

Cubic B-spline based walk generator

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1 Cubic B splines

Let

- m be an integer bigger than 7,
- $t_0 \leq t_1 \leq \dots \leq t_{m-1}$ an increasing sequence of real values,
- $\mathbf{x}_i \in \mathbb{R}^2, 0 \leq i \leq m-5$ control points in the plane.

We define the curve \mathbf{x} from interval $[t_3, t_{m-4}]$ in \mathbb{R}^2 as

$$\mathbf{x} = \sum_{i=0}^{m-5} b_{i,3} \mathbf{x}_i \quad (1)$$

where $b_{i,3}$ are the basis function of cubic B splines:

$$b_{i,3} = (B_{i,i} \mathbb{I}_{[t_i, t_{i+1})} + B_{i,i+1} \mathbb{I}_{[t_{i+1}, t_{i+2})} + B_{i,i+2} \mathbb{I}_{[t_{i+2}, t_{i+3})} + B_{i,i+3} \mathbb{I}_{[t_{i+3}, t_{i+4})})$$

$$i = 0, \dots, m-5$$

with

$$\begin{aligned} B_{i,i}(t) &= \frac{(t - t_i)^3}{(t_{i+3} - t_i)(t_{i+2} - t_i)(t_{i+1} - t_i)} \\ B_{i,i+1}(t) &= \frac{(t - t_i)^2(t_{i+2} - t)}{(t_{i+3} - t_i)(t_{i+2} - t_{i+1})(t_{i+2} - t_i)} + \frac{(t - t_i)(t_{i+3} - t)(t - t_{i+1})}{(t_{i+3} - t_i)(t_{i+3} - t_{i+1})(t_{i+2} - t_{i+1})} \\ &\quad + \frac{(t_{i+4} - t)(t - t_{i+1})^2}{(t_{i+4} - t_{i+1})(t_{i+3} - t_{i+1})(t_{i+2} - t_{i+1})} \\ B_{i,i+2}(t) &= \frac{(t - t_i)(t_{i+3} - t)^2}{(t_{i+3} - t_i)(t_{i+3} - t_{i+1})(t_{i+3} - t_{i+2})} + \frac{(t_{i+4} - t)(t - t_{i+1})(t_{i+3} - t)}{(t_{i+4} - t_{i+1})(t_{i+3} - t_{i+1})(t_{i+3} - t_{i+2})} \\ &\quad + \frac{(t_{i+4} - t)^2(t - t_{i+2})}{(t_{i+4} - t_{i+1})(t_{i+4} - t_{i+2})(t_{i+3} - t_{i+2})} \\ B_{i,i+3}(t) &= \frac{(t_{i+4} - t)^3}{(t_{i+4} - t_{i+1})(t_{i+4} - t_{i+2})(t_{i+4} - t_{i+3})} \end{aligned}$$

and for any interval I of \mathbb{R} , \mathbb{I}_I is the function equal to 1 over I and to 0 outside I .

2 Walk generator

2.1 Input

The input of the walk generator is a sequence of time-stamped steps defined as follows:

1. p is a positive integer not smaller than 3,
2. $\tau_0, \tau_1, \dots, \tau_{2p-3}$ is an increasing sequence of real values,
3. $\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_{p-1}$ is a sequence of p elements of \mathbb{R}^2 representing the successive positions of the foot centers.
4. $\mathbf{c}_{init} \in \mathbb{R}^2$ is the initial position of the center of mass at time τ_0 ,
5. $\mathbf{c}_{final} \in \mathbb{R}^2$ is the final position of the center of mass at time τ_{2p-3} .

2.2 Reference trajectory of the center of pressure

We define a reference trajectory of the center of pressure called \mathbf{zmp}_{ref} as a continuous piecewise affine curve as follows:

$$\begin{aligned}
 \mathbf{zmp}_{ref}(\tau_0) &= \mathbf{c}_{init} \\
 \mathbf{zmp}_{ref}(\tau_1) &= \mathbf{s}_1 \\
 \mathbf{zmp}_{ref}(\tau_2) &= \mathbf{s}_1 \\
 \mathbf{zmp}_{ref}(\tau_3) &= \mathbf{s}_2 \\
 &\vdots \\
 \mathbf{zmp}_{ref}(\tau_{2p-5}) &= \mathbf{s}_{p-2} \\
 \mathbf{zmp}_{ref}(\tau_{2p-4}) &= \mathbf{s}_{p-2} \\
 \mathbf{zmp}_{ref}(\tau_{2p-3}) &= \mathbf{c}_{final}
 \end{aligned}$$

such that \mathbf{zmp}_{ref} restricted to each interval $[\tau_i, \tau_{i+1}]$ with $i \in \{0, 1, \dots, 2p-4\}$ is affine.

