Review Problems
Calculus AB – 2019

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August 19, 2019

1 Algebraic Laws

Statement of laws for the field of real numbers.

1.1 Associative Law

Multiplication is associative:

$$(ab)c = a(bc)$$

Addition is associative:

$$(a + b) + c = a + (b + c)$$

1.2 Commutative Law

Multiplication is commutative:

$$ab = ba$$

Addition is commutative:

$$a + b = b + a$$

1.3 Distributive Law

Distributivity of multiplication over addtion:

$$a(b + c) = ab + ac$$

1.4 Identity Operators

Multiplicative identity is 1:

$$1 \cdot x = x$$

Additive identity is 0:

$$0 + x = x$$

1.5 Inverse Operators

For $x \neq 0$, the multiplicative inverse of x is $\frac{1}{x}$:

$$x \cdot \frac{1}{x} = 1$$

Additive inverse of x is $-x$:

$$x - x = 0$$

2 Expand and Simplify

Problems marked with an asterisk ($*$) are stated in the simplest form and no further simplification is needed.

2.1 Problem 1

$$(-6ab)(0.5ac) = -3a^2bc$$

2.2 Problem 3*

$$2x(x-5) = 2x^2 - 10x$$

2.3 Problem 5

$$-2(4-3a) = 6a - 8$$

Simplest form: $2(3a-4)$

2.4 Problem 7

$$4\left(x^2 - x + 2\right) - 5\left(x^2 - 2x + 1\right) = -x^2 + 6x + 3$$

2.5 Problem 9*

$$(4x-1)(3x+7) = 12x^2 + 25x - 7$$

2.6 Problem 11*

$$(2x-1)^2 = 4x^2 - 4x + 1$$

2.7 Problem 13*

$$y^6(6-y)(5+y) = -y^8 + y^7 + 30y^6$$

2.8 Problem 15*

$$(1+2x)(x^2-3x+1) = 2x^3 - 5x^2 - x + 1$$

3 Expand and Simplify

Problems marked with an asterisk ($*$) are stated in the simplest form and no further simplification is needed.

3.1 Problem 17

$$\frac{2+8x}{2} = \frac{1}{2}(8x+2) = 4x+1$$

3.2 Problem 19

$$\frac{1}{x+5} + \frac{2}{x-3} = \frac{1}{x+5}\left(\frac{x-3}{x-3}\right) + \frac{2}{x-3}\left(\frac{x+5}{x+5}\right) = \frac{x-3+2(x+5)}{(x-3)(x+5)} = \frac{3x+7}{(x-3)(x+5)}$$

Restriction: $x \notin \{3, -5\}$

3.3 Problem 21*

$$u+1+\frac{u}{u+1} = u\left(\frac{u+1}{u+1}\right) + 1\left(\frac{u+1}{u+1}\right) + \frac{u}{u+1} = \frac{u^2+3u+1}{u+1}$$

Restriction: $u \neq -1$

3.4 Problem 23

$$\frac{x/y}{z} = \frac{x \cdot \frac{1}{y}}{z} = \frac{x}{z} \cdot \frac{1}{y} = \frac{x}{yz}$$

Restrictions: $y \neq 0, z \neq 0$

3.5 Problem 25

$$\left(\frac{-2r}{s}\right)\left(\frac{s^2}{-6t}\right) = \frac{rs}{3t}$$

Restriction: $t \neq 0$

3.6 Problem 27

$$\frac{1+\frac{1}{c-1}}{1-\frac{1}{c-1}} = \frac{1+\frac{1}{c-1}}{1-\frac{1}{c-1}}\left(\frac{c-1}{c-1}\right) = \frac{c-1+1}{c-1-1} = \frac{c}{c-2}$$

Restriction: $c \neq 2$

4 Factor the expression

4.1 Problem 29

$$2x + 12x^3 = 2x\left(6x^2 + 1\right)$$

4.2 Problem 31

$$x^2 + 7x + 6 = (x + 1)(x + 6)$$

Method: Look at *constant* term 6. Factors are either (a) $6 = 6 \cdot 1$ or (b) $6 = 3 \cdot 2$. Use these factors to check *linear* term: (a) $6 + 1 = 7$ (winner), (b) $3 + 2 = 5$.

