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Chapter 4

The Soundness of PL

Chapter 0 provided an informal account of logic as the study of formal reasoning which was glossed as *what follows from what in virtue of logical form.* Rather than attempting to describe all of what follows from what in English which lacks a precise definition of a grammatical sentence, Chapter 1 introduced an artificial language $\mathcal{L}^{\text{PL}}$ in which we stipulated a definition of the well-formed sentences (wfss) of $\mathcal{L}^{\text{PL}}$. Given the definition of the wfss of $\mathcal{L}^{\text{PL}}$, Chapter 2 defined the interpretations of $\mathcal{L}^{\text{PL}}$ in order to provide a theory of logical consequence $\vDash$ which answered the question of what follows from what in $\mathcal{L}^{\text{PL}}$. Nevertheless, we did not say *how* one wfs follows from a set of wfss in $\mathcal{L}^{\text{PL}}$ since nothing in the theory of logical consequence described how to reason with the wfss of $\mathcal{L}^{\text{PL}}$. Chapter 3 filled this lacuna by identifying a collection of deduction rules which were both basic and natural, allowing us to draw inferences between the wfss of $\mathcal{L}^{\text{PL}}$. By considering any way of chaining together individual applications of these rules into a finite sequence of inferences, we defined what it is to derive a wfs from a set of wfss in $\mathcal{L}^{\text{PL}}$ where the derivation relation $\vdash$ asserts that there is at least one derivation of the wfs on the right from the set of wfss on the left.

Despite providing a completely different account of formal reasoning, Chapter 3 closed by asserting that the derivation relation $\vdash$ and the logical consequence relation $\vDash$ have the same extension. Given the extensional equivalence of these two relations together with the naturalness of the basic rules for PL, we have every right to take PL to provide an adequate natural deduction system for $\mathcal{L}^{\text{PL}}$. In order to establish the extensional equivalence of the derivation and logical consequence relations, this chapter will prove PL Soundness and the next chapter will prove PL Completeness. Although these results may appear to be very similar in form, they differ considerably in significance. For instance, suppose that we had provided a logic X where X Soundness failed to hold. This means that there is some line of reasoning that we could carry out in X, i.e., $\Gamma \vdash_X \varphi$, where the conclusion fails to be a logical consequence of the premises, i.e., $\Gamma \nvDash \varphi$. Assuming that logical consequence provides an accurate guide to *what* follows from what even if it does not say how, we may take a failure of X Soundness to disqualify X from providing an adequate logic for $\mathcal{L}^{\text{PL}}$. Put otherwise, X could not be relied upon to reason in $\mathcal{L}^{\text{PL}}$ since it is possible to begin with certain premises and reason to a conclusion that does not follow as a logical consequence.

Establishing PL SOUNDNESS shows that PL can be relied on to reason in $\mathcal{L}^{\text{PL}}$ without ever deriving a conclusion that does not follow as a logical consequence of the premises with which one begins. In order to appreciate the importance of PL SOUNDNESS, it is helpful to compare PL SOUNDNESS to an analogous property that we may expect any calculator to satisfy. In particular, any calculator must have the property that no matter what arithmetical operations you enter, if it gives you an answer, that answer is guaranteed to be a truth of arithmetic. Otherwise the calculator is not really a calculator at all but rather something more like a magic eight ball, turning up incorrect answers in an unpredictable manner.

Since we should like to be able to rely on PL in order to carry out reasoning in $\mathcal{L}^{\text{PL}}$, it is important to establish PL SOUNDNESS. This result belongs to METALOGIC insofar as it concerns the properties that our logical system PL may be said to possess. Given that our present aim is to show that PL can be relied upon by proving PL SOUNDNESS, it does not make sense to to use PL in order to prove PL SOUNDNESS since this would beg the question. Put otherwise, we cannot rely on PL to show that PL can be relied upon. Rather, the proofs of metalogic are developed in mathematical English in a similar manner to the semantic proofs that we provided in Chapter 2. That is the proofs in metalogic are INFORMAL in contrast to the FORMAL derivations in PL that we presented in Chapter 3.

In order to establish that PL SOUNDNESS holds for any set of wfss Γ and wfs φ of $\mathcal{L}^{\text{PL}}$, it is natural to consider an arbitrary Γ and φ for which $\Gamma \vdash \varphi$. It follows from the definition of the derivation relation that there is some PL derivation X where φ is the conclusion and Γ is the set of premises. Despite knowing that there is such a PL derivation as X, we cannot conclude much more than that, and so it is hard to see how we might show that $\Gamma \vDash \varphi$. In particular, we do not know how the derivation X proceeds, and so cannot say which wfss of $\mathcal{L}^{\text{PL}}$ are on which lines of X nor can we appeal to any of their justifications. Although X is finite, we do not know how long X is and so are left contemplating an infinite number of proofs of finite length, any one of which X might be. This is a common predicament.

One thought is to attempt a *reductio* style proof by assuming that $\Gamma \nvDash \varphi$ and attempting to derive a contradiction. Even so, we still do not have much to work with. In particular, we do not know what φ is or what Γ includes, and so the *reductio* assumption is of little help.

In order to overcome these challenges we will employ MATHEMATICAL INDUCTION which uses a recursive strategy for showing that $\Gamma \vDash \varphi$. In particular, we will show that every line of X is a logical consequence of the premises and undischarged assumptions of X at that line. Since the last line cannot have any undischarged assumptions, it follows that the last line of X is a logical consequence of just the premises. Before presenting the details of this proof, the following section will provide a detailed guide for writing clear and concise induction proofs. In addition to helping you to write your own induction proofs, this guide will help you to understand how the proof of PL SOUNDNESS works. If you are already familiar with mathematical induction, consider the following section a review.

4.1 Mathematical Induction

Step 1: Whenever a domain of objects has a recursive definition, it is natural to appeal to an induction proof in order to show that every object in that domain has a given property. Accordingly, we must identify the relevant domain of objects and the property which we are attempting to show is had by every object in that domain. In the case of PL Soundness, we will begin by assuming that $\Gamma \vdash \varphi$, and so there is a PL derivation X of φ from Γ. We will then present an induction argument that each line is a logical consequence of the premises and undischarged assumptions at that line. So the domain of objects in question are the lines of the derivation X of φ from Γ, and the property in question is *being a logical consequence of its premises and undischarged assumptions*. This brings to light what can be one of the trickiest part of an induction proof: not only is it important to accurately identify the relevant domain of objects, the property of interest must also be carefully chosen.

Step 2: We must now provide some way of organizing the domain into a sequence of stages. For instance, if our domain was the set of natural numbers, we might consider their natural ordering where every number in the sequence is followed by its successor. In the case of the PL derivation X, we will consider the sequence of lines that constitute X. Sometimes the ordering is not so obvious, or else one must reconsider the domain of objects such that they may be ordered in an manner which is advantageous.

