Mecánica Estadística
Tarea 2: Mecánica Estadística Cuántica

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Límite clásico y efectos de intercambio

1. Por definición del operador de momento tenemos la ecuación diferencial

$$-i\hbar\boldsymbol{\nabla}\langle\mathbf{r}|\mathbf{p}\rangle=\langle\mathbf{r}|\hat{\mathbf{p}}|\mathbf{p}\rangle=\mathbf{p}\langle\mathbf{r}|\mathbf{p}\rangle. \tag{1}$$

Entonces a los largo de un camino γ que empieza en $\mathbf{r}_1$ y termina en $\mathbf{r}_2$ se tiene

$$\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{r}_2-\mathbf{r}_1)=\int_\gamma\frac{1}{\langle\mathbf{r}|\mathbf{p}\rangle}\boldsymbol{\nabla}\langle\mathbf{r}|\mathbf{p}\rangle\cdot d\mathbf{r}=\int_\gamma\boldsymbol{\nabla}(\ln\langle\mathbf{r}|\mathbf{p}\rangle)\cdot d\mathbf{r}=\ln\left(\frac{\langle\mathbf{r}_2|\mathbf{p}\rangle}{\langle\mathbf{r}_1|\mathbf{p}\rangle}\right). \tag{2}$$

Se concluye que existe una constante $A\in\mathbb{C}$ tal que

$$\langle\mathbf{r}|\mathbf{p}\rangle=Ae^{i\mathbf{p}\cdot\mathbf{r}/\hbar} \tag{3}$$

Nuestra convención de normalización exige que para todo $\mathbf{p}\in\mathbb{R}^3$

$$|\mathbf{p}\rangle=\int\frac{d^3\mathbf{p}'}{h}|\mathbf{p}'\rangle\langle\mathbf{p}'|\mathbf{p}\rangle. \tag{4}$$

Por lo tanto $\langle\mathbf{p}'|\mathbf{p}\rangle=h\delta(\mathbf{p}'-\mathbf{p})$. Se concluye que

$$\begin{aligned}1&=\int\frac{d^3\mathbf{p}'}{h}\langle\mathbf{p}'|\mathbf{p}\rangle=\int\frac{d^3\mathbf{p}'}{h}\langle\mathbf{p}'|\int d^3\mathbf{r}\,|\mathbf{r}\rangle\langle\mathbf{r}|\mathbf{p}\rangle=\int d^3\mathbf{r}\frac{d^3\mathbf{p}'}{h}\langle\mathbf{p}'|\mathbf{r}\rangle\langle\mathbf{r}|\mathbf{p}\rangle\\&=\int d^3\mathbf{r}\frac{d^3\mathbf{p}'}{h}|A|^2e^{-i\mathbf{p}'\cdot\mathbf{r}/\hbar}e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}=\int d^3\mathbf{p}'\frac{d^3\mathbf{r}}{h}|A|^2e^{2\pi i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}/h}\\&=2\pi|A|^2\int d^3\mathbf{p}'\delta(2\pi(\mathbf{p}-\mathbf{p}'))=|A|^2.\end{aligned} \tag{5}$$

Por lo tanto podemos escoger $A=1$ y tenemos

$$\langle\mathbf{r}|\mathbf{p}\rangle=e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}. \tag{6}$$

2. Tomando la traza en el espacio simetrizado o antisimetrizado tenemos

$$
\begin{aligned}
Z &= \frac{1}{N!}\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N{}^{S,A}\left\langle \mathbf{r}_1,\ldots,\mathbf{r}_N\right| e^{-\beta H}\left|\mathbf{r}_1,\ldots,\mathbf{r}_N\right\rangle^{S,A} \\
&= \frac{1}{N!^2}\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N\sum_{\sigma\in S_N}\sum_{\sigma'\in S_N}\epsilon(\sigma)\epsilon(\sigma') \\
&\left\langle \mathbf{r}_{\sigma(1)},\ldots,\mathbf{r}_{\sigma(N)}\middle|e^{-\beta H}\middle|\mathbf{r}_{\sigma'(1)},\ldots,\mathbf{r}_{\sigma'(N)}\right\rangle .
\end{aligned} \tag{7}
$$

Recordando que en un grupo la multiplicación por un elemento es una biyección podemos cambiar la suma por

$$
\begin{aligned}
Z &= \frac{1}{N!^2}\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N\sum_{\sigma\in S_N}\sum_{\sigma'\in S_N}\epsilon(\sigma\circ\sigma'^{-1})\epsilon(\sigma'\circ\sigma'^{-1}) \\
&\left\langle \mathbf{r}_{\sigma\circ\sigma'^{-1}(1)},\ldots,\mathbf{r}_{\sigma\circ\sigma'^{-1}(N)}\middle|e^{-\beta H}\middle|\mathbf{r}_{\sigma'\circ\sigma'^{-1}(1)},\ldots,\mathbf{r}_{\sigma'\circ\sigma'^{-1}(N)}\right\rangle \\
&= \frac{1}{N!^2}\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N\sum_{\sigma\in S_N}\sum_{\sigma'\in S_N}\epsilon(\sigma) \\
&\left\langle \mathbf{r}_{\sigma(1)},\ldots,\mathbf{r}_{\sigma(N)}\middle|e^{-\beta H}\middle|\mathbf{r}_1,\ldots,\mathbf{r}_N\right\rangle \\
&= \frac{N!}{N!^2}\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N\sum_{\sigma\in S_N}\epsilon(\sigma)\left\langle \mathbf{r}_{\sigma(1)},\ldots,\mathbf{r}_{\sigma(N)}\middle|e^{-\beta H}\middle|\mathbf{r}_1,\ldots,\mathbf{r}_N\right\rangle \\
&= \frac{1}{N!}\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N\sum_{\sigma\in S_N}\epsilon(\sigma)\left\langle \mathbf{r}_{\sigma(1)},\ldots,\mathbf{r}_{\sigma(N)}\middle|e^{-\beta H}\middle|\mathbf{r}_1,\ldots,\mathbf{r}_N\right\rangle .
\end{aligned} \tag{8}
$$

3. Podemos aplicar la fórmula de Baker-Campbell-Hausdorff de manera truncada pues los terminos superiores dependen de potencias del operador de momento y por lo tanto de $\hbar$. En efecto

$$
\begin{aligned}
&\exp\left(-\beta\sum_{j=1}^{N}\frac{\hat{\mathbf{p}}_j^2}{2m}\right)\exp(-\beta V(\hat{\mathbf{r}}_1,\ldots,\hat{\mathbf{r}}_N)) \\
&= \exp\Bigg(-\beta\sum_{j=1}^{N}\frac{\hat{\mathbf{p}}_j^2}{2m} - \beta V(\hat{\mathbf{r}}_1,\ldots,\hat{\mathbf{r}}_N) \\
&+\frac{1}{2}\left[-\beta\sum_{j=1}^{N}\frac{\hat{\mathbf{p}}_j^2}{2m}, -\beta V(\hat{\mathbf{r}}_1,\ldots,\hat{\mathbf{r}}_N)\right]+\cdots\Bigg) \\
&= \exp\Bigg(-\beta\left(\sum_{j=1}^{N}\frac{\hat{\mathbf{p}}_j^2}{2m} + V(\hat{\mathbf{r}}_1,\ldots,\hat{\mathbf{r}}_N)\right) \\
&+\frac{\beta^2\hbar^2}{4m}\sum_{j=1}^{N}\left[\Delta_j^2, V(\hat{\mathbf{r}}_1,\ldots,\hat{\mathbf{r}}_N)\right]+\mathcal{O}(\hbar^4)\Bigg) \\
&= \exp\Big(-\beta\hat{H}+\mathcal{O}(\hbar^2)\Big)\cong e^{-\beta\hat{H}}
\end{aligned} \tag{9}
$$

en el límite clásico, es decir, $\hbar \to 0$.

