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Chapter 3

Natural Deduction in PL

This chapter introduces a natural deduction proof system for $\mathcal{L}^{\text{PL}}$ which we will refer to as *Propositional Logic* (PL). This is the logic of sentences which aims to describe formal reasoning which is valid in virtue of the logical form of the sentences involved where the syntactic primitives are all sentences or sentential operators. In a future chapter, we will introduce *First-Order Logic* (FOL) for $\mathcal{L}^{\text{FOL}}$ which includes predicates, constants, variables, and quantifiers. Until then, we will continue to restrict attention to what can be expressed with the resources included in $\mathcal{L}^{\text{PL}}$. Accordingly, PL does not aim to describe *all* of formal reasoning, but rather only the formal reasoning that can be carried out in $\mathcal{L}^{\text{PL}}$.

In Chapter 2, the logical consequence relation $\vDash$ provided a first answer to the question of what logic aims to study. Additionally, we presented the truth table method and semantic proof method for establishing which logical consequences hold and which do not. Nevertheless, these methods leave something to be desired. To begin with, we saw just how poorly the truth table method scales with the number of sentence letters, making this method practically infeasible for more than four sentence letters. Although the semantic proof method did not face this same problem, semantic proofs were often cumbersome to write where their construction was completely unconstrained. That is, we never said, and indeed cannot say, what counts as an adequate semantic proof. Rather, these proofs took place in our metalanguage mathematical English which does not have clear cut boundaries or rules.

Recall the strategies for writing semantic proofs that we began to describe in the previous chapter. For instance, these included proof by contradiction and proof by cases. This raises a question about what are *all* of the strategies that one might employ along these lines. More than strategies, we want to know what are *all* of the moves that we can make when writing a proof, and what inferences are absolutely basic and cannot be subdivided into further steps. Questions of these kinds lead to a completely different approach to our present inquiry into the nature of formal reasoning. Instead of asking what is a logical consequence of what by quantifying over the interpretations of a language, we may seek to describe a collection of basic inferences, chaining these together in order to say what can be inferred from what. This chapter will be concerned to answer this question by providing a proof system for $\mathcal{L}^{\text{PL}}$.

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After considering a number of arguments in English in Chapter 0, we observed that natural languages do not have well-defined boundaries, frustrating any attempt to say something completely general about all sentences and arguments in English. Chapter 1 avoided this problem by presenting the artificial language $\mathcal{L}^{\text{PL}}$ which has a well-defined notion of a wfs that we may use to regiment English sentences and arguments. Although regimentation itself remains a matter of judgment with no definite answers, this method nevertheless provided a way to identify the logical forms that explain why certain patterns of reasoning in English are especially compelling. In particular, we defined the interpretations of $\mathcal{L}^{\text{PL}}$ to be functions from the wfss of $\mathcal{L}^{\text{PL}}$ to truth-values, drawing on this definition in order to introduce logical consequence along with a number of other logical properties and relations. However, none of this would have been possible were we to attempt these definitions for English.

In just the same way that it was important to work with an artificial language in order to provide a mathematically precise definition of logical consequence, it will also be important to draw on a well-defined language in order to describe the basic inferences that hold in virtue of their logical forms. Rather than introducing another artificial language, we will continue to work with $\mathcal{L}^{\text{PL}}$ maintaining all of the definitions from before in order to provide the proof system PL, also called a *logic*, for reasoning in $\mathcal{L}^{\text{PL}}$. Functionally, you can think of PL as including rules somewhat akin to the proof strategies and steps that we used in writing informal proofs before, only now they will take a precise mathematical form.

What of our informal proofs from before? Are they to be trusted given that they are written in the vague natural language English together with some mathematical conventions? What of mathematics proofs in general which are also written in this kind of language? Are these really mere approximations whose validity is only to be accounted for by regimenting them in some more precise language? You might be surprised to know that the answer is 'No'.

Rather than encoding some final truth, the logical systems that we will present are better understood to be abstractions from the intuitive bedrock from which we must begin: natural language, and in our case, English. After all, how would you ever hope to learn what the sentential operators (much less the sentence letters) of $\mathcal{L}^{\text{PL}}$ mean? The semantic answer we provided above used mathematical English to do so, and this was no mistake since meanings have to get going somewhere and in this respect $\mathcal{L}^{\text{PL}}$ is no place to begin.

Instead of undercutting the meanings that you understand in English, introducing formal languages provides a way to distill certain elements of meaning that we have reason to care about even if they depart from their correlates in natural language. In analogy, you can think of this like refining the raw materials found in nature into the sorts of materials that are of considerable use to us in constructing the build environment. Rather than the material world, our concern is with the conceptual world, and what we are doing here is a kind of conceptual engineering. Although these are only metaphors, hopefully they will help to shed some light on what we have been doing and will continue to do throughout this course. In particular, it is important to appreciate that English cannot be given up any more than the natural world around us. Accordingly, we will continue to write informal proofs to establish claims about our object language $\mathcal{L}^{\text{PL}}$. Soon we will have an analogue for also proving things *in* $\mathcal{L}^{\text{PL}}$.

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Whereas the semantic clauses for $\mathcal{L}^{\text{PL}}$ drew on our grasp of certain elements of mathematical English in order to provide a systematic way to *interpret* the wfss of $\mathcal{L}^{\text{PL}}$, this chapter will also draw on mathematical English in order describe how to *reason* in $\mathcal{L}^{\text{PL}}$. One way to think about our target here is to contemplate the extension of the logical consequence relation $\vDash$. That is, think of the set of ordered pairs which relate any set of wfss of $\mathcal{L}^{\text{PL}}$ to a further wfs of $\mathcal{L}^{\text{PL}}$ where the latter is a logical consequence of the former, or in set notation: $\{\langle \Gamma, \varphi \rangle : \Gamma \vDash \varphi\}$. Needless to say, this is a large, though not unruly space. Although our definition of logical consequence $\vDash$ provides an essential account of this space of logical consequences, it is hard work to check which wfss are logical consequences of which sets of wfss of $\mathcal{L}^{\text{PL}}$. Thus it would be convenient to streamline the process by which we may determine whether $\Gamma \vDash \varphi$.

As with all of our methods, writing formal proofs in $\mathcal{L}^{\text{PL}}$ has its range of natural applications where sometimes it is more trouble than it is worth and a semantic proof would have been better. Nevertheless, convenience is not our only motivation. Rather, what we should like to describe are the most basic inferences that make up the practice of formal reasoning, composing those inferences in order to not only say what follows from what in virtue of logical form, but also how. Of course, not any rational seeming maneuvers ought to be included. In particular, we will require the basic inferences that make up PL to be valid. This is referred to as *soundness*: if φ can be inferred from Γ, then φ is a logical consequence of Γ. However, this is not all of what we want. Rather, we also want our basic inferences to be inherently compelling. Put otherwise, we are looking to find the atoms that make up formal reasoning.

