the numerical integration of differential equations of the boundary value problem. These operations are carried out with the purpose of the numerical formation, at each iteration, of the solution of the boundary value problem for a matrix of partial derivatives of discrepancies with respect to the initial values of conjugate variables (in the formulated two-point boundary value problem the discrepancies are represented by conditions (1.5), (1.19) on the right end of the trajectory).

Since the quantities $(\mathbf{r}(t_0), \mathbf{v}(t_0)$ are specified, then, solving the Cauchy problem over the time interval $[t_0 = 0, t_1^0]$, where t_1^0 is some chosen quantity, for the system of differential equations (1.2), (1.11) (taking into account expressions (1.13)–(1.16) determining the optimum control), with some chosen initial values of conjugate variables $\psi_r(t_0) = \psi_r^0$, $\psi_v(t_0) = \psi_v^0$, we determine the quantities $\Delta\mathbf{r}(t_1^0) = \mathbf{r}(t_1^0) - \mathbf{r}^*$, $\Delta\mathbf{v}(t_1^0) = \mathbf{v}(t_1^0) - \mathbf{v}^*$, $\Delta H(t_1^0) = H(t_1^0)$ as functions of vectors $\psi_r(t_0) = \psi_r^0$, $\psi_v(t_0) = \psi_v^0$ and time t_1^0:

$$\Delta\mathbf{r}(t_1^0) = \mathbf{f}_r(\psi_r^0, \psi_v^0, t_1^0), \quad \Delta\mathbf{v}(t_1^0) = \mathbf{f}_v(\psi_r^0, \psi_v^0, t_1^0),$$
$$\Delta H(t_1^0) = f_H(\psi_r^0, \psi_v^0, t_1^0),$$

which are called discrepancies, as we have already noted.

The problem of determining the optimum control $\mathbf{p}^0$ to bring the SC center of mass to the specified point of the phase space $\mathbf{r}(t_1) = \mathbf{r}^*$, $\mathbf{v}(t_1) = \mathbf{v}^*$ is reduced to finding (by Newton's method, for instance) the zeroes of vector functions $\mathbf{f}_r(\psi_r(t_0), \psi_v(t_0), t_1)$, $\mathbf{f}_v(\psi_r(t_0), \psi_v(t_0), t_1)$ and the scalar function $f_H(\psi_r(t_0), \psi_v(t_0), t_1)$. For this purpose, for each iteration of the numerical solution of the boundary value problem, a small increment is successively imparted to each component of the chosen initial values of vector-conjugate variables ψ_r^0, ψ_v^0 and time t_1^0, and the Cauchy problem is solved seven times for the purpose of forming the matrix of partial derivatives of discrepancies with respect to the initial values of conjugate variables and time t_1. The success of the solution to the boundary value problem by Newton's method is largely determined by the accuracy of the first approximation ψ_r^0, ψ_v^0, t_1^0 for the initial values of conjugate variables and time t_1. However, as a rule, the choice of approximation is not immediately correct, and, so, for the numerical solution of the problem one uses the combination of the gradient descent and Newton's (Newton's modified) method and the laborious "trial and error" technique.

It should be noted that, as is well known in optimum control theory [45], the great difficulty in solution of boundary value problems of optimum control is related to the computational instability of solutions of differential equations for the boundary value problems. This instability takes place in the studied problem as well, and it is manifested in the fact that small changes in the vector $(\psi_r^0, \psi_v^0, t_1^0)$ usually result in quite considerable changes in quantities $\Delta\mathbf{r}(t_1^0)$, $\Delta\mathbf{v}(t_1^0)$, $\Delta H(t_1^0)$. Moreover, in the problem of optimum control of motion of the SC's center of mass, for a wide class of trajectories the Lyapunov instability of solutions of differential equations for conjugate variables occurs, that is, the instability of solutions of conjugate equations with respect to initial conditions of integration, which are to be found in the course of the numerical solution of the problem.

We will now focus on this aspect of the problem under consideration in more detail. The differential equations of the disturbed motion of SC center of mass, obtained by varying equations (1.2) under the assumption that $\mathbf{p} = \mathbf{p}^0(t)$ (assuming that the optimum control is found as an explicit function of time), in the first (linear) approximation have the form of the well-known differential equations of disturbed motion of a material point

$$d(\delta\mathbf{r})/dt = \delta\mathbf{v},$$
$$d(\delta\mathbf{v})/dt + fMr^{-3}\delta\mathbf{r} - 3fMr^{-5}(\mathbf{r}\cdot\delta\mathbf{r})\mathbf{r} = 0 \tag{3.1}$$

or

$$d^2(\delta\mathbf{r})/dt^2 + fMr^{-3}\delta\mathbf{r} - 3fMr^{-5}(\mathbf{r}\cdot\delta\mathbf{r})\mathbf{r} = 0. \tag{3.2}$$

Here $\delta\mathbf{r}$ and $\delta\mathbf{v}$ are variations of vectors $\mathbf{r}$ and $\mathbf{v}$: $\delta\mathbf{r} = \mathbf{r}_p - \mathbf{r}(t)$, $\delta\mathbf{v} = \mathbf{v}_p - \mathbf{v}(t)$; $\mathbf{r} = \mathbf{r}(t)$, $\mathbf{v} = \mathbf{v}(t)$ is the solution of equations (1.2) corresponding to the initial conditions of motion $\mathbf{r}(t_0)$, $\mathbf{v}(t_0)$ and to the control $\mathbf{p} = \mathbf{p}^0(t)$, which is accepted as undisturbed motion of SC center of mass; $\mathbf{r}_p$, $\mathbf{v}_p$ is the solution of equations (1.2) corresponding to the disturbed initial conditions of motion $\mathbf{r}_p(t_0) = \mathbf{r}(t_0) + \delta\mathbf{r}(t_0)$, $\mathbf{v}_p(t_0) = \mathbf{v}(t_0) + \delta\mathbf{v}(t_0)$ and to the same control $\mathbf{p} = \mathbf{p}^0(t)$, which is accepted as a disturbed motion of SC center of mass.

We will write the system of differential equations (1.11) for the conjugate variables in the form of a single vector differential equation of the second order

$$d^2\psi_v/dt^2 + fMr^{-3}\psi_v - 3fMr^{-5}(\psi_v\cdot\mathbf{r})\mathbf{r} = 0. \tag{3.3}$$

It follows from the comparison of conjugate equations (1.11) and (3.3) with equations of disturbed motion (3.1) and (3.2) that equations (1.11) transfer into equations (3.1), if one lets in them $\psi_r = -\delta\mathbf{v}$, $\psi_v = \delta\mathbf{r}$, and the vector differential equation (3.3) for the conjugate variable ψ_v, corresponding to the velocity vector $\mathbf{v}$ of SC center of mass, and the vector differential equation (3.2) for the variation $\delta\mathbf{r}$ of the radius vector of SC center of mass have the same appearance and fully coincide for $\psi_v = \delta\mathbf{r}$.

In material point dynamics and in inertial navigation the problem of stability of undisturbed motion of a material point

$$\delta\mathbf{r} = 0, \quad \delta\mathbf{v} = d(\delta\mathbf{r})/dt = 0, \tag{3.4}$$

which represents a trivial (zero) solution of equations (3.1) or equation (3.2), is of importance.

This solution is unstable for a wide class of motions of a material point [46–48]. Thus, it is unstable for the Keplerian motion of a material point [46, 47], for motion with a constant velocity over an arc of the great circle on a motionless sphere surrounding the Earth, and for motion along a latitude parallel [47], as well as in the case of any point motions restricted by some limits [48].

In the problems of the optimum control of motion of the SC's center of mass, we are interested in the properties (in particular the stability) of the solution satisfying the conditions of the principle of maximum and meeting some certain, a priori unknown (nonzero, in the general case) initial conditions, rather than the properties of a trivial solution of equations (1.11) or equation (3.3).

