# Computation of the exact inverse kinematics solution of a 6D robotic arm

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## 1 Introduction

This document explains how to implement exact inverse kinematics of a Stäubli robotic arm as an explicit constraint in Humanoid Path Planner software. Notation and definitions are the same as in [1].

## 2 Notation and Definitions

The constraint is defined as a 6 dimensional *grasp* constraint between a *gripper* and a *handle*. The *gripper* is attached to the robotic arm end-effector (`joint1`) and the *handle* is attached to `joint2` on the composite kinematic chain. `root` is the joint that holds the robot arm or the global frame (`"universe"` in `pinocchio` software).

We denote by

- $\mathbf{q}_{in}$ the input variables of the explicit constraint,
- $\mathbf{q}_{out}$ the output variables of the explicit constraint,
- ${}^0M_1$ the pose of `joint1` in the world frame,
- ${}^0M_2$ the pose of `joint2` in the world frame,
- ${}^0M_r$ the pose of `root` in the world frame,
- ${}^2M_h$ the pose of the handle in `joint2` frame,
- ${}^1M_g$ the pose of the gripper in `joint1` frame,
- ${}^rM_b$ the pose of the robot arm origin (`base_link` in URDF description) in the `root` frame.

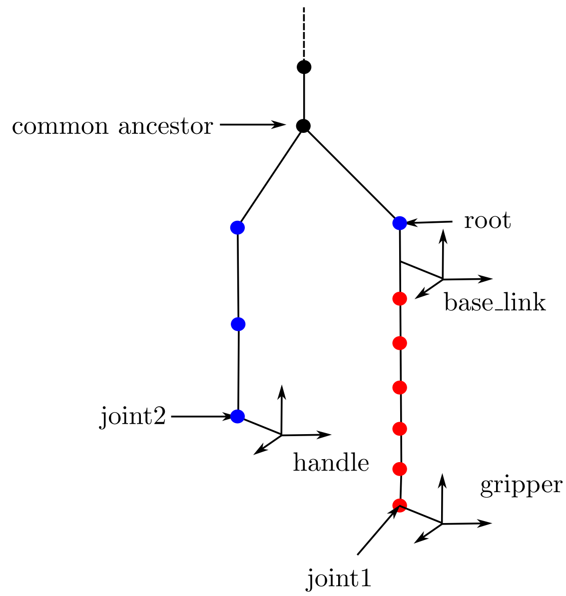


Figure 1: Input (blue) and output (red) variables of the explicit constraint that computes the robot arm configuration with respect to the handle pose.

# 3 Inverse kinematics

Exact inverse kinematics computes the 6 joint values of the robotic arm with respect to the input configuration variables:

$$\mathbf{q}_{out} = f(\mathbf{q}_{in}) \tag{1}$$

Note that the input variables include a extra degree of freedom that is interpreted as an integer to select among the various solutions of the inverse kinematics.

For the Staubli TX90, we have the following cotation in meters:

$$\begin{aligned} r1 &= 0.478 \\ r2 &= 0.050 \\ r3 &= 0.05 \\ r4 &= 0.425 \\ r5 &= 0.425 \\ r6 &= 0.1 \end{aligned}$$

We are going to use the following schematic to determinate each transformation matrix for each frame. We denote by:

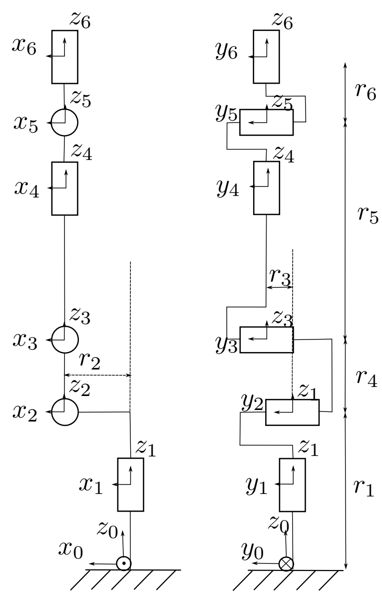


Figure 2: Kinematic diagram of the Stäubli TX2-90 robot.

