Contents

Chapter 11

The Soundness of $\text{FOL}^=$

In Chapter 4, we showed that PL was sound over its semantics. In particular, soundness showed that whenever a wfs φ of $\mathcal{L}^{\text{PL}}$ is derivable in PL from some set of wfss Γ, we may conclude that φ is a logical consequence of Γ which we expressed as $\Gamma \vDash \varphi$. As a result, we could rely on derivations in PL in order to evaluate whether a conclusion of an argument was a logical consequence of its premises, making the argument valid.

Soundness was an important result to establish for PL where similar considerations extend to $\text{FOL}^=$. If $\text{FOL}^=$ were to fail to be sound, we would have little reason to care about $\text{FOL}^=$ for we could not rely on it to establish valid arguments. Showing that $\text{FOL}^=$ is sound will be the focus of this chapter where we will extend the proof that PL is sound from before.

As in the soundness proof for PL, the soundness proof for $\text{FOL}^=$ will go by induction on the length of proof. If we can show that soundness holds for proofs of length 1 and that soundness holds for proofs of length $n + 1$ whenever soundness holds for proofs of length n or less, then we may conclude by induction that soundness holds for proofs of any length. Almost everything in the proof will remain the same as before, though now we need to check that a few extra rules also preserve logical consequence.

Since we will at times need to talk about derivability or logical consequence in different systems, subscripts will help to avoid ambiguity. Given that our primary concern is with $\text{FOL}^=$, the default is to assume that $\vdash$ means derivability in $\text{FOL}^=$ and $\vDash$ means logical consequence in $\mathcal{L}^=$. Occasionally it will improve readability to subscript these as well, writing $\vdash_{\text{FOL}^=}$ and $\vDash_{\mathcal{L}^=}$ as needed. With these details in place, we now turn to the proof, beginning as before with the global argument which we will then turn to support be filling in the details.

If you get lost, or forget what was happening or why, it can help to return to this first part of the proof to regain your bearings, reflecting on what has previously been established.

11.1 Soundness

Assume $\Gamma \vdash_{\text{FOL}^=} \varphi$, written $\Gamma \vdash \varphi$ for readability. By definition, there is some proof X in FOL$^=$ of φ from the premises Γ. As in the proof of PL SOUNDNESS, it will help to introduce some notation that we will use throughout. In particular, φ_i is the sentence on the i-th line of the proof X and Γ_i includes all and only the premises and undischarged assumptions at the i-th line in X. We may then prove the following, writing $\vDash$ in place of $\vDash_{\mathcal{L}^=}$ for readability:

FOL$^=$ SOUNDNESS: Assume that $\Gamma \vdash \varphi$ for an arbitrary set Γ of wfss of $\mathcal{L}^=$ and wfs φ of $\mathcal{L}^=$. It follows that there is some FOL$^=$ derivation X of φ from Γ. Letting φ_i be the sentence on the i-th line of the derivation X and Γ_i be the set of premises that occur on any line $j \leqslant i$ of X together with the assumptions that are undischarged at line i, we may seek to prove:

Lemma 11.1: (Base Step) $\Gamma_1 \vDash \varphi_1$.

Lemma 11.13: (Induction Step) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if $\Gamma_k \vDash \varphi_k$ for every $k \leqslant n$.

Given the lemmas above, it follows by strong induction that $\Gamma_n \vDash \varphi_n$ for all n. Since every proof is finite in length, there is a last line m of X where $\varphi_m = \varphi$ is the conclusion. By the definition of a FOL$^=$ derivation, we know that every assumption in X is eventually discharged, and so $\Gamma_m = \Gamma$ is the set of premises. Thus we may conclude that $\Gamma \vDash \varphi$. Discharging the assumption that $\Gamma \vdash \varphi$ and generalizing on Γ and φ completes the proof. □

Given the lemmas cited above, this proof establishes FOL$^=$ SOUNDNESS. We may now prove the supporting lemmas in a similar manner to before.

Lemma 11.1 (Base Step) $\Gamma_1 \vDash \varphi_1$.

Proof: By the definition of a FOL$^=$ derivation, φ_1 is either a premise or follows by one of the natural deduction rules for FOL$^=$. Since φ_1 is the first line of the proof, there are no earlier lines to be cited, and so φ_1 is either a premise, assumption, or follows by =I. In the first two cases, $\Gamma_1 = \{\varphi_1\}$ since φ_1 is not discharged at the first line, and so $\Gamma_1 \vDash \varphi_1$ is immediate. Thus it remains to show that $\Gamma_1 \vDash \varphi_1$ in the final case where φ_1 is $\alpha = \alpha$ for some constant α and $\Gamma_1 = \varnothing$.

Assume φ_1 is $\alpha = \alpha$ and let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be an arbitrary model of $\mathcal{L}^=$. It follows that $\mathcal{I}(\alpha) \in \mathbb{D}$ where trivially $\mathcal{I}(\alpha) = \mathcal{I}(\alpha)$. Letting $\hat{a}$ be any variable assignment defined over the domain $\mathbb{D}$, it follows by definition that $v_{\mathcal{I}}^{\hat{a}}(\alpha) = v_{\mathcal{I}}^{\hat{a}}(\alpha)$, and so $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha = \alpha) = 1$. Since $\hat{a}$ was arbitrary, $\mathcal{V}_{\mathcal{I}}(\alpha = \alpha) = 1$, and so $\vDash \alpha = \alpha$ follows be generalizing on $\mathcal{M}$. Thus $\Gamma_1 \vDash \varphi_1$ given the case assumption. □

We have already considered two proof rules in proving **Lemma 11.1** by showing that AS and =I preserve logical consequence at least in the special case as $\Gamma_1 \vDash \varphi_1$. More generally, we should like to show that all of the rules preserve logical consequence, and not just in the case of proofs with one line. Thus we will seek to establish the following:

FOL$^=$ RULES: If $\Gamma_k \vDash \varphi_k$ for every $k \leqslant n$ and φ_{n+1} follows by the proof rules for FOL$^=$, then $\Gamma_{n+1} \vDash \varphi_{n+1}$.

In order to divide the proof of FOL$^=$ RULES into more manageable parts, the following section will focus on the proof rules for PL. More specifically, we will aim to show:

PL RULES: If $\Gamma_k \vDash \varphi_k$ for every $k \leqslant n$ and φ_{n+1} follows by the proof rules for PL, then $\Gamma_{n+1} \vDash \varphi_{n+1}$.

In §11.4, we will extend the same strategy to the remaining proof rules that belong to FOL$^=$ in order to establish FOL$^=$ RULES. It is this latter result which will play a critical role in the proof of **Lemma 11.13** cited in the proof of FOL$^=$ SOUNDNESS above.

11.2 PL Rules

You might recognize PL RULES from Chapter 4, wondering why we can't simply cite this previous result. Despite the superficial similarities, PL RULES stated above says something about the wfss of $\mathcal{L}^=$, a language we had not introduced in Chapter 4. Even though PL RULES only concerns proofs rules that occur in PL, the semantic turnstile $\vDash$ used above quantifies over the models of $\mathcal{L}^=$, not the interpretations of $\mathcal{L}^{\text{PL}}$. Were we to disambiguate, we may replace the turnstiles above with $\vDash_{\mathcal{L}^=}$ and not $\vDash_{\mathcal{L}^{\text{PL}}}$, however tempting.

Given these caveats, you still might wonder why we can't just cite our previous result. After all, we showed that PL is sound over its semantics. Shouldn't the result somehow carry over to allow us to assert PL RULES without saying much more?

The answer is that there are proof strategies that go this way, though they typically go one of two ways. Either they merely wave their hands, suppressing the details that make the proof worth reading, or they define an injection from $\mathcal{L}^=$ into $\mathcal{L}^{\text{PL}}$ in order to make use of PL SOUNDNESS. Since this latter strategy is abstract and cumbersome, and the former is pointless, we will follow the much more concrete approach of simply revising our former proofs. In addition to providing the opportunity to review how the proof of PL SOUNDNESS worked before, we will also be in a position to omit certain elements when the details are very similar to the proofs that we already provided above. Nevertheless, by referring to the proofs in Chapter 4, it should be possible to reconstruct every element of the proof in rigorous detail.

11.2.1 Assumption and Reiteration

Before attending to the introduction and elimination rules for each of the logical operators included in PL, this section will focus on the assumption and reiteration rules. Whereas the proofs for most of the rules will appeal to the induction hypothesis given above, the proof for the assumption rule AS is an exception and is similar to what was given in **Lemma 11.1**.

Rule 1 (AS) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule AS.

Proof: Assume that φ_{n+1} follows by the assumption rule AS from the wfss in Γ_{n+1}. Since φ_{n+1} is an undischarged assumption, it follows that $\varphi_{n+1} \in \Gamma_{n+1}$, and so $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$ for any model $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ where $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma_{n+1}$. By definition, it follows that $\Gamma_{n+1} \vDash \varphi_{n+1}$ as desired. $\square$

The proof above does not require the induction hypothesis or any additional results. By contrast, it will help to establish the reiteration rule by first recalling the following lemmas.

