

# 1 7 January 2014

## 1.1 Stable Marriage Problem

Gale-Shapley, 1962

Residents list hospital preferences, hospitals list resident preferences.

Try to find a stable arrangement. That is, there does not exist a resident  $R$  or hospital  $H$  such that  $R$  and  $H$  are not currently assigned to each other, but would prefer to be assigned to each other over their current assignments.

	1	2	3
A	Y	X	Z
B	X	Y	Z
C	Z	X	Y

(a) Amy, Bertha, Clare

	1	2	3
X	A	B	C
Y	B	C	A
Z	B	A	C

(b) Xavier, Yancey, Zeus

Do stable matchings exist? If so, how do we find them?

(X, A) (Y, B): stable

(X, A) (Z, C): stable

(Y, B) (Z, C): stable

(X, C), (Y, B), and (Z, C) is unstable: B prefers X over Y, X prefers B over C.

### Possible approaches

**Approach 1** Assign a value to each pair and pick match with largest value

**Approach 2** If a man and woman prefer each other, match them and repeat

**Approach 3** If exists unique top choice, match them. If two or more have same top choice, go through the others and identify stable pairs.

### Gale-Shapley Method

```
while there exists unmatched man  $m$  who has not proposed to all women do  
   $m$  proposes to next woman  $w$  on his list  
  if  $w$  is unmatched or  $w$  is engaged to  $m'$  but prefers  $m$  to  $m'$  then  
     $w$  gets matched to  $m$  (dumping  $m'$  if necessary)  
  else  $w$  rejects  $m$   
  end if  
end while
```

### Example

X proposes to A	(X, A)
Z proposes to B	(X, A), (Z, B)
Y proposes to B	(X, A), <del>(Z, B)</del> (Y, B)
Z proposes to A, A rejects Z	(X, A), (Y, B)
Z proposes to C	(X, A), (Y, B), (Z, C)

**Lemma 1** : G-S always terminates.

**Lemma 2** : At termination, the matching is stable.

### Stable roommates

	1	2	3
A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C

Never stable, because no one wants to live with D.

Observation 1: Once a woman is proposed to, she is always matched.

Proof: First time a woman is proposed to (line 2) she gets engaged (line 4). Thereafter she breaks off the engagement only to get engaged to someone else.

Observation 2: At termination, every man is either matched or has proposed to every woman.

Proof of Lemma 1:

- In each iteration of the while loop, some man  $m$  proposes to some woman  $w$ .
- No man proposes to the same woman twice.
- No more than  $n^2$  proposals.
- No more than  $n^2$  iterations.
- G-S terminates in  $\leq n^2$  iterations.