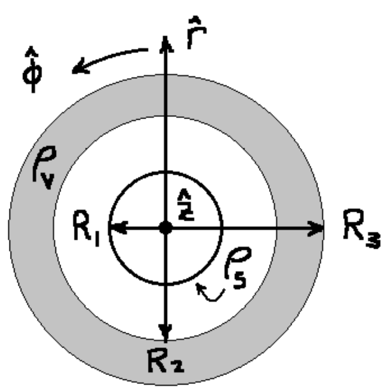
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.221/6.621 – Electromagnetic Fields, Forces & Motion
Fall 2025

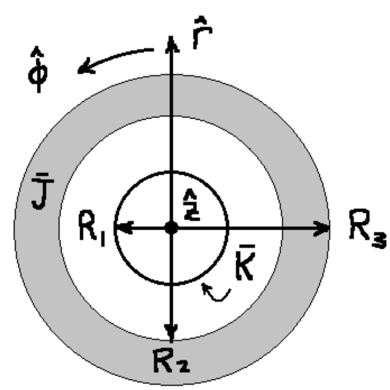
Problem Set #2
(6.221 Only: Problem 2.4)
(6.621 Only: Problem 2.3)

Issued 9/10/25 – Due 9/17/25

Problem 2.1: Consider the EQS system show below in which ρ_V is a uniform volume charge density and ρ_S is a uniform surface charge density defined by $\rho_S \equiv \rho_V(R_2^2 - R_3^2)/(2R_1)$. Determine the electric potential $\Phi(r)$ that accompanies this system. In doing so, define $\Phi(r = R_1) \equiv 0$. Note that this system was studied in Problem 1.1. The electric field solution found there might be useful here since $\bar{E} = -\nabla\Phi$. Or, to solve this problem you could use the superposition integral.



Problem 2.1



Problem 2.2

Problem 2.2: Determine the magnetic vector potential $\bar{A}(r)$ that accompanies the MQS system shown above. In doing so, assume $\bar{A}(r = R_1) \equiv 0$, and determine $\bar{A}$ such that $\nabla \cdot \bar{A} = 0$. In the system, the region $R_2 \leq r \leq R_3$ contains the uniform volume current density $\bar{J} = J_\circ \hat{z}$, and the cylindrical surface $r = R_1$ contains the uniform surface current density $\bar{K} \equiv J_\circ(R_2^2 - R_3^2)/(2R_1)$. Note that this system, with slightly different labeling, was studied in Problem 1.1. The magnetic field solution found there might be useful here since $\bar{B} = \nabla \times \bar{A}$. Or, to solve this problem you could use the superposition integral.

Problem 2.3 (6.621 Only): The main objective of this problem is to explore the use of the superposition integral for numerical analysis by implementing the analysis of HM Section 4.8. That analysis computes the electric potential in the vicinity of a two-dimensional parallel-plate capacitor such as that shown in Figure 4.8.3. The practical objective of this problem is to create electric-potential contour plots similar to those shown in HM Figure 4.8.4. Carrying out the following steps using Matlab, for example, should lead to the desired result.

Note that HM uses σ for surface charge, as opposed to ρ_S as used in class.

(A) This part of the problem concerns finding the surface charge distribution on the two capacitor plates. Note that the surface charge is assumed to be constant over the width of a plate segment. When uploading your solution to Gradescope, identify this part of your solution separately.

- Refer to HM Figure 4.8.3. Using any Cartesian coordinate system that you find convenient, define a two-dimensional capacitor like that shown in the figure. Segment its top and bottom plates as shown in the figure, and thereby define the plate segments. Then determine the location of the centers of each segment.

  Hint: using variables instead of hard-coded numbers for the capacitor width and gap separation, and the number of segments per plate, will allow you to most easily try different numerical combinations.

  Further hint: you may find it convenient to choose a coordinate system such that the coordinates of the centers of the plate segments are easily determined. For example, following Figure 4.8.3 you could put the coordinate-system origin in the center of the capacitor as shown. Then, given a segment width a, a plate separation $2d$, and segment numbering from 1 to N, left to right along the top and bottom rows, the position x_n of the centers of top and bottom segments n would be $x_n = (n-1)a - (N-1)a/2$, the position y_n of the top-row segment centers would be d, and the position y_n of the bottom-row segment centers would be $-d$.

- Define the ordering of the elements within the surface charge density vector $\bar{\sigma}$, the voltage vector $\bar{V}$, and correspondingly the elastance matrix (inverse capacitance matrix) $\boldsymbol{S}$ given in (6) of HM Section 4.8. Note that $\bar{\sigma}$ and $\bar{V}$ contain the surface charge densities and voltages, respectively, of the individual segments on the top and bottom plates. Thus, $\bar{V} = \boldsymbol{S}\bar{\sigma}$. In this way, (6) of HM Section 4.8 carries out the superposition of the contributions to each segment potential from each segment charge.

  Using the potential function given below, which implements (2) of HM Section 4.8, numerically fill in the elastance matrix $\boldsymbol{S}$ given the chosen capacitor geometry. Note that the function assumes unity charge density. Note further that if you write a .m Matlab script for this problem, you need only include the function at the end of the script. In that case, consider copying and pasting the function.

  ```
  function p = potential(x,y,a)
    p = (a + (x-a/2) * log(sqrt((x-a/2)^2 + y^2)) ...
           - (x+a/2) * log(sqrt((x+a/2)^2 + y^2)) ...
           + y * atan((x-a/2)/y) ...
           - y * atan((x+a/2)/y)) / (2*pi*8.854e-12);
  end
  ```

- Assign the voltage +0.5 [Volts] to the elements of $\bar{V}$ on the top plate, and the voltage -0.5 [Volts] to the elements of $\bar{V}$ on the bottom plate. Thus, the capacitor is driven with 1 V across it. Solve for the charge density vector $\bar{\sigma}$ in (6) as a function of the voltage vector $\bar{V}$ using the Matlab backslash operator. That is, execute $\bar{\sigma} = \boldsymbol{S}\backslash\bar{V}$ in Matlab. The vector $\bar{\sigma}$ now contains the (segmented approximate) charge distribution that would reside on the capacitor plates if the two plates were charged to ± 0.5 V.

