# Cubic B-spline based walk generator

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## 1 Cubic B splines

Let

- $m$ be an integer bigger than 7,
- $t_0 \leq t_1 \leq \cdots \leq t_{m-1}$ an increasing sequence of real values,
- $\mathbf{x}_i \in \mathbb{R}^2, 0 \leq i \leq m-5$ control points in the plane.

We define the curve $\mathbf{x}$ from interval $[t_3, t_{m-4}]$ in $\mathbb{R}^2$ as

$$\mathbf{x} = \sum_{i=0}^{m-5} b_{i,3}\mathbf{x}_i \quad (1)$$

where $b_{i,3}$ are the basis function of cubic B splines:

$$\begin{aligned} b_{i,3} &= (B_{i,i}\mathbb{I}_{[t_i,t_{i+1}]} + B_{i,i+1}\mathbb{I}_{[t_{i+1},t_{i+2}]} + B_{i,i+2}\mathbb{I}_{[t_{i+2},t_{i+3}]} + B_{i,i+3}\mathbb{I}_{[t_{i+3},t_{i+4}]}) \\ & \quad i = 0, \ldots, m-5 \end{aligned}$$

with

$$\begin{aligned} B_{i,i}(t) &= \frac{(t-t_i)^3}{(t_{i+3}-t_i)(t_{i+2}-t_i)(t_{i+1}-t_i)} \\ B_{i,i+1}(t) &= \frac{(t-t_i)^2(t_{i+2}-t)}{(t_{i+3}-t_i)(t_{i+2}-t_{i+1})(t_{i+2}-t_i)} + \frac{(t-t_i)(t_{i+3}-t)(t-t_{i+1})}{(t_{i+3}-t_i)(t_{i+3}-t_{i+1})(t_{i+2}-t_{i+1})} \\ &\quad + \frac{(t_{i+4}-t)(t-t_{i+1})^2}{(t_{i+4}-t_{i+1})(t_{i+3}-t_{i+1})(t_{i+2}-t_{i+1})} \\ B_{i,i+2}(t) &= \frac{(t-t_i)(t_{i+3}-t)^2}{(t_{i+3}-t_i)(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})} + \frac{(t_{i+4}-t)(t-t_{i+1})(t_{i+3}-t)}{(t_{i+4}-t_{i+1})(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})} \\ &\quad + \frac{(t_{i+4}-t)^2(t-t_{i+2})}{(t_{i+4}-t_{i+1})(t_{i+4}-t_{i+2})(t_{i+3}-t_{i+2})} \\ B_{i,i+3}(t) &= \frac{(t_{i+4}-t)^3}{(t_{i+4}-t_{i+1})(t_{i+4}-t_{i+2})(t_{i+4}-t_{i+3})} \end{aligned}$$

and for any interval $I$ of $\mathbb{R}$, $\mathbb{I}_I$ is the function equal to 1 over $I$ and to 0 outside $I$.

## 2 Walk generator

### 2.1 Input

The input of the walk generator is a sequence of time-stamped steps defined as follows:

1. $p$ is a positive integer not smaller than 2,
2. $\tau_0, \tau_1, \cdots, \tau_{2p-1}$ is an increasing sequence of real values,
3. $\mathbf{s}_0, \mathbf{s}_1, \cdots, \mathbf{s}_{p-1}$ is a sequence of $p$ elements of $\mathbb{R}^2$ representing the successive positions of the foot centers.
4. $\mathbf{c}_{init} \in \mathbb{R}^2$ is the initial position of the center of mass at time $\tau_0$,
5. $\mathbf{c}_{final} \in \mathbb{R}^2$ is the final position of the center of mass at time $\tau_p$.

### 2.2 Reference trajectory of the center of pressure

We define a reference trajectory of the center of pressure called $\mathbf{zmp}_{ref}$ as a continuous piecewise affine curve as follows:

$$\begin{array}{rcl}
\mathbf{zmp}_{ref}(\tau_0) & = & \mathbf{c}_{init} \\
\mathbf{zmp}_{ref}(\tau_1) & = & \mathbf{s}_1 \\
\mathbf{zmp}_{ref}(\tau_2) & = & \mathbf{s}_1 \\
\mathbf{zmp}_{ref}(\tau_3) & = & \mathbf{s}_2 \\
\vdots & \vdots & \vdots \\
\mathbf{zmp}_{ref}(\tau_{2p-3}) & = & \mathbf{s}_{p-1} \\
\mathbf{zmp}_{ref}(\tau_{2p-2}) & = & \mathbf{s}_{p-1} \\
\mathbf{zmp}_{ref}(\tau_{2p-1}) & = & \mathbf{c}_{final}
\end{array}$$

such that $\mathbf{zmp}_{ref}$ restricted to each interval $[\tau_i, \tau_{i+1}]$ with $i \in \{0, 1, \cdots, 2p-2\}$ is affine.

