1	2	3	$\sum$

ML: Algo & Theory
SS 18

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Assignment 10

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Exercise 1

a.) We have to prove that $\lambda_n \leq 2$. From the Rayleigh principle we know that $\lambda_n = \max_v v^T Lv$, where $v \in \mathbb{R}^n$ is normalized, meaning that is has length $= 1$. Because L is a normalized Laplacian, we get the following equations:
$v^T Lv = \frac{1}{2}\sum_{i,j=1}^n a_{ij}(\frac{v_i}{\sqrt{d_i}} - \frac{v_j}{\sqrt{d_j}})^2$
$= \frac{1}{2}\sum_{i,j=1}^n a_{ij}((\frac{v_i}{\sqrt{d_i}})^2 - 2\frac{v_i}{\sqrt{d_i}}\frac{v_j}{\sqrt{d_j}} + (\frac{v_j}{\sqrt{d_j}})^2) = \frac{1}{2}\sum_{i,j=1}^n a_{ij}(\frac{v_i^2}{d_i} - 2\frac{v_i v_j}{\sqrt{d_i d_j}} + \frac{v_j^2}{d_j})$
$= \frac{1}{2}\sum_{i,j=1}^n a_{ij}\frac{v_i^2}{d_i} - \sum_{i,j=1}^n a_{ij}\frac{v_i v_j}{\sqrt{d_i d_j}} + \frac{1}{2}\sum_{i,j=1}^n a_{ij}\frac{v_j^2}{d_j} = \sum_{i=1}^n(\sum_{j=1}^n a_{ij})\frac{v_i^2}{d_i} - \sum_{i,j=1}^n a_{ij}\frac{v_i v_j}{\sqrt{d_i d_j}}$
$= \sum_{i=1}^n v_i^2 - \sum_{i,j=1}^n a_{ij}\frac{v_i v_j}{\sqrt{d_i d_j}} = 1 - \sum_{i,j=1}^n \frac{a_{ij}}{\sqrt{d_i d_j}} v_i v_j$.
Since v is normalized, the following to relations hold: $-1 \leq v \leq 1$ and $-1 \leq v_i v_j \leq 1$. We can now assume that $\sum_{i,j=1}^n \frac{a_{ij}}{\sqrt{d_i d_j}} \leq 1$, since the sum of the adjacency entries is divided by other sums of the same value. By multiplying $\sum_{i,j=1}^n \frac{a_{ij}}{\sqrt{d_i d_j}}$ with some number $v_i v_j$ between -1 and 1, we would still get a number between -1 and 1, leading to $\sum_{i,j=1}^n \frac{a_{ij}}{\sqrt{d_i d_j}} v_i v_j \geq -1$. With all those relations, we can now conclude that:
$\lambda_n = \max_v v^T Lv = \max_v 1 - \sum_{i,j=1}^n \frac{a_{ij}}{\sqrt{d_i d_j}} \leq 1 - (-1) = 2$.

b.) If G is a complete graph on n vertices, we have $A = 1 - I$, which is a matrix that has 1 in every entry, except on the diagonal. Then $D = diag(n-1, \dots, n-1)$, which leads to $D - A$ being $(n-1)$ on the diagonal and -1 everywhere else. If we multiply $D^{-\frac{1}{2}}$ on the left and on the right, we divide each entry by $(n-1)$, which leads to L being 1 on the diagonal and $\frac{-1}{n-1}$ everywhere else.
If we set the eigenvector v_1 to $(1, \dots, 1)$, we get $Lv_1 = 0v_1$, which means that we have an eigenvalue of 0. If we set the eigenvector to one of the other $n-1$ possible ones, we get $v_i = (0, \dots, 0, 1, -1, 0, \dots, 0)$. The 1 here stands at position $i-1$ and the -1 stands at position i. Then we would have $Lv_i = \frac{n}{n-1}v_i$, because L would contain 0 at every place, except for $i-1$ and i, which leads to $1 \cdot 1 - 1 \cdot \frac{-1}{n-1} = \frac{n}{n-1}$ and $1 \cdot \frac{-1}{n-1} - 1 \cdot 1 = \frac{-n}{n-1} = (-1) \cdot \frac{n}{n-1}$.

c.) If G is a complete bipartite graph on $n+m$ vertices, than A consists of a $n \times n$ block of zeros on the top left, of a $m \times m$ block of zeros on the bottom right and two $n \times m$ and $m \times n$ blocks of non-zeros on the remaining places. Then D has a m on the diagonal for the first n vertices and a n for the remaining m vertices. Therefore $D - A$ has m or n written on its

diagonal and -1 in the upper right and lower left blocks and 0 on all other positions. If we now multiply $D^{-\frac{1}{2}}$ on the left and on the right, we would now divide each entry by either m, n or $\sqrt{mn}$, so L would be 1 on the diagonal and $\frac{-1}{\sqrt{nm}}$ on the upper right blocks and lower left blocks and 0 everywhere else.
If we set the eigenvector to v_1 consisting of n times $\sqrt{m}$ and m times $\sqrt{n}$, we would get $Lv_1 = 0v_1$, since each entry is $\sqrt{m} \cdot 1 + m \cdot \sqrt{n} \cdot \frac{-1}{\sqrt{mn}} = \sqrt{m} - \frac{\sqrt{n}\sqrt{m}^2}{\sqrt{nm}} = \sqrt{m} - \sqrt{m} = 0$ and the analogous for n and m changed. This means we have an eigenvalue of 0.
If we set the eigenvector to v_n consisting of n times $-\sqrt{m}$ and m times $\sqrt{n}$, we would get $Lv_n = 2v_n$, since each entry is $-\sqrt{m} \cdot 1 + m \cdot \sqrt{n} \cdot \frac{-1}{\sqrt{mn}} = -\sqrt{m} - \frac{\sqrt{n}\sqrt{m}^2}{\sqrt{nm}} = 2(-\sqrt{m})$ and the analogous for n and m changed. This means we have an eigenvalue of 2.
For the remaining $n-1+m-1$ eigenvectors, we set $v_i = (0, \cdots, 0, 1, -1, 0, \cdots, 0)$ with a 1 at position $i-1$ and -1 at position i. We would then get $Lv_i = 1v_i$, since Lv_i contains 0 at every position, except for position $i-1$ and i, where we would have values 1 and -1. This means we would have $(n+m-2)$ eigenvalues with value 1.

d.) -

Exercise 2

see code.

Exercise 3