Problem Set 1

- **Due Date:** noon on **Tuesday 10$^{\text{th}}$ September, 2024**
- **Days Covered**: 01 (including Lecture, Warm-Up, and Recitation)

Please make note of the following instructions:

- Your solutions must be submitted to Gradescope as a single PDF file. While we allow handwritten solutions, we **strongly suggest** that you **typeset in LaTeX**, using the template available on Canvas.
- **Each problem** must be accompanied by a **collaboration statement**: the students you collaborated with, and any external resources you consulted when solving the problem. Do not leave this blank—instead write "collaborators: none" if appropriate. See the Course Information handout for more details on the collaboration policy.

Problem 1. Collaboration Policy [5 points]

Consider the scenarios described below, and discuss whether and why each one does or does not violate the course homework policies. If the issue is ambiguous, explain how you would resolve it. Simply answering "I'd check with the TA" for each answer is not sufficient.

(a) You host a pset party where ten of you work together to solve the problems before separating to write up your own solutions.

(b) You work with a friend on a problem, taking turns writing ideas and calculations jointly in the same notebook as you work out the solution, then photograph those notes so you can each have a copy of your sketch before you separate to write up your own solutions.

(c) You work together to figure out a detailed solution; one of you then dictates it to the other so that each of you can turn in a solution you've written yourself.

(d) After you've solved a problem, your friend who is still working on the problem asks you for feedback on a solution idea. Leveraging your knowledge of the right answer, you identify and point out a flaw in their proposed solution.

(e) After you've solved a problem, you try to help your friends by acting as a sounding board while they work towards the solution. With the deadline approaching, since they are still stuck, you allow them to skim your writeup to get a general idea of how to approach the problem, which allows them to go off and independently figure out their solution and write it up.

(f) After you've worked with your friend to solve a problem and each done your own writeup, you exchange writeups and examine them to identify and correct any flaws in one or the other.

(g) You remember seeing the homework problem in a problem set of a class you took previously. You look at your notes, see that you got a perfect grade, and copy your solution.

(h) You know the answer to the problem because you read it in a book you are reading to complement the required textbook, so you write it down that way.

(i) You know the answer to the problem because you read it in a book you are reading to complement the required textbook, so you explain the solution to your pset collaborators.

(j) While completing a late problem set for partial credit, you read the published solutions as you type your own answers, making sure to adjust the word choice and sentence structure as you go.

Problem 2. Proving Equivalence [10 points]

Prove that the formula $P := A$ IMPLIES (B IMPLIES C) is equivalent to $Q :=$ (A IMPLIES B) IMPLIES (A IMPLIES C), in two ways:

(a) [5 pts] Using a truth table, and

(b) [5 pts] using a proof by cases, based on the value of A.

Problem 3. Predicate Practice [10 points]

Translate the statements below into predicate logic, while preserving the meaning as closely as possible. In addition to boolean logic symbols like AND, OR, and NOT, and quantifiers $\exists$ and $\forall$, you may build predicates using arithmetic, inequalities, and constants.

For this problem, you should use $\mathbb{N} = \{0, 1, 2, \ldots\}$ as your domain of discourse: all variables will be assumed to belong to $\mathbb{N}$, so uses such as $\exists x$ and $\forall x$ are understood to mean $\exists x \in \mathbb{N}$ and $\forall x \in \mathbb{N}$, respectively, and any predicate such as $S(y)$ will be defined for all inputs $y \in \mathbb{N}$, and no others. For this problem we'd like you to avoid quantifying over other sets, such as "$\forall p \in$ Primes" or "$\exists n \geq 1$".

For example, the statement "n is odd" (recall that $n \in \mathbb{N}$ is assumed for this problem) could be translated into

$$\text{isOdd}(n) := \exists m. \, (2m + 1 = n)$$

or perhaps, relatedly,

$$\text{isNotEven}(n) ::= \text{NOT}\,(\exists a.\, n = 2a)\,.$$

As another example, the predicate "p is a prime number" could be translated to

$$\text{isPrime}(p) ::= (p > 1) \;\text{ AND }\; \text{NOT}\,(\exists m\, \exists n.\, (m > 1 \text{ AND } n > 1 \text{ AND } mn = p))\,.$$

(In more literal english, this says: p is greater than 1, and there do not exist two integers greater than 1 whose product is p.) You may use these predicates isOdd(n) and isPrime(p) by name in your answers below.

(a) (Lagrange's Four-Square Theorem) Every nonnegative integer is expressible as the sum of four perfect squares.

(b) (Goldbach Conjecture) Every even integer greater than two is the sum of two primes.

Note: We don't know whether this is true or not, but that doesn't stop us from translating and discussing it!

Note: Recall that, for this problem, you shouldn't use quantifier shorthand like $\forall n > 2.\, V(n)$. Instead, use the more basic $\forall n.\, W(n)$ (where $n \in \mathbb{N}$ is implied), and include all necessary logic inside the predicate $W(n)$.

Hint: "**If** n is even and greater than 4, **then** ...".

(c) (Fermat's Last Theorem) There are no nontrivial solutions to the equation:

$$x^n + y^n = z^n$$

over the nonnegative integers when $n > 2$.

(d) There is no largest prime number.

(e) (Bertrand's Postulate) If $n > 1$, then there is always at least one prime p such that $n < p < 2n$.