Mecánica Estadística
Tarea 5: Teoría de Landau y Renormalización

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1. Teoría de Landau

1.1. Parámetro de orden

1. La derivada funcional de S es

$$\begin{aligned}\frac{\delta S}{\delta\phi(\mathbf{r})}[\phi] =& \frac{\delta H}{\delta\phi(\mathbf{r})}[\phi] - \frac{\delta}{\delta\phi(\mathbf{r})}\int \mathrm{d}^d\mathbf{r}'\, h(\mathbf{r}')\phi(\mathbf{r}') \\ =& \frac{\delta}{\delta\phi(\mathbf{r}')}\int \mathrm{d}^d\mathbf{r}'\left(\frac{1}{2}\|\boldsymbol{\nabla}\phi(\mathbf{r}')\|^2 + A\phi(\mathbf{r}')^2 + B\phi(\mathbf{r}')^4\right) \\ &- \int \mathrm{d}^d\mathbf{r}'\, h(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r}) \\ =& \int \mathrm{d}^d\mathbf{r}'\, \boldsymbol{\nabla}\phi(\mathbf{r}')\cdot\boldsymbol{\nabla}\delta(\mathbf{r}'-\mathbf{r}) \\ &+ \int \mathrm{d}^d\mathbf{r}'\left(2A\phi(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r}) + 4B\phi(\mathbf{r}')^3\delta(\mathbf{r}'-\mathbf{r})\right) - h(\mathbf{r}) \\ =& \int \mathrm{d}^d\mathbf{r}'\, \boldsymbol{\nabla}\phi(\mathbf{r}')\cdot\boldsymbol{\nabla}\delta(\mathbf{r}'-\mathbf{r}) + 2A\phi(\mathbf{r}) + 4B\phi(\mathbf{r})^3 - h(\mathbf{r}).\end{aligned} \tag{1}$$

Note que según la definición de derivada de una distribución se tiene

$$\begin{aligned}\boldsymbol{\nabla}\phi(\mathbf{r}')\cdot\boldsymbol{\nabla}\delta(\mathbf{r}'-\mathbf{r}) =& \sum_{i=1}^d \partial_i\phi(\mathbf{r}')\partial_i\delta(\mathbf{r}'-\mathbf{r}) = -\sum_{i=1}^d \partial_i^2\phi(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r}) \\ =& -\Delta\phi(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r}).\end{aligned} \tag{2}$$

Se concluye entonces que

$$\begin{aligned}\frac{\delta S}{\delta\phi(\mathbf{r})}[\phi] =& \int \mathrm{d}^d\mathbf{r}'\, \boldsymbol{\nabla}\phi(\mathbf{r}')\cdot\boldsymbol{\nabla}\delta(\mathbf{r}'-\mathbf{r}) + 2A\phi(\mathbf{r}) + 4B\phi(\mathbf{r})^3 - h(\mathbf{r}) \\ =& -\int \mathrm{d}^d\mathbf{r}'\, \Delta\phi(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r}) + 2A\phi(\mathbf{r}) + 4B\phi(\mathbf{r})^3 - h(\mathbf{r}) \\ =& -\Delta\phi(\mathbf{r}) + 2A\phi(\mathbf{r}) + 4B\phi(\mathbf{r})^3 - h(\mathbf{r}).\end{aligned} \tag{3}$$

En efecto, la condición de minimización $0 = \frac{\delta S}{\delta\phi(\mathbf{r})}[\phi_0]$ toma la forma

$$h(\mathbf{r}) = 2A\phi_0(\mathbf{r}) + 4B\phi_0(\mathbf{r})^3 - \Delta\phi_0(\mathbf{r}). \tag{4}$$

2. En el caso $h = 0$ en el que ϕ_0 es constante, $\Delta\phi = 0$ y por lo tanto

$$0 = 2A\phi_0 + 4B\phi_0^3 = 2\phi_0(A + 2B\phi_0^2). \tag{5}$$

   Se obtienen las soluciones $\phi_0 = 0$ y, si A y B tienen signos opuestos, $\phi_0 = \pm\sqrt{-\frac{A}{2B}}$.

3. En este caso, si $T > T_c$, se tiene $A > 0$. Por lo tanto, la única solución posible es $\phi_0 = 0$. Por el otro lado, si $T < T_c$, tanto la solución $\phi_0 = 0$ como las soluciones

$$\phi_0 = \pm\sqrt{\frac{A_1(T_c - T)}{B}} \tag{6}$$

   son permitidas. Sin embargo, solo consideramos la solución (6) ya que esta minimiza la energía libre. En efecto, ya que $h = 0$, se tiene

$$\begin{aligned} F =& H[\phi_0] = \int \mathrm{d}^d\mathbf{r}\,\left(A\phi^2 + B\phi^4\right) = \int \mathrm{d}^d\mathbf{r}\,\left(A + B\phi^2\right)\phi^2 \\ =& \begin{cases} 0, & \phi_0 = 0 \\ -\int \mathrm{d}^d\mathbf{r}\,(A - B\frac{A}{2B})\frac{A}{2B} = -\int \mathrm{d}^d\mathbf{r}\,\frac{A^2}{4B} < 0, & \phi_0 = \pm\sqrt{-\frac{A}{2B}}. \end{cases} \end{aligned} \tag{7}$$