We denote by $\psi_{v\,\mathrm{opt}}(t)$, $d\psi_{v\,\mathrm{opt}}(t)/dt$ the solution of equation (3.3) meeting the optimum control and the optimum phase trajectory $\mathbf{r}(t)$, $\mathbf{v}(t)$. This solution corresponds to some initial conditions $\psi_{v\,\mathrm{opt}}(t_0)$, $\dot{\psi}_{v\,\mathrm{opt}}(t_0)$, which must be determined when solving the optimum control problem. We denote by $\psi_v = \psi_{v\,\mathrm{opt}} + \delta\psi_v$ and $\dot{\Psi}_v = \dot{\Psi}_{v\,\mathrm{opt}} + \delta\dot{\Psi}_v$ the solution of equation (3.3) meeting the same optimum trajectory $\mathbf{r}(t)$, $\mathbf{v}(t)$ and corresponding to other (disturbed) initial conditions

$$\psi_v(t_0) = \psi_{v\,\mathrm{opt}}(t_0) + \delta\psi_v(t_0),$$
$$\dot{\Psi}_v(t_0) = \dot{\Psi}_{v\,\mathrm{opt}}(t_0) + \delta\dot{\Psi}_v(t_0).$$

In virtue of the linearity of equation (3.3), the differential equation for the deviation $\delta\psi_v$ will have the same form as equation (3.3):

$$d^2(\delta\psi_v)/dt^2 + fMr^{-3}\delta\psi_v - 3fMr^{-5}(\delta\psi_v\cdot\mathbf{r})\mathbf{r} = 0. \tag{3.5}$$

Obviously, the zero solution

$$\delta\psi_v = 0, \ \ \delta\dot{\Psi}_v = 0 \tag{3.6}$$

of equation (3.5) meets the optimum control and optimum trajectory.

The full analogy of differential equations of disturbed motion (3.2) and (3.5) and undisturbed motions (3.4) and (3.6) suggests the instability of solutions of equation (3.3) for the wide class of optimum motions of the SC's center of mass, which makes the problem of the solution of the boundary value optimization problem (i.e., the problem of searching for initial conditions for conjugate equations (1.11) or (3.3)) even more complicated.

Note that for optimum trajectories of SC center of mass, which include sections of passive (Keplerian) motion, the instability of the solutions of conjugate equations (1.11) follows also from the general solution of equation (3.3) written in projections on the axes of the orbital coordinate system [5, 49]. So, for the motion over the arcs of circles this solution has the form [5]

$$\psi_{vr} = c_1\cos\varphi_{tr} + c_2\sin\varphi_{tr} + 2c_3,$$
$$\psi_{v\tau} = 2c_2\cos\varphi_{tr} + 2c_1\sin\varphi_{tr} - 3c_3\varphi_{tr} + c_4,$$
$$\psi_{vz} = c_5\cos\varphi_{tr} + c_6\sin\varphi_{tr}.$$

Here ψ_{vr}, $\psi_{v\tau}$, ψ_{vz} are projections of the vector conjugate variable ψ_v on the radial, transversal (with respect to the radius vector of the SC's center of mass) and normal to the plane of the SC's orbit; $c_i (i = \overline{1,6})$ are arbitrary constants.

It is seen from this solution that the solution with respect to variable $\psi_{v\tau}$ (the transversal component of vector ψ_v) is unstable (see also paragraph 4.1 of this paper).

Consider now the strict formulation [10] of the problem of stability of solutions of differential equations for the boundary value problem of optimum control of motion of the SC's center of mass in the central Newtonian gravitational field, which were obtained by using the principle of the maximum. Assume that the optimum control $\mathbf{p}^0$ is found as an explicit function of time: $\mathbf{p}^0 = \mathbf{p}^0(t)$. Denote by $\mathbf{r}_{\text{opt}}(t)$, $\mathbf{v}_{\text{opt}}(t)$ the solution of equations (1.2) meeting the given initial conditions of motion $\mathbf{r}(t_0)$, $\mathbf{v}(t_0)$, passing through the specified end point $\mathbf{r}(t_1) = \mathbf{r}^*$, $\mathbf{v}(t_1) = \mathbf{v}^*$ and meeting the found optimum control $\mathbf{p}^0(t)$, that is, we will assume that $\mathbf{r}_{\text{opt}}(t)$, $\mathbf{v}_{\text{opt}}(t)$ is the optimum (undisturbed) motion of the SC's center of mass. We denote by $\psi_{v\,\text{opt}}(t)$, $\dot{\Psi}_{v\,\text{opt}}(t)$ the solution of equation (3.3) corresponding to the optimum trajectory of the SC's center of mass (or, more precisely, to the optimum law of the variation of radius vector $\mathbf{r} = \mathbf{r}_{\text{opt}}(t)$ and to the optimum control $\mathbf{p}^0(t)$. This solution corresponds to some initial conditions $\psi_{v\,\text{opt}}(t_0)$, $\dot{\Psi}_{v\,\text{opt}}(t_0)$ found in the solution of a boundary value problem. Here, in virtue of (1.13), the values of $\psi_{v\,\text{opt}}(t_0)$, and $\mathbf{p}^0(t_0)$ are related by the equation

$$\mathbf{p}^0(t_0) = p^0(t_0)\psi_{v\,\text{opt}}(t_0)/|\psi_{v\,\text{opt}}(t_0)|.$$

We denote by $\mathbf{r}_p = \mathbf{r}_{\text{opt}}(t) + \delta\mathbf{r}$ and $\mathbf{v}_p = \mathbf{v}_{\text{opt}}(t) + \delta\mathbf{v}$ the solution of equations (1.2) meeting the specified initial conditions of motion $\mathbf{r}(t_0)$, $\mathbf{v}(t_0)$ and the disturbed control $\mathbf{p}_p = \mathbf{p}^0(t) + \delta\mathbf{p}$, and by $\psi_{vp} = \psi_{v\,\text{opt}}(t) + \delta\psi_v$, $\dot{\Psi}_{vp} = \dot{\Psi}_{v\,\text{opt}}(t) + \delta\dot{\Psi}_v$ the solution of equation (3.3) meeting the disturbed initial conditions with respect to conjugate variables $\psi_{vp}(t_0) = \psi_{v\,\text{opt}}(t_0) + \delta\psi_v(t_0)$, $\dot{\Psi}_{vp}(t_0) = \dot{\Psi}_{v\,\text{opt}}(t_0) + \delta\dot{\Psi}_v(t_0)$ and the disturbed control $\mathbf{p}_p = \mathbf{p}^0(t) + \delta\mathbf{p}$.

Taking into account that variables $\mathbf{r}_{\text{opt}}$, $\mathbf{r}_p$ and $\psi_{v\,\text{opt}}$, ψ_{vp} satisfy equations (1.2) (or (1.1)) and (1.11) (or (3.3)), we obtain the following nonlinear differential equations of disturbed motion of the principle of maximum for the problem of optimum control of motion of the SC's center of mass in the central Newtonian gravitational field (the equations in deviations $\delta\mathbf{r}$, $\delta\psi_v$) [10]:

$$d^2(\delta\mathbf{r})/dt^2 + fM[(r_p^{-3} - r_{\text{opt}}^{-3})\mathbf{r}_{\text{opt}} + r_p^{-3}\delta\mathbf{r}] = \delta\mathbf{p}, \tag{3.7}$$

$$\begin{aligned} d^2(\delta\psi_v)/dt^2 &+ fM[(r_p^{-3} - r_{\text{opt}}^{-3})\psi_{v\,\text{opt}} + r_p^{-3}\delta\psi_v \\ &- 3r_p^{-5}((\psi_{v\,\text{opt}} + \delta\psi_v)\cdot(\mathbf{r}_{\text{opt}} + \delta\mathbf{r}))(\mathbf{r}_{\text{opt}} + \delta\mathbf{r}) \\ &+ 3r_{\text{opt}}^{-5}(\psi_{v\,\text{opt}}\cdot\mathbf{r}_{\text{opt}})\mathbf{r}_{\text{opt}}] = 0. \end{aligned} \tag{3.8}$$

The first (linear) approximation of the differential equations of disturbed motion has the form:

$$\begin{aligned} d^2(\delta\mathbf{r})/dt^2 &+ fMr_{\text{opt}}^{-3}\delta\mathbf{r} \\ &- 3fMr_{\text{opt}}^{-5}(\mathbf{r}_{\text{opt}}\cdot\delta\mathbf{r})\mathbf{r}_{\text{opt}} = \delta\mathbf{p}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} d^2(\delta\psi_v)/dt^2 &+ fMr_{\text{opt}}^{-3}\delta\psi_v - 3fMr_{\text{opt}}^{-5} \\ &\times(\mathbf{r}_{\text{opt}}\cdot\delta\psi_v)\mathbf{r}_{\text{opt}} - 3fMr_{\text{opt}}^{-5}\big[(\mathbf{r}_{\text{opt}}\cdot\delta\mathbf{r})\psi_{v\,\text{opt}} \\ &+ (\psi_{v\,\text{opt}}\cdot\delta\mathbf{r})\mathbf{r}_{\text{opt}} + (\psi_{v\,\text{opt}}\cdot\mathbf{r}_{\text{opt}})\delta\mathbf{r}\big] \\ &+ 15fMr_{\text{opt}}^{-7}(\mathbf{r}_{\text{opt}}\cdot\delta\mathbf{r})(\psi_{v\,\text{opt}}\cdot\mathbf{r}_{\text{opt}})\mathbf{r}_{\text{opt}} = 0. \end{aligned} \tag{3.10}$$

