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Chapter 9

Identity

The last two chapters introduced the syntax and semantics for $\mathcal{L}^{\text{FOL}}$. In this chapter, we will extend both the syntax and semantics of $\mathcal{L}^{\text{FOL}}$ to accommodate the IDENTITY predicate '=', referring to this extended language as $\mathcal{L}^{=}$.

It is important to emphasize that to say that **x** and **y** are identical is different from saying that **x** and **y** are *duplicates* though this is a common way of using the word 'identical' in English. For instance, consider the use of the word 'identical' in the following case:

> `Spheres`: Consider a possible world in which there is nothing but two identical spheres, similar in every way to each other, separated by just one meter. Although there is no property that they do not share in common, the two spheres are distinct. After all, there are two spheres, not just one.

Insofar as the spheres are distinct— there are two of them, not one— we will say that they are not *numerically identical*, or just not *identical* for short. If 'a' names one sphere and 'b' names the other, we may express this with the sentence '$\neg{=}ab$' where '=' is a two place predicate, or with '$a \neq b$' for the sake of readability and familiarity. In this sense of identity, it is not true that the two spheres are identical as claimed in `Spheres`. Indeed, no two things whatsoever are identical in our sense since if they were, then there would not be two of them but rather just one thing perhaps with different names.

Before showing how to include a designated predicate for identity in the syntax and semantics for $\mathcal{L}^{=}$, it will help to guide our ambitions by considering some of what motivates this addition. After all, $\mathcal{L}^{\text{FOL}}$ is a very powerful language, at least by comparison to $\mathcal{L}^{\text{PL}}$. Why should we need to further extend $\mathcal{L}^{\text{FOL}}$? Can't we get by without including identity in the language? In particular, we could take a 2-place predicate in the language (e.g., I) to symbolize identity in just the same way as we would for any other 2-place predicate. Why should identity deserve special treatment, and how should we think about the difference between identity and the other predicates that we will included in the language $\mathcal{L}^{=}$?

9.1 Identity and Logic

It turns out that there is a lot that cannot be said without an identity predicate. You might be wondering why we can't just declare that a certain predicate be used to express identity the way that we do in regimenting other predicates in $\mathcal{L}^{\text{FOL}}$. For instance, suppose we were to regiment 'Hesperus is Phosphorus' as 'Ihp' given the following symbolization key:

Ixy: x is y
h: Hesperus
p: Phosphorus
v: Venus

One might take the regimentation given above to do as good a job as any of our regimentations in $\mathcal{L}^{\text{FOL}}$. Why does identity deserve special treatment?

Consider the following English argument regimented with the symbolization given above:

A1. Hesperus is Phosphorus.	B1. Ihp
A2. Phosphorus is Venus.	B2. Ipv
A3. Hesperus is Venus.	B3. Ihv

This argument is invalid. For instance, consider the following countermodel:

$$\begin{aligned} \mathbb{D} &= \{h, p, v\} \\ \mathcal{I}(I) &= \{\langle h, p\rangle, \langle p, v\rangle\} \\ \mathcal{I}(h) &= h \\ \mathcal{I}(p) &= p \\ \mathcal{I}(v) &= v \end{aligned}$$

Since $\langle h, p\rangle, \langle p, v\rangle \in \mathcal{I}(I)$ but $\langle h, v\rangle \notin \mathcal{I}(I)$, it follows that $\langle \mathcal{I}(h), \mathcal{I}(p)\rangle, \langle \mathcal{I}(p), \mathcal{I}(v)\rangle \in \mathcal{I}(I)$ but $\langle \mathcal{I}(h), \mathcal{I}(v)\rangle \notin \mathcal{I}(I)$. Given any variable assignment $\hat{a}$, it follows by definition that both $\langle \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(h), \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(p)\rangle, \langle \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(p), \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(v)\rangle \in \mathcal{I}(I)$ while $\langle \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(h), \boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(v)\rangle \notin \mathcal{I}(I)$. It follows from the semantics that $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(Ihp) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(Ipv) = 1$ and yet $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(Ihv) \neq 1$. Since $\hat{a}$ was an arbitrary v.a. and Ihp, Ipv, and Ihv are all wfss of $\mathcal{L}^{\text{FOL}}$, we may conclude that $\mathcal{V}_{\mathcal{I}}(Ihp) = \mathcal{V}_{\mathcal{I}}(Ipv) = 1$ and yet $\mathcal{V}_{\mathcal{I}}(Ihv) \neq 1$. Having produced a model which makes the premises true and the conclusion false, it follows that the argument is not valid.

An analogous argument shows that the following argument is also invalid:

Txy: x is taller than y
k: Kate
s: Sam
l: Lu

C1. Kate is taller than Sam.
C2. Sam is taller than Lu.
C3. Kate is taller than Lu.

$$\begin{aligned}\mathbb{D} &= \{k, s, l\}\\ \mathcal{I}(T) &= \{\langle k, s\rangle, \langle s, l\rangle\}\\ \mathcal{I}(k) &= k\\ \mathcal{I}(s) &= s\\ \mathcal{I}(l) &= l\end{aligned}$$

D1. Tks
D2. $\underline{Tsl\quad\quad}$
D3. Tkl

Replacing 'I' with 'T' and similarly replacing 'h' with 'k' and so on for the other constants, the semantic proof given above could be adapted to show that the premises in the previous argument do not entail the conclusion. Here we may ask if this is right, and if so, why we shouldn't say the very same thing about the identity argument.

Certainly it should be admitted that the taller-than argument is a very strong argument in ordinary contexts. After all, given the intended interpretation of English, any *possibility* in which Kate is taller than Sam and in which Sam is taller than Lu is also a possibility in which Kate is taller than Lu. The reason the argument is invalid is that nothing forces us to interpret the predicate 'is taller than' as meaning what it usually means. Put otherwise, the dyadic predicate 'is taller than' is a NON-LOGICAL term of our language, and is to be regimented by a non-logical dyadic predicate in $\mathcal{L}^{\text{FOL}}$ which we may interpret by any subset of $\mathbb{D}^2$ given any domain $\mathbb{D}$ whatsoever. This makes the argument easy to invalidate.

In order to make the taller-than argument valid, we would have to add an additional premise. For instance, here are two ways that we might make the argument valid:

E1. $\forall x\forall y\forall z((Txy \wedge Tyz) \rightarrow Txz)$
E2. Tks
E3. $\underline{Tsl\quad\quad}$
E4. Tkl

F1. $(Tks \wedge Tsl) \rightarrow Tkl$
F2. Tks
F3. $\underline{Tsl\quad\quad}$
F4. Tkl

Both of the arguments above are valid. Whereas the argument on the left starts off by asserting that the taller-than relation is transitive, the argument on the right appeals to a particular instances of the transitivity of the taller-than relation.

It is natural to assume that on the intended interpretation of 'is taller than' in English, we mean to express a transitive relation since this is how heights behave. In reasoning from Tks and Tsl to Tkl, we are implicitly relying on our intuitive grasp of a particular interpretation rather than general logical features of the sentences involved. That is, the argument is convincing not because of its logical form, but because of the particular interpretation that we are assuming, i.e., where 'is taller than' expresses a transitive relation.

When we add a premises which requires T to be transitive (or else add the relevant instance), we are making our assumptions about how to interpret 'is taller than' explicit in a way that avoids reliance on a particular intended interpretation of our language. The amended arguments are valid since the conclusions are true in any model in which all of the premises true. By showing that the conclusion is true when the premises are true in any model of $\mathcal{L}^{\text{FOL}}$ whatsoever, we are interpreting the non-logical terms in all possible ways while holding the meanings of the logical terms fixed. Accordingly, we may conclude that the conclusion follows by virtue of the logical forms of the sentences involved and not a particular interpretation.