Insofar as the proof systems that we will be concerned with in this course aim to encode the natural patterns of formal reasoning, we will refer to these systems as NATURAL DEDUCTION systems. In particular, PL is a natural deduction system for $\mathcal{L}^{\text{PL}}$ where later chapters will consider a natural deduction system for $\mathcal{L}^{\text{FOL}}$. These systems provide a way to argue from the premises to a conclusion in logically valid ways while resembling natural forms of reasoning. In addition to being familiar, reasoning in this way helps to illustrate the logical connections between various claims in a way that is compelling *on its own terms*. That is, you don't have to take a course in logic or learn the semantics for an artificial language in order to appreciate the patterns of reasoning that we will consider, finding them compelling.

In what follows, we will introduce ten basic derivation rules for the five sentential operators in $\mathcal{L}^{\text{PL}}$. Each operator will have an introduction and an elimination rule which, taken together, describe a certain dimension of the *meaning* of that operator. In particular, the introduction and elimination rules describe how to reason with the operators in $\mathcal{L}^{\text{PL}}$. There is also a rule for introducing assumptions and a trivial rule for reiterating earlier lines of a proof. Given these twelve rules in all, we will be in a position to provide a precise definition of a proof in PL, where it is for this reason that PL is referred to as a proof system, or *logic* for $\mathcal{L}^{\text{PL}}$.

As finite and familiar as all of this will turn out to be, PL may nevertheless be shown to have some remarkable properties. In addition to showing that φ is a logical consequence of Γ whenever φ can be proved from Γ, we will also show that nothing is missing: our proof system is capable of deriving *all* logical consequences whatsoever. That is, PL is *complete* in addition to being *sound*. We will prove these results in later chapters.

3.1 Premises and Assumptions

Before introducing the rules, it will help to get a sense of what PL proofs look like in order to articulate some important constraints on the lines of a proof to which a rule may appeal.

A PL proof begins with a (possibly empty) list of premises, where these will be indicated by writing ':PR' on the right. It is often helpful to include a note of what you intend to derive at this point. For instance, consider the following list of premises:

$$\begin{array}{r|ll}
1 & A \to (B \to C) & :\text{PR} \\
2 & A & :\text{PR} \\ \cline{2-2}
\end{array}$$

The horizontal line indicates where the premises end and the rest of the derivation begins. For instance, we may apply conditional elimination (discussed below) to derive the following:

$$\begin{array}{r|ll}
1 & A \to (B \to C) & :\text{PR} \\
2 & A & :\text{PR} \\ \cline{2-2}
3 & B \to C & :1,\ 2 \to\text{E}
\end{array}$$

Note that we appealed to lines 1 and 2 in order to derive line 3, indicating as much in the justification of line 3. If $B \to C$ is all that we wanted to derive, then we would be done. However, suppose that we were to continue by adding a new assumption.

$$\begin{array}{r|ll}
1 & A \to (B \to C) & :\text{PR} \\
2 & A & :\text{PR} \\ \cline{2-2}
3 & B \to C & :1,\ 2 \to\text{E} \\
4 & \quad \underline{\mid\, B\ } & :\text{AS}
\end{array}$$

At any point in a proof, we can introduce a new assumption on an indented line and starting a new vertical line. More precisely, consider the *assumption rule* (AS) below:

$$\begin{array}{|ll}
\quad \underline{\mid\, \varphi\ } & :\text{AS}
\end{array}$$

We will refer to the process of adding an assumption as one of OPENING A SUBPROOF. Subproofs are what they sound like: a proof within a proof, starting from a single assumption added anywhere in a proof rather than the premises with which we began the proof. For instance, we might add the following lines to the proof above:

$$
\begin{array}{r|l|l}
1 & A \to (B \to C) & :\text{PR} \\
2 & A & :\text{PR} \\
\hline
3 & B \to C & :1,\ 2\ {\to}\text{E} \\
4 & \quad \vert\ B & :\text{AS} \\
5 & \quad \vert\ C & :3,\ 4\ {\to}\text{E} \\
6 & \quad \vert\ C \land A & :2,\ 5\ {\land}\text{I} \\
7 & B \to (C \land A) & :4\text{–}6\ {\to}\text{I}
\end{array}
$$

Line 5 applies conditional elimination (discussed below) on lines 3 and 4, and then line 6 applies conjunction introduction (also discussed below) to lines 2 and 5. We then close the subproof, where this may take place at any point in the subproof by ending the vertical line and stepping back one level of indentation. Once a subproof closes, the lines of that closed subproof are DEAD, and so cannot be appealed to individually. Nevertheless, we may appeal to the subproof in its entirety, where line 7 does just this, using conditional introduction (discussed below) which cites lines 4–6 (note the hyphen in place of the comma).

Every line of a proof that is not dead is referred to as LIVE, where rules that cite individual lines (as opposed to subproofs) can only appeal to lines that are live at that point in the proof. For instance, were we to continue our proof a little further, we could not appeal to lines 4, 5, or 6 since these lines are dead. Thus we stipulate the following restriction:

> CITATION: For a rule to cite a single line, that line must not occur within a subproof that has been closed before the line where the rule is being applied.

Closing a subproof is also called DISCHARGING the assumption of that subproof. Subproofs allow us to think about what we could show if we made a further assumption. Accordingly, we have to be careful to keep track of what assumptions we are making and when it is and is not permitted to appeal to an assumption or the wfs of $\mathcal{L}^{\text{PL}}$ that we can derive from that assumption. Our Fitch-style proof system accomplishes this task graphically by indenting assumptions and drawing vertical lines along the length of the resulting subproof. The details for each rule which makes use of this feature of our proof system will be discussed below, but it is important to have some sense of all of this before introducing the rules.