2.3 Trajectory of the center of mass

We restrict the trajectory of the center of mass to be a cubic-spline defined by Equation (1). We want

- the whole center of mass trajectory to be defined on interval $[\tau_0, \tau_{2p-3}]$, and
- $l \geq 1$ knots on each interval $[\tau_i, \tau_{i+1})$, $i \in \{0, \dots, 2p-3\}$.

We get the following relations between the various parameters:

$$\begin{aligned} m &= (2p-3)l+7 \\ t_3 &= \tau_0 \\ t_{m-4} &= \tau_{2p-3} \end{aligned}$$

$$\begin{array}{lll} \text{We set} & \begin{aligned} t_0 &= \tau_0 - 3 \\ t_1 &= \tau_0 - 2 \\ t_2 &= \tau_0 - 1 \end{aligned} & \begin{aligned} t_{m-3} &= \tau_{2p-3} + 1 \\ t_{m-2} &= \tau_{2p-3} + 2 \\ t_{m-1} &= \tau_{2p-3} + 3 \end{aligned} \end{array}$$

and

$$t_{3+k\,l+j} = \frac{l-j}{l} \tau_k + \frac{j}{l} \tau_{k+1} \quad \text{for } j \in \{0, \dots, l-1\}, \quad k \in \{0, \dots, 2p-4\}$$

2.4 Trajectory of the center of pressure

Let g be the gravity constant, and z the constant height of the center of mass. By denoting $\omega = \sqrt{g/z}$, we get the simplified formula for the center of pressure of the robot:

$$\mathbf{zmp} = \mathbf{x} - \frac{1}{\omega^2} \ddot{\mathbf{x}}$$

By setting

$$z_{i,3} \triangleq b_{i,3} - \frac{1}{\omega^2} \ddot{b}_{i,3}$$

we get an expression of the center of pressure with respect to the control points of the cubic B spline:

$$\mathbf{zmp} = \sum_{i=0}^{m-5} z_{i,3} \mathbf{x}_i = \sum_{i=0}^{2p-1} z_{i,3} \mathbf{x}_i \quad (2)$$

2.5 Optimal control problem

We denote by $X = (\mathbf{x}_0, \dots, \mathbf{x}_{m-5})$ the vector of control points. We wish to find the cubic B spline trajectory that minimizes the following cost function:

$$C(X) \triangleq \frac{1}{2} \int_{\tau_0}^{\tau_{2p-3}} \|\mathbf{zmp}(t) - \mathbf{zmp}_{ref}(t)\|^2 dt \quad (3)$$

Let us expand this formula using (2):

$$\begin{aligned}
C(X) &= \frac{1}{2} \int_{\tau_0}^{\tau_{2p-3}} \left(\sum_{i=0}^{2p-4} z_{i,3} \mathbf{x}_i - \mathbf{zmp}_{ref}(t) \right)^T \left(\sum_{i=0}^{2p-4} z_{i,3} \mathbf{x}_i - \mathbf{zmp}_{ref}(t) \right) dt \\
&= \frac{1}{2} \sum_{i=0}^{m-5} \sum_{j=0}^{m-5} \int_{\tau_0}^{\tau_{2p-3}} z_{i,3} z_{j,3}(t) dt \mathbf{x}_i^T \mathbf{x}_j - \sum_{i=0}^{m-5} \int_{\tau_0}^{\tau_{2p-3}} z_{i,3} \mathbf{zmp}_{ref}^T(t) dt \mathbf{x}_i \\
&\quad + \int_{\tau_0}^{\tau_{2p-3}} \mathbf{zmp}_{ref}^T \mathbf{zmp}_{ref}(t) dt
\end{aligned}$$

The cost function can be rewritten as

$$C(X) = \frac{1}{2} X^T H X - b^T X + C_0$$

with

$$\begin{aligned}
H &= \left(\int_{\tau_0}^{\tau_{2p-3}} z_{i,3} z_{j,3}(t) dt \ I_2 \right)_{i,j=0,\dots,m-5} \\
b &= \left(\int_{\tau_0}^{\tau_{2p-3}} z_{i,3} \mathbf{zmp}_{ref}(t) dt \right)_{i=0,\dots,m-5} \\
C_0 &= \int_{\tau_0}^{\tau_{2p-3}} \mathbf{zmp}_{ref}^T \mathbf{zmp}_{ref}(t) dt
\end{aligned}$$