4. Tenemos en la aproximación clásica

$$Z = \frac{1}{N!}\int d^3\mathbf{r}_1 \cdots d^3\mathbf{r}_N \sum_{\sigma\in S_N} \epsilon(\sigma)$$

$$\left\langle \mathbf{r}_{\sigma(1)}, \ldots, \mathbf{r}_{\sigma(N)} \middle| \exp\left(-\beta \sum_{j=1}^{N} \frac{\hat{\mathbf{p}}_j^2}{2m}\right) \exp(-\beta V(\hat{\mathbf{r}}_1, \ldots, \hat{\mathbf{r}}_N)) \middle| \mathbf{r}_1, \ldots, \mathbf{r}_N \right\rangle$$

$$= \frac{1}{N!}\int d^3\mathbf{r}_1 \cdots d^3\mathbf{r}_N \sum_{\sigma\in S_N} \epsilon(\sigma) \exp(-\beta V(\mathbf{r}_1, \ldots, \mathbf{r}_N))$$

$$\left\langle \mathbf{r}_{\sigma(1)}, \ldots, \mathbf{r}_{\sigma(N)} \middle| \exp\left(-\beta \sum_{j=1}^{N} \frac{\hat{\mathbf{p}}_j^2}{2m}\right) \middle| \mathbf{r}_1, \ldots, \mathbf{r}_N \right\rangle$$

$$= \frac{1}{N!}\int d^3\mathbf{r}_1 \cdots d^3\mathbf{r}_N \sum_{\sigma\in S_N} \epsilon(\sigma) \exp(-\beta V(\mathbf{r}_1, \ldots, \mathbf{r}_N))$$

$$\left\langle \mathbf{r}_{\sigma(1)}, \ldots, \mathbf{r}_{\sigma(N)} \right| \exp\left(-\beta \sum_{j=1}^{N} \frac{\hat{\mathbf{p}}_j^2}{2m}\right)$$

$$\int \frac{d^3\mathbf{p}_1 \cdots d^3\mathbf{p}_N}{h^{3N}} \left|\mathbf{p}_1, \ldots, \mathbf{p}_N\right\rangle \left\langle \mathbf{p}_1, \ldots, \mathbf{p}_N \middle| \mathbf{r}_1, \ldots, \mathbf{r}_N \right\rangle$$

$$= \frac{1}{h^{3N}N!}\int d^3\mathbf{r}_1 \cdots d^3\mathbf{r}_N d^3\mathbf{p}_1 \cdots d^3\mathbf{p}_N \sum_{\sigma\in S_N} \epsilon(\sigma) \tag{10}$$

$$\exp(-\beta V(\mathbf{r}_1, \ldots, \mathbf{r}_N)) \exp\left(-\beta \sum_{j=1}^{N} \frac{\mathbf{p}_j^2}{2m}\right) \left\langle \mathbf{r}_{\sigma(1)}, \ldots, \mathbf{r}_{\sigma(N)} \middle| \mathbf{p}_1, \ldots, \mathbf{p}_N \right\rangle$$

$$\left\langle \mathbf{p}_1, \ldots, \mathbf{p}_N \middle| \mathbf{r}_1, \ldots, \mathbf{r}_N \right\rangle$$

$$= \frac{1}{h^{3N}N!} \sum_{\sigma\in S_N} \epsilon(\sigma) \int d^3\mathbf{r}_1 \cdots d^3\mathbf{r}_N d^3\mathbf{p}_1 \cdots d^3\mathbf{p}_N e^{-\beta H_{\mathrm{clas}}}$$

$$\left\langle \mathbf{r}_{\sigma(1)}, \ldots, \mathbf{r}_{\sigma(N)} \middle| \mathbf{p}_1, \ldots, \mathbf{p}_N \right\rangle \left\langle \mathbf{p}_1, \ldots, \mathbf{p}_N \middle| \mathbf{r}_1, \ldots, \mathbf{r}_N \right\rangle$$

$$= \frac{1}{h^{3N}N!} \sum_{\sigma\in S_N} \epsilon(\sigma) \int d^3\mathbf{r}_1 \cdots d^3\mathbf{r}_N d^3\mathbf{p}_1 \cdots d^3\mathbf{p}_N e^{-\beta H_{\mathrm{clas}}}$$

$$\prod_{n=1}^{N} e^{i\mathbf{p}_n \cdot \mathbf{r}_{\sigma(n)}/\hbar} \prod_{m=1}^{N} e^{-i\mathbf{p}_m \cdot \mathbf{r}_m/\hbar}$$

$$= \frac{1}{h^{3N}N!} \sum_{\sigma\in S_N} \epsilon(\sigma) \int d^3\mathbf{r}_1 \cdots d^3\mathbf{r}_N d^3\mathbf{p}_1 \cdots d^3\mathbf{p}_N e^{-\beta H_{\mathrm{clas}}}$$

$$\exp\left(-i \sum_{n=1}^{N} \left(\mathbf{p}_n \cdot (\mathbf{r}_m - \mathbf{r}_{\sigma(n)})\right)/\hbar\right)$$

5.

$$
\begin{aligned}
Z = & \frac{1}{h^{3N}N!}\sum_{\sigma\in S_N}\epsilon(\sigma)\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N d^3\mathbf{p}_1\cdots d^3\mathbf{p}_N e^{-\beta H_{\mathrm{clas}}} \\
& \exp\left(-i\sum_{n=1}^{N}\mathbf{p}_n\cdot\big(\mathbf{r}_n-\mathbf{r}_{\sigma(n)}\big)/\hbar\right) \\
& = \frac{1}{h^{3N}N!}\sum_{\sigma\in S_N}\epsilon(\sigma)\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N d^3\mathbf{p}_1\cdots d^3\mathbf{p}_N e^{-\beta V(\mathbf{r}_1,\ldots,\mathbf{r}_N)} \\
& \exp\left(\sum_{n=1}^{N}\left(-\beta\frac{\mathbf{p}_n^2}{2m}-i\mathbf{p}_n\cdot\big(\mathbf{r}_n-\mathbf{r}_{\sigma(n)}\big)/\hbar\right)\right) \\
& = \frac{1}{h^{3N}N!}\sum_{\sigma\in S_N}\epsilon(\sigma)\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N d^3\mathbf{p}_1\cdots d^3\mathbf{p}_N e^{-\beta V(\mathbf{r}_1,\ldots,\mathbf{r}_N)} \\
& \prod_{n=1}^{N}\prod_{j=1}^{3}\exp\left(-\beta\frac{(\mathbf{p}_n)_j^2}{2m}-i(\mathbf{p}_n)_j\big((\mathbf{r}_n)_j-(\mathbf{r}_{\sigma(n)})_j\big)/\hbar\right) \\
& = \frac{1}{h^{3N}N!}\sum_{\sigma\in S_N}\epsilon(\sigma)\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N e^{-\beta V(\mathbf{r}_1,\ldots,\mathbf{r}_N)} \\
& \prod_{n=1}^{N}\prod_{j=1}^{3}\int du\exp\left(-\beta\frac{u^2}{2m}-iu\big((\mathbf{r}_n)_j-(\mathbf{r}_{\sigma(n)})_j\big)/\hbar\right) \\
& = \frac{1}{h^{3N}N!}\sum_{\sigma\in S_N}\epsilon(\sigma)\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N e^{-\beta V(\mathbf{r}_1,\ldots,\mathbf{r}_N)} \\
& \prod_{n=1}^{N}\prod_{j=1}^{3}\sqrt{\frac{2m\pi}{\beta}}\exp\left(-\frac{m((\mathbf{r}_n)_j-(\mathbf{r}_{\sigma(n)})_j)^2}{2\beta\hbar^2}\right) \\
& = \frac{1}{h^{3N}N!}\sum_{\sigma\in S_N}\epsilon(\sigma)\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N e^{-\beta V(\mathbf{r}_1,\ldots,\mathbf{r}_N)} \\
& \left(\frac{h}{\lambda}\right)^{3N}\exp\left(-\frac{\pi}{\lambda^2}\sum_{n=1}^{N}(\mathbf{r}_n-\mathbf{r}_{\sigma(n)})^2\right) \\
& = \frac{1}{\lambda^{3N}N!}\sum_{\sigma\in S_N}\epsilon(\sigma)\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N e^{-\beta V(\mathbf{r}_1,\ldots,\mathbf{r}_N)}e^{-\frac{\pi}{\lambda^2}\sum_{n=1}^{N}(\mathbf{r}_n-\mathbf{r}_{\sigma(n)})^2}.
\end{aligned}
\tag{11}
$$

6. Vemos que el término en la suma que corresponde a la premutación

$\mathrm{id}_{\{1,\ldots,n\}}$ hace que

$$
\begin{aligned}
Z &= \frac{1}{\lambda^{3N}N!}\epsilon(\mathrm{id}_{\{1,\ldots,n\}})\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N e^{-\beta V(\mathbf{r}_1,\ldots,\mathbf{r}_N)} \\
&\quad e^{-\frac{\pi}{\lambda^2}\sum_{n=1}^{N}(\mathbf{r}_n-\mathbf{r}_{\mathrm{id}_{\{1,\ldots,n\}}(n)})^2} \\
&= \frac{1}{\lambda^{3N}N!}\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N e^{-\beta V(\mathbf{r}_1,\ldots,\mathbf{r}_N)} e^{-\frac{\pi}{\lambda^2}\sum_{n=1}^{N}(\mathbf{r}_n-\mathbf{r}_n)^2} \\
&= \frac{1}{\lambda^{3N}N!}\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N e^{-\beta V(\mathbf{r}_1,\ldots,\mathbf{r}_N)}.
\end{aligned}
\tag{12}
$$

lo cual corresponde a la función de partición clásica.