4.3 Problem 33

$$x^2 - 2x - 8 = (x - 4)(x + 2)$$

Look at *constant* term -8. Factors are either (a) $-8 = \pm 8 \cdot \mp 1$ or (b) $-8 = \pm 4 \cdot \mp 2$. Check *linear* term: (a) $\pm 8 + \mp 1 = \pm 7$, (b) $\pm 4 + \mp 2 = \pm 2$,

4.4 Problem 35

$$9x^2 - 36 = 9(x^2 - 4) = 9(x - 2)(x + 2)$$

4.5 Problem 37

$$6x^2 - 5x - 6 = (2x - 3)(3x + 2)$$

Method: Look for integer roots a and b.
I. Try $6x^2 - 5x - 6 = (6x + a)(x + b)$ (Does not work.)
II. Try $6x^2 - 5x - 6 = (3x - a)(2x + b) = 6x^2 + x(3b - 2a) - ab$. The choices $(a, b) = (6, 1)$ and $(a, b) = (1, 6)$ do not work. The choice $(a, b) = (3, 2)$ does not work. But the choice $(a, b) = (2, 3)$ works. Linear term $(3b - 2a) = 9 - 4 = 5$.

Your teach may want to see synthetic division.

4.6 Problem 39

$$4t^2 - 12t + 9 = (2t - 3)^2$$

Method: Memorization of special cases

$$t^3 + 1 = (t + 1)\left(t^2 - t + 1\right)$$

$$t^3 - 1 = (t - 1)\left(t^2 + t + 1\right)$$

4.7 Problem 41

$$4t^2 - 12t + 9 = (2t - 3)^2$$

Method: Try two cases after observing roots are both negative (constant term is positive, linear term is negative).
I. $4t^2 - 12t + 9 = (4t + a)(t + b) = 4t^2 + t(4b + a) + 9$. Choices $(a, b) = (-9, -1)$, $(a, b) = (-3, -3)$ do not work.
II. Try $4t^2 - 12t + 9 = (2t + a)(2t + b) = 4t^2 + 2t(a + b) + 9$. Choice $(a, b) = (-9, -1)$ does not work. Choice $(a, b) = (-3, -3)$ does work.

4.8 Problem 43

$$x^3 + 2x^2 + x = x\left(x^2 + 2x + 1\right) = x(x + 1)^2$$

Method: Memorization of special cases

$$(x + 1)^2 = x^2 + 2x + 1$$

$$(x - 1)^2 = x^2 + 1$$

4.9 Problem 45

$$x^3 + 3x^2 - x - 3 = (x - 1)(x + 1)(x + 3)$$

Method: Hope for simple problem where roots are ± 3 and 1 and -1 to match constant term.

$$x^3 + 3x^2 - x - 3 = (x + a)(x + b)(x + c) = x^3 + x^2(a + b + c) + x(ab + ac + bc) + abc$$

where

$$a + b + c = 3,\ ab + ac + bc = -1, abc = 3$$

4.10 Problem 47

$$x^3 + 5x^2 - 2x - 24 = (x - 2)(x + 3)(x + 4)$$

Method: $x^3+5x^2-2x-24 = (x+a)(x+b)(x+c) = x^3+x^2(a+b+c)+x(ab+ac+bc)+abc$ such that $abc = 24$.

Again, try simplest case first with the triplet $(a, b, c) = (4, 3, 2)$ where one is negative or two are negative. Constraints are

$$a + b + c = 5,\ ab + ac + bc = -2, abc = 24$$

From $a + b + c = 5$ we must have $\pm(a, b, c) = \mp(4, -3, 2)$.

5 Simplify the expression

5.1 Problem 49

$$\frac{x^2+x-2}{x^2-3x+2} = \frac{(x-1)(x+2)}{(x-1)(x-2)} = \frac{x+2}{x-2}$$

Restriction: $x \neq 2$

5.2 Problem 51

$$\frac{x^2-1}{x^2-9x+8} = \frac{(x-1)(x+1)}{(x-1)(x-8)} = \frac{x+1}{x-8}$$

Restriction: $x \notin \{8, -1\}$

5.3 Problem 53

$$\frac{1}{x^2-9}+\frac{1}{x+3} = \frac{1}{(x-3)(x+3)}+\frac{1}{x+3} = \frac{1}{(x-3)(x+3)}+\frac{1}{x+3}\left(\frac{x-3}{x-3}\right) = \frac{x-2}{(x-3)(x+3)}$$

Restriction: $x \notin \{3, -3\}$

6 Complete the square

General method:

$$ax^2+bx+c = a(x+d)^2+e, \qquad d = \frac{b}{2a},\ e = c - \frac{b^2}{4a}$$

6.1 Problem 55

$$x^2+2x+5 = (x+1)^2+4$$

$$\{a,b,c\} = \{1,-5,10\} \implies \{d,e\} = \{1,4\}$$

6.2 Problem 57

$$x^2-5x+10 = \left(x-\frac{5}{2}\right)^2+\frac{15}{4}$$

$$\{a,b,c\} = \{1,-5,10\} \implies \{d,e\} = \left\{-\frac{5}{2}, \frac{15}{4}\right\}$$

6.3 Problem 59

$$4x^2 + 4x - 2 = 4\left(x + \frac{1}{2}\right)^2 - 3$$

$$\{a, b, c\} = \{4, 4, -2\} \implies \{d, e\} = \left\{\frac{1}{2}, -3\right\}$$

7 Solve the equation

Find the roots of the expression.