$C(X)$ is the sum of two terms that respectively depend on the x and y coordinates of the control points. Let us denote $\mathbf{x}_{i,0}$, $\mathbf{x}_{i,1}$ the abscissa and the ordinate of \mathbf{x}_i , $\mathbf{zmp}_{ref\ 0}$, $\mathbf{zmp}_{ref\ 1}$ the abscissa and ordinate of \mathbf{zmp}_{ref} , and let us define

$$\begin{aligned}
X_0 &= (\mathbf{x}_{0,0}, \dots, \mathbf{x}_{m-5,0}) \\
X_1 &= (\mathbf{x}_{0,1}, \dots, \mathbf{x}_{m-5,1}) \\
b_0 &= \left(\int_{\tau_0}^{\tau_{2p-3}} z_{i,3} \mathbf{zmp}_{ref\ 0}(t) dt \right)_{i=0,\dots,m-5} \\
b_1 &= \left(\int_{\tau_0}^{\tau_{2p-3}} z_{i,3} \mathbf{zmp}_{ref\ 1}(t) dt \right)_{i=0,\dots,m-5} \\
H_0 = H_1 &= \left(\int_{\tau_0}^{\tau_{2p-3}} z_{i,3} z_{j,3}(t) dt \right)_{i,j=0,\dots,m-5}
\end{aligned}$$

Then,

$$C(X) = \frac{1}{2} X_0^T H_0 X_0 - b_0^T X_0 + \frac{1}{2} X_1^T H_1 X_1 - b_1^T X_1 + C_0$$

The problem can therefore be decomposed into two decoupled sub-problems, one in X_0 and the other one in X_1 .

2.5.1 Linear constraints

Boundary conditions can be added as a constraint on the value of the trajectory of the center of mass and its derivative at a given parameter – usually at the

beginning or at the end of the definition interval. Each of these constraints is defined by a tuple $(t, \mathbf{y}, \dot{\mathbf{y}}) \in [\tau_0, \tau_{2p-3}] \times \mathbb{R}^2 \times \mathbb{R}^2$ and is linear in the vector of control points.

$$\sum_{i=0}^{m-5} b_{i,3}(t) \mathbf{x}_i = \mathbf{y} \quad (4)$$

$$\sum_{i=0}^{m-5} \dot{b}_{i,3}(t) \mathbf{x}_i = \dot{\mathbf{y}} \quad (5)$$

These constraints can be translated to each sub-problem as follows:

$$\begin{aligned} A_0 X_0 &= c_0 \\ A_1 X_1 &= c_1 \end{aligned}$$

with

$$\begin{aligned} A_0 = A_1 &\triangleq \begin{pmatrix} b_{0,3}(t) & b_{1,3}(t) & \cdots & b_{m-5,3}(t) \\ \dot{b}_{0,3}(t) & \dot{b}_{1,3}(t) & \cdots & \dot{b}_{m-5,3}(t) \end{pmatrix} \\ c_0 &= \begin{pmatrix} \mathbf{y}_0 \\ \dot{\mathbf{y}}_0 \end{pmatrix} \\ c_1 &= \begin{pmatrix} \mathbf{y}_1 \\ \dot{\mathbf{y}}_1 \end{pmatrix} \end{aligned}$$

2.5.2 Resolution of the quadratic program

The constrained problem can be expressed as follows for $i \in 0, 1$:

$$\min_{X_i} \frac{1}{2} X_i^T H_i X_i - b_i^T X_i \quad \text{such that} \quad A_i X = c_i$$

using the singular value decomposition of A_i

$$A_i = \begin{pmatrix} U_1 & U_0 \end{pmatrix} \Sigma \begin{pmatrix} V_1 & V_0 \end{pmatrix}^T$$

we get a parameterization of the affine sub-space defined by the constraint:

$$X_i = X_{i0} + V_0 \mathbf{u} \quad \mathbf{u} \in \mathbb{R}^{m-4-\text{rank}(A_i)}$$

where $X_{i0} = A_i^+ c_i$. Solving the constrained QP consists in finding \mathbf{u} that minimizes

$$\begin{aligned} &\frac{1}{2} (X_{i0} + V_0 \mathbf{u})^T H_i (X_{i0} + V_0 \mathbf{u}) - b_i^T (X_{i0} + V_0 \mathbf{u}) \\ &= \frac{1}{2} \mathbf{u}^T V_0^T H_i V_0 \mathbf{u} + X_{i0}^T H_i V_0 \mathbf{u} - b_i^T V_0 \mathbf{u} + Cste \\ &= \frac{1}{2} \mathbf{u}^T V_0^T H_i V_0 \mathbf{u} + (V_0^T H_i X_{i0} - V_0^T b_i)^T \mathbf{u} + Cste \end{aligned}$$

