

Computation of the exact inverse kinematics solution of a 6D robotic arm

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1 Introduction

This document explains how to implement exact inverse kinematics of a Stäubli robotic arm as an explicit constraint in Humanoid Path Planner software. Notation and definitions are the same as in [1].

2 Notation and Definitions

The constraint is defined as a 6 dimensional *grasp* constraint between a *gripper* and a *handle*. The *gripper* is attached to the robotic arm end-effector (`joint1`) and the *handle* is attached to `joint2` on the composite kinematic chain. `root` is the joint that holds the robot arm or the global frame ("`universe`" in `pinocchio` software).

We denote by

- \mathbf{q}_{in} the input variables of the explicit constraint,
- \mathbf{q}_{out} the output variables of the explicit constraint,
- 0M_1 the pose of `joint1` in the world frame,
- 0M_2 the pose of `joint2` in the world frame,
- 0M_r the pose of `root` in the world frame,
- 2M_h the pose of the handle in `joint2` frame,
- 1M_g the pose of the gripper in `joint1` frame,
- rM_b the pose of the robot arm origin (`base.link` in URDF description) in the `root` frame.

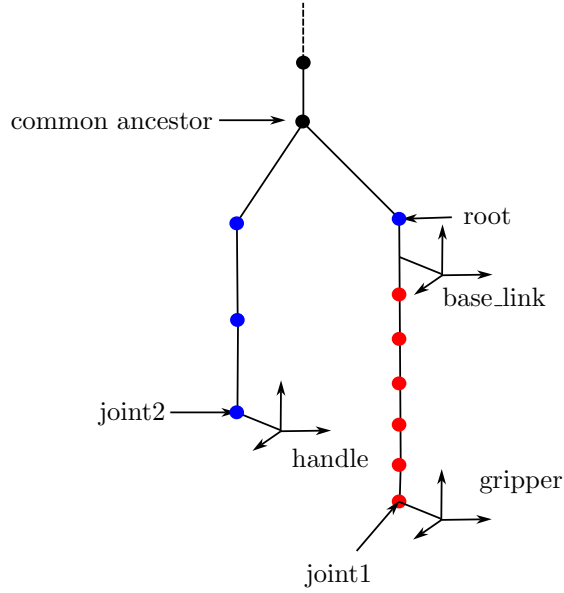


Figure 1: Input (blue) and output (red) variables of the explicit constraint that computes the robot arm configuration with respect to the handle pose.

3 Inverse kinematics

Exact inverse kinematics computes the 6 joint values of the robotic arm with respect to the input configuration variables:

$$\mathbf{q}_{out} = f(\mathbf{q}_{in}) \quad (1)$$

Note that the input variables include a extra degree of freedom that is interpreted as an integer to select among the various solutions of the inverse kinematics.

4 Jacobian

In order to implement exact inverse kinematics as an explicit constraint, we need to compute the Jacobian of f . For that, let us consider of motion of the kinematic chain that keeps the gripper and handle in the same pose:

$$\forall t \in \mathbb{R}, {}^0M_2(t) {}^2M_h = {}^0M_r(t) {}^rM_1(t) {}^1M_g \quad (2)$$

Moreover

$$\begin{pmatrix} {}^r\mathbf{v}_{1/r} \\ {}^r\omega_{1/r} \end{pmatrix} = J_{out} \dot{\mathbf{q}}_{out}$$

Using homogeneous matrix notation and derivating with respect to time, Equation (2) can be written as $\forall t \in \mathbb{R}$,

