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Chapter 5

The Completeness of PL

PL COMPLETENESS is not mandatory in the same way as PL SOUNDNESS. Continuing the analogy above, we may observe that no calculator is complete for the simple reason that every calculator has a finite amount of memory which is exhausted by arithmetical operations with sufficiently large operands. Even so, this does not stop calculators from being of considerable utility. These considerations might lead one to give up any hope of finding a complete logic for $\mathcal{L}^{\text{PL}}$, settling for a logic that is at least sound and so consistent. However, logical systems are not finite mechanisms made up of material elements such as the bits inside a calculator. For this reason, logics don't face the same constraints that a calculator does, and so the analogy breaks down. In this chapter, we will prove PL COMPLETENESS thereby establishing that PL does not leave any room for logical consequences that cannot be derived within PL. This is a beautiful result and a great achievement of twentieth century logic.

5.1 Introduction

Completeness asserts that φ is derivable from Γ whenever φ is a logical consequence of Γ, or more compactly: if $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$. Instead of beginning with $\Gamma \vDash \varphi$ as an assumption and arguing to the conclusion $\Gamma \vdash \varphi$, we will focus on establishing a closely related result:

> **Theorem 5.1** Every consistent set Γ of wfss of $\mathcal{L}^{\text{PL}}$ is satisfiable.

You might recognize this as the converse of **Corollary 4.2** from Chapter 4. Indeed, **Theorem 5.1** will bear a close connection to PL COMPLETENESS in a way that is related to the connection between **Corollary 4.2** and PL SOUNDNESS. However, instead of proving **Theorem 5.1** from PL COMPLETENESS, the proof will work in the reverse direction so that officially PL COMPLETENESS will be a corollary of **Theorem 5.1**. Before diving into the proof in earnest, this section will present an overview of the proof so that you can find your bearings. If you get lost, think of this section as a map to which you can return.

Recall that a set of $\mathcal{L}^{\text{PL}}$ wfss Γ is inconsistent if $\bot$ is derivable from Γ, and consistent otherwise. If $\Gamma \vDash \varphi$, we know that $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable by **Lemma 2.3**, and so $\Gamma \cup \{\neg\varphi\}$ is inconsistent by **Theorem 5.1**. We will then show that it follows that $\Gamma \vdash \neg\neg\varphi$ by appealing to **Lemma 5.2** below. Given the derived rule DN for double negation elimination, we may conclude that $\Gamma \vdash \varphi$ from the assumption that $\Gamma \vDash \varphi$. This provides a sketch of the completeness proof for PL given the theorem above (see **Corollary 12.6** for the full proof).

It remains to establish **Theorem 5.1**. We begin by assuming that Γ is a consistent set of wfss of $\mathcal{L}^{\text{PL}}$. We will then extend Γ so that it includes every wfs or its negation but not both, calling this *maximal* set Δ_Γ, or often just Δ for short. We will show in §5.2 that Δ is consistent, where it follows that Δ is *deductively closed* insofar as it contains every wfs that is derivable from Δ. Deductive closure is a very important and convenient property which will play a critical role in the later stages of the proof.

Having extended Γ to a much bigger set of wfss Δ that is consistent, maximal, and deductively closed, we will proceed to use this set of wfss to construct an interpretation $\mathcal{I}_\Delta$ of $\mathcal{L}^{\text{PL}}$ that satisfies Δ, and so satisfies Γ as a result. This may sound strange since we are starting with a set Γ of wfss of $\mathcal{L}^{\text{PL}}$ to ultimately interpret the wfss in Γ itself. Strange as this may seem, there is no circularity here: to interpret our language $\mathcal{L}^{\text{PL}}$, we need a systematic way to assign each sentence letter to exactly one truth-value. So long as we achieve this without assuming that such an assignment has already been given, no questions will have been begged.