3.2 Reiteration

The first rule was already eluded to above. Given any wfs of $\mathcal{L}^{\text{PL}}$ on a live line of a proof, the *reiteration rule* (R) allows you to repeat that wfs on a new line.

$$
\begin{array}{r|ll}
\vdots & \vdots & \\
4 & A \wedge B & \\
\vdots & \vdots & \\
10 & A \wedge B & :4\ \text{R}
\end{array}
$$

Given that we have written '$A \wedge B$' on line 4, we may repeat this wfs at some later line, e.g., line 10. We also add a citation which justifies what we have written. In this case, we write 'R', to indicate that we are using the reiteration rule, and we write '4', to indicate that we have applied it to line 4. Here is the general expression of the reiteration rule R:

$$
\boxed{
\begin{array}{r|ll}
m & \varphi & \\
 & \varphi & :m\ \text{R}
\end{array}
}
$$

If φ occurs on any line m within the scope of application, we can reiterate φ, justifying this addition by writing ':m R' to indicate that reiteration was applied to line m. Of course, in an actual proof, the lines are numbered, and so m will take on a numerical value.

Here is an example of three legal applications of rule R followed by an illegal application:

$$
\begin{array}{r|lll}
1 & P & & :\text{PR} \\
\cline{2-2}
2 & & \vert\ \neg Q & :\text{AS} \\
3 & & \vert\ \neg Q & :2\ \text{R (LEGAL)} \\
4 & & \vert\ P & :1\ \text{R (LEGAL)} \\
5 & \neg Q \rightarrow P & & :2\text{–}4 \rightarrow\text{I} \\
6 & P & & :1\ \text{R (LEGAL)} \\
7 & \neg Q & & :2\ \text{R, (ILLEGAL)}
\end{array}
$$

On the second line, we begin a subproof by assuming $\neg Q$. You can reiterate $\neg Q$ within the subproof as in line 3, but not when you leave the subproof as in line 7. On line 4, we reiterate P on line 1, maintaining the indentation of the subproof. We then close the subproof, citing the subproof in line 5. At line 6, we can reiterate line 1 which is live, but at line 7 we cannot appeal to line 2 since this line is now dead. Even if we were to open another subproof, we still could not appeal to line 2. Rather, the lines of a closed subproof are forever dead. Even so, this does not stop us from appealing to the subproof as a whole as we do in line 5.

3.3 Conjunction

Consider the rule for *conjunction introduction* ($\wedge$I):

m	φ	
n	ψ	
	$\varphi \wedge \psi$	:m, n $\wedge$I

This rule says that given any wfss φ and ψ of $\mathcal{L}^{\text{PL}}$ on live lines, you may derive their conjunction $\varphi \wedge \psi$. It is worth noting that m and n need not be consecutive lines, nor do they need to appear in the order listed. We require only that each line has been established somewhere in the proof, and that both lines are live at the line in which we are applying the rule.

Whereas conjunction introduction licenses the derivation of a conjunction from any two wfss of $\mathcal{L}^{\text{PL}}$, conjunction elimination lets us do the opposite. Given any live conjunction, we may derive either of its conjuncts. For instance, if $A \wedge (P \vee Q)$ is live, we may derive A, or we may derive $P \vee Q$, but we must choose which. If we wish to derive both, then two applications of the rule is required, though the order does not matter.

Here are the left and right *conjunction elimination* ($\wedge$E) rules:

m	$\varphi \wedge \psi$	
	φ	:m $\wedge$E
	ψ	:m $\wedge$E

These rules allow us to derive either conjunct. Although we will often end up deriving both conjunts, we need not do so. For instance, this is a perfectly acceptable proof:

1	$A \wedge B$	:PR
2	B	:1 $\wedge$E

Note that the $\wedge$E rule only requires one wfs of $\mathcal{L}^{\text{PL}}$, and so there is only one line number in the justification. In order to see the conjunction rules to work together, consider the argument:

A1. $[(A \vee B) \rightarrow (C \vee D)] \wedge [(E \vee F) \rightarrow (G \vee H)]$
A2. $[(E \vee F) \rightarrow (G \vee H)] \wedge [(A \vee B) \rightarrow (C \vee D)]$

The main logical operator in both the premise and conclusion is conjunction. Since conjunction is commutative, the argument is obviously valid since the two conjunctions have the same two conjuncts in the opposite order. In order to provide a proof, we begin by writing down the premise on a numbered line indicating that it is a premise. Since this is the only premise, we draw a horizontal line where everything below this line must be justified by a proof rule.

1	$[(A \vee B) \rightarrow (C \vee D)] \wedge [(E \vee F) \rightarrow (G \vee H)]$	:PR

From the premise, we can separate the conjuncts with $\wedge$E. This yields the following:

1	$[(A \vee B) \rightarrow (C \vee D)] \wedge [(E \vee F) \rightarrow (G \vee H)]$	:PR
2	$[(A \vee B) \rightarrow (C \vee D)]$	:1 $\wedge$E
3	$[(E \vee F) \rightarrow (G \vee H)]$	:1 $\wedge$E

The $\wedge$I rule requires that each of the conjuncts is live somewhere in the proof from the current line, though their order and distance from each other does not matter. By applying the $\wedge$I rule to lines 3 and 2, we may arrive at the desired conclusion.

1	$[(A \vee B) \rightarrow (C \vee D)] \wedge [(E \vee F) \rightarrow (G \vee H)]$	:PR
2	$[(A \vee B) \rightarrow (C \vee D)]$	:1 $\wedge$E
3	$[(E \vee F) \rightarrow (G \vee H)]$	:1 $\wedge$E
4	$[(E \vee F) \rightarrow (G \vee H)] \wedge [(A \vee B) \rightarrow (C \vee D)]$	:3, 2 $\wedge$I

This proof may not look terribly interesting or surprising, but it shows how we can use the proof rules together to demonstrate the validity of an argument. Note that using a truth table to show that this argument is valid would have required a staggering 256 lines, since there are eight sentence letters in the argument. A semantic proof would be less unwieldy, but would not have been as simple or natural of an argument. At the very least, you would have to know a bit about the semantics for our language $\mathcal{L}^{\text{PL}}$. By contrast, the steps in the proof above are already pretty compelling given that '$\wedge$' expresses conjunction.

3.4 Disjunction

Suppose Ludwig is reactionary.[1] Then Ludwig is either reactionary or libertarian. Trivial as this may seem, it speaks to the logic of disjunction. Just as we may derive either conjunct from a conjunction, we may derive a disjunction from either of its disjuncts.

Thus the *disjunction introduction* ($\vee$I) rule may be stated as follows:

m	φ	
	$\varphi \vee \psi$	$:m \vee$I
	$\psi \vee \varphi$	$:m \vee$I

As above, the line m must be live, where we cite this line in the justification of the rule application. Since ψ can be *any* wfs of $\mathcal{L}^{\text{PL}}$, the following is a perfectly acceptable proof:

1	M	:PR
2	$M \vee ([(A \leftrightarrow B) \to (C \wedge D)] \leftrightarrow [E \wedge F])$	:1 $\vee$I

Using a truth table to show this would have taken 128 lines.