Lemma 2.1: If $\Gamma \vDash \varphi$, then $\Gamma \cup \Sigma \vDash \varphi$.

Lemma 4.3: If φ_k is live at line n of an FOL$^=$ derivation where $k \leqslant n$, then $\Gamma_k \subseteq \Gamma_n$.

The proofs for these lemmas is very similar to what it was before though Γ is now permitted to be any set of wfss of $\mathcal{L}^=$, where similarly, φ is any wfs of $\mathcal{L}^=$. It is nevertheless worth looking back to confirm that the old proofs continue to apply. Given these two lemmas, we may now proceed to establish the reiteration rule R in the same manner as before.

Rule 2 (R) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule R.

Proof: Assume that φ_{n+1} follows by the reiteration rule R from the sentences in Γ_{n+1}. It follows that $\varphi_{n+1} = \varphi_k$ for some $k \leqslant n$, and so $\Gamma_k \vDash \varphi_k$ by hypothesis. Since φ_k is live at line $n+1$, we know by **Lemma 4.3** that $\Gamma_k \subseteq \Gamma_{n+1}$, and so $\Gamma_{n+1} \vDash \varphi_k$ by **Lemma 2.1**. Thus $\Gamma_{n+1} \vDash \varphi_{n+1}$ given the identity above. $\square$

Given that nothing needs to change about the proof of **Rule 2**, you might suspect that all of the proofs for the PL rules will be unchanged. This is true in some cases and not in others where the same may be said for some of the lemmas established before.

11.2.2 Negation Rules

In order to show that the negation rules preserve logical consequence, Chapter 4 appealed to two supporting lemmas. Since these lemmas will continue to be important for what follows, we prove that they continue to hold given the semantics for $\mathcal{L}^=$ which, unlike before, includes variable assignments. In particular, consider the following proof:

Lemma 11.2 If $\Gamma \vDash \varphi$ and $\Gamma \vDash \neg\varphi$, then Γ is unsatisfiable.

Proof: Assume $\Gamma \vDash \varphi$ and $\Gamma \vDash \neg\varphi$. Assume for contradiction that Γ is satisfiable, and so there is some model $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ where $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma$. It follows from the assumption that both $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\neg\varphi) = 1$, and so $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\neg\varphi) = 1$ follows from the latter for some $\hat{a}$ by **Lemma 9.2**. By the semantics for negation, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) \neq 1$, and so $\mathcal{V}_{\mathcal{I}}(\varphi) \neq 1$ contradicting the above. Thus Γ is unsatisfiable. □

The lemma above is much as it was before save for some extra details to do with variable assignments. Nevertheless, the result holds for the same basic reason. Something similar may be said for the following lemma which has been left as an exercise:

Lemma 11.3 $\Gamma \cup \{\varphi\}$ is unsatisfiable just in case $\Gamma \vDash \neg\varphi$.

Proof: This proof is left as an exercise for the reader.

Given the updated lemmas above, the proof of $\neg$I is exactly the same as it was before. Omitting the details of the proof here, it worth reviewing the proof given in Chapter 4.

> **Rule 3 ($\neg$I)** $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule $\neg$I.

Despite how similar the introduction and elimination rules are for negation, a few minor amendments are required in order to extend the proof of **Rule 4** to hold for the wfss of $\mathcal{L}^=$.

> **Rule 4 ($\neg$E)** $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule $\neg$E.

Proof: Assume φ_{n+1} follows from Γ_{n+1} by negation elimination $\neg$E. Thus there is some subproof on lines i-j where $i < j \leqslant n$ and $\varphi_i = \neg\varphi_{n+1}$, $\psi = \varphi_h$, and $\neg\psi = \varphi_k$ for $i \leqslant h \leqslant j$ and $i \leqslant k \leqslant j$. By parity of reasoning, we may assume that $h < k = j$. Thus we may represent the subproof as follows:

i		$\neg\varphi$	:AS for $\neg$E
h		ψ	
j		$\neg\psi$	
$n+1$	φ		:i–j $\neg$E

By hypothesis, $\Gamma_h \vDash \psi$ and $\Gamma_j \vDash \neg\psi$. With the exception of $\varphi_i = \neg\varphi$, the undischarged assumptions at lines h and j are also undischarged at line $n+1$. It follows that $\Gamma_h, \Gamma_j \subseteq \Gamma_{n+1} \cup \{\varphi_i\}$, and so $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \psi$ and $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \neg\psi$ by **Lemma 2.1**. Thus $\Gamma_{n+1} \cup \{\varphi_i\}$ is unsatisfiable by **Lemma 11.2**, and so $\Gamma_{n+1} \vDash \neg\varphi_i$ by **Lemma 11.3**. Equivalently, $\Gamma_{n+1} \vDash \neg\neg\varphi_{n+1}$. Given any model $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ where $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma_{n+1}$, it follows that $\mathcal{V}_{\mathcal{I}}(\neg\neg\varphi_{n+1}) = 1$, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\neg\neg\varphi_{n+1}) = 1$ for all $\hat{a}$. By two applications of the semantics for negation, it follows that $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi_{n+1}) = 1$ all $\hat{a}$, and so $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$. Thus we may conclude by generalizing on $\mathcal{M}$ that $\Gamma_{n+1} \vDash \varphi_{n+1}$ as desired. □

This proof is almost identical to the **Rule 3** except that an additional negation sign is introduced before eliminating the double negation by appealing to the semantics.

11.2.3 Conjunction and Disjunction

The following rule is established in much the same way as before save for minor details particular to the semantics for $\mathcal{L}^=$ and so has been left as an exercise for the reader.

Rule 5 (&I) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule $\wedge$I.

Proof: This proof is left as an exercise for the reader. □

Since the details for the following proof rule were omitted before, they will be provided here for completeness. Nevertheless, few changes are required for the proof to go through.

Rule 6 (&E) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule $\wedge$E.

Proof: Assuming φ_{n+1} follows from Γ_{n+1} by conjunction elimination $\wedge$E, there is some $i \leqslant n$ where either $\varphi_i = \varphi_{n+1} \wedge \psi$ or $\varphi_i = \psi \wedge \varphi_{n+1}$ is live at line $n+1$. By hypothesis, $\Gamma_i \vDash \varphi_i$ where $\Gamma_i \subseteq \Gamma_{n+1}$ by **Lemma 4.3**, and so $\Gamma_{n+1} \vDash \varphi_i$ by **Lemma 2.1**. Letting $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be a model of $\mathcal{L}^=$ where $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma_{n+1}$, it follows that $\mathcal{V}_{\mathcal{I}}(\varphi_i) = 1$, and so either $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1} \wedge \psi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\psi \wedge \varphi_{n+1}) = 1$. By **Lemma 9.2**, either $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi_{n+1} \wedge \psi) = 1$ or $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi \wedge \varphi_{n+1}) = 1$ for some v.a. $\hat{a}$, and so either way $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi_{n+1}) = 1$ by the semantics for conjunction. Thus $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$ again by **Lemma 9.2**, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$. □

Rule 7 ($\vee$I) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule $\vee$I.

Proof: Assume that φ_{n+1} follows from Γ_{n+1} by disjunction introduction $\vee$I. Thus $\varphi_{n+1} = \varphi_i \vee \psi$ or $\varphi_{n+1} = \psi \vee \varphi_i$ for some line $i \leqslant n$ that is live at line $n+1$. By hypothesis, $\Gamma_i \vDash \varphi_i$ where $\Gamma_i \subseteq \Gamma_{n+1}$ by **Lemma 4.3**, and so $\Gamma_{n+1} \vDash \varphi_i$ by **Lemma 2.1**. Letting $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be a model where $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma_{n+1}$, it follows that $\mathcal{V}_{\mathcal{I}}(\varphi_i) = 1$, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi_i) = 1$ for some variable assignment $\hat{a}$ by **Lemma 9.2**. By the semantics for disjunction, both $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi_i \vee \psi) = 1$ and $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi \vee \varphi_i) = 1$, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi_{n+1}) = 1$. Thus $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$ again by **Lemma 9.2**, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$ follows from generalizing on $\mathcal{M}$. □

Neither of the rule proofs above should surprise, amounting to little more than applications of the semantic clauses for conjunction and disjunction respectively. The only differences with the proofs given before concern the way that the wfss of $\mathcal{L}^=$ are assigned truth-values relative to models instead of interpretations and the way that variable assignments are negotiated throughout. Since the proof rules for PL do not appeal to variables, little turns on these extra details. Rather, they are included only to conform to the strict letter of the definitions.

Something similar may be said for the following proof though a little more care is required to keep track of all of the moving parts in the proof rule for disjunction elimination.

Rule 8 ($\vee$E) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule $\vee$E.