Step 3: Next we will establish that the first stage has the property in question. This step is often called the *base case* of our induction proof. For instance, we may show that the first line of the derivation X follows from the premises and undischarged assumptions at that first line. Although the base case is often easy— sometimes so obvious it is hard to know what to write— this is not always the true, and so should not be dismissed.

Step 4: We will then help ourselves to an important assumption called the *induction hypothesis*. This assumption can come in both weak and strong varieties. Whereas *weak induction* assumes that the property in question holds for the n-th stage, *strong induction* assumes that the property in question holds for the n-th stage and all previous stages. For instance, below we will make the stronger assumption that every line $k \leq n$ is a logical consequence of the premises and undischarged assumptions at k.

Step 5: We will complete the induction proof by showing that the property in question also holds for the $n + 1$-th stage. If we can establish this claim, then it follows that the property in question holds for every stage. After all, we have shown that the property in question holds for the first stage, and that if property holds at (or up through) the n-th stage, then it holds for the $n + 1$-th stage. In the case of PL Soundness, we show that the $n + 1$-th line of X is a logical consequence of its premises and undischarged assumptions at that line.

This provides the rough outline of proof by induction with some reference to the induction proof for soundness. In actual practice, the hardest part about induction proofs is staying organized and figuring out which properties to focus on, since sometimes you can make things a lot easier by proving something related to what you really want to show.

4.2 Soundness

Although we could attempt to prove PL Soundness in one shot, it is common to break up long proofs into parts by establishing a number of supporting lemmas. In addition to making the over all structure of a proof easier to read, it is common for certain lemmas to be used again and again throughout different parts of a proof, or else in other proofs entirely, thereby reducing redundancy. Were a lemma to have significant and far reaching consequences that are of interest in their own right, we would do better to call it a *proposition* or even a *theorem*. For instance, it would be inappropriate to refer to PL Soundness as a lemma given the significance of this result. Although lemmas can often help to streamline the presentation of a proof, too many lemmas can clutter a proof that would have been better to present all at once. Knowing when to carve off a lemma to establish separately from a proof of primary interest is a skill in its own right, one that takes lots of practice to cultivate.

PL Soundness: Assume that $\Gamma \vdash \varphi$ for an arbitrary set Γ of wfss of $\mathcal{L}^{\text{PL}}$ and wfs φ of $\mathcal{L}^{\text{PL}}$. It follows that there is some PL derivation X of φ from Γ. Letting φ_i be the sentence on the i-th line of the derivation X and Γ_i be the set of premises that occur on any line $j \leq i$ of X together with the assumptions that are undischarged at line i, we may prove the following:

Lemma 4.1 (Base Step) $\Gamma_1 \vDash \varphi_1$.

> *Proof:* By the definition of a PL derivation, φ_1 is either a premise, an assumption that is eventually discharged, or follows by one of the natural deduction rules for PL besides AS. Since φ_1 is the first line of the proof, there are no earlier lines to be cited, and so φ_1 is either a premise or an assumption. Either way, $\Gamma_1 = \{\varphi_1\}$ since φ_1 is not discharged at the first line. As a result, $\Gamma_1 \vDash \varphi_1$ is immediate. □

Lemma 4.2 (Induction Step) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if $\Gamma_k \vDash \varphi_k$ for every $k \leq n$.

Given the lemmas above, it follows by strong induction that $\Gamma_n \vDash \varphi_n$ for all n. Since every proof is finite in length, there is a last line m of X where $\varphi_m = \varphi$ is the conclusion. By the definition of a PL derivation, we know that every assumption in X is eventually discharged, and so $\Gamma_m = \Gamma$ is the set of premises. Thus we may conclude that $\Gamma \vDash \varphi$. Discharging the assumption that $\Gamma \vdash \varphi$ and generalizing on Γ and φ completes the proof. □

Whereas **Lemma 4.1** is easy to prove, **Lemma 4.2** requires checking that all of the natural deduction rules for PL preserve logical consequence. Since there are twelve rules, this proof will require quite a bit more work. Having presented the over all structure of the proof of PL Soundness, the following section will fill in the missing details by proving **Lemma 4.2**.

4.3 Induction Step

In order to prove **Lemma 4.2**, assume for strong induction that $\Gamma_k \vDash \varphi_k$ for every $k \leq n$. It remains to show that $\Gamma_{n+1} \vDash \varphi_{n+1}$. By the definition of a PL derivation, φ_{n+1} is either a premise, assumption that is eventually discharged, or follows by a PL deduction rule besides AS. If φ_{n+1} is a premise, then $\Gamma_{n+1} \vDash \varphi_{n+1}$ for the same reason given in **Lemma 4.1**. Thus it remains to show that $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} has been justified by a deduction rule for PL. There are twelve rule in all, and so we must check each case. The following subsections will attend to this task, establishing a number of supporting lemmas along the way.

4.3.1 Assumption and Reiteration

Before attending to the introduction and elimination rules for each of the sentential operators included in PL, this section focuses on the assumption and reiteration rules. Whereas the proofs for most of the rules will appeal to the induction hypothesis assumed above, the proof for the assumption rule is an exception, employing the same reasoning given in **Lemma 4.1**.

Rule 1 (AS) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by AS.

Proof: Assume that φ_{n+1} is justified by AS. Since φ_{n+1} is an undischarged assumption at line $n+1$, it follows from the definition of Γ_{n+1} that $\varphi_{n+1} \in \Gamma_{n+1}$, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$ follows immediately. □

The proof above does not require the induction hypothesis or any additional results. By contrast, it will help to establish the reiteration rule by first proving the following lemma.

Lemma 4.3 If φ_k is live at line n of a PL derivation where $k \leq n$, then $\Gamma_k \subseteq \Gamma_n$.

Proof: Let X be a PL derivation where Γ_k is the set of premises and undischarged assumptions at line k. Assume there is some $\psi \in \Gamma_k$ where $\psi \notin \Gamma_n$ for $n > k$. It follows that ψ has been discharged at a line $j > k$ where $j \leq n$, and so φ_k is dead at n. By contraposition, if φ_k is live at line $n > k$, then $\Gamma_k \subseteq \Gamma_n$ as desired. □

Although the proof is short, the lemma above makes an important observation about how the undischarged assumptions of live lines are inherited. As we will see, this lemma plays a critical role throughout many of the following proofs and so is important to understand. Given this lemma, we may now move to establish the reiteration rule R.

Rule 2 (R) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by R.