- $T_{i-1,i}$ the homogeneous transformation matrix from frame $i-1$ to frame $i$,
- $T_{0,1}$ the transformation matrix from base link to the first joint frame,
- $T_{5,6}$ the transformation matrix from joint 5 frame to the gripper frame,

Homogeneous transformation matrix $T_{i-1,i}$ from frame $i-1$ to frame $i$ :

$$T_{0,1} = \begin{bmatrix} \cos(q_1) & -\sin(q_1) & 0 & 0 \\ \sin(q_1) & \cos(q_1) & 0 & 0 \\ 0 & 0 & 1 & r_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{1,2} = \begin{bmatrix} \cos(q_2) & 0 & \sin(q_2) & r_2 \\ 0 & 1 & 0 & 0 \\ -\sin(q_2) & 0 & \cos(q_2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{2,3} = \begin{bmatrix} \cos(q_3) & 0 & \sin(q_3) & 0 \\ 0 & 1 & 0 & r_3 \\ -\sin(q_3) & 0 & \cos(q_3) & r_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{3,4} = \begin{bmatrix} \cos(q_4) & -\sin(q_4) & 0 & 0 \\ \sin(q_4) & \cos(q_4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{4,5} = \begin{bmatrix} \cos(q_5) & 0 & \sin(q_5) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(q_5) & 0 & \cos(q_5) & r_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{5,6} = \begin{bmatrix} \cos(q_6) & -\sin(q_6) & 0 & 0 \\ \sin(q_6) & \cos(q_6) & 0 & 0 \\ 0 & 0 & 1 & r_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let us notice that the last three axes are concurrent. The concurrent point is the center of axis 5 and can be obtained by applying $T_{0,6}$ to vector $(0\ 0\ -r_6)$. Calculation of $T_{0,5}$ and $T_{3,6}$ are given below. We denote $sin(q_i) = s_i$ for each $i$ between 1 and 6.

$$T_{0,5} = T_{0,1}T_{1,2}T_{2,3}T_{3,4}T_{4,5} = \begin{bmatrix} X & X & X & r_5c_1s_{2+3} - r_3s_1 + r_4c_1s_2 + c_1r_2 \\ X & X & X & r_5s_1s_{2+3} + r_3c_1 + r_4s_1s_2 + r_2s_1 \\ X & X & X & r_5c_{2+3} + r_4c_2 + r_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Computation of $q_1$**

Let us rewrite

$$\begin{aligned} X &= r_5c_1s_{2+3} - r_3s_1 + r_4c_1s_2 + c_1r_2 \\ Y &= r_5s_1s_{2+3} + r_3c_1 + r_4s_1s_2 + r_2s_1 \\ Z &= r_5c_{2+3} + r_4c_2 + r_1 \end{aligned}$$

the coordinates of the translation part of $T_{0,5}$. Importantly, this point is the intersection of the last 3 axes. Given the geometry of the robot, we search $(X, Y)$ in the form:

$$\begin{bmatrix} c1 & -s1 \\ s_1 & c_1 \end{bmatrix} \begin{bmatrix} x \\ r_3 \end{bmatrix} = \begin{bmatrix} c_1x - s_1r_3 \\ s_1x + c_1r_3 \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} \tag{2}$$

Replacing $X$ and $Y$ by their expressions, we get:

$$\begin{aligned} c_1x - \cancel{s_1r_3} &= r_5c_1s_{2+3} - \cancel{s_1r_3} + c_1s_2r_4 + c_1r_2 \\ s_1x + \cancel{c_1r_3} &= r_5s_1s_{2+3} + \cancel{c_1r_3} + s_1s_2r_4 + s_1r_2 \end{aligned}$$

we simplify by $c_1$ and $s_1$ :

$$x = r_5s_{2+3} + s_2r_4 + r_2 \tag{3}$$

We need to determine $q_1$ from these expressions. the $x$ and $y$ components of the translation part of $T_{0,5}$. Multiplying the first line of (2) by $s_1$ and the second line by $c_1$, we get

$$\begin{aligned} s_1c_1x - s_1^2r_3 &= s_1X \\ s_1c_1x + c_1^2r_3 &= c_1Y \end{aligned}$$

Substracting the second line to the first one, we get

$$r_3 = c_1Y - s_1X.$$

Let us express $(X, Y)$ in polar coordinates:

$$X = \rho\cos\theta \tag{4}$$

$$Y = \rho\sin\theta \tag{5}$$

$$\rho \geq 0, -\pi \leq \theta \leq \pi. \tag{6}$$

We get

$$r_3 = \rho(\cos q_1 \sin\theta - \sin q_1 \cos\theta) = \rho \sin(\theta - q_1).$$