The disjunction elimination rule is slightly trickier. Suppose that either Ludwig is reactionary or he is libertarian. It does not follow that Ludwig is reactionary, for he might be a libertarian. Equally, we cannot conclude that Ludwig is libertarian, since he might be reactionary. Given that we don't know which disjunct is true, it is difficult to deduce anything from a disjunction on its own. The elimination rule for disjunction provides a workaround.

Suppose that we could show that if Ludwig's is reactionary, then he is an Austrian economist. Suppose that we could also show that if Ludwig's is a libertarian, then he is also an Austrian economist. Even though we don't know whether Ludwig is reactionary or a libertarian, it doesn't matter: in either case he is an Austrian economist. This is a natural way to make use of a disjunction even when we don't know which disjunct is true. Indeed, we employed reasoning of this kind in the semantic proof by cases that we gave in Chapter 2. Generalizing on this line of reasoning, consider the following *disjunction elimination* ($\vee$E) rule:

[1] This section has been adapted from the Calgary remix §16.7.

m	$\varphi \vee \psi$	
i	$\quad \vert\ \varphi$	:AS for $\vee$E
$\vdots$	$\quad \vert\ \vdots$	
j	$\quad \vert\ \chi$	
k	$\quad \vert\ \psi$	:AS for $\vee$E
$\vdots$	$\quad \vert\ \vdots$	
l	$\quad \vert\ \chi$	
	χ	:m, i–j, k–l $\vee$E

This rule is somewhat clunkier to write down than our previous rules, but the idea is a natural one. Suppose that we have some disjunction $\varphi \vee \psi$. Suppose that we can also construct two subproofs showing that χ can be derived from the assumption that φ, and that χ can be derived from the assumption that ψ. We can then infer χ from the original disjunction $\varphi \vee \psi$ together with our two subproofs. As usual, there can be as many lines as you like between i and j, and as many lines as you like between k and l. Moreover, the subproofs and the disjunction can come in any order, and do not have to be adjacent to each other as above. Although the lines i–j and k–l belong to closed subproofs and so dead, line m must be live.

Some examples will help illustrate. Consider the following argument:

B1. $(P \wedge Q) \vee (P \wedge R)$
B2. P

A proof might run like this, adding the notes 'for $\vee$E' to improve readability:

1	$(P \wedge Q) \vee (P \wedge R)$	:PR
2	$\quad \vert\ P \wedge Q$	:AS for $\vee$E
3	$\quad \vert\ P$	:2 $\wedge$E
4	$\quad \vert\ P \wedge R$	:AS for $\vee$E
5	$\quad \vert\ P$	:4 $\wedge$E
6	P	:1, 2–3, 4–5 $\vee$E

Here is a slightly harder example. Consider the following argument:

C1. $\underline{A \wedge (B \vee C)}$
C2. $(A \wedge B) \vee (A \wedge C)$

We may then construct the following proof:

1	$A \wedge (B \vee C)$	:PR
2	A	:1 $\wedge$E
3	$B \vee C$	:1 $\wedge$E
4	$\quad B$	:AS for $\vee$E
5	$\quad A \wedge B$	:2, 4 $\wedge$I
6	$\quad (A \wedge B) \vee (A \wedge C)$	:5 $\vee$I
7	$\quad C$	:AS for $\vee$E
8	$\quad A \wedge C$	:2, 7 $\wedge$I
9	$\quad (A \wedge B) \vee (A \wedge C)$	:8 $\vee$I
10	$(A \wedge B) \vee (A \wedge C)$	:3, 4–6, 7–9 $\vee$E

As natural as the rules may seem in isolation, it is not always obvious how to put them together to get from some premises to a conclusion. Like any skill, the ability to construct PL proofs requires practice. To help, we will cover some strategies for finding proofs at the end of the chapter. Nevertheless, once a natural deduction proof has been constructed, each step is easy to justify, making the derivation in total impervious to doubts. Moreover, this certainty does not stem from any semantic considerations. Rather, the proof rules are directly justified by our intuitive understanding of how to use the sentential operators in our language.

3.5 Conditional Introduction

The rule for conditional introduction has already been used in the examples used to first set up the proof system, and should have felt both compelling an familiar. The idea here is that you help yourself to something that you may not know is true, do some reasoning to arrive at some further claim, then conclude by asserting a conditional claim: if the assumption is true, then the further claim is true. This conclusion follows despite not knowing if the original assumption is true. Here is a concrete example of this type of reasoning:

> Ludwig is reactionary. Therefore if Ludwig is libertarian, then Ludwig is both reactionary and libertarian.

We may regiment this argument as a natural deduction proof by starting with one premise R for 'Ludwig is reactionary':

$$\begin{array}{r|ll} 1 & \underline{R} & :\text{PR} \end{array}$$

We may now make an additional assumption L for 'Ludwig is libertarian'. In common parlance, we might use the turn of phrase 'Suppose for the sake of argument that…', or when writing informal proofs, we might start off with 'Assume R for conditional proof'. In PL, we will indicate that we are adding an assumption by writing 'AS' on the right, where it is often helpful to also include 'for →Intro' as a note to yourself or your reader.

$$\begin{array}{r|ll} 1 & \underline{R} & :\text{PR} \\ 2 & \quad \underline{L} & :\text{AS for } \to\text{Intro} \end{array}$$

Note that we are not claiming to have proved L from line 1. Accordingly, we do not write any justification for the additional assumption on line 2. Rather, we have started a new subproof by indenting the wfs L and starting a new vertical line. We have also underlined L since it is playing a role analogous to a premise in our new subproof.

With this extra assumption in place, we are now in a position to use ∧I from before.

$$\begin{array}{r|ll} 1 & \underline{R} & :\text{PR} \\ 2 & \quad \underline{L} & :\text{AS for } \to\text{Intro} \\ 3 & \quad R \wedge L & :1,\ 2\ \wedge\text{I} \end{array}$$

Given the assumption L, we have deduced $R \wedge L$. We may now discharge our assumption, closing the subproof and adding an appropriate conditional on the next line.

$$\begin{array}{r|ll} 1 & \underline{R} & :\text{PR} \\ 2 & \quad \underline{L} & :\text{AS for } \to\text{Intro} \\ 3 & \quad R \wedge L & :1,\ 2\ \wedge\text{I} \\ 4 & L \to (R \wedge L) & :2\text{–}3\ \to\text{I} \end{array}$$

Whereas the indented subproof carries out reasoning under the assumption of L, line 4 reverts back to our original proof which carries out reasoning under the assumption of our single premise R. Accordingly, we cannot conclude $R \wedge L$ merely under the assumption of R by writing $R \wedge L$ at the original level of indenting. Nevertheless, we can assert the conditional $L \to (R \wedge L)$ as given in 4, justifying this line by referring to the entire subproof rather than to individual lines of our proof. In this case, there are only two lines in the subproof, but in general there may be many more. Even in the case where the subproof only consists of two lines, we must use a hyphen to indicate that we are citing a subproof instead of two lines.