What about the identity argument? Certainly we could reproduce a similar story, claiming that as it stands, the identity argument we started off with is not valid but could be made valid by adding a premise that requires identity to be transitive. The question is whether this would be appropriate in the case of the identity predicate. More specifically, is it permissible to interpret identity as *any* subset of $\mathbb{D}^2$ over any domain $\mathbb{D}$?

The answer is certainly 'Yes' since this is exactly how we would interpret the identity argument in $\mathcal{L}^{\text{FOL}}$. Nevertheless, there is good reason not to go this way, choosing instead to include a designated predicate for identity in $\mathcal{L}^{=}$. Recast in $\mathcal{L}^{=}$, the identity argument becomes:

G1. $h = p$
G2. $p = v$
G3. $h = v$

Instead of taking this argument to only be convincing when we restrict consideration to an intended interpretation where '=' means identity, we are taking identity to be a logical notion akin to negation, conjunction, and the quantifiers. Rather than relying on the intended interpretation of our language to tell us what identity means, we are going to provide a semantic clause for identity which holds its meaning fixed across all interpretations in just the same way that we did for the other logical terms included in our language. As a result, the argument above will turn out to be valid as it stands.

You might be wondering why we don't do something similar for the 'is taller than' predicate, and so on for other notions like 'between', or 'is older than', etc. There are two reasons worth considering. The first is that there is no clear stopping point. Were we to start expanding the range of logical predicates whose interpretation we hold fixed by providing semantic clauses, we could go and go forever. This in itself does not require that we do so— we could just choose to include certain predicates in the logical vocabulary of our language and not others given our purposes. The second reason is more forceful: in order to provide a semantic clause for the taller-than predicate 'T', we would have to provide a theory of what it is for one thing to be taller than another. Without providing such a theory, nothing guarantees that T is transitive, and so the taller-than argument would remain invalid.

Although one might attempt to provide a theory of the taller-than relation, doing so reaches beyond the subject-matter of logic. Moreover, it would be natural to use a language like $\mathcal{L}^{=}$ in order to develop such a theory. The same cannot so easily be said for identity. Instead of falling outside the subject-matter of logic, identity is taken to fit squarely within our present aim to develop the most basic conceptual resources that we need to articulate theories. Instead of requiring that we develop an independent theory of identity, the semantics for identity will rely on our understanding of identity in the metalanguage in the same way that the semantics for negation relied on an understanding of negation in the metalanguage. In doing so, we are making our intuitive grasp on the logical terms in English explicit.

Before pressing on, it is worth considering three more cases involving identity. To begin with, consider the following example originally presented by Gottlob Frege:

Rx: x is rising.
h: Hesperus
p: Phosphorus

H1. Hesperus is rising.
H2. Hesperus is Phosphorus.
H3. Phosphorus is rising.

As specified below, identity is a primitive symbol of $\mathcal{L}^{=}$. Accordingly, we do not need to include identity in the symbolization key given above to provide the following regimentation.

I1. Rh
I2. $h = p$
I3. Rp

As we will soon see, this is a perfectly valid argument. Instead of restricting consideration to an intended interpretation, or else adding some further assumptions, the conclusion is entailed by the premises given the semantics that we will provide for $\mathcal{L}^{=}$.

Next consider the regimentation of the following arguments:

Lxy: x loves y
c: Cara
p: Pedro
d: DJ Faro

J1. Only Cara loves Pedro.
J2. DJ Faro loves Pedro.
J3. Cara is DJ Faro.

K1. $\forall x(Lxp \leftrightarrow x = c)$
K2. Ldp
K3. $c = d$

This is a valid argument. Although we could say that Cara loves Pedro in $\mathcal{L}^{\text{FOL}}$, we could not say that *only* Cara loves Pedro in $\mathcal{L}^{\text{FOL}}$ since to do so we would need to say that anything that loves Pedro is identical to Cara in addition to saying that Cara loves Pedro. Here we may accomplish both claims at once by saying that for anything, it loves Pedro just in case it is identical to Cara. Since Cara is identical to herself, she must love Pedro, and moreover, for anything that loves Pedro, it must be identical to Cara. Since DJ Faro loves Pedro, we may conclude that DJ Faro must be identical to Cara. Reasoning in this way requires that we extend the expressive power of $\mathcal{L}^{\text{FOL}}$ by including identity in the language.

The example above suggests the manner in which certain properties may serve to uniquely identify a given object. In doing so, we are effectively saying there is one thing which has a given property. More generally, identity can be used to say how many things have a given property, or else satisfy some combination of properties. For instance, consider the following sentence and its various competing regimentations:

L1. Mozart composed at least two things.

Cxy:	x composed y	M1.	$\exists x Cmx \wedge \exists y Cmy$.
m:	Mozart	M2.	$(\exists x Cmx \wedge \exists y Cmy) \wedge x \neq y$.
		M3.	$\exists x \exists y((Cmx \wedge Cmy) \wedge x \neq y)$.

Although M1 regiments the sentence L1 in $\mathcal{L}^{\text{FOL}}$, this regimentation does not require that there are at least two things that Mozart composed. This is because both conjuncts could be satisfied by the same thing, and so L1 would be true if there was just one thing that Mozart composed. The regimentation given in M2 is worse since this is not a wfs of $\mathcal{L}^{\text{FOL}}$. Rather, M2 is an open sentence since it includes two free variables which fall outside of the scope of the quantifiers. By contrast, sentence M3 provides an adequate regimentation, though does so by making the quantifiers have scope over all instances of x and y.

The success of M3 as a regimentation of L1 has profound consequences for it means that we can regiment 'at least two' in $\mathcal{L}^{=}$. As we will soon see, we may also regiment 'at most two', where 'exactly two' will be regimented by their conjunction. This means that cardinality may be expressed in $\mathcal{L}^{=}$ by answering questions of the form "How many things are such that φ" with wfss that regiment claims of the form "Exactly n things are such that φ". By contrast, such expressions cannot be regimented in $\mathcal{L}^{\text{FOL}}$ without identity.

The cases considered above demonstrate that if we want to capture the logical relationships having to do with identity, we need designated logical vocabulary to do so. Just as we introduced the '$\forall$' and '$\exists$' to regiment quantified claims, we also need a special symbol '$=$' for identity. Whereas the following section will specify a syntax for our expanded language, we will then proceed to provide its semantics the section after.

9.2 The Syntax for $\mathcal{L}^{=}$

The primitive symbols included in $\mathcal{L}^{=}$ are exactly the same as those included in $\mathcal{L}^{\text{FOL}}$ with the single addition of the identity predicate '$=$'. Thus we have the following:

n-place predicates for $n \geqslant 0$ with subscripts, as needed	$A^n, B^n, C^n, \ldots, Z^n$ $A_1^n, B_1^n, Z_1^n, A_2^n, A_{25}^n, J_{375}^n, \ldots$
constants with subscripts, as needed	$a, b, c, \ldots, v$ $a_1, w_4, h_7, m_{32}, \ldots$
variables with subscripts, as needed	w, x, y, z $x_1, y_1, z_1, x_2, \ldots$
sentential connectives	$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
identity	$=$
quantifiers	$\forall, \exists$
parentheses	(,)

Here we have included '=' in our alphabet of primitive symbols. This may seem like a small change and given the examples above, it may be obvious to you how to build up sentences in $\mathcal{L}^=$. Nevertheless, we need to define the wffs of $\mathcal{L}^=$ afresh, where something similar will be repeated for our other recursive definitions that we provided before.

Although much will be as it was before, it is important to attend to the differences that occur throughout the definitions given in the following two sections. In addition to reviewing the details that remain the same, hopefully our methodology should be beginning to feel familiar. As we expand the expressive power of our language, we need to say what counts as a wfs of that language. If the language includes variables, we will need to first say what counts as a wff of the language. By defining what it is for a variable to be free in a wff, we may specify that the wfss of the language are those wffs without free variables. In just the same way that we did for $\mathcal{L}^{\text{FOL}}$, we will now provide these definitions for $\mathcal{L}^=$.