7. Recordemos que el significado físico de la longitud de onda termal de Broglie λ es la de el tamaño promedio de la función de onda de una partícula en nuestro sistema. Los efectos de intercambio solo deberían ser notables entre partículas cuyas funciones de onda se encuentran superpuestas. Por lo tanto se espera que solo aporten a este efecto las partículas que esten más cercanas. En el límite clásico $\lambda \to 0$ se tiene que eventualmente la distancia media entre partículas se va a hacer mucho mayor que λ. Esto significa que si $i \neq j$ e $i, j \in \{1, \ldots, n\}$ entonces

$$
\frac{\|\mathbf{r}_i - \mathbf{r}_j\|}{\lambda} \gg 1. \tag{13}
$$

Definiendo $S_N|_m = \{\sigma \in S_N || \{k \in \{1,\ldots,N\} | \sigma(k) \neq k\}| = m\}$ para $m \in \{1,\ldots,n\}$ vemos que forman una familia disjunta cuya union es S_N. Luego

$$
\begin{aligned}
Z &= \frac{1}{\lambda^{3N}N!}\sum_{\sigma\in S_N}\epsilon(\sigma)\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N e^{-\beta V(\mathbf{r}_1,\ldots,\mathbf{r}_N)} e^{-\frac{\pi}{\lambda^2}\sum_{n=1}^{N}(\mathbf{r}_n-\mathbf{r}_{\sigma(n)})^2} \\
&= \frac{1}{\lambda^{3N}N!}\sum_{m=1}^{N}\sum_{\sigma\in S_N|_m}\epsilon(\sigma)\int d^3\mathbf{r}_1\cdots d^3\mathbf{r}_N e^{-\beta V(\mathbf{r}_1,\ldots,\mathbf{r}_N)} \\
&\quad e^{-\frac{\pi}{\lambda^2}\sum_{n=1}^{N}(\mathbf{r}_n-\mathbf{r}_{\sigma(n)})^2}.
\end{aligned}
\tag{14}
$$

Ahora bien, es claro que a medida que aumenta m se tiene que

$$
\sum_{n=1}^{N}(\mathbf{r}_n - \mathbf{r}_{\sigma(n)})^2/\lambda^2 \tag{15}
$$

con $\sigma \in S_N|_m$ aumenta pues el número de $n \in \{1,\ldots,n\}$ tal que $n \neq \sigma(n)$ aumenta. Por lo tanto, a medida que aumenta m se tiene que el termino

$$
\sum_{\sigma\in S_N|_m}\epsilon(\sigma)e^{-\frac{\pi}{\lambda^2}\sum_{n=1}^{N}(\mathbf{r}_n-\mathbf{r}_{\sigma(n)})^2} \tag{16}
$$

disminuye. Por lo tanto, si se quieren estudiar las primeras correcciones cuánticas se puede empezar considerando $m \in \{0, 1, 2\}$, es decir, las permutaciones de 2 partículas.

8. En tal caso tenemos notando que para transposiciones σ se tiene $\epsilon(\sigma) = 1$ para bosones y -1 para fermiones y que $S_N|_1 = \varnothing$, y denotando $k_\sigma = \text{máx}\{n \in \{1, \ldots, N\} | \sigma(n) \neq n\}$ para todo $\sigma \in S_N|_2$

$$\begin{aligned}
Z = &\frac{1}{\lambda^{3N} N!} \sum_{m=0}^{2} \sum_{\sigma \in S_N|_m} \epsilon(\sigma) \int d^3\mathbf{r}_1 \cdots d^3\mathbf{r}_N e^{-\beta V(\mathbf{r}_1, \ldots, \mathbf{r}_N)} \\
&e^{-\frac{\pi}{\lambda^2} \sum_{n=1}^{N} (\mathbf{r}_n - \mathbf{r}_{\sigma(n)})^2} \\
&= \frac{1}{\lambda^{3N} N!} \int d^3\mathbf{r}_1 \cdots d^3\mathbf{r}_N e^{-\beta V(\mathbf{r}_1, \ldots, \mathbf{r}_N)} \\
&\left(1 \pm \sum_{\sigma \in S_N|_2} e^{-\frac{\pi}{\lambda^2} \sum_{n=1}^{N} (\mathbf{r}_n - \mathbf{r}_{\sigma(n)})^2} \right) \\
&= \frac{1}{\lambda^{3N} N!} \int d^3\mathbf{r}_1 \cdots d^3\mathbf{r}_N e^{-\beta V(\mathbf{r}_1, \ldots, \mathbf{r}_N)} \\
&\left(1 \pm \sum_{\sigma \in S_N|_2} e^{-\frac{2\pi}{\lambda^2} (\mathbf{r}_{k_\sigma} - \mathbf{r}_{\sigma(k_\sigma)})^2} \right) \\
&= \frac{1}{\lambda^{3N} N!} \int d^3\mathbf{r}_1 \cdots d^3\mathbf{r}_N e^{-\beta V(\mathbf{r}_1, \ldots, \mathbf{r}_N)} \left(1 \pm \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j<i}}^{N} e^{-\frac{2\pi}{\lambda^2} (\mathbf{r}_i - \mathbf{r}_j)^2} \right) \\
&= \frac{1}{\lambda^{3N} N!} \int d^3\mathbf{r}_1 \cdots d^3\mathbf{r}_N e^{-\beta V(\mathbf{r}_1, \ldots, \mathbf{r}_N)} \\
&\left(1 - \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j<i}}^{N} \beta v_{\text{exch}}(\|\mathbf{r}_i - \mathbf{r}_j\|) \right)
\end{aligned} \tag{17}$$

Donde $\beta v_{\text{exch}}(r) = \pm e^{-2\pi r^2/\lambda^2}$ con $+$ para fermiones y $-$ para bosones. En la aproximación $\lambda \to 0$ el punto anterior justifica que $v_{\text{exch}}(\|\mathbf{r}_i - \mathbf{r}_j\|)$ es muy pequeño. Por lo tanto podemos hacer la aproximación

$$1 - \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j<i}}^{N} \beta v_{\text{exch}}(\|\mathbf{r}_i - \mathbf{r}_j\|) \cong e^{-\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j<i}}^{N} \beta v_{\text{exch}}(\|\mathbf{r}_i - \mathbf{r}_j\|)}. \tag{18}$$

Concluimos que

$$Z = \frac{1}{\lambda^{3N} N!} \int d^3\mathbf{r}_1 \cdots d^3\mathbf{r}_N e^{-\beta \left(V(\mathbf{r}_1, \ldots, \mathbf{r}_N) + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j<i}}^{N} v_{\text{exch}}(\|\mathbf{r}_i - \mathbf{r}_j\|) \right)}. \tag{19}$$

9. Dado que es exponencial, nuestras experiencias con profundidades de piel en electromagnetismo o del estudio de la interacción de Yukawa hacen natural caracterizar el alcance con el factor que hace adimensional el exponente. Concluimos entonces que el alcance es del orden de $\lambda/\sqrt{\pi}$.

10. El principio de exclusión de Pauli garantiza que dos fermiones no se pueden encontrar en el mismo estado. En la aproximación clásica, el potential de intercambio es un potencial repelente para fermiones (gracias al signo positivo) que nos permite modelar tal repulsión cuántica de manera clásica.

Formalismo de la segunda cuantización

1. Se tiene que

$$\begin{aligned}\hat{N} &= \sum_{\alpha} a_{\alpha}^{\dagger} a_{\alpha} = \sum_{\alpha}\sum_{\beta} a_{\alpha}^{\dagger} a_{\beta}\delta_{\alpha\beta} = \sum_{\alpha}\sum_{\beta} a_{\alpha}^{\dagger} a_{\beta} \int d^3\mathbf{r}\phi_{\alpha}^{*}(\mathbf{r})\phi_{\beta}(\mathbf{r}) \\ &= \int d^3\mathbf{r} \sum_{\alpha} \phi_{\alpha}^{*}(\mathbf{r}) a_{\alpha}^{\dagger} \sum_{\beta} \phi_{\beta}(\mathbf{r}) a_{\beta} = \int d^3\mathbf{r}\psi^{\dagger}(\mathbf{r})\psi(\mathbf{r}).\end{aligned} \tag{20}$$

2. Para bosones

$$\begin{aligned}[\psi(\mathbf{r}), \hat{N}] &= \psi(\mathbf{r})\hat{N} - \hat{N}\psi(\mathbf{r}) = \psi(\mathbf{r})\hat{N} - \int d^3\mathbf{r'}\psi^{\dagger}(\mathbf{r'})\psi(\mathbf{r'})\psi(\mathbf{r}) \\ &= \psi(\mathbf{r})\hat{N} - \int d^3\mathbf{r'}\psi^{\dagger}(\mathbf{r'})\psi(\mathbf{r})\psi(\mathbf{r'}) \\ &= \psi(\mathbf{r})\hat{N} - \int d^3\mathbf{r'}(\psi(\mathbf{r})\psi^{\dagger}(\mathbf{r'}) - \delta(\mathbf{r}-\mathbf{r'}))\psi(\mathbf{r'}) \\ &= \psi(\mathbf{r})\hat{N} - \psi(\mathbf{r})\hat{N} + \psi(\mathbf{r}) = \psi(\mathbf{r})\end{aligned} \tag{21}$$

y

$$\begin{aligned}[\psi^{\dagger}(\mathbf{r}), \hat{N}] &= \psi^{\dagger}(\mathbf{r})\hat{N} - \hat{N}\psi^{\dagger}(\mathbf{r}) = \psi^{\dagger}(\mathbf{r})\hat{N} - \int d^3\mathbf{r'}\psi^{\dagger}(\mathbf{r'})\psi(\mathbf{r'})\psi^{\dagger}(\mathbf{r}) \\ &= \psi^{\dagger}(\mathbf{r})\hat{N} - \int d^3\mathbf{r'}\psi^{\dagger}(\mathbf{r'})(\psi^{\dagger}(\mathbf{r})\psi(\mathbf{r'}) + \delta(\mathbf{r'}-\mathbf{r})) \\ &= \psi^{\dagger}(\mathbf{r})\hat{N} - \psi^{\dagger}(\mathbf{r})\hat{N} - \psi^{\dagger}(\mathbf{r}) = -\psi^{\dagger}(\mathbf{r}).\end{aligned} \tag{22}$$

