

Tarea 3: Aplicaciones de mecánica estadística cuántica

Iván Mauricio Burbano Aldana
Universidad de los Andes

5 de mayo de 2018

1. El modelo XY cuántico en una dimensión

1.1. El modelo y preliminares

De la forma explícita de los operadores de spin vemos que podemos tomar como espacio de Hilbert de una partícula \mathbb{C}^2 . El espacio de Hilbert total es $\mathcal{H} = \bigotimes_{n=1}^N \mathbb{C}^2$ y los operadores de spin son

$$S_n^i = \text{id}_{\mathbb{C}^2} \otimes \cdots \otimes \text{id}_{\mathbb{C}^2} \otimes \underbrace{S^i}_{\text{n-ésimo termino}} \otimes \text{id}_{\mathbb{C}^2} \otimes \cdots \otimes \text{id}_{\mathbb{C}^2} \quad (1)$$

para todo $n \in \{1, \dots, N\}$ e $i \in \{x, y, z, +, -\}$. Entonces podemos escoger $|+\rangle = (1, 0)$ y $|-\rangle = (0, 1)$. Denotaremos para $s_1, \dots, s_N \in \{+, -\}$

$$|s_1 \cdots s_N\rangle = |s_1\rangle \otimes \cdots \otimes |s_N\rangle. \quad (2)$$

1. Calculando se obtiene

$$\begin{aligned} S^+ |-\rangle &= (S^x + iS^y) |-\rangle = \frac{1}{2} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |+\rangle \\ S^- |-\rangle &= (S^x - iS^y) |-\rangle = \frac{1}{2} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \\ S^+ |+\rangle &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \\ S^- |+\rangle &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |-\rangle. \end{aligned} \quad (3)$$

1.2. Desarrollo

1. Se tiene

$$\begin{aligned} \{S^+, S^-\} |+\rangle &= (S^+ S^- + S^- S^+) |+\rangle = |+\rangle + 0 = |+\rangle \\ \{S^+, S^-\} |-\rangle &= (S^+ S^- + S^- S^+) |-\rangle = 0 + |-\rangle = |-\rangle, \end{aligned} \quad (4)$$

es decir, $\{S^+, S^-\} = \text{id}_{\mathbb{C}^2}$. Entonces para todo $n \in \{1, \dots, N\}$ se tiene

$$\begin{aligned} \{S_n^+, S_n^-\} &= (S_n^+ S_n^- + S_n^- S_n^+) \\ &= \text{id}_{\mathbb{C}^2} \otimes \dots \otimes \text{id}_{\mathbb{C}^2} \otimes \underbrace{S^+ S^- + S^- S^+}_{\text{n-ésimo termino}} \otimes \text{id}_{\mathbb{C}^2} \otimes \dots \otimes \text{id}_{\mathbb{C}^2} \\ &= \text{id}_{\mathcal{H}}. \end{aligned} \quad (5)$$

2. Se tiene

$$2S^+ S^- - \text{id}_{\mathbb{C}^2} = 2S^+ S^- - \{S^+, S^-\} = [S^+, S^-] \quad (6)$$

y

$$\begin{aligned} [S^+, S^-] |+\rangle &= (S^+ S^- - S^- S^+) |+\rangle = |+\rangle \\ [S^+, S^-] |-\rangle &= (S^+ S^- - S^- S^+) |-\rangle = -|-\rangle. \end{aligned} \quad (7)$$

Se concluye entonces que $2S^+ S^- - \text{id}_{\mathbb{C}^2} = [S^+, S^-] = 2S^z$. Tensorizando la ecuación a ambos lados con identidades es claro que

$$2S_n^+ S_n^- - \text{id}_{\mathcal{H}} = [S_n^+, S_n^-] = 2S_n^z. \quad (8)$$

3. Note que para todo $n \in \{1, \dots, N-1\}$

$$\begin{aligned} S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+ &= (S_n^x + iS_n^y)(S_{n+1}^x - iS_{n+1}^y) + (S_n^x - iS_n^y)(S_{n+1}^x + iS_{n+1}^y) \\ &= S_n^x S_{n+1}^x - iS_n^x S_{n+1}^y + iS_n^y S_{n+1}^x + S_n^y S_{n+1}^y \\ &\quad + S_n^x S_{n+1}^x + iS_n^x S_{n+1}^y - iS_n^y S_{n+1}^x + S_n^y S_{n+1}^y \\ &= 2S_n^x S_{n+1}^x + 2S_n^y S_{n+1}^y. \end{aligned} \quad (9)$$

Entonces, en vista de (??), es claro que

$$H = \frac{J}{2} \sum_{i=1}^{N-1} (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) - h \sum_{n=1}^N [S_n^+, S_n^-]. \quad (10)$$