Proof: Assume φ_{n+1} follows from Γ_{n+1} by disjunction elimination $\vee$I. Thus there is some line $\varphi_i = \varphi_j \vee \varphi_k$ which is live at $n+1$ and subproofs on lines j-h and k-l where $i < j, k, h, l \leqslant n$ and $\varphi_h = \varphi_l = \varphi_{n+1}$. By parity of reasoning, we may assume that $h < k$, and so represent the proof as follows:

$$
\begin{array}{l|l}
i & \varphi \vee \psi \\
j & \quad \begin{array}{|l} \varphi \quad \text{:AS for } \vee\text{E} \\ \hline \end{array} \\
h & \quad \begin{array}{|l} \chi \end{array} \\
k & \quad \begin{array}{|l} \psi \quad \text{:AS for } \vee\text{E} \\ \hline \end{array} \\
l & \quad \begin{array}{|l} \chi \end{array} \\
n+1 & \chi \qquad :i,\ j\text{–}h,\ k\text{–}l\ \vee\text{E}
\end{array}
$$

By hypothesis, $\Gamma_i \vDash \varphi_i$, $\Gamma_h \vDash \varphi_h$, and $\Gamma_l \vDash \varphi_l$ where $\Gamma_i \subseteq \Gamma_{n+1}$ all follow by **Lemma 4.3**, and so $\Gamma_{n+1} \vDash \varphi_i$ by **Lemma 2.1**. With the exception of $\varphi_j = \varphi$, every assumption that is undischarged at line h is also undischarged at line $n+1$, and so $\Gamma_h \subseteq \Gamma_{n+1} \cup \{\varphi_j\}$. Similarly, we may conclude that $\Gamma_l \subseteq \Gamma_{n+1} \cup \{\varphi_k\}$, and so $\Gamma_{n+1} \cup \{\varphi_j\} \vDash \varphi_h$ and $\Gamma_{n+1} \cup \{\varphi_k\} \vDash \varphi_l$ by **Lemma 2.1**.

Letting $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be any model where $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma_{n+1}$, it follows from above that $\mathcal{V}_{\mathcal{I}}(\varphi_i) = 1$. Equivalently, $\mathcal{V}_{\mathcal{I}}(\varphi_j \vee \varphi_k) = 1$, and so $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi_j \vee \varphi_k) = 1$ for some v.a. $\hat{a}$ defined over $\mathbb{D}$ by **Lemma 9.2**. By the semantics for disjunction, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi_j) = 1$ or $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi_k) = 1$, and so $\mathcal{V}_{\mathcal{I}}(\varphi_j) = 1$ or $\mathcal{V}_{\mathcal{I}}(\varphi_k) = 1$ by **Lemma 9.2**. If $\mathcal{V}_{\mathcal{I}}(\varphi_j) = 1$, then $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma_{n+1} \cup \{\varphi_j\}$, and so $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$ since $\Gamma_{n+1} \cup \{\varphi_j\} \vDash \varphi_h$ and $\varphi_h = \varphi_{n+1}$. If $\mathcal{V}_{\mathcal{I}}(\varphi_k) = 1$, then $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma_{n+1} \cup \{\varphi_k\}$, and so $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$ since $\Gamma_{n+1} \cup \{\varphi_k\} \vDash \varphi_l$ and $\varphi_l = \varphi_{n+1}$. Either way, $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$ by generalizing on $\mathcal{M}$. □

As with the previous two proof rules for conjunction and disjunction, the proof above turns on little more than an application of the semantics for disjunction.

11.2.4 Conditional Rules

The elimination rules for the conditional and the biconditional are also straightforward applications of the semantics. By contrast, the introduction rules for the conditional and biconditional benefit from the following analogue of **Lemma 4.6** from Chapter 4. Since the proofs are very similar, the details have been left as an exercise for the reader.

Lemma 11.4 If $\Gamma \cup \{\varphi\} \vDash \psi$, then $\Gamma \vDash \varphi \to \psi$.

Proof: This proof is left as an exercise for the reader. □

The application of the previous lemma in the proof of the following rule is unchanged from before, and so the details of the proof will be omitted.

Rule 9 (→I) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule →I.

Having previously left the proof for the following proof rule as an exercise for the reader, we now provide the following details for completeness:

Rule 10 (→E) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule →E.

Proof: Assume φ_{n+1} follows from Γ_{n+1} by conditional introduction →E. Thus there are some lines $\varphi_i = \varphi_j \to \varphi_{n+1}$ and φ_j for $i, j \leqslant n$ which are live at $n+1$, and so $\Gamma_i, \Gamma_j \subseteq \Gamma_{n+1}$ by **Lemma 4.3**. By hypothesis, $\Gamma_i \vDash \varphi_i$ and $\Gamma_j \vDash \varphi_j$, and so both $\Gamma_{n+1} \vDash \varphi_i$ and $\Gamma_{n+1} \vDash \varphi_j$ by **Lemma 2.1**.

Letting $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be any model where $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma_{n+1}$, it follows that $\mathcal{V}_{\mathcal{I}}(\varphi_i) = \mathcal{V}_{\mathcal{I}}(\varphi_j) = 1$. It follows that $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi_i) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi_j \to \varphi_{n+1}) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi_j) = 1$ for all variable assignment $\hat{a}$ over $\mathbb{D}$, and so for some $\hat{a}$ in particular.

By the semantics for the conditional, either $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi_j) \neq 1$ or $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi_{n+1}) = 1$. Given the above, we may conclude that $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi_{n+1}) = 1$, and so $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$ follows by **Lemma 9.2** . Generalizing on $\mathcal{M}$, it follows that $\Gamma_{n+1} \vDash \varphi_{n+1}$ as desired. □

Rule 11 (↔I) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule ↔I.

Proof: Assume φ_{n+1} follows from Γ_{n+1} by biconditional introduction ↔E. Thus there are some subproofs on lines i-j and h-k for some $i, j < h, k \leqslant n$ where $\varphi_i = \varphi_h = \varphi$, $\varphi_j = \varphi_k = \psi$, and either $\varphi_{n+1} = \varphi \leftrightarrow \psi$ or $\varphi_{n+1} = \psi \leftrightarrow \varphi$. By parity of reasoning, we may assume that $\varphi_{n+1} = \varphi \leftrightarrow \psi$. Thus we have:

i	$\quad\mid\ \varphi$	:AS for ∨E
j	$\quad\mid\ \psi$	
h	$\quad\mid\ \psi$	:AS for ∨E
k	$\quad\mid\ \varphi$	
$n+1$	$\varphi \leftrightarrow \psi$	:i–j, h–k ↔I

By hypothesis, $\Gamma_j \vDash \varphi_j$, $\Gamma_k \vDash \varphi_k$, and $\Gamma_{n+1} \vDash \varphi_{n+1}$. With the exception of φ_i, every assumption that is undischarged at line j is also undischarged at line $n+1$, and so $\Gamma_j \subseteq \Gamma_{n+1} \cup \{\varphi_i\}$. Similarly, we may conclude that $\Gamma_k \subseteq \Gamma_{n+1} \cup \{\varphi_h\}$, and so $\Gamma_{n+1} \cup \{\varphi_i\} \vDash \varphi_j$ and $\Gamma_{n+1} \cup \{\varphi_h\} \vDash \varphi_k$ by **Lemma 2.1**.

By **Lemma 11.4**, both $\Gamma_{n+1} \vDash \varphi_i \rightarrow \varphi_j$ and $\Gamma_{n+1} \vDash \varphi_h \rightarrow \varphi_k$. Equivalently, $\Gamma_{n+1} \vDash \varphi \rightarrow \psi$ and $\Gamma_{n+1} \vDash \psi \rightarrow \varphi$. Letting $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be any model where $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma_{n+1}$, it follows that $\mathcal{V}_{\mathcal{I}}(\varphi \rightarrow \psi) = \mathcal{V}_{\mathcal{I}}(\psi \rightarrow \varphi) = 1$ given the results above. Thus $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi \rightarrow \psi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi \rightarrow \varphi) = 1$ for all variable assignments $\hat{a}$, and so for some $\hat{a}$ in particular. By the semantics for the conditional, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) \neq 1$ or $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi) = 1$, and $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi) \neq 1$ or $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$.

As a result, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi) = 1$ if $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$, and similarly, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ if $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi) = 1$, and so $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi)$. Thus $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi \leftrightarrow \psi) = 1$ by the semantics for the biconditional, and so $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1})$ by **Lemma 9.2**. Generalizing on $\mathcal{M}$, we know $\Gamma_{n+1} \vDash \varphi_{n+1}$. □

Rule 12 (↔E) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule ↔E.

