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1.1 Stable Marriage Problem

Gale-Shapley, 1962

Residents list hospital preferences, hospitals list resident preferences.

Try to find a stable arrangement. That is, there does not exist a resident R or hospital H such that R and H are not currently assigned to each other, but would prefer to be assigned to each other over their current assignments.

	1	2	3
A	Y	X	Z
B	X	Y	Z
C	Z	X	Y

(a) Amy, Bertha, Clare

	1	2	3
X	A	B	C
Y	B	C	A
Z	B	A	C

(b) Xavier, Yancey, Zeus

Do stable matchings exist? If so, how do we find them?

(X, A) (Y, B): stable
(X, A) (Z, C): stable
(Y, B) (Z, C): stable

(X, C), (Y, B), and (Z, C) is unstable: B prefers X over Y, X prefers B over C.

Possible approaches

Approach 1 Assign a value to each pair and pick match with largest value

Approach 2 If a man and woman prefer each other, match them and repeat

Approach 3 If exists unique top choice, match them. If two or more have same top choice, go through the others and identify stable pairs.

Gale–Shapley Method

while there exists unmatched man m who has not proposed to all women **do**
  m proposes to next woman w on his list
  if w is unmatched or w is engaged to m' but prefers m to m' **then**
    w gets matched to m (dumping m' if necessary)
  else$w$ rejects m
  end if
end while

Example

X proposes to A	(X, A)
Z proposes to B	(X, A), (Z, B)
Y proposes to B	(X, A), ~~(Z, B)~~ (Y, B)
Z proposes to A, A rejects Z	(X, A), (Y, B)
Z proposes to C	(X, A), (Y, B), (Z, C)

Lemma 1 : G–S always terminates.

Lemma 2 : At termination, the matching is stable.

Stable roommates

	1	2	3
A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C

Never stable, because no one wants to live with D.

Observation 1: Once a woman is proposed to, she is always matched.
Proof: First time a woman is proposed to (line 2) she gets engaged (line 4). Thereafter she breaks off the engagement only to get engaged to someone else.

Observation 2: At termination, every man is either matched or has proposed to every woman.
Proof of Lemma 1:

- In each iteration of the while loop, some man m proposes to some woman w.
- No man proposes to the same woman twice.
- No more than n^2 proposals.
- No more than n^2 iterations.
- G–S terminates in $\leq n^2$ iterations.

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Lemma 2: G–S returns a stable matching

a Every man is matched to a woman

b There does not exist an unstable pair, i.e. pairs (m, 2) and (m', w') such that m prefers w' over w and w' prefers m over m'.

(a): By contradiction
Suppose there exists a man m who is not matched to any woman. There must exist an unmatched woman.

- m must have gone through his list
- at some point proposed to w
- certainly from that point on, w is matched

Contradicts the fact that w is unmatched at the end. Therefore **(a)** holds.

(b): By contradiction
Suppose two pairs exist (m, w) and (m', w'). Their preferences are:
$m : w'...w$
$w' : m...m'$
We know that at step 1, m proposed to w' at some instant t. After the proposal, w' is matched to some man whom she likes at least as much as m. At termination, w' is engaged to a man she likes at least as much as m. Contradicting matching (m', w').

Total number of steps $\leq 10n^2$
List for woman w, where the ith value is her ith preferred man:

1	2	3	4	5	6	7	8	9	10
5	3	1	9	10	8	6	7	2	4

Invert, so the ith value is the rank of man i:

1	2	3	4	5	6	7	8	9	10
3	9	2	10	1	7	8	6	4	5

Suppose woman is engaged to man #5 and man #3 proposes. She'll look up 5 and 3 in the inverted list. If we didn't reverse the list, the total number of steps would be n^3.

2.1 Array of numbers

A[1, ..., n]

3	4	13	28	30
0	1	10	25	27
0	0	9	24	26
0	0	0	15	17
0	0	0	0	2

For rows i and columns j:
$B[i/j] = \begin{cases} \text{sum of } A[i...j] & : i \leq j \\ 0 & : i > j \end{cases}$
Where the sum of $A[i...j]$ is $A[i] + A[i+1] + ... + A[j]$

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1:  Zero the array by setting B[i/j] to 0 for all i, j
2:  for i from 1 to n do
3:      for j from i to n do
4:          set B[i/j] to the sum of A[i], A[i+1], ..., A[j]
5:          sum ← 0
6:          for k ← i to j do
7:              sum ← sum + A[k]
8:          end for
9:          B[i, j] ← sum
10:     end for
11: end for
```

1 n^2

2 n

3 For iteration i: $n - i + 1$ times $= \frac{n(n+1)}{2}$

5 For fixed i, j: $1 \leftarrow \frac{n(n+1)}{2}$

6 For fixed i, j: $j - i + 1$

7 For fixed i, j: $j - i + 1$

9 For fixed i, j: 1

Running time: $n^2 + n + \frac{n(n+1)}{2} + \frac{n(n+1)}{2} + \frac{n(n+1)}{2} + n^3 + n^3 \leq 2n^3 + \frac{5n^2}{2} + \frac{5n}{2}$