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1.1 Stable Marriage Problem

Gale-Shapley, 1962

Residents list hospital preferences, hospitals list resident preferences.

Try to find a stable arrangement. That is, there does not exist a resident R or hospital H such that R and H are not currently assigned to each other, but would prefer to be assigned to each other over their current assignments.

	1	2	3
A	Y	X	Z
B	X	Y	Z
C	Z	X	Y

(a) Amy, Bertha, Clare

	1	2	3
X	A	B	C
Y	B	C	A
Z	B	A	C

(b) Xavier, Yancey, Zeus

Do stable matchings exist? If so, how do we find them?

(X, A) (Y, B): stable
(X, A) (Z, C): stable
(Y, B) (Z, C): stable

(X, C), (Y, B), and (Z, C) is unstable: B prefers X over Y, X prefers B over C.

Possible approaches

Approach 1 Assign a value to each pair and pick match with largest value

Approach 2 If a man and woman prefer each other, match them and repeat

Approach 3 If exists unique top choice, match them. If two or more have same top choice, go through the others and identify stable pairs.

Gale–Shapley Method

while there exists unmatched man m who has not proposed to all women **do**
  m proposes to next woman w on his list
  if w is unmatched or w is engaged to m' but prefers m to m' **then**
    w gets matched to m (dumping m' if necessary)
  else$w$ rejects m
  end if
end while

Example

X proposes to A	(X, A)
Z proposes to B	(X, A), (Z, B)
Y proposes to B	(X, A), ~~(Z, B)~~ (Y, B)
Z proposes to A, A rejects Z	(X, A), (Y, B)
Z proposes to C	(X, A), (Y, B), (Z, C)

Lemma 1 : G–S always terminates.

Lemma 2 : At termination, the matching is stable.

Stable roommates

	1	2	3
A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C

Never stable, because no one wants to live with D.

Observation 1: Once a woman is proposed to, she is always matched.
Proof: First time a woman is proposed to (line 2) she gets engaged (line 4). Thereafter she breaks off the engagement only to get engaged to someone else.

Observation 2: At termination, every man is either matched or has proposed to every woman.
Proof of Lemma 1:

- In each iteration of the while loop, some man m proposes to some woman w.
- No man proposes to the same woman twice.
- No more than n^2 proposals.
- No more than n^2 iterations.
- G–S terminates in $\leq n^2$ iterations.