(B) This part of the problem concerns checking your numerical analysis against the theoretical analysis of an ideal parallel-plate capacitor in which field fringing is ignored at the edges. When uploading your solution to Gradescope, identify this part of your solution separately.

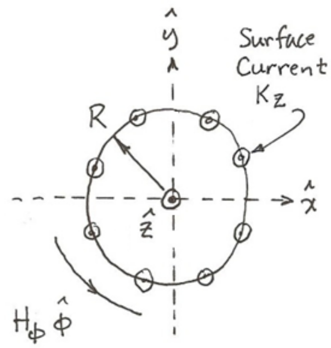
- Integrate (sum) the charge on the top segmented plate (per unit depth) and divide it by 1 V to get the (approximate) capacitance per unit depth. As a sanity check, compare this capacitance to that from an ideal theoretical analysis that ignores field fringing.

(C) This part of the problem concerns using the results of your numerical analysis to draw electric equipotential contours in the vicinity of the parallel-plate capacitor. When uploading your solution to Gradescope, identify this part of your solution separately.

- Define a matrix of potentials that is centered around the capacitor. Extending the matrix beyond the capacitor by one full width on each side, and by several gap spacings above and below should make for a good plot of the equipotential contours. Using the potential function given above, and the segment charge densities from the vector σ, fill in the potential matrix.

- Plot the electric equipotential contours using the Matlab *contour* function. The Matlab command *contour(matrix,n)* will plot n constant-valued contours of the data in *matrix.*

Problem 2.4 (6.221 Only): A $\hat{z}$-directed surface current of strength K_z flows uniformly along the cylindrical surface at $r = R$ as shown below. It sources the $\hat{\phi}$-directed magnetic field H_ϕ.



(A) Using Ampere's Law in integral form, determine the magnetic field both inside ($r < R$) and outside ($r > R$) the cylindrical surface current.

(B) Set up a superposition integral that can be used to determine the magnetic vector potential $A_z(r)$ outside the cylindrical surface current. In doing so, subtract a constant from the integral so that $A_z(R) = 0$. Hint: the appropriate constant is the original integral evaluated at $r = R$.

(C) Evaluate the superposition integral to determine $A_z(r)$. The indefinite integral

$$\int \frac{dx}{\sqrt{x^2 + a}} = \ln(x + \sqrt{x^2 + a})$$

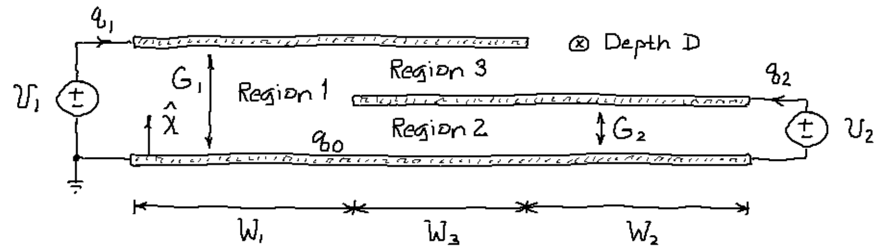
and the definite integral

$$\int_0^\pi \ln(a \pm b\cos(x))dx = \pi \ln\left(\frac{a + \sqrt{a^2 - b^2}}{2}\right) \quad \text{for} \quad a \geq b$$

should both help.

(D) Using the magnetic vector potential $A_z(r)$, determine the magnetic field outside the cylindrical surface current. It should match that found in Part (A).

Problem 2.5: A capacitive EQS system comprises three partially-overlapping very thin plates stacked in free space at $x = 0$, $x = G_1$ and $x = G_2$, as shown below. The plates have varying widths defined with W_1, W_2 and W_3, and common depth D. They are driven by the two voltage sources v_1 and v_2 as shown, giving rise to the potentials $\Phi_1(x)$, $\Phi_2(x)$ and Φ_3 in Regions 1, 2 and 3 between the plates, respectively, and total charges q_0, q_1 and q_2 on the plates. Assume that W_1, W_2, W_3 and D are all much bigger than G_1 and G_2 so that fringing can be ignored at the plate edges.



(A) Determine the potentials $\Phi_1(x)$, $\Phi_2(x)$ and $\Phi_3(x)$. Again, ignore fringing at the plate edges. (As a consequence of the negligible-fringing approximation, the electric potential will not be continuous between Regions 1 and 2, and Regions 1 and 3, as required by the boundary condition for Faradays's Law.)

(B) Determine the electric fields $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$ in Regions 1, 2 and 3 between the plates, respectively.

(C) Determine the total charges q_0, q_1 and q_2 on the plates.

(D) Let the relation between q_1 and q_2, and v_1 and v_2, be given by

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} .$$

Determine the capacitance coefficients C_{11}, C_{12}, C_{21}, and C_{22}.

(E) Suppose the plates are excited below by the single voltage source v_{21} with the bottom plate left open-circuited and uncharged. Determine v_1 and v_2. Hint: are the results from Parts A-D helpful?