Whereas there was one way to form atomic wffs in $\mathcal{L}^{\text{FOL}}$, we now have two ways to form atomic wffs. Thus we will define the WELL-FORMED FORMULAS (wffs) of $\mathcal{L}^=$ as follows:

1. $\mathcal{F}^n\alpha_1, \ldots, \alpha_n$ is a wff of $\mathcal{L}^=$ if $\mathcal{F}^n$ is an n-place predicate of $\mathcal{L}^=$ and $\alpha_1, \ldots, \alpha_n$ are singular terms (i.e., variables or constants) of $\mathcal{L}^=$.
2. $\alpha = \beta$ is a wff of $\mathcal{L}^=$ if α and β are singular terms of $\mathcal{L}^=$.
3. For any wffs φ and ψ of $\mathcal{L}^=$ and variable α of $\mathcal{L}^=$:
   (a) $\exists\alpha\varphi$ is a wff of $\mathcal{L}^=$;
   (b) $\forall\alpha\varphi$ is a wff of $\mathcal{L}^=$;
   (c) $\neg\varphi$ is a wff of $\mathcal{L}^=$;
   (d) $(\varphi \wedge \psi)$ is a wff of $\mathcal{L}^=$;
   (e) $(\varphi \vee \psi)$ is a wff of $\mathcal{L}^=$;
   (f) $(\varphi \to \psi)$ is a wff of $\mathcal{L}^=$; and
   (g) $(\varphi \leftrightarrow \psi)$ is a wff of $\mathcal{L}^=$.
4. Nothing else is a wff of $\mathcal{L}^=$.

Officially, the clauses above are non-sense, and can only be made sense of by adding corner quotes in appropriate places. Having explained how to do this above, we will rely on the reader to know where these corner quotes are implicitly intended as we did before.

We may either form atomic wffs as we did in $\mathcal{L}^{\text{FOL}}$, or we may form wffs with the identity predicate together with two singular terms. Nevertheless, nothing requires identity wffs to be wfss since they may include free variables in just the same way that 2-place predicates may combine with free variables. This means that there is a new way for free variables to occur in wffs and so we will have to extend our definition of free variables accordingly:

1. α is free in $\mathcal{F}^n\alpha_1, \ldots, \alpha_n$ if $\alpha = \alpha_i$ for some $1 \leqslant i \leqslant n$ where α is a variable, $\mathcal{F}^n$ is an n-place predicate, and $\alpha_1, \ldots, \alpha_n$ are singular terms.
2. α is free in $\beta = \gamma$ if $\alpha = \beta$ or $\alpha = \gamma$ where α is a variable.
3. If φ and ψ are wffs of $\mathcal{L}^=$ and α and β are variables, then:
   (a) α is free in $\exists\beta\varphi$ if α is free in φ and $\alpha \neq \beta$;
   (b) α is free in $\forall\beta\varphi$ if α is free in φ and $\alpha \neq \beta$;
   (c) α is free in $\neg\varphi$ if α is free in φ;
   (d) α is free in $(\varphi \wedge \psi)$ if α is free in φ or α is free in ψ;
   (e) α is free in $(\varphi \vee \psi)$ if α is free in φ or α is free in ψ;
   (f) α is free in $(\varphi \to \psi)$ if α is free in φ or α is free in ψ;
   (g) α is free in $(\varphi \leftrightarrow \psi)$ if α is free in φ or α is free in ψ;
4. Nothing else is a free variable.

Given the definition of free variables in $\mathcal{L}^=$, we may define a WELL-FORMED SENTENCE (wfs) of $\mathcal{L}^=$ to be any wffs of $\mathcal{L}^=$ which does not include any free variables, and an OPEN SENTENCE of $\mathcal{L}^=$ is any wff of $\mathcal{L}^=$ which does include free variables.

In order to establish that all wfss of $\mathcal{L}^=$ have certain properties, it will be important to be able to organize the wfss of $\mathcal{L}^=$ into a countable sequence of stages. To do so, we will define the COMPLEXITY $\texttt{Comp}(\varphi) \in \mathbb{N}$ of any wff of $\mathcal{L}^=$ to be the number of occurrences of sentential operators that belong to $\mathcal{L}^=$, where both $\forall\alpha$ and $\exists\alpha$ are unary operators for any variable α.

1. $\texttt{Comp}(\mathcal{F}^n\alpha_1, \ldots, \alpha_n) = 0$ if $\mathcal{F}^n$ is an n-place predicate of $\mathcal{L}^=$ and $\alpha_1, \ldots, \alpha_n$ are singular terms (i.e., variables or constants) of $\mathcal{L}^=$.
2. $\texttt{Comp}(\alpha = \beta) = 0$ if α and β are singular terms of $\mathcal{L}^=$.
3. For any wffs φ and ψ of $\mathcal{L}^=$ and variable α of $\mathcal{L}^=$:
   (a) $\texttt{Comp}(\exists\alpha\varphi) = \texttt{Comp}(\varphi) + 1$;
   (b) $\texttt{Comp}(\forall\alpha\varphi) = \texttt{Comp}(\varphi) + 1$;
   (c) $\texttt{Comp}(\neg\varphi) = \texttt{Comp}(\varphi) + 1$;
   (d) $\texttt{Comp}(\varphi \wedge \psi) = \texttt{Comp}(\varphi) + \texttt{Comp}(\psi) + 1$;
   (e) $\texttt{Comp}(\varphi \vee \psi) = \texttt{Comp}(\varphi) + \texttt{Comp}(\psi) + 1$;
   (f) $\texttt{Comp}(\varphi \to \psi) = \texttt{Comp}(\varphi) + \texttt{Comp}(\psi) + 1$;
   (g) $\texttt{Comp}(\varphi \leftrightarrow \psi) = \texttt{Comp}(\varphi) + \texttt{Comp}(\psi) + 1$;

Given these definitions, we may now turn to interpret the sentences of $\mathcal{L}^=$.

9.3 The Semantics for $\mathcal{L}^=$

Since the identity predicate '=' belongs to the logical vocabulary of $\mathcal{L}^=$, including identity in the list or primitive symbols does not effect the manner in which the non-logical symbols are interpreted (i.e., the constants or predicates), no change is required to the definition of a model. Accordingly, $\mathcal{L}^=$ and $\mathcal{L}^{\text{FOL}}$ have precisely the same models. As in Chapter 8, we may define a MODEL of $\mathcal{L}^=$ to be any ordered pair $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ where the DOMAIN $\mathbb{D}$ is any nonempty set and the INTERPRETATION $\mathcal{I}$ over $\mathbb{D}$ satisfies the following conditions:

Constants: $\mathcal{I}(\alpha) \in \mathbb{D}$ for every constant α of $\mathcal{L}^=$.

Predicates: $\mathcal{I}(\mathcal{F}^n) \subseteq \mathbb{D}^n$ for every n-place predicate $\mathcal{F}^n$ of $\mathcal{L}^=$ where $n \geqslant 0$.

Recall the manner in which VARIABLE ASSIGNMENTS were defined over a domain $\mathbb{D}$ to be any function $\hat{a}$ from the variables in $\mathcal{L}^{\text{FOL}}$ to elements of $\mathbb{D}$. Again, no change is required since neither the variables nor the domains that we might consider are effected by the addition of the identity predicate to the language. For a similar reason, we may also preserve the definition of a α-VARIANT of $\hat{a}$ as any variable assignment $\hat{c}$ where $\hat{c}(\beta) = \hat{a}(\beta)$ for every variable $\beta \neq \alpha$. Lastly, we define the VALUE (REFERENCE) of singular terms as before:

$$v_{\mathcal{I}}^{\hat{a}}(\alpha) = \begin{cases} \mathcal{I}(\alpha) & \text{if } \alpha \text{ is a constant} \\ \hat{a}(\alpha) & \text{if } \alpha \text{ is a variable.} \end{cases}$$

So far, all of the semantic definitions have remained just as they were in $\mathcal{L}^{\text{FOL}}$. Nevertheless, the semantics for $\mathcal{L}^=$ will differ insofar as it includes an extra clause for identity, mirroring the changes we made to the definition of the wffs of $\mathcal{L}^=$ given above.