The disturbance of control $\delta\mathbf{p}$, appearing in the differential equations of disturbed motion (3.7), (3.8) and (3.9), (3.10), is determined, in virtue of (1.13)–(1.16), by the following relations:

(1) on the trajectory sections on which the value of control does not reach its maximum admissible value,

$$\delta\mathbf{p} = (2\alpha_2)^{-1}\delta\psi_v; \tag{3.11}$$

(2) on the trajectory sections on which SC center of mass is moving with the maximum control,

$$\begin{aligned} \delta\mathbf{p} = p_{\max}\big(&|\psi_{v\,\text{opt}} + \delta\psi_v|^{-1}(\psi_{v\,\text{opt}} + \delta\psi_v) \\ &- |\psi_{v\,\text{opt}}|^{-1}\psi_{v\,\text{opt}}\big) \end{aligned} \tag{3.12}$$

or, in the first approximation,

$$\begin{aligned} \delta\mathbf{p} = p_{\max}|\psi_{v\,\text{opt}}|^{-3}&[(\psi_{v2\,\text{opt}}^2 + \psi_{v3\,\text{opt}}^2)\delta\psi_{v1}\mathbf{x}_1 \\ &+ (\psi_{v1\,\text{opt}}^2 + \psi_{v3\,\text{opt}}^2)\delta\psi_{v2}\mathbf{x}_2 \\ &+ (\psi_{v1\,\text{opt}}^2 + \psi_{v2\,\text{opt}}^2)\delta\psi_{v3}\mathbf{x}_3], \end{aligned} \tag{3.13}$$

where $\mathbf{x}_1$, $\mathbf{x}_2$,$\mathbf{x}_3$ are unit vectors (orts) of the coordinate system X;

(3) on the passive trajectory sections (where control is zero),

$$\delta\mathbf{p} = 0. \tag{3.14}$$

On the passive trajectory sections, the differential equations of disturbed motion (3.9), (3.10) take the form

$$d^2(\delta\mathbf{r})/dt^2 + fMr_{\text{opt}}^{-3}\delta\mathbf{r} - 3fMr_{\text{opt}}^{-5}(\mathbf{r}_{\text{opt}}\cdot\delta\mathbf{r})\mathbf{r}_{\text{opt}} = 0, \tag{3.15}$$

$$\begin{aligned} d^2(\delta\psi_v)/dt^2 &+ fMr_{\text{opt}}^{-3}\delta\psi_v \\ - 3fMr_{\text{opt}}^{-5}(\mathbf{r}_{\text{opt}}\cdot\delta\psi_v)\mathbf{r}_{\text{opt}} &= 3fMr_{\text{opt}}^{-5}[(\mathbf{r}_{\text{opt}}\cdot\delta\mathbf{r}(t)) \\ \times\psi_{v\,\text{opt}} + (\psi_{v\,\text{opt}}\cdot\delta\mathbf{r}(t))\mathbf{r}_{\text{opt}} &+ (\psi_{v\,\text{opt}}\cdot\mathbf{r}_{\text{opt}})\delta\mathbf{r}(t)] \\ &- 15fMr_{\text{opt}}^{-7}(\mathbf{r}_{\text{opt}}\cdot\delta\mathbf{r}(t))(\psi_{v\,\text{opt}}\cdot\mathbf{r}_{\text{opt}})\mathbf{r}_{\text{opt}}. \end{aligned} \tag{3.16}$$

The vector function $\mathbf{r}_{\text{opt}} = \mathbf{r}_{\text{opt}}(t)$ on these sections describes Keplerian motion. Therefore, in this case equation (3.15) has a well-known general analytical solution $\delta\mathbf{r} = \delta\mathbf{r}(\mathbf{t})$. Equation (3.16) is a linear nonstationary inhomogeneous equation, whose left-hand part coincides with the left-hand part of homogeneous equation (3.15). Its solution can be constructed using the Poincaré theorem: the independent partial solutions of the equation in variations (3.16) are found in the form of derivatives of the general solution (3.15) with respect to arbitrary constants.

Thus, in order to study the stability of solutions of differential equations of the boundary value problem of optimum control of motion of the SC's center of mass in the central Newtonian gravitational field, using the Cartesian coordinates as variables, it is necessary to consider the stability of the zero solution

$$\delta\mathbf{r} = 0,\quad d(\delta\mathbf{r})/dt = 0,\quad \delta\psi_v = 0,\quad d(\delta\psi_v)/dt = 0$$

of nonlinear equations of the disturbed motion (3.7), (3.8), or of the equations of the first approximation (3.9), (3.10) supplemented by relations (3.11)–(3.14).

We note, however, that, as A.M. Lyapunov stated (see [50, p. 20] and also V.V. Rumyantsev's paper [51]): "If a material point, attracted by a motionless center inversely proportional to the square of the distance, describes a circular trajectory, then its motion with respect to the radius vector, drawn from the center of attraction, as well as with respect to its velocity, is stable. The same motion with respect to rectangular coordinates is unstable. If, however, the considered point describes an elliptical trajectory, then its motion is unstable not only with respect to rectangular coordinates, but with respect to the radius vector and velocity as well."[3] Therefore, in order to increase the efficiency of the numerical solution of the problem of constructing the optimum trajectories and laws of optimum control, it is expedient to use the equations of motion of SC center of mass in such variables, the behavior of which (or, at least, of a part of which) is stable (nonasymptotically) for important particular cases of motion of SC center of mass. One of these cases is, for example, motion over a circular orbit (in the absence of a vector of acceleration due to jet thrust). Such equations of motion include equations (5.4)–(5.7) ((5.4), (5.5), (5.8)) [3], as well as (8.3)–(8.6) ((8.3), (8.4), (8.7)) [3] in quaternion osculating elements, whose partial solutions, which meet the motion of the SC's center of mass over a circular orbit, are stable with respect to variables r, v_1, c, $\Lambda_j (j = \overline{0,3})$; r, v_1, c, $\Lambda_{orj} (j = \overline{0,3})$, respectively, which appear in these solutions. Here the scalar variables Λ_j and Λ_{orj} represent the components of quaternion osculating elements Λ and Λ_{or}, respectively. In this case an instability of motion takes place with respect to the generalized true anomaly φ described by equation (5.5) [3], or with respect to the true anomaly φ_{tr} described by equation (8.4) [3]. Each of these variables characterizes the position of SC center of mass in orbit.

Note also that in the problem of the optimum reorientation of SC orbit [33–37, 41–43], under an effect of jet thrust orthogonal to the orbital plane, the differential quaternion equations of orientation of the SC orbit and the orbital coordinate system, as well as the quaternion equations conjugate to them, have, in the cases of high-speed response and characteristic velocity minimization, nonasymptotically stable solutions.

Thus, the improvement of the computational stability of the solution of boundary value optimization problems can be achieved by using quaternion variables having the sense of quaternions of rotations. When these variables are used as components of differential equations of boundary value problems, there appear quaternion phase and conjugate equations, which have an identical compact structure and have, in some important particular cases of motion, nonasymptotically stable solutions.

4. THE ELIMINATION OF SINGULAR POINTS IN THE DIFFERENTIAL EQUATIONS OF ORBITAL MOTION AND IN THEIR PARTIAL SOLUTIONS

4.1. The singular point $r = 0$ (singularity at coordinate origin) in the Newtonian equations of motion of SC center of mass (1.1) (1.2) is most efficiently eliminated by the regularizing the Kustaanheimo–Stiefel transformation, which is directly related to the quaternion method of describing the motion [1, 2]. Here, the regular quaternion equations of orbital motion in *KS*-variables (16) [2] represent the most convenient form of regular equations of motion of the SC's center of mass in the Newtonian gravitational field.