If $\rho < r_3$, then there is no solution. Otherwise,

$$\sin(\theta - q_1) = \frac{r_3}{\rho}$$

Then

$$\theta - q_1 = \arcsin\frac{r_3}{\rho} + 2k\pi, \text{ or } \theta - q_1 = \pi - \arcsin\frac{r_3}{\rho} + 2k\pi, k \in \mathbb{Z}.$$

and

$$\begin{aligned} q_1 &= \theta - \arcsin\frac{r_3}{\rho} + 2k\pi \text{ or} \\ q_1 &= \theta + \pi + \arcsin\frac{r_3}{\rho} + 2k\pi, k \in \mathbb{Z}. \end{aligned}$$

**Possible solutions.**

$$0 < \arcsin\frac{r_3}{\rho} \leq \frac{\pi}{2}$$

so from (6),

$$\begin{aligned} -\frac{3\pi}{2} &\leq \theta - \arcsin\frac{r_3}{\rho} \leq \pi \\ 0 &\leq \theta + \pi + \arcsin\frac{r_3}{\rho} \leq 2\pi + \frac{\pi}{2} \end{aligned}$$

As $-\pi < q_1 < \pi$, there is possibly 4 solutions:

$$\begin{aligned} q_1 &= \theta - \arcsin\frac{r_3}{\rho} \\ q_1 &= \theta - \arcsin\frac{r_3}{\rho} + 2\pi \\ q_1 &= \theta + \pi + \arcsin\frac{r_3}{\rho} \\ q_1 &= \theta - \pi + \arcsin\frac{r_3}{\rho} \end{aligned}$$

**Computation of $q_3$**

From $q_1$ by inverting (2), we get the value of $x$:

$$x = c_1 X + s_1 Y.$$

Using (3), we can write

$$x = r_5 \sin(q_2 + q_3) + r_4 \sin q_2 + r_2 \tag{7}$$

$$Z = r_5 \cos(q_2 + q_3) + r_4 \cos q_2 + r_1 \tag{8}$$

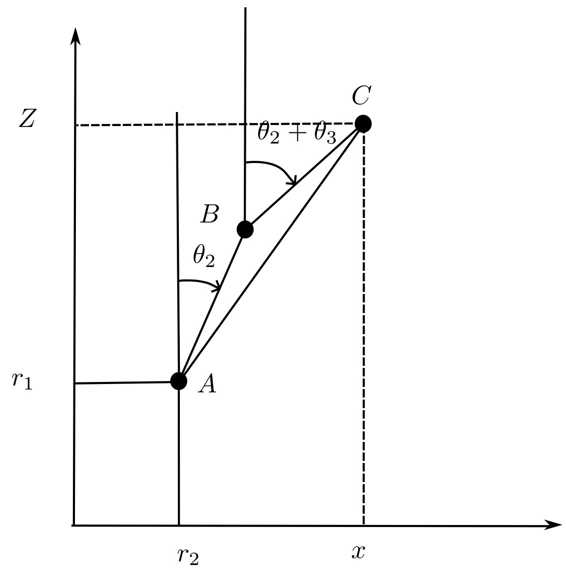


Figure 3: From the lengths of each side of triangle ABC, we can compute $\theta_3$

These equations define a triangle denoted ABC in Figure 3. Thus we can write

$$\begin{aligned}\|\vec{AC}\|^2 &= \|\vec{AB}+\vec{BC}\|^2 = \|\vec{AB}\|^2 + \|\vec{BC}\|^2 + 2\vec{AB}.\vec{BC}\\ (x-r_2)^2+(Z-r_1)^2 &= r_4^2+r_5^2+2r_4r_5\cos q_3\\ \cos q_3 &= \frac{(x-r_2)^2+(Z-r_1)^2-r_4^2-r_5^2}{2r_4r_5}\end{aligned}$$

If the right hand side of the latter is bigger than 1, there is no solution. Otherwise

$$q_3 = \pm\arccos\frac{(x-r_2)^2+(Z-r_1)^2-r_4^2-r_5^2}{2r_4r_5} + 2k\pi, k\in\mathbb{Z} \tag{9}$$

