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Contact Geometry: the Geometrical Method of Gibbs's Thermodynamics

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Every mathematician knows that it is impossible to understand any elementary course in thermodynamics. The reason is that the thermodynamics is based—as Gibbs has explicitly proclaimed—on a rather complicated mathematical theory, on the contact geometry. Contact geometry is one of the few "simple geometries" of the so-called Cartan's list, but it is still mostly unknown to the physicist (unlike the Riemannian geometry and the symplectic or Poisson geometries, whose fundamental role in physics is today generally accepted).

1. Contact structures and Legendre submanifolds. A contact structure on an odd-dimensional manifold M^{2n+1} is a field of hyperplanes (of linear subspaces of codimension 1) in the tangent spaces to M at all its points. All the generic fields of hyperplanes of a manifold of a fixed dimension are locally equivalent; they define the (local) contact structures. The other (degenerate) fields are exceptional[1]). (See Figure 1, p. 164.)

EXAMPLE. A *1-jet of a function* $y = f(x_1, \cdots, x_n)$ at point x of manifold V^n is defined by the point $(x, y, p) \in \mathbb{R}^{2n+1}$, where $p_i = \partial f/\partial x_i$.

The 1-jets of all functions $f\colon V \to \mathbb{R}$ in all points x of V form a $2n+1$-dimensional *1-jets space* $J^1(V^n, \mathbb{R})$ (of the Taylor polynomials of degree 1).

The *natural contact structure* of this space is defined by the following condition: the 1-graphs $\{x, y = f(x), p = \partial f/\partial x\} \subset J^1(V^n, \mathbb{R})$ of all the functions on V should be tangent to the contact structure hyperplane at every point. In coordinates this condition means that the 1-form $dy - p\,dx$ should vanish on the hyperplanes of the contact field. (See Figure 2, p. 164.)

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58A99, 58F05.

[1]Technically speaking, the nondegeneracy condition for the structure $\alpha = 0$ is the nondegeneracy of the bilinear 2-form $d\alpha$ on the plane $\alpha = 0$. This explains why the dimension of M should be odd.

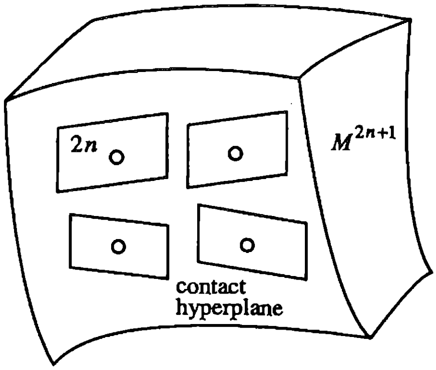


FIGURE 1. A contact structure.

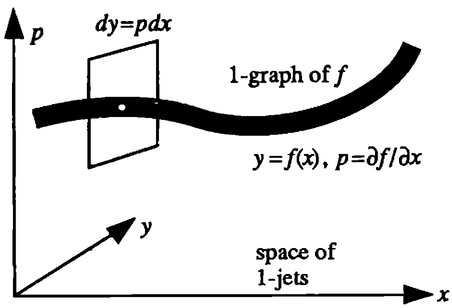


FIGURE 2. The contact structure of the 1-jets manifold.

The maximal integral submanifolds of a contact manifold are called *Legendre submanifolds*. For instance, the 1-graph of any function is a Legendre submanifold in the space of 1-jets. More general Legendre submanifolds may be considered as the "multivalued" functions on V like the Riemannian surfaces define "multivalued" holomorphic functions.

Even in the simplest case of a 3-dimensional contact manifold $(n = 1)$ we obtain a nontrivial theory: that of implicit ordinary differential equations $F(x, y, p) = 0$. Geometrically this equation defines a surface in the space of 1-jets. The planes of the contact structure $dy = p\,dx$ intersect this surface along the directions of the integral curves of the equation $F(x, y, dy/dx) = 0$. These integral curves are called the *characteristics* of the surface.

The study of the characteristics of implicit differential equations (in a neighborhood of a singular point) was one of the four problems, proposed by the Swedish king Oskar II in 1884 as a prize problem. The winning paper was the memoir of H. Poincaré on the 3-body problem (which was one of the other problems on the king's list). The implicit equation problem was solved only a hundred years later, when A. A. Davydov gave the list of normal forms

$$(p + kx)^2 = y$$

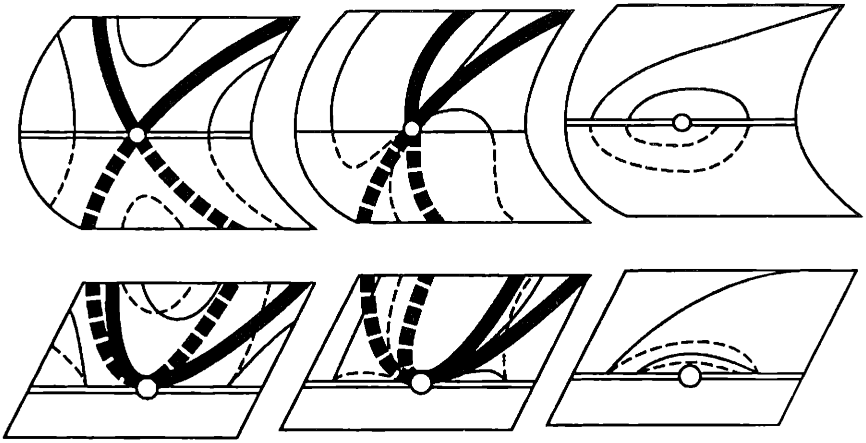


FIGURE 3. The folded saddle, node and focus.

