KRR - Conditional Independence in Bayesian Networks

Tudor Berariu, Alexandru Sorici

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1 Conditional Independence and D-separability

This tutorial focuses on the concepts of *conditional independence* and *d-separabilty* in Bayesian Networks. The definitions below skip a lot of details and do not offer proofs as their role is only to bridge the aforementioned notions.

A Bayesian Network is a probabilistic graphical model that compactly represents a joint distribution over a set of random variables $\mathcal{X}$. More precisely, a Bayesian Network is a directed acyclic graph $\mathcal{G}$ having a node for each random variable in $\mathcal{X}$. Each edge in $\mathcal{G}$ corresponds to a direct causal influence from one variable to another. One of the strengths of this representation is that it captures conditional independence assumptions about the random variables in $\mathcal{X}$.

We are interested in evaluating *conditional independence* assertions of the form $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$ where $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are all disjoint subsets of $\mathcal{X}$. The semantics of such an expression is *Assuming that* $\mathbf{Z}$ *is known would observing* $\mathbf{X}$ *bring any information (change our belief) about the variables in* $\mathbf{Y}$*?*.

The conditional independencies in $\mathcal{G}$ correspond to *d-separation* relations between variables: $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z}) \iff dsep_{\mathcal{G}}(\mathbf{X}; \mathbf{Y} \mid \mathbf{Z})$.

We state that $\mathbf{X}$ and $\mathbf{Y}$ are d-separated given $\mathbf{Z}$ if there are no *active paths* between any node $X \in \mathbf{X}$ and $Y \in \mathbf{Y}$ given $\mathbf{Z}$. In other words, $\mathbf{X}$ and $\mathbf{Y}$ are d-separated given $\mathbf{Z}$ if information does not flow between the two sets of random variables.

A path in $\mathcal{G}$: $X_1 - X_2 - X_3 - \ldots - X_N$ is active given $\mathbf{Z}$ if all its trails of two consecutive edges $X_{i-1} - X_i - X_{i+1}$ are active given $\mathbf{Z}$.

Simple rules can be applied to analyse if a trail $X_{i-1} - X_i - X_{i+1}$ is active given $\mathbf{Z}$:

- Causal trail: $X_{i-1} \rightarrow X_i \rightarrow X_{i+1}$ is active if $X_i \notin \mathbf{Z}$ (X_i is not observed);
- Evidential trail: $X_{i-1} \leftarrow X_i \leftarrow X_{i+1}$ is active if $X_i \notin \mathbf{Z}$ (X_i is not observed);
- Common cause: $X_{i-1} \leftarrow X_i \rightarrow X_{i+1}$ is active if $X_i \notin \mathbf{Z}$ (X_i is not observed);
- Common effect: $X_{i-1} \rightarrow X_i \leftarrow X_{i+1}$ is active if $X_i \in \mathbf{Z}$ or any of the descendants $D \in desc_{\mathcal{G}}(X_i) \in \mathbf{Z}$ (X_i or any of its descendants is observed);

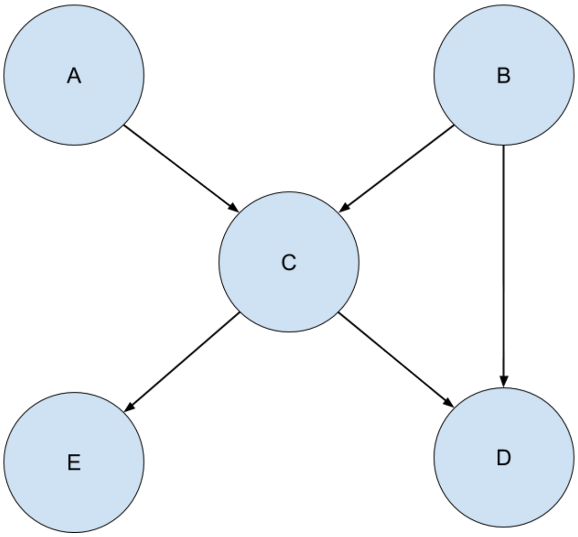
2 Tasks

Design an algorithm to check independence assumptions $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$ in a Bayesian Network. A simple idea would be to explore the graph $\mathcal{G}$ starting from the nodes in $\mathbf{X}$ following active trails given $\mathbf{Z}$. If at any point a node from $\mathbf{Y}$ is reached, then the independence does not hold. Otherwise, if $\mathbf{Z}$ blocks all the paths from $\mathbf{X}$ to $\mathbf{Y}$, d-separating the two sets of nodes, then the conditional independence holds.