**Possible solutions.**

$$0 \leq \arccos\frac{(x-r_2)^2+(Z-r_1)^2-r_4^2-r_5^2}{2r_4r_5} \leq \pi$$

and $-\pi < q_3 < \pi$. Thus, there are 2 solutions:

$$\begin{aligned}q_3 &= +\arccos\frac{(x-r_2)^2+(Z-r_1)^2-r_4^2-r_5^2}{2r_4r_5}\\ q_3 &= -\arccos\frac{(x-r_2)^2+(Z-r_1)^2-r_4^2-r_5^2}{2r_4r_5}\end{aligned}$$

**Computation of $q_2$**

If we develop (7)-(8), we get

$$
\begin{aligned}
x - r_2 &= r_5(\sin q_2 \cos q_3 + \sin q_3 \cos q_2) + r_4 \sin q_2 \\
Z - r_1 &= r_5(\cos q_2 \cos q_3 - \sin q_2 \sin q_3) + r_4 \cos q_2 \\
x - r_2 &= r_5 \sin q_3 \cos q_2 + (r_5 \cos q_3 + r_4) \sin q_2 \\
Z - r_1 &= (r_5 \cos q_3 + r_4) \cos q_2 - r_5 \sin q_3 \sin q_2
\end{aligned}
$$

Using Kramer's formula, we have

$$
\begin{aligned}
\cos q_2 &= \frac{r_5 \sin q_3 (x - r_2) + (r_5 \cos q_3 + r_4)(Z - r_1)}{r_4^2 + r_5^2 + 2 r_4 r_5 \cos q_3} \\
\sin q_2 &= \frac{(r_5 \cos q_3 + r_4)(x - r_2) - r_5 \sin q_3 (Z - r_1)}{r_4^2 + r_5^2 + 2 r_4 r_5 \cos q_3}
\end{aligned}
$$

As $-\pi < q_2 < \pi$, this system of equations has a unique solution.

**Computation of $q_5$**

With the first 3 angles, we can compute the orientation matrix $R_{03}$ of axis 3:

$$
R_{0,3} = \begin{bmatrix} c_1c_2c_3 - c_1s_2s_3 & -s_1 & c_1c_2s_3 + c_1s_2c_3 \\ s_1c_2c_3 - s_1s_2s_3 & c_1 & s_1c_2s_3 - s_1s_2c_3 \\ -s_2c_3 - c_2s_3 & 0 & -s_2s_3 + c_2c_3 \end{bmatrix}
$$

From this matrix, we can compute $R_{3,6} = R_{0,3}^T R_{0,6}$ where $R_{0,6}$ is the orientation of the end effector that is given as input. Using the expressions of each axis rotation matrix given at the beginning if this section, we get an expression of $R_{3,6}$ with respect to the last three angles:

$$
R \triangleq R_{3,6} = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & -s_5s_6 & c_5 \end{bmatrix}
$$

The coefficient of the last row and of the last column enables us to compute $q_5$:

$$
q_5 = \pm \arccos R_{3\,3} + 2k\pi.
$$

**Possible solutions.**

$$
0 \leq \arccos R_{3\,3} \leq \pi
$$

and $-\pi < q_5 < \pi$. Thus, $q_5$ admits at most two solutions:

$$
\begin{aligned}
q_5 &= \arccos R_{3\,3} \\
q_5 &= -\arccos R_{3\,3}
\end{aligned}
$$

**Computation of $q_4$**

If $s_5 \neq 0$, we get $q_4$ from coefficients $R_{1\,3}$ and $R_{2\,3}$:

$$q_4 = \arctan2(\frac{R_{2\,3}}{\sin q_5}, \frac{R_{1\,3}}{\sin q_5}) + 2k\pi,$$

**Possible solutions.**

$$-\pi \leq \arctan2(\frac{R_{2\,3}}{\sin q_5}, \frac{R_{1\,3}}{\sin q_5}) < \pi.$$

As $-\frac{3\pi}{2} \leq q_4 \leq \frac{3\pi}{2}$, $q_4$ admits 3 solutions:

$$q_4 = \arctan2(\frac{R_{2\,3}}{\sin q_5}, \frac{R_{1\,3}}{\sin q_5}) - 2\pi$$
$$q_4 = \arctan2(\frac{R_{2\,3}}{\sin q_5}, \frac{R_{1\,3}}{\sin q_5})$$
$$q_4 = \arctan2(\frac{R_{2\,3}}{\sin q_5}, \frac{R_{1\,3}}{\sin q_5}) + 2\pi$$