Proof: Assume φ_{n+1} follows from Γ_{n+1} by biconditional introduction →E. Thus there are some lines $i, j \leqslant n$ that are live at $n+1$ where either $\varphi_i = \varphi_j \leftrightarrow \varphi_{n+1}$ or $\varphi_i = \varphi_{n+1} \leftrightarrow \varphi_j$. By parity of reasoning, we may assume that $\varphi_i = \varphi_j \leftrightarrow \varphi_{n+1}$ where $\Gamma_i, \Gamma_j \subseteq \Gamma_{n+1}$ follows by **Lemma 4.3**. By hypothesis, $\Gamma_i \vDash \varphi_i$ and $\Gamma_j \vDash \varphi_j$, and so $\Gamma_{n+1} \vDash \varphi_i$ and $\Gamma_{n+1} \vDash \varphi_j$ by **Lemma 2.1**.

Letting $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be any model where $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma_{n+1}$, it follows that $\mathcal{V}_{\mathcal{I}}(\varphi_i) = \mathcal{V}_{\mathcal{I}}(\varphi_j) = 1$. Thus $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi_i) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi_j \leftrightarrow \varphi_{n+1}) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi_j) = 1$ for every v.a. $\hat{a}$ defined over $\mathbb{D}$, and so for some $\hat{a}$ in particular. By the semantics for the biconditional, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi_j) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi_{n+1})$, and so $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi_{n+1}) = 1$. Thus $\mathcal{V}_{\mathcal{I}}(\varphi_{n+1}) = 1$ by **Lemma 9.2**, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$ follows by generalizing on $\mathcal{M}$. □

These final results complete the last of the proofs for all of the rules included in PL. Given **Rule 1** – **Rule 12**, we may report the following preliminary result:

PL RULES: If $\Gamma_k \vDash \varphi_k$ for every $k \leqslant n$ and φ_{n+1} follows by the proof rules for PL, then $\Gamma_{n+1} \vDash \varphi_{n+1}$.

It remains to extend this result to include the remaining proof rules in FOL$^=$. In order to do so, the following section will prove two important supporting lemmas. Whereas the lemmas above merely adapted the lemmas already given in Chapter 4, the lemmas proven in the following section are entirely novel to $\mathcal{L}^=$. We will then put these lemmas to work in order to establish FOL$^=$ RULES in the following section.

11.3 Substitution and Model Lemmas

This section establishes two closely related results, both of which show that the truth-value of a wff is preserved by specific changes to that wff or to the model in which it is evaluated. These results will play a crucial role in proving the lemmas that we will need to show that the remaining proof rules that belong to FOL$^=$ preserve logical consequence.

In slightly greater detail, the following lemma shows that replacing α with β in a wff φ does not effect its truth-value when evaluated at a model and variable assignment so long as α and β refer to the same element of the domain on that model and variable assignment.

Lemma 11.5 $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\beta/\alpha])$ if $\boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\alpha) = \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\beta)$ and β is free for α in φ.

Proof: The proof goes by induction on the wff of $\mathcal{L}^=$.

Base: Let φ be a wff of $\mathcal{L}^=$ where $\texttt{Comp}(\varphi) = 0$. Assume that $\boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\alpha) = \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\beta)$. It follows that φ is either $\mathcal{F}^n\alpha_1, \ldots \alpha_n$ or $\alpha_1 = \alpha_2$. If φ is $\mathcal{F}^n\alpha_1, \ldots \alpha_n$ where $\gamma_i = \beta$ if $\alpha_i = \alpha$ and otherwise $\gamma_i = \alpha_i$, then we have:

$$\begin{array}{rll} \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1 & \textit{iff} & \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\mathcal{F}^n\alpha_1, \ldots, \alpha_n) = 1 \\ & \textit{iff} & \langle \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\alpha_1), \ldots, \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\alpha_n) \rangle \in \mathcal{I}(\mathcal{F}^n) \\ (\star) & \textit{iff} & \langle \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\gamma_1), \ldots, \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\gamma_n) \rangle \in \mathcal{I}(\mathcal{F}^n) \\ & \textit{iff} & \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\mathcal{F}^n\gamma_1, \ldots, \gamma_n) = 1 \\ & \textit{iff} & \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\beta/\alpha]) = 1. \end{array}$$

Whenever $\alpha_i = \alpha$, it follows that $\boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\alpha_i) = \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\alpha) = \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\beta)$ by assumption. Since $\boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\beta) = \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\gamma_i)$ by definition, we may conclude that $\boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\alpha_i) = \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\gamma_i)$. If $\alpha_i \neq \alpha$, then $\alpha_i = \gamma_i$ by definition, and so $\boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\alpha_i) = \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\gamma_i)$ is immediate. It follows that $\boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\alpha_i) = \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\gamma_i)$ for all $1 \leqslant i \leqslant n$, thereby justifying $(\star)$. The other biconditionals hold by definition or the semantics for atomic wffs of $\mathcal{L}^=$.

If instead φ is $\alpha_1 = \alpha_n$, then assuming for all $1 \leqslant i \leqslant n$ as before that $\gamma_i = \beta$ if $\alpha_i = \alpha$ and otherwise $\gamma_i = \alpha_i$, we have the following biconditionals:

$$\begin{array}{rll} \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1 & \textit{iff} & \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha_1 = \alpha_2) = 1 \\ & \textit{iff} & \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\alpha_1) = \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\alpha_n) \\ (*) & \textit{iff} & \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\gamma_1) = \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\gamma_n) \\ & \textit{iff} & \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\gamma_1 = \gamma_n) = 1 \\ & \textit{iff} & \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\beta/\alpha]) = 1. \end{array}$$

We may justify $(*)$ in an analogous manner to $(\star)$, where the justifications for the other biconditionals is the same as before. It follows that $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\beta/\alpha])$ whenever $\boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\alpha) = \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\beta)$ and $\texttt{Comp}(\varphi) = 0$.

Induction: Assume that if $\texttt{Comp}(\varphi) \leqslant n$, then $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\beta/\alpha])$ whenever $\upsilon_{\mathcal{I}}^{\hat{a}}(\alpha) = \upsilon_{\mathcal{I}}^{\hat{a}}(\beta)$. Letting $\texttt{Comp}(\varphi) = n+1$, there are seven cases to consider corresponding to the operators $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall\gamma$, and $\exists\gamma$.

Case 1: Assume that $\varphi = \neg\psi$ where $\upsilon_{\mathcal{I}}^{\hat{a}}(\alpha) = \upsilon_{\mathcal{I}}^{\hat{a}}(\beta)$. Since $\texttt{Comp}(\varphi) = n+1$ and $\texttt{Comp}(\neg\psi) = \texttt{Comp}(\psi) + 1$, it follows that $\texttt{Comp}(\psi) \leqslant n$. It follows by hypothesis that $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi[\beta/\alpha])$, and so $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\neg\psi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\neg\psi[\beta/\alpha])$ by the semantics for negation. Thus $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\beta/\alpha])$ as desired.

Case 6: Assume $\varphi = \forall\gamma\psi$ where $\upsilon_{\mathcal{I}}^{\hat{a}}(\alpha) = \upsilon_{\mathcal{I}}^{\hat{a}}(\beta)$. If $\gamma = \alpha$, if follows that α is not free in φ, and so trivially $\varphi = \varphi[\beta/\alpha]$. Thus $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\beta/\alpha])$ is immediate. Assume instead that $\gamma \neq \alpha$ and consider the following biconditionals:

$$\begin{array}{rll}
\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1 & \textit{iff} & \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall\gamma\psi) = 1 \\
& \textit{iff} & \mathcal{V}_{\mathcal{I}}^{\hat{e}}(\psi) = 1 \text{ for all } \gamma\text{-variants } \hat{e} \text{ of } \hat{a} \\
(\dagger) & \textit{iff} & \mathcal{V}_{\mathcal{I}}^{\hat{e}}(\psi[\beta/\alpha]) = 1 \text{ for all } \gamma\text{-variants } \hat{e} \text{ of } \hat{a} \\
& \textit{iff} & \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall\gamma\psi[\beta/\alpha]) = 1 \\
& \textit{iff} & \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\beta/\alpha]) = 1.
\end{array}$$

Let $\hat{e}$ be an arbitrary γ-variant of $\hat{a}$. Since $\gamma \neq \alpha$, it follows that $\hat{e}(\alpha) = \hat{a}(\alpha)$ if α is a variable, and so $\upsilon_{\mathcal{I}}^{\hat{e}}(\alpha) = \upsilon_{\mathcal{I}}^{\hat{a}}(\alpha)$ regardless of whether α is a variable or a constant. Given the starting assumption, $\upsilon_{\mathcal{I}}^{\hat{e}}(\alpha) = \upsilon_{\mathcal{I}}^{\hat{a}}(\beta)$. Since β is free for α in φ, we know that $\gamma \neq \beta$. If β is a variable, then $\hat{e}(\beta) = \hat{a}(\beta)$ since $\hat{e}$ is a γ-variant of $\hat{a}$, and so $\upsilon_{\mathcal{I}}^{\hat{a}}(\beta) = \upsilon_{\mathcal{I}}^{\hat{e}}(\beta)$ regardless of whether β is a variable or a constant. Thus $\upsilon_{\mathcal{I}}^{\hat{e}}(\alpha) = \upsilon_{\mathcal{I}}^{\hat{e}}(\beta)$. As in *Case 1*, $\texttt{Comp}(\psi) \leqslant n$, and so $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(\psi) = \mathcal{V}_{\mathcal{I}}^{\hat{e}}(\psi[\beta/\alpha])$ by hypothesis. Since $\hat{e}$ was any γ-variant of $\hat{a}$, it follows that $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(\psi) = \mathcal{V}_{\mathcal{I}}^{\hat{e}}(\psi[\beta/\alpha])$ for all γ-variants $\hat{e}$ of $\hat{a}$, thereby establishing ($\dagger$). The other biconditionals follow from the definitions and the semantics for the universal quantifier.