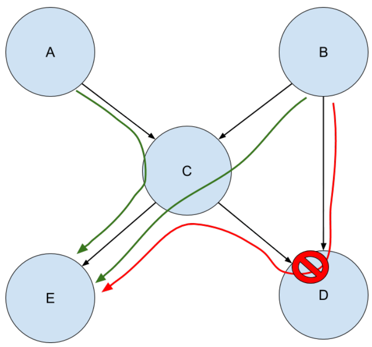
You are given test files containing the graph structure, various queries, and the correct answers. Such a file has the following structure ($1 + N + 2M$ lines):

- two positive numbers on the first line: N - the number of nodes in $\mathcal{G}$, and M - the number of queries;
- N lines with the name of each variable followed by the names of its parents;
- M lines containing queries expressed as a sequence of names from $\mathbf{X}$, the `‘‘;’’` symbol, the names from $\mathbf{Y}$, the `‘‘|’’` symbol and then the names of all observed variables from $\mathbf{Z}$;
- M lines containing either `‘‘true’’` or `‘‘false’’` corresponding to the correct answers of the M independence queries.

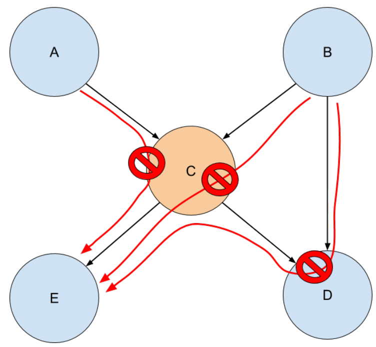
3 Example



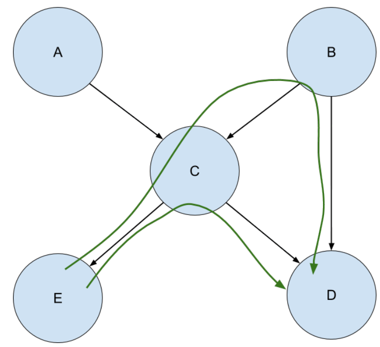

```
5 8
A
B
C A B
D C B
E C
A B ; E |
A B ; E | C
E ; D |
E ; D | B
E ; D | C
A ; B |
A ; B | C
A ; B | D
false
true
false
false
true
true
false
false
```



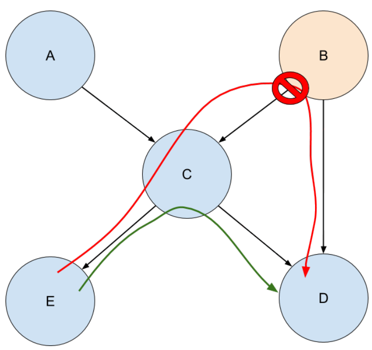
(a) $(\{A, B\} \perp \{E\})$



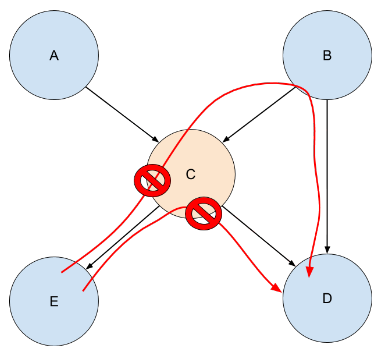
(b) $(\{A, B\} \perp \{E\} \mid \{C\})$



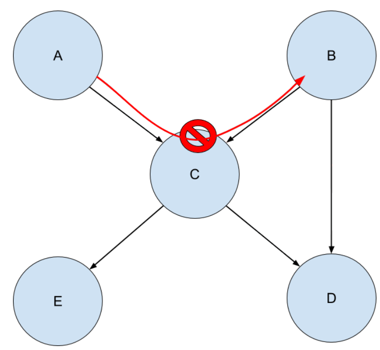
(c) $(\{E\} \perp \{D\})$



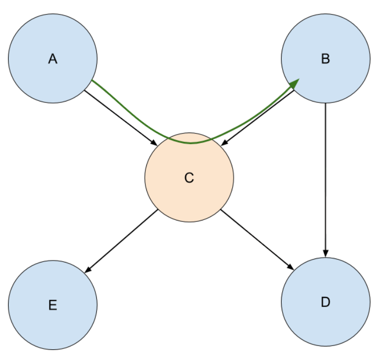
(d) $(\{E\} \perp \{D\} \mid \{B\})$



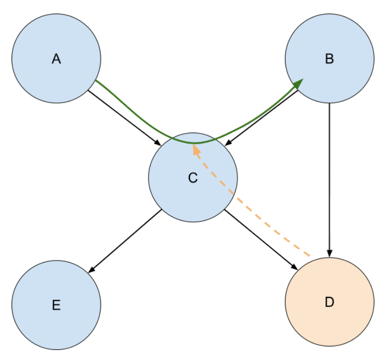
(e) $(\{E\} \perp \{D\} \mid \{C\})$



(f) $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$



(g) $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$



(h) $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$