Generalising on this pattern, consider the *conditional introduction* rule ($\to$I):

i	$\quad \varphi$	:AS
$\vdots$	$\quad \vdots$	
k	$\quad \psi$	
	$\varphi \to \psi$	:i–k $\to$I

By appealing to the subproof as a whole for justification, we may write a conditional in a new line stepping back one level of indentation where the assumption of the subproof occurs as the antecedent of the conditional and the conclusion of the subproof occurs as the consequent. As we will see, knowing what the rule is one thing and knowing when to use it is another.

3.6 Conditional Elimination

Many different arguments demonstrate the classic inference *modus ponens*:

D1. $P \to \neg Q$	E1. $\neg P \to (A \leftrightarrow B)$	F1. $(P \vee Q) \to A$
D2. P	E2. $\neg P$	F2. $P \vee Q$
D3. $\neg Q$	E3. $A \leftrightarrow B$	F3. A

The natural deduction system of this chapter will include a rule of inference corresponding to *modus ponens* which goes by the name *conditional elimination* ($\to$E). Here is the rule:

m	$\varphi \to \psi$	
n	φ	
	ψ	:m, n $\to$E

What this rule says is that if you have a conditional $\varphi \to \psi$ on a live line number m, and you also have the antecedent φ of that conditional on a live line n, you can write the consequent ψ on a new line. In order to justify this inference, we will list the line numbers m and n as well as '$\to$E' to specify the rule. Given the conditional elimination rule, we can prove that the arguments given above are valid. Here are proofs of two of them:

1	$P \to \neg Q$	:PR
2	P	:PR
3	$\neg Q$	:1, 2 $\to$E

1	$(P \lor Q) \to A$	:PR
2	$P \lor Q$	:PR
3	A	:1, 2 $\to$E

Notice that these proofs share the same structure. We start by listing the premises followed by a horizontal line, where subsequent lines will need to be derived with the rules. We then apply the conditional elimination rule to get the conclusion, citing the appropriate lines. One can produce more complicated proofs with the same rule.

G1. A
G2. $A \to B$
G3. $B \to C$
G4. $C \to [\neg P \leftrightarrow (Q \lor R)]$
G5. $\neg P \leftrightarrow (Q \lor R)$

We begin by writing our four premises on numbered lines:

1	A	:PR
2	$A \to B$	:PR
3	$B \to C$	:PR
4	$C \to [\neg P \leftrightarrow (Q \lor R)]$	:PR (Want $\neg P \leftrightarrow (Q \lor R)$)

The parenthetical off to the right is optional, but can help to keep track of the conclusion that we are attempting to establish. The proof will be complete once we derive $\neg P \leftrightarrow (Q \lor R)$ by applying the rules to the premises or lines that result from doing so. Since we cannot use conditional elimination to get to our desired conclusion directly from our premises, it is worth considering what we can do. For instance, we can use conditional elimination on lines 1 and 2 to get B on a new line, and then repeat using our new line together with line 3 to get C on yet another new line. Continuing in this manner gives us the following proof:

1	A	:PR
2	$A \to B$	:PR
3	$B \to C$	:PR
4	$C \to [\neg P \leftrightarrow (Q \vee R)]$	:PR want ¬P ↔(Q ∨R)
5	B	:1, 2 →E
6	C	:3, 5 →E
7	$\neg P \leftrightarrow (Q \vee R))$	:4, 6 →E

Having derive line 5 from lines 1 and 2, we may derive 6 from 3 and 5, and then conclude by deriving 7 from 4 and 6. In general, each time that we appeal to earlier lines in a proof in order to apply a rule, we must check to see if those lines are live. However, in this case, we have not introduced any assumptions, and so there is no risk that any lines fail to be live.

In order to see the conditional introduction and elimination rules work together, consider:

H1. $P \to Q$
H2. $\underline{Q \to R}$
H3. $P \to R$

We start by listing the premises— this much is automatic requiring no thinking whatsoever. But now we have to think about where we are going, i.e., we want to conclude with the conditional $P \to R$. A great way to do this is by conditional introduction and so, to use this rule, we must begin by assuming the antecedent P of the conditional we want to conclude.

1	$P \to Q$	:PR
2	$Q \to R$	:PR
3	$\quad P$	:AS for →I

Note that there is nothing preventing us from appealing to P in the course of our subproof since before we have closed the subproof, all of its lines are still live. Thus we have:

1	$P \rightarrow Q$	:PR
2	$Q \rightarrow R$	:PR
3	$\quad P$	:AS for $\rightarrow$I
4	$\quad Q$	:1, 3 $\rightarrow$E
5	$\quad R$	:2, 4 $\rightarrow$E
6	$P \rightarrow R$	:3–5 $\rightarrow$I

Whereas line 4 derives Q from lines 1 and 3 by conditional elimination, we may apply the same rule to derive R on line 5 from the lines 2 and 4. Finally, we may close our subproof, and conclude $P \rightarrow R$ on line 6 while citing the subproof on lines 3–5. Knowing when exactly to open a subproof can take some practice, but a good rule of thumb is that if you want to establish a conditional either at the end of a proof or along the way, you may well need to assume its antecedent and reason your way to the consequent.

3.7 The Biconditional

The rules for the biconditional will be like double-barrelled versions of the rules for the conditional. In order to prove $F \leftrightarrow G$ you must be able to prove G on the assumption F, and separately, prove F on the assumption G. The *biconditional introduction* rule ($\leftrightarrow$I) therefore requires two subproofs. Schematically, the rule looks like this:

i	$\quad \varphi$	:AS for $\leftrightarrow$I
$\vdots$	$\quad \vdots$	
j	$\quad \psi$	
k	$\quad \psi$	:AS for $\leftrightarrow$I
$\vdots$	$\quad \vdots$	
l	$\quad \varphi$	
	$\varphi \leftrightarrow \psi$	:i–j, k–l $\leftrightarrow$I

There can be as many lines as you like between i and j, and as many lines as you like between k and l. Moreover, the subproofs can come in any order, and the second subproof does not need to come immediately after the first.