The cases for $\wedge, \vee, \rightarrow, \leftrightarrow$, and $\exists\gamma$ are similar and so will be left as exercises for the reader. It follows that $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\beta/\alpha])$ whenever $\upsilon_{\mathcal{I}}^{\hat{a}}(\alpha) = \upsilon_{\mathcal{I}}^{\hat{a}}(\beta)$ and $\texttt{Comp}(\varphi) = n+1$. Thus the lemma follows by induction. □

The proof above works by induction on complexity where the only tricky cases are for the quantifiers. In a similar manner to **Lemma 9.1**, we avoided assuming the antecedent of the claim to be proved at the outset so that the induction hypothesis took a general form. In particular, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\beta/\alpha])$ whenever $\upsilon_{\mathcal{I}}^{\hat{a}}(\alpha) = \upsilon_{\mathcal{I}}^{\hat{a}}(\beta)$. Since this holds for any variable assignment $\hat{a}$, we were able to apply the induction hypothesis in order to prove ($\dagger$).

The next lemma proves something similar, this time holding the wff φ fixed and varying the model. In particular, any model that agrees with $\mathcal{M}$ on all constants and predicates which occur in φ will yield the same truth-value at any given variable assignment.

Lemma 11.6 If $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ and $\mathcal{M}' = \langle \mathbb{D}, \mathcal{I}' \rangle$ share the domain $\mathbb{D}$ where $\mathcal{I}(\mathcal{F}^n) = \mathcal{I}'(\mathcal{F}^n)$ and $\mathcal{I}(\alpha) = \mathcal{I}'(\alpha)$ for every n-place predicate $\mathcal{F}^n$ and constant α that occurs in a wff φ, then $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}'}^{\hat{a}}(\varphi)$ for any variable assignment $\hat{a}$ over $\mathbb{D}$.

Proof: Assume that $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ and $\mathcal{M}' = \langle \mathbb{D}, \mathcal{I}' \rangle$ where $\mathcal{I}(\mathcal{F}^n) = \mathcal{I}'(\mathcal{F}^n)$ and $\mathcal{I}(\alpha) = \mathcal{I}'(\alpha)$ for every n-place predicate $\mathcal{F}^n$ and constant α that occurs in φ. The proof goes by induction on the complexity of φ.

Base: Assume $\texttt{Comp}(\varphi) = 0$ where $\hat{a}$ is any variable assignment over $\mathbb{D}$. It follows that φ is either $\mathcal{F}^n\alpha_1, \ldots \alpha_n$ or $\alpha_1 = \alpha_2$. Consider the following cases:

$$\begin{aligned} \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\mathcal{F}^n\alpha_1, \ldots, \alpha_n) = 1 \quad & \textit{iff} \;\; \langle v_{\mathcal{I}}^{\hat{a}}(\alpha_1), \ldots, v_{\mathcal{I}}^{\hat{a}}(\alpha_n) \rangle \in \mathcal{I}(\mathcal{F}^n) \\ (\star) \quad & \textit{iff} \;\; \langle v_{\mathcal{I}'}^{\hat{a}}(\alpha_1), \ldots, v_{\mathcal{I}'}^{\hat{a}}(\alpha_n) \rangle \in \mathcal{I}'(\mathcal{F}^n) \\ & \textit{iff} \;\; \mathcal{V}_{\mathcal{I}'}^{\hat{a}}(\mathcal{F}^n\alpha_1, \ldots, \alpha_n) = 1. \end{aligned}$$

$$\begin{aligned} \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha_1 = \alpha_n) = 1 \quad & \textit{iff} \;\; v_{\mathcal{I}}^{\hat{a}}(\alpha_1) = v_{\mathcal{I}}^{\hat{a}}(\alpha_n) \\ (*) \quad & \textit{iff} \;\; v_{\mathcal{I}'}^{\hat{a}}(\alpha_1) = v_{\mathcal{I}'}^{\hat{a}}(\alpha_n) \\ & \textit{iff} \;\; \mathcal{V}_{\mathcal{I}'}^{\hat{a}}(\alpha_1 = \alpha_n) = 1. \end{aligned}$$

Whereas $\mathcal{I}(\mathcal{F}^n) = \mathcal{I}'(\mathcal{F}^n)$ is immediate from the assumption, given any $1 \leqslant i \leqslant n$, observe that $v_{\mathcal{I}}^{\hat{a}}(\alpha_i) = \mathcal{I}(\alpha_i) = \mathcal{I}'(\alpha_i) = v_{\mathcal{I}'}^{\hat{a}}(\alpha_i)$ if α_i is a constant. If instead α_i is a variable, then $v_{\mathcal{I}}^{\hat{a}}(\alpha_i) = \hat{a}(\alpha_i) = v_{\mathcal{I}'}^{\hat{a}}(\alpha_i)$, thereby establishing $(\star)$ and $(*)$. It follows that $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}'}^{\hat{a}}(\varphi)$ for any variable assignment $\hat{a}$ over $\mathbb{D}$ if $\texttt{Comp}(\varphi) = 0$.

Induction: Assume that if $\texttt{Comp}(\varphi) \leqslant n$, then $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}'}^{\hat{a}}(\varphi)$ for all variable assignments $\hat{a}$ over $\mathbb{D}$. Letting $\texttt{Comp}(\varphi) = n + 1$, there are seven cases to consider corresponding to the operators $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall\gamma$, and $\exists\gamma$.

Case 1: Assume $\varphi = \neg\psi$. Since $\texttt{Comp}(\varphi) = n+1$ and $\texttt{Comp}(\neg\psi) = \texttt{Comp}(\psi)+1$, it follows that $\texttt{Comp}(\psi) \leqslant n$. By hypothesis, we know $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi) = \mathcal{V}_{\mathcal{I}'}^{\hat{a}}(\psi)$ for all variable assignments $\hat{a}$ over $\mathbb{D}$, and so by the semantics for negation, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\neg\psi) = \mathcal{V}_{\mathcal{I}'}^{\hat{a}}(\neg\psi)$ for all variable assignments $\hat{a}$ over $\mathbb{D}$. Equivalently, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}'}^{\hat{a}}(\varphi)$ for all variable assignments $\hat{a}$ over $\mathbb{D}$ as desired. The cases for $\wedge, \vee, \rightarrow$, and $\leftrightarrow$ are similar.

Case 6: Assume $\varphi = \forall\gamma\psi$. For the same reasons given above, $\texttt{Comp}(\psi) \leqslant n$. We may then consider the following biconditionals:

$$\begin{aligned} \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall\gamma\psi) = 1 \quad & \textit{iff} \;\; \mathcal{V}_{\mathcal{I}}^{\hat{e}}(\psi) = 1 \text{ for all } \gamma\text{-variants } \hat{e} \text{ of } \hat{a} \\ (\dagger) \quad & \textit{iff} \;\; \mathcal{V}_{\mathcal{I}'}^{\hat{e}}(\psi) = 1 \text{ for all } \gamma\text{-variants } \hat{e} \text{ of } \hat{a} \\ & \textit{iff} \;\; \mathcal{V}_{\mathcal{I}'}^{\hat{a}}(\forall\gamma\psi) = 1. \end{aligned}$$

By hypothesis, $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(\psi) = \mathcal{V}_{\mathcal{I}'}^{\hat{e}}(\psi)$ for any variable assignment $\hat{e}$, thereby establishing $(\dagger)$. The other biconditionals follow from the semantics for the universal quantifier. Thus $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}'}^{\hat{a}}(\varphi)$ for all variable assignments $\hat{a}$ over $\mathbb{D}$.

Since the cases for $\wedge, \vee, \rightarrow, \leftrightarrow$, and $\exists\gamma$ are similar to those above, they will be left as exercises for the reader. It follows that $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}'}^{\hat{a}}(\varphi)$ for all variable assignments $\hat{a}$ if $\texttt{Comp}(\varphi) = n + 1$. Thus the lemma follows by induction. □

Although by no means surprising, the lemma above plays a crucial role in a number of the proofs below. We may now turn to consider the remaining proof rules for FOL$^=$.