VALUATION FUNCTION: For any wffs φ and ψ of $\mathcal{L}^=$, n-place predicate $\mathcal{F}^n$ of $\mathcal{L}^=$, and n singular terms $\alpha_1, \ldots, \alpha_n$ of $\mathcal{L}^=$:

$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\mathcal{F}^n\alpha_1, \ldots, \alpha_n) = 1$ *iff* $\langle v_{\mathcal{I}}^{\hat{a}}(\alpha_1), \ldots, v_{\mathcal{I}}^{\hat{a}}(\alpha_n) \rangle \in \mathcal{I}(\mathcal{F}^n)$.

$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\alpha = \beta) = 1$ just in case $v_{\mathcal{I}}^{\hat{a}}(\alpha) = v_{\mathcal{I}}^{\hat{a}}(\beta)$.

$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall\alpha\varphi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 1$ for every α-variant $\hat{c}$ of $\hat{a}$.

$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\exists\alpha\varphi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 1$ for some α-variant $\hat{c}$ of $\hat{a}$.

$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\neg\varphi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) \neq 1$.

$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi \vee \psi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ or $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi) = 1$ (or both).

$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi \wedge \psi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ and $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi) = 1$.

$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi \to \psi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 0$ or $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi) = 1$ (or both).

$\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi \leftrightarrow \psi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\psi)$.

Having added the clause for identity, the other clauses continue to apply. It is worth comparing the semantic clause for identity to the clause for negation and considering the following worry.

> `Homophonic`: The semantic clause for identity doesn't tell us anything because we have used identity— indeed the same symbol— on both sides of the semantic clause. So in order to know about whether an identity sentence such as $\alpha = \beta$ is true in a model on an assignment, we already need to know what is identical to what. Thus the semantics does not tell us anything we didn't already know.

If we were attempting to understand what '=' means without drawing on any previous understanding, then certainly we should agree that the semantic clauses given above would be a complete failure. As brought out before, the very same thing may be said for negation, conjunction, disjunction, and the quantifiers. In each of these cases, analogues of the terms with which we are concerned appear in the metalanguage and play a critical role in stating the semantic clauses. Thus we cannot lean on our semantics to learn what these terms mean without any prior understanding of at least their analogues in the metalanguage.

All of this we must learn to accept. Where the complaint above goes wrong is in assuming that semantics ought to draw on independently understood terms in order to shed light on new terms. Instead of constructing something out nothing, the semantic clauses allow us to use the meanings we already grasp in English to interpret a simplified formal language in a way that is both systematic and explicit. Identity is no exception, though perhaps even more poignant given that we have used the same symbol in the metalanguage for identity.

Having defined truth relative to a model and a v.a., we are now in a position to specify what it is for a wfs of $\mathcal{L}^=$ to be true in a model independent of an v.a.:

> Theory of Truth: For any wfs φ and model $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ of $\mathcal{L}^=$:
>
> $\mathcal{V}_\mathcal{I}(\varphi) = 1$ just in case $\mathcal{V}_\mathcal{I}^{\hat{a}}(\varphi) = 1$ for every v.a. $\hat{a}$ defined over $\mathbb{D}$.

This definition is just as it was before save that '$\mathcal{L}^{\text{FOL}}$' has been replaced by '$\mathcal{L}^=$'. For completeness, we will copy over the definitions of logical consequence $\vDash$ given its importance, though the other semantic definitions are just as they were in $\mathcal{L}^{\text{FOL}}$ and $\mathcal{L}^{\text{PL}}$.

> Logical Consequence: For any set of wfss $\Gamma \cup \{\varphi\}$ of $\mathcal{L}^=$:
>
> $\Gamma \vDash \varphi$ *iff* for any model $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ of $\mathcal{L}^=$, if $\mathcal{V}_\mathcal{I}(\gamma) = 1$ for all $\gamma \in \Gamma$, then $\mathcal{V}_\mathcal{I}(\varphi) = 1$.

Although there is a lot of redundancy with the syntax and semantics that we provided for $\mathcal{L}^{\text{FOL}}$, hopefully providing the details in full helps to give you a good overview of all of the working piece that make up these theories. In the remaining sections of this chapter, we will put these theories to work in order to evaluate sentences and arguments in $\mathcal{L}^{=}$ that we could not adequate regiment in $\mathcal{L}^{\text{FOL}}$. As we will see, the language $\mathcal{L}^{=}$ is very powerful and perhaps for this reason has become a *lingua franca* by which a wide range of theories have been developed. One especially prominent example is set theory where the dyadic predicate '$\in$' for set-membership may be axiomatized in $\mathcal{L}^{=}$.

As already evident from the previous chapter, the semantics for $\mathcal{L}^{\text{FOL}}$ and $\mathcal{L}^{=}$ has a lot of moving parts and it can be very easy to get tied up in knots while attempting to write semantic proofs. The good news is that soon we will introduce a sound and complete proof system for $\mathcal{L}^{=}$, limiting the need to writing semantic proofs that $\Gamma \vDash \varphi$, or that φ is a tautology or, similarly, a contradiction. Nevertheless, we will need to write semantic proofs to show that $\Gamma \nvDash \varphi$, or φ is not a tautology or, not a contradiction. In order to streamline the semantic proofs that we will write, it will help to establish two supporting lemmas which will help us simplify the use of v.a.s in our proofs. We will establish these results in the following section before presenting a number applications in later sections.

9.4 Assignment Lemmas

Recall that a wfs φ of $\mathcal{L}^{=}$ is true in a model $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ just in case it is true in true in that model relative to all v.a.s defined over the domain $\mathbb{D}$. This definition captures the intuition that the truth-value of a wfs φ in a given model of $\mathcal{L}^{=}$ should be independent of the v.a.s defined over the domain of that model since φ does not have any free variables. Nevertheless, it is not always convenient to be told that φ is true in a model $\mathcal{M}$ relative to *all* v.a.s defined over the domain of that model, and similarly, it is not always convenient to show that φ is true in a model $\mathcal{M}$ relative to *all* v.a.s defined over the domain of that model. Since a wfs has no free variables, if a wfs φ is true in all v.a.s defined over the domain of the model in question, then φ should be true in *some* v.a. defined over the domain, and *vice versa*. In order to establish this equivalence, we may now provide proofs of the following lemmas.

Lemma 9.1 If $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in a wff φ of $\mathcal{L}^{=}$, then $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi)$.

Letting $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be a model of $\mathcal{L}^{=}$, the proof proceeds by induction on the complexity of wffs of $\mathcal{L}^{=}$ where now there are two cases in the base step.

