1	2	3	$\sum$

ML: Algo & Theory
SS 18

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Assignment 10

(Abgabe am 03. Juli 2018)

Exercise 1

a.) We have to prove that $\lambda_n \leq 2$. From the Rayleigh principle we know that $\lambda_n = \max_v v^T L v$, where $v \in \mathbb{R}^n$ is normalized, meaning that is has length $= 1$. Because L is a normalized Laplacian, we get the following equations:
$v^T L v = \frac{1}{2} \sum_{i,j=1}^n a_{ij} (\frac{v_i}{\sqrt{d_i}} - \frac{v_j}{\sqrt{d_j}})^2$
$= \frac{1}{2} \sum_{i,j=1}^n a_{ij} ((\frac{v_i}{\sqrt{d_i}})^2 - 2\frac{v_i}{\sqrt{d_i}} \frac{v_j}{\sqrt{d_j}} + (\frac{v_j}{\sqrt{d_j}})^2) = \frac{1}{2} \sum_{i,j=1}^n a_{ij} (\frac{v_i^2}{d_i} - 2\frac{v_i v_j}{\sqrt{d_i d_j}} + \frac{v_j^2}{d_j})$
$= \frac{1}{2} \sum_{i,j=1}^n a_{ij} \frac{v_i^2}{d_i} - \sum_{i,j=1}^n a_{ij} \frac{v_i v_j}{\sqrt{d_i d_j}} + \frac{1}{2} \sum_{i,j=1}^n a_{ij} \frac{v_j^2}{d_j} = \sum_{i=1}^n (\sum_{j=1}^n a_{ij}) \frac{v_i^2}{d_i} - \sum_{i,j=1}^n a_{ij} \frac{v_i v_j}{\sqrt{d_i d_j}}$
$= \sum_{i=1}^n v_i^2 - \sum_{i,j=1}^n a_{ij} \frac{v_i v_j}{\sqrt{d_i d_j}} = 1 - \sum_{i,j=1}^n \frac{a_{ij}}{\sqrt{d_i d_j}} v_i v_j$.
Since v is normalized, the following to relations hold: $-1 \leq v \leq 1$ and $-1 \leq v_i v_j \leq 1$. We can now assume that $\sum_{i,j=1}^n \frac{a_{ij}}{\sqrt{d_i d_j}} \leq 1$, since the sum of the adjacency entries is divided by other sums of the same value. By multiplying $\sum_{i,j=1}^n \frac{a_{ij}}{\sqrt{d_i d_j}}$ with some number $v_i v_j$ between -1 and 1, we would still get a number between -1 and 1, leading to $\sum_{i,j=1}^n \frac{a_{ij}}{\sqrt{d_i d_j}} v_i v_j \geq -1$. With all those relations, we can now conclude that:
$\lambda_n = \max_v v^T L v = \max_v 1 - \sum_{i,j=1}^n \frac{a_{ij}}{\sqrt{d_i d_j}} \leq 1 - (-1) = 2$.

b.) If G is a complete graph on n vertices, we have $A = 1 - I$, which is a matrix that has 1 in every entry, except on the diagonal. Then $D = diag(n-1, \ldots, n-1)$, which leads to $D - A$ being $(n-1)$ on the diagonal and -1 everywhere else. If we multiply $D^{-\frac{1}{2}}$ on the left and on the right, we divide each entry by $(n-1)$, which leads to L being 1 on the diagonal and $\frac{-1}{n-1}$ everywhere else.
If we set the eigenvector v_1 to $(1, \ldots, 1)$, we get $Lv_1 = 0v_1$

c.)

d.)

Exercise 2

see code.

Exercise 3