	1	2	3	4	5	$\sum$

ML: Algo & Theory
SS 18

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Assignment 1

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Exercise 1

a.) In the following we will abbreviate small with s, medium with me, large with l, female with f and male with m:

- $P(Y = s) = P(X = m, Y = s) + P(X = f, Y = s) = 0.1 + 0.3 = 0.4$
- $P(Y = me) = P(X = m, Y = me) + P(X = f, Y = me) = 0.15 + 0.1 = 0.25$
- $P(Y = l) = P(X = m, Y = l) + P(X = f, Y = l) = 0.25 + 0.1 = 0.35$
- $P(X = m) = P(X = m, Y = s) + P(X = m, Y = me) + P(X = m, Y = l) = 0.1 + 0.15 + 0.25 = 0.5$
- $P(X = f) = P(X = f, Y = s) + P(X = f, Y = me) + P(X = f, Y = l) = 0.3 + 0.1 + 0.1 = 0.5$

b.) Beweis.

Exercise 2

a.) $P(X = me|Y = f) = \frac{P(X=me \,\cap\, Y=f)}{P(Y=f)} = \frac{0.1}{0.5} = 0.02$

b.) Two random variables X, Y are indepent if and only if $P(X \cap Y) = P(X) \cdot P(Y)$ or if $P(X|Y) = P(X)$. This means that the random variable Y has nor influence on the probability of X and vice-versa.

c.) In the following we will abbreviate a positive test with $+$, a negative test with $-$, cancer with c and no cancer with $\not c$:

- $P(A = +|B = c) = 0.95$
  $P(A = -|B = \not c) = 0.95$
  $P(B = c) = 0.01$
- $P(B = c|A = +) = \frac{P(A=+|B=c) \cdot P(B=c)}{P(A=+)}$
- $P(A = +)$ can be calculated with marginalisation:
  $P(A = +) = P(A = +|B = c) \cdot P(B = c) + P(A = +|B = \not c) \cdot P(B = \not c)$, with $P(A = +|B = \not c)$ being 0.05. This results in $P(A = +) = 0.95 \cdot 0.01 + 0.05 \cdot 0.99 = 0.059$

– This results in: $P(B = c|A = +) = \frac{0.95 \cdot 0.01}{0.059} = 0.16$

Exercise 3

a.) $A \cdot x = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 5 & 7 & 8 \end{pmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1 \end{bmatrix} \cdot x_1 + \begin{bmatrix} a_2 \end{bmatrix} \cdot x_2 + \begin{bmatrix} a_3 \end{bmatrix} \cdot x_3 =$
$\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \cdot x_1 + \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} \cdot x_2 + \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix} \cdot x_3 = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + 2x_2 + x_3 \\ 5x_1 + 7x_2 + 8x_3 \end{bmatrix}$

b.) Columns: If the columns of A should form a Basis of $\mathbb{R}^3$, one needs to test if the column-vectors are linearly independent and if they are able to produce every single possible vector of $\mathbb{R}^3$. This is done in the following:
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + \lambda_2 \cdot \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} + \lambda_3 \cdot \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}.$

This results in the following LGS:

$$\begin{array}{rr} \text{I)} & \lambda_1 + \lambda_2 + 2\lambda_3 = x \\ \text{II)} & \lambda_1 + 2\lambda_2 + \lambda_3 = y \\ \text{III)} & 5\lambda_1 + 7\lambda_2 + 8\lambda_3 = z \\ \hline \text{I)} & x - \lambda_2 - 2\lambda_3 = \lambda_1 \\ \lambda_1 \text{ in II)} & (x - \lambda_2 - 2\lambda_3) + 2\lambda_2 + \lambda_3 = y \\ \Leftrightarrow & x + \lambda_2 - \lambda_3 = y \\ \Leftrightarrow & y + \lambda_3 - x = \lambda_2 \\ \hline \lambda_1, \lambda_2 \text{ in III)} & 5(x - \lambda_2 - 2\lambda_3) + 7(y + \lambda_3 - x) + 8\lambda_3 = z \\ \Leftrightarrow & 3x + 2y + 10\lambda_3 = z \\ \Leftrightarrow & (z - 3x - 2y)/10 = \lambda_3 \end{array}$$

$\lambda_1, \lambda_2, \lambda_3$ are therefore all three $\in \mathbb{R}^3$ and explicit, which means that all three column vectors are linearly indepent and also able to build all possible vectors in $\mathbb{R}^3$, which means that there are a basis of $\mathbb{R}^3$.