$$\begin{aligned} {}^0M_2 \begin{pmatrix} [{}^2\omega_{2/0}]_{\times} & {}^2\mathbf{v}_{2/0} \\ 0 & 0 \end{pmatrix} {}^2M_h &= {}^0M_r \begin{pmatrix} [{}^r\omega_{r/0}]_{\times} & {}^r\mathbf{v}_{r/0} \\ 0 & 0 \end{pmatrix} {}^rM_1 {}^1M_g \quad (3) \\ &+ {}^0M_r {}^rM_1 \begin{pmatrix} [{}^1\omega_{1/r}]_{\times} & {}^1\mathbf{v}_{1/r} \\ 0 & 0 \end{pmatrix} {}^1M_g \quad (4) \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} {}^0R_2[{}^2\omega_{2/0}]_{\times} & {}^0R_2 {}^2\mathbf{v}_{2/0} \\ 0 & 0 \end{pmatrix} {}^2M_h &= \begin{pmatrix} {}^0R_r[{}^r\omega_{r/0}]_{\times} & {}^0R_r {}^r\mathbf{v}_{r/0} \\ 0 & 0 \end{pmatrix} {}^rM_1 {}^1M_g \\ &+ {}^0M_r \begin{pmatrix} {}^rR_1[{}^1\omega_{1/r}]_{\times} & {}^rR_1 {}^1\mathbf{v}_{1/r} \\ 0 & 0 \end{pmatrix} {}^1M_g \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} {}^0R_2[{}^2\omega_{2/0}]_{\times} {}^2R_h & {}^0R_2[{}^2\omega_{2/0}]_{\times} {}^2\mathbf{t}_h + {}^0R_2 {}^2\mathbf{v}_{2/0} \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} {}^0R_r[{}^r\omega_{r/0}]_{\times} {}^rR_g & {}^0R_r[{}^r\omega_{r/0}]_{\times} {}^r\mathbf{t}_g + {}^0R_r {}^r\mathbf{v}_{r/0} \\ 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} {}^0R_r {}^rR_1[{}^1\omega_{1/r}]_{\times} {}^1R_g & {}^0R_r {}^rR_1[{}^1\omega_{1/r}]_{\times} {}^1\mathbf{t}_g + {}^0R_r {}^rR_1 {}^1\mathbf{v}_{1/r} \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Extracting the upper blocks of this matrix equality, we get

$$\begin{aligned} {}^0R_2[{}^2\omega_{2/0}]_{\times} {}^2R_h &= {}^0R_r[{}^r\omega_{r/0}]_{\times} {}^rR_g + {}^0R_r {}^rR_1[{}^1\omega_{1/r}]_{\times} {}^1R_g \\ {}^0R_2[{}^2\omega_{2/0}]_{\times} {}^2\mathbf{t}_h + {}^0R_2 {}^2\mathbf{v}_{2/0} &= {}^0R_r[{}^r\omega_{r/0}]_{\times} {}^r\mathbf{t}_g + {}^0R_r {}^r\mathbf{v}_{r/0} + {}^0R_r {}^rR_1[{}^1\omega_{1/r}]_{\times} {}^1\mathbf{t}_g + {}^0R_r {}^rR_1 {}^1\mathbf{v}_{1/r} \\ {}^0R_2[{}^2\omega_{2/0}]_{\times} {}^2R_h &= {}^0R_r[{}^r\omega_{r/0}]_{\times} {}^rR_g + {}^0R_1[{}^1\omega_{1/r}]_{\times} {}^1R_g \\ {}^0R_2[{}^2\omega_{2/0}]_{\times} {}^2\mathbf{t}_h + {}^0R_2 {}^2\mathbf{v}_{2/0} &= {}^0R_r[{}^r\omega_{r/0}]_{\times} {}^r\mathbf{t}_g + {}^0R_r {}^r\mathbf{v}_{r/0} + {}^0R_1[{}^1\omega_{1/r}]_{\times} {}^1\mathbf{t}_g + {}^0R_1 {}^1\mathbf{v}_{1/r} \\ [{}^0\omega_{2/0}]_{\times} {}^0R_h &= [{}^0\omega_{r/0}]_{\times} {}^0R_g + [{}^0\omega_{1/r}]_{\times} {}^0R_g \\ {}^0R_2[{}^2\omega_{2/0}]_{\times} {}^2\mathbf{t}_h + {}^0R_2 {}^2\mathbf{v}_{2/0} &= {}^0R_r[{}^r\omega_{r/0}]_{\times} {}^r\mathbf{t}_g + {}^0R_r {}^r\mathbf{v}_{r/0} + {}^0R_1[{}^1\omega_{1/r}]_{\times} {}^1\mathbf{t}_g + {}^0R_1 {}^1\mathbf{v}_{1/r} \end{aligned}$$

References

- [1] Florent Lamiraux and Joseph Mirabel. Prehensile Manipulation Planning: Modeling, Algorithms and Implementation. *IEEE Transactions on Robotics*, 38(4):2370–2388, August 2022.