**Computation of $q_6$**

If $s_5 \neq 0$, $q_6$ from coefficients $R_{3\,1}$ and $R_{3\,2}$:

$$q_6 = \arctan2(\frac{R_{3\,2}}{-\sin q_5}, \frac{R_{3\,1}}{-\sin q_5}) + 2k\pi.$$

**Possible solutions.**

$$-\pi \leq \arctan2(\frac{R_{3\,2}}{-\sin q_5}, \frac{R_{3\,1}}{-\sin q_5}) < \pi.$$

As $-\frac{3\pi}{2} \leq q_6 \leq \frac{3\pi}{2}$, $q_6$ admits 3 solutions:

$$q_6 = \arctan2(\frac{R_{3\,2}}{-\sin q_5}, \frac{R_{3\,1}}{-\sin q_5}) - 2\pi$$
$$q_6 = \arctan2(\frac{R_{3\,2}}{-\sin q_5}, \frac{R_{3\,1}}{-\sin q_5})$$
$$q_6 = \arctan2(\frac{R_{3\,2}}{-\sin q_5}, \frac{R_{3\,1}}{-\sin q_5}) + 2\pi$$

**Encoding of the solutions**

To encode the solution, we use an extra degree of freedom that stores an integer. The table below gathers the possible solutions for all joint angles

| configuration variable | solutions | numbering |
|---|---|---|
| $q_1$ | $\theta - \arcsin \frac{r_3}{\rho}$ | $i_1 = 0$ |
| | $\theta - \arcsin \frac{r_3}{\rho} + 2\pi$ | $i_1 = 1$ |
| | $\theta + \pi + \arcsin \frac{r_3}{\rho}$ | $i_1 = 2$ |
| | $\theta - \pi + \arcsin \frac{r_3}{\rho}$ | $i_1 = 3$ |
| $q_3$ | $+ \arccos \frac{(x-r_2)^2+(Z-r_1)^2-r_4^2-r_5^2}{2r_4r_5}$ | $i_3 = 0$ |
| | $- \arccos \frac{(x-r_2)^2+(Z-r_1)^2-r_4^2-r_5^2}{2r_4r_5}$ | $i_3 = 1$ |
| $q_5$ | $\arccos R_{3\,3}$ | $i_5 = 0$ |
| | $- \arccos R_{3\,3}$ | $i_5 = 1$ |
| $q_4$ | $\operatorname{arctan2}(\frac{R_{2\,3}}{\sin q_5}, \frac{R_{1\,3}}{\sin q_5}) - 2\pi$ | $i_4 = 0$ |
| | $\operatorname{arctan2}(\frac{R_{2\,3}}{\sin q_5}, \frac{R_{1\,3}}{\sin q_5})$ | $i_4 = 1$ |
| | $\operatorname{arctan2}(\frac{R_{2\,3}}{\sin q_5}, \frac{R_{1\,3}}{\sin q_5}) + 2\pi$ | $i_4 = 2$ |
| $q_6$ | $\operatorname{arctan2}(\frac{R_{3\,2}}{-\sin q_5}, \frac{R_{3\,1}}{-\sin q_5}) - 2\pi$ | $i_6 = 0$ |
| | $\operatorname{arctan2}(\frac{R_{3\,2}}{-\sin q_5}, \frac{R_{3\,1}}{-\sin q_5})$ | $i_6 = 1$ |
| | $\operatorname{arctan2}(\frac{R_{3\,2}}{-\sin q_5}, \frac{R_{3\,1}}{-\sin q_5}) + 2\pi$ | $i_6 = 2$ |

The indices of each configuration variables are linked to the extra degree of freedom $i$ by the following formula:

$$i = i_1 + 4i_3 + 8i_5 + 16i_4 + 48i_6$$

$i_1, i_3, i_5, i_4, i_6$ are retrieved from $i$ by computing successive Euclidean divisions.

- $i_1$ is the remainder of $i$ divided by 4,
- $i_3$ is the remainder of $\frac{i-i_1}{4}$ divided by 8,...