So, if the vector equation of motion of the SC's center of mass in the central Newtonian gravitational field

$$d^2\mathbf{r}/dt^2 + fMr^{-3}\mathbf{r} = 0, \qquad (4.1)$$

written on a section of passive motion (in the absence of control), contains a specific point $r = 0$ of the singularity type and represents a nonlinear differential equation, and the equation

$$d^2\psi_v/dt^2 + fMr^{-3}\psi_v - 3fMr^{-5}(\psi_v \cdot \mathbf{r})\mathbf{r} = 0 \qquad (4.2)$$

which is conjugate to this nonlinear equation, contains the same specific point $r = 0$ and represents a linear, but nonstationary, differential equation, then the quaternion equation of motion of the SC's center of mass in *KS*-variables [2, 11–13]

$$d^2\mathbf{u}/d\tau^2 - (h/2)\mathbf{u} = 0,\quad h = \text{const} \qquad (4.3)$$

[3] We clarify that in this statement that by the radius vector, A.M. Lyapunov means its magnitude.

and the quaternion equation [11–13]

$$d^2\psi_s/d\tau^2 - (h/2)\psi_s = a\mathbf{u}, \quad h = \text{const}, \quad a = \text{const} \tag{4.4}$$

which is conjugate to it, in this important case do not contain singular points and represent linear differential equations with constant coefficients.

Here $\mathbf{u} = u_0 + u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3$ is the regular quaternion phase variable (u_j are Kustaanheimo–Stiefel's variables), $\psi_s = \psi_{s0} + \psi_{s1}\mathbf{i}_1 + \psi_{s2}\mathbf{i}_2 + \psi_{s3}\mathbf{i}_3$ is the quaternion variable conjugate to the phase variable $\mathbf{s} = d\mathbf{u}/d\tau$, h is the full mechanical energy of the SC's unit mass (the Keplerian energy), τ is the new independent variable associated with time t by the differential relation $dt/d\tau = r = u_0^2 + u_1^2 + u_2^2 + u_3^2$, $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are vector imaginary Hamilton's units.

The general solution of equation (4.4), corresponding to the Keplerian elliptic motion of the SC's center of mass when $h < 0$, has the form

$$\begin{aligned} \psi_s(\tau) &= \cos(k\tau)\psi_{s0} + (1/k)\sin(k\tau)(d\psi_s/d\tau)_0 \\ &+ (a/(2k))[\tau\sin(k\tau)\mathbf{u}_0 + (1/k)((1/k)\sin(k\tau) \\ &- \tau\cos(k\tau))(du/d\tau)_0], \\ d\psi_s/d\tau &= -k\sin(k\tau)\psi_{s0} + \cos(k\tau)(d\psi_s/d\tau)_0 \\ &+ (a/(2k))[(\sin(k\tau) + k\tau\cos(k\tau))\mathbf{u}_0 \\ &+ \tau\sin(k\tau)(d\mathbf{u}/d\tau)_0], \quad k = (-h/2)^{1/2}. \end{aligned} \tag{4.5}$$

It is seen from this solution that it contains terms proportional to the first power of "time" τ for the initial values of phase variables $\mathbf{u}_0$, $(d\mathbf{u}/d\tau)_0$, which are specified accurately on the first section of active motion only. The frequency $k = (-h/2)^{1/2}$ is also specified accurately on the first section of active motion only. On all other possible sections of passive motion, the initial values of these quantities will already be obtained (in the iteration numerical solution of a boundary value problem) as inaccurate ones. This results in the instability of the solutions of conjugate equations in variables ψ_{sj} $(j = \overline{0,3})$ with respect to the initial conditions of motion on all sections of passive motion.

4.2. The singular points at which the orbit inclination angle $I = 0, \pi$, and the cumbersome trigonometric expressions in classical differential equations of SC orbit orientation [52] (in equations (1.1) [3]) (as well as in the other differential equations of orientation of often used rotating coordinate systems, written in the angular variables) and in the equations conjugate to them, are eliminated by using quaternion differential equations of instantaneous SC orbit orientation [10, 11, 22, 27, 44] (equations (8.7) [3])

$$\begin{aligned} 2d\Lambda_{or}/dt &= \Lambda_{or} \circ \Omega_{or\zeta}, \\ \Omega_{or\zeta} &= rc^{-1}p_3(\cos\varphi_{tr}\mathbf{i}_1 + \sin\varphi_{tr}\mathbf{i}_2) \\ &- r(c^2 - fMr)^{-1}\cos\varphi_{tr}(cp_1\cos\varphi_{tr} \\ &- (c + fMrc^{-1})p_2\sin\varphi_{tr})\mathbf{i}_3; \end{aligned} \tag{4.6}$$

of the nonholonomic (azimuthally free) coordinate system (equations (3.6) [3])

$$2d\lambda/dt = r^{-2}\lambda \circ \mathbf{c}_\eta, \quad \mathbf{c}_\eta = c_2\mathbf{i}_2 + c_3\mathbf{i}_3; \tag{4.7}$$

of the orbital coordinate system [10, 11, 19, 22, 23] (equations (4.9) [3])

$$\begin{aligned} 2d\lambda/dt &= \lambda \circ \omega_\eta, \\ \omega_\eta &= \omega_1\mathbf{i}_1 + \omega_3\mathbf{i}_3 = (r/c)p_3\mathbf{i}_1 + (c/r^2)\mathbf{i}_3; \end{aligned} \tag{4.8}$$

of the ideal coordinate system [10, 11, 22, 23] (equations (5.8) [3])

$$\begin{aligned} 2d\Lambda/dt &= \Lambda \circ \Omega_\xi, \\ \Omega_\xi &= \Omega_1\mathbf{i}_1 + \Omega_2\mathbf{i}_2 = (r/c)p_3(\cos\varphi\mathbf{i}_1 + \sin\varphi\mathbf{i}_2) \end{aligned} \tag{4.9}$$

and quaternion equations conjugate to them which have a form similar to equations (4.6)–(4.9):

$$\begin{gathered} 2d\mathbf{M}_{or}/dt = \mathbf{M}_{or} \circ \Omega_{or\zeta}, \\ 2d\mu/dt = r^{-2}\mu \circ \mathbf{c}_\eta, \\ 2d\mu/dt = \mu \circ \omega_\eta, \quad 2d\mathbf{M}/dt = \mathbf{M} \circ \Omega_\xi. \end{gathered}$$

Here Λ_{or}, λ, and Λ are quaternion phase variables characterizing instantaneous orientation of the orbit, nonholonomic (or orbital) and ideal coordinate systems, respectively; $\mathbf{M}_{or}$, μ, and $\mathbf{M}$ are quaternion conjugate variables corresponding to phase variables Λ_{or}, λ, and Λ, $\Omega_{or\zeta}$, $r^{-2}\mathbf{c}_\eta$, ω_η, Ω_ξ are quaternions of instantaneous absolute angular velocities of SC orbit, of nonholonomic, orbital and ideal coordinate systems, respectively; φ_{tr} is the true anomaly characterizing the position of the SC's center of mass at the instantaneous orbit, φ is the generalized anomaly; $c = |\mathbf{r} \times \mathbf{v}|$ is the magnitude of vector $\mathbf{c}$ of the moment of orbital velocity of the SC's center of mass; $c_1 = 0$, c_2, c_3 are projections of vector $\mathbf{c}$ on the axes of the nonholonomic coordinate system; $p_k (k = 1, 2, 3)$ are projections of vector $\mathbf{p}$ of acceleration of the SC's center of mass on the axes of the orbital coordinate system η (of the control); symbol $\circ$ indicates quaternion multiplication.

We indicate the basic properties of quaternion differential equations of the SC orbit orientation, of nonholonomic, orbital and perfect coordinate systems: (1) self-contingency (quaternion phase and conjugate-to-them equations fully coincide in their form); (2) absence of singular points generated by the use of angular osculating orbital elements and other angular variables; (3) nonasymptotical stability of solutions of phase and conjugate equations in the problem of optimum reorientation of SC orbit under the effect of jet thrust orthogonal to the orbital plane; in the case of high-speed response and characteristic velocity minimization (under such a control SC orbit does not change its shape and its size in the process of controlled motion [22, 33–43]).

4.3. The special point $e_{or} = 0$ of a singularity type is inherent in the differential equation for the true anomaly [52]

$$\begin{gathered}\dot{\varphi}_{tr} = c/r^2 + (c/(fMe_{or})) \\ \times [p_1\cos\varphi_{tr} - (1 + fMr/c^2)p_2\sin\varphi_{tr}],\end{gathered} \tag{4.10}$$

when the instantaneous eccentricity of SC orbit e_{or} becomes equal to zero in the process of controlled motion (instantaneous orbit assumes a circular shape).