(where k is a parameter), to which the equation may be generically reduced by a smooth (holomorphic) change of variables in a neighborhood of a critical point.

The corresponding patterns of integral curves are represented in Figure 3 (both on the surface defining the equation in the contact 3-manifold and projected to the plane (x, y) along the p-axis). They are called the *folded saddles, folded nodes* and *folded foci.* It is interesting to note, that the same patterns occur in different branches of mathematics and of physics: they describe the asymptotical lines' net on a generic 2-surface in euclidean 3-space and the characteristics' net of a hyperbolic equation at the boundary of hyperbolicity, etc. Let us consider, for instance, a linear (or nonlinear) oscillator with friction, say

$$\ddot{x} = F(x) - k\dot{x}.$$

Let us introduce the total energy $E = (\dot{x}^2/2) + U(x)$, $F = -dU/dx$. The net of the trajectories on the plane (x, E) is diffeomorphic to the net of integral curves of the standard folded singularities (represented in Figure 3) in the neighborhoods of the (generic) equilibria (maxima and minima of the potential U).

2. Contact elements spaces and the Legendre fibrations. The name "contact manifolds" comes from the following example. Let B^m be any smooth manifold. A *contact element* on B is a hyperplane (that is, a plane of dimension $m-1$) in a tangent space to B. All the contact elements on B form a manifold of dimension $2m-1$ (since the space of contact elements, centered at the same point of B^m, has dimension $m-1$, being the projective space $\mathbb{R}P^{m-1}$).

Since we have obtained an odd-dimensional manifold, we may conjecture that it carries a natural contact structure. Indeed, let us restrict the velocity

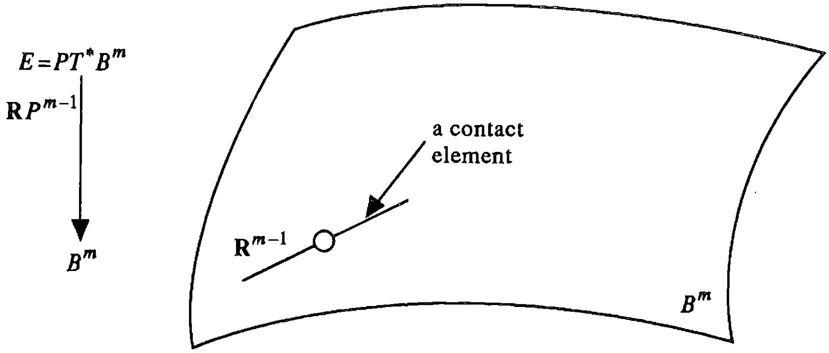


FIGURE 4. The space of contact elements.

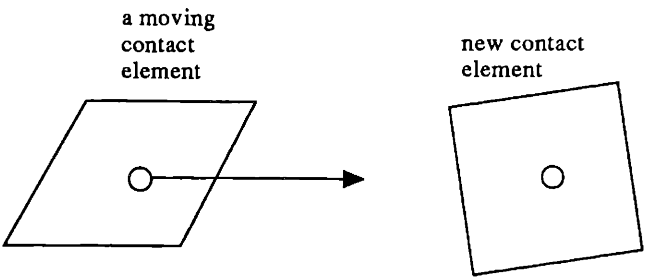


FIGURE 5. The contact structure of the contact element's space.

of a contact element (moving in the space of all the contact elements of B) by the following condition: the velocity of the point of contact should belong to the element; the rotation of the element is unrestricted.

It is a joyful exercise to prove (geometrically) that this restriction defines a genuine contact structure on the space of all contact elements of B. This space E^{2m-1} fibered over B^m, is also called *the projectivized cotangent bundle's space* $E = PT^*B$ of B. Indeed, if B is the configuration space, and the cotangent bundle space T^*B is the corresponding phase space, then E consists of the projectivizations

$$\mathbb{R}P_b^{m-1} = (T_b^*B^m \backslash 0)/(\mathbb{R}\backslash 0)$$

of the "momenta spaces" $T_b^*B^m$ at all the points of the configuration space. (See Figures 4 and 5.)

The contact elements, tangent to a submanifold C^k of B^m (of *any* given dimension k) form a Legendre submanifold L^{m-1} of E^{2m-1}.

EXAMPLE. C = point, $k = 0$. L is the fiber of the projectivized Legendre fibration. Hence these fibers are Legendrian submanifolds.

A fibration into Legendre submanifolds is called *a Legendre fibration*. Thus the projectivized cotangent bundle is a Legendre fibration.

EXAMPLE. C = hypersurface, $k = m - 1$. L is the set of tangent hyperplanes of C. Hence any smooth hyperplane many be lifted from B to a smooth Legendre submanifold of the contact space E (the lifted Legendre submanifold may have self-intersection points).

THEOREM (Darboux). *All the contact structures (on a manifold of a given dimension) are locally equal (may be transformed into each other by the diffeomorphical local changes of variables).*

COROLLARY. *Any contact structure is locally reducible to the Darboux normal form*

$$dy - p\,dx = 0$$

by a suitable choice of the local Darboux coordinates.

