Computation of the exact inverse kinematics solution of a 6D robotic arm

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1 Introduction

This document explains how to implement exact inverse kinematics of a Stäubli robotic arm as an explicit constraint in Humanoid Path Planner software. Notation and definitions are the same as in [1].

2 Notation and Definitions

The constraint is defined as a 6 dimensional *grasp* constraint between a *gripper* and a *handle*. The *gripper* is attached to the robotic arm end-effector (`joint1`) and the *handle* is attached to `joint2` on the composite kinematic chain. `root` is the joint that holds the robot arm or the global frame (`"universe"` in `pinocchio` software).

We denote by

- $\mathbf{q}_{in}$ the input variables of the explicit constraint,
- $\mathbf{q}_{out}$ the output variables of the explicit constraint,
- ${}^0M_1$ the pose of `joint1` in the world frame,
- ${}^0M_2$ the pose of `joint2` in the world frame,
- ${}^0M_r$ the pose of `root` in the world frame,
- ${}^2M_h$ the pose of the handle in `joint2` frame,
- ${}^1M_g$ the pose of the gripper in `joint1` frame,
- ${}^rM_b$ the pose of the robot arm origin (`base_link` in URDF description) in the `root` frame.

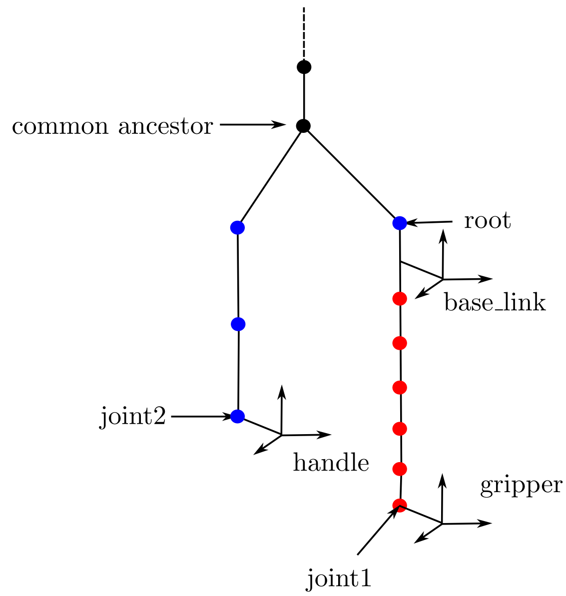

Figure 1: Input (blue) and output (red) variables of the explicit constraint that computes the robot arm configuration with respect to the handle pose.

3 Inverse kinematics

Exact inverse kinematics computes the 6 joint values of the robotic arm with respect to the input configuration variables:

$$\mathbf{q}_{out} = f(\mathbf{q}_{in}) \tag{1}$$

Note that the input variables include a extra degree of freedom that is interpreted as an integer to select among the various solutions of the inverse kinematics.

4 Jacobian

In order to implement exact inverse kinematics as an explicit constraint, we need to compute the Jacobian of f. For that, let us consider of motion of the kinematic chain that keeps the gripper and handle in the same pose:

$$\forall t \in \mathbb{R},\ {}^0M_2(t)\ {}^2M_h = {}^0M_r(t)\ {}^rM_1(t)\ {}^1M_g \tag{2}$$

Moreover

$$\left(\begin{array}{c} {}^r\mathbf{v}_{1/r} \\ {}^r\omega_{1/r} \end{array} \right) = J_{out}\dot{\mathbf{q}}_{out} \tag{3}$$

where J_{out} is the 6x6 matrix composed of the columns of Jacobian of `joint1` corresponding to the arm degrees of freedom. We denote respectively by $J_{out\,\mathbf{v}}$ and $J_{out\,\omega}$ the first 3 and the last 3 lines of this matrix.