Base Step: Letting φ be a wff of $\mathcal{L}^{=}$ where $\varphi(\texttt{Comp}) = 0$ and both $\hat{a}$ and $\hat{c}$ are v.a.s defined over $\mathbb{D}$, we may assume that $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in φ. It follows that either $\varphi = \ulcorner \mathcal{F}^n \alpha_1, \ldots, \alpha_n \urcorner$ or $\varphi = \ulcorner \alpha_1 = \alpha_n \urcorner$ where in the latter case $n = 2$ and using corner quotes to improve clarity. In either case, $\boldsymbol{v}_{\mathcal{I}}^{\hat{a}}(\alpha_i) = \boldsymbol{v}_{\mathcal{I}}^{\hat{c}}(\alpha_i)$ for all $1 \leqslant i \leqslant n$. We may then consider the following biconditionals:

$$\begin{array}{rll} \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1 & \textit{iff} & \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\mathcal{F}\alpha_1, \ldots, \alpha_n) = 1 \\ & \textit{iff} & \langle \boldsymbol{v}^{\hat{a}}_{\mathcal{I}}(\alpha_1), \ldots, \boldsymbol{v}^{\hat{a}}_{\mathcal{I}}(\alpha_n) \rangle \in \mathcal{I}(\mathcal{F}^n) \\ (\star) & \textit{iff} & \langle \boldsymbol{v}^{\hat{c}}_{\mathcal{I}}(\alpha_1), \ldots, \boldsymbol{v}^{\hat{c}}_{\mathcal{I}}(\alpha_n) \rangle \in \mathcal{I}(\mathcal{F}^n) \\ & \textit{iff} & \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\mathcal{F}\alpha_1, \ldots, \alpha_n) = 1 \\ & \textit{iff} & \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1. \end{array}$$

$$\begin{array}{rll} \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1 & \textit{iff} & \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\alpha_1 = \alpha_n) = 1 \\ & \textit{iff} & \boldsymbol{v}^{\hat{a}}_{\mathcal{I}}(\alpha_1) = \boldsymbol{v}^{\hat{a}}_{\mathcal{I}}(\alpha_n) \\ (\star) & \textit{iff} & \boldsymbol{v}^{\hat{c}}_{\mathcal{I}}(\alpha_1) = \boldsymbol{v}^{\hat{c}}_{\mathcal{I}}(\alpha_n) \\ & \textit{iff} & \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\alpha_1 = \alpha_n) = 1 \\ & \textit{iff} & \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1. \end{array}$$

Whereas the starred biconditionals follow by the identities above, the other biconditionals are immediate from case assumptions and semantics for $\mathcal{L}^=$. Thus the lemma holds for any wff φ of $\mathcal{L}^=$ where $\texttt{Comp}(\varphi) = 0$.

Induction Step: Assume for induction that the lemma holds for every wff φ of $\mathcal{L}^=$ where $\texttt{Comp}(\varphi) \leqslant n$. Letting φ be a wff of $\mathcal{L}^=$ where $\texttt{Comp}(\varphi) = n + 1$ and both $\hat{a}$ and $\hat{c}$ are v.a.s defined over $\mathbb{D}$, we may assume that $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in φ. It remains to show that $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi)$.

Case 1: Assume $\varphi = \neg\psi$. Since $\texttt{Comp}(\neg\psi) = \texttt{Comp}(\psi) + 1 = n + 1$ by assumption and the definition of complexity, we know that $\texttt{Comp}(\psi) = n$, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\psi)$ by hypothesis. We may then observe the following biconditionals:

$$\begin{array}{rll} \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1 & \textit{iff} & \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\neg\psi) = 1 \\ & \textit{iff} & \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = 0 \\ (\star) & \textit{iff} & \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\psi) = 0 \\ & \textit{iff} & \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\neg\psi) = 1 \\ & \textit{iff} & \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1. \end{array}$$

Whereas the starred biconditional follows from the induction hypothesis, the other biconditionals hold by the semantics and case assumption.

Case 2: Assume $\varphi = \psi \wedge \chi$. Since $\texttt{Comp}(\psi \wedge \chi) = \texttt{Comp}(\psi) + \texttt{Comp}(\chi) + 1 = n + 1$ by assumption and the definition of complexity, we know that $\texttt{Comp}(\psi), \texttt{Comp}(\chi) \leqslant n$, and so $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\psi)$ and $\mathcal{V}^{\hat{a}}_{\mathcal{I}}(\chi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\chi)$ by hypothesis. Now consider:

$$\begin{array}{rll} \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\varphi) = 1 & \textit{iff} & \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi \wedge \chi) = 1 \\ & \textit{iff} & \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\psi) = \mathcal{V}^{\hat{a}}_{\mathcal{I}}(\chi) = 1 \\ (\star) & \textit{iff} & \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\psi) = \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\chi) = 1 \\ & \textit{iff} & \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\psi \wedge \chi) = 1 \\ & \textit{iff} & \mathcal{V}^{\hat{c}}_{\mathcal{I}}(\varphi) = 1. \end{array}$$

Whereas the starred biconditional follows from the induction hypothesis, the other biconditionals hold by the semantics and case assumption.

Case 3: Assume $\varphi = \psi \vee \chi$. (Exercise for the reader.)

Case 4: Assume $\varphi = \psi \rightarrow \chi$. (Exercise for the reader.)

Case 5: Assume $\varphi = \psi \leftrightarrow \chi$. (Exercise for the reader.)

Case 6: Assume $\varphi = \forall\gamma\psi$. Since $\texttt{Comp}(\forall\gamma\psi) = \texttt{Comp}(\psi)+1 = n+1$ by assumption and the definition of complexity, we know that $\texttt{Comp}(\psi) = n$, and so by hypothesis, $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(\psi) = \mathcal{V}_{\mathcal{I}}^{\hat{g}}(\psi)$ for any v.a.s $\hat{e}$ and $\hat{g}$ defined over $\mathbb{D}$ where $\hat{e}(\alpha) = \hat{g}(\alpha)$ for all free variables α in ψ. We may then observe the following biconditionals:

$$\begin{array}{rll} \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1 & \textit{iff} & \mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall\gamma\psi) = 1 \\ & \textit{iff} & \mathcal{V}_{\mathcal{I}}^{\hat{e}}(\psi) = 1 \text{ for every } \gamma\text{-variant } \hat{e} \text{ of } \hat{a} \\ (\star) & \textit{iff} & \mathcal{V}_{\mathcal{I}}^{\hat{e}}(\psi) = 1 \text{ for every } \gamma\text{-variant } \hat{e} \text{ of } \hat{c} \\ & \textit{iff} & \mathcal{V}_{\mathcal{I}}^{\hat{c}}(\forall\gamma\psi) = 1 \\ & \textit{iff} & \mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 1. \end{array}$$

Letting $\hat{e}$ be any γ-variant of $\hat{a}$, it follows that $\hat{e}(\beta) = \hat{a}(\beta)$ for every variable $\beta \neq \gamma$. By assumption, $\hat{a}(\alpha) = \hat{c}(\alpha)$ for all free variables α in φ. Since $\varphi = \forall\gamma\psi$, we know by the definition of free variables that γ is not free in φ, and so $\hat{e}(\beta) = \hat{c}(\beta)$ for every variable $\beta \neq \gamma$. By generalizing on $\hat{e}$, it follows that every γ-variant $\hat{e}$ of $\hat{a}$ is also a γ-variant of $\hat{c}$, where the converse holds by parity of reasoning. This establishes the stared biconditional. As before, the other biconditionals given above follow by the semantics and case assumption.

Case 7: Assume $\varphi = \exists\gamma\psi$. (Exercise for the reader.) □

Lemma 9.2 If φ is a wfs of $\mathcal{L}^=$: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ *iff* $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ for some v.a. $\hat{a}$ over $\mathbb{D}$.

Letting φ be a wfs and $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ of $\mathcal{L}^=$, we aim to prove both directions of the biconditional: $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$ just in case $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ for some v.a. $\hat{a}$ over $\mathbb{D}$.

Assume $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$. By definition, $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ for every v.a. $\hat{a}$ defined over $\mathbb{D}$. Since $\mathbb{D} \neq \varnothing$, we know that there is some $d \in \mathbb{D}$. We may then consider the constant v.a. where $\hat{c}(\alpha) = d$ for every variable α of $\mathcal{L}^=$. Given the above, $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 1$, and so we may conclude that $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ for some v.a. $\hat{a}$ defined over $\mathbb{D}$.

