Electrodynamics
Third Exam
2018-I

Iván Mauricio Burbano Aldana

May 3, 2018

(i) As we have seen in class the potential A^μ is given by

$$A^\mu(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\,\frac{j^\mu(\mathbf{r}',t-\|\mathbf{r}-\mathbf{r}'\|/c)}{\|\mathbf{r}-\mathbf{r}'\|} \tag{1}$$

ignoring all background waves, that is, solutions of $\Box A^\mu=0$. Therefore, its temporal Fourier transform, if we assume exists, is given by

$$\begin{aligned}A^\mu(\mathbf{r},\omega)&=\int \mathrm{d}t\,e^{i\omega t}A^\mu(\mathbf{r},t)\\&=\frac{\mu_0}{4\pi}\int \mathrm{d}t\,e^{i\omega t}\int \mathrm{d}^3\mathbf{r}'\,\frac{j^\mu(\mathbf{r}',t-\|\mathbf{r}-\mathbf{r}'\|/c)}{\|\mathbf{r}-\mathbf{r}'\|}\\&=\frac{\mu_0}{4\pi}\int \mathrm{d}t\int \mathrm{d}^3\mathbf{r}'\,\frac{1}{\|\mathbf{r}-\mathbf{r}'\|}e^{i\omega t}j^\mu(\mathbf{r}',t-\|\mathbf{r}-\mathbf{r}'\|/c).\end{aligned} \tag{2}$$

Assuming the proper conditions are met to apply Fubini's theorem we have

$$\begin{aligned}A^\mu(\mathbf{r},\omega)&=\frac{\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\int \mathrm{d}t\,\frac{1}{\|\mathbf{r}-\mathbf{r}'\|}e^{i\omega t}j^\mu(\mathbf{r}',t-\|\mathbf{r}-\mathbf{r}'\|/c)\\&=\frac{\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\,\frac{1}{\|\mathbf{r}-\mathbf{r}'\|}\int \mathrm{d}t\,e^{i\omega t}j^\mu(\mathbf{r}',t-\|\mathbf{r}-\mathbf{r}'\|/c)\end{aligned} \tag{3}$$

Through the change of variables

$$\begin{aligned}&\mathbb{R}\to\mathbb{R}\\&t\mapsto t+\|\mathbf{r}-\mathbf{r}'\|/c\end{aligned} \tag{4}$$

which keeps the domain $\mathbb{R}$ invariant we obtain

$$\begin{aligned}A^\mu(\mathbf{r},\omega)&=\frac{\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\,\frac{1}{\|\mathbf{r}-\mathbf{r}'\|}\int \mathrm{d}t\,e^{i\omega(t+\|\mathbf{r}-\mathbf{r}'\|/c)}j^\mu(\mathbf{r}',t)\\&=\frac{\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\,\frac{1}{\|\mathbf{r}-\mathbf{r}'\|}\int \mathrm{d}t\,e^{i\omega\|\mathbf{r}-\mathbf{r}'\|/c}e^{i\omega t}j^\mu(\mathbf{r}',t)\\&=\frac{\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\,\frac{e^{iK\|\mathbf{r}-\mathbf{r}'\|}}{\|\mathbf{r}-\mathbf{r}'\|}\int \mathrm{d}t\,e^{i\omega t}j^\mu(\mathbf{r}',t).\end{aligned} \tag{5}$$

We now recognize the Fourier transform

$$j^\mu(\mathbf{r},\omega) = \int \mathrm{d}t\, e^{i\omega t} j^\mu(\mathbf{r},t) \tag{6}$$

concluding

$$A^\mu(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\, \frac{e^{iK\|\mathbf{r}-\mathbf{r}'\|}}{\|\mathbf{r}-\mathbf{r}'\|} j^\mu(\mathbf{r},\omega). \tag{7}$$

Separating this equation by remembering that $A^\mu = (\phi/c, \mathbf{A})$, $j^\mu = (c\rho, \mathbf{J})$, and $c^2\mu_0 = \frac{\mu_0}{\mu_0\epsilon_0} = \frac{1}{\epsilon_0}$ we obtain

$$\begin{aligned}
\phi(\mathbf{r},\omega) =& \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3\mathbf{r}'\, \frac{e^{iK\|\mathbf{r}-\mathbf{r}'\|}}{\|\mathbf{r}-\mathbf{r}'\|}\rho(\mathbf{r},\omega),\\
\mathbf{A}(\mathbf{r},\omega) =& \frac{\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\, \frac{e^{iK\|\mathbf{r}-\mathbf{r}'\|}}{\|\mathbf{r}-\mathbf{r}'\|}\mathbf{J}(\mathbf{r},\omega).
\end{aligned} \tag{8}$$