### 2.3 Trajectory of the center of mass

We restrict the trajectory of the center of mass to be a cubic-spline defined by Equation (1). As we want the whole center of mass trajectory to be defined on interval $[\tau_0, \tau_{2p-1}]$, we get the following relations between the various parameters:

$$\begin{array}{rcl}
m & = & 2p+6 \\
t_3 & = & \tau_0 \\
t_{m-4} & = & \tau_{2p-1}
\end{array}$$

We need to fix the undefined knots. We arbitrarily set

$$\begin{aligned} t_0 &= \tau_0 - 3 & \qquad t_{m-3} &= \tau_{2p-1} + 1 \\ t_1 &= \tau_0 - 2 & \qquad t_{m-2} &= \tau_{2p-1} + 2 \\ t_2 &= \tau_0 - 1 & \qquad t_{m-1} &= \tau_{2p-1} + 3 \end{aligned}$$

## 2.4 Trajectory of the center of pressure

Let $g$ be the gravity constant, and $z$ the constant height of the center of mass. By denoting $\omega = \sqrt{g/z}$, we get the simplified formula for the center of pressure of the robot:

$$\mathbf{zmp} = \mathbf{x} - \frac{1}{\omega^2}\ddot{\mathbf{x}}$$

By setting

$$z_{i,3} \triangleq b_{i,3} - \frac{1}{\omega^2}\ddot{b}_{i,3}$$

we get an expression of the center of pressure with respect to the control points of the cubic B spline:

$$\mathbf{zmp} = \sum_{i=0}^{m-5} z_{i,3}\mathbf{x}_i = \sum_{i=0}^{2p-2} z_{i,3}\mathbf{x}_i \tag{2}$$

## 2.5 Optimal control problem

We denote by $X = (\mathbf{x}_0, \cdots, \mathbf{x}_{m-5})$ the vector of control points. We wish to find the cubic B spline trajectory that minimizes the following cost function:

$$C(X) \triangleq \frac{1}{2}\int_{\tau_0}^{\tau_{2p-1}} \|\mathbf{zmp}(t) - \mathbf{zmp}_{ref}(t)\|^2 dt \tag{3}$$

Let us expand this formula using (2):

$$\begin{aligned} C(X) &= \frac{1}{2}\int_{\tau_0}^{\tau_{2p-1}} (\sum_{i=0}^{2p-2} z_{i,3}\mathbf{x}_i - \mathbf{zmp}_{ref}(t))^T (\sum_{i=0}^{2p-2} z_{i,3}\mathbf{x}_i - \mathbf{zmp}_{ref}(t)) dt \\ &= \frac{1}{2}\sum_{i=0}^{m-5}\sum_{j=0}^{m-5}\int_{\tau_0}^{\tau_{2p-1}} z_{i,3}z_{j,3}(t)dt\ \mathbf{x}_i^T\mathbf{x}_j - \sum_{i=0}^{m-5}\int_{\tau_0}^{\tau_{2p-1}} z_{i,3}\mathbf{zmp}_{ref}^T(t)dt\ \mathbf{x}_i \\ &\quad + \int_{\tau_0}^{\tau_{2p-1}} \mathbf{zmp}_{ref}^T\mathbf{zmp}_{ref}(t)dt \end{aligned}$$

The cost function can be rewritten as

$$C(X) = \frac{1}{2}X^T H X - b^T X + C_0$$

with

$$
\begin{aligned}
H &= \left( \int_{\tau_0}^{\tau_{2p-1}} z_{i,3} z_{j,3}(t) dt \ I_2 \right)_{i,j=0,\cdots,m-5} \\
b &= \left( \int_{\tau_0}^{\tau_{2p-1}} z_{i,3} \mathbf{zmp}_{ref}(t) dt \right)_{i=0,\cdots,m-5} \\
C_0 &= \int_{\tau_0}^{\tau_{2p-1}} \mathbf{zmp}_{ref}^T \mathbf{zmp}_{ref}(t) dt
\end{aligned}
$$