4. Note que para todo $n \in \{1, \dots, N\}$ el operador ϵ_n está bien definido ya que en vista de (??) los terminos en la productoria son $2S_1^z, \dots, 2S_{n-1}^z$ y conmutan. Entonces se tiene para $n \in \{1, \dots, N\}$ y $s_1, \dots, s_N \in \{+, -\}$

$$\begin{aligned} \epsilon_n |s_1 \dots s_N\rangle &= \prod_{j=1}^{n-1} S_j^z |s_1 \dots s_N\rangle = \prod_{j=1}^{n-1} \begin{Bmatrix} 1 & s_j = + \\ -1 & s_j = - \end{Bmatrix} |s_1 \dots s_N\rangle \\ &= (-1)^{|\{j \in \{1, \dots, n-1\} | s_j = -\}|} |s_1 \dots s_N\rangle = (-1)^A |s_1 \dots s_N\rangle. \end{aligned} \quad (11)$$

5. Se tiene que para todo $n \in \{1, \dots, N\}$ y $s_1, \dots, s_N \in \{+, -\}$

$$\epsilon_n^2 |s_1 \cdots s_N\rangle = (-1)^{2A} |s_1 \cdots s_N\rangle = |s_1 \cdots s_N\rangle. \quad (12)$$

Ya que $\{|s_1 \cdots s_N\rangle | s_1, \dots, s_N \in \{+, -\}\}$ genera \mathcal{H} se concluye que $\epsilon_n^2 = \text{id}_{\mathcal{H}}$. Por otra parte, se tiene que

$$\begin{aligned} 2S^z 2S^z |+\rangle &= |+\rangle \\ 2S^z 2S^z |-\rangle &= (-1)^2 |-\rangle = |-\rangle. \end{aligned} \quad (13)$$

Entonces es claro que $(2S^z)^2 = \text{id}_{\mathbb{C}^2}$ y por lo tanto $(2S_j^+ S_j^- - \text{id}_{\mathcal{H}})^2 = (2S_j^z)^2 = \text{id}_{\mathcal{H}}$ para todo $j \in \{1, \dots, N\}$.

6. Note que

$$\begin{aligned} (S^+)^2 |-\rangle &= S^+ |+\rangle = 0 \\ (S^-)^2 |-\rangle &= 0 \\ (S^+)^2 |+\rangle &= 0 \\ (S^-)^2 |+\rangle &= S^- |-\rangle = 0, \end{aligned} \quad (14)$$

es decir, $(S^+)^2 = (S^-)^2 = 0$. Por otra parte

$$\begin{aligned} \{2S^z, S^+\} |+\rangle &= (2S^z S^+ + 2S^+ S^z) |+\rangle = 0 \\ \{2S^z, S^+\} |-\rangle &= (2S^z S^+ + 2S^+ S^z) |-\rangle = |+\rangle - |+\rangle = 0 \\ \{2S^z, S^-\} |+\rangle &= (2S^z S^- + 2S^- S^z) |+\rangle = -|-\rangle + |-\rangle = 0 \\ \{2S^z, S^-\} |-\rangle &= (2S^z S^- + 2S^- S^z) |-\rangle = 0, \end{aligned} \quad (15)$$

es decir, para todo $n \in \{1, \dots, N\}$ se tiene $\{2S_n^z, S_n^{\pm}\} = 0$.

Dado que operadores en distintos sitios conmutan, los operadores S_n^i conmutan con ϵ_m para $n, m \in \{1, \dots, N\}$, $m \geq n$ e $i \in \{x, y, z, +, -\}$. Se tiene entonces $c_n^2 = \epsilon_n^2 (S_n^+)^2 = 0 = (S_n^-)^2 \epsilon_n^2 = (c_n^\dagger)^2$, $c_n^\dagger c_n = S_n^- S_n^+ \epsilon_n^2 = S_n^- S_n^+$, $c_n c_n^\dagger = S_n^+ S_n^- \epsilon_n^2 = S_n^+ S_n^-$ y $\{c_n^\dagger, c_n\} = \{S_n^-, S_n^+\} = \text{id}_{\mathcal{H}}$ para todo $n \in \{1, \dots, N\}$.

Tome $n, m \in \{1, \dots, N\}$ y suponga que $n > m$. Luego

$$\begin{aligned}
\{c_n^\dagger, c_m\} &= S_n^- \epsilon_n \epsilon_m S_m^+ + \epsilon_m S_m^+ S_n^- \epsilon_n \\
&= S_n^- \left(\prod_{j=1}^{n-1} 2S_j^z \right) S_m^+ \epsilon_m + \epsilon_m S_n^- S_m^+ \left(\prod_{j=1}^{n-1} 2S_j^z \right) \\
&= S_n^- \left(\prod_{\substack{j=1 \\ j \neq m}}^{n-1} 2S_j^z \right) 2S_m^z S_m^+ \epsilon_m + \epsilon_m S_n^- S_m^+ 2S_m^z \left(\prod_{\substack{j=1 \\ j \neq m}}^{n-1} 2S_j^z \right) \\
&= S_n^- \left(\prod_{\substack{j=1 \\ j \neq m}}^{n-1} 2S_j^z \right) \epsilon_m 2S_m^z S_m^+ + S_n^- \left(\prod_{\substack{j=1 \\ j \neq m}}^{n-1} 2S_j^z \right) \epsilon_m S_m^+ 2S_m^z \\
&= S_n^- \left(\prod_{\substack{j=1 \\ j \neq m}}^{n-1} 2S_j^z \right) \epsilon_m \{2S_m^z, S_m^+\} = 0
\end{aligned} \tag{16}$$