Rows: If the rows of A should form a Basis of $\mathbb{R}^3$, one needs to test if the row-vectors are linearly independent and if they are able to produce every single possible vector of $\mathbb{R}^3$. This is done in the following:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \lambda_2 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \lambda_3 \cdot \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}.$$

This results in the following LGS:

$$\begin{aligned}
\text{I)} &\qquad & \lambda_1 + \lambda_2 + 5\lambda_3 &= x \\
\text{II)} &\qquad & \lambda_1 + 2\lambda_2 + 7\lambda_3 &= y \\
\text{III)} &\qquad & 2\lambda_1 + 1\lambda_2 + 8\lambda_3 &= z \\
\hline
\text{I)} &\qquad & x - \lambda_2 - 5\lambda_3 &= \lambda_1 \\
\lambda_1 \text{ in II)} &\qquad & (x - \lambda_2 - 5\lambda_3) + 2\lambda_2 + 7\lambda_3 &= y \\
\Leftrightarrow &\qquad & x + \lambda_2 + 2\lambda_3 &= y \\
\Leftrightarrow &\qquad & y - x - 2\lambda_3 &= \lambda_2 \\
\hline
\lambda_1, \lambda_2 \text{ in III)} &\qquad & 2(x - \lambda_2 - 5\lambda_3) + (y - x - 2\lambda_3) + 8\lambda_3 &= z \\
\Leftrightarrow &\qquad & x - 2\lambda_2 - 4\lambda_3 + y &= z \\
\Leftrightarrow &\qquad & 3x - y &= z
\end{aligned}$$

Since λ_3 can be chosen freely the row vectors of A are not linearly indepent from each other, which means that they are not building a Basis of the $\mathbb{R}^3$.

c.) Considering $b = (2, 3, 12)$:

$A \cdot x = b \Leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 5 & 7 & 8 \end{pmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 12 \end{bmatrix}$. This results in the following LGS:

$$\begin{aligned}
\text{I)} &\qquad & x_1 + x_2 + 2x_3 &= 2 \\
\text{II)} &\qquad & x_1 + 2x_2 + x_3 &= 3 \\
\text{III)} &\qquad & 5x_1 + 7x_2 + 8x_3 &= 12 \\
\hline
\text{I)} &\qquad & 2 - x_2 - 2x_3 &= x_1 \\
x_1 \text{ in II)} &\qquad & 2 - x_2 - 2x_3 + 2x_2 + x_3 &= 3 \\
\Leftrightarrow &\qquad & 2 + x_2 - x_3 &= 3 \\
\Leftrightarrow &\qquad & 1 + x_3 &= x_2 \\
\hline
x_1, x_2 \text{ in III)} &\qquad & 5(2 - x_2 - 2x_3) + 7(1 + x_3) + 8x_3 &= 12 \\
\Leftrightarrow &\qquad & 10 - 5x_2 - 10x_3 + 7 + 7x_3 + 8x_3 &= 12 \\
\Leftrightarrow &\qquad & 5 + 7 &= 12 \ \surd
\end{aligned}$$

This means that x_3 can be chosen freely. We chose $x_3 = 1$, which leads to $x_1 = -2$ and $x_2 = 2$. This leads to $x = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$

d.)
- Column rank: The column rank is 3, since A has 3 linearly indepent column vectors.
- Row rank: The row rank is 2, since there are only 2 linearly indepent row vectors, namely $r_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $r_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.
- Rank of A: The rank of A is 3, since the maximum rank of column and row rank is 3.

Exercise 4

Exercise 5