7.1 Problem 61

$$x^2 + 9x - 10 = (x - 10)(x + 1)$$

The quadratic function is 0 when $x = 10$ and $x = -1$.

7.2 Problem 63

Complete the square:

$$x^2 + 9x - 1 = \left(x + \frac{9}{2}\right)^2 - \frac{85}{4}$$

$$\{a, b, c\} = \{1, 9, -1\} \implies \{d, e\} = \left\{\frac{9}{2}, -\frac{85}{4}\right\}$$

The quadratic function is 0 when $x = -\frac{9}{2} \pm \sqrt{\frac{85}{4}} = \frac{1}{2}\left(-9 \pm \sqrt{85}\right)$.

7.3 Problem 65

Complete the square:

$$3x^2 + 5x + 1 = \left(x + \frac{5}{6}\right)^2 - \frac{13}{12}$$

$$\{a, b, c\} = \{3, 5, 1\} \implies \{d, e\} = \left\{\frac{5}{6}, -\frac{13}{12}\right\}$$

The quadratic function is 0 when $x = -\frac{5}{6} \pm \sqrt{\frac{13}{3 \cdot 12}} = \frac{1}{6}\left(-5 \pm \sqrt{13}\right)$.

7.4 Problem 67

$$x^3 - 2x + 1 = (x-1)\left(x^2 + x - 1\right)$$

Complete the square for the quadratic term $x^2 + x - 1$:

$$\{a, b, c\} = \{1, 1, -1\} \implies \{d, e\} = \left\{\frac{1}{2}, -\frac{5}{4}\right\}$$

$$3x^2 + 5x + 1 = (x-1)\left(3\left(x + \frac{1}{2}\right)^2 - \frac{5}{4}\right)$$

The cubic function is $x^3 - 2x + 1 = 0$ when $x = 1$ and when $x = -\frac{1}{2} \pm \sqrt{\frac{5}{4}} = -\frac{1}{2}\left(1 \pm \sqrt{5}\right)$.

8 Which of the quadratics are irreducible?

Complete the square. If the term e is positive, there are no real roots, and the function is not reducible over the field of real numbers.

8.1 Problem 69

$$2x^2 + 3x - 4 = 2\left(x + \frac{3}{4}\right)^2 - \frac{41}{8}$$

$$\{a, b, c\} = \{2, 3, -4\} \implies \{d, e\} = \left\{\frac{3}{4}, -\frac{41}{8}\right\}$$

Because $e = -\frac{41}{8} \leq 0$ this quadratic function is reducible.

8.2 Problem 71

$$3x^2 + x - 6 = 2\left(x + \frac{3}{4}\right)^2 - \frac{41}{8}$$

$$\{a, b, c\} = \{3, 1, -6\} \implies \{d, e\} = \left\{\frac{1}{6}, -\frac{73}{12}\right\}$$

Because $e = -\frac{73}{12} \leq 0$ this quadratic function is reducible.

9 Use the Binomial Theorem to expand the expression

The binomial theorem states

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y Y^k$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Example:

$$\binom{6}{3} = \frac{6!}{3!(3)!} = \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(3\cdot 2\cdot 1)(3\cdot 2\cdot 1)} = 20$$

Or use ...

Table 1: Pascal's triangle

degree						coefficients						
0						1						
1					1		1					
2				1		2		1				
3			1		3		3		1			
4		1		4		6		4		1		
5	1		5		10		10		5		1	
6	1	6		15		20		15		6		1

9.1 Problem 73

$$(a+b)^6 = a^6 + 6a^5 b + 15a^4 b^2 + 20a^3 b^3 + 15a^2 b^4 + 6ab^5 + b^6$$

9.2 Problem 75

Start with

$$(a+b)^4 = a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$$

Let $b \to -1$:

$$a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4 \to a^4 - 4a^3 + 6a^2 - 4a + 1$$

Let $a \to x^2$:

$$a^4 - 4a^3 + 6a^2 - 4a + 1 \to x^8 - 4x^6 + 6x^4 - 4x^2 + 1$$

Therefore,

$$(x^2-1)^4 = x^8 - 4x^6 + 6x^4 - 4x^2 + 1$$

10 Simplify the radicals

10.1 Problem 77

$$\sqrt{32}\sqrt{2} = \sqrt{64} = 8$$

10.2 Problem 79

$$\frac{\sqrt[4]{32x^4}}{\sqrt[4]{2}} = \sqrt[4]{\frac{32x^4}{2}} = \sqrt[4]{16x^4} = \sqrt[4]{16}\sqrt[4]{x^4} = 2\,|x|$$

10.3 Problem 81

$$\sqrt{16a^4b^3} = 4a^2\sqrt{b^3}$$

To stay in the field of reals $b \geq 0$.

