1 9 January 2014

<u>Lemma 2:</u> G–S returns a stable matching

a Every man is matched to a woman

b There does not exist an unstable pair, i.e. pairs (m, 2) and (m', w') such that m prefers w' over w and w' prefers m over m'.

(a): By contradiction
Suppose there exists a man m who is not matched to any woman. There must exist an unmatched woman.

- m must have gone through his list
- at some point proposed to w
- certainly from that point on, w is matched

Contradicts the fact that w is unmatched at the end. Therefore **(a)** holds.

(b): By contradiction
Suppose two pairs exist (m, w) and (m', w'). Their preferences are:
$m : w'...w$
$w' : m...m'$
We know that at step 1, m proposed to w' at some instant t. After the proposal, w' is matched to some man whom she likes at least as much as m. At termination, w' is engaged to a man she likes at least as much as m. Contradicting matching (m', w').

Total number of steps $\leq 10n^2$
List for woman w, where the ith value is her ith preferred man:

1	2	3	4	5	6	7	8	9	10
5	3	1	9	10	8	6	7	2	4

Invert, so the ith value is the rank of man i:

1	2	3	4	5	6	7	8	9	10
3	9	2	10	1	7	8	6	4	5

Suppose woman is engaged to man #5 and man #3 proposes. She'll look up 5 and 3 in the inverted list. If we didn't reverse the list, the total number of steps would be n^3.

1.1 Array of numbers

A[1, ..., n]

3	4	13	28	30
0	1	10	25	27
0	0	9	24	26
0	0	0	15	17
0	0	0	0	2

For rows i and columns j:
$B[i/j] = \begin{cases} \text{sum of } A[i...j] & : i \leq j \\ 0 & : i > j \end{cases}$
Where the sum of $A[i...j]$ is $A[i] + A[i+1] + ... + A[j]$

1: Zero the array by setting $B[i/j]$ to 0 for all i, j
2: **for** i from 1 to n **do**
3:     **for** j from i to n **do**
4:         set $B[i/j]$ to the sum of $A[i], A[i+1], ..., A[j]$
5:         sum $\leftarrow 0$
6:         **for** $k \leftarrow i$ to j **do**
7:             sum $\leftarrow$ sum $+A[k]$
8:         **end for**
9:         $B[i, j] \leftarrow$ sum
10:     **end for**
11: **end for**

1 n^2

2 n

3 For iteration i: $n - i + 1$ times $= \frac{n(n+1)}{2}$

5 For fixed i, j: $1 \leftarrow \frac{n(n+1)}{2}$

6 For fixed i, j: $j - i + 1$

7 For fixed i, j: $j - i + 1$

9 For fixed i, j: 1

Running time: $n^2 + n + \frac{n(n+1)}{2} + \frac{n(n+1)}{2} + \frac{n(n+1)}{2} + n^3 + n^3 \leq 2n^3 + \frac{5n^2}{2} + \frac{5n}{2}$