Para fermiones se tiene

$$\begin{aligned}[\psi(\mathbf{r}), \hat{N}] &= \psi(\mathbf{r})\hat{N} - \hat{N}\psi(\mathbf{r}) = \psi(\mathbf{r})\hat{N} - \int d^3\mathbf{r'}\psi^{\dagger}(\mathbf{r'})\psi(\mathbf{r'})\psi(\mathbf{r}) \\ &= \psi(\mathbf{r})\hat{N} + \int d^3\mathbf{r'}\psi^{\dagger}(\mathbf{r'})\psi(\mathbf{r})\psi(\mathbf{r'}) \\ &= \psi(\mathbf{r})\hat{N} + \int d^3\mathbf{r'}(-\psi(\mathbf{r})\psi^{\dagger}(\mathbf{r'}) + \delta(\mathbf{r}-\mathbf{r'}))\psi(\mathbf{r'}) \\ &= \psi(\mathbf{r})\hat{N} - \psi(\mathbf{r})\hat{N} + \psi(\mathbf{r}) = \psi(\mathbf{r})\end{aligned} \tag{23}$$

y

$$
\begin{aligned}
[\psi^\dagger(\mathbf{r}),\hat{N}] &= \psi^\dagger(\mathbf{r})\hat{N}-\hat{N}\psi^\dagger(\mathbf{r}) = \psi^\dagger(\mathbf{r})\hat{N}-\int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}')\psi^\dagger(\mathbf{r}) \\
&= \psi^\dagger(\mathbf{r})\hat{N}-\int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')(-\psi^\dagger(\mathbf{r})\psi(\mathbf{r}')+\delta(\mathbf{r}'-\mathbf{r})) \\
&= \psi^\dagger(\mathbf{r})\hat{N}+\int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r})\psi(\mathbf{r}')-\psi^\dagger(\mathbf{r}) \\
&= \psi^\dagger(\mathbf{r})\hat{N}-\psi^\dagger(\mathbf{r})\int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}')-\psi^\dagger(\mathbf{r}) \\
&= \psi^\dagger(\mathbf{r})\hat{N}-\psi^\dagger(\mathbf{r})\hat{N}-\psi^\dagger(\mathbf{r}) = -\psi^\dagger(\mathbf{r}).
\end{aligned} \tag{24}
$$

3. Se tiene

$$
\begin{aligned}
\hat{N}\psi(\mathbf{r})\left|E,N\right\rangle &= (\psi(\mathbf{r})\hat{N}-\psi(\mathbf{r}))\left|E,N\right\rangle = (\psi(\mathbf{r})N-\psi(\mathbf{r}))\left|E,N\right\rangle \\
&= (N-1)\psi(\mathbf{r})\left|E,N\right\rangle.
\end{aligned} \tag{25}
$$

Se concluye que $\psi(\mathbf{r})\left|E,N\right\rangle$ corresponde a un estado con un número definitivo de $N-1$ partículas. De manera análoga

$$
\begin{aligned}
\hat{N}\psi^\dagger(\mathbf{r})\left|E,N\right\rangle &= (\psi^\dagger(\mathbf{r})\hat{N}+\psi^\dagger(\mathbf{r}))\left|E,N\right\rangle = (\psi^\dagger(\mathbf{r})N+\psi^\dagger(\mathbf{r}))\left|E,N\right\rangle \\
&= (N+1)\psi^\dagger(\mathbf{r})\left|E,N\right\rangle.
\end{aligned} \tag{26}
$$

Se concluye que $\psi^\dagger(\mathbf{r})\left|E,N\right\rangle$ corresponde a un estado con un número definitivo de $N+1$ partículas.

4. Siguiendo a Altland y Simons considere un operador o que actua sobre el espacio de Hilbert de una partícula y el operador $O=\sum_i o_i$ que actua sobre el espacio de Fock, donde $o_i = \mathrm{id}_{\mathcal{H}}\otimes\mathrm{id}_{\mathcal{H}}\otimes\cdots\otimes o\otimes\mathrm{id}_{\mathcal{H}}\otimes\cdots$. Suponga aún más que o admite $\{|\lambda\rangle\}$ un conjunto contable completo ortogonal de vectores propios con valores propios o_λ. Entonces se tiene

$$
\begin{aligned}
\left\langle n'_{\lambda_1},n'_{\lambda_2},\ldots\middle|O\middle|n_{\lambda_1},n_{\lambda_2},\ldots\right\rangle &= \sum_i o_{\lambda_i}n_{\lambda_i}\left\langle n'_{\lambda_1},n'_{\lambda_2},\ldots\middle|n_{\lambda_1},n_{\lambda_2},\ldots\right\rangle \\
&= \left\langle n'_{\lambda_1},n'_{\lambda_2},\ldots\middle|\sum_i o_{\lambda_i}\hat{n}_{\lambda_i}\middle|n_{\lambda_1},n_{\lambda_2},\ldots\right\rangle
\end{aligned} \tag{27}
$$

concluyendo

$$
O=\sum_\lambda o_\lambda\hat{n}_\lambda = \sum_\lambda o_\lambda a^\dagger_\lambda a_\lambda = \sum_\lambda \left\langle\lambda|o|\lambda\right\rangle a^\dagger_\lambda a_\lambda. \tag{28}
$$

Ahora bien, si $|\mu\rangle$ es una base arbitraria, se tiene que

$$
a^\dagger_\mu\left|0\right\rangle = \left|\mu\right\rangle = \sum_\lambda \left\langle\lambda|\mu\right\rangle\left|\lambda\right\rangle = \sum_\lambda\left\langle\lambda|\mu\right\rangle a^\dagger_\lambda\left|0\right\rangle \tag{29}
$$

de lo que se obtiene que

$$
a^\dagger_\mu = \sum_\lambda\left\langle\lambda|\mu\right\rangle a^\dagger_\lambda. \tag{30}
$$

Por lo tanto, en una base arbitraria

$$\begin{aligned} O &= \sum_{\lambda\lambda'} \langle\lambda'|o|\lambda\rangle\, a^\dagger_{\lambda'} a_\lambda = \sum_{\mu,\mu'}\sum_{\lambda\lambda'} \langle\lambda'|\mu\rangle\, \langle\mu|o|\mu'\rangle\, \langle\mu'|\lambda\rangle\, a^\dagger_{\lambda'} a_\lambda \\ &= \sum_{\mu,\mu'} \langle\mu|o|\mu'\rangle\, a^\dagger_\mu a_{\mu'}. \end{aligned} \tag{31}$$

Ahora, podemos introducir los operadores de campo

$$\begin{aligned} O &= \int d^3\mathbf{r} d^3\mathbf{r}' \sum_{\mu,\mu'} \langle\mu|\mathbf{r}\rangle\, \langle\mathbf{r}|o|\mathbf{r}'\rangle\, \langle\mathbf{r}'|\mu'\rangle\, a^\dagger_\mu a_{\mu'} \\ &= \int d^3\mathbf{r} d^3\mathbf{r}' \sum_{\mu,\mu'} \phi^*_\mu(\mathbf{r})\, \langle\mathbf{r}|o|\mathbf{r}'\rangle\, \phi_{\mu'}(\mathbf{r}') a^\dagger_\mu a_{\mu'} \\ &= \int d^3\mathbf{r} d^3\mathbf{r}' \psi^\dagger(\mathbf{r})\, \langle\mathbf{r}|o|\mathbf{r}'\rangle\, \psi(\mathbf{r}'). \end{aligned} \tag{32}$$

En nuestro caso particular se tiene

$$\begin{aligned} \sum_i \frac{\mathbf{p}_i^2}{2m} + \sum_i u(\mathbf{r}_i) &= \int d^3\mathbf{r} d^3\mathbf{r}' \psi^\dagger(\mathbf{r})\, \langle\mathbf{r}|\left(\frac{\mathbf{p}^2}{2m} + u(\mathbf{r}')\right)|\mathbf{r}'\rangle\, \psi(\mathbf{r}') \\ &= \int d^3\mathbf{r} d^3\mathbf{r}' \psi^\dagger(\mathbf{r})\left(\frac{-\hbar^2}{2m}\Delta_{'} + u(\mathbf{r}')\right)\delta(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}') \\ &= \int d^3\mathbf{r} \psi^\dagger(\mathbf{r})\left(\frac{-\hbar^2}{2m}\Delta + u(\mathbf{r})\right)\psi(\mathbf{r}). \end{aligned} \tag{33}$$