This equation supplements the classical differential equations of SC orbit orientation in the angular osculating elements of orbit [52] or the differential equations of SC orbit orientation in the Euler (Rodrigues–Hamilton) parameters [27, 44], as well as the quaternion differential equation of SC orbit orientation in Euler's parameters [27, 44].

When the differential equations of orbit orientation in the angular osculating orbital elements or in Euler's parameters, supplemented by equation (4.10) for the true anomaly, are used for solving problems of controlling SC orbital motion, computational difficulties may arise because of the presence of the mentioned singular point. For example, in paper [27] it was shown that, in the numerical solution of the problem of controlled SC rendezvous with an uncontrolled vehicle moving over an elliptical Keplerian orbit with particular parameters, using the quaternion differential equation of orbit orientation in Euler's parameters and the differential equation for the true anomaly, the orbit of a controlled SC at some time instant becomes nearly circular (the eccentricity assumes a value of about zero), which results in the degeneracy of the equations of motion (in the interruption of the numerical solution of a boundary value problem). To escape from such situations, the authors of [27] have used a transition, in the neighborhood of a singular point, to another model of orbital motion [19, 22, 53] that uses the quaternion differential equation of the ideal coordinate system orientation and the equation for a generalized true anomaly, and after passing this point a return to the former model was accomplished.

The presence of singular point $e_{or} = 0$ makes inconsistent the solution of the problem of optimum reorientation of a circular SC orbit as a deformable figure using the general equations of motion of SC center of mass containing the differential equations of orbit orientation in the angular variables or in Euler's parameters and the differential equation for the true anomaly, with an arbitrary direction of the jet thrust vector in inertial space. However, this singular point is eliminated if the jet thrust vector is directed orthogonally to SC orbital plane. With such a direction of the jet thrust vector, orbital motion equations describing the shape and size of the orbit are integrated, resulting in an equation of a conic section. SC orbit, while keeping its shape and size unchanged, turns in inertial space as an unchanged (nondeformable) figure. In this case, differential equations describing orbit orientation and SC position in orbit form a closed system of regular differential equations that do not contain the singular point $e_{or} = 0$. These equations (equations in Euler's parameters especially) allow an effective solution to the problem of optimum reorientation of both circular and elliptical SC orbits [33–43].

4.4. Analytical solutions of quaternion regular differential equations of motion of SC center of mass in the Kustaanheimo–Stiefel variables, obtained for some partial cases of its motion, also do not contain singular points, unlike solutions obtained by means of traditionally used equations. So, for example, it is known [5, 49] that integrals appearing in the solution of a conjugate system of equations (1.11) written in an orbital coordinate system, contain on Keplerian elliptical arcs singularities for $\varphi_{tr} = 0, \pi$ (in this case similar singular points [17] appear in the solution of the differential equation for a conjugate variable meeting the projection of the velocity vector of the SC's center of mass on the direction of its radius vector **r**), whereas the corresponding analytical solution (4.5) of the quaternion conjugate equation (4.4), presented above, does not contain singularities.

5. FIRST INTEGRALS, DECREASE OF DIMENSIONS OF SYSTMS OF DIFFERENTIAL EQUATIONS OF BOUNDARY VALUE OPTIMIZATION PROBLEMS AND THEIR SIMPLIFICATION

The solution of problems of the optimum control of motion of SC center of mass is largely complicated by the nonlinearity and high dimensionality of systems of differential equations of boundary value problems (from the 12th to 20th orders of magnitude and higher, depending on the problem solved and the model of motion used).

The decrease of the dimensions of boundary value problems can be achieved through the use of first integrals of differential equations of boundary value problems. In the problems of optimum control of motion of SC center of mass in the central Newtonian gravitational field there exists (for the optimum control and optimum trajectory) the first integral of the vector (1.17). However, its direct use for lowering dimensions of the systems of differential equations of boundary value problems of the optimum control of motion of the SC's center of mass is not effective because of the impossibility of resolving relation (1.17) with respect to the vector conjugate variable ψ_v or the variable $\psi_r = -d\psi_v/dt$, because the matrix analogue of the first integral of this vector contains skew-symmetric matrices of third-order coefficients whose determinants are equal to zero.

The differential equations of the boundary value problems of the optimum control of motion of SC center of mass, written in any rotating coordinate system and using quaternions of turns to describe the angular motion of these coordinate systems, have quaternion first integrals, which exist for any control [11, 16, 17, 19, 22, 27, 44]. The equations of the boundary value problem of optimum rendezvous of a controlled SC with an uncontrolled vehicle moving over a specified Keplerian orbit in the central Newtonian gravitational field, written in the orbital coordinate system using the quaternion differential equation of this coordinate system orientation, have the first quaternion integral [22]

$$\begin{gathered}\mu \circ \bar{\lambda} = \nu^* = \text{const}, \\ \nu^* = \nu_0^* + \nu_v^* = \nu_0^* + \nu_1^*\mathbf{i}_1 + \nu_2^*\mathbf{i}_2 + \nu_3^*\mathbf{i}_3,\end{gathered} \tag{5.1}$$

which is equivalent to four scalar equations

$$\begin{aligned}\mu_0\lambda_0 + \mu_1\lambda_1 + \mu_2\lambda_2 + \mu_3\lambda_3 &= \nu_0^* = \text{const}, \\ -\mu_0\lambda_1 + \mu_1\lambda_0 - \mu_2\lambda_3 + \mu_3\lambda_2 &= \nu_1^* = \text{const}, \\ -\mu_0\lambda_2 + \mu_1\lambda_3 + \mu_2\lambda_0 - \mu_3\lambda_1 &= \nu_2^* = \text{const}, \\ -\mu_0\lambda_3 - \mu_1\lambda_2 + \mu_2\lambda_1 + \mu_3\lambda_0 &= \nu_3^* = \text{const}.\end{aligned}$$

Here λ is the quaternion of orbital coordinate system orientation in the inertial coordinate system, $\lambda_j (j = 0, 1, 2, 3)$ are the components of the quaternion λ (the Euler (Rodrigues–Hamilton) parameters), μ and μ_j are conjugate quaternion variables corresponding to the phase variable λ, and its components; the upper dash means a symbol of conjugation.

Similar quaternion first integrals appear in the equations of problems of the optimum control of motion of the SC's center of mass, written in the nonholonomic (azimuthal-free) coordinate system [16, 17], in the ideal coordinate system [22], and in the coordinate system linked with an instantaneous SC orbit and characterizing its orientation in the inertial coordinate system [27, 44]. These quaternion first integrals are similar to the quaternion first integrals, which were stated in the problems of optimum control of rotational (angular) motion of a solid body (SC) by V.N. Branets and I.P. Shmylevskii [54]. The existence of these first integrals is conditioned by the property of the self-conjugation of quaternion kinematic equations of angular motion of rotating coordinate systems that we have used in the problems of optimum control of the SC orbital motion. We emphasize that the quaternion first integrals, in contrast to the classical vector first integral (1.17), exist for any control. In paper [19] it was shown that the vector first integral of form (1.17) is obtained from the quaternion first integral (5.1) in the case of optimum control (it represents a partial case).