Given that $\mathcal{I}_\Delta$ satisfies Γ, we may conclude that Γ is satisfiable. Following tradition, we will refer to this cleverly constructed interpretation $\mathcal{I}_\Delta$ as a Henkin interpretation after Leon Henkin who first presented this proof strategy in 1949. In Chapter 12, we will extend this same method to show that the first-order logic FOL that we will provide in Chapter 10 is complete with respect to the theory of logical consequence given in Chapter 8. Not only will we repeat the same methodology that we have followed so far in developing a semantics, logic, and metalogic for $\mathcal{L}^{\text{PL}}$ for our first-order language $\mathcal{L}^{\text{FOL}}$, many of the same results will carry over. For the time being, we will continue to restrict attention to $\mathcal{L}^{\text{PL}}$, it's semantics, and the proof system PL with which we are presently concerned.

This provides a rough overview of the proof strategy that will be deployed below. If you get lost along the way, continuing to slog on in the dark is not advisable. Rather, it is better to keep zooming out so that you can keep track of where you are and where you are headed to next. In addition to returning to this section, you may need to scan back through the proof multiple times, unlocking each piece and slowly watching them come together.

Without further ado, we may make the first assumption indicated above that Γ is an arbitrary set of wfss of $\mathcal{L}^{\text{PL}}$. Although it may help to hold onto Γ throughout the course of the following sections, many of the lemmas that we establish will be completely general in form, holding for any arbitrary set of wfss of $\mathcal{L}^{\text{PL}}$. We will then put these general results together to apply to Γ in order to show that Γ is indeed satisfiable.

5.2 Maximal Consistency

A set of wfss Δ is MAXIMAL in $\mathcal{L}^{\text{PL}}$ just in case as either $\psi \in \Delta$ or $\neg\psi \in \Delta$ for every wfs ψ in $\mathcal{L}^{\text{PL}}$. Having assumed that Γ is consistent in $\mathcal{L}^{\text{PL}}$, we may maximize Γ by adding every wfs of $\mathcal{L}^{\text{PL}}$ that we can without inconsistency. To do so, we will begin enumerating all wfss $\psi_0, \psi_1, \psi_2, \ldots$ in $\mathcal{L}^{\text{PL}}$ whatsoever in order to present the following recursive construction:

$$\Delta_0 = \Gamma$$
$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\psi_n\} & \text{if } \Gamma_n \cup \{\psi_n\} \text{ is consistent} \\ \Delta_n \cup \{\neg\psi_n\} & \text{otherwise.} \end{cases}$$
$$\Delta_\Gamma = \bigcup_{i \in \mathbb{N}} \Delta_n.$$

If Γ is consistent, we may show that Δ_Γ is both consistent and maximal. Given these two properties, we may then show that Δ_Γ is deductively closed. Moreover, we may show $\Gamma \subseteq \Delta_\Gamma$. These properties will form the basis upon which we will construct the Henkin interpretation in §5.4. In order to establish these results, we will begin by proving some supporting lemmas.

Lemma 5.1 If $\Lambda \cup \{\varphi\}$ is inconsistent, then $\Lambda \vdash \neg\varphi$.

Proof: Assume $\Lambda \cup \{\varphi\}$ is inconsistent. Thus $\Lambda \cup \{\varphi\} \vdash \bot$, and so there is a derivation X of $A \wedge \neg A$ from $\Lambda \cup \{\varphi\}$ given the definition of $\bot$. Let X' be the result of replacing the premise φ with φ as an assumption and adding lines for A and $\neg A$ by $\wedge$E. We may then discharge the assumption of φ by $\neg$I in order to derive $\neg\varphi$ from Λ. Thus we may conclude that $\Lambda \vdash \neg\varphi$. $\square$

Lemma 5.2 If $\Lambda \cup \{\varphi\}$ and $\Lambda \cup \{\neg\varphi\}$ are both inconsistent, then Λ is inconsistent.