Proof: Assume that φ_{n+1} is justified by R. It follows that $\varphi_{n+1} = \varphi_k$ for some $k \leq n$, and so $\Gamma_k \vDash \varphi_k$ by hypothesis. Since φ_k is live at line $n+1$, $\Gamma_k \subseteq \Gamma_{n+1}$ by **Lemma 4.3**, and so $\Gamma_{n+1} \vDash \varphi_k$ by **Lemma 2.1**. Thus $\Gamma_{n+1} \vDash \varphi_{n+1}$.

By contrast with the assumption rule, the reiteration makes an essential appeal to the induction hypothesis. We will see something similar in all of the rule proofs given below.

4.3.2 Negation Rules

The negation rules are much more complicated than the assumption and reiteration rules on account of citing subproofs rather than individual lines. Accordingly, it will help to establish two supporting lemmas before presenting the proofs for the negation rules. Whereas the first lemma asserts that a satisfiable set of wfss of $\mathcal{L}^{\text{PL}}$ cannot have both a wfs of $\mathcal{L}^{\text{PL}}$ and its negation logical consequences, the second lemma draws a connection between logical consequence and unsatisfiability. These lemmas work nicely together and will reoccur in a number of rule proofs besides the negation rule proofs given below.

Lemma 4.4 If $\Gamma \vDash \varphi$ and $\Gamma \vDash \neg\varphi$, then Γ is unsatisfiable.

Proof: Assume $\Gamma \vDash \varphi$ and $\Gamma \vDash \neg\varphi$. Assume for contradiction that Γ is satisfiable, and so there is some $\mathcal{L}^{\text{PL}}$ interpretation $\mathcal{I}$ where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$. By assumption, it follows that $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$. By the semantics for negation, $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$, contradicting the above. Thus Γ is unsatisfiable. □

Lemma 4.5 $\Gamma \cup \{\varphi\}$ is unsatisfiable just in case $\Gamma \vDash \neg\varphi$.

Proof: Assume $\Gamma \cup \{\varphi\}$ is unsatisfiable and let $\mathcal{I}$ be an arbitrary $\mathcal{L}^{\text{PL}}$ interpretation where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$. Assume for contradiction that $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 0$. It follows that $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$, and so $\Gamma \cup \{\varphi\}$ is satisfiable contrary to assumption. Thus $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$. Generalizing on $\mathcal{I}$, it follows that $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$ for any $\mathcal{I}$ where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$. By definition, $\Gamma \vDash \neg\varphi$.

Assume instead that $\Gamma \cup \{\varphi\}$ is satisfiable. It follows that there is some $\mathcal{L}^{\text{PL}}$ interpretation $\mathcal{I}$ where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma \cup \{\varphi\}$, and so $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$ by the semantics for negation. Since $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$, it follows that $\Gamma \nvDash \neg\varphi$. Thus we may conclude by contraposition that $\Gamma \cup \{\varphi\}$ is unsatisfiable if $\Gamma \vDash \neg\varphi$. □

Given the lemmas above, we may provide the following negation rule proofs. It will be important to study this proof carefully, observing how all the working parts come together.

> **Rule 3 (¬I)** $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by ¬I.

Proof: Assume that φ_{n+1} follows by ¬I. Thus there is some subproof on lines i-j where $i < j \leq n$ and $\varphi_{n+1} = \neg\varphi_i$, $\psi = \varphi_h$, and $\neg\psi = \varphi_k$ for $i \leq h \leq j$ and $i \leq k \leq j$. By parity of reasoning, we may assume that $h < k = j$. Thus we may represent the subproof as follows:

$$
\begin{array}{l|l|ll}
i & & \underline{\varphi} & :\text{AS for } \neg\text{I} \\
h & & \psi & \\
j & & \neg\psi & \\
n+1 & \neg\varphi & & :i\text{–}j\ \neg\text{I}
\end{array}
$$

By hypothesis, $\Gamma_h \vDash \psi$ and $\Gamma_j \vDash \neg\psi$. With the exception of $\varphi_i = \varphi$, every assumption that is undischarged at lines h and j are also undischarged at line $n+1$. It follows that $\Gamma_h, \Gamma_j \subseteq \Gamma_{n+1} \cup \{\varphi_i\}$, and so $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \psi$ and $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \neg\psi$ by **Lemma 2.1**. By **Lemma 4.4**, $\Gamma_{n+1} \cup \{\varphi_i\}$ is unsatisfiable, and so $\Gamma_{n+1} \vDash \neg\varphi_i$ by **Lemma 4.5**. Equivalently, $\Gamma_{n+1} \vDash \varphi_{n+1}$. □

The proof begins by assuming φ_{n+1} follows from Γ_{n+1} by negation introduction ¬I and unpacking the consequences. This provides a number of details about the proof that are required for $\neg\varphi$ to be derived on line $n+1$ by ¬I. In particular, we know that ψ and $\neg\psi$ must occur on earlier lines in a subproof. After appealing to the induction hypothesis to conclude $\Gamma_h \vDash \psi$ and $\Gamma_j \vDash \neg\psi$, the proof observes that although φ has been discharged by line $n+1$, this is the only difference between the sets of undischarged sentences for lines i–j and line $n+1$. Thus the logical consequences $\Gamma_h \vDash \psi$ and $\Gamma_j \vDash \neg\psi$ may be related to the undischarged assumptions at line $n+1$ together with the assumption φ which has been discharged at $n+1$. The core of the proof follows from the two lemmas given above which show that the undischarged assumptions at $n+1$ together with φ are unsatisfiable, and so $\neg\varphi$ is a logical consequences of those undischarged assumptions.

Before moving on to consider the rest of the rule proofs, it can help to try writing the proof for yourself. You might also give the following proof a try which works in a similar manner. Getting a good understanding of how these proofs work will make reading the rest of the proofs in this chapter a lot easier and more meaningful than they would be otherwise.

Rule 4 (¬E) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by ¬E.

Proof: This proof is left as an exercise for the reader. □

4.3.3 Conjunction and Disjunction

Whereas the rule proofs given above for negation drew on two lemmas established for just this purpose, the rule proof for conjunction introduction is straightforward:

Rule 5 (&I) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by ∧I.