$C(X)$ is the sum of two terms that respectively depend on the $x$ and $y$ coordinates of the control points. Let us denote $\mathbf{x}_{i,0}$, $\mathbf{x}_{i,1}$ the abscissa and the ordinate of $\mathbf{x}_i$, $\mathbf{zmp}_{ref,0}$, $\mathbf{zmp}_{ref,1}$ the abscissa and ordinate of $\mathbf{zmp}_{ref}$, and let us define

$$
\begin{aligned}
X_0 &= (\mathbf{x}_{0,0}, \cdots, \mathbf{x}_{m-5,0}) \\
X_1 &= (\mathbf{x}_{0,1}, \cdots, \mathbf{x}_{m-5,1}) \\
b_0 &= \left( \int_{\tau_0}^{\tau_{2p-1}} z_{i,3} \ \mathbf{zmp}_{ref,0}(t) dt \right)_{i=0,\cdots,m-5} \\
b_1 &= \left( \int_{\tau_0}^{\tau_{2p-1}} z_{i,3} \ \mathbf{zmp}_{ref,1}(t) dt \right)_{i=0,\cdots,m-5} \\
H_0 = H_1 &= \left( \int_{\tau_0}^{\tau_{2p-1}} z_{i,3} z_{j,3}(t) dt \right)_{i,j=0,\cdots,m-5}
\end{aligned}
$$

Then,

$$
C(X) = \frac{1}{2} X_0^T H_0 X_0 - b_0^T X_0 + \frac{1}{2} X_1^T H_1 X_1 - b_1^T X_1 + C_0
$$

The problem can therefore be decomposed into two decoupled sub-problems, one in $X_0$ and the other one in $X_1$.

## 2.6 Computation of the coefficients

$$
\begin{aligned}
z_{i,3} = {} & (Z_{i,i} \mathbb{I}_{[t_i,t_{i+1}]} + Z_{i,i+1} \mathbb{I}_{[t_{i+1},t_{i+2}]} + Z_{i,i+2} \mathbb{I}_{[t_{i+2},t_{i+3}]} + Z_{i,i+3} \mathbb{I}_{[t_{i+3},t_{i+4}]}) \\
& i = 0, \ldots, m-5
\end{aligned}
$$

with

$$
\begin{aligned}
Z_{i,i}(t) &= B_{i,i} - \frac{1}{\omega^2} \ddot{B}_{i,i} \\
Z_{i,i+1}(t) &= B_{i,i+1} - \frac{1}{\omega^2} \ddot{B}_{i,i+1} \\
Z_{i,i+2}(t) &= B_{i,i+2} - \frac{1}{\omega^2} \ddot{B}_{i,i+2} \\
Z_{i,i+3}(t) &= B_{i,i+3} - \frac{1}{\omega^2} \ddot{B}_{i,i+3}
\end{aligned}
$$

Matrix $H_0$ is symmetric. For any integer $i$ such that $0 \leq i \leq m-5$, and any non-negative integer $k$, such that $k \leq 3$ and $i+k \leq m-5$, The coefficient

$(i, i+k)$ of matrix $H_0$, with $k \in \{0,1,2,3\}$ is equal to

$$\begin{aligned} H_{0\ i,i+k} &= \int_{\tau_0}^{\tau_{2p-1}} z_{i,3} z_{i+k,3}(t) dt \\ &= \sum_{j=0}^{3-k} \int_{t_{i+k+j}}^{t_{i+k+j+1}} Z_{i,i+k+j} Z_{i+k,i+k+j}(t) dt \end{aligned}$$

The coefficients of vector $b_0$ are equal to:

$$\begin{aligned} b_{0\,i} &= \int_{\tau_0}^{\tau_{2p-1}} \left( Z_{i,i} \mathbb{I}_{[t_i, t_{i+1}]} + Z_{i,i+1} \mathbb{I}_{[t_{i+1}, t_{i+2}]} + Z_{i,i+2} \mathbb{I}_{[t_{i+2}, t_{i+3}]} + Z_{i,i+3} \mathbb{I}_{[t_{i+3}, t_{i+4}]} \right) \mathbf{zmp}_{ref,0}(t) dt \\ &= \sum_{j=\max(0,3-i)}^{\min(3,2p+1-i)} \int_{t_{i+j}}^{t_{i+j+1}} Z_{i,i+j} \mathbf{zmp}_{ref,0}(t)\, dt \end{aligned}$$