Proof: Assume that $\Lambda \cup \{\varphi\}$ and $\Lambda \cup \{\neg\varphi\}$ are both inconsistent. It follows that $\Lambda \vdash \neg\varphi$ and $\Lambda \vdash \neg\neg\varphi$ by **Lemma 5.1**, and so there is a derivation X of $\neg\varphi$ from Λ, and a derivation Y of $\neg\neg\varphi$ from Λ. Let Z be the result of concatenating X and Y and using EFQ from §4.4.8 on the last lines of X and Y to derive $A \wedge \neg A$. Since the only premises in Z are the premises in X and Y, we may conclude that $\Lambda \vdash A \wedge \neg A$, and so Λ is inconsistent. $\square$

Lemma 5.3 If Γ is consistent in $\mathcal{L}^{\text{PL}}$, then Δ_Γ is maximal consistent.

Proof: Assume Γ is consistent and let φ be any wfs of $\mathcal{L}^{\text{PL}}$. Thus $\varphi = \psi_i$ for some $i \in \mathbb{N}$ given the enumeration above where either $\psi_i \in \Delta_{i+1}$ or $\neg\psi_i \in \Delta_{i+1}$. Since $\Delta_{i+1} \subseteq \Delta_\Gamma$, either $\varphi \in \Delta_\Gamma$ or $\neg\varphi \in \Delta_\Gamma$, and so Δ_Γ is maximal.

The proof that Δ_Γ is consistent goes by induction on the construction of Δ_Γ, where we know by assumption that $\Gamma = \Delta_0$ is consistent. Assume for induction that Δ_n is consistent. There are two cases to consider.

Case 1: $\Delta_n \cup \{\psi_n\}$ is consistent, and so $\Delta_{n+1} = \Delta_n \cup \{\psi_n\}$ is consistent.

Case 2: $\Delta_n \cup \{\psi_n\}$ is not consistent, and so $\Delta_{n+1} = \Delta_n \cup \{\neg\psi_n\}$. Assume for contradiction that $\Delta_n \cup \{\neg\psi_n\}$ is not consistent. By **Lemma 5.2**, Δ_n is inconsistent, contradicting the hypothesis. Thus Γ_{n+1} is consistent.

Since Γ_{n+1} is consistent, it follows by induction that Γ_n is consistent for all $n \in \mathbb{N}$. Assume for contradiction that Δ_Γ is inconsistent. Thus $\Delta_\Gamma \vdash \bot$, and so there is a proof Y of $A \wedge \neg A$ from Δ_Γ given the definition of $\bot$. Since Y is finite, there is a finite number of premises cited in Y, and so there is some $k \in \mathbb{N}$ where every premise cited in Y belongs to Δ_k. As a result, Y is also a proof of $A \wedge \neg A$ from Δ_k, and so Δ_k is inconsistent, contradicting the above. Thus Δ_Γ is consistent. □

5.3 Deductive Closure

We begin by defining a set Δ of wfss of $\mathcal{L}^{\text{PL}}$ to be DEDUCTIVELY CLOSED in PL just in case for any wfs φ of $\mathcal{L}^{\text{PL}}$, if $\Delta \vdash \varphi$, then $\varphi \in \Delta$. Accordingly, deductively closed sets of wfss are identical to the set of wfss which they derive. In order to show that every maximal consistent set of wfss of $\mathcal{L}^{\text{PL}}$ is deductively closed, we begin by establishing the following lemma.

Lemma 5.4 If $\Lambda \vdash \varphi$ and $\Lambda \vdash \neg\varphi$, then Λ is inconsistent.

Proof: Assume $\Lambda \vdash \varphi$ and $\Lambda \vdash \neg\varphi$. Thus there is a PL derivation X of φ from Λ as well as a PL derivation Y of $\neg\varphi$ from Λ. Letting Z be the result of concatenating X and Y and renumbering lines. We may then extend the derivation by using EFQ from §4.4.8 to derive $A \wedge \neg A$, observing that Z is a derivation of $A \wedge \neg A$ from Λ. By definition, Λ is inconsistent. □

Lemma 5.5 If Δ is maximal consistent, then Δ is deductively closed.