Assume instead that $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = 1$ for some v.a. $\hat{a}$ defined over $\mathbb{D}$. Let $\hat{c}$ be any v.a. defined over $\mathbb{D}$. Since φ has no free variables, $\hat{a}(\alpha) = \hat{c}(\alpha)$ holds vacuously for all free variables α in φ, and so $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\varphi) = \mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi)$ follows by **Lemma 9.1**. Since $\hat{c}$ was arbitrary, $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(\varphi) = 1$ for all v.a. $\hat{c}$ over $\mathbb{D}$, and so $\mathcal{V}_{\mathcal{I}}(\varphi) = 1$. □

Given **Lemma 9.1**, the lemma given above follows easily. As we will see, **Lemma 9.2** will never be entirely necessary, but often convenient. After all, we made do just fine without **Lemma 9.2** until now. Nevertheless, it will help to streamline our semantic proofs to be able to appeal to these lemmas when convenient since it is often easier to work with a particular v.a. rather than a claim about all v.a.s defined over a given domain.

9.5 Uniqueness

Recall the following argument from before:

J1. Only Cara loves Pedro.
J2. DJ Faro loves Pedro.
J3. DJ Faro is Cara.

K1. $\forall x(Lxp \leftrightarrow x = c)$
K2. Ldp
K3. $d = c$

We are now in a position to show that this argument is valid.

Proof: Let $\mathcal{M} = \langle \mathbb{D}, \mathcal{I} \rangle$ be a $\mathcal{L}^{=}$ model where: (1) $\mathcal{V}_{\mathcal{I}}(\forall x(Lxp \leftrightarrow x = c)) = 1$; and (2) $\mathcal{V}_{\mathcal{I}}(Ldp) = 1$. It follows from the latter assumption by **Lemma 9.2** that $\mathcal{V}_{\mathcal{I}}^{\hat{c}}(Ldp) = 1$ for some $\hat{c}$ defined over $\mathbb{D}$, and so $\langle v_{\mathcal{I}}^{\hat{c}}(d), v_{\mathcal{I}}^{\hat{c}}(p) \rangle \in \mathcal{I}(L)$. Since d and p are constants, we know by definition that $\langle \mathcal{I}(d), \mathcal{I}(p) \rangle \in \mathcal{I}(L)$.

Let $\hat{e}$ be a v.a. defined over $\mathbb{D}$ where $\hat{e}(x) = \mathcal{I}(d)$. Since $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(\forall x(Lxp \leftrightarrow x = c)) = 1$ for every $\hat{a}$ defined over $\mathbb{D}$ by (1), we know that $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(\forall x(Lxp \leftrightarrow x = c)) = 1$ in particular. By the semantics, $\mathcal{V}_{\mathcal{I}}^{\hat{e}}(Lxp) = \mathcal{V}_{\mathcal{I}}^{\hat{e}}(x = c)$. Thus $\langle v_{\mathcal{I}}^{\hat{e}}(x), v_{\mathcal{I}}^{\hat{e}}(p) \rangle \in \mathcal{I}(L)$ just in case $v_{\mathcal{I}}^{\hat{e}}(x) = v_{\mathcal{I}}^{\hat{e}}(c)$, and so $\langle \hat{e}(x), \mathcal{I}(p) \rangle \in \mathcal{I}(L)$ just in case $\hat{e}(x) = \mathcal{I}(c)$. Since $\hat{e}(x) = \mathcal{I}(d)$, we know that $\langle \mathcal{I}(d), \mathcal{I}(p) \rangle \in \mathcal{I}(L)$ just in case $\mathcal{I}(d) = \mathcal{I}(c)$.

Given the above, we may conclude that $\mathcal{I}(d) = \mathcal{I}(c)$. Thus $v_{\mathcal{I}}^{\hat{g}}(d) = v_{\mathcal{I}}^{\hat{g}}(c)$ where $\hat{g}$ is any v.a., and so $\mathcal{V}_{\mathcal{I}}^{\hat{g}}(d = c) = 1$. Since $d = c$ is a wfs, we may conclude that $\mathcal{V}_{\mathcal{I}}(d = c) = 1$ by **Lemma 9.2**. Hence $\forall x(Lxp \leftrightarrow x = c), Ldp \vDash d = c$. □

The proof begins by considering an arbitrary model in which the premises are true, showing that the conclusion must also be true in that model. The idea is to extend the general claim given in K1 to the particular claim given in K2 by considering an assignment of the variable x to whatever the constant d happens to refer to in the model at hand. Accordingly, it makes sense to begin by unpacking what we know about K2 before making use of K1.

Once we have worked out that $\langle \mathcal{I}(d), \mathcal{I}(p) \rangle \in \mathcal{I}(L)$, we are now in a position to make a strategic choice. Although it follows that $\mathcal{V}_{\mathcal{I}}^{\hat{a}}(Lxp \leftrightarrow x = c) = 1$ for any v.a. $\hat{a}$ defined over the domain, we were careful to consider a v.a. $\hat{e}$ where $\hat{e}(x) = \mathcal{I}(d)$. This is akin to instantiating x by d, resulting in the sentence $Ldp \leftrightarrow d = c$ which, together with Ldp entails $d = c$. Instead of replacing 'x' with 'd', we chose the x-variant $\hat{e}$ where $\hat{e}(x) = \mathcal{I}(d)$. This sort of reasoning will reappear once we introduce a proof system for $\mathcal{L}^{=}$.

Given that only Cara loves Pedro, we may think of her as the unique-Pedro-lover. That is, not only is there something out there that loves Pedro, Cara is *the* Pedro-lover. Suppose we forget Cara's name, but remember this prominent fact about her. We might then ask: is the Pedro-lover the same as DJ Faro? Instead of using her name, we are using this distinguishing feature to refer to her. This is a common practice since we don't have names for everything in English, and even when we do, we don't always know everything name.

Whereas Cara's distinguishing feature was loving Pedro, in general we may appeal to any condition however complex so long as it succeeds in picking out exactly one individual. That is, if we know that there is some particular way that one thing happens to be where only that thing is that way, then we may use that way of being to pick out that particular thing. For instance, perhaps many people love Pedro, but Cara is the only DJ to love Pedro. We may express this with $\forall x((Dx \wedge Lxp) \leftrightarrow x = c)$. Replacing the constant 'c' with a variable as in $\forall x((Dx \wedge Lxp) \leftrightarrow x = y)$ returns an open sentence which corresponds to the condition of *being the only DJ to love Pedro*. This brings us to the topic of definite descriptions.

9.6 Definite Descriptions

In 1905, Bertrand Russell famously characterized *definite descriptions* in terms of identity. In the paradigm cases, definite descriptions use the definite article 'the'. Suppose one were to hear crying in the next room, saying 'The baby is hungry'. This is to claim that the *one and only baby* (in the vicinity) is hungry. Russell was motivated in part by the apparent fact that one can use this sort of language in a meaningful way even if one is wrong about whether there is anything fitting the description. If there is no baby— the crying is a recording — the statement is false, but it's still meaningful. For this reason, Russell was reluctant to suppose that we should understand 'the baby' as a name. Instead, the sentence can be understood to be an existentially quantified claim about a unique baby in the vicinity though there may not be a baby in the vicinity. According to Russell, saying 'The baby is hungry' is to say three things: there is a baby, there's no other baby than that one, and that baby is hungry. As brought out above, uniqueness can be expressed in $\mathcal{L}^=$ with the help of identity.

In order to contrast names with definite descriptions, consider the symbolization key and regimentations for 'Jonathan is hungry' and 'The baby is hungry':

Bx: x is a baby	N1. Hj
Hx: x is hungry	N2. $\exists x((Bx \wedge \forall y(By \to y = x)) \wedge Hx)$.
j: Jonathan	N3. $\exists x(\forall y(By \leftrightarrow y = x) \wedge Hx)$.