The use of the new quaternion variable $\nu = \bar{\lambda} \circ \mu$ or similar quaternion variables $\mathbf{N} = \bar{\Lambda} \circ \mathbf{M}$, $\mathbf{N}_{or} = \bar{\Lambda}_{or} \circ \mathbf{M}_{or}$, related to quaternion first integrals (more precisely, to quaternion constants of motion ν^*, $\mathbf{N}^*$, $\mathbf{N}^*_{or}$) by the transformations of rotation

$$\begin{gathered}\nu = \nu_0 + \nu_v = \bar{\lambda} \circ \nu^* \circ \lambda, \\ \mathbf{N} = N_0 + \mathbf{N}_v = \bar{\Lambda} \circ \mathbf{N}^* \circ \Lambda, \\ \mathbf{N}_{or} = N_{0or} + \mathbf{N}_{vor} = \bar{\Lambda}_{or} \circ \mathbf{N}^*_{or} \circ \Lambda_{or}\end{gathered}$$

allows the lowering by 5 units of the order of systems of differential equations of boundary value problems for any quality functional with their simultaneous simplification. In so doing, the quaternion differential equations for the phase variable λ (or Λ, or Λ_{or}) and for the conjugate variable μ (or $\mathbf{M}$, or $\mathbf{M}_{or}$), which are equivalent to eight scalar differential equations, are replaced by a single vector differential equation for the variable $\nu_v = \nu_1\mathbf{i}_1 + \nu_2\mathbf{i}_2 + \nu_3\mathbf{i}_3$:

$$\begin{gathered}\dot{\nu}_v = \nu_v \times \omega_\eta, \\ \omega_\eta = \omega_1\mathbf{i}_1 + \omega_3\mathbf{i}_3 = (r/c)p_3\mathbf{i}_1 + (c/r^2)\mathbf{i}_3,\end{gathered}$$

which has (for any control), the first integral

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = \nu_v^2 = \nu_v^{*2} = \text{const}, \tag{5.2}$$

or by the equation for variable $\mathbf{N}_v$

$$\begin{gathered}\dot{\mathbf{N}}_v = \mathbf{N}_v \times \Omega_\xi, \\ \Omega_\xi = \Omega_1\mathbf{i}_1 + \Omega_2\mathbf{i}_2 = (r/c)p_3(\cos\varphi\mathbf{i}_1 + \sin\varphi\mathbf{i}_2),\end{gathered} \tag{5.3}$$

or by the equation for variable $\mathbf{N}_{vor}$

$$\begin{aligned}\dot{\mathbf{N}}_{vor} &= \mathbf{N}_{vor} \times \Omega_{or\zeta}, \\ \Omega_{or\zeta} &= rc^{-1}p_3(\cos\varphi_{tr}\mathbf{i}_1 + \sin\varphi_{tr}\mathbf{i}_2) \\ &- r(c^2 - fMr)^{-1}\cos\varphi_{tr}(cp_1\cos\varphi_{tr} \\ &- (c + fMrc^{-1})p_2\sin\varphi_{tr})\mathbf{i}_3,\end{aligned} \tag{5.4}$$

each of which is equivalent to three scalar differential equations in variables $\nu_k (k = 1, 2, 3)$ (or N_k, or N_{ork}).

Note that each of the vector equations (5.3) and (5.4) has a first integral similar to integral (5.2).

The use of new quaternion variables ν, $\mathbf{N}$, $\mathbf{N}_{or}$ and, also, the exclusion, from the set of differential equations of boundary value optimization problems, of the differential equation for the variable conjugate with respect to the true anomaly of an uncontrolled SC, in virtue of its integrability, allows one, in such problems of optimum control of motion of SC center of mass, as the problem of (soft or hard) rendezvous of a controlled SC with an uncontrolled vehicle moving over the Keplerian trajectory, to lower the 16th or 18th order of the initial boundary value problem (with simultaneous simplification of equations) down to the 10th or 11th order [11, 16, 17, 19, 22], and in the problem of optimum reorientation of SC orbit by means of jet thrust orthogonal to SC orbital plane, from the 10th down to the 3rd order [34–37, 41–43]. In the problem of the optimum rendezvous of two SCs, the dimension of boundary value problems can be further lowered by 3 units due to: (a) the exclusion of the differential equation for the true anomaly of an uncontrolled SC from the set of differential equations of boundary value problems, and (b) taking into account the first integral that reflects zeroing of the Hamilton–Pontryagin function for the optimum control and the first scalar integral for new variables $\nu_k (k = 1, 2, 3)$ (or N_k, or N_{ork}) having the form of (5.2). However, such a lowering of the dimension is not appropriate, because of complication of remained equations and complication of the algorithms for numerical solution of boundary value optimization problems.

Consider now the relation of the first quaternion integral (5.1) and variables ν_k $(k = 1, 2, 3)$ with the known vector first integral (1.17) of the problem of the optimum control of motion of the SC's center of mass and its projections on the axes of the orbital coordinate system η. It follows from the principle of the maximum that the optimum control vector $\mathbf{p}^0$ is collinear to the vector conjugate variable ψ_v corresponding to the vector phase variable $\mathbf{v}$ (to the velocity vector of the SC's center of mass). The classical equations (1.2), (1.11) of the problem for $\mathbf{p} = \mathbf{p}^0$ have the first integral of the vector (1.17). In the case of the solution to the problem using the equations of motion of the SC's center of mass, written in the orbital coordinate system, the optimum control vector $\mathbf{p}^0$ is collinear to the vector $\mathbf{n}$, whose projections n_k in the orbital coordinate system have the form [19]:

$$n_1 = s_1, \quad n_2 = er, \quad n_3 = (1/2)(r/c)\nu_1.$$

Here s_1, e are scalar conjugate variables corresponding to the projection v_1 of vector $\mathbf{v}$ of velocity of SC center of mass onto the direction of its radius vector $\mathbf{r}$ and to the magnitude c of vector c of the moment of velocity of the SC's center of mass, v_1 is the component of new quaternion variable $\nu = \overline{\lambda} \circ \mu$.

Consider vector $\mathbf{I}$ defined by a relation similar to the expression contained in parentheses of the vector first integral (1.17):

$$\mathbf{l} = (d\mathbf{r}/dt) \times \mathbf{n} + (d\mathbf{n}/dt) \times \mathbf{r}.$$

It can be shown [19] that, for any control, vector parts ν_v and ν_v^* of quaternions ν and ν^* are related to vector $\mathbf{I}$ by the relationships

$$\nu_v = 2\mathbf{l}_\eta + 2r^3c^{-2}(l_1p_2 - cep_3)\mathbf{i}_2, \tag{5.5}$$

$$\begin{gathered} \nu_v^* = \lambda \circ \nu_v \circ \overline{\lambda} \\ = 2\mathbf{l}_x + 2r^3c^{-2}(l_1p_2 - cep_3)\lambda \circ \mathbf{i}_2 \circ \overline{\lambda} = \text{const}, \end{gathered} \tag{5.6}$$

where $\mathbf{l}_\eta = l_1\mathbf{i}_1 + l_2\mathbf{i}_2 + l_3\mathbf{i}_3$ and $\mathbf{l}_x = l_1^*\mathbf{i}_1 + l_2^*\mathbf{i}_2 + l_3^*\mathbf{i}_3 = \lambda \circ \mathbf{l}_\eta \circ \overline{\lambda}$ are the maps of vector $\mathbf{I}$ on the orbital and inertial bases η and X.

For optimum control $\mathbf{p}^0$ we have [19]:

$$l_1p_2^0 - cep_3^0 = (1/2)\nu_1p_2^0 - cep_3^0 \equiv 0.$$

So,

$$\nu_v^0 = 2\mathbf{I}_\eta^0, \; \nu_v^{*0} = 2\mathbf{I}_x^0 = \text{const}; \;\; \nu_k^0 = 2l_k^0,$$

$$\nu_k^{*0} = 2l_k^{*0} = \text{const} \quad (k = 1, 2, 3),$$

where upper index 0 means that the corresponding quantity meets optimum control.

It follows from (5.6) that our analogue $\mathbf{l}_x^0 = \mathbf{l}_x|_{\mathbf{p}=\mathbf{p}^0} =$ const of the classical first integral of the vector (1.17) represents a partial case of the first integral of the vector $\nu_v^* = \text{const}$, which represents the vector part of the first integral of the quaternion (5.1), and is obtained from it in the case of optimum control. The introduced new variables $\nu_k (k = 1, 2, 3)$, are related (for any control $\mathbf{p}$) with the projections of vector $\mathbf{I}$ onto the axes of the orbital coordinate system η by relationship (5.5), for the optimum control $\nu_k^0 = 2l_k^0$. In the initial variables for the optimum control we have:

$$\begin{aligned} \nu_v^* &= \text{vect}(\mu \circ \overline{\lambda}) = \lambda_0\mu_v - \mu_0\lambda_v + \lambda_v \times \mu_v \\ &= ((d\mathbf{r}/dt) \times \mathbf{n} + (d\mathbf{n}/dt) \times \mathbf{r})_x = \text{const}, \\ \nu_v &= \text{vect}(\overline{\lambda} \circ \mu) = \lambda_0\mu_v - \mu_0\lambda_v - \lambda_v \times \mu_v \\ &= ((d\mathbf{r}/dt) \times \mathbf{n} + (d\mathbf{n}/dt) \times \mathbf{r})_\eta = \text{var}, \end{aligned}$$

where λ_v and μ_v are vector parts of quaternion variables λ and μ.