Proof: Assume Δ is maximal consistent. Let φ be a wfs of $\mathcal{L}^{\text{PL}}$ where $\Delta \vdash \varphi$. If $\Delta \vdash \neg\varphi$, then Δ is inconsistent by **Lemma 5.4**, contradicting the assumption. Thus $\Delta \nvdash \neg\varphi$, and so it follows that $\neg\varphi \notin \Delta$ since otherwise $\Delta \vdash \neg\varphi$ follows by the reiteration rule R. Since Δ is maximal, we may conclude that $\varphi \in \Delta$. □

Given these results, we are now ready to construct the Henkin interpretation of $\mathcal{L}^{\text{PL}}$.

5.4 Henkin Interpretation

Having extended the consistent set of wfss Γ in $\mathcal{L}^{\text{PL}}$ to a maximal consistent set of wfss Δ_Γ in $\mathcal{L}^{\text{PL}}$ and showing that Δ_Γ is deductively closed, we may use Δ_Γ to construct a Henkin interpretation that satisfies Δ_Γ, and so also satisfies Γ. For ease of exposition, we will drop the subscripts, assuming $\Delta = \Delta_\Gamma$ throughout what follows.

We may now proceed to draw on the definition of Δ in order to specify an especially natural interpretation of $\mathcal{L}^{\text{PL}}$ which will guarantee that the resulting interpretation satisfies all of the wfss in Δ. In particular, consider the following definitions:

$$\text{For all sentence letters } \varphi \text{ of } \mathcal{L}^{\text{PL}},\ \text{let } \mathcal{I}_\Delta(\varphi) = \begin{cases} 1 & \text{if } \varphi \in \Delta \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{I}_\Delta$ assigns every sentence letter of $\mathcal{L}^{\text{PL}}$ to a truth-value, $\mathcal{I}_\Delta$ satisfies the definition of a $\mathcal{L}^{\text{PL}}$ interpretation. Since this construction was introduced by Leon Henkin (1949), we will refer to $\mathcal{M}_\Delta$ as the HENKIN INTERPRETATION of $\mathcal{L}^{\text{PL}}$ for Γ (recall that $\Delta = \Delta_\Gamma$).

It remains to show that $\mathcal{I}_\Delta$ satisfies Δ, and so satisfies Γ as a result. To do so, we will begin by proving the following lemmas where the first is a proof theoretic analogue of **Lemma 2.1**.

Lemma 5.6 If $\Lambda \vdash \varphi$, then $\Lambda \cup \Pi \vdash \varphi$.

Proof: Assuming that $\Lambda \vdash \varphi$, there is a derivation X of φ from Λ in PL. Since $\Lambda \subseteq \Lambda \cup \Pi$, it follows that X is also a derivation of φ from $\Lambda \cup \Pi$ in PL. Thus we may conclude that $\Lambda \cup \Pi \vdash \varphi$. □

Lemma 5.7 If Δ is a maximal consistent set of wfss of $\mathcal{L}^{\text{PL}}$, then for every wfs φ of $\mathcal{L}^{\text{PL}}$: $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$.

Proof: Assume Δ is a maximal consistent set of $\mathcal{L}^{\text{PL}}$ wfss. We will show by induction on complexity that for any wfs φ of $\mathcal{L}^{\text{PL}}$, $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$. After the base case, there are five cases in the induction step.

Base Case: Let φ be an arbitrary wfs of $\mathcal{L}^{\text{PL}}$ where $\texttt{Comp}(\varphi) = 0$. It follows that φ is a sentence letter of $\mathcal{L}^{\text{PL}}$. We may then consider the following biconditionals:

$$\begin{aligned} \mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1 \quad & \textit{iff} \quad \mathcal{I}_\Delta(\varphi) = 1 \\ & \textit{iff} \quad \varphi \in \Delta. \end{aligned}$$

Since φ is a sentence letter, the first biconditional follows by the semantics for $\mathcal{L}^{\text{PL}}$, and the second biconditional follows from the definition of $\mathcal{I}_\Delta$. It follows that for any wfs φ of $\mathcal{L}^{\text{PL}}$ where $\texttt{Comp}(\varphi) = 0$, $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$.

Induction: Assume for induction that for every wfs φ of $\mathcal{L}^{\text{PL}}$, if $\texttt{Comp}(\varphi) \leq n$, then $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$. Let φ be a wfs of $\mathcal{L}^{\text{PL}}$ where $\texttt{Comp}(\varphi) = n + 1$.