11.4 FOL$^{=}$ Rules

By drawing on the previous lemmas, we may prove a number of much more usable results. In particular, the following lemma provides a semantic analogue for universal introduction whereby we may assert the logical consequence of a universal claim given only the logical consequence of a sufficiently arbitrary instance.

11.4.1 Universal Quantifier Rules

Lemma 11.7 For any constant β that does not occur in $\forall\alpha\varphi$ or in any sentence $\chi \in \Gamma$, if $\Gamma \vDash \varphi[\beta/\alpha]$, then $\Gamma \vDash \forall\alpha\varphi$.

Proof: Assume $\Gamma \vDash \varphi[\beta/\alpha]$ where β is a constant that does not occur in $\forall\alpha\varphi$ or in any sentence $\chi \in \Gamma$. Assume for contradiction that $\Gamma \nvDash \forall\alpha\varphi$, and so there is some model $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ where $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma$ but $\mathcal{V}_{\mathcal{I}}(\forall\alpha\varphi) = 1$. Thus $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\forall\alpha\varphi) \neq 1$ for some v.a. $\hat{a}$. By the semantics for the universal quantifier, $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) \neq 1$ for some α-variant $\hat{c}$ of $\hat{a}$. Let $\mathcal{M}'$ be the same as $\mathcal{M}$ with the exception that $\mathcal{I}'(\beta) = \hat{c}(\alpha)$. The following biconditionals hold for every $\psi \in \Gamma$:

$$\begin{aligned} \mathcal{V}_{\mathcal{I}}(\psi) = 1 \quad & \textit{iff} \;\; \mathcal{V}^{\hat{e}}_{\mathcal{I}}(\psi) = 1 \text{ for every variable assignment } \hat{e} \\ (\star) \quad & \textit{iff} \;\; \mathcal{V}^{\hat{e}}_{\mathcal{I}'}(\psi) = 1 \text{ for every variable assignment } \hat{e} \\ & \textit{iff} \;\; \mathcal{V}_{\mathcal{I}'}(\psi) = 1. \end{aligned}$$

By construction, $\mathcal{M}$ and $\mathcal{M}'$ have the same domain $\mathbb{D}$ where $\mathcal{I}(\mathcal{F}^n) = \mathcal{I}'(\mathcal{F}^n)$ and $\mathcal{I}(\alpha) = \mathcal{I}'(\alpha)$ for every n-place predicate $\mathcal{F}^n$ and every constant $\alpha \neq \beta$. Since β does not occur in any $\psi \in \Gamma$, we know $(\star)$ follows from **Lemma 11.6**. Thus $\mathcal{V}_{\mathcal{I}}(\psi) = \mathcal{V}_{\mathcal{I}'}(\psi)$ for all $\psi \in \Gamma$, and so $\mathcal{V}_{\mathcal{I}'}(\chi) = 1$ for all $\chi \in \Gamma$. By the starting assumption, $\mathcal{V}_{\mathcal{I}'}(\varphi[\beta/\alpha]) = 1$, and so $\mathcal{V}^{\hat{g}}_{\mathcal{I}'}(\varphi[\beta/\alpha]) = 1$ for every v.a. defined over $\mathbb{D}$. It follows that $\mathcal{V}^{\hat{c}}_{\mathcal{I}'}(\varphi[\beta/\alpha]) = 1$ in particular.

Recall $\mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) \neq 1$ from above. Since β does not occur in $\forall\alpha(\varphi)$, it follows that β does not occur in φ, and so $\mathcal{V}^{\hat{c}}_{\mathcal{I}'}(\varphi) \neq 1$ follows by **Lemma 11.6**. However, $\hat{c}(\alpha) = \mathcal{I}'(\beta)$ where β is a constant, and so $\boldsymbol{v}^{\hat{c}}_{\mathcal{I}'}(\alpha) = \boldsymbol{v}^{\hat{c}}_{\mathcal{I}'}(\beta)$ where β is free for α in φ. Thus $\mathcal{V}^{\hat{c}}_{\mathcal{I}'}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}'}(\varphi[\beta/\alpha])$ follows from **Lemma 11.5**, and so $\mathcal{V}^{\hat{c}}_{\mathcal{I}'}(\varphi[\beta/\alpha]) \neq 1$, contradicting the above. We may then conclude that $\Gamma \vDash \forall\alpha\varphi$. □

Given the lemma above, it easy to prove that the universal introduction proof rule preserves logical consequence in a similar manner to proof rules above. Consider the following proof.

> **Rule 13 (∀I)** $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule ∀I.

Proof: Assume that φ_{n+1} follows from Γ_{n+1} by universal introduction $\forall$I. Thus there is some $i \leqslant n$ where $\varphi_i = \varphi[\beta/\alpha]$ is live at $n+1$ and β does not occur in $\varphi_{n+1} = \forall\alpha\varphi$ or in undischarged assumptions in Γ_{n+1}. By **Lemma 4.3**, $\Gamma_i \subseteq \Gamma_{n+1}$ where $\Gamma_i \vDash \varphi_i$ by hypotheses, and so $\Gamma_{n+1} \vDash \varphi_i$ by **Lemma 2.1**. Equivalently, $\Gamma_{n+1} \vDash \varphi[\beta/\alpha]$. Since β does not occur in $\forall\alpha\varphi$ or any undischarged assumptions in Γ_{n+1}, it follows by **Lemma 11.7** that $\Gamma_{n+1} \vDash \forall\alpha\varphi$, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$. $\square$

This proof amounts to little more than an application of **Lemma 11.7**. In particular, there is no mention of the semantics for the universal quantifiers in the proof of **Rule 13** since all of these details are already contained in the supporting lemma. The following lemma will play an analogous role for universal elimination.

Lemma 11.8 $\forall\alpha\varphi \vDash \varphi[\beta/\alpha]$ where α is a variable and $\varphi[\beta/\alpha]$ is a wfs of $\mathcal{L}^=$.

Proof: Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be any model where $\mathcal{V}_{\mathcal{I}}(\forall\alpha\varphi) = 1$. Thus $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall\alpha\varphi) = 1$ for every v.a. $\hat{a}$ defined over $\mathbb{D}$, and so $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 1$ for every α-variant $\hat{c}$ of $\hat{a}$ by the semantics for the universal quantifier. Letting $\hat{e}$ be an α-variant of $\hat{a}$ where $\hat{e}(\alpha) = \mathcal{I}(\beta)$, it follows that $\boldsymbol{v}_{\mathcal{I}}^{\hat{e}}(\alpha) = \boldsymbol{v}_{\mathcal{I}}^{\hat{e}}(\beta)$. Since there are no free variables in $\varphi[\beta/\alpha]$, we know that β is a constant, and so β is free for α in φ. Thus $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{e}}(\varphi[\beta/\alpha])$ follows by **Lemma 11.5**, and so $\mathcal{V}_{\mathcal{I}}(\varphi[\beta/\alpha]) = 1$ by **Lemma 9.2** since $\varphi[\beta/\alpha]$ is a wfs of $\mathcal{L}^=$. It follows that $\forall\alpha\varphi \vDash \varphi[\beta/\alpha]$. $\square$

Whereas **Lemma 11.7** made use of the particular constraints that must hold for the universal introduction rule to be applied, the lemma above is much less constrained. This corresponds to the fact that universal claims entail all of their substitution instances.

We now turn to provide another supporting lemma which will help further streamline the proof for the universal elimination rule as well as a number of other proofs below.

Lemma 11.9 If $\Gamma \vDash \varphi$ and $\Sigma \cup \{\varphi\} \vDash \psi$, then $\Gamma \cup \Sigma \vDash \psi$.

Proof: Assume $\Gamma \vDash \varphi$ and $\Sigma \cup \{\varphi\} \vDash \psi$. Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be a model where $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma \cup \Sigma$. Since $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma$, we know by the starting assumptions that $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$. Since $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Sigma$, it follows that $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Sigma \cup \{\varphi\}$, and so $\mathcal{V}_{\mathcal{I}}(\psi) = 1$ follows be the starting assumptions. Generalizing on $\mathcal{M}$, we may conclude that $\Gamma \cup \Sigma \vDash \psi$. $\square$

The proof above is a semantic analogue of a metarule that goes by the name 'Cut' since it allows us to cut out intermediaries. This will play a helpful role in the following proof.

Rule 14 (∀E) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule ∀E.

Proof: Assume that φ_{n+1} follows from Γ_{n+1} by universal elimination ∀E. Thus there is some $i \leqslant n$ where $\varphi_i = \forall\alpha\varphi$ is live at $n+1$ and $\varphi_{n+1} = \varphi[\beta/\alpha]$ for some variable α and constant β. By **Lemma 4.3**, $\Gamma_i \subseteq \Gamma_{n+1}$ where $\Gamma_i \vDash \varphi_i$ by hypotheses, and so $\Gamma_{n+1} \vDash \varphi_i$ by **Lemma 2.1**. Equivalently, $\Gamma_{n+1} \vDash \forall\alpha\varphi$. By **Lemma 11.8** that $\forall\alpha\varphi \vDash \varphi[\beta/\alpha]$, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$ by **Lemma 11.9**. □

This proof turns on **Lemma 11.8** where the other lemmas only play a supporting role.