All the Legendre submanifolds are (locally) equal (contactomorphic), and all the Legendre fibrations are locally equal (contactomorphic in a neighborhood of any point of the ambient contact manifold).

3. The Gibbs contact structure. According to Gibbs, the geometrical structure of thermodynamics is described by a contact manifold, equipped with the contact form, whose zeroes define the laws of thermodynamics:

$$d\varepsilon = t\,d\eta - p\,dv,$$

where ε is the energy, t the temperature, η the entropy, p the pressure, and v the volume. Let us call this contact 5-manifold *the Gibbs manifold*.

GIBBS THESIS. *Substances are Legendre submanifolds of the Gibbs manifold.*

For different chemical substances these submanifolds are different, but they are always Legendre submanifolds of the universal Gibbs contact manifold.

I think Gibbs was the first to understand the crucial role of the contact structure for physics and for thermodynamics.

4. Wavefronts and Legendre singularities. Let us consider the propagation of any perturbation (light, sound, epidemia ...) in any medium. The fronts of the propagating singularities are described mathematically as the projections of the corresponding Legendre submanifolds of the space of contact elements.

Let us consider, for example, the propagation of some perturbation inside a domain bounded by a parabola in a euclidean plane.

The fronts are the equidistant curves of the initial parabola. If t is small, the equidistant curve is smooth, but for larger t it acquires singularities, namely, two cusp points of semicubical type (defined by $x^2 = y^3$ for a suitable choice of coordinates).

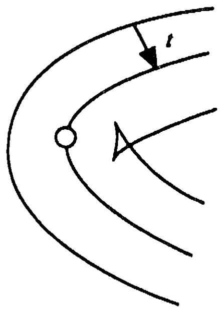


FIGURE 6. Wave fronts propagating in a plane.

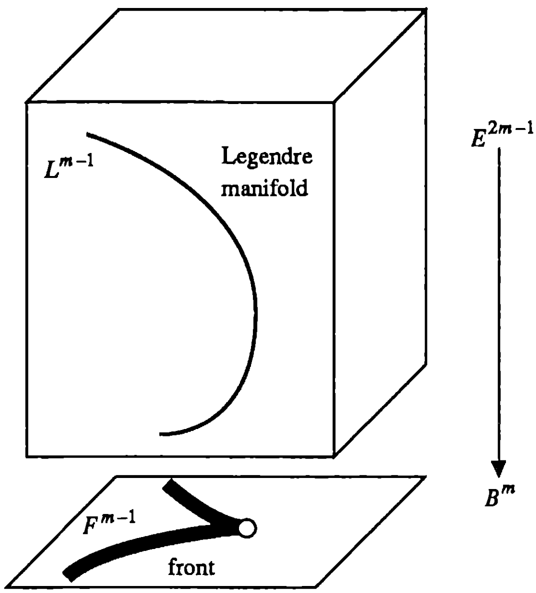


FIGURE 7. A Legendre mapping and its front.

The contact elements of all these fronts are smooth Legendre submanifolds of the space of contact elements. The fronts are the projections of those Legendre submanifolds to the base space.

The projection of a Legendre submanifold L^{m-1} of a space of Legendre fibration $\pi: E^{2m-1} \to B^m$ to the base space B is called *a Legendre mapping* and its image $\pi(L^{m-1}) = F^{m-1}$ is called *the front* of this mapping. The front of a smooth Legendre submanifold may have singularities. Generically it is a hypersurface with very special singularities. (See Figures 6 and 7.)

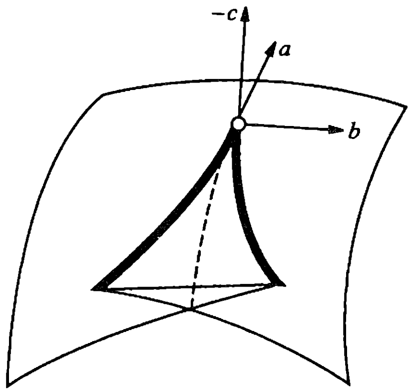


FIGURE 8. The swallowtail.

One of the ways to understand the occurrence of singularities on the wave fronts is the study to their *perestroikas.* (The Russian word perestroika was always used in mathematics to denote any qualitative change, like in the Russian "Morse perestroika" for the English "Morse surgery". In the past it was translated into English as "metamorphosis." Now the Russian word "perestroika" has become international and I see no reason not to use it here.)

Let us describe the propagation of waves in the plane with the help of a 3-dimensional space-time. Each momentary wave front, lifted to the corresponding isochrone plane $t = \text{const}$, will belong to its own horizontal plane isochrone. Together these curves form a surface in the 3-dimensional space-time. The perestroika of the wave fronts, shown in Figure 6, will be described by a surface shown in Figure 8. This surface was studied by Kroneker and is called *the swallowtail* surface. It has hundreds of different definitions. One of the simplest definitions of the swallowtail surface is:

$$\text{swallowtail} = \left\{(a, b, c) \in \mathbb{R}^3 : x^4 + ax^2 + bx + c \text{ has a multiple root}\right\}.$$

The isochrone surfaces are represented by the planes $a = \text{const}$ (the time function on the space-time may be reduced to the form $t = \pm a + \text{const}$ by a swallowtail-preserving local diffeomorphism of the space-time). The perestroika moment corresponds to $a = 0$.