Case 1: Assume $\varphi = \neg\psi$. Since $\texttt{Comp}(\neg\psi) = \texttt{Comp}(\psi) + 1$ and $\texttt{Comp}(\varphi) = n + 1$, it follows that $\texttt{Comp}(\psi) = n$. We may then reason as follows:

$$
\begin{aligned}
\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1 \quad &\textit{iff} \;\; \mathcal{V}_{\mathcal{I}_\Delta}(\neg\psi) = 1 \\
&\textit{iff} \;\; \mathcal{V}_{\mathcal{I}_\Delta}(\psi) = 0 && (1)\\
&\textit{iff} \;\; \psi \notin \Delta && (2)\\
&\textit{iff} \;\; \neg\psi \in \Delta && (3)\\
&\textit{iff} \;\; \varphi \in \Delta.
\end{aligned}
$$

Whereas (1) follows from the semantics for negation and (2) holds by hypothesis, (3) follows from the maximality of Δ. The other biconditionals follow from the case assumption. Thus $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$, completing the case.

Case 2: Assume $\varphi = \psi \wedge \chi$. Since $\texttt{Comp}(\psi \wedge \chi) = \texttt{Comp}(\psi) + \texttt{Comp}(\chi) + 1$ and $\texttt{Comp}(\varphi) = n + 1$, it follows that $\texttt{Comp}(\psi), \texttt{Comp}(\chi) \leq n$. Thus we have:

$$
\begin{aligned}
\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1 \quad &\textit{iff} \;\; \mathcal{V}_{\mathcal{I}_\Delta}(\psi \wedge \chi) = 1 \\
&\textit{iff} \;\; \mathcal{V}_{\mathcal{I}_\Delta}(\psi) = \mathcal{V}_{\mathcal{I}_\Delta}(\chi) = 1 && (1)\\
&\textit{iff} \;\; \psi, \chi \in \Delta && (2)\\
&\textit{iff} \;\; \psi \wedge \chi \in \Delta && (3)\\
&\textit{iff} \;\; \varphi \in \Delta.
\end{aligned}
$$

Whereas (1) follows from the semantics for conjunction, (2) holds by hypothesis. In order to establish (3), assume that $\psi \wedge \chi \in \Delta$, it follows that $\Delta \vdash \psi$ and $\Delta \vdash \chi$ by $\wedge$E, and so $\psi, \chi \in \Delta$ by **Lemma 5.5**. Assuming instead that $\psi, \chi \in \Delta$, we know that $\Delta \vdash \psi \wedge \chi$ by $\wedge$I, and so $\psi \wedge \chi \in \Delta$ by **Lemma 5.5**. This proves (3) where the other biconditionals follow from the case assumption.

Case 3: Assume $\varphi = \psi \vee \chi$. (Exercise for the reader.)

Case 4: Assume $\varphi = \psi \rightarrow \chi$. Since $\texttt{Comp}(\psi \rightarrow \chi) = \texttt{Comp}(\psi) + \texttt{Comp}(\chi) + 1$ and $\texttt{Comp}(\varphi) = n + 1$, it follows that $\texttt{Comp}(\psi), \texttt{Comp}(\chi) \leq n$. Thus we have:

$$
\begin{aligned}
\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1 \quad &\textit{iff} \;\; \mathcal{V}_{\mathcal{I}_\Delta}(\psi \rightarrow \chi) = 1 \\
&\textit{iff} \;\; \mathcal{V}_{\mathcal{I}_\Delta}(\psi) = 0 \text{ or } \mathcal{V}_{\mathcal{I}_\Delta}(\chi) = 1 && (1)\\
&\textit{iff} \;\; \psi \notin \Delta \text{ or } \chi \in \Delta && (2)\\
&\textit{iff} \;\; \psi \rightarrow \chi \in \Delta && (3)\\
&\textit{iff} \;\; \varphi \in \Delta.
\end{aligned}
$$

Whereas (1) follows from the semantics for conjunction, (2) holds by hypothesis.