Para un operador de dos cuerpos que es sumado sobre todas las posibles combinaciones tenemos una fórmula similar. En efecto, note que un sistema de dos cuerpos puede ser visto como un sistema de un cuerpo con un espacio de Hilbert generado por productos tensoriales. Luego si $\{|\lambda\rangle\}$ es una base para el espacio de un cuerpo, $\{|\lambda,\lambda'\rangle\}$ es una base para el espacio de dos. Es fácil convenserse entonces que el operador de creación es $a^\dagger_{\lambda,\lambda'} = a^\dagger_\lambda a^\dagger_{\lambda'}$. Entonces extendemos nuestro resultado previo de manera obvia para un operador $O = \frac{1}{2}\sum_{i,j} o_{i,j}$ donde o es un operador en el espacio de dos partículas

$$O = \sum_{\lambda,\lambda',\lambda'',\lambda'''} \langle\lambda\lambda'|o|\lambda''\lambda'''\rangle\, a^\dagger_\lambda a^\dagger_{\lambda'} a_{\lambda''} a_{\lambda'''} \tag{34}$$

Introduciendo los operadores de campo tenemos

$$\begin{aligned} O = \int & d^3\mathbf{r} d^3\mathbf{r}' d^3\mathbf{r}'' d^3\mathbf{r}''' \sum_{\lambda,\lambda',\lambda'',\lambda'''} \\ & \langle\lambda\lambda'|\mathbf{r}\mathbf{r}'\rangle\, \langle\mathbf{r}\mathbf{r}'|o|\mathbf{r}''\mathbf{r}'''\rangle\, \langle\mathbf{r}''\mathbf{r}'''|\lambda''\lambda'''\rangle\, a^\dagger_\lambda a^\dagger_{\lambda'} a_{\lambda''} a_{\lambda'''} \\ = \int & d^3\mathbf{r} d^3\mathbf{r}' d^3\mathbf{r}'' d^3\mathbf{r}''' \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')\, \langle\mathbf{r}\mathbf{r}'|o|\mathbf{r}''\mathbf{r}'''\rangle\, \psi(\mathbf{r}'')\psi(\mathbf{r}''') \\ = \int & d^3\mathbf{r} d^3\mathbf{r}' \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}') o \psi(\mathbf{r}')\psi(\mathbf{r}) \end{aligned} \tag{35}$$

Se concluye que el Hamiltoniano es

$$
\begin{aligned}
H = &-\frac{\hbar^2}{2m}\int d^3\mathbf{r}\psi^\dagger(\mathbf{r})\Delta\psi(\mathbf{r}) + \int d^3\mathbf{r}\psi^\dagger(\mathbf{r})u(\mathbf{r})\psi(\mathbf{r}) \\
&+ \frac{1}{2}\int d^3\mathbf{r}d^3\mathbf{r}'\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r}-\mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})
\end{aligned}
\tag{36}
$$

5. Se tiene

$$
\begin{aligned}
[\hat{N}, H] =& \hat{N} \left(-\frac{\hbar^2}{2m} \int d^3\mathbf{r} \psi^\dagger(\mathbf{r}) \Delta \psi(\mathbf{r}) + \int d^3\mathbf{r} \psi^\dagger(\mathbf{r}) u(\mathbf{r}) \psi(\mathbf{r}) \right. \\
& \left. + \frac{1}{2} \int d^3\mathbf{r} d^3\mathbf{r}' \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') v(\|\mathbf{r}-\mathbf{r}'\|) \psi(\mathbf{r}') \psi(\mathbf{r}) \right) \\
& - \left(-\frac{\hbar^2}{2m} \int d^3\mathbf{r} \psi^\dagger(\mathbf{r}) \Delta \psi(\mathbf{r}) + \int d^3\mathbf{r} \psi^\dagger(\mathbf{r}) u(\mathbf{r}) \psi(\mathbf{r}) \right. \\
& \left. + \frac{1}{2} \int d^3\mathbf{r} d^3\mathbf{r}' \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') v(\|\mathbf{r}-\mathbf{r}'\|) \psi(\mathbf{r}') \psi(\mathbf{r}) \right) \hat{N} \\
& = -\frac{\hbar^2}{2m} \int d^3\mathbf{r} (\hat{N} \psi^\dagger(\mathbf{r}) \Delta \psi(\mathbf{r}) - \psi^\dagger(\mathbf{r}) \Delta \psi(\mathbf{r}) \hat{N}) \\
& + \int d^3\mathbf{r} (\hat{N} \psi^\dagger(\mathbf{r}) u(\mathbf{r}) \psi(\mathbf{r}) - \psi^\dagger(\mathbf{r}) u(\mathbf{r}) \psi(\mathbf{r}) \hat{N}) \\
& + \frac{1}{2} \int d^3\mathbf{r} d^3\mathbf{r}' (\hat{N} \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') v(\|\mathbf{r}-\mathbf{r}'\|) \psi(\mathbf{r}') \psi(\mathbf{r}) \\
& - \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') v(\|\mathbf{r}-\mathbf{r}'\|) \psi(\mathbf{r}') \psi(\mathbf{r}) \hat{N}) \\
& = -\frac{\hbar^2}{2m} \int d^3\mathbf{r} ((\psi^\dagger(\mathbf{r}) \hat{N} + \psi^\dagger(\mathbf{r})) \Delta \psi(\mathbf{r}) - \psi^\dagger(\mathbf{r}) \Delta \psi(\mathbf{r}) \hat{N}) \\
& + \int d^3\mathbf{r} ((\psi^\dagger(\mathbf{r}) \hat{N} + \psi^\dagger(\mathbf{r})) u(\mathbf{r}) \psi(\mathbf{r}) - \psi^\dagger(\mathbf{r}) u(\mathbf{r}) \psi(\mathbf{r}) \hat{N}) \\
& + \frac{1}{2} \int d^3\mathbf{r} d^3\mathbf{r}' ((\psi^\dagger(\mathbf{r}) \hat{N} + \psi^\dagger(\mathbf{r})) \psi^\dagger(\mathbf{r}') v(\|\mathbf{r}-\mathbf{r}'\|) \psi(\mathbf{r}') \psi(\mathbf{r}) \\
& - \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') v(\|\mathbf{r}-\mathbf{r}'\|) \psi(\mathbf{r}') \psi(\mathbf{r}) \hat{N}) \\
& = -\frac{\hbar^2}{2m} \int d^3\mathbf{r} (\psi^\dagger(\mathbf{r}) \Delta \hat{N} \psi(\mathbf{r}) + \psi^\dagger(\mathbf{r}) \Delta \psi(\mathbf{r}) - \psi^\dagger(\mathbf{r}) \Delta \psi(\mathbf{r}) \hat{N}) \\
& + \int d^3\mathbf{r} (\psi^\dagger(\mathbf{r}) u(\mathbf{r}) \hat{N} \psi(\mathbf{r}) + \psi^\dagger(\mathbf{r}) u(\mathbf{r}) \psi(\mathbf{r}) - \psi^\dagger(\mathbf{r}) u(\mathbf{r}) \psi(\mathbf{r}) \hat{N}) \\
& + \frac{1}{2} \int d^3\mathbf{r} d^3\mathbf{r}' (\psi^\dagger(\mathbf{r}) \hat{N} \psi^\dagger(\mathbf{r}') v(\|\mathbf{r}-\mathbf{r}'\|) \psi(\mathbf{r}') \psi(\mathbf{r}) \\
& + \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') v(\|\mathbf{r}-\mathbf{r}'\|) \psi(\mathbf{r}') \psi(\mathbf{r}) \\
& - \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') v(\|\mathbf{r}-\mathbf{r}'\|) \psi(\mathbf{r}') \psi(\mathbf{r}) \hat{N}) \\
& = -\frac{\hbar^2}{2m} \int d^3\mathbf{r} (\psi^\dagger(\mathbf{r}) \Delta (\psi(\mathbf{r}) \hat{N} - \psi(\mathbf{r})) \\
& + \psi^\dagger(\mathbf{r}) \Delta \psi(\mathbf{r}) - \psi^\dagger(\mathbf{r}) \Delta \psi(\mathbf{r}) \hat{N}) \\
& + \int d^3\mathbf{r} (\psi^\dagger(\mathbf{r}) u(\mathbf{r}) (\psi(\mathbf{r}) \hat{N} - \psi(\mathbf{r})) \\
& + \psi^\dagger(\mathbf{r}) u(\mathbf{r}) \psi(\mathbf{r}) - \psi^\dagger(\mathbf{r}) u(\mathbf{r}) \psi(\mathbf{r}) \hat{N}) \\
& + \frac{1}{2} \int d^3\mathbf{r} d^3\mathbf{r}' (\psi^\dagger(\mathbf{r}) (\psi^\dagger(\mathbf{r}') \hat{N} + \psi^\dagger(\mathbf{r}')) v(\|\mathbf{r}-\mathbf{r}'\|) \psi(\mathbf{r}') \psi(\mathbf{r}) \\
& + \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') v(\|\mathbf{r}-\mathbf{r}'\|) \psi(\mathbf{r}') \psi(\mathbf{r}) \\
& - \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') v(\|\mathbf{r}-\mathbf{r}'\|) \psi(\mathbf{r}') \psi(\mathbf{r}) \hat{N})
\end{aligned}
\tag{37}
$$