Using homogeneous matrix notation and derivating with respect to time, Equation (2) can be written as $\forall t \in \mathbb{R}$,

$$\begin{aligned} {}^0M_2 \begin{pmatrix} [\,{}^2\omega_{2/0}]_\times & {}^2\mathbf{v}_{2/0} \\ 0 & 0 \end{pmatrix} {}^2M_h = {}^0M_r \begin{pmatrix} [\,{}^r\omega_{r/0}]_\times & {}^r\mathbf{v}_{r/0} \\ 0 & 0 \end{pmatrix} {}^rM_1\,{}^1M_g \quad (4)\\ + \ {}^0M_r\,{}^rM_1 \begin{pmatrix} [\,{}^1\omega_{1/r}]_\times & {}^1\mathbf{v}_{1/r} \\ 0 & 0 \end{pmatrix} {}^1M_g \\ (5) \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} {}^0R_2[{}^2\omega_{2/0}]_\times & {}^0R_2\,{}^2\mathbf{v}_{2/0} \\ 0 & 0 \end{pmatrix} {}^2M_h = \begin{pmatrix} {}^0R_r[{}^r\omega_{r/0}]_\times & {}^0R_r\,{}^r\mathbf{v}_{r/0} \\ 0 & 0 \end{pmatrix} {}^rM_1\,{}^1M_g \\ + \ {}^0M_r \begin{pmatrix} {}^rR_1[{}^1\omega_{1/r}]_\times & {}^rR_1\,{}^1\mathbf{v}_{1/r} \\ 0 & 0 \end{pmatrix} {}^1M_g \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} {}^0R_2[{}^2\omega_{2/0}]_\times\,{}^2R_h & {}^0R_2[{}^2\omega_{2/0}]_\times\,{}^2\mathbf{t}_h +{}^0 R_2\,{}^2\mathbf{v}_{2/0} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} {}^0R_r[{}^r\omega_{r/0}]_\times\,{}^rR_g & {}^0R_r[{}^r\omega_{r/0}]_\times\,{}^r\mathbf{t}_g +{}^0 R_r\,{}^r\mathbf{v}_{r/0} \\ 0 & 0 \end{pmatrix} \\ + \begin{pmatrix} {}^0R_r\,{}^rR_1[{}^1\omega_{1/r}]_\times\,{}^1R_g & {}^0R_r\,{}^rR_1[{}^1\omega_{1/r}]_\times\,{}^1\mathbf{t}_g +{}^0 R_r\,{}^rR_1\,{}^1\mathbf{v}_{1/r} \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Extracting the upper blocks of this matrix equality, we get

$$\begin{aligned} {}^0R_2[{}^2\omega_{2/0}]_\times\,{}^2R_h &={}^0 R_r[{}^r\omega_{r/0}]_\times\,{}^rR_g +{}^0 R_r\,{}^rR_1[{}^1\omega_{1/r}]_\times\,{}^1R_g \\ {}^0R_2[{}^2\omega_{2/0}]_\times\,{}^2\mathbf{t}_h +{}^0 R_2\,{}^2\mathbf{v}_{2/0} &={}^0 R_r[{}^r\omega_{r/0}]_\times\,{}^r\mathbf{t}_g +{}^0 R_r\,{}^r\mathbf{v}_{r/0} +{}^0 R_r\,{}^rR_1[{}^1\omega_{1/r}]_\times\,{}^1\mathbf{t}_g +{}^0 R_r\,{}^rR_1\,{}^1\mathbf{v}_{1/r} \\ {}^0R_2[{}^2\omega_{2/0}]_\times\,{}^2R_h &={}^0 R_r[{}^r\omega_{r/0}]_\times\,{}^rR_g +{}^0 R_1[{}^1\omega_{1/r}]_\times\,{}^1R_g \\ {}^0R_2[{}^2\omega_{2/0}]_\times\,{}^2\mathbf{t}_h +{}^0 R_2\,{}^2\mathbf{v}_{2/0} &={}^0 R_r[{}^r\omega_{r/0}]_\times\,{}^r\mathbf{t}_g +{}^0 R_r\,{}^r\mathbf{v}_{r/0} +{}^0 R_1[{}^1\omega_{1/r}]_\times\,{}^1\mathbf{t}_g +{}^0 R_1\,{}^1\mathbf{v}_{1/r} \\ [{}^0\omega_{2/0}]_\times\,{}^0R_h &= [{}^0\omega_{r/0}]_\times\,{}^0R_g + [{}^0\omega_{1/r}]_\times\,{}^0R_g \\ {}^0R_2[{}^2\omega_{2/0}]_\times\,{}^2\mathbf{t}_h +{}^0 R_2\,{}^2\mathbf{v}_{2/0} &={}^0 R_r[{}^r\omega_{r/0}]_\times\,{}^r\mathbf{t}_g +{}^0 R_r\,{}^r\mathbf{v}_{r/0} +{}^0 R_1[{}^1\omega_{1/r}]_\times\,{}^1\mathbf{t}_g +{}^0 R_1\,{}^1\mathbf{v}_{1/r} \end{aligned}$$