Thus, the classical first integral of the vector represents a partial case of the first integral representing the vector part of the first integral of the quaternion (5.1) that is valid for any control, and is obtained from it in the case of optimum control.

The quaternion regular equations of boundary value problems of optimum control of motion of the SC's center of mass in the Newtonian gravitational field, written using the Kustaanheimo–Stiefel variables u_j $(j = \overline{0,3})$ and variables $s_j = du_j/d\tau$, have the first integrals [12, 31]

$$\text{scal}(\overline{\mathbf{u}} \circ \mathbf{i}_1 \circ \psi_u) + \text{scal}(\overline{\mathbf{s}} \circ \mathbf{i}_1 \circ \psi_s) = a_0 = \text{const},$$

$$\text{vect}(\overline{\mathbf{u}} \circ \psi_u + \overline{\mathbf{s}} \circ \psi_s) = \mathbf{a} = \text{const},$$

$$\bar{\mathbf{u}} = u_0 - \mathbf{u}_v, \quad \psi_u = \psi_{u0} + \psi_{uv},$$
$$\bar{\mathbf{s}} = s_0 - \mathbf{s}_v, \quad \psi_s = \psi_{s0} + \psi_{sv},$$

where ψ_u and ψ_s are the quaternion conjugate variables corresponding to quaternion phase variables $\mathbf{u}$ and $\mathbf{s}$; scal() and vect() are the scalar and vector parts of the quaternion concluded in parentheses.

In the scalar recording, these integrals take the form

$$\begin{aligned} &u_0\psi_{u1} - u_1\psi_{u0} + u_2\psi_{u3} - u_3\psi_{u2} + s_0\psi_{s1} \\ &- s_1\psi_{s0} + s_2\psi_{s3} - s_3\psi_{s2} = a_0 = \text{const}; \end{aligned} \tag{5.7}$$

$$\begin{aligned} &u_0\psi_{u1} - u_1\psi_{u0} - u_2\psi_{u3} + u_3\psi_{u2} + s_0\psi_{s1} \\ &- s_1\psi_{s0} - s_2\psi_{s3} + s_3\psi_{s2} = a_1 = \text{const}, \\ &u_0\psi_{u2} + u_1\psi_{u3} - u_2\psi_{u0} - u_3\psi_{u1} + s_0\psi_{s2} \\ &+ s_1\psi_{s3} - s_2\psi_{s0} - s_3\psi_{s1} = a_2 = \text{const}, \\ &u_0\psi_{u3} - u_1\psi_{u2} + u_2\psi_{u1} - u_3\psi_{u0} + s_0\psi_{s3} \\ &- s_1\psi_{s2} + s_2\psi_{s1} - s_3\psi_{s0} = a_3 = \text{const}, \end{aligned} \tag{5.8}$$

where ψ_{uj} and ψ_{sj} $(j = \overline{0,3})$ are the components of quaternion conjugate variables ψ_u and ψ_s.

The last three integrals can be written in the following vector form:

$$\begin{aligned} &u_0\psi_{uv} + s_0\psi_{sv} - \psi_{u0}\mathbf{u}_v - \psi_{s0}\mathbf{s}_v \\ &- \mathbf{u}_v \times \psi_{uv} - \mathbf{s}_v \times \psi_{sv} = \mathbf{a} = \text{const}. \end{aligned}$$

Integral (5.7) takes place for any control, and integrals (5.8) are valid in the case of optimum control. In the problem of soft rendezvous of two SCs the constant $a_0 = 0$. Presented integrals can be used for transferring the boundary conditions from the right edge to the left one, which is important for the numerical solution of the boundary value problem.

6. THE EXTENSION OF POSSIBILITIES OF ANALYTICAL STUDY OF THE DIFFERENTIAL EQUATIONS OF BOUNDARY VALUE PROBLEMS

6.1. The possibility of constructing the general solutions of the differential equations for the phase and conjugate variables on the sections of passive motion of the SC's center of mass in the most simple and convenient form is important for solving the problems of optimum pulse transfers of space vehicles. To solve these problems by the variation of methods of calculus, one uses general solutions of differential equations of free motion of SC center of mass in the central Newtonian gravitational field (when the control is zero) and the general solutions of corresponding conjugate equations. The differential equations of free motion of SC center of mass in the central Newtonian gravitational field in the Cartesian and other traditionally used coordinates are nonlinear, and the differential equations for conjugate variables, corresponding to them, are linear but nonstationary (see, e.g., equations (4.1) and (4.2)). The analytical general solutions of these and other equations have a complicated structure; so, the solution of problems of pulse transfers on their basis in the nonlinear formulation can be obtained only numerically, using computer technology.

The differential equations for phase and conjugate variables in the problems of optimum pulse transfers, in the case of using the quaternion equations of motion of the SC's center of mass in the regular Kustaanheimo–Stiefel variables on the sections of passive motion of the SC's center of mass, are linear and stationary (see equations (4.3) and (4.4)). General analytical solutions for them are presented in a simple form (see solution (4.5) of the equations for conjugate variables), which allows the hope of more effective solution of problems. Similar possibilities in the problems of optimum pulse transfers are opened when using the equations of orbital motion of SC center of mass, which contain in their structure the quaternion equations of SC orbit orientation, of the nonholonomic, orbital and ideal coordinate systems [3]. So the quaternion equation (4.8) of motion of the orbital coordinate system assumes in this case ($p_3 = 0$) the form of the equation

$$2\dot{\lambda} = \lambda \circ \omega_\eta, \quad \omega_\eta = \omega_3\mathbf{i}_3 = (c/r^2)\mathbf{i}_3 = \dot{\varphi}\mathbf{i}_3,$$
$$c = \text{const},$$

which has the general solution

$$\lambda(t) = \lambda(t_0) \circ \left[\cos\left(\frac{1}{2}(\varphi(t) - \varphi(t_0))\right) + i_3 \sin\left(\frac{1}{2}(\varphi(t) - \varphi(t_0))\right)\right].$$

Note that these analytical solutions take place also on sections of the passive motion of SC center of mass in the case of the minimization of its characteristic velocity.

6.2. The use of quaternion models of SC orbital motion given in [2, 3] extends the possibilities of analytical studying the differential equations of boundary value optimization problems with the purpose of identifying basic regularities of the optimum motion of SC center of mass. So, in the problem of optimum reorientation of SC orbit [36, 37, 42, 43] by applying the jet thrust orthogonal to the orbital plane, the quaternion equations of boundary value problems of the principle of maximum, forming systems of nonlinear differential 10th-order equations, are reduced, for any quality functional, to a nonlinear stationary 4th-order system or to a nonlinear, nonstationary system of the 3rd order with respect to variables ν_k, which has the simple form

$$\begin{aligned} &\nu_1' = \nu_2, \quad \nu_2' = -\nu_1 + \kappa\nu_3, \quad \nu_3' = -\kappa\nu_2, \\ &\kappa = (r^3/c^2)p_3, \quad r = p_{or}/(1 + e_{or}\cos\varphi), \\ &\nu_1^2 + \nu_2^2 + \nu_3^2 = \text{const}, \end{aligned} \tag{6.1}$$

where the upper prime means differentiation with respect to the true anomaly φ, defined in this case by the differential equation $\dot{\varphi} = c/r^2$, ν_1 is the control switching function, p_{or} and e_{or} are the parameter and eccentricity of the orbit, $c = \text{const}$ is the constant of areas.

In the case of a high-speed response or characteristic velocity minimization, the integration of these systems of equations is reduced to the integration of a single scalar Riccati equation, and in the case of circular orbit reorientation, with the minimization of the integral quadratic (with respect to control) quality functional, it is reduced to the integration of the Duffing equation. In the problem of SC circular orbit reorientation, with the minimization of time or characteristic velocity, these systems of equations are integrated in elementary (trigonometric) functions. Note that in this problem equations (6.1) represent differential equations of the control switching line. Note also that the system of equations of the form (6.1) appears in many boundary value problems of optimum control of SC orbital motion, constructed using the quaternion models.

The quaternion models of SC orbital motion, presented in [2, 3], are integrated in some cases in a closed form, which can be used in studying the controlled and uncontrolled motions of the SC's center of mass. We will present below some partial cases of integrability of these equations.