One may consider the swallowtail surface as the graph of the (multivalued) function t, whose values represent the time moments, when the front passes through a given point of the physical space (of the plane in our example). The graph of the multivalued time function is a front of a Legendre singularity of higher dimension; the corresponding Legendre submanifold lives in the space of contact elements of the space-time.

5. Legendre transformation and projective duality. The name "Legendre submanifold" comes from the theory of the Legendre transformation, which is a special case of the Legendre singularities construction.

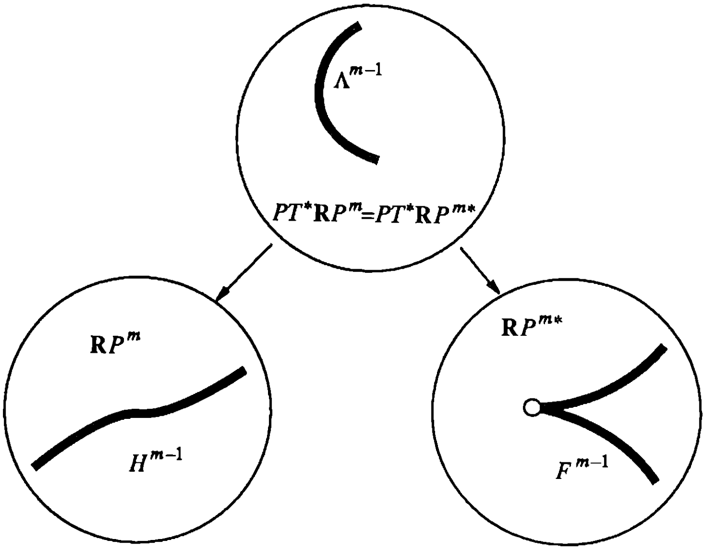


FIGURE 9. The Legendre transformation.

Let P^n be a projective space, and let P^{n*} be the dual projective space (whose points are hyperplanes of P^n). The space of contact elements of P^{n*} is not different from the space of contact elements of P^n: both consist of the pairs (a point of P^n, a hyperplane of P^n, containing this point). Hence the spaces of contact elements of both projective spaces coincide: $PT^*P^n = PT^*P^{n*} = E^{2n-1}$. But these two spaces of contact elements might have different natural contact structures: that defined by the projection to P^n and that defined by the projection to P^{n*}.

An important geometrical theorem (containing essentially all the theory of projective duality, Legendre transformation, Young duality, and so on) reads:

THEOREM. *The natural contact structures of the space of contact elements of a projective space and of the same space, considered as the space of contact elements of the dual space, coincide.*

(Since this fact is geometrical, it has a simple (calculation free) geometrical proof, which I leave for the freshmen who would wish to understand the contact geometry.)

Hence the space E of contact elements of a projective space has *two* natural Legendre fibrations: $E \to P^n$ and $E \to P^{n*}$.

Every smooth hypersurface H^{n-1} of our projective space P^n may be lifted to a smooth Legendre submanifold of E^{2n-1} and then projected to its front F^{n-1} in P^{n*} (Figure 9).

The transformation of H^{n-1} into F^{n-1} is called *the projective duality*, and written in the affine coordinates, it is called *the Legendre transformation* (of the functions, whose graphs are H^{n-1} and F^{n-1}).

Hence the graph of a Legendre transformation of a nonconvex function may have singularities. They are the Legendre singularities. To generic functions there correspond generic Legendre submanifolds and singularities. Hence:

THEOREM. *The singularities of the (generic) wave fronts and of the (generic) graphs of the multivalued time function are diffeomorphic to the singularities of the Legendre transformations of the generic functions (and the same is true for the perestroikas in the generic families).*

6. The classification of the Legendre singularities. The list of the generic Legendre singularities starts with the set of discriminants of the following simple Lie algebras:

$$\begin{tikzcd}
A_1 & A_2 \arrow[l] & A_3 \arrow[l] & A_4 \arrow[l] & A_5 \arrow[l] & A_6 \arrow[l] & A_7 \arrow[l] & A_8 \arrow[l] & \cdots \arrow[l] \\
 & & & D_4 \arrow[ul] & D_5 \arrow[l] \arrow[ul] & D_6 \arrow[l] \arrow[ul] & D_7 \arrow[l] \arrow[ul] & D_8 \arrow[l] \arrow[ul] & \\
 & & & & & E_6 \arrow[ul] \arrow[uul] & E_7 \arrow[l] \arrow[ul] \arrow[uul] & E_8 \arrow[l] \arrow[ul] \arrow[uul] & \\
 & & & & & & (P_8) \arrow[ul] & (X_9) \arrow[ul] & (J_{10}) \arrow[ul]
\end{tikzcd}$$

For instance, A_2 represents the cusp point on the wave front, and A_3 represents the swallowtail (Figure 8).

The arrow $A_3 \to A_2$ signifies the adjacency of the swallowtail singularity to the cusp singularity, represented in Figure 10 (see p. 172) by the cusped edge of the swallowtail. In general, $X \to Y$ means that the Legendre singularity X may be transformed into Y by an arbitrary small variation of the Legendre manifold.

