Problem Set 8

Due Friday, November 22nd by 5pm

(10 points per question. Please scan and upload to Canvas as a PDF)

Collaborators
Felipe Abreu
Tyler Proctor

Lemma 11.5 $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi[\beta/\alpha])$ if $\mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\alpha) = \mathfrak{v}_{\mathcal{I}}^{\hat{a}}(\beta)$ and β is free for α in φ.

1. Construct an example to show what can go wrong in **Lemma 11.5** if β is not free for α in φ.

Proof

Let $\alpha = x$ be a variable, and let $\beta = y$ be another variable. We could then create the formula:

$$\phi = \forall x(P(x) \to Q(x, y))$$

Now, suppose we want to substitute $\beta = y$ for $\alpha = x$ in ϕ. If y is not free for x in ϕ, then when the variable y now appears inside the scope of the quantifier $\forall x$ which is not always true.

2. In the induction step of the proof of **Lemma 11.5**, fill in the details for the conditional case where $\varphi = (\psi \to \chi)$.

Proof

Case 4: $(\psi \to \chi)$ Assume $\varphi = \psi \to \chi$, where $v_I^{\hat{a}}(\alpha) = v_I^{\hat{a}}(\beta)$. Since $\text{Comp}(\psi \to \chi) = \text{Comp}(\psi) + \text{Comp}(\chi) + 1$, it follows that $\text{Comp}(\psi), \text{Comp}(\chi) \leq n$. By the induction hypothesis:

$$v_I^{\hat{a}}(\psi) = v_I^{\hat{a}}(\psi[\beta/\alpha]) \quad \text{and} \quad v_I^{\hat{a}}(\chi) = v_I^{\hat{a}}(\chi[\beta/\alpha]).$$

Using the semantics of implication, we have:

$$v_I^{\hat{a}}(\psi \to \chi) = v_I^{\hat{a}}(\psi[\beta/\alpha] \to \chi[\beta/\alpha]).$$

Thus:

$$v_I^{\hat{a}}(\varphi) = v_I^{\hat{a}}(\varphi[\beta/\alpha]),$$

as desired.

3. In the induction step of the proof of **Lemma 11.5**, fill in the details for the existential case where $\varphi = \exists\gamma\varphi$.

Proof

Case 7: $(\exists\gamma\varphi)$ Assume $\varphi = \exists\gamma\varphi'$, where $v_I^{\hat{a}}(\alpha) = v_I^{\hat{a}}(\beta)$. If $\gamma = \alpha$, then α is not free in φ, so $\varphi = \varphi[\beta/\alpha]$, and:

$$v_I^{\hat{a}}(\varphi) = v_I^{\hat{a}}(\varphi[\beta/\alpha]).$$

Otherwise, assume $\gamma \neq \alpha$. By the semantics of $\exists$:

$$v_I^{\hat{a}}(\exists\gamma\varphi') = v_I^{\hat{e}}(\varphi') \quad \text{for some } \gamma \text{ variant } \hat{e} \text{ of } \hat{a}.$$

By the induction hypothesis:

$$v_I^{\hat{e}}(\varphi') = v_I^{\hat{e}}(\varphi'[\beta/\alpha]) \quad \text{for all } \gamma \text{ variants } \hat{e} \text{ of } \hat{a}.$$

Therefore:

$$v_I^{\hat{a}}(\exists\gamma\varphi') = v_I^{\hat{e}}(\varphi'[\beta/\alpha]) \quad \text{for some } \gamma \text{ variant } \hat{e} \text{ of } \hat{a}.$$

Thus:

$$v_I^{\hat{a}}(\varphi) = v_I^{\hat{a}}(\varphi[\beta/\alpha]),$$

as desired.

4. Explain the role that **Lemma 11.6** plays in the proof of **Lemma 11.7**.

Proof

Lemma 11.7 uses **Lemma 11.6** with the assumption that they have the same domain $\mathbb{D}$ where $\mathcal{I}(F^n) = \mathcal{I}'(F^n)$ and $\mathcal{I}(a) = \mathcal{I}'(a)$ for every n-place predicate F^n and every constant $\alpha \neq \beta$. This relates $\mathcal{V}^e_{\mathcal{I}}(\psi) = 1$ to get $\mathcal{V}^e_{\mathcal{I}'}(\psi) = 1$ for every v.a. e. This then allows the step $\mathcal{V}_{\mathcal{I}}(\psi) = \mathcal{V}_{\mathcal{I}'}(\psi)$ for all $\psi \in \Gamma$. This also later sets $\mathcal{V}^c_{\mathcal{I}'}(\varphi) \neq 1$ since we know $\mathcal{V}^c_{\mathcal{I}'}(\varphi) \neq 1$ so we can prove **Lemma 11.7**.

5. Explain the role that **Lemma 9.2** plays in the proof of **Lemma 11.12**.

Proof

Lemma 9.2 lets us introduce a variable assignment $\hat{a}$ from the statement $v_I(\varphi[\alpha/\gamma])$ to then use "$v_I^{\hat{a}}(\varphi[\alpha/\gamma])$ for some v.a. in $\hat{a}$ over $\mathbb{D}$".