The value of \mathbf{u} that minimizes the above expression is given by

$$\mathbf{u}_i^* = (V_0^T H_i V_0)^{-1} (V_0^T b_i - V_0^T H_i X_{i0})$$

2.6 Computation of the coefficients

$$\begin{aligned} z_{i,3} &= (Z_{i,i} \mathbb{I}_{[t_i, t_{i+1})} + Z_{i,i+1} \mathbb{I}_{[t_{i+1}, t_{i+2})} + Z_{i,i+2} \mathbb{I}_{[t_{i+2}, t_{i+3})} + Z_{i,i+3} \mathbb{I}_{[t_{i+3}, t_{i+4})}) \\ i &= 0, \dots, m-5 \end{aligned}$$

with

$$\begin{aligned} Z_{i,i}(t) &= B_{i,i} - \frac{1}{\omega^2} \ddot{B}_{i,i} \\ Z_{i,i+1}(t) &= B_{i,i+1} - \frac{1}{\omega^2} \ddot{B}_{i,i+1} \\ Z_{i,i+2}(t) &= B_{i,i+2} - \frac{1}{\omega^2} \ddot{B}_{i,i+2} \\ Z_{i,i+3}(t) &= B_{i,i+3} - \frac{1}{\omega^2} \ddot{B}_{i,i+3} \end{aligned}$$

Matrix H_0 is symmetric. For any integer i such that $0 \leq i \leq m-5$, and any non-negative integer k , such that $k \leq 3$ and $i+k \leq m-5$, The coefficient $(i, i+k)$ of matrix H_0 , with $k \in \{0, 1, 2, 3\}$ is equal to

$$\begin{aligned} H_{0 \ i, i+k} &= \int_{\tau_0}^{\tau_{2p-3}} z_{i,3} z_{i+k,3}(t) dt \\ &= \sum_{j=0}^{3-k} \int_{t_{i+k+j}}^{t_{i+k+j+1}} Z_{i,i+k+j} Z_{i+k,i+k+j}(t) dt \end{aligned}$$

The coefficients of vector b_0 are equal to:

$$\begin{aligned} b_{0i} &= \int_{t_3}^{t_{m-4}} (Z_{i,i} \mathbb{I}_{[t_i, t_{i+1})} + Z_{i,i+1} \mathbb{I}_{[t_{i+1}, t_{i+2})} + Z_{i,i+2} \mathbb{I}_{[t_{i+2}, t_{i+3})} + Z_{i,i+3} \mathbb{I}_{[t_{i+3}, t_{i+4})}) \mathbf{zmp}_{ref\ i}(t) dt \\ &= \sum_{j=\max(0, 3-i)}^{\min(3, m-5-i)} \int_{t_{i+j}}^{t_{i+j+1}} Z_{i,i+j} \mathbf{zmp}_{ref\ i}(t) dt \end{aligned}$$

Special cases for boundary conditions

If $t = \tau_0 = t_3$,

$$A_0 = A_1 = \begin{pmatrix} B_{0,0+3}(t_3) & B_{1,1+2}(t_3) & B_{2,2+1}(t_3) & 0 & \cdots & 0 \\ \dot{B}_{0,0+3}(t_3) & \dot{B}_{1,1+2}(t_3) & \dot{B}_{2,2+1}(t_3) & 0 & \cdots & 0 \end{pmatrix}$$

If $t = \tau_{2p-3} = t_{m-4}$

$$A_0 = A_1 = \begin{pmatrix} 0 & \cdots & 0 & B_{m-7, m-4}(t_{m-4}) & B_{m-6, m-4}(t_{m-4}) & B_{m-5, m-4}(t_{m-4}) \\ 0 & \cdots & 0 & \dot{B}_{m-7, m-4}(t_{m-4}) & \dot{B}_{m-6, m-4}(t_{m-4}) & \dot{B}_{m-5, m-4}(t_{m-4}) \end{pmatrix}$$