The fronts, corresponding to these singularities, are defined as the *bifurcation diagrams*: $\Sigma = \{\lambda \colon V_\lambda$ is singular, where $V_\lambda = \{(x, y) \in \mathbb{C}^2 \colon f = 0\}$ *of the corresponding families of functions* $f(x, y)$, depending on the parameters $\lambda = (\lambda_1, \ldots, \lambda_\mu) \in \mathbb{C}^\mu$:

$$A_\mu \colon f = x^{\mu+1} + y^2 + \lambda_1 x^{\mu-1} + \cdots + \lambda_\mu,$$
$$D_\mu \colon f = x^2 y + y^{\mu-1} + \lambda_1 y^{\mu-2} + \cdots + \lambda_{\mu-1} + \lambda_\mu x,$$
$$E_6 \colon f = x^3 + y^4 + \lambda_1 y^2 + \lambda_2 y + \lambda_3 + \lambda_4 x y^2 + \lambda_5 x y + \lambda_6 x,$$
$$E_7 \colon f = x^3 + x y^3 + \lambda_1 y^4 + \cdots + \lambda_5 + \lambda_6 x + \lambda_7 x y,$$
$$E_8 \colon f = x^3 + y^5 + \lambda_1 y^3 + \cdots + \lambda_4 + \lambda_5 x y^3 + \lambda_6 x y^2 + \lambda_7 x y + \lambda_8 x.$$

The fronts of the generic Legendre manifolds of dimension $n \le 6$ have no other local singularities $(\mu \le n)$. They are *stable* and *simple* in the sense that each of them is adjacent to only a finite set of nonequivalent singularities (simplicity) and persists under any small variation of the Legendre submanifold or of the Legendre projection (stability).

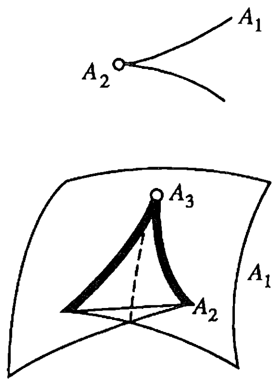


FIGURE 10. The adjacency of the simplest singularities.

Our list contains all the simple stable Legendre singularities (up to a trivial addition to f of the nondegenerate quadratic forms of new variables, which only multiply the front by a linear space).

Some of the (generic) Legendre singularities in higher dimension are neither simple nor stable; they have moduli (continuous parameters). All such singularities belong to the three classes P_8, X_9, J_{10}. Their topological classification is finite, but unknown. For instance, the simplest of the P_8 singularities $x^3 + y^3 + z^3 + 3axyz$ contains a module a. Some of the values of a are exceptional, but this finite set of exceptional elliptic curves still is not completely known.

The notations in our list of simple singularities are those of the theory of simple Lie algebras or of the linear Coxeter groups. Thus,

$$A_\mu \sim SU(\mu + 1), \quad D_\mu \sim O(2\mu).$$

To explain the relation of the Legendre singularities with the finite reflection groups let us start with a finite set of hyperplanes (mirrors) containing the origin of a euclidean space. For instance, let us consider the action of the group S_3 of permutations of the three coordinates in a 3-dimensional space with coordinates (Z_0, Z_1, Z_2). We shall think of complex coordinates Z_j, hence our space is $\mathbb{C}^3$. The group S_3 (consisting of all six permutations) acts orthogonally on $\mathbb{C}^3$, leaving invariant the diagonal $Z_0 = Z_1 = Z_2$ and hence its orthocomplement, the plane $\mathbb{C}^2 = \{Z : Z_0 + Z_1 + Z_2 = 0\}$. The action of the group on this 2-plane is irreducible. It is generated by two reflections (in any two of the three mirrors $Z_i = Z_j$). An orbit of our group consists generically of six different points (Figure 11). Such orbits are called *regular*. The *irregular orbits*, consisting of fewer points, are the images of the mirrors under the natural projection (ordered triples) $\longrightarrow$ (unordered triples) of roots of polynomials

$$Z^3 + \lambda_1 Z + \lambda_2.$$

(See Figure 11.)

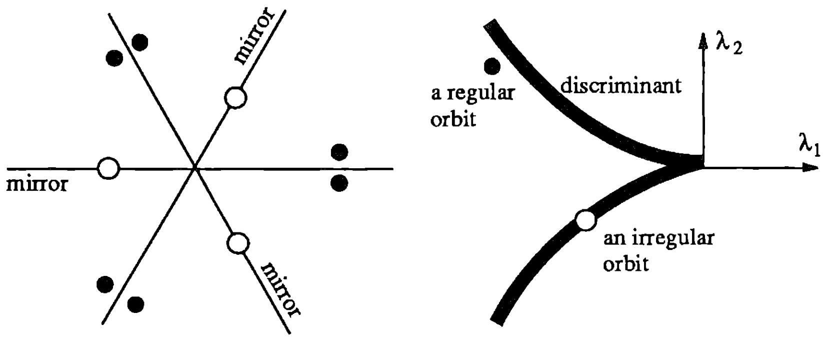
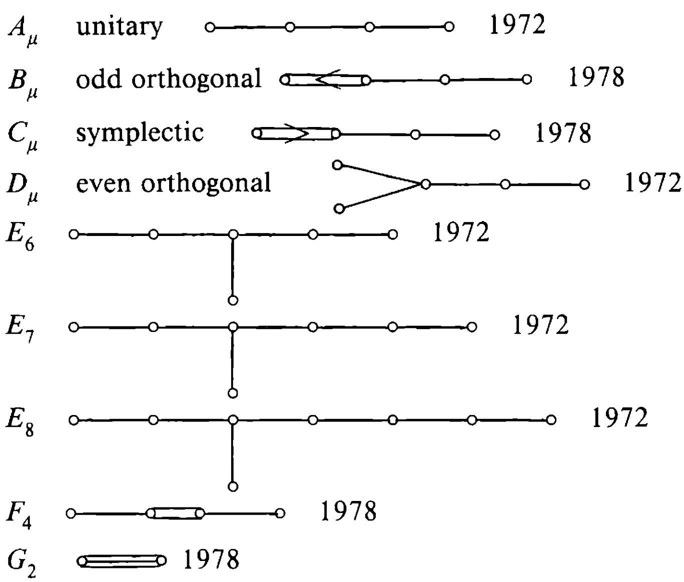