# 4 Jacobian

In order to implement exact inverse kinematics as an explicit constraint, we need to compute the Jacobian of $f$. For that, let us consider of motion of the kinematic chain that keeps the gripper and handle in the same pose:

$$\forall t \in \mathbb{R},\ {}^0M_2(t)\ {}^2M_h = {}^0M_r(t)\ {}^rM_1(t)\ {}^1M_g \tag{10}$$

Moreover

$$\begin{pmatrix} {}^r\mathbf{v}_{1/r} \\ {}^r\omega_{1/r} \end{pmatrix} = J_{out}\dot{\mathbf{q}}_{out} \tag{11}$$

where $J_{out}$ is the 6x6 matrix composed of the columns of Jacobian of `joint1` corresponding to the arm degrees of freedom. We denote respectively by $J_{out\ \mathbf{v}}$ and $J_{out\ \omega}$ the first 3 and the last 3 lines of this matrix.

Using homogeneous matrix notation and derivating with respect to time, Equation (10) can be written as $\forall t \in \mathbb{R}$,

$$
{}^0M_2 \begin{pmatrix} [\,{}^2\omega_{2/0}]_\times & {}^2\mathbf{v}_{2/0} \\ 0 & 0 \end{pmatrix} {}^2M_h = {}^0M_r \begin{pmatrix} [\,{}^r\omega_{r/0}]_\times & {}^r\mathbf{v}_{r/0} \\ 0 & 0 \end{pmatrix} {}^rM_1\, {}^1M_g \tag{12}
$$

$$
+\ {}^0M_r\, {}^rM_1 \begin{pmatrix} [\,{}^1\omega_{1/r}]_\times & {}^1\mathbf{v}_{1/r} \\ 0 & 0 \end{pmatrix} {}^1M_g \tag{13}
$$

$$
\begin{pmatrix} {}^0R_2[{}^2\omega_{2/0}]_\times & {}^0R_2\, {}^2\mathbf{v}_{2/0} \\ 0 & 0 \end{pmatrix} {}^2M_h = \begin{pmatrix} {}^0R_r[{}^r\omega_{r/0}]_\times & {}^0R_r\, {}^r\mathbf{v}_{r/0} \\ 0 & 0 \end{pmatrix} {}^rM_1\, {}^1M_g + {}^0M_r \begin{pmatrix} {}^rR_1[{}^1\omega_{1/r}]_\times & {}^rR_1\, {}^1\mathbf{v}_{1/r} \\ 0 & 0 \end{pmatrix} {}^1M_g
$$

$$
\begin{pmatrix} {}^0R_2[{}^2\omega_{2/0}]_\times\, {}^2R_h & {}^0R_2[{}^2\omega_{2/0}]_\times\, {}^2\mathbf{t}_h +{}^0 R_2\, {}^2\mathbf{v}_{2/0} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} {}^0R_r[{}^r\omega_{r/0}]_\times\, {}^rR_g & {}^0R_r[{}^r\omega_{r/0}]_\times\, {}^r\mathbf{t}_g +{}^0 R_r\, {}^r\mathbf{v}_{r/0} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} {}^0R_r\, {}^rR_1[{}^1\omega_{1/r}]_\times\, {}^1R_g & {}^0R_r\, {}^rR_1[{}^1\omega_{1/r}]_\times\, {}^1\mathbf{t}_g +{}^0 R_r\, {}^rR_1\, {}^1\mathbf{v}_{1/r} \\ 0 & 0 \end{pmatrix}
$$