In order to establish (3), assume that $\psi \notin \Delta$. It follows by **Lemma 5.3** that $\neg\psi \in \Delta$. We may then derive $\neg\psi \vdash \psi \rightarrow \chi$ since given $\neg\psi$ as a premise, we may use the rule AS to write ψ on a second line, deriving χ by EFQ from §4.4.8

and using $\to$I to discharge the assumption. Thus $\Delta \vdash \psi \to \chi$ by **Lemma 5.6**, and so $\psi \to \chi \in \Delta$ by **Lemma 5.5**. Next we may assume that $\chi \in \Delta$, we may derive $\chi \vdash \psi \to \chi$ since given χ as a premise, we may use the rule AS to write ψ on a second line. By then using the rule R, we may rewrite the premise χ, discharging our assumption with the rule $\to$I in order to derive $\psi \to \chi$ from χ. Thus $\Delta \vdash \psi \to \chi$ by **Lemma 5.6**, and so $\psi \to \chi \in \Delta$ by **Lemma 5.5**. We may then conclude that $\psi \to \chi \in \Delta$ if either $\psi \notin \Delta$ or $\chi \in \Delta$.

Assume instead that $\psi \to \chi \in \Delta$. If $\psi \notin \Delta$, then $\psi \notin \Delta$ or $\chi \in \Delta$. If $\psi \in \Delta$, then $\Delta \vdash \chi$ by the rule $\to$E, and so $\chi \in \Delta$ by **Lemma 5.5**. Thus $\psi \notin \Delta$ or $\chi \in \Delta$ if $\psi \to \chi \in \Delta$ which, given the above, establishes (3).

The other biconditionals follow from the case assumption.

Case 5: Assume $\varphi = \psi \leftrightarrow \chi$. (Exercise for the reader.)

Conclusion: It follows by induction that for every wfs φ of $\mathcal{L}^{\text{PL}}$ of any complexity, $\mathcal{V}_{\mathcal{I}_\Delta}(\varphi) = 1$ just in case $\varphi \in \Delta$. This completes the proof. □

5.5 Completeness and Compactness

Having constructed a maximal consistent set Δ of wfss of $\mathcal{L}^{\text{PL}}$ from the consistent set Γ of wfs of $\mathcal{L}^{\text{PL}}$ and defined the Henkin interpretation $\mathcal{I}_\Delta$ as above, we are now ready to draw on the lemmas above in order to show that Δ is satisfiable. In order to extend this result to Γ, we may begin with the trivial lemma given below.

Lemma 5.8 $\Gamma \subseteq \Delta_\Gamma$.

Proof: By definition, $\Gamma = \Delta_0$ where $\Delta_0 \subseteq \Delta_\Gamma$. □

This lemma amounts to little more than an observation, but will be convenient to reference below. We may now move to draw the following conclusion:

Theorem 5.1 Every consistent set Γ of wfss of $\mathcal{L}^{\text{PL}}$ is satisfiable.

Proof: Let Γ be a consistent set of wfss of $\mathcal{L}^{\text{PL}}$. By **Lemma 5.3**, Δ_Γ is a maximal consistent set of wfss in $\mathcal{L}^{\text{PL}}$. Letting $\Delta = \Delta_\Gamma$ and $\mathcal{I}_\Delta$ be the Henkin interpretation of $\mathcal{L}^{\text{PL}}$ defined above, **Lemma 5.7** shows that for every wfs φ of $\mathcal{L}^{\text{PL}}$, $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ just in case $\varphi \in \Delta$. Thus $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all $\varphi \in \Delta$. Since $\Gamma \subseteq \Delta$ by **Lemma 5.8**, $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ for all $\varphi \in \Gamma$. By definition, Γ is satisfiable. □

Given this result, the completeness of PL over the semantics for $\mathcal{L}^{\text{PL}}$ follows as a corollary.