Whereas 'Jonathan is hungry' is easilyt regimented by N1, according to Russell's theory of definite descriptions, 'The baby is hungry' has a much more complex logical form. Whereas the regimentation given in N2 says that there is some baby which is the only baby and is hungry, N3 collapses the first two parts of N2, claiming that the unique baby is hungry.

As mentioned in the previous section, the distinguishing feature by which an individual is uniquely identified need not be expressed by a single predicate. For instance, suppose there is a baby who is sleeping right in front of us, but we hear crying from the other room. One may then be a little more specific by saying 'The crying baby is hungry'. Accordingly, we may expand our symbolization key to regiment this more specific claim.

Cx: x is crying

O1. $\exists x(\forall y((Cy \land By) \leftrightarrow y = x) \land Hx)$.

Instead of the single predicate 'B', we have used 'C' together with 'B' in order to form the open sentence '$Cy \land By$' which describes the unique individual to which we intend to refer.

In order to speak generally about the means by which we may refer to some unique individual satisfying a certain condition, let $\varphi(\alpha)$ be any wff of $\mathcal{L}^{=}$ in which the variable α is free. If α is the only free variable in $\varphi(\alpha)$, we may take $\varphi(\alpha)$ to be a DESCRIPTION. Moreover, $\varphi(\alpha)$ provides a DEFINITE DESCRIPTION in a model $\mathcal{M}$ just in case $\varphi(\alpha)$ is a description which just one thing satisfies, i.e., $\exists\beta\forall\alpha(\varphi(\alpha) \leftrightarrow \alpha = \beta)$ where $\alpha \neq \beta$ are distinct variables. Given a definite description $\varphi(\alpha)$, we may make claims about the object satisfying that description by conjoining another description $\psi(\beta)$ within the scope of the existential quantifier as follows:

$$\exists\beta(\forall\alpha(\varphi(\alpha) \leftrightarrow \alpha = \beta) \land \psi(\beta)).$$

This reads, the unique thing for which φ is such that ψ. The sentence O1 is an instance of this general recipe, and reads: the unique thing for which it is a crying baby, is hungry. Russell's idea is that this is what is going on when we use the definite article 'the' since we may say the same thing much more naturally with: *the* crying baby is hungry.

One of the interesting features of Russell's theory is that 'The baby is not hungry' is not the negation of 'The baby is hungry'. Instead, the negation applies only to the last conjunct:

P1. $\exists x(\forall y((Cy \land By) \leftrightarrow y = x) \land \neg Hx)$.

The reason Russell designed his theory this way was that he thought that both of these sentences equally implied that there is a baby. If there is no baby, then you'd be mistaken in asserting either 'The baby is hungry' or 'The baby is not hungry'. Consequently, one can't be the negation of the other, but rather requires the analysis given above.

As a treatment of the truth conditions of English sentences, Russell's theory is controversial. For instance, consider the following case:

Kxy: x is king of y
Bx: x is Bald
f: France

1. The king of France is bald.
2. $\exists x(\forall y(Kyf \leftrightarrow y = x) \land Bx)$.

Some philosophers of language think that sentences that seem to presuppose the existence of something that isn't there aren't straightforwardly false, but are rather defective in some other way— perhaps they fail to be meaningful, or perhaps they take on some truth value other than true or false. These matters are beyond the scope of this book, and so we will remain neutral on whether Russell's theory is an accurate treatment of English. Nevertheless, without including identity in the language, this question would not even arise. This helps to bring out what is distinctive about the expressive power of $\mathcal{L}^{=}$ in contrast to $\mathcal{L}^{\text{FOL}}$.

9.7 Quantities

Including identity in $\mathcal{L}^{=}$ permits us to express claims about quantities that we couldn't in $\mathcal{L}^{\text{FOL}}$. In §9.1 we considered the sentence 'Mozart composed at least two things' where identity was found to play a critical role. In particular, we provided the following regimentation:

M3. $\exists x \exists y((Cmx \wedge Cmy) \wedge x \neq y)$.

Given that we were able to regiment 'at least two', you might suspect that we can also regiment 'at least three', and so on for the other natural numbers. Consider the following:

R1. $\exists x \exists y \exists z(((((Cmx \wedge Cmy) \wedge Cmz) \wedge x \neq y) \wedge x \neq z) \wedge y \neq z)$.

Though it is a lot longer than sentence M3, the sentence above says that there are at least three things that Mozart composed. Given that conjunction is associative and commutative, all but the outermost parentheses are more trouble than they are worth. In general, we will indulge in the convention of dropping the parentheses that occur in long conjunctions and long disjunctions. Thus we may rewrite sentence R1, as follows:

S1. $\exists x \exists y \exists z(Cmx \wedge Cmy \wedge Cmz \wedge x \neq y \wedge x \neq z \wedge y \neq z)$.

This is a lot easier to read and nothing significant is lost. It is important to stress that we can only drop parentheses in sentences which *only* include conjunction, or *only* include disjunction. Even so, these sentences are bound to get very long for large values of n.

In order to characterize quantities in a more general way, it can be useful to introduce some abbreviations for what we will refer to as the *inequality quantifiers*. Instead of adding new primitive symbols to $\mathcal{L}^{=}$, we are only providing conventions for abbreviating long expression with much shorter expressions for the sake of readability.

In order to state these abbreviations in a general way, we will take β to be FREE FOR α in φ just in case there is no free occurrence of α in φ in the scope of a quantifier that binds β. For instance, y is not free for x in '$\forall yFxy$' since replacing 'x' with 'y' would yield '$\forall yFyy$' where the quantifier '$\forall y$' would end up binding an extra variable. Roughly speaking, you can take 'β is free for α' to mean 'β can replace α without leading to extra binding'.

We may then define $\varphi[\beta/\alpha]$ to be the SUBSTITUTION that results from replacing all free occurrences of α in φ with β where β is required to be free for α in φ. For instance, $\forall yFxy[z/x]$ is the wff $\forall yFzy$, and $\forall yFxy[y/x]$ is undefined since y is not free for x. Given this notation, we may define the following abbreviations for quantifiers of the form 'at least n things are such that φ':

$$\exists_{\geqslant 1}\alpha\varphi := \exists\alpha\varphi(\alpha)$$
$$\exists_{\geqslant n+1}\alpha\varphi := \exists\alpha(\varphi(\alpha) \wedge \exists_{\geqslant n}\beta(\alpha \neq \beta \wedge \varphi[\beta/\alpha])) \text{ where } \beta \text{ is free for } \alpha \text{ in } \varphi.$$

As above ':=' represents that the left side is merely an abbreviation for the right side. These abbreviations have a recursive structure which defines $\exists_{\geqslant n}\alpha$ for all n. Even in the base clause, it is important to require α to be free in φ where we have achieved this by writing $\varphi(\alpha)$. Otherwise, claims like $\exists_{\geqslant n}xHj$ would be read as 'at least n things are such that Jonathan is hungry' despite the fact that Hj does not include any free variables for the quantifiers to bind. We can put these quantifiers to work in order to construct sets of sentences like:

$$\Gamma_\infty := \{\exists_{\geqslant n}x(x = x) : n \in \mathbb{N}\}.$$

For any natural number n, the set Γ_∞ includes a sentence that says at least n things are self-identical. Not only may we show that there are models which satisfy Γ, these models must have infinite domains. That we can begin to express claims about the infinite further demonstrates just how much more expressive power $\mathcal{L}^=$ has than $\mathcal{L}^{\text{FOL}}$.

In addition to being able to say that there are at least n things that satisfy a certain condition, $\mathcal{L}^=$ permits us to say that that there are at most n things that satisfy a certain condition. Consider the following sentence and its regimentation:

T1. Mozart composed at most two things.
T2. $\forall x\forall y\forall z((Cmx \wedge Cmy \wedge Cmz) \to (x = y \vee x = z \vee y = z))$.