FIGURE 11. Regular and irregular orbits and the discriminant of group A_2.

This plane with coordinates (λ_1, λ_2) is the manifold of orbits of our reflection group. The variety of irregular orbits is a hypersurface (in our case a curve with a semicubical cusp). This hypersurface of irregular orbits in the manifold of orbits is called the *discriminant* of the corresponding reflection group (Figure 11).

All the fronts of simple stable Legendre singularities are the discriminant varieties of the corresponding irreducible finite euclidean reflections groups (up to diffeomorphic changes of variables). This fact is crucial in the theory of Legendre singularities; it allows one to use for the study of the fronts and of their perestroikas the heavy artillery of Lie group theory, invariant theory, algebraic geometry, and so on.

7. Legendre singularities and the icosahedron. The Coxeter list of finite euclidean reflection groups contains the Weyl groups of the simple Lie algebras:



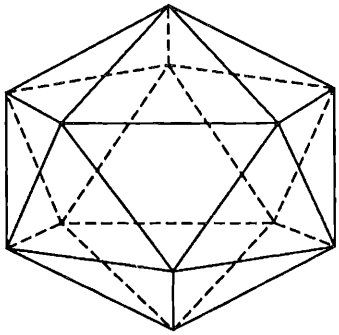


FIGURE 12. The icosahedron.

Each dot signifies here a basic vector, orthogonal to the mirror. The angles between the vectors, connected by one line, are $120°$, by two— $135°$, by three— $150°$, disconnected— $90°$.

The Weyl groups preserve "crystallographical" lattices of full dimension (μ in $\mathbb{C}^{\mu}$). They are characterized by this property. The mirrors and groups B_μ and C_μ coincide, but the corresponding lattices differ.

The noncrystallographical Coxeter groups contain one more infinite series:

$$I_2(p) = \text{ symmetry group of a regular plane } p\text{-gone}$$

and two exceptional groups H_3, H_4($I_2(3) = A_2$, $I_2(4) = B_2$, $I_2(5) = H_2$, $I_2(6) = G_2$).

H_3 is the symmetry group of a regular icosahedron in the 3-space (Figure 12).

It is generated by the reflection in three of its 15 planes of symmetry. The discriminant of this group defines an interesting Legendre singularity: it describes the propagation of rays on a 2-manifold with boundary.

The fronts of the perturbation, propagating in a manifold with boundary, are, in the 2-dimensional case, the Huygens *involutes* of the boundary curve (Figure 13).

It was discovered in 1982 that the perestroika of the family of the wave fronts at a generic point of inflection of the obstacle curve is governed by a small invisible icosahedron. O. V. Lyashko, using a computer, has drawn the discriminant hypersurface of H_3 (Figure 14).

The comparison with the pattern of the perestroika of the system of involutes of a plane curve at a generic inflection point has led A. B. Givental to the conjecture that the multivalued time-function graph is diffeomorphic to the discriminant hypersurface of H_3. This conjecture was then proved by O. P. Shcherbak. The patterns of the involutes of a cubical parabola were later discovered in the first textbook on calculus, published by L'Hôspital[2]). I doubt whether modern students in mathematics are able to investigate these

[2]This was explained in D. Bennequin's talk at the seminar Bourbaki "Caustique mistique".

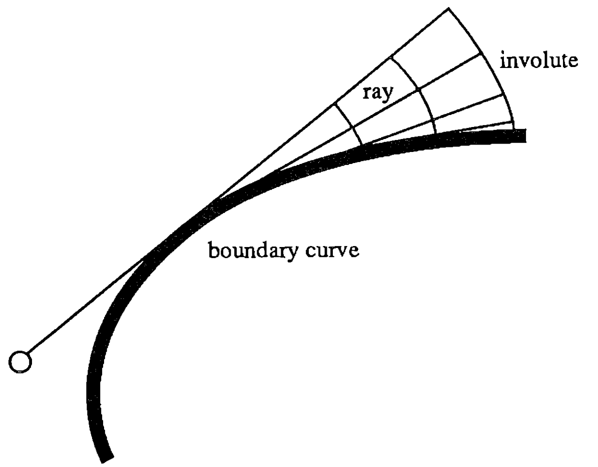


FIGURE 13. The involutes as wavefronts at an obstacle.