Corollary 5.3 (Completeness) If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

Proof: Assume $\Gamma \vDash \varphi$. By **Lemma 2.3**, $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable, and so $\Gamma \cup \{\neg\varphi\}$ is inconsistent by **Theorem 5.1**. Thus $\Gamma \vdash \neg\neg\varphi$ by **Lemma 5.1**. Thus there is some PL derivation X of $\neg\neg\varphi$ from Γ. Given the rule DN derived in §4.4.7, we may extend X to derive φ from Γ, and so conclude that $\Gamma \vdash \varphi$. $\square$

Completeness may seems like a good property for any proof system to have. In particular, the completeness of PL shows that there is no (extensionally) better proof system which allows us to derive a valid inference that PL leaves out. However, there is another perspective which takes completeness to describe a certain limitation on what sorts of entailments hold between wfss in $\mathcal{L}^{\text{PL}}$, calling the notion of entailment in $\mathcal{L}^{\text{PL}}$ into question. We will close the chapter with an important consequence of completeness.

Corollary 5.4 If $\Gamma \vDash \varphi$, then there is a finite subset $\Lambda \subseteq \Gamma$ where $\Lambda \vDash \varphi$.

Proof: Assume $\Gamma \vDash \varphi$. It follows by completeness that $\Gamma \vdash \varphi$, and so there is a derivation X of φ from Γ. Letting Γ_X be the set of premises which appear in X, it follows that $\Gamma_X \vdash \varphi$, and so $\Gamma_X \vDash \varphi$. Since X is finite, Γ_X is also finite, and so whenever $\Gamma \vDash \varphi$ there is a finite subset $\Lambda \subseteq \Gamma$ where $\Lambda \vDash \varphi$. $\square$

Corollary 5.5 (Compactness) If every finite subset $\Lambda \subseteq \Gamma$ is satisfiable, then Γ is satisfiable.

Proof: Assume for contraposition that Γ is unsatisfiable. It follows vacuously that $\Gamma \vDash \bot$, and so $\Lambda \vDash \bot$ by **Corollary 12.7** for some finite subset $\Lambda \subseteq \Gamma$. Thus there is some finite subset $\Lambda \subseteq \Gamma$ that is unsatisfiable. By contraposition, if every finite subset $\Lambda \subseteq \Gamma$ is satisfiable, then Γ is satisfiable. $\square$

This property is referred to as COMPACTNESS. Recall that arguments were required to be finite sequences of wfss of $\mathcal{L}^{\text{PL}}$ where an argument is valid just in case its conclusion is a logical consequence of its premises. When we defined what it is for φ to be a logical consequence of Γ, we permitted Γ to be any set of wfss of $\mathcal{L}^{\text{PL}}$ including infinite sets. What compactness shows is that this additional permission does not add anything that would have been lost were logical consequence restricted to finite sets of wfss of $\mathcal{L}^{\text{PL}}$.

It is important to emphasize that PL Completeness does not entail that PL provides an exhaustive description of formal reasoning. Rather, PL only claims to exhaustively describe logical consequence in $\mathcal{L}^{\text{PL}}$ which in turn is constrained by the expressive resources which $\mathcal{L}^{\text{PL}}$ includes. At most, we may think of PL as providing a complete description of formal reasoning in $\mathcal{L}^{\text{PL}}$ given the logical forms that $\mathcal{L}^{\text{PL}}$ is able to capture.

In the following chapter we will extend the expressive resources of $\mathcal{L}^{\text{PL}}$ by adding constants, variables, predicates, and quantifiers, referring to this first-order language as $\mathcal{L}^{\text{FOL}}$. These additions will make it possible to regiment a host of valid arguments that we are unable to capture in $\mathcal{L}^{\text{PL}}$. After stipulating the syntax of $\mathcal{L}^{\text{FOL}}$ in a similar manner to $\mathcal{L}^{\text{PL}}$, we will also provide a semantics and natural deduction system FOL, developing its metalogic by establishing both the soundness and completeness of FOL. Despite the increase in expressive power of $\mathcal{L}^{\text{FOL}}$ and the logical strength of FOL, the methodology that we will follow will be the same as what we have already provided for $\mathcal{L}^{\text{PL}}$.