11.4.2 Existential Quantifier Rules

Just as universal elimination is an easier rule to apply with fewer constraints, something similar may be said for existential introduction. Nevertheless, the following lemma will help to show that the proof rule for existential introduction preserves logical consequence.

Lemma 11.10 $\varphi[\beta/\alpha] \vDash \exists\alpha\varphi$ where α is a variable and $\varphi[\beta/\alpha]$ is a wfs of $\mathcal{L}^=$.

Proof: Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be a model where $\mathcal{V}_\mathcal{I}(\varphi[\beta/\alpha]) = 1$, and so $\mathcal{V}_\mathcal{I}^{\hat{a}}(\varphi[\beta/\alpha]) = 1$ for every v.a. $\hat{a}$ defined over $\mathbb{D}$. Letting $\hat{c}$ be a v.a. where $\hat{c}(\alpha) = \mathcal{I}(\beta)$, it follows that $v_\mathcal{I}^{\hat{c}}(\alpha) = v_\mathcal{I}^{\hat{c}}(\beta)$ where β is free for α in φ, and so $\mathcal{V}_\mathcal{I}^{\hat{c}}(\varphi[\beta/\alpha]) = \mathcal{V}_\mathcal{I}^{\hat{c}}(\varphi)$ by **Lemma 11.5**. By the semantics for the existential quantifier, $\mathcal{V}_\mathcal{I}^{\hat{c}}(\exists\alpha\varphi) = 1$ since $\hat{c}$ is an α-variant of itself. Since $\varphi[\beta/\alpha]$ is a wfs $\mathcal{L}^=$, at most α is free in φ, and so $\exists\alpha\varphi$ is a wfs. Hence $\mathcal{V}_\mathcal{I}(\exists\alpha\varphi) = 1$ by **Lemma 9.2**, and so $\varphi[\beta/\alpha] \vDash \exists\alpha\varphi$. □

This lemma follows easily from **Lemma 11.5** where most of the work was already accomplished save for one critical appeal to the semantics for the existential quantifier. We may now turn to provide a proof for the existential introduction rule given below:

Rule 15 (∃I) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule ∃I.

Proof: Assume that φ_{n+1} follows from Γ_{n+1} by existential introduction ∃I. Thus there is some $i \leqslant n$ where $\varphi_i = \varphi[\beta/\alpha]$ is live at $n+1$ and $\varphi_{n+1} = \exists\alpha\varphi$ for some variable α and constant β. By **Lemma 4.3**, $\Gamma_i \subseteq \Gamma_{n+1}$ where $\Gamma_i \vDash \varphi_i$ by hypotheses, and so $\Gamma_{n+1} \vDash \varphi_i$ by **Lemma 2.1**. Equivalently, $\Gamma_{n+1} \vDash \varphi[\beta/\alpha]$. Since $\varphi[\beta/\alpha] \vDash \exists\alpha\varphi$ by **Lemma 11.10**, $\Gamma_{n+1} \vDash \varphi_{n+1}$ by **Lemma 11.9**. □

Like the proof for universal elimination, this proof amounts to little more than an application of **Lemma 11.10** where most of the work was already completed there. Whereas universal elimination and existential introduction are easy to apply and relatively unconstrained, the existential elimination rule is much more restricted. Accordingly, the following lemma makes use of these restrictions in order to establish a semantic analogue of the existential elimination rule in a similar manner to the supporting lemma for universal introduction.

Lemma 11.11 For any constant β that does not occur in $\exists\alpha\varphi$, ψ, or in any sentence $\chi \in \Gamma$, if $\Gamma \vDash \exists\alpha\varphi$ and $\Gamma \cup \{\varphi[\beta/\alpha]\} \vDash \psi$, then $\Gamma \vDash \psi$.

Proof: Assume $\Gamma \vDash \exists\alpha\varphi$ and $\Gamma \cup \{\varphi[\beta/\alpha]\} \vDash \psi$ where β is a constant that does not occur in $\exists\alpha\varphi$, ψ, or in any sentence $\chi \in \Gamma$. Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be a model where $\mathcal{V}_{\mathcal{I}}(\chi) = 1$ for all $\chi \in \Gamma$. It follows that $\mathcal{V}_{\mathcal{I}}(\exists\alpha\varphi) = 1$, and so $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\exists\alpha\varphi) = 1$ for some v.a. $\hat{a}$ defined over $\mathbb{D}$ by **Lemma 9.2**. Thus $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 1$ for some α-variant $\hat{c}$ of $\hat{a}$ by the semantics for the existential quantifier.

Let $\mathcal{M}'$ be the same as $\mathcal{M}$ with the only possible exception being that $\mathcal{I}'(\beta) = \hat{c}(\alpha)$ so that $\boldsymbol{v}_{\mathcal{I}'}^{\hat{c}}(\beta) = \boldsymbol{v}_{\mathcal{I}'}^{\hat{c}}(\alpha)$. Letting $\chi \in \Gamma$, we know that $\mathcal{V}_{\mathcal{I}}(\chi) = 1$, and so $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(\chi) = 1$ for some v.a. $\hat{e}$ by **Lemma 9.2**. Given the assumptions about β, it follows from **Lemma 11.6** that $\mathcal{V}_{\mathcal{I}'}^{\hat{e}}(\chi) = 1$, and so $\mathcal{V}_{\mathcal{I}'}(\chi) = 1$ again by **Lemma 9.2**. By generalizing on χ, we may conclude that $\mathcal{V}_{\mathcal{I}'}(\chi) = 1$ for all $\chi \in \Gamma$.

Since β does not occur in $\exists\alpha\varphi$, it follows that β does not occur in φ, and so $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = \mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi)$ by **Lemma 11.6**. Moreover, $\boldsymbol{v}_{\mathcal{I}'}^{\hat{c}}(\beta) = \boldsymbol{v}_{\mathcal{I}'}^{\hat{c}}(\alpha)$ where β is free for α in φ on account of being a constant, and so $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi) = \mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi[\beta/\alpha])$ by **Lemma 11.5**. Given the identities above, $\mathcal{V}_{\mathcal{I}'}^{\hat{c}}(\varphi[\beta/\alpha]) = 1$, and so $\mathcal{V}_{\mathcal{I}'}(\varphi[\beta/\alpha]) = 1$ since $\varphi[\beta/\alpha]$ is a wfs of $\mathcal{L}^=$. Thus $\mathcal{V}_{\mathcal{I}'}(\chi) = 1$ for all $\chi \in \Gamma \cup \{\varphi[\beta/\alpha]\}$.

It follows by the starting assumption that $\mathcal{V}_{\mathcal{I}'}(\psi) = 1$, and so $\mathcal{V}_{\mathcal{I}'}^{\hat{g}}(\psi) = 1$ for every v.a. $\hat{g}$ defined over $\mathbb{D}$. Since β does not occur in ψ, we may conclude by **Lemma 11.6** that $\mathcal{V}_{\mathcal{I}}^{\hat{g}}(\psi) = 1$ for every v.a. $\hat{g}$ defined over $\mathbb{D}$. Thus $\mathcal{V}_{\mathcal{I}}(\psi) = 1$, and so $\Gamma \vDash \psi$ follows by generalizing on $\mathcal{M}$. □

Given a model $\mathcal{M}$ which makes all of the premises in Γ, it follows that $\exists\alpha\varphi$ is true in $\mathcal{M}$ on some variable assignment $\hat{c}$ by **Lemma 9.2**. The proof then draws on **Lemma 11.6** in order to introduce a model variant $\mathcal{M}'$ which assigns the constant β to whatever the variable α happens to be assigned by $\hat{c}$. The variable α in the wff φ is then replaced with β where **Lemma 11.5** is used to show that the truth-value of $\varphi[\beta/\alpha]$ remains unaffected in the model variant and variable assignment in question.

Since β does not occur in the premises, the premises are also true on the model variant, and since $\varphi[\beta/\alpha]$ is true in the model variant, ψ is true in in the model variant given the starting assumption. Since β does not occur in the conclusion ψ, we may conclude by **Lemma 11.6** that ψ is true in the original model $\mathcal{M}$. Generalizing on $\mathcal{M}$ completes the proof.

Given the previous lemma, we may proceed to show that the proof rule for existential elimination preserves logical consequence as desired.

Rule 16 ($\exists$E) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule $\exists$E.