(ii) Recall that the law of charge current conservation is given by

$$\partial_\mu j^\mu(\mathbf{r},t) = 0. \tag{9}$$

Assuming that the Fourier transformation (6) is invertible, we must have

$$j^\mu(\mathbf{r},t) = \frac{1}{2\pi}\int \mathrm{d}\omega\, e^{-i\omega t} j^\mu(\mathbf{r},\omega) \tag{10}$$

and

$$0 = \frac{1}{2\pi}\partial_\mu \int \mathrm{d}\omega\, e^{-i\omega t} j^\mu(\mathbf{r},\omega). \tag{11}$$

Assuming that the right conditions are met to interchange differentiation with integration we have

$$\begin{aligned}
0 &= \int \mathrm{d}\omega\, \partial_\mu\big(e^{-i\omega t} j^\mu(\mathbf{r},\omega)\big)\\
&= \int \mathrm{d}\omega\, \left(\frac{1}{c}\frac{\partial e^{-i\omega t}}{\partial t} c\rho(\mathbf{r},\omega) + e^{-i\omega t}\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},\omega)\right)\\
&= \int \mathrm{d}\omega\, \big(-i\omega e^{-i\omega t}\rho(\mathbf{r},\omega) + e^{-i\omega t}\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},\omega)\big)\\
&= \int \mathrm{d}\omega\, e^{-i\omega t}(-i\omega\rho(\mathbf{r},\omega) + \boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},\omega)).
\end{aligned} \tag{12}$$

Under the right conditions the uniqueness theorem is valid and a function is null if and only if is Fourier transform is. We thus conclude that the law of charge current conservation can be expressed as

$$\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},\omega) = i\omega\rho(\mathbf{r},\omega). \tag{13}$$

(iii) We recall the definition of the electric dipole moment and magnetic dipole moment

$$\begin{aligned}\mathbf{p}(\mathbf{r}) &= \int \mathrm{d}^3\mathbf{r}'\,(\mathbf{r}'-\mathbf{r})\rho(\mathbf{r}'),\\ \mathbf{m}(\mathbf{r}) &= \int \mathrm{d}^3\mathbf{r}'\,(\mathbf{r}'-\mathbf{r})\times\mathbf{J}(\mathbf{r}').\end{aligned} \tag{14}$$

By using the product rule of the divergence

$$\begin{aligned}\int \mathrm{d}^3\mathbf{r}\,\mathbf{J}(\mathbf{r},\omega) &= \sum_{i=1}^{3}\hat{\mathbf{e}}_i\int \mathrm{d}^3\mathbf{r}\,\mathbf{J}(\mathbf{r},\omega)\cdot\hat{\mathbf{e}}_i\\ &= \sum_{i=1}^{3}\hat{\mathbf{e}}_i\int \mathrm{d}^3\mathbf{r}\,\mathbf{J}(\mathbf{r},\omega)\cdot\boldsymbol{\nabla}x^i\\ &= \sum_{i=1}^{3}\hat{\mathbf{e}}_i\int \mathrm{d}^3\mathbf{r}\,\left(\boldsymbol{\nabla}\cdot\big(x^i\mathbf{J}\big)(\mathbf{r},\omega)-x^i\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},\omega)\right)\\ &= \sum_{i=1}^{3}\hat{\mathbf{e}}_i\left(\int \mathrm{d}^3\mathbf{r}\,\boldsymbol{\nabla}\cdot\big(x^i\mathbf{J}\big)(\mathbf{r},\omega)-\int \mathrm{d}^3\mathbf{r}\,x^i\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},\omega)\right).\end{aligned} \tag{15}$$

The first integral vanishes since it is a total differential on a region without a boundary, namely $\mathbb{R}^3$. Therefore, by using (13) we obtain

$$\begin{aligned}\int \mathrm{d}^3\mathbf{r}\,\mathbf{J}(\mathbf{r},\omega) &= -\sum_{i=1}^{3}\hat{\mathbf{e}}_i\int \mathrm{d}^3\mathbf{r}\,x^i\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},\omega)\\ &= -\int \mathrm{d}^3\mathbf{r}\sum_{i=1}^{3}\hat{\mathbf{e}}_i x^i i\omega\rho(\mathbf{r},\omega) = -\int \mathrm{d}^3\mathbf{r}\,\mathbf{r}i\omega\rho(\mathbf{r},\omega)\\ &= -\,i\omega\mathbf{p}(0)\end{aligned} \tag{16}$$