1.2. Correlaciones

1. Tomando la derivada funcional y utilizando la relación vista en clase $G(\mathbf{r}', \mathbf{r}) = \frac{\delta}{\delta h(\mathbf{r})}\phi_0(\mathbf{r}')$ se tiene

$$\begin{aligned} \delta(\mathbf{r}' - \mathbf{r}) =& \frac{\delta}{\delta h(\mathbf{r})} h(\mathbf{r}') = \frac{\delta}{\delta h(\mathbf{r})}\big(2A\phi_0(\mathbf{r}') + 4B\phi_0(\mathbf{r}')^3 - \Delta\phi_0(\mathbf{r}')\big) \\ =& 2A\frac{\delta}{\delta h(\mathbf{r})}\phi_0(\mathbf{r}') + 12B\phi_0(\mathbf{r}')^2\frac{\delta}{\delta h(\mathbf{r})}\phi_0(\mathbf{r}') - \Delta\frac{\delta}{\delta h(\mathbf{r})}\phi_0(\mathbf{r}') \\ =& 2AG(\mathbf{r}', \mathbf{r}) + 12B\phi_0(\mathbf{r}')^2 G(\mathbf{r}', \mathbf{r}) - \Delta_{\mathbf{r}'}G(\mathbf{r}', \mathbf{r}). \end{aligned} \tag{8}$$

   Si $h = 0$ y el sistema es homogeneo, ϕ_0 es constante y $G(\mathbf{r}', \mathbf{r}) = G(\mathbf{r}' - \mathbf{r})$. Es claro entonces por la regla de la cadena que

$$\Delta_{\mathbf{r}'}G(\mathbf{r}', \mathbf{r}) = \Delta_{\mathbf{r}'}G(\mathbf{r}' - \mathbf{r}) = \Delta G(\mathbf{r}' - \mathbf{r}). \tag{9}$$

   Tomando

$$\xi^{-2} = 2A + 12B\phi_0^2 \tag{10}$$

   y haciendo el cambio de variable $\mathbf{r}' - \mathbf{r} \mapsto \mathbf{r}$ se concluye

$$\delta(\mathbf{r}) = (-\Delta + \xi^{-2})G(\mathbf{r}). \tag{11}$$

Podemos determinar

$$\begin{aligned}\xi =& \frac{1}{\sqrt{2A+12B\phi^2}} \\ =& \begin{cases} \frac{1}{\sqrt{2A-12B\frac{A}{2B}}} = \frac{1}{2\sqrt{-A}} = \frac{1}{2\sqrt{A_1(T_c-T)}}, & T < T_c \\ \frac{1}{\sqrt{2A}} = \frac{1}{\sqrt{2A_1(T-T_c)}}, & T > T_c. \end{cases}\end{aligned} \tag{12}$$

2. Utilizando la forma integral de la distribución delta e insertando la transformada de Fourier en (11) se tiene

$$\begin{aligned}\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{r}} =& \delta(\mathbf{r}) = (-\Delta+\xi^{-2}) \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{r}} \tilde{G}(\mathbf{k}) \\ =& \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} (-\Delta+\xi^{-2}) e^{i\mathbf{k}\cdot\mathbf{r}} \tilde{G}(\mathbf{k}) \\ =& \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} (\mathbf{k}^2+\xi^{-2}) e^{i\mathbf{k}\cdot\mathbf{r}} \tilde{G}(\mathbf{k})\end{aligned} \tag{13}$$

en vista de que $\Delta e^{i\mathbf{k}\cdot\mathbf{r}} = i\mathbf{k}\cdot i\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}} = -\mathbf{k}^2 e^{i\mathbf{k}\cdot\mathbf{r}}$. Comparando se obtiene que

$$\tilde{G}(\mathbf{k}) = \frac{1}{\mathbf{k}^2+\xi^{-2}}. \tag{14}$$

3. Utilizando el resultado anterior y restringiendo a $d = 3$ tenemos

$$\begin{aligned}G(\mathbf{r}) =& \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2+\xi^{-2}} \\ =& \frac{1}{(2\pi)^3} \int_0^\infty \mathrm{d}u \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, u^2 \sin(\theta) \frac{e^{iu\|\mathbf{r}\|\cos(\theta)}}{u^2+\xi^{-2}} \\ =& \frac{1}{(2\pi)^2} \int_0^\infty \mathrm{d}u \int_1^{-1} \mathrm{d}v\, (-1) u^2 \frac{e^{iu\|\mathbf{r}\|v}}{u^2+\xi^{-2}} \\ =& \frac{1}{i\|\mathbf{r}\|(2\pi)^2} \int_0^\infty \mathrm{d}u\, u \frac{e^{iu\|\mathbf{r}\|} - e^{-iu\|\mathbf{r}\|}}{u^2+\xi^{-2}}\end{aligned} \tag{15}$$