11 Laws of exponents

11.1 Problem 83

$$3^{10} \times 9^8 = 3^{10} \times (3 \times 3)^8 = 3^{10} \times 3^8 \times 3^8 = 3^{10+8+8} = 3^{26}$$

11.2 Problem 85

$$\frac{x^9(2x)^4}{x^3} = \frac{x^9 \cdot 2^4x^4}{x^3} = 16\frac{x^{9+4}}{x^3} = 16x^{9+4-3} = 16x^{10}$$

11.3 Problem 87

$$\frac{a^{-3}b^4}{a^{-5}b^5} = a^{-3-(-5)}b^{4-5} = a^2b^{-1} = \frac{a^2}{b}$$

Restrictions: $b \neq 0$

11.4 Problem 89

$$3^{-\frac{1}{2}} = \frac{1}{\sqrt{3}}$$

11.5 Problem 91

$$125^{\frac{2}{3}} = \left(\sqrt[3]{125}\right)^2 = 5^2 = 25$$

11.6 Problem 93

$$\left(2x^2y^4\right)^{\frac{3}{2}} = \left(\sqrt{2}\,|x|\,y^2\right)^3 = 2\sqrt{2}\,|x|^3\,y^6$$

11.7 Problem 95

$$\sqrt[5]{y^6} = y^{\frac{5}{6}}$$

11.8 Problem 97

$$\frac{1}{\left(\sqrt{t}\right)^5} = \frac{1}{t^{-\frac{5}{2}}} = t^{-\frac{5}{2}}$$

Restrictions: $t \geq 0$

11.9 Problem 99

$$\sqrt[4]{\frac{t^{\frac{1}{2}}\sqrt{st}}{s^{\frac{2}{3}}}} = \sqrt[4]{\frac{t^{\frac{1}{2}}s^{\frac{1}{2}}t^{\frac{1}{2}}}{s^{\frac{2}{3}}}} = \sqrt[4]{\frac{ts^{\frac{1}{2}}}{s^{\frac{2}{3}}}} = \sqrt[4]{ts^{\frac{1}{2}-\frac{2}{3}}} = \sqrt[4]{ts^{-\frac{1}{6}}} = t^{\frac{1}{4}}s^{-\frac{1}{24}}$$

Restrictions: $s > 0,\ t > 0$

12 Rationalize the expression

Not clear on the context. Making a guess and rationalizing either numerator or denominator depending upon where the radical is.

12.1 Problem 101

$$\frac{\sqrt{x}-3}{x-9} = \frac{\sqrt{x}-3}{x-9}\frac{\sqrt{x}+3}{\sqrt{x}+3} = \frac{x-9}{(x-9)\left(\sqrt{x}+3\right)} = \frac{1}{\sqrt{x}+3}$$

12.2 Problem 103

$$\frac{x\sqrt{x}-8}{x-4} = \frac{x\sqrt{x}-8}{x-4}\frac{x\sqrt{x}+8}{x\sqrt{x}+8} = \frac{x^3-64}{x^{5/2}-4x^{3/2}+8x-32} = \frac{x+2\sqrt{x}+4}{\sqrt{x}+2}$$

12.3 Problem 105

$$\frac{2}{3-\sqrt{5}} = \frac{2}{3-\sqrt{5}}\frac{3+\sqrt{5}}{3+\sqrt{5}} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$$

12.4 Problem 107

$$\sqrt{x^2+3x+4}-x = \left(\sqrt{x^2+3x+4}-x\right)\frac{\sqrt{x^2+3x+4}+x}{\sqrt{x^2+3x+4}+x} = \frac{3x+4}{\sqrt{x^2+3x+4}+x}$$

13 State whether or not the expression is true for all values of the variable

13.1 Problem 109

$$\sqrt{x^2} = x$$

No. Only true when $x > 0$.

13.2 Problem 111

$$\frac{16+a}{16} = 1 + \frac{a}{16}$$

Yes. Valid for all real numbers a.

13.3 Problem 113

$$\frac{x}{x+y} = \frac{1}{1+y}$$

Hell no. Only valid when $x = 1$ and $y \neq -1$.

13.4 Problem 115

$$\left(x^3\right)^4 = x^7$$

Hell no. Only valid when $x = 1$. The power law for exponents reveals $\left(x^3\right)^4 = x^{12}$.