The biconditional elimination rule (↔E) lets you do a bit more than the conditional rule. If you have the left-hand subsentence of the biconditional, you can obtain the right-hand subsentence. If you have the right-hand subsentence, you can obtain the left-hand subsentence.

m	$\varphi \leftrightarrow \psi$	
n	φ	
	ψ	:m, n ↔E

Equally, we may work in the reverse direction:

m	$\varphi \leftrightarrow \psi$	
n	ψ	
	φ	:m, n ↔E

Note that in the citation for ↔E, we always cite the biconditional first and either the left or right argument depending as the second argument.

3.8 Negation

Here is a simple mathematical argument:

I1. Assume there is some greatest natural number, call it n.
I2. Now consider its successor $n + 1$ which is also a natural number.
I3. Since $n + 1 > n$, we may conclude that n is not the greatest natural number.
I4. But this contradicts our assumption.
I5. Thus there is no greatest natural number.

We used *reductio* style arguments of this kind in some of the semantic proofs in Chapter 2. The full Latin name *reductio ad absurdum* means "reduction to absurdity." Proofs of this form are also sometimes called *indirect proofs*. A *reductio* argument assumes something which we would like to show is false and aims to derive a contradiction. For instance, we might end up reaching the negation of the *reductio* assumption, or else two wfss of the form ψ and $\neg\psi$. Given such a contradiction, we may assert the negation of the original assumption.

In mathematics, *reductio* arguments often lead to conclusions like $0 = 1$ that contradict something that is already known more generally though the negation $0 \neq 1$ might not show up anywhere in the proof. Whether stated or not, what is going on here is that we really have two contradictory claims: $0 = 1$ and $0 \neq 1$, or to be even more explicit, $\neg(0 = 1)$. Mathematical proofs typically suppress many of the obvious details, and so do not take the form of fully explicit valid arguments of the kind with which we will be concerned.

The negation rules will allow us to write *reductio* style arguments. Like the conditional introduction rule ($\rightarrow$I), the negation rules require a new assumption on an indented line, starting a new vertical line. If this assumption can be shown to lead to both a wfs of $\mathcal{L}^{\text{PL}}$ as well as its negation within the course of the subproof, then we may write the negation of the assumption of this subproof on a new line, stepping back one level of indentation. Schematically, this is what the *negation introduction* ($\neg$I) rule looks like:

m	$\quad\mid\ \varphi$	:AS for $\neg$I
n	$\quad\mid\ \psi$	
o	$\quad\mid\ \neg\psi$	
	$\neg\varphi$	:m–o $\neg$I

On line m, we assume φ for *reductio*. Our goal is to derive a contradiction, represented by two wfss ψ and $\neg\psi$ of $\mathcal{L}^{\text{PL}}$ on separate lines in any order. Accordingly, it is often convenient to include a note to ourselves and our readers that we are trying to introduce a negation by reaching a contradiction. Observe that ψ could be the same wfs as φ, e.g. both could be P, but this need not always be the case. Once we have derived a contradictory pair of wfss of $\mathcal{L}^{\text{PL}}$, we may close the subproof, moving to the left one level of indentation. We may then write the negation of the assumption in the subproof $\neg\varphi$ on a new line, citing the whole subproof by using a hyphen and indicating the negation introduction rule $\neg$I.

Suppose that we want to derive an instance of the law of non-contradiction: $\neg(G \wedge \neg G)$. A decent rule of thumb is that if you want to conclude a negated wfs of $\mathcal{L}^{\text{PL}}$, it is natural to assume the negand and see if you can reach a contradiction, though this may not always be the best strategy. However, in the case of $\neg(G \wedge \neg G)$, this is just what we will do, starting a subproof by adding the assumption $G \wedge \neg G$ to a proof without any premises.

1	$\quad G \wedge \neg G$	:AS for ¬I
2	$\quad G$	:1 ∧E
3	$\quad \neg G$	:1 ∧E
4	$\neg(G \wedge \neg G)$	:1–3 ¬I

Although some proofs require some real creativity, this one is pretty obvious once we make the right assumption. After all, the only rule we could apply to our assumption is ∧E, where two applications give us a contradiction. By applying ¬I, we may conclude the proof.

The *negation elimination* (¬E) rule works in much the same way. If we assume $\neg\varphi$ and show that it leads to a wfs of $\mathcal{L}^{\text{PL}}$ and its negation, we may conclude φ. So the rule looks like this:

m	$\quad \neg\varphi$	:AS for ¬E
n	$\quad \psi$	
o	$\quad \neg\psi$	
	φ	:m–o ¬E

As in the case of negation introduction, it is important that the justification of an application of negation elimination cite an entire subproof, indicating as much with a hyphen between the first and last lines. Additionally, it is important that the contradictory pair of wfss of $\mathcal{L}^{\text{PL}}$ occur in the subproof itself rather than elsewhere in the proof. Below is an example which makes an essential appeal to the reiteration rule in order to apply negation elimination:

1	P	:PR
2	$\neg Q \rightarrow \neg P$	:PR want Q
3	$\quad \neg Q$	:AS for ¬E
4	$\quad \neg P$	:2, 3 →E
5	$\quad P$	:1 R
6	Q	:3–5 ¬E

Negation elimination requires that one show that some wfs of $\mathcal{L}^{\text{PL}}$ and its negation are derivable given the assumption of a negated wfs of $\mathcal{L}^{\text{PL}}$. In this case, we establish that $\neg P$ follows from the assumption that $\neg Q$ by conditional elimination. Even though P occurs on a live line, we must use the reiteration rule in order to include P in our subproof. Only then may we draw Q as a conclusion by way of negation elimination.

3.9 Proof Strategy

The examples have been relatively simple so far, but perhaps you can already get a sense of the kinds of strategic thinking that natural deduction proofs sometimes require. For instance, although it is always permissible to open a subproof with any assumption, knowing which assumption to introduce and when to do so can require some care. Starting a subproof with any arbitrary assumption may clutter your proof. In order to obtain a conditional by →I, for example, it makes sense to assume the antecedent of the conditional in a subproof and see if you can derive the consequent. This is an example of a good proof strategy.

It is also always permissible to close a subproof, discharging its assumptions. However, it will not be helpful to do so until you have reached something useful. Once the subproof is closed, you can only cite the entire subproof in a justification for a line following that subproof. Those rules that call for a subproof, or multiple subproofs, require that the last line of the subproof is a wfs of $\mathcal{L}^{\text{PL}}$ of some form or other. For instance, you are only allowed to cite a subproof for →I if the line you are justifying is of the form $\varphi \rightarrow \psi$ where φ is the assumption of your subproof and ψ is the last line of your subproof. This constrains the strategies that one might hope to employ in attempting to construct proofs in PL.