Extracting the upper blocks of this matrix equality, we get

$$
\begin{aligned}
{}^0R_2[{}^2\omega_{2/0}]_\times\, {}^2R_h &={}^0 R_r[{}^r\omega_{r/0}]_\times\, {}^rR_g +{}^0 R_r\, {}^rR_1[{}^1\omega_{1/r}]_\times\, {}^1R_g \\
{}^0R_2[{}^2\omega_{2/0}]_\times\, {}^2\mathbf{t}_h +{}^0 R_2\, {}^2\mathbf{v}_{2/0} &={}^0 R_r[{}^r\omega_{r/0}]_\times\, {}^r\mathbf{t}_g +{}^0 R_r\, {}^r\mathbf{v}_{r/0} +{}^0 R_r\, {}^rR_1[{}^1\omega_{1/r}]_\times\, {}^1\mathbf{t}_g +{}^0 R_r\, {}^rR_1\, {}^1\mathbf{v}_{1/r} \\
{}^0R_2[{}^2\omega_{2/0}]_\times\, {}^2R_h &={}^0 R_r[{}^r\omega_{r/0}]_\times\, {}^rR_g +{}^0 R_1[{}^1\omega_{1/r}]_\times\, {}^1R_g \\
{}^0R_2[{}^2\omega_{2/0}]_\times\, {}^2\mathbf{t}_h +{}^0 R_2\, {}^2\mathbf{v}_{2/0} &={}^0 R_r[{}^r\omega_{r/0}]_\times\, {}^r\mathbf{t}_g +{}^0 R_r\, {}^r\mathbf{v}_{r/0} +{}^0 R_1[{}^1\omega_{1/r}]_\times\, {}^1\mathbf{t}_g +{}^0 R_1\, {}^1\mathbf{v}_{1/r} \\
[{}^0\omega_{2/0}]_\times\, {}^0R_h &= [{}^0\omega_{r/0}]_\times\, {}^0R_g + [{}^0\omega_{1/r}]_\times\, {}^0R_g \\
{}^0R_2[{}^2\omega_{2/0}]_\times\, {}^2\mathbf{t}_h +{}^0 R_2\, {}^2\mathbf{v}_{2/0} &={}^0 R_r[{}^r\omega_{r/0}]_\times\, {}^r\mathbf{t}_g +{}^0 R_r\, {}^r\mathbf{v}_{r/0} +{}^0 R_1[{}^1\omega_{1/r}]_\times\, {}^1\mathbf{t}_g +{}^0 R_1\, {}^1\mathbf{v}_{1/r}
\end{aligned}
$$

As ${}^0R_h = {}^0R_g$ all along the motion,

$$
\begin{aligned}
{}^0\omega_{2/0} &= {}^0\omega_{r/0} + {}^0\omega_{1/r} \\
-{}^0R_2[\,{}^2\mathbf{t}_h]_\times\, {}^2\omega_{2/0} +{}^0 R_2\, {}^2\mathbf{v}_{2/0} &= -{}^0R_r[\,{}^r\mathbf{t}_g]_\times\, {}^r\omega_{r/0} +{}^0 R_r\, {}^r\mathbf{v}_{r/0} -{}^0 R_1[\,{}^1\mathbf{t}_g]_\times\, {}^1\omega_{1/r} + {}^0R_1\, {}^1\mathbf{v}_{1/r} \\
{}^0R_2\, {}^2\omega_{2/0} &= {}^0R_r\, {}^r\omega_{r/0} + {}^0R_r\, {}^r\omega_{1/r} \\
-{}^0R_2[\,{}^2\mathbf{t}_h]_\times\, {}^2\omega_{2/0} +{}^0 R_2\, {}^2\mathbf{v}_{2/0} &= -{}^0R_r[\,{}^r\mathbf{t}_g]_\times\, {}^r\omega_{r/0} +{}^0 R_r\, {}^r\mathbf{v}_{r/0} -{}^0 R_1[\,{}^1\mathbf{t}_g]_\times\, {}^1R_r\, {}^r\omega_{1/r} + {}^0R_1\, {}^1\mathbf{v}_{1/r}
\end{aligned}
$$

Using (11), we can write

$$
\begin{aligned}
{}^r\omega_{1/r} &= {}^rR_2\, {}^2\omega_{2/0} - {}^r\omega_{r/0} \\
-{}^0R_2[\,{}^2\mathbf{t}_h]_\times\, {}^2\omega_{2/0} + {}^0R_2\, {}^2\mathbf{v}_{2/0} &= -\ {}^0R_r[\,{}^r\mathbf{t}_g]_\times\, {}^r\omega_{r/0} + {}^0R_r\, {}^r\mathbf{v}_{r/0} \\
&\quad -\ {}^0R_1[\,{}^1\mathbf{t}_g]_\times(\,{}^1R_2\, {}^2\omega_{2/0} - {}^1R_r\, {}^r\omega_{r/0}) + {}^0R_1\, {}^1\mathbf{v}_{1/r}
\end{aligned}
$$