Proof: Assume that φ_{n+1} follows from Γ_{n+1} by existential elimination $\exists$E. Thus there is some $i < j < k \leqslant n$ where $\varphi_i = \exists\alpha\varphi$ is live at $n+1$, $\varphi_j = \varphi[\beta/\alpha]$ for some constant β that does not occur in φ_i, φ_k, or any $\psi \in \Gamma_i$. Thus we have:

$$\begin{array}{l|lll}
i & \exists\alpha\varphi & & \\
j & & \varphi[\beta/\alpha] & :\text{AS for } \exists\text{E} \\
 & & \vdots & \\
k & & \psi & \\
n+1 & \psi & & :i,\ j\text{–}k\ \exists\text{E}
\end{array}$$

By hypothesis, $\Gamma_i \vDash \varphi_i$ and $\Gamma_k \vDash \varphi_k$ where $\Gamma_i \subseteq \Gamma_{n+1}$ by **Lemma 4.3**. With the exception of φ_j, every assumption that is undischarged at line k is also undischarged at line $n+1$, and so $\Gamma_k \subseteq \Gamma_{n+1} \cup \{\varphi_j\}$. It follows by **Lemma 2.1** that $\Gamma_{n+1} \vDash \varphi_i$ and $\Gamma_{n+1} \cup \{\varphi_j\} \vDash \varphi_k$, and so $\Gamma_{n+1} \vDash \exists\alpha\varphi$ and $\Gamma_{n+1} \cup \{\varphi[\beta/\alpha]\} \vDash \psi$. Thus $\Gamma_{n+1} \vDash \psi$ by **Lemma 11.11**, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$. $\square$

Since **Lemma 11.11** already does most of the heavy lifting, the proof above is the result of carefully setting up a generic scenario in which the existential elimination rule is applied, using the lemmas cited above to draw out the resulting consequences.

11.4.3 Identity Rules

Recall from the proof of **Lemma 11.1** that we have already considered identity introduction in the case of a one line proof. All that remains is to generalize this proof to the present setting where the $n+1$ line is the result of identity introduction.

Rule 17 (=I) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule =I.

Proof: Assume that φ_{n+1} follows from Γ_{n+1} by existential introduction =I. Thus φ_{n+1} is $\alpha = \alpha$ for some constant α. Letting $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be any model, it follows that $\mathcal{I}(\alpha) = \mathcal{I}(\alpha)$, and so $\boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\alpha) = \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\alpha)$ for any variable assignment $\hat{a}$. By the semantics for identity, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha = \alpha) = 1$, and so $\vDash \alpha = \alpha$ by generalizing on $\mathcal{M}$. Equivalently $\vDash \varphi_{n+1}$, and so $\Gamma_{n+1} \vDash \varphi_{n+1}$ follows by **Lemma 2.1**. □

The proof above follows the same line of reasoning given in **Lemma 11.1**. In order to provide a proof for identity elimination, the following lemma establishes a semantic analogue of the identity elimination rule where this proof will draw on the substitution lemma given above.

Lemma 11.12 If α and β are constants, then $\varphi[\alpha/\gamma], \alpha = \beta \vDash \varphi[\beta/\gamma]$.

Proof: Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be a model where $\mathcal{V}_{\mathcal{I}}(\varphi[\alpha/\gamma]) = \mathcal{V}_{\mathcal{I}}(\alpha = \beta) = 1$ where α and β are both constants. By **Lemma 9.2**, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\alpha/\gamma]) = 1$ for some variable assignment $\hat{a}$ over $\mathbb{D}$, where $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\alpha = \beta) = 1$ for all variable assignments $\hat{c}$ over $\mathbb{D}$, and so $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha = \beta) = 1$ in particular. Thus $\boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\alpha) = \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\beta)$.

Since β is a constant, β is free for α in $\varphi[\alpha/\gamma]$, and so it follows by **Lemma 11.5** that $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\alpha/\gamma]) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}((\varphi[\alpha/\gamma])[\beta/\alpha])$. However, $(\varphi[\alpha/\gamma])[\beta/\alpha] = \varphi[\beta/\gamma]$, and so $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\beta/\gamma]) = 1$. Since $\varphi[\beta/\gamma]$ is a wfs of $\mathcal{L}^{=}$, $\mathcal{V}_{\mathcal{I}}(\varphi[\beta/\gamma]) = 1$. By generalizing on $\mathcal{M}$ we may conclude that $\varphi[\alpha/\gamma], \alpha = \beta \vDash \varphi[\beta/\gamma]$. □

This lemma amounts to little more than an application of **Lemma 11.5** together with the observation that $(\varphi[\alpha/\gamma])[\beta/\alpha] = \varphi[\beta/\gamma]$. We may then provide the following proof:

Rule 18 (=E) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if φ_{n+1} follows from Γ_{n+1} by the rule =E.

Proof: Assume that φ_{n+1} follows from Γ_{n+1} by existential elimination =E. Thus there are some live lines $i, j \leqslant n$ at $n+1$ where φ_i is $\alpha = \beta$ for some constants α and β and either $\varphi_j = \varphi[\alpha/\gamma]$ and $\varphi_{n+1} = \varphi[\beta/\gamma]$ or else $\varphi_j = \varphi[\beta/\gamma]$ and $\varphi_{n+1} = \varphi[\beta/\gamma]$. By parity of reasoning, we may assume that $\varphi_j = \varphi[\alpha/\gamma]$ and $\varphi_{n+1} = \varphi[\beta/\gamma]$ which we may represent as follows:

i	$\alpha = \beta$	
j	$\varphi[\alpha/\gamma]$	
$n+1$	$\varphi[\beta/\gamma]$	:i, j =E

By **Lemma 4.3**, $\Gamma_i, \Gamma_j \subseteq \Gamma_{n+1}$ where $\Gamma_i \vDash \varphi_i$ and $\Gamma_j \vDash \varphi_j$ by hypotheses, and so $\Gamma_{n+1} \vDash \varphi_i$ and $\Gamma_{n+1} \vDash \varphi_j$ by **Lemma 2.1**. Equivalently, $\Gamma_{n+1} \vDash \alpha = \beta$ and

$\Gamma_{n+1} \vDash \varphi[\alpha/\gamma]$. Since α and β are constants, we know by **Lemma 11.12** that $\varphi[\alpha/\gamma], \alpha = \beta \vDash \varphi[\beta/\gamma]$. By two applications of **Lemma 11.9**, we may conclude that $\Gamma_{n+1} \vDash \varphi[\beta/\gamma]$, or equivalently, $\Gamma_{n+1} \vDash \varphi_{n+1}$. □

Since **Lemma 11.12** does most of the work above and **Lemma 11.5** made it easy to prove **Lemma 11.12**, identity elimination can be viewed as an application of **Lemma 11.5**. Put otherwise, **Lemma 11.5** is what explains why the identity elimination rule preserves validity.

11.5 Conclusion

Given PL RULES together with **Rule 13** – **Rule 18**, we may now assert the following:

FOL$^=$ RULES: If $\Gamma_k \vDash \varphi_k$ for every $k \leqslant n$ and φ_{n+1} follows by the proof rules for FOL$^=$, then $\Gamma_{n+1} \vDash \varphi_{n+1}$.

Having done most of work required, we are now in a position to establish the induction lemma cited in the proof of FOL$^=$ SOUNDNESS above.

Lemma 11.13 (Induction Step) $\Gamma_{n+1} \vDash \varphi_{n+1}$ if $\Gamma_k \vDash \varphi_k$ for every $k \leqslant n$.

Assume that $\Gamma_k \vDash \varphi_k$ for every $k \leqslant n$. It remains to show that $\Gamma_{n+1} \vDash \varphi_{n+1}$. By the definition of a proof in FOL$^=$, we know that φ_{n+1} is either a premise or follows by one of the proof rules for FOL$^=$. If φ_{n+1} is a premise, then $\varphi_{n+1} \in \Gamma_{n+1}$ and so $\Gamma_{n+1} \vDash \varphi_{n+1}$ is immediate. If φ_{n+1} follows from the previous lines by one of the proof rules for FOL$^=$, then given our starting assumption that $\Gamma_k \vDash \varphi_k$ for every $k \leqslant n$, it follows from FOL$^=$ RULES that $\Gamma_{n+1} \vDash \varphi_{n+1}$. Thus $\Gamma_{n+1} \vDash \varphi_{n+1}$ in either case. Discharging our assumption completes the proof. □

Not only does the soundness of FOL$^=$ tell us that we can rely on our natural deduction systems in order to construct valid arguments in which the conclusion is a logical consequence of the premises, soundness begins to close the gap between two very different approaches to logic. Whereas the logical consequence relation $\vDash$ for $\mathcal{L}^=$ describes what follows from what in virtue of logical form by quantifying over all models of $\mathcal{L}^=$, the derivation relation $\vdash$ aims to directly encode natural patterns of reasoning in $\mathcal{L}^=$. What soundness shows is that our purely proof-theoretic descriptions of logical reasoning in $\mathcal{L}^=$ does not diverge from our model-theoretic descriptions of logical reasoning in $\mathcal{L}^=$.

In the following chapter, we will consider the converse, showing that in addition to being sound, FOL$^=$ is also complete. By contrast with soundness which one might insist any proof system must satisfy over a reasonable semantics, completeness is a powerful and deeply surprising result. As before, our approach will be to build on what we have already established.