Note que bajo un cambio de variable

$$\begin{aligned}\int_0^\infty \mathrm{d}u\, u \frac{e^{-iu\|\mathbf{r}\|}}{u^2+\xi^{-2}} =& \int_0^{-\infty} \mathrm{d}u\, (-1)(-u) \frac{e^{iu\|\mathbf{r}\|}}{(-u)^2+\xi^{-2}} \\ =& -\int_{-\infty}^0 \mathrm{d}u\, u \frac{e^{iu\|\mathbf{r}\|}}{u^2+\xi^{-2}}.\end{aligned} \tag{16}$$

Por lo tanto

$$G(\mathbf{r}) = \frac{1}{i\|\mathbf{r}\|(2\pi)^2} \int_{-\infty}^\infty \mathrm{d}u\, \frac{u e^{iu\|\mathbf{r}\|}}{u^2+\xi^{-2}}. \tag{17}$$

Sea $R > i\xi^{-1}$. Considere bajo la orientación en contra de las manecillas del reloj los conjuntos

$$\begin{aligned} S_R^+ :=&\{Re^{i\theta} \in \mathbb{C} | \theta \in [0, \pi]\} \subseteq D_R := S_R^+ \cup [-R, R] \times \{0\} \\ \subseteq&\mathbb{C} \setminus \{i\xi^{-1}, -i\xi^{-1}\}, \end{aligned} \tag{18}$$

el dominio de definición de la función holomorfa

$$\begin{aligned} f : \mathbb{C} \setminus \{i\xi^{-1}, -i\xi^{-1}\} &\to \mathbb{C} \\ z &\mapsto \frac{ze^{iz\|\mathbf{r}\|}}{z^2 + \xi^{-2}} = \frac{ze^{iz\|\mathbf{r}\|}}{(z + i\xi^{-1})(z - i\xi^{-1})}. \end{aligned} \tag{19}$$

D_R solo encierra el polo $i\xi^{-1}$ y en este punto el residuo de f claramente es

$$\frac{i\xi^{-1}e^{ii\xi^{-1}\|\mathbf{r}\|}}{2i\xi^{-1}} = \frac{1}{2}e^{-\|\mathbf{r}\|/\xi}. \tag{20}$$

Por lo tanto por el teorema del residuo

$$\int_{D_R} dz f(z) = 2\pi i \frac{1}{2} e^{-\|\mathbf{r}\|/\xi} = i\pi e^{-\|\mathbf{r}\|/\xi}, \tag{21}$$

y

$$\begin{aligned} \int_{-\infty}^{\infty} du f(u) =& \lim_{R\to\infty} \left(\int_{D_R} dz f(z) - \int_{S_R^+} dz f(z) \right) \\ =&i\pi e^{-\|\mathbf{r}\|/\xi} - \lim_{R\to\infty} \int_{S_R^+} dz f(z) = i\pi e^{-\|\mathbf{r}\|/\xi}. \end{aligned} \tag{22}$$

Se concluye que

$$G(\mathbf{r}) = \frac{1}{i\|\mathbf{r}\|(2\pi)^2} i\pi e^{-\|\mathbf{r}\|/\xi} = \frac{e^{-\|\mathbf{r}\|/\xi}}{4\pi\|\mathbf{r}\|} \tag{23}$$

4. En el caso $d = 2$ tenemos tomando coordenadas polares con los ángulos medidos desde $\mathbf{r}$

$$\begin{aligned} G(\mathbf{r}) =& \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2 + \xi^{-2}} \\ =&\frac{1}{(2\pi)^2} \int_0^{\infty} \mathrm{d}u \int_0^{2\pi} \mathrm{d}\theta \, u \frac{e^{iu\|\mathbf{r}\| \cos(\theta)}}{u^2 + \xi^{-2}}. \end{aligned} \tag{24}$$

Recordando la expresión integral para la función de Bessel de orden 0

$$\begin{aligned} J_0(x) =& \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\tau \, e^{-ix\sin(\tau)} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\tau \, e^{-ix(-\cos(\tau))} \\ =& \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\tau \, e^{ix\cos(\tau)}, \end{aligned} \tag{25}$$

se tiene

$$G(\mathbf{r}) = \frac{1}{2\pi}\int_0^\infty \mathrm{d}u\,\frac{uJ_0(u\|\mathbf{r}\|)}{u^2+\xi^{-2}}. \tag{26}$$

Defina

$$\begin{aligned} g : \mathbb{R}_0^+ := \{x \in \mathbb{R} | x \geq 0\} &\to \mathbb{C} \\ x &\mapsto \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}u\,\frac{uJ_0(x)}{u^2+\xi^{-2}}. \end{aligned} \tag{27}$$

Entonces se tiene para todo $x \in \mathbb{R}^+ := \mathbb{R}_0^+ \setminus \{0\}$

$$g''(x) + \frac{1}{x}g'(x) - g(x) = \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}u\,\frac{u}{u^2+\xi^{-2}}\left(J_0''(x) + \frac{1}{x}J_0'(x) - J_0(x)\right) \tag{28}$$