Proof: Assume that φ_{n+1} is justified by ∧I. Thus $\varphi_{n+1} = \varphi_i \wedge \varphi_j$ for some lines $i, j \leq n$ that are live at line $n+1$. By hypothesis, $\Gamma_i \vDash \varphi_i$ and $\Gamma_j \vDash \varphi_j$ where $\Gamma_i, \Gamma_j \subseteq \Gamma_{n+1}$ by **Lemma 4.3**. Thus $\Gamma_{n+1} \vDash \varphi_i$ and $\Gamma_{n+1} \vDash \varphi_j$ by **Lemma 2.1**. Letting $\mathcal{I}$ be an arbitrary $\mathcal{L}^{\text{PL}}$ interpretation where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma_{n+1}$, it follows that $\mathcal{V}_{\mathcal{I}}(\varphi_i) = \mathcal{V}_{\mathcal{I}}(\varphi_j) = 1$ and so $\mathcal{V}_{\mathcal{I}}(\varphi_i \wedge \varphi_j) = \mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$. By generalizing on $\mathcal{I}$, we may conclude that $\Gamma_{n+1} \vDash \varphi_{n+1}$. □

In addition to drawing on the induction hypothesis, the proof above makes an essential appeal to the semantic clause for conjunction. The rule proofs for conjunction elimination and disjunction introduction work by similar reasoning, and so have been left as exercises.

Rule 6 (&E) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by ∧E.

Proof: This proof is left as an exercise for the reader. □

Rule 7 (∨I) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule ∨I.

Proof: This proof is left as an exercise for the reader. □

Given the induction hypothesis, the rule proofs above amount to little more than applications of the semantic clauses for conjunction and disjunction respectively. Something similar may be said for the rule proof for disjunction elimination though a little more care is required to keep track of all of the moving parts, and so the details have been provided in full.

Rule 8 ($\vee$E) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule $\vee$E.

Proof: Assume that φ_{n+1} is justified by $\vee$I. Thus there is some $\varphi_i = \varphi_j \vee \varphi_h$ which is live at $n+1$ and subproofs on lines j-h and k-l where $i < j, k, h, l \leq n$ and $\varphi_k = \varphi_l = \varphi_{n+1}$. By parity of reasoning, we represent the proof as follows:

i	$\varphi \vee \psi$	
j	$\quad\varphi$	:AS for $\vee$E
k	$\quad\chi$	
h	$\quad\psi$	:AS for $\vee$E
l	$\quad\chi$	
$n+1$	χ	:i, j–k, h–l $\vee$E

By hypothesis, $\Gamma_i \vDash \varphi_i$, $\Gamma_k \vDash \varphi_k$, and $\Gamma_l \vDash \varphi_l$. Given **Lemma 4.3**, $\Gamma_i \subseteq \Gamma_{n+1}$, and so $\Gamma_{n+1} \vDash \varphi_i$ by **Lemma 2.1**. With the exception of $\varphi_j = \varphi$, every assumption that is undischarged at line k is also undischarged at line $n+1$, and so $\Gamma_k \subseteq \Gamma_{n+1} \cup \{\varphi_j\}$. By the similar reasoning, we may conclude that $\Gamma_l \subseteq \Gamma_{n+1} \cup \{\varphi_h\}$, and so $\Gamma_{n+1} \cup \{\varphi_j\} \vDash \varphi_k$ and $\Gamma_{n+1} \cup \{\varphi_h\} \vDash \varphi_l$ follows by **Lemma 2.1**.

Letting $\mathcal{I}$ be an arbitrary $\mathcal{L}^{\text{PL}}$ interpretation where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma_{n+1}$, it follows from the above that $\mathcal{V}_{\mathcal{I}}(\varphi_i) = \mathcal{V}_{\mathcal{I}}(\varphi_j \vee \varphi_h) = 1$, and so either $\mathcal{V}_{\mathcal{I}}(\varphi_j) = 1$ or $\mathcal{V}_{\mathcal{I}}(\varphi_h) = 1$ by the semantics for disjunction.

Case 1: If $\mathcal{V}_{\mathcal{I}}(\varphi_j) = 1$, then $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma_{n+1} \cup \{\varphi_j\}$, and so $\mathcal{V}_{\mathcal{I}}(\varphi_k) = 1$ since $\Gamma_{n+1} \cup \{\varphi_j\} \vDash \varphi_k$. Thus $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$ since $\varphi_k = \varphi_{n+1}$.

Case 2: If $\mathcal{V}_{\mathcal{I}}(\varphi_h) = 1$, then $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma_{n+1} \cup \{\varphi_h\}$, and so $\mathcal{V}_{\mathcal{I}}(\varphi_l) = 1$ since $\Gamma_{n+1} \cup \{\varphi_h\} \vDash \varphi_l$. Thus $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$ since $\varphi_l = \varphi_{n+1}$.

In either case, $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$ by generalizing on $\mathcal{I}$. □

As with the previous two deduction rules for conjunction and disjunction, the proof above turns on little more than an application of the semantics for disjunction given the induction hypothesis. Nevertheless, it is very easy for parts to become tangled and a lot of care is required to write a proof that is both clear and concise for your reader.

4.3.4 Conditional Rules

In order to streamline the rule proof for $\rightarrow$I, it will help to prove the following.

Lemma 4.6 If $\Gamma \cup \{\varphi\} \vDash \psi$, then $\Gamma \vDash \varphi \supset \psi$.

Proof: Assume $\Gamma \cup \{\varphi\} \vDash \psi$ and let $\mathcal{I}$ be an arbitrary $\mathcal{L}^{\text{PL}}$ interpretation where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$. Since $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ or not, there are two cases to consider.

Case 1: If $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$, then $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma \cup \{\varphi\}$, and so $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ given the starting assumption. Thus $\mathcal{V}_{\mathcal{I}}(\varphi \rightarrow \psi) = 1$ by the semantics for the conditional.

Case 2: If $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$, then $\mathcal{V}_{\mathcal{I}}(\varphi \rightarrow \psi) = 1$ by the semantics for the conditional.

Since $\mathcal{V}_{\mathcal{I}}(\varphi \rightarrow \psi) = 1$ in both cases, $\Gamma \vDash \varphi \supset \psi$ follows by generalizing on $\mathcal{I}$. □

Rule 9 ($\rightarrow$I) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by $\rightarrow$I.

Proof: Assume that φ_{n+1} is justified by $\rightarrow$I. Thus there is a subproof on lines i-j where $i < j \leq n$ and $\varphi_{n+1} = \varphi_i \rightarrow \varphi_j$. We may represent the subproof as follows:

i	$\quad\vert\ \varphi$	:AS for $\rightarrow$I
j	$\quad\vert\ \psi$	
$n+1$	$\varphi \rightarrow \psi$	:i–j $\rightarrow$I

By hypothesis, we know that $\Gamma_j \vDash \varphi_j$. With the exception of φ_i, every assumption that is undischarged at line j is also undischarged at line $n+1$. It follows that $\Gamma_j \subseteq \Gamma_{n+1} \cup \{\varphi_i\}$, and so $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \varphi_j$ by **Lemma 2.1**. Thus $\Gamma_{n+1} \vDash \varphi_i \rightarrow \varphi_j$ by **Lemma 4.6**. Equivalently, $\Gamma_{n+1} \vDash \varphi_{n+1}$. □

Whereas the proof above appealed to **Lemma 4.6**, the following proof proceeds in a similar manner to the proofs given above, and so the details have been left as an exercise.