$$= -\frac{\hbar^2}{2m}\int d^3\mathbf{r}(\psi^\dagger(\mathbf{r})\Delta(\psi(\mathbf{r})\hat{N} - \psi(\mathbf{r}))$$
$$+ \psi^\dagger(\mathbf{r})\Delta\psi(\mathbf{r}) - \psi^\dagger(\mathbf{r})\Delta\psi(\mathbf{r})\hat{N})$$
$$+ \int d^3\mathbf{r}(\psi^\dagger(\mathbf{r})u(\mathbf{r})(\psi(\mathbf{r})\hat{N} - \psi(\mathbf{r}))$$
$$+ \psi^\dagger(\mathbf{r})u(\mathbf{r})\psi(\mathbf{r}) - \psi^\dagger(\mathbf{r})u(\mathbf{r})\psi(\mathbf{r})\hat{N})$$
$$+ \frac{1}{2}\int d^3\mathbf{r}d^3\mathbf{r}'(\psi^\dagger(\mathbf{r})(\psi^\dagger(\mathbf{r}')\hat{N} + \psi^\dagger(\mathbf{r}'))v(\|\mathbf{r} - \mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})$$
$$+ \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})$$
$$- \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})\hat{N})$$
$$= -\frac{\hbar^2}{2m}\int d^3\mathbf{r}(\psi^\dagger(\mathbf{r})\Delta\psi(\mathbf{r})\hat{N} - \psi^\dagger(\mathbf{r})\Delta\psi(\mathbf{r})$$
$$+ \psi^\dagger(\mathbf{r})\Delta\psi(\mathbf{r}) - \psi^\dagger(\mathbf{r})\Delta\psi(\mathbf{r})\hat{N})$$
$$+ \int d^3\mathbf{r}(\psi^\dagger(\mathbf{r})u(\mathbf{r})\psi(\mathbf{r})\hat{N} - \psi^\dagger(\mathbf{r})u(\mathbf{r})\psi(\mathbf{r})$$
$$+ \psi^\dagger(\mathbf{r})u(\mathbf{r})\psi(\mathbf{r}) - \psi^\dagger(\mathbf{r})u(\mathbf{r})\psi(\mathbf{r})\hat{N})$$
$$+ \frac{1}{2}\int d^3\mathbf{r}d^3\mathbf{r}'(\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)\hat{N}\psi(\mathbf{r}')\psi(\mathbf{r})$$
$$+ \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})$$
$$+ \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})$$
$$- \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})\hat{N})$$
$$= \frac{1}{2}\int d^3\mathbf{r}d^3\mathbf{r}'(\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)(\psi(\mathbf{r}')\hat{N} - \psi(\mathbf{r}'))\psi(\mathbf{r})$$
$$+ 2\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})$$
$$- \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})\hat{N})$$
$$= \frac{1}{2}\int d^3\mathbf{r}d^3\mathbf{r}'(\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)\psi(\mathbf{r}')\hat{N}\psi(\mathbf{r})$$
$$- \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})$$
$$+ 2\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})$$
$$- \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})\hat{N})$$
$$= \frac{1}{2}\int d^3\mathbf{r}d^3\mathbf{r}'(\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)\psi(\mathbf{r}')(\psi(\mathbf{r})\hat{N} - \psi(\mathbf{r}))$$
$$+ \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})$$
$$- \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})\hat{N})$$

$$
\begin{aligned}
&= \frac{1}{2}\int d^3\mathbf{r}d^3\mathbf{r}'(\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r}-\mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})\hat{N}\\
&- \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r}-\mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})\\
&+ \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r}-\mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})\\
&- \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r}-\mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})\hat{N}) = 0.
\end{aligned}
$$

Note que el cálculo es idéntico para bosones y fermiones pues las relaciones de conmutación entre los operadores de campo y el operador número de partículas lo son. Este conmutador tiene el significado físico de que el número de partículas se conserva. En efecto, ya que el Hamiltoniano es el generador de translaciones temporales, cualquier observable que conmuta con el es conservado.

6. En general tenemos

$$
\begin{aligned}
[\psi(\mathbf{r}), H] =&\psi(\mathbf{r})\left(-\frac{\hbar^2}{2m}\int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')\Delta\psi(\mathbf{r}') + \int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')u(\mathbf{r}')\psi(\mathbf{r}')\right.\\
&\left.+\frac{1}{2}\int d^3\mathbf{r}'d^3\mathbf{r}''\psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\right)\\
&-\left(-\frac{\hbar^2}{2m}\int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')\Delta\psi(\mathbf{r}') + \int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')u(\mathbf{r}')\psi(\mathbf{r}')\right.\\
&\left.+\frac{1}{2}\int d^3\mathbf{r}'d^3\mathbf{r}''\psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\right)\psi(\mathbf{r})\\
&= -\frac{\hbar^2}{2m}\int d^3\mathbf{r}'(\psi(\mathbf{r})\psi^\dagger(\mathbf{r}')\Delta\psi(\mathbf{r}') - \psi^\dagger(\mathbf{r}')\Delta\psi(\mathbf{r}')\psi(\mathbf{r}))\\
&+ \int d^3\mathbf{r}'(\psi(\mathbf{r})\psi^\dagger(\mathbf{r}')u(\mathbf{r}')\psi(\mathbf{r}') - \psi^\dagger(\mathbf{r}')u(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}))\\
&+ \frac{1}{2}\int d^3\mathbf{r}'d^3\mathbf{r}''(\psi(\mathbf{r})\psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\\
&- \psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\psi(\mathbf{r})).
\end{aligned}
\tag{38}
$$

En el caso particular de bosones se tiene

$$
\begin{aligned}
[\psi(\mathbf{r}), H] = &-\frac{\hbar^2}{2m}\int d^3\mathbf{r}'((\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}) + \delta(\mathbf{r}-\mathbf{r}'))\Delta\psi(\mathbf{r}')\\
&- \psi^\dagger(\mathbf{r}')\Delta\psi(\mathbf{r}')\psi(\mathbf{r}))\\
&+ \int d^3\mathbf{r}'((\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}) + \delta(\mathbf{r}-\mathbf{r}'))u(\mathbf{r}')\psi(\mathbf{r}')\\
&- \psi^\dagger(\mathbf{r}')u(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}))\\
&+ \frac{1}{2}\int d^3\mathbf{r}'d^3\mathbf{r}''((\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}) + \delta(\mathbf{r}-\mathbf{r}'))\\
&\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\\
&- \psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\psi(\mathbf{r}))
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
&= -\frac{\hbar^2}{2m}\int d^3\mathbf{r}'(\psi^\dagger(\mathbf{r}')\Delta\psi(\mathbf{r}')\psi(\mathbf{r}) + \delta(\mathbf{r}-\mathbf{r}')\Delta\psi(\mathbf{r}') \\
&- \psi^\dagger(\mathbf{r}')\Delta\psi(\mathbf{r}')\psi(\mathbf{r})) \\
&+ \int d^3\mathbf{r}'(\psi^\dagger(\mathbf{r}')u(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}) + \delta(\mathbf{r}-\mathbf{r}')u(\mathbf{r}')\psi(\mathbf{r}') \\
&- \psi^\dagger(\mathbf{r}')u(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r})) \\
&+ \frac{1}{2}\int d^3\mathbf{r}'d^3\mathbf{r}''(\psi^\dagger(\mathbf{r}')\psi(\mathbf{r})\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}') \\
&+ \delta(\mathbf{r}-\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}') \\
&- \psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\psi(\mathbf{r})) \\
&= -\frac{\hbar^2}{2m}\Delta\psi(\mathbf{r}) + u(\mathbf{r})\psi(\mathbf{r}) \\
&+ \frac{1}{2}\int d^3\mathbf{r}'d^3\mathbf{r}''(\psi^\dagger(\mathbf{r}')(\psi^\dagger(\mathbf{r}'')\psi(\mathbf{r}) + \delta(\mathbf{r}-\mathbf{r}'')) \\
&v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}') \\
&- \psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\psi(\mathbf{r})) \\
&+ \frac{1}{2}\int d^3\mathbf{r}''\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}) \\
&= -\frac{\hbar^2}{2m}\Delta\psi(\mathbf{r}) + u(\mathbf{r})\psi(\mathbf{r}) \\
&+ \frac{1}{2}\int d^3\mathbf{r}'d^3\mathbf{r}''(\psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')\psi(\mathbf{r})v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}') \\
&+ \psi^\dagger(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}') \\
&- \psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\psi(\mathbf{r})) \\
&+ \frac{1}{2}\int d^3\mathbf{r}''\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}) \\
&= -\frac{\hbar^2}{2m}\Delta\psi(\mathbf{r}) + u(\mathbf{r})\psi(\mathbf{r}) \\
&+ \frac{1}{2}\int d^3\mathbf{r}'d^3\mathbf{r}''(\psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\psi(\mathbf{r}) \\
&- \psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\psi(\mathbf{r})) \\
&+ \int d^3\mathbf{r}''\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}) \\
&= -\frac{\hbar^2}{2m}\Delta\psi(\mathbf{r}) + u(\mathbf{r})\psi(\mathbf{r}) + \int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')v(\|\mathbf{r}-\mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r}).
\end{aligned}
$$