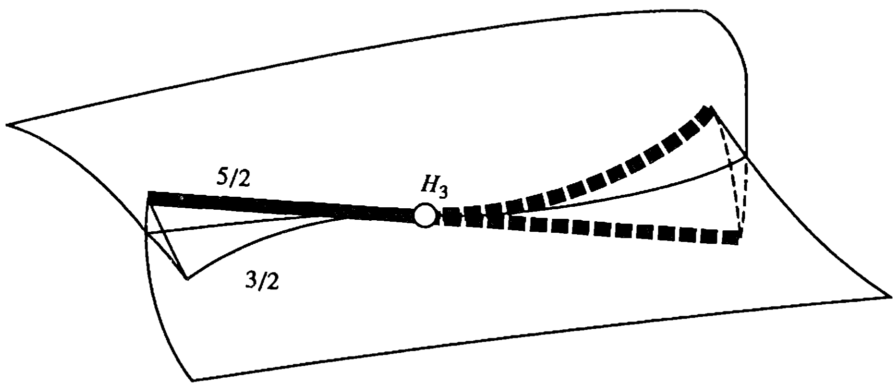


FIGURE 14. The discriminant of the group of symmetries of an icosahedron.

involutes, which have cusps of order $3/2$ on the obstacle curve and of order $5/2$ on its inflectional tangent (Figure 15, p. 176).

The last Coxeter group H_4 is related in the same way to the problem of bypassing a generic obstacle in a 3-space (the discriminant of H_3 occurs in this case as one of the typical singularities of wave fronts, the discriminant of H_4—as a singularity of a graph of the time function).

H_4 is the symmetry group of a "120-gon" in euclidean 4-space. Let us start with the rotation group of an icosahedron. This group contains sixty elements and lives in $SO(3)$. The "spin" double covering $SU(2) \to SO(3)$ maps to

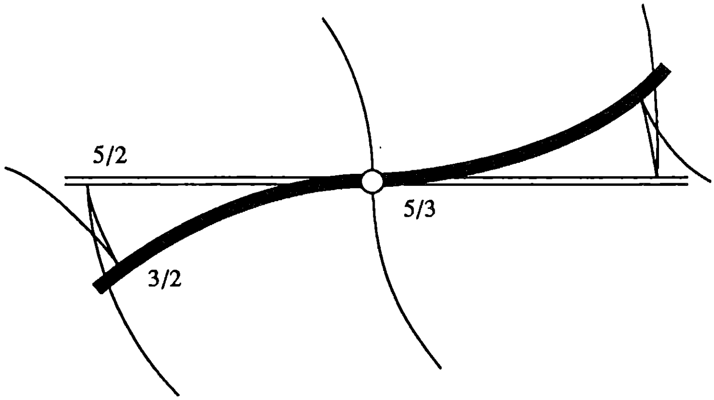


FIGURE 15. Perestroika of the involutes of a cubical parabola (accordingly to L'Hôspital).

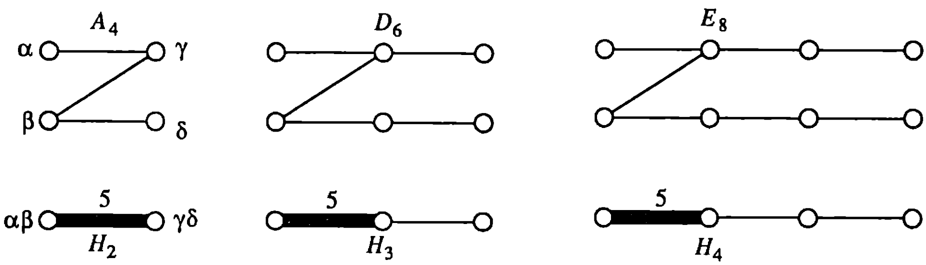


FIGURE 16. The products of commuting reflections in each vertical generate the (reducible) representations of the H_i groups and hence the "quasicrystallic" irreducible representation.

this rotation group a group of order 120, called the *binary icosahedron group.* The 120 elements of this group lie on $SU(2) = S^3$ in $\mathbb{R}^4$. The convex hull of these 120 vertices is the desired regular polyhedron in $\mathbb{R}^4$ (it is formed by 600 tetrahedral faces).

The groups H_2, H_3 and H_4 are closely related to the groups A_4, D_6, E_8 (Figure 16). For more details see the article of O. Shcherbak, *Russian Math. Surveys,* (3) 1988, *The Encyclopaedia of Mathematical Sciences,* vol. 4, Springer-Verlag, and the Givental thesis, published in *Itogi Nauki, Sovremennye problemy mathematiki, Noveishie dostijenia,* vol. 33, 1988 (translated in the Journal of Soviet Mathematics).

8. Wave transformations. The contact geometry equips us with the general methods of investigation of the wave propagation. Waves of different kinds (say, the longitudinal one and the oblique ones in the elasticity theory) travel generically independently of each other. However in nonhomogeneous media the interior transformation of the waves of one kind into the waves of another kind at some interior points of space-time becomes possible.

The simplest example of this phenomenon (related to the nonstrict hyperbolicity or to singularities of dispersion relations, or to the crossing of energy levels) is the conical refraction, predicted by Hamilton essentially using the contact geometry method. Hamilton considered the homogeneous media case. It happens, that the generic case leads to a different (and complicated) topology of rays and wave fronts, which I shall briefly describe here.

It is well known, that the generic Hamilton ODES are locally "rectifiable" (at a neighborhood of a noncritical point) and present in this sense no difference from the generic, dissipative ODE. However the global behavior of the Hamilton equations and of the generic equations is enormous and this global difference is manifested in a neighborhood of a critical point.