$$
\begin{aligned}
{}^1\mathbf{v}_{1/r} &= {}^1R_0 \big( - {}^0R_2 [\,{}^2\mathbf{t}_h]_\times \, {}^2\omega_{2/0} + \, {}^0R_2 \, {}^2\mathbf{v}_{2/0} + \, {}^0R_r [\,{}^r\mathbf{t}_g]_\times \, {}^r\omega_{r/0} - \, {}^0R_r \, {}^r\mathbf{v}_{r/0} \\
&\quad + {}^0R_1 [\,{}^1\mathbf{t}_g]_\times ( \, {}^1R_2 \, {}^2\omega_{2/0} - \, {}^1R_r \, {}^r\omega_{r/0}) \big) \\
{}^r\omega_{1/r} &= {}^rR_2 \, {}^2\omega_{2/0} - \, {}^r\omega_{r/0}
\end{aligned}
$$

$$
\begin{aligned}
{}^1\mathbf{v}_{1/r} &= - \, {}^1R_2 [\,{}^2\mathbf{t}_h]_\times \, {}^2\omega_{2/0} + \, {}^1R_2 \, {}^2\mathbf{v}_{2/0} + \, {}^1R_r [\,{}^r\mathbf{t}_g]_\times \, {}^r\omega_{r/0} - \, {}^1R_r \, {}^r\mathbf{v}_{r/0} && (14) \\
&\quad + \, {}^0R_1 [\,{}^1\mathbf{t}_g]_\times ( \, {}^1R_2 \, {}^2\omega_{2/0} - \, {}^1R_r \, {}^r\omega_{r/0}) && (15) \\
{}^r\omega_{1/r} &= {}^rR_2 \, {}^2\omega_{2/0} - \, {}^r\omega_{r/0} && (16)
\end{aligned}
$$

We denote

- $J_{2\,in}$ the columns of the Jacobian of `joint2` corresponding to the input variables,
- $J^{\mathbf{v}}_{2\,in}$, $J^{\omega}_{2\,in}$, respectively the first 3 and last 3 lines of the latter,
- $J_{r\,in}$ the columns of the Jacobian of `root` corresponding to the input variables,
- $J^{\mathbf{v}}_{r\,in}$, $J^{\omega}_{r\,in}$, respectively the first 3 and last 3 lines of the latter,

With this notation, (15-16) become

$$
\begin{aligned}
{}^1\mathbf{v}_{1/r} &= \big( - \, {}^1R_2 [\,{}^2\mathbf{t}_h]_\times J^{\omega}_{2\,in} + \, {}^1R_2 J^{\mathbf{v}}_{2\,in} + \, {}^1R_r [\,{}^r\mathbf{t}_g]_\times J^{\omega}_{r\,in} - \, {}^1R_r J^{\mathbf{v}}_{r\,in} \\
&\quad + {}^0R_1 [\,{}^1\mathbf{t}_g]_\times ( \, {}^1R_2 J^{\omega}_{2\,in} - \, {}^1R_r J^{\omega}_{r\,in}) \big) \, \dot{\mathbf{q}}_{in} \\
{}^r\omega_{1/r} &= ( \, {}^rR_2 J^{\omega}_{2\,in} - J^{\omega}_{r\,in}) \, \dot{\mathbf{q}}_{in}
\end{aligned}
$$

Let $J$ be the 6 matrix the first 3 lines of which are

$$
{}^1R_2(-[\,{}^2\mathbf{t}_h]_\times J^{\omega}_{2\,in} + J^{\mathbf{v}}_{2\,in}) + {}^1R_r([\,{}^r\mathbf{t}_g]_\times J^{\omega}_{r\,in} - J^{\mathbf{v}}_{r\,in}) + {}^0R_1 [\,{}^1\mathbf{t}_g]_\times ( \, {}^1R_2 J^{\omega}_{2\,in} - \, {}^1R_r J^{\omega}_{r\,in})
$$

and the last 3 lines of which are

$$
{}^rR_2 J^{\omega}_{2\,in} - J^{\omega}_{r\,in}
$$

Using (11), we can write

$$
\dot{\mathbf{q}}_{out} = J_{out}^{-1} J \dot{\mathbf{q}}_{in} \;\text{ and }\; \frac{\partial f}{\partial \mathbf{q}_{in}} = J_{out}^{-1} J
$$

# References


[1] Florent Lamiraux and Joseph Mirabel. Prehensile Manipulation Planning: Modeling, Algorithms and Implementation. *IEEE Transactions on Robotics*, 38(4):2370–2388, August 2022.