Rule 10 ($\rightarrow$E) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by $\rightarrow$E.

Proof: This proof is left as an exercise for the reader. □

Rule 11 ($\leftrightarrow$I) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by $\leftrightarrow$I.

Proof: Assume φ_{n+1} is justified by $\leftrightarrow$E. Thus there are some subproofs on lines i-j and h-k for some $i < j \leq n$ and $h < k \leq n$ where $\varphi_i = \varphi_k = \varphi$, $\varphi_j = \varphi_h = \psi$, and either $\varphi_{n+1} = \varphi \leftrightarrow \psi$ or $\varphi_{n+1} = \psi \leftrightarrow \varphi$. By parity of reasoning, we may assume that $\varphi_{n+1} = \varphi \leftrightarrow \psi$. Thus we have:

$$
\begin{array}{l|l|l}
i & \quad \varphi & \text{:AS for } \vee\text{E} \\
j & \quad \psi & \\
h & \quad \psi & \text{:AS for } \vee\text{E} \\
k & \quad \varphi & \\
n+1 & \varphi \leftrightarrow \psi & :i\text{–}j,\ h\text{–}k \leftrightarrow\text{I}
\end{array}
$$

By hypothesis, $\Gamma_j \vDash \varphi_j$, $\Gamma_k \vDash \varphi_k$, and $\Gamma_{n+1} \vDash \varphi_{n+1}$. With the exception of φ_i, every assumption that is undischarged at line j is also undischarged at line $n+1$, and so $\Gamma_j \subseteq \Gamma_{n+1} \cup \{\varphi_i\}$. Similarly, we may conclude that $\Gamma_k \subseteq \Gamma_{n+1} \cup \{\varphi_h\}$, and so $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \varphi_j$ and $\Gamma_{n+1} \cup \{\varphi_h\} \vDash \varphi_k$ by **Lemma 2.1**.

Let $\mathcal{I}$ be an arbitrary $\mathcal{L}^{\text{PL}}$ interpretation where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma_{n+1}$. Assuming $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$, it follows that $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma_{n+1} \cup \{\varphi_i\}$, and so $\mathcal{V}_{\mathcal{I}}(\varphi_j) = 1$ given that $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \varphi_j$. Thus $\mathcal{V}_{\mathcal{I}}(\psi) = 1$. Assuming instead that $\mathcal{V}_{\mathcal{I}}(\psi) = 1$, it follows that $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma_{n+1} \cup \{\varphi_h\}$, and so $\mathcal{V}_{\mathcal{I}}(\varphi_k) = 1$ given that $\Gamma_{n+1} \cup \{\varphi_h\} \vDash \varphi_k$. Thus $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$. We may then conclude that $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ if and only if $\mathcal{V}_{\mathcal{I}}(\psi) = 1$, and so $\mathcal{V}_{\mathcal{I}}(\varphi) = \mathcal{V}_{\mathcal{I}}(\psi)$. By the semantics for the biconditional, $\mathcal{V}_{\mathcal{I}}(\varphi \leftrightarrow \psi) = 1$, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$ by generalizing on $\mathcal{I}$. □

Rule 12 ($\leftrightarrow$E) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} is justified by $\leftrightarrow$E.

Proof: This proof is left as an exercise for the reader. □

Given **Rule 1** – **Rule 12**, it follows that $\Gamma_{n+1} \vDash \varphi_{n+1}$ no matter how φ_{n+1} has been derived, thereby completing the proof of **Lemma 4.2** as well as the proof of PL SOUNDNESS. Accordingly, we may carry out reasoning in PL while remaining confident that the our derivations preserve logical consequence, and so there is no risk of using PL to reason from some premises to a conclusion that does not follow as a logical consequence.

4.4 Derived Rules

Having established PL Soundness, we may now proceed to put this theorem to work. In particular, we may explore the range of logical consequences without having to write semantic proofs. Rather, we can use PL in order to write derivations where each conclusion follows as a logical consequences from its premises given PL Soundness.

Suppose that we have managed to construct a PL derivation, for instance that $\neg A \vdash \neg(A \wedge B)$. Even though this derivation is written in terms of particular wfss of $\mathcal{L}^{\text{PL}}$, we could have written a similar derivation by substituting any wfss of $\mathcal{L}^{\text{PL}}$ for 'A' and 'B'. Thus instead of merely asserting that $\neg A \vdash \neg(A \wedge B)$, we may wish to assert the schema $\neg\varphi \vdash \neg(\varphi \wedge \psi)$ where φ and ψ are any wfss of $\mathcal{L}^{\text{PL}}$ whatsoever. More generally, given any particular derivation, we may assert a generalization by replacing the sentence letters with schematic variables, referring to the result as a rule schema or, what is also often called a derived rule.

The reason it makes sense to refer to the schematic generalizations of particular derivations as *rules* at all is that although they have not been included as basic rules of the proof system PL, they may be used in much the same way as the basic rules are used. This is because anything that can be proven with a derived rule can also be proven using just the basic rules included in PL. Accordingly, we may think of the derived rules as abbreviating subroutines which only appeal to the basic rules of PL. Derived rules can then be used to shorten proofs, making some proofs easier to write and more intuitive to read.

Given PL Soundness, derived rules may also be used to indicate logical consequences that are of interest, bringing the vast range of logical consequences that there are into better view. Nevertheless, little is to be gained be restating every derived rule of the form $\Gamma \vdash \varphi$ as a logical consequence of the form $\Gamma \vDash \varphi$. Rather, this much is understood given PL Soundness. Moreover, as brought out above, soundness is an absolutely essential property of any proof system of interest, and so it goes without saying that the derivations in a proof system indicate a corresponding range of logical consequences.

With these general points in order, we may now turn to provide a range of derived rules in PL. Despite being derived rather than basic, many of the derived rules will look familiar, capturing standard ways of reasoning. In addition to shedding light on the logical consequence relation for $\mathcal{L}^{\text{PL}}$, these rules will help to write tricky proofs since they may be cited much like the basic rules, vastly simplifying otherwise lengthy derivations within PL.

4.4.1 *Modus Tollens*

Modus tollens is an extremely important and common inference rule in ordinary reasoning. Here is the derived rule for *modus tollens* (MT):

m	$\varphi \to \psi$	
n	$\neg\psi$	
	$\neg\varphi$	:m, n MT

If you have a conditional on one numbered line and the negation of its consequent on another line, you may derive the negation of its antecedent on a new line. We abbreviate the justification for this rule as 'MT' for *modus tollens*. For instance, if you know that if Sue found the treasure, then she is happy, and you also know that Sue isn't happy, then you can infer that Sue didn't find the treasure. Inferences of this form should feel familiar.