The situation is similar for the general hyperbolical PDE and for those defined by variational principles. Microlocally, in a neighborhood of a ray of strict hyperbolicity, any hyperbolical equation may be symmetrized and so written in a variational form (at least at the level of the highest order terms).

In a neighborhood of a multiple root of the characteristic equation the situation is quite different. We consider the hyperbolical systems defined by the variational principles. The zeroes of the principal symbol define a "light hypersurface" in the space of contact elements of the space-time, $E = PT^*(V^d + \mathbb{R})$. The generic singularities of this light hypersurface (defined by a variational principle) will be the conical point, where the hypersurface may be defined by an equation $x^2 + y^2 = z^2$, where $(x, y, z \dots)$ are the coordinates in E^{2d+1}. In the case of just one space variable $(d = 1)$ this hypersurface is a genuine quadratic cone, but for higher d the hypersurface's singularities form a set of codimension 2 in this hypersurface.

The contact geometry leads to the

THEOREM. *The hypersurface $x^2+y^2 = z^2$, generic with respect to a contact structure of the ambient space, may be reduced to the normal form $H = 0$, where*

$$\begin{aligned} H &= p_1^2 \pm q_1^2 - w^2 + cw^3 \quad \text{if } d = 1, \\ &= p_1^2 \pm q_1^2 - p_2^2 \quad \text{if } d > 1 \end{aligned}$$

by a formal diffeomorphism, reducing the contact structure to the Darboux type form

$$\alpha = dw + \frac{pdq - qdp}{2}$$

(The reduction may perhaps be performed by a C^∞-diffeomorphism, but the series, reducing the pairs (H, α) to the normal form in the analytical case are generically divergent).

To explain the physical meaning of these results (which it is possible and should be useful to confirm experimentally), let us consider the simplest case $d = 1$.

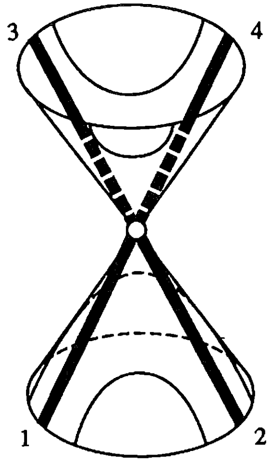


FIGURE 17. The light surface and its characteristics in the space of contact elements of the space-time.

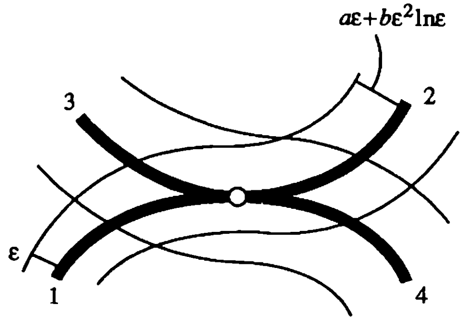


FIGURE 18. The scattering of a family of ray by a ray of a different kind.

The light surface in the space-time contact elements' space is represented in Figure 17.

The characteristic directions (corresponding to the rays) look like the intersections with the parallel planes (the sign $+$ in H contradicts the casuality principle and is impossible for globally hyperbolical systems).

The projections of these characteristics to the space-time plane are shown in Figure 18. They form two families of the characteristics transversal everywhere but at the origin. The plane curves 14 and 23 are smooth (analytic)

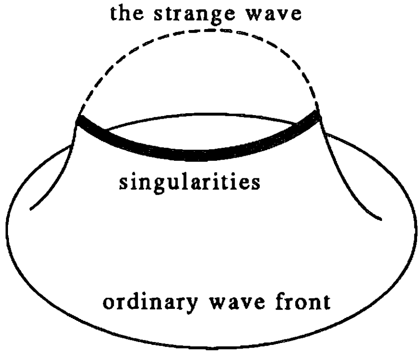


FIGURE 19. Two points of nonstrict hyperbolicity on a wave in 3-space.

and tangent to each other quadratically at the origin. But the characteristics are the curves 12 and 34, they are singular.

The neighboring characteristics experience some scattering at the origin. Namely, a ray starting at distance ε of the ray 12 at the point 1 will at point 2 be at a distance $a\varepsilon + b\varepsilon^2 \ln \varepsilon + \dots$ from the ray 2. This logarithm represents the effect of scattering of the rays of the first family by the collision with the ray of the second family at the origin.

When the dimension d of the physical space is higher than 1, the situation becomes even more complicated. For instance, in the case $d = 3$ a generic wave front contains isolated points of nonstrict hyperbolicity, connected by the logarithmically singular lines (Figure 19).

For more details on the wave transformation see *Mathematicheskie Zametki* (Notes in Math.), vol. 44, no. 1, 1988 and *Journal of Geometry and Physics*, vol. 5, no. 4, 1988, dedicated to I. M. Gelfand.

9. Singularities and the conformal fields theories. The manifestation of the scaling exponents of the A-D-E singularities in the modern (1987–89 ...) conformal superstring theory (Martinez, Wafa, Suzuki ...) suggest its direct relations to the Legendre singularities. Since I do not understand the reasons of this coincidence, I simply reproduce here the words of E. Witten, according to whom the superdimension 2.2 is large enough to use the Landau phase transition theory, which is directly related to the classification of critical points of functions (of some "order parameter"). I hope that the coming years will introduce more order into this domain of superstring phase transitions.

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