En el caso de fermiones

$$
\begin{aligned}
[\psi(\mathbf{r}), H] = &-\frac{\hbar^2}{2m}\int d^3\mathbf{r}'((-\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}) + \delta(\mathbf{r}-\mathbf{r}'))\Delta\psi(\mathbf{r}') \\
&- \psi^\dagger(\mathbf{r}')\Delta\psi(\mathbf{r}')\psi(\mathbf{r})) \\
&+ \int d^3\mathbf{r}'((-\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}) + \delta(\mathbf{r}-\mathbf{r}'))u(\mathbf{r}')\psi(\mathbf{r}') \\
&- \psi^\dagger(\mathbf{r}')u(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r})) \\
&+ \frac{1}{2}\int d^3\mathbf{r}'d^3\mathbf{r}''((-\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}) + \delta(\mathbf{r}-\mathbf{r}')) \\
&\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}') \\
&- \psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\psi(\mathbf{r})) \\
&= -\frac{\hbar^2}{2m}\int d^3\mathbf{r}'(-\psi^\dagger(\mathbf{r}')\Delta(-\psi(\mathbf{r}')\psi(\mathbf{r})) + \delta(\mathbf{r}-\mathbf{r}')\Delta\psi(\mathbf{r}') \\
&- \psi^\dagger(\mathbf{r}')\Delta\psi(\mathbf{r}')\psi(\mathbf{r})) \\
&+ \int d^3\mathbf{r}'(-\psi^\dagger(\mathbf{r}')u(\mathbf{r}')(-\psi(\mathbf{r}')\psi(\mathbf{r})) + \delta(\mathbf{r}-\mathbf{r}')u(\mathbf{r}')\psi(\mathbf{r}') \\
&- \psi^\dagger(\mathbf{r}')u(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r})) \\
&+ \frac{1}{2}\int d^3\mathbf{r}'d^3\mathbf{r}''(-\psi^\dagger(\mathbf{r}')\psi(\mathbf{r})\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}') \\
&+ \delta(\mathbf{r}-\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}') \\
&- \psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\psi(\mathbf{r})) \\
&= -\frac{\hbar^2}{2m}\Delta\psi(\mathbf{r}) + u(\mathbf{r})\psi(\mathbf{r}) \\
&+ \frac{1}{2}\int d^3\mathbf{r}'d^3\mathbf{r}''(-\psi^\dagger(\mathbf{r}')(-\psi^\dagger(\mathbf{r}'')\psi(\mathbf{r}) + \delta(\mathbf{r}-\mathbf{r}'')) \\
&v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}') \\
&- \psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\psi(\mathbf{r})) \\
&+ \frac{1}{2}\int d^3\mathbf{r}''\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}) \\
&= -\frac{\hbar^2}{2m}\Delta\psi(\mathbf{r}) + u(\mathbf{r})\psi(\mathbf{r}) \\
&+ \frac{1}{2}\int d^3\mathbf{r}'d^3\mathbf{r}''(\psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')\psi(\mathbf{r})v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}') \\
&- \psi^\dagger(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}') \\
&- \psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}'-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\psi(\mathbf{r})) \\
&+ \frac{1}{2}\int d^3\mathbf{r}''\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}-\mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r})
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
&= -\frac{\hbar^2}{2m}\Delta\psi(\mathbf{r}) + u(\mathbf{r})\psi(\mathbf{r}) \\
&+ \frac{1}{2}\int d^3\mathbf{r}' d^3\mathbf{r}'' (\psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}' - \mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\psi(\mathbf{r}) \\
&+ \psi^\dagger(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}'')v(\|\mathbf{r}' - \mathbf{r}''\|)\psi(\mathbf{r}')\psi(\mathbf{r}'') \\
&- \psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r}' - \mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}')\psi(\mathbf{r})) \\
&+ \frac{1}{2}\int d^3\mathbf{r}''\psi^\dagger(\mathbf{r}'')v(\|\mathbf{r} - \mathbf{r}''\|)\psi(\mathbf{r}'')\psi(\mathbf{r}) \\
&= -\frac{\hbar^2}{2m}\Delta\psi(\mathbf{r}) + u(\mathbf{r})\psi(\mathbf{r}) + \int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')v(\|\mathbf{r} - \mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r}).
\end{aligned}
$$

7. Tenemos

$$
\begin{aligned}
E\psi_E(\mathbf{r}_1, \ldots \mathbf{r}_N) &= E\,\langle 0|\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N)|E, N\rangle \\
&= \langle 0|\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N)E|E, N\rangle \\
&= \langle 0|\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N)H|E, N\rangle \\
&= \langle 0|(H\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N) + [\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N), H])|E, N\rangle \\
&= \langle 0|[\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N), H]|E, N\rangle
\end{aligned} \tag{41}
$$

Este conmutador es la suma de elementos que genéricamente son de la forma

$$
\begin{aligned}
&\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_{i-1})[\psi(\mathbf{r}_i),H]\psi(\mathbf{r}_{i+1})\cdots\psi(\mathbf{r}_N)\\
=&\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_{i-1})\\
&\left(-\frac{\hbar^2}{2m}\Delta_i\psi(\mathbf{r}_i)+u(\mathbf{r}_i)\psi(\mathbf{r}_i)+\int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')v(\|\mathbf{r}_i-\mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r}_i)\right)\\
&\psi(\mathbf{r}_{i+1})\cdots\psi(\mathbf{r}_N)\\
=&\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_{i-2})\\
&\left(-\frac{\hbar^2}{2m}\Delta_i\psi(\mathbf{r}_{i-1})\psi(\mathbf{r}_i)+u(\mathbf{r}_i)\psi(\mathbf{r}_{i-1})\psi(\mathbf{r}_i)\right.\\
&\left.+\int d^3\mathbf{r}'\psi(\mathbf{r}_{i-1})\psi^\dagger(\mathbf{r}')v(\|\mathbf{r}_i-\mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r}_i)\right)\\
&\psi(\mathbf{r}_{i+1})\cdots\psi(\mathbf{r}_N)\\
=&\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_{i-2})\\
&\left(-\frac{\hbar^2}{2m}\Delta_i\psi(\mathbf{r}_{i-1})\psi(\mathbf{r}_i)+u(\mathbf{r}_i)\psi(\mathbf{r}_{i-1})\psi(\mathbf{r}_i)\right.\\
&\left.+\int d^3\mathbf{r}'((-1)^\rho\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}_{i-1})+\delta(\mathbf{r}'-\mathbf{r}_{i-1}))v(\|\mathbf{r}_i-\mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r}_i)\right)\\
&\psi(\mathbf{r}_{i+1})\cdots\psi(\mathbf{r}_N)\\
=&\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_{i-2})\\
&\left(-\frac{\hbar^2}{2m}\Delta_i\psi(\mathbf{r}_{i-1})\psi(\mathbf{r}_i)+u(\mathbf{r}_i)\psi(\mathbf{r}_{i-1})\psi(\mathbf{r}_i)\right.\\
&+v(\|\mathbf{r}_i-\mathbf{r}_{i-1}\|)\psi(\mathbf{r}_{i-1})\psi(\mathbf{r}_i)\\
&\left.+\int d^3\mathbf{r}'(-1)^\rho\psi^\dagger(\mathbf{r}')v(\|\mathbf{r}_i-\mathbf{r}'\|)(-1)^\rho\psi(\mathbf{r}')\psi(\mathbf{r}_{i-1})\psi(\mathbf{r}_i)\right)\\
&\psi(\mathbf{r}_{i+1})\cdots\psi(\mathbf{r}_N)\\
=&\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_{i-2})\\
&\left(-\frac{\hbar^2}{2m}\Delta_i\psi(\mathbf{r}_{i-1})\psi(\mathbf{r}_i)+u(\mathbf{r}_i)\psi(\mathbf{r}_{i-1})\psi(\mathbf{r}_i)\right.\\
&+v(\|\mathbf{r}_i-\mathbf{r}_{i-1}\|)\psi(\mathbf{r}_{i-1})\psi(\mathbf{r}_i)\\
&\left.+\int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')v(\|\mathbf{r}_i-\mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r}_{i-1})\psi(\mathbf{r}_i)\right)\\
&\psi(\mathbf{r}_{i+1})\cdots\psi(\mathbf{r}_N).
\end{aligned}
\tag{42}
$$

donde

$$
\rho=\begin{cases}0 & \text{bosones}\\ 1 & \text{fermiones.}\end{cases}
\tag{43}
$$

Continuando con este cálculo es claro que

$$
\begin{aligned}
&\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_{i-1})[\psi(\mathbf{r}_i),H]\psi(\mathbf{r}_{i+1})\cdots\psi(\mathbf{r}_N) \\
=&-\frac{\hbar^2}{2m}\Delta_i\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N)+u(\mathbf{r}_i)\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N) \\
&+\sum_{j=1}^{i-1}v(\|\mathbf{r}_i-\mathbf{r}_j\|)\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N) \\
&+\int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')v(\|\mathbf{r}_i-\mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N).
\end{aligned}
\tag{44}
$$