In order to derive MT from our basic rules, we will construct a derivation in the manner above while using schematic variables instead of wfss of $\mathcal{L}^{\text{PL}}$. Consider the following:

1	φ	
2	$\psi \to \neg\varphi$	(Want: $\neg\psi$)
3	$\quad \psi$	:AS for $\neg$I
4	$\quad \neg\varphi$	:2, 3 $\to$E
5	$\quad \varphi$	:1 R
6	$\neg\psi$	:3–5 $\neg$I

Since φ and ψ are schematic variables, the lines above do not constitute a PL derivation. Rather, what we have above is a DERIVATION SCHEMA which is a kind of recipe for constructing derivations. Given any wfss φ and ψ, the derivation schema for MT returns a PL derivation as an instance. Accordingly, applications of MT can always be replaced with an appropriate instance of the derivation schema for MT which only refers to the basic rules included in PL. Nevertheless, MT is a convenient shortcut and so we will add it to our list of derived rules.

Here is a simple example that would have been much more cumbersome without using MT:

1	$A \to B$	
2	$B \to C$	
3	$C \to D$	
4	$\neg D$	(Want: $\neg A$)
5	$\neg C$	:3, 4 MT
6	$\neg B$	:2, 5 MT
7	$\neg A$	:1, 6 MT

4.4.2 Dilemma

One of the most difficult deduction rules to apply is disjunction elimination, and so it will be convenient to derive deduction rules that streamline arguments from disjunctive sentences. Consider the *dilemma rule* (DL):

m	$\varphi \vee \psi$	
n	$\varphi \to \chi$	
o	$\psi \to \chi$	
	χ	:m, n, o DL

If you know that two conditionals are true, and they have the same consequent, and you also know that one of the two antecedents is true, then the conclusion is true no matter which antecedent is true. We may derive this rule as follows:

1	$\varphi \vee \psi$	
2	$\varphi \to \chi$	
3	$\psi \to \chi$	want χ
4	$\quad \varphi$	:AS
5	$\quad \chi$	:2, 4 $\to$E
6	$\quad \psi$	:AS
7	$\quad \chi$	:3, 6 $\to$E
8	χ	:1, 4–5, 6–7 $\vee$E

Whereas $\vee$E cites subproofs, DL only appeals to live lines in a proof, and so may be easier to apply in certain contexts. For example, suppose you know all of the following:

A1. If it is raining, the car is wet.
A2. If it is snowing, the car is wet.
A3. It is raining or it is snowing.

From these premises, you can definitely establish that the car is wet. This is an example of the argument form that DL captures, nicely describing a common way of reasoning.

As in the case of MT, the DL rule doesn't allow us to prove anything we couldn't prove via basic rules. Anytime you wanted to use the DL rule, you could always include a few extra steps to prove the same result without DL. Nevertheless, DL captures an natural form of reasoning in its own right, and so is well worth including in our stock of derived rules.

4.4.3 Disjunctive Syllogism

Although DL is occasionally useful, there other common forms of reasoning from a disjunction which DL does not capture. In particular, consider the following argument.

B1. $P \vee Q$
B2. $\neg P$
B3. Q

Even small children and non-human animals can engage in reasoning of the form given above. For instance, if a ball is under one of two cups but you don't know which, and then it is revealed that it is not under one of the cups, it is natural to conclude that the ball must be under the other cup. This inference is called *disjunctive syllogism* (DS):

m	$\varphi \vee \psi$	
n	$\neg\psi$	
	φ	:m, n DS

m	$\varphi \vee \psi$	
n	$\neg\varphi$	
	ψ	:m, n DS

We represent two different inference patterns here, because the rule allows you to conclude *either* disjunct from the negation of the other. Nevertheless, both go by the same name as is the case for other symmetrical rules like $\wedge$E. The derivations for DS go as follows:

1	$\varphi \vee \psi$	
2	$\neg\varphi$	
3	$\quad \varphi$	:AS for $\vee$E
4	$\quad\quad \neg\psi$	:AS for $\neg$E
5	$\quad\quad \neg\varphi$	:2 R
6	$\quad\quad \varphi$	:3 R
7	$\quad \psi$	:4–6 $\neg$I
8	$\quad \psi$	:AS for $\vee$E
9	$\quad \psi$	:8 R
10	ψ	:1, 3–7, 8–9 $\vee$E

1	$\varphi \vee \psi$	
2	$\neg\psi$	
3	$\quad \psi$	:AS for $\vee$E
4	$\quad\quad \neg\varphi$	:AS for $\neg$E
5	$\quad\quad \neg\psi$	:2 R
6	$\quad\quad \psi$	:3 R
7	$\quad \varphi$	:4–6 $\neg$I
8	$\quad \varphi$	:AS for $\vee$E
9	$\quad \varphi$	:8 R
10	φ	:1, 3–7, 8–9 $\vee$E

Like DL, the derived rule DS makes it easier to write derivations while capturing a natural way of reasoning. In order to put DS to work, consider the following derivation:

1	$\neg L \to (J \vee L)$	
2	$\neg L$	(Want: J)
3	$J \vee L$	:1, 2 $\to$E
4	J	:2, 3 DS

It is easy to see that $J \vee L$ follows by $\to$E from the two premises, but it is difficult to see how the proof will go next were we constrained to the basic rules. However, given DS, it is plain to see that J follows immediately from $J \vee L$ and $\neg L$. So the proof is easy.

4.4.4 Hypothetical Syllogism

We also add *hypothetical syllogism* (HS) as a derived rule:

m	$\varphi \to \psi$	
n	$\psi \to \chi$	
	$\varphi \to \chi$	:m, n HS

Note that HS does not cite any subproofs, and so makes for elegant proofs that are easy to read. The same cannot be said for the derivation schema for HS:

1	$\varphi \to \psi$	
2	$\psi \to \chi$	
3	$\quad \varphi$	:AS
4	$\quad \psi$	:1, 3 $\to$E
5	$\quad \chi$	:2, 4 $\to$E
6	$\varphi \to \chi$	:3–5 $\to$I

4.4.5 Contraposition

Next we may add *contraposition* (CP) as a derived rule:

m	$\varphi \to \psi$	
	$\neg\varphi \to \neg\psi$	:m CP

Not only is this inference natural, it is extremely useful. We have had various occasions to use CP in the informal proofs given above. The derivation in PL goes as follows:

1	$\varphi \to \psi$	
2	$\quad \neg\psi$	:AS
3	$\quad\quad \varphi$	:AS
4	$\quad\quad \psi$	:1, 3 $\to$E
5	$\quad\quad \neg\psi$	:2 R
6	$\quad \neg\varphi$	:3–5 $\neg$I
7	$\neg\psi \to \neg\varphi$	:2–6 $\to$I

Whereas the proof above involves two subproofs, one embedded in the other, applications of CP directly cite live lines of a proof, greatly simplifying the resulting argument.