This says that for any x, y, and z which Mozart composed, at least two of them are identical. So far, nothing prevents all of them from being identical or requires there to be something which Mozart composed. More generally, we may say that there are at most n things which satisfy a given condition φ. Although we could define this recursively in a similar fashion to what was given above, we may avoid doing so by adopting the following convention for all n.

$$\exists_{\leqslant n}\alpha\varphi := \neg\exists_{\geqslant n+1}\alpha\varphi.$$

This says that it is not the case that there are at least $n + 1$ things that are φ, and so no more than n things that are φ. We may combine these two types of quantifiers to say that there are between n and m things that are φ as given by the following abbreviation:

$$\exists_{[n,m]}\alpha\varphi := \exists_{\geqslant n}\alpha\varphi \wedge \exists_{\leqslant m}\alpha\varphi.$$

Whereas $\exists_{[n,m]}\alpha\varphi$ says that between n and m things are φ, in the special case where $n = m$, the statement $\exists_{[n,n]}\alpha\varphi$ says that exactly n things are φ. We have already seen instances of this above with uniqueness. After all, saying that *only* Cara loves Pedro is like saying there is *exactly one* thing that loves Pedro where this entails both that there is *at least* one thing that loves Pedro and that there is *at most* one thing that loves Pedro, namely Cara.

Suppose that we want to say that there are exactly two things that Mozart composed. One way to do this is to conjoin sentences M3 and T2 since this amounts to saying that there is at least two things that Mozart composed and at most two things that Mozart composed. However, the result is long and difficult to parse. Alternatively, we could use the notation $\exists_{[2,2]}\alpha$ introduced above, though this notation abbreviates something just as complicated.

Instead of employing the quantifier $\exists_{[2,2]}\alpha$, we can simplify the regimentation as follows:

U1. Mozart composed exactly two things.
U2. $\exists x \exists y(x \neq y \wedge \forall z(Cmz \leftrightarrow (z = x \vee z = y)))$.

More generally, consider the following definitions:

$$\exists_0 \alpha\varphi := \forall\alpha\neg\varphi(\alpha)$$
$$\exists_{n+1}\alpha\varphi := \exists\alpha(\varphi(\alpha) \wedge \exists_n\beta(\alpha \neq \beta \wedge \varphi[\beta/\alpha])) \text{ where } \beta \text{ is free for } \alpha \text{ in } \varphi.$$

What it is for no α to be φ is for everything to not be φ. What it is for exactly $n+1$ things to be φ is for something to be φ and exactly n other things to be φ. Given these recursive definitions, we may work out the following logical equivalences:

$$\exists_0\alpha\varphi \mathrel{=\!\!|\!\!\models} \forall\alpha\neg\varphi(\alpha)$$
$$\exists_1\alpha\varphi \mathrel{=\!\!|\!\!\models} \exists\alpha\forall\beta(\varphi[\beta/\alpha] \leftrightarrow \beta = \alpha)$$
$$\exists_2\alpha\varphi \mathrel{=\!\!|\!\!\models} \exists\alpha\exists\beta(\alpha \neq \beta \wedge \forall\gamma(\varphi(\gamma/\alpha) \leftrightarrow (\gamma = \alpha \vee \gamma = \beta)))$$
$$\exists_3\alpha\varphi \mathrel{=\!\!|\!\!\models} \exists\alpha\exists\beta\exists\gamma(\alpha \neq \beta \wedge \alpha \neq \gamma \wedge \beta \neq \gamma \wedge \forall\delta(\varphi(\delta/\alpha) \leftrightarrow (\delta = \alpha \vee \delta = \beta \vee \delta = \gamma)))$$
$$\vdots$$

Although the results differ in logical form from those that we may derive from $\exists_{[n,n]}\alpha$ for different values of n, they are logically equivalent insofar as $\exists_n\alpha\varphi \mathrel{=\!\!|\!\!\models} \exists_{[n,n]}\alpha\varphi$.

Whereas $\exists_{\geqslant n}\alpha$ and $\exists_{\leqslant n}\alpha$ are referred to as INEQUALITY QUANTIFIERS, we will refer to $\exists_{[n,m]}\alpha$ and $\exists_n\alpha$ as CARDINALITY QUANTIFIERS. Although the inequality operators are often useful, the cardinality operators express something very specific, further demonstrating the expressive power of $\mathcal{L}^{=}$. After all, how would one attempt any of the definitions given above in $\mathcal{L}^{\text{FOL}}$?

9.8 Leibniz's Law

In §9.1, we saw that treating identity as any other predicate invalidated the argument:

H1. Hesperus is rising.	I1. Rh
H2. Hesperus is Phosphorus.	I2. Ihp
H3. Phosphorus is rising.	I3. Rp

The problem was that if the extension of the identity predicate can be assigned to any subset of $\mathbb{D}^2$, there was no guarantee that h and p refer to the same object. The problem was avoided by including identity among the primitive symbols of the language and providing its semantics as above. We may replace 'I' with '$=$' in the argument above where the result is valid. This is because if '$h = p$' is true in a model, then 'h' and 'p' refer to the very same object in the domain, and so 'Rh' and 'Rp' will have the same truth-value. More generally, we may show that all instances of the following schema are valid where α and β are constants and φ is a sentence, referring to this as a version of *Leibniz's Law*:

$$\alpha = \beta \models \varphi \leftrightarrow \varphi[\beta/\alpha].$$

The idea behind this schema is that if we are given two names α and β for the same object, then whatever we can say about that object with one name will have the same truth-value if we use the other name. This might seem natural in the case above. After all, if Hesperus is Phosphorus, how could it be that Hesperus is rising without Phosphorus rising?

However compelling this particular instance may be, Leibniz's Law admits of a wide range of exceptions. For instance, consider the following argument:

Bxy:	x believes that y is rising.	V1.	Bth
t:	Thales	V2.	$h = p$
h:	Hesperus	V3.	Btp
p:	Phosphorus		

Since the identity of Hesperus and Phosphorus was not always known, we might imagine a time when Thales believed that Hesperus is rising without also believing that Phosphorus is rising. As a result, the argument above may have true premises and a false conclusion. To take another case, we may imagine that Lois loves Clark Kent and Lois does not love Superman despite the fact that Clark Kent is Superman. Nevertheless, both of these arguments are valid given the semantics for $\mathcal{L}^=$. Has something gone wrong?

The constants α and β may be said to *co-refer* just in case they name the same thing, i.e., $\ulcorner\alpha = \beta\urcorner$ is true. One common response holds that some claims are *opaque* insofar as we cannot freely substitute co-referring constants. Belief claims are paradigmatic of opacity: just because one believes that φ doesn't mean that one must believe that $\varphi[\beta/\alpha]$ whenever $\ulcorner\alpha = \beta\urcorner$ is true. For instance, suppose that Kaya is learning arithmetic and believes that 2 is even. Nevertheless, she hasn't learned anything about prime numbers and so does not believe that the first prime number is even despite the fact that 2 is the first prime number.

Given that Leibniz's Law is valid given the semantics for $\mathcal{L}^=$, restricting Leibniz's Law requires significant revisions to the present semantic theory. For our purposes here, we will assume that co-referring terms can always be substituted for each other as asserted by *Leibniz's Law*. This amounts to the assumption that none of the sentences with which we will be concerned are opaque. This is a significant limitation since it is easy to introduce predicates like B given above for belief ascriptions. Nevertheless, overcoming this limitation is far from straightforward and lies outside the scope of our present concern.

Although $\mathcal{L}^=$ is a flexible and expressive powerful language, every language has its limits. Nevertheless, $\mathcal{L}^=$ is perfectly adequate for a wide range of applications. In particular, it is natural to assume that mathematics is *transparent* insofar as it excludes consideration of opaque claims in which *Leibniz's Law* fails to hold.