(1) The quaternion differential equation of motion of the orbital coordinate system (4.8) and the quaternion differential equation of SC orbit orientation (4.6) are integrated in trigonometric functions in the case of motion of SC center of mass over a circular orbit under the effect of constant control orthogonal to SC orbital plane. In this case $r = \text{const}$, $c = \text{const}$, and the controls $p_1 = p_2 = 0$, $p_3 = \text{const}$. In this case the vector of instantaneous absolute angular velocity of the orbit is directed along the radius vector of SC center of mass, and its algebraic value is constant and equal to $(r/c)p_3$. The quaternion differential equation of motion of the orbital coordinate system in this case has the general solution

$$\lambda(t) = \lambda(t_0) \circ \left(\cos(\frac{1}{2}|\omega_\eta| dt\right) + \frac{\omega_\eta}{|\omega_\eta|}\sin\left(\frac{1}{2}|\omega_\eta| dt)\right),$$

$$\omega_\eta = \omega_1\mathbf{i}_1 + \omega_3\mathbf{i}_3 = (r/c)p_3\mathbf{i}_1 + (c/r^2)\mathbf{i}_3,$$

and the quaternion differential equation of SC orbit orientation has the general solution

$$\begin{aligned}\Lambda(t) &= \lambda(t) \circ \left(\cos\left(\frac{1}{2}\varphi(t)\right) - \mathbf{i}_3 \sin\left(\frac{1}{2}\varphi(t)\right)\right) \\ &= \Lambda(t_0) \circ \left(\cos\left(\frac{1}{2}\varphi(t_0)\right) + \mathbf{i}_3 \sin\left(\frac{1}{2}\varphi(t_0)\right)\right) \\ &\circ \left(\cos\left(\frac{1}{2}|\omega_\eta| dt\right) + \frac{\omega_\eta}{|\omega_\eta|}\sin\left(\frac{1}{2}|\omega_\eta| dt\right)\right) \\ &\circ \left(\left(\cos\frac{1}{2}\varphi(t)\right) - \mathbf{i}_3 \sin\left(\frac{1}{2}\varphi(t)\right)\right).\end{aligned}$$

(2) The quaternion equations of SC orbit and orbital coordinate system orientation are integrated for $p_1 = p_2 = 0$ in trigonometric functions in the case of synchronous rotation of SC center of mass over the orbit and orbit rotation in the space. In this case, SC center of mass rotates over the orbit with the angular velocity $\dot{\varphi} = c/r^2$, and the orbit turns in space under an effect of control action $p_3 = c^2/r^3$ around the instantaneous axis of rotation, coinciding with the radius vector of SC center of mass, with the same angular velocity $\dot{\varphi}$. The vectors of angular velocities of rotation of the orbital coordinate system and SC orbit are determined by the relations

$$\omega_\eta = \dot{\varphi}(\mathbf{i}_1 + \mathbf{i}_3), \;\; \Omega_\xi = \dot{\varphi}(\cos\varphi\mathbf{i}_1 + \sin\varphi\mathbf{i}_2),$$
$$\dot{\varphi} = c/r^2,$$

and the quaternion differential equations of the orbital coordinate system and SC orbit orientation have the general analytical solutions

$$\begin{aligned}\lambda(t) = \lambda(t_0) \circ &\left[\cos\left(\frac{1}{2}(\varphi(t) - \varphi(t_0))\right)\right. \\ &\left. + \sin\left(\frac{1}{2}(\varphi(t) - \varphi(t_0))\right)(\mathbf{i}_1 + \mathbf{i}_3)\right],\end{aligned}$$

$$\Lambda(t) = \lambda(t) \circ \left(\cos\left(\frac{1}{2}\varphi(t)\right) - \mathbf{i}_3 \sin\left(\frac{1}{2}\varphi(t)\right)\right), \;\; \dot{\varphi} = c/r^2.$$

These equations are also integrated in the more general case, where $p_3 = ac^2/r^3$, $a = \text{const}$.

Note that, in virtue of the self-contingency of quaternion kinematic equations of rotational motion, all aforementioned analytical solutions have also conjugate quaternion differential equations corresponding to phase quaternion differential equations of the orbital coordinate system and SC orbit orientation. Note also that the issues of the integrability of a quaternion kinematic equation of rotational motion have been considered in papers [54–57].

(3) The quaternion equations of the orbital motion of SC center of mass (1.16) [2] (see also [10–12]) in the regular Kustaanheimo–Stiefel variables are integrated [10] in the case of motion of SC center of mass in the central gravitational field under the effect of a external force that is constant in magnitude and direction (in the inertial coordinate system).

In the case under consideration, directing the X_1 axis of the coordinate system X along the acceleration vector $\mathbf{p}$, we have:

$$\begin{aligned}\mathbf{q} &= -\mathbf{i}_1 \circ \mathbf{u} \circ \mathbf{p}_x = -p\mathbf{i}_1 \circ \mathbf{u} \circ \mathbf{i}_1 \\ &= p(u_0 + u_1\mathbf{i}_1 - u_2\mathbf{i}_2 - u_3\mathbf{i}_3), \quad p = |\mathbf{p}| = \text{const},\end{aligned}$$

$$h = p(u_0^2 + u_1^2 - u_2^2 - u_3^2) + h^*,$$
$$h^* = h_0 - p(u_{00}^2 + u_{10}^2 - u_{20}^2 - u_{30}^2) = \text{const}.$$

The equations of motion of the SC's center of mass take the form of:

$$\begin{aligned} &d^2u_0/d\tau^2 - (h^*/2)u_0 - p(u_0^2 + u_1^2)u_0 = 0, \\ &d^2u_1/d\tau^2 - (h^*/2)u_1 - p(u_0^2 + u_1^2)u_1 = 0, \\ &d^2u_2/d\tau^2 - (h^*/2)u_2 + p(u_2^2 + u_3^2)u_2 = 0, \\ &d^2u_3/d\tau^2 - (h^*/2)u_3 + p(u_2^2 + u_3^2)u_3 = 0. \end{aligned} \tag{6.2}$$

The system of equations (6.2) breaks into two independent subsystems consisting of the first two and last two equations of this system. Replacing the variables

$$u_0 = a\cos\alpha, \quad u_1 = a\sin\alpha;$$
$$a^2 = u_0^2 + u_1^2, \quad \tan\alpha = u_1/u_0$$

the equations of the first subsystem in new variables a and α take the form

$$\begin{gathered} d^2a/d\tau^2 - a(d\alpha/d\tau)^2 - (1/2)(h^* + 2pa^2)a = 0, \\ ad^2\alpha/d\tau^2 + 2(da/d\tau)\,d\alpha/d\tau = 0. \end{gathered} \tag{6.3}$$

These equations have the first integrals

$$a^2(d\alpha/d\tau) = c = \text{const},$$
$$(da/d\tau)^2 + c/a^2 - (1/2)(h^* + pa^2)a^2 = \eta = \text{const},$$

Taking into account these expressions, the integration of equations (6.3) is reduced to the integration of the second-order equation with respect to variable a

$$\begin{gathered} d^2a/d\tau^2 - c^2/a^3 - (1/2)(h^* + 2pa^2)a = 0, \\ (c, h^*, p - \text{const}) \end{gathered} \tag{6.4}$$

or to the integration (in elliptic integrals) of the first-order equation

$$\begin{gathered} (da/d\tau)^2 + c/a^2 - (1/2)(h^* + pa^2)a^2 = \eta, \\ (c, h^*, p, \eta - \text{const}). \end{gathered} \tag{6.5}$$

Instead of the integration of equation (6.4) or (6.5), one can integrate (with respect to variable $b = a^2$) one of the equations

$$d^2b/d\tau^2 - 2h^*b - 3pb^2 = 2\eta,$$
$$(db/d\tau)^2 - 4\eta b - 2(h^* + pb)b^2 - 4c,$$
$$(c, h^*, p, \eta - \text{const}).$$

In conclusion, we note that the improvement of the computational stability of the solution of boundary value problems and the decrease of the required volume of calculations when using quaternion models of the orbital motion of the SC's center of mass are achieved due to the appearance of quaternion equations of orbit orientation (or introduced rotating coordinate systems) having nonasymptotically stable solutions, in the set of differential equations of boundary value problems, as well as due to quaternion regular, compact, and symmetrical structures of equations. The use of regular equations of the two-body problem in the Kustaanheimo–Stiefel variables for numerical construction of optimum controls and trajectories of SC motion allows one to use the numerical integration techniques, which are superior to the classical methods in terms of both accuracy, and volume of computer calculations [58–60], as well as to improve the convergence of the numerical solution of boundary value optimization problems.

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