4.4.6 Negative Biconditionals

Biconditional elimination only works when we have a biconditional together with one of the arguments of the biconditional on live lines. However, it in cases where we have the negation of one of the arguments of a biconditional, it is convenient to make use of the following derived rule for *negative biconditionals* (NB):

m	$\varphi \leftrightarrow \psi$	
n	$\neg\varphi$	
	$\neg\psi$	:m, n NB

m	$\varphi \leftrightarrow \psi$	
n	$\neg\psi$	
	$\neg\varphi$	:m, n NB

The derivations for NB go as follows:

1	$\varphi \leftrightarrow \psi$	
2	$\neg\varphi$	
3	$\quad \psi$	:AS
4	$\quad \varphi$	:1, 3 $\leftrightarrow$E
5	$\quad \neg\varphi$	:2 R
6	$\neg\psi$	:3–5 $\neg$I

1	$\varphi \leftrightarrow \psi$	
2	$\neg\psi$	
3	$\quad \varphi$	:AS
4	$\quad \psi$	:1, 3 $\leftrightarrow$E
5	$\quad \neg\psi$	:2 R
6	$\neg\varphi$	:3–5 $\neg$I

4.4.7 Double Negation

Whereas we have included two similar rules for negation introduction and elimination, some texts only include negation introduction together with the following rule for *double negation elimination* (DN):

$$
\begin{array}{l|ll}
m & \neg\neg\varphi & \\
 & \varphi & :m \text{ DN}
\end{array}
$$

Although some philosophers of logic contest DN, arguing instead for *intuitionistic logics* in which DN is neither basic nor derivable, most take DN to be a useful and extremely natural inference to draw. After all, what is meant by saying that it is not the case that the ball is not round, and yet it fails to be the case that the ball is round? Or to take the converse, what is meant by saying that the ball is round, but it fails to be the case that the ball is not not round. The classical logician may claim that there is no difference at all here by accepting DN. Although, DN is a derived rule in PL rather than basic, this much is only a difference in convention. Here is the derivation of DN in the present system PL:

$$
\begin{array}{l|ll}
1 & \neg\neg\varphi & \\
\hline
3 & \quad \neg\varphi & :\text{AS for } \neg\text{E} \\
4 & \quad \neg\neg\varphi & :1 \text{ R} \\
5 & \varphi & :3\text{–}4\ \neg\text{E}
\end{array}
$$

As in the other case, our derived rule DN allows us to draw natural inferences with minimal complexity, avoiding the need to open any subproofs.

4.4.8 Ex Falso Quodlibet

From a falsehood anything follows, or in Latin, *ex falso quodlibet.* For instance, if A is false, then $\neg A$ is true, and so if we were to take A to also be true, then together we may derive B from this contradiction. More generally, we have the following rule (EFQ):

$$
\begin{array}{l|ll}
m & \varphi & \\
n & \neg\varphi & \\
 & \psi & :m,\, n \text{ EFQ}
\end{array}
$$

This inference is occasionally convenient since, given φ and $\neg\varphi$ on live lines we may draw any conclusion that we might happen to want on the next line. Here is the derivation of EFQ.

$$
\begin{array}{lll}
1 & \mid\ \varphi & \\
2 & \mid\ \neg\varphi & \\
\hline
3 & \mid\ \mid\ \neg\psi & \text{:AS for } \neg\text{E} \\
4 & \mid\ \mid\ \varphi & \text{:1 R} \\
5 & \mid\ \mid\ \neg\varphi & \text{:2 R} \\
6 & \mid\ \psi & \text{:3–5 } \neg\text{E}
\end{array}
$$

This puts a syntactic spin on a semantic idea that we considered before: just as every wfs of $\mathcal{L}^{\text{PL}}$ is a logical consequence of an unsatisfiable sets of wfss of $\mathcal{L}^{\text{PL}}$, every wfs of $\mathcal{L}^{\text{PL}}$ can be derived from any wfss φ and $\neg\varphi$ of $\mathcal{L}^{\text{PL}}$, and indeed from any set Γ containing φ and $\neg\varphi$.

The explosion of wfss of $\mathcal{L}^{\text{PL}}$ that can be derived from a set containing φ and $\neg\varphi$ helps to shed light on why a set Γ of wfss of $\mathcal{L}^{\text{PL}}$ was said to be inconsistent just in case $\Gamma \vdash \bot$. Since $\bot \coloneqq A \wedge \neg A$, both $\bot \vdash A$ and $\bot \vdash \neg A$ by $\wedge$E, and so by EFQ, any ψ can be derived from $\bot$. Thus if $\Gamma \vdash \bot$, it follows that $\Gamma \vdash \psi$ for any wfs ψ of $\mathcal{L}^{\text{PL}}$ whatsoever.

4.4.9 Law of Excluded Middle

Recall from Chapter 2 that the $\mathcal{L}^{\text{PL}}$ interpretations assign every sentence letter of $\mathcal{L}^{\text{PL}}$ to exactly one of just two truth-values 1 and 0. It follows that every wfs φ of $\mathcal{L}^{\text{PL}}$ is assigned to either 1 or 0 and not both, i.e., $\mathcal{V}_{\mathcal{I}}(\varphi) \in \{1, 0\}$. Thus $\mathcal{V}_{\mathcal{I}}(\varphi \vee \neg\varphi) = 1$ for any wfs φ and interpretation $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$, and so every instance of $\varphi \vee \neg\varphi$ is a tautology. The syntactic analogue of this semantic claim asserts that every instance of $\varphi \vee \neg\varphi$ is a theorem of PL which, given its central place within classical logic, is referred to as *the law of excluded middle*:

$$
\mid\ \varphi \vee \neg\varphi \qquad \text{:LEM}
$$

$$
\begin{array}{lll}
1 & \mid\ \mid\ \neg(\varphi \vee \neg\varphi) & \text{:AS for } \neg\text{E} \\
2 & \mid\ \mid\ \mid\ \varphi & \text{:AS for } \neg\text{I} \\
3 & \mid\ \mid\ \mid\ \varphi \vee \neg\varphi & \text{:2 } \vee\text{I} \\
4 & \mid\ \mid\ \mid\ \neg(\varphi \vee \neg\varphi) & \text{:1 R} \\
5 & \mid\ \mid\ \neg\varphi & \text{:2–4 } \neg\text{I} \\
6 & \mid\ \mid\ \varphi \vee \neg\varphi & \text{:5 } \vee\text{I} \\
7 & \mid\ \varphi \vee \neg\varphi & \text{:1–6 } \neg\text{E}
\end{array}
$$

By contrast with the basic and derived rules given above, theorems do not cite previous lines of the proofs in which they occur, though they are justified all the same. This is because applications of LEM abbreviate proofs of the form given above on the right.