As ${}^0R_h = {}^0R_g$ all along the motion,

$$\begin{aligned} {}^0\omega_{2/0} &= {}^0\omega_{r/0} + {}^0\omega_{1/r} \\ -{}^0R_2[\,{}^2\mathbf{t}_h]_\times\,{}^2\omega_{2/0} +{}^0 R_2\,{}^2\mathbf{v}_{2/0} &= -{}^0R_r[\,{}^r\mathbf{t}_g]_\times\,{}^r\omega_{r/0} +{}^0 R_r\,{}^r\mathbf{v}_{r/0} -{}^0 R_1[\,{}^1\mathbf{t}_g]_\times\,{}^1\omega_{1/r} + {}^0R_1\,{}^1\mathbf{v}_{1/r} \\ {}^0R_2\,{}^2\omega_{2/0} &= {}^0R_r\,{}^r\omega_{r/0} + {}^0R_r\,{}^r\omega_{1/r} \\ -{}^0R_2[\,{}^2\mathbf{t}_h]_\times\,{}^2\omega_{2/0} +{}^0 R_2\,{}^2\mathbf{v}_{2/0} &= -{}^0R_r[\,{}^r\mathbf{t}_g]_\times\,{}^r\omega_{r/0} +{}^0 R_r\,{}^r\mathbf{v}_{r/0} -{}^0 R_1[\,{}^1\mathbf{t}_g]_\times\,{}^1R_r\,{}^r\omega_{1/r} + {}^0R_1\,{}^1\mathbf{v}_{1/r} \end{aligned}$$

Using (3), we can write

$$\begin{aligned} {}^r\omega_{1/r} &= {}^rR_2\,{}^2\omega_{2/0} - {}^r\omega_{r/0} \\ -{}^0R_2[\,{}^2\mathbf{t}_h]_\times\,{}^2\omega_{2/0} + {}^0R_2\,{}^2\mathbf{v}_{2/0} &= - {}^0R_r[\,{}^r\mathbf{t}_g]_\times\,{}^r\omega_{r/0} + {}^0R_r\,{}^r\mathbf{v}_{r/0} \\ &\quad - {}^0R_1[\,{}^1\mathbf{t}_g]_\times(\,{}^1R_2\,{}^2\omega_{2/0} - {}^1R_r\,{}^r\omega_{r/0}) + {}^0R_1\,{}^1\mathbf{v}_{1/r} \end{aligned}$$

$$
\begin{aligned}
{}^1\mathbf{v}_{1/r} &= {}^1R_0 \big(-\,{}^0R_2 [\,{}^2\mathbf{t}_h]_\times\,{}^2\omega_{2/0} + \,{}^0R_2\,{}^2\mathbf{v}_{2/0} + \,{}^0R_r [\,{}^r\mathbf{t}_g]_\times\,{}^r\omega_{r/0} - \,{}^0R_r\,{}^r\mathbf{v}_{r/0} \\
&\quad + {}^0R_1 [\,{}^1\mathbf{t}_g]_\times (\,{}^1R_2\,{}^2\omega_{2/0} - \,{}^1R_r\,{}^r\omega_{r/0}) \big) \\
{}^r\omega_{1/r} &= {}^rR_2\,{}^2\omega_{2/0} - \,{}^r\omega_{r/0}
\end{aligned}
$$