Getting good at natural deduction will take some practice. The good news is that natural deduction proofs are a lot more interesting to construct than truth tables, and a much more beneficial skill: practicing natural deduction will streamline your reasoning well beyond the scope of this course, where the same cannot be said for filling out arrays with 1s and 0s. Although there are no fail-safe methods, and certainly no substitute for practice, there are some general rules of thumb and strategies that are worth keeping in mind.

Work backwards from what you want. The ultimate goal is to derive the conclusion. Look at the conclusion and ask what the introduction rule is for its main operator. This gives you an idea of what should happen just before the last line of the proof. Then you can treat this line as if it were your goal, asking what you could do to derive this new goal.

For example, if your conclusion is a conditional $\varphi \rightarrow \psi$, plan to use the →I rule. This requires starting a subproof in which you assume φ. In the subproof, you want to derive ψ.

Work forwards from what you have. When you are starting a proof, look at the premises and consider what implications they might have, or what you would need to derive in order to make use of the premises. It can help to think about the elimination rules for the main operators of the premises, or the wfss that you have derived so far.

For example, if you have a conditional $\varphi \rightarrow \psi$, and you also have φ on a line, →E is a pretty natural move to make. Sometimes it is a lot trickier to know what to do next, but not always.

Repeat as necessary. A long proof is just a number of short proofs linked together, so you can fill the gap by alternately working back from the conclusion and forward from the premises. Once you have decided how you might be able to get to the conclusion, ask what you might be able to do with the premises. Then consider these new targets, asking how you might reach each them, slowly connecting the dots of your proof.

Try a reductio when nothing else works. If you cannot find a way to show something directly, try assuming its negation and see where this leads. Sometimes this can help unlock a proof, perhaps even leading you to a direct line of argument.

Persist. Try different things. If one approach fails, then try something else. In general, there are typically many ways to construct a proof.

Although these guidelines can help if you get stuck, it is worth mentioning some of the mistakes that are easy to make. In particular, we will review some of the common errors that can occur when using subproofs. Consider the following example:

1	A	:PR
2	$\quad$ B	:AS for $\rightarrow$I
3	$\quad$ B	:2 R
4	$B \rightarrow B$	:2–3 $\rightarrow$I

This is perfectly in keeping with the rules that we have laid down above, and it should not seem particularly surprising. After all, $B \rightarrow B$ is a tautology, and so follows from no premises. Thus it is just as easy to derive $B \rightarrow B$ from some starting premises.

But now suppose that tried to continue the proof as follows:

1	A	:PR
2	$\quad$ B	:AS for $\rightarrow$I
3	$\quad$ B	:2 R
4	$B \rightarrow B$	:2–3 $\rightarrow$I
5	B	:3, 4 $\rightarrow$E (illegal)

If we were allowed to do this, we could derive any wfs of $\mathcal{L}^{\text{PL}}$ from any other. However, if you tell me that Anne is fast (symbolized by A), we shouldn't be able to conclude that Queen

Boudica stood twenty-feet tall (symbolized by B). Thankfully we are prohibited from making this move since our rules only permit us to draw on live lines.

Once a subproof closes, the wfss of $\mathcal{L}^{\text{PL}}$ in that proof are dead, and so we cannot appeal to them individually at a later point in the proof. This does not mean that we can't appeal to their results, or to the subproof as a whole. For instance, we could appeal to $B \to B$ on a later line since this wfs is live throughout the proof. Indentation has been included in the proof system to help keep track of what we can and cannot appeal to while writing proofs since it is easy to see which subproofs are closed. In particular, once you step back one level of indentation, the indented lines of the subproof above are dead, and so can only be cited by certain rules which appeal to the entire subproof, not any one of its lines.

Once we have started thinking about what we can derive from additional assumptions, nothing stops us from asking what we can derive from adding even more assumptions. Instead of doing this all at once the way that we may begin with many premises, we will do so by opening subproofs within subproofs. For instance, here is a proof of contraposition:

1	$\vert\ Q \to P$	:PR want $\neg P \to \neg Q$
2	$\vert\quad\vert\ \neg P$	:AS for $\to$I
3	$\vert\quad\vert\quad\vert\ Q$	:AS for $\neg$I
4	$\vert\quad\vert\quad\vert\ P$	:1, 3 $\to$E
5	$\vert\quad\vert\quad\vert\ \neg P$	:2 R
6	$\vert\quad\vert\ \neg Q$	:3–5 $\neg$I
7	$\vert\ \neg P \to \neg Q$	:2–6 $\to$I

Since we can't do anything with a conditional by itself, line 2 introduces the assumption $\neg P$. This is a natural choice given that we want to conclude $\neg P \to \neg Q$. Even so, there is not much more that we can do than before, and so we are forced to introduce yet another assumption Q on line 3. This is also a natural choice given that we would like to conclude $\neg Q$, and we know that we can use $\neg$I to do so if we reach a contradictory pair of wfss of $\mathcal{L}^{\text{PL}}$ from assuming Q. Given our assumptions, we may then derive P in line 4 by appealing to lines 1 and 3, both of which are live. Since line 2 is still live, we may derive $\neg P$ on line 5 by reiteration. Closing the second subproof, we may justify $\neg Q$ on line 6 by citing the lines 3–5. Now can close the first subproof, using these lines to justify $\neg P \to \neg Q$ on line 7.

For contrast, here is a proof where things go awry:

1	A	:PR
2	B	:AS for $\rightarrow$I
3	C	:AS for $\rightarrow$I
4	$A \wedge B$	:1, 2 $\wedge$I
5	$C \rightarrow (A \wedge B)$	:3–4 $\rightarrow$I
6	$B \rightarrow (C \rightarrow (A \wedge B))$	:2–5 $\rightarrow$I
7	$C \rightarrow (A \wedge B)$	:3–4 $\rightarrow$I (illegal)

The problem is that the subproof that began with the assumption C was under the assumption of B on line 2. By lines 6 and 7, we have discharged the assumption B, and so are no longer asking what we could show if we assumed B. Although it was perfectly legitimate to draw this same inference on line 5, by the time we are at line 7 we cannot appeal to lines 2–5.