4.4.10 Law of Non-Contradiction

Given that $\mathcal{V}_{\mathcal{I}}(\varphi) \in \{1, 0\}$ for any wfs φ of $\mathcal{L}^{\text{PL}}$, it also follows by the semantics for negation and conjunction that $\mathcal{V}_{\mathcal{I}}(\varphi \wedge \neg\varphi) = 0$ for any wfs φ and interpretation $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$, and so every instance of $\varphi \wedge \neg\varphi$ is a contradiction. Equivalently, all instances of $\neg(\varphi \wedge \neg\varphi)$ are tautologies, and so we may expect $\neg(\varphi \wedge \neg\varphi)$ to be a theorem for any wfs φ of $\mathcal{L}^{\text{PL}}$. In order to cover all instances, we may provide the following derivation:

$\neg(\varphi \wedge \neg\varphi)$	:LNC

1	$\quad\varphi \wedge \neg\varphi$	:AS for $\neg$I
2	$\quad\varphi$	:1 $\wedge$E
3	$\quad\neg\varphi$	:1 $\wedge$E
4	$\neg(\varphi \wedge \neg\varphi)$	:1–3 $\neg$I

Having observed that every instance of $\varphi \vee \neg\varphi$ and $\neg(\varphi \wedge \neg\varphi)$ are tautologies, one might reasonably expect these to be derivable in PL though nothing so far allows us to jump to this conclusion. Rather, the derivations above do important work, indicating that PL is doing what it should do by allowing us to reason our way to the logical consequences of any set of premises where the logical consequences of the empty set are a special case. Nevertheless, we should like to know if there is anything missing. That is, we may ask whether there are logical consequences of $\mathcal{L}^{\text{PL}}$ which PL is unable to derive. It turns out that this is not the case: every logical consequences of $\mathcal{L}^{\text{PL}}$ whatsoever is derivable in PL. In a word, PL is *complete*.

We will turn to prove PL Completeness in the following chapter. For the time being, there is an important consequence of PL Soundness that we are now in a position to draw.

4.5 Consistency

In the previous section we set about deriving a host of rules and theorems. You might begin to wonder just how many derived rules and theorems there are where it might be natural to think that the more the better. Another way to put this point is in terms of the strength of PL as a proof system where this refers to how much we can derive with the basic rules that PL provides. Accordingly, one might be tempted to think that stronger logics are better. After all, what could be bad about being able to derive more rather than less?

Tempting as it may be to think that strength is only a good thing, we have already seen some cases where being able to derive too much is not a good thing. In particular, we saw that everything can be derived from a set containing a wfs of $\mathcal{L}^{\text{PL}}$ and its negation. More generally, all wfss of $\mathcal{L}^{\text{PL}}$ are derivable from an inconsistent set of wfss of $\mathcal{L}^{\text{PL}}$. We may then prove:

Corollary 4.1 If Γ is inconsistent, then Γ is unsatisfiable.

Having established PL Soundness and **Lemma 2.2**, the proof of **Corollary 4.1** follows easily and so has been left as an exercise for the reader. By contrast with lemmas which are used to establish important results, corollaries are the consequences of important results. What the corollary above shows is that any set of wfss of $\mathcal{L}^{\text{PL}}$ that is strong enough to be able to derive all wfss of $\mathcal{L}^{\text{PL}}$ is also unsatisfiable. Thus we also have the immediate consequence:

Corollary 4.2 If Γ is satisfiable, then Γ is consistent.

Proof: Follows immediately from **Corollary 4.1** by contraposition. □

Whereas inconsistency has witnesses, consistency does not. That is, although you might show how to derive $\bot$ from some set Γ of wfss of $\mathcal{L}^{\text{PL}}$ by providing a particular derivation in PL, to claim that Γ is consistent is to say that there is no way to derive $\bot$ from Γ in PL. This might seem like a hard thing to show since how should we expect to survey the entire space of possible derivations in order to claim that there are none in which $\bot$ is the conclusion and Γ is the set of premises? Were one to proceed by *reductio*, it is not clear how to derive a contradiction from the assumption that there is a derivation of $\bot$ from Γ in PL.

Given **Corollary 4.2**, there is a much easier way to show that a set Γ of wfss of $\mathcal{L}^{\text{PL}}$ is consistent: simply show that it is satisfiable. Whereas consistency does not have witnesses on account of asserting something general, satisfiability does have witnesses. That is, given any satisfiable set Γ, there is a least one particular interpretation $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$ where $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$. Assuming that we can identify an interpretation $\mathcal{I}$ which witnesses the satisfiability of Γ, we may draw on **Corollary 4.2** in order to conclude that Γ is consistent.

There is a particularly important application of this general procedure. That is, something we should like to know is whether the theorems of PL are consistent, since if the theorems of PL turned out to be inconsistent, then PL would be so strong as to be able to derive anything from nothing. But that is not what we want. Rather, our hope in setting up PL was to describe what follows from what in virtue of logical form where we had previously characterized this by defining logical consequence. If it turns out that everything is derivable from nothing, then all our hard work will have been for nothing since PL will have been shown to massively overshoot its intended target: formal reasoning.

Given our present strategy, all that remains is to find an interpretation of $\mathcal{L}^{\text{PL}}$ that satisfies all of the theorems of PL. But how shall we choose? The answer is that we don't need to: any interpretation at all will do. Since we know by PL Soundness that every theorem of PL is a $\mathcal{L}^{\text{PL}}$ tautology, $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for any theorem φ of PL and interpretation $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$ whatsoever. As our witness, suppose we choose the $\mathcal{L}^{\text{PL}}$ interpretation $\mathcal{I}^+$ where $\mathcal{V}_{\mathcal{I}^+}(\psi) = 1$ for every sentence letter ψ of $\mathcal{L}^{\text{PL}}$. Since, like any $\mathcal{L}^{\text{PL}}$ interpretation, $\mathcal{V}_{\mathcal{I}^+}(\varphi) = 1$ for every theorem φ of PL, it follows that the theorems of PL are indeed satisfiable, and so consistent by **Corollary 4.2** above. Thus we may conclude that despite all of the rules and theorems we derived, PL is not so strong as to be able to derive everything from nothing.