$$
\begin{aligned}
{}^1\mathbf{v}_{1/r} &= -\,{}^1R_2 [\,{}^2\mathbf{t}_h]_\times\,{}^2\omega_{2/0} + \,{}^1R_2\,{}^2\mathbf{v}_{2/0} + \,{}^1R_r [\,{}^r\mathbf{t}_g]_\times\,{}^r\omega_{r/0} - \,{}^1R_r\,{}^r\mathbf{v}_{r/0} && (6) \\
&\quad + \,{}^0R_1 [\,{}^1\mathbf{t}_g]_\times (\,{}^1R_2\,{}^2\omega_{2/0} - \,{}^1R_r\,{}^r\omega_{r/0}) && (7) \\
{}^r\omega_{1/r} &= {}^rR_2\,{}^2\omega_{2/0} - \,{}^r\omega_{r/0} && (8)
\end{aligned}
$$

We denote

- $J_{2\,in}$ the columns of the Jacobian of `joint2` corresponding to the input variables,
- $J^{\mathbf{v}}_{2\,in}$, $J^{\omega}_{2\,in}$, respectively the first 3 and last 3 lines of the latter,
- $J_{r\,in}$ the columns of the Jacobian of `root` corresponding to the input variables,
- $J^{\mathbf{v}}_{r\,in}$, $J^{\omega}_{r\,in}$, respectively the first 3 and last 3 lines of the latter,

With this notation, (7-8) become

$$
\begin{aligned}
{}^1\mathbf{v}_{1/r} &= \big(-\,{}^1R_2 [\,{}^2\mathbf{t}_h]_\times J^{\omega}_{2\,in} + \,{}^1R_2 J^{\mathbf{v}}_{2\,in} + \,{}^1R_r [\,{}^r\mathbf{t}_g]_\times J^{\omega}_{r\,in} - \,{}^1R_r J^{\mathbf{v}}_{r\,in} \\
&\quad + {}^0R_1 [\,{}^1\mathbf{t}_g]_\times (\,{}^1R_2 J^{\omega}_{2\,in} - \,{}^1R_r J^{\omega}_{r\,in}) \big)\, \dot{\mathbf{q}}_{in} \\
{}^r\omega_{1/r} &= (\,{}^rR_2 J^{\omega}_{2\,in} - J^{\omega}_{r\,in})\, \dot{\mathbf{q}}_{in}
\end{aligned}
$$

Let J be the 6 matrix the first 3 line of which are

$$
-\,{}^1R_2 [\,{}^2\mathbf{t}_h]_\times J^{\omega}_{2\,in} + {}^1R_2 J^{\mathbf{v}}_{2\,in} + {}^1R_r [\,{}^r\mathbf{t}_g]_\times J^{\omega}_{r\,in} - {}^1R_r J^{\mathbf{v}}_{r\,in} + {}^0R_1 [\,{}^1\mathbf{t}_g]_\times (\,{}^1R_2 J^{\omega}_{2\,in} - {}^1R_r J^{\omega}_{r\,in})
$$

and the last 3 line of which are

$$
{}^rR_2 J^{\omega}_{2\,in} - J^{\omega}_{r\,in}
$$

Using (3), we can write

$$
\dot{\mathbf{q}}_{out} = J_{out}^{-1} J \dot{\mathbf{q}}_{in} \;\text{ and }\; \frac{\partial f}{\partial \mathbf{q}_{in}} = J_{out}^{-1} J
$$

References

[1] Florent Lamiraux and Joseph Mirabel. Prehensile Manipulation Planning: Modeling, Algorithms and Implementation. *IEEE Transactions on Robotics*, 38(4):2370–2388, August 2022.