Here is one further mistake worth watching out for:

1	A	:PR
2	B	:AS for $\rightarrow$I
3	C	:AS for $\rightarrow$I
4	$B \wedge C$	:2, 3 $\wedge$I
5	$B \rightarrow (B \wedge C)$	:2–4 $\rightarrow$I (illegal)

Line 5 tries to cite a subproof that begins on line 2 and ends on line 4, but the wfs on line 4 depends not only on the assumption on line 2, but also on another assumption (line 3) which we have not discharged at the end of the subproof. Put otherwise, the subproof which starts by assuming B does not end with a wfs at all, but rather ends with a subproof. Although we can close both subproofs at once, doing so wouldn't be strategic since line 5 cannot then cite lines 2–4 to justify $B \rightarrow (B \wedge C)$ in the manner presented above.

It is worth further stressing the difference between citing a single line and citing a subproof with a further example. In particular, when a rule requires you to cite a subproof, you cannot cite an individual line instead, nor *vice versa*. So for instance, this is incorrect:

1	A	:PR
2	$\quad B$	:AS for $\rightarrow$I
3	$\quad\quad C$	:AS for $\rightarrow$I
4	$\quad\quad B \wedge C$	:2, 3 $\wedge$I
5	$\quad\quad C$	:4 $\wedge$E
6	$\quad C$	:3–5 R (illegal)
7	$B \rightarrow C$	:2–6 $\rightarrow$I

Here, we have tried to justify C on line 6 by the reiteration rule, but we have done so by citing the subproof on lines 3–5. Although that subproof could in principle be cited on line 6, the reiteration rule does not permit us to do so. Rather, we could have used $\rightarrow$I to derive $C \rightarrow C$ while citing that subproof. By contrast, the reiteration rule R requires you to cite an individual line that is live, so citing the entire subproof is not permissible.

However obvious these mistakes may seem, it can be tempting to bend the rules when writing natural deduction proofs. This is like bending the rules while playing chess: you simply are no longer playing chess, but rather moving chess pieces around a boards in a manner that is no longer constrained by the rules of chess, or any other game for that matter. So in writing your own proofs in PL, keep these rules in mind, sticking to them precisely.

3.10 Derivability

Given the natural deduction rules specified above, we may present the following definition:

> A derivation (or proof) of φ from Γ in PL is any finite sequence of wfs of $\mathcal{L}^{\text{PL}}$ ending in φ where every wfs in the sequence is either: (1) a premise in Γ; (2) an assumption which is eventually discharged; or (3) follows from previous lines by a natural deduction rule for PL besides AS.

A wfs φ of $\mathcal{L}^{\text{PL}}$ is derivable (or provable) from Γ in PL, i.e., $\Gamma \vdash \varphi$, just in case there is a natural deduction derivation (proof) of φ from Γ in PL. Whereas logical consequence provides a semantic answer to the question of what follows from what in virtue of logical form by quantifying over interpretations, the derivation relation $\vdash$ provides a purely syntactic answer to this question by specifying which wfs of $\mathcal{L}^{\text{PL}}$ can be written after which in a manner which constitutes a derivation. Perhaps surprisingly, these two relations will be shown to have the same extension, describing formal reasoning in $\mathcal{L}^{\text{PL}}$ in two different ways.

Given the definition of the derivation relation above, we are now in a position to introduce a number of other important definitions. Two sentences φ and ψ of $\mathcal{L}^{\text{PL}}$ are INTERDERIVABLE (or PROVABLY EQUIVALENT) in PL— i.e., $\varphi \dashv\vdash \psi$— just in case both $\varphi \vdash \psi$ and $\psi \vdash \varphi$. Letting BOTTOM (also called the FALSUM) be the arbitrary contradiction $\bot := A \wedge \neg A$, we may take a set of sentences Γ to be INCONSISTENT just in case $\Gamma \vdash \bot$.

Provability is relative to a proof system. Whereas the meaning of the '$\vdash$' symbol featured in this chapter concerns PL, later we will use the same symbol for the derivation relation in FOL, distinguishing these with subscripts as in '$\vdash_{\text{PL}}$' and '$\vdash_{\text{FOL}}$' if need be. As with logical consequence, it is often convenient to write '$\varphi_1, \varphi_2, \ldots \vdash \psi$' as a shorthand for $\{\varphi_1, \varphi_2, \ldots\} \vdash \psi$. A wfs φ of $\mathcal{L}^{\text{PL}}$ is a THEOREM of PL just in case $\vdash \varphi$. As with the derivations in PL, it is important to note that the wfss of $\mathcal{L}^{\text{PL}}$ are only theorems relative to a proof system, and so there is no sense in which $A \vee \neg A$ is a theorem full stop. Nevertheless, it is natural to expect $A \vee \neg A$ to be a theorem of most any proof system for $\mathcal{L}^{\text{PL}}$.

In order to show that something is a theorem we have to derive it from no premises. But how could we show that something is *not* a theorem? More generally, how could we show that $\Gamma \nvdash \varphi$? Showing that there is no proof of φ from Γ would seem to require searching the space of all natural deduction proofs, and this is not bound to be easy. For instance, even if you (or a computer program) failed to derive φ from Γ in a thousand different ways, perhaps their is a proof that has not yet been considered. This brings out an important difference between our natural deduction system PL and the truth table method presented above for deciding whether $\Gamma \vDash \varphi$. In particular, PL does not provide an effective method for determining whether $\Gamma \vDash \varphi$. However, consider the following:

PL Soundness: If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.

Given the above, we may show that $\Gamma \nvdash \varphi$ by proving that $\Gamma \nvDash \varphi$ by finding an interpretation $\mathcal{I}$ of $\mathcal{L}^{\text{PL}}$ in which $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for all $\gamma \in \Gamma$ but $\mathcal{V}_{\mathcal{I}}(\varphi) = 0$. The reasoning goes by contraposition. After all, if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$ follows by Soundness, and so $\Gamma \vdash \varphi$ cannot be the case so long as we have shown that $\Gamma \nvDash \varphi$. Thus we may conclude that $\Gamma \nvdash \varphi$.

Suppose instead that we want to show that $\Gamma \vdash \varphi$ but cannot seem to find a proof however hard we look. Here is another important principle to which we might appeal:

PL Completeness: If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

If we can show that $\Gamma \vDash \varphi$ either by constructing a truth table or writing a semantic proof, Completeness permits us to conclude that $\Gamma \vdash \varphi$, and so we know that there is a derivation of φ from Γ even if we haven't managed to find one. This is different from knowing *how* to derive φ from Γ, but valuable information nonetheless. In order to make use of the principles above, the following two chapters will establish the soundness and completeness of PL.