Por lo tanto

$$
\begin{aligned}
E\psi_E(\mathbf{r}_1,\ldots,\mathbf{r}_N) &= \langle 0|\sum_{i=1}^{N}\Bigg(-\frac{\hbar^2}{2m}\Delta_i\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N) \\
&+ u(\mathbf{r}_i)\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N) \\
&+ \sum_{j=1}^{i-1} v(\|\mathbf{r}_i-\mathbf{r}_j\|)\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N) \\
&+ \int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')v(\|\mathbf{r}_i-\mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N)\Bigg) \\
&|E,N\rangle \\
&= \langle 0|\Bigg(-\frac{\hbar^2}{2m}\sum_{i=1}^{N}\Delta_i\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N) \\
&+ \sum_{i=1}^{N} u(\mathbf{r}_i)\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N) \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{i-1} v(\|\mathbf{r}_i-\mathbf{r}_j\|)\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N) \\
&+ \sum_{i=1}^{N}\int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')v(\|\mathbf{r}_i-\mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N)\Bigg) \\
&|E,N\rangle \\
&= \langle 0|\Bigg(-\frac{\hbar^2}{2m}\sum_{i=1}^{N}\Delta_i\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N) \\
&+ \sum_{i=1}^{N} u(\mathbf{r}_i)\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N) \\
&+ \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} v(\|\mathbf{r}_i-\mathbf{r}_j\|)\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N) \\
&+ \int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')\sum_{i=1}^{N} v(\|\mathbf{r}_i-\mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N)\Bigg) \\
&|E,N\rangle \\
&= \left(-\frac{\hbar^2}{2m}\sum_{i=1}^{N}\Delta_i + \sum_{i=1}^{N}u(\mathbf{r}_i) + \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} v(\|\mathbf{r}_i-\mathbf{r}_j\|)\right) \\
&\langle 0|\psi(\mathbf{r}_1)\cdots\psi(\mathbf{r}_N)|E,N\rangle
\end{aligned}
\tag{45}
$$

$$=\left(-\frac{\hbar^2}{2m}\sum_{i=1}^{N}\Delta_i+\sum_{i=1}^{N}u(\mathbf{r}_i)+\frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}v(\|\mathbf{r}_i-\mathbf{r}_j\|)\right)$$
$$\psi_E(\mathbf{r}_1,\ldots,\mathbf{r}_N).$$

8. Podemos interpretar ambos lados de la ecuación (2.8) del taller. El módulo al cuadrado del lado izquierdo representa la densidad de probabilidad de hallar N partículas en $\mathbf{r}_1,\ldots,\mathbf{r}_N$. Por otr parte, el módulo al cuadrado del lado derecho representa la densidad de probabilidad de que al quitar particulas en las posiciones $\mathbf{r}_1,\ldots,\mathbf{r}_N$ se obtenga el vacío. Estos dos son replanteamientos de la misma proposición.

9. Se tiene

$$\begin{aligned}
i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} &= i\hbar\frac{\partial}{\partial t}\left(e^{iHt/\hbar}\psi(\mathbf{r})e^{-iHt/\hbar}\right)\\
&= i\hbar\left(\frac{i}{\hbar}He^{iHt/\hbar}\psi(\mathbf{r})e^{-iHt/\hbar}-\frac{i}{\hbar}e^{iHt/\hbar}\psi(\mathbf{r})He^{-iHt/\hbar}\right)\\
&= e^{iHt/\hbar}[\psi(\mathbf{r}),H]e^{-iHt/\hbar}\\
&= e^{iHt/\hbar}\left(-\frac{\hbar^2}{2m}\Delta\psi(\mathbf{r})+u(\mathbf{r})\psi(\mathbf{r})\right.\\
&\left.+\int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}')v(\|\mathbf{r}-\mathbf{r}'\|)\psi(\mathbf{r}')\psi(\mathbf{r})\right)e^{-iHt/\hbar}\\
&= -\frac{\hbar^2}{2m}\Delta e^{iHt/\hbar}\psi(\mathbf{r})e^{-iHt/\hbar}+u(\mathbf{r})e^{iHt/\hbar}\psi(\mathbf{r})e^{-iHt/\hbar}\\
&+\int d^3\mathbf{r}'e^{iHt/\hbar}\psi^\dagger(\mathbf{r}')e^{-iHt/\hbar}v(\|\mathbf{r}-\mathbf{r}'\|)e^{iHt/\hbar}\psi(\mathbf{r}')e^{-iHt/\hbar}\\
&e^{iHt/\hbar}\psi(\mathbf{r})e^{-iHt/\hbar}\\
&= -\frac{\hbar^2}{2m}\Delta\psi(\mathbf{r},t)+u(\mathbf{r})\psi(\mathbf{r},t)\\
&+\int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}',t)v(\|\mathbf{r}-\mathbf{r}'\|)\psi(\mathbf{r}',t)\psi(\mathbf{r},t)
\end{aligned}\tag{46}$$

10. Se tiene

$$
\begin{aligned}
\frac{\partial \hat{n}(\mathbf{r},t)}{\partial t} &= \frac{\partial \psi^\dagger(\mathbf{r},t)}{\partial t}\psi(\mathbf{r},t) + \psi^\dagger(\mathbf{r},t)\frac{\partial \psi(\mathbf{r},t)}{\partial t} \\
&= \left(\frac{\partial \psi(\mathbf{r},t)}{\partial t}\right)^\dagger \psi(\mathbf{r},t) + \psi^\dagger(\mathbf{r},t)\frac{\partial \psi(\mathbf{r},t)}{\partial t} \\
&= \Bigg(\frac{i\hbar}{2m}\Delta\psi(\mathbf{r},t) - \frac{i}{\hbar}u(\mathbf{r})\psi(\mathbf{r},t) \\
&\quad -\frac{i}{\hbar}\int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}',t)v(\|\mathbf{r}-\mathbf{r}'\|)\psi(\mathbf{r}',t)\psi(\mathbf{r},t)\Bigg)^\dagger \psi(\mathbf{r},t) \\
&\quad + \psi^\dagger(\mathbf{r},t)\Bigg(\frac{i\hbar}{2m}\Delta\psi(\mathbf{r},t) - \frac{i}{\hbar}u(\mathbf{r})\psi(\mathbf{r},t) \\
&\quad -\frac{i}{\hbar}\int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}',t)v(\|\mathbf{r}-\mathbf{r}'\|)\psi(\mathbf{r}',t)\psi(\mathbf{r},t)\Bigg)
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
&= \Bigg(-\frac{i\hbar}{2m}\Delta\psi^\dagger(\mathbf{r},t) + \frac{i}{\hbar}u(\mathbf{r})\psi^\dagger(\mathbf{r},t) \\
&+\frac{i}{\hbar}\int d^3\mathbf{r}'\psi^\dagger(\mathbf{r},t)\psi^\dagger(\mathbf{r}',t)v(\|\mathbf{r}-\mathbf{r}'\|)\psi(\mathbf{r}',t)\Bigg)\psi(\mathbf{r},t) \\
&+ \psi^\dagger(\mathbf{r},t)\Bigg(\frac{i\hbar}{2m}\Delta\psi(\mathbf{r},t) - \frac{i}{\hbar}u(\mathbf{r})\psi(\mathbf{r},t) \\
&-\frac{i}{\hbar}\int d^3\mathbf{r}'\psi^\dagger(\mathbf{r}',t)v(\|\mathbf{r}-\mathbf{r}'\|)\psi(\mathbf{r}',t)\psi(\mathbf{r},t)\Bigg) \\
&= -\frac{i\hbar}{2m}\big((\Delta\psi^\dagger(\mathbf{r},t))\psi(\mathbf{r},t) - \psi^\dagger(\mathbf{r},t)\Delta\psi(\mathbf{r},t)\big) \\
&= -\frac{i\hbar}{2m}((\Delta\psi^\dagger(\mathbf{r},t))\psi(\mathbf{r},t) + \boldsymbol{\nabla}\big(\psi^\dagger(\mathbf{r},t)\big)\cdot\boldsymbol{\nabla}(\psi(\mathbf{r},t)) \\
&- \boldsymbol{\nabla}\big(\psi^\dagger(\mathbf{r},t)\big)\cdot\boldsymbol{\nabla}(\psi(\mathbf{r},t)) - \psi^\dagger(\mathbf{r},t)\Delta\psi(\mathbf{r},t)) \\
&= -\boldsymbol{\nabla}\cdot\left(\frac{i\hbar}{2m}((\boldsymbol{\nabla}\psi^\dagger(\mathbf{r},t))\psi(\mathbf{r},t) - \psi^\dagger(\mathbf{r},t)\boldsymbol{\nabla}\psi(\mathbf{r},t))\right).
\end{aligned}
$$

Se concluye entonces que

$$
\frac{\partial \hat{n}(\mathbf{r},t)}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r},t) = 0 \tag{48}
$$

donde

$$
\mathbf{j}(\mathbf{r},t) = \frac{i\hbar}{2m}\big((\boldsymbol{\nabla}\psi^\dagger(\mathbf{r},t))\psi(\mathbf{r},t) - \psi^\dagger(\mathbf{r},t)\boldsymbol{\nabla}\psi(\mathbf{r},t)\big). \tag{49}
$$