$$\begin{aligned}
\{c_n^\dagger, c_m^\dagger\} &= S_n^- \epsilon_n S_m^- \epsilon_m + S_m^- \epsilon_m S_n^- \epsilon_n \\
&= S_n^- \left(\prod_{j=1}^{n-1} 2S_j^z \right) S_m^- \epsilon_m + \epsilon_m S_n^- S_m^- \left(\prod_{j=1}^{n-1} 2S_j^z \right) \\
&= S_n^- \left(\prod_{\substack{j=1 \\ j \neq m}}^{n-1} 2S_j^z \right) 2S_m^z S_m^- \epsilon_m + \epsilon_m S_n^- S_m^- 2S_m^z \left(\prod_{\substack{j=1 \\ j \neq m}}^{n-1} 2S_j^z \right) \\
&= S_n^- \left(\prod_{\substack{j=1 \\ j \neq m}}^{n-1} 2S_j^z \right) \epsilon_m 2S_m^z S_m^- + S_n^- \left(\prod_{\substack{j=1 \\ j \neq m}}^{n-1} 2S_j^z \right) \epsilon_m S_m^- 2S_m^z \\
&= S_n^- \left(\prod_{\substack{j=1 \\ j \neq m}}^{n-1} 2S_j^z \right) \epsilon_m \{2S_m^z, S_m^-\} = 0
\end{aligned} \tag{17}$$

$$\begin{aligned}
\{c_n, c_m\} &= \epsilon_n S_n^+ \epsilon_m S_m^+ + \epsilon_m S_m^+ \epsilon_n S_n^+ \\
&= \left(\prod_{j=1}^{n-1} 2S_j^z \right) S_m^+ S_n^+ \epsilon_m + \epsilon_m S_m^+ \left(\prod_{j=1}^{n-1} 2S_j^z \right) S_n^+ \\
&= \left(\prod_{\substack{j=1 \\ j \neq m}}^{n-1} 2S_j^z \right) 2S_m^z S_m^+ S_n^+ \epsilon_m + \epsilon_m S_m^+ 2S_m^z \left(\prod_{\substack{j=1 \\ j \neq m}}^{n-1} 2S_j^z \right) S_n^+ \\
&= S_n^+ \left(\prod_{\substack{j=1 \\ j \neq m}}^{n-1} 2S_j^z \right) \epsilon_m 2S_m^z S_m^+ + S_n^+ \left(\prod_{\substack{j=1 \\ j \neq m}}^{n-1} 2S_j^z \right) \epsilon_m S_m^+ 2S_m^z \\
&= S_n^+ \left(\prod_{\substack{j=1 \\ j \neq m}}^{n-1} 2S_j^z \right) \epsilon_m \{2S_m^z, S_m^+\} = 0.
\end{aligned} \tag{18}$$

Las mismas relaciones son validas si $m > n$ pues

$$\{c_n^\dagger, c_m\} = (c_n^\dagger c_m + c_m c_n^\dagger) = ((c_m^\dagger c_n)^\dagger + (c_n c_m^\dagger)^\dagger) = \{c_m^\dagger, c_n\}^\dagger = 0 \tag{19}$$

y el anticonmutador es simétrico. Se concluye entonces que se satisfacen la relaciones de anticonmutación canónicas para $n, m \in \{1, \dots, N\}$

$$\begin{aligned}
\{c_n^\dagger, c_m\} &= \delta_{nm} \text{id}_{\mathcal{H}} \\
\{c_n^\dagger, c_m^\dagger\} &= 0 = \{c_n, c_m\}.
\end{aligned} \tag{20}$$

7. En el punto anterior se demostró que para todo $n \in \{1, \dots, N\}$ se tiene $c_n c_n^\dagger = S_n^+ S_n^-$ debido a que S_n^\pm conmuta con ϵ_n . Entonces

$$\left(\prod_{j=1}^{n-1} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) c_n = \left(\prod_{j=1}^{n-1} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) \left(\prod_{k=1}^{n-1} (2c_k c_k^\dagger - \text{id}_{\mathcal{H}}) \right) S_n^+. \tag{21}$$

Note que para $j, k \in \{1, \dots, N\}$ si $j \neq k$

$$\begin{aligned}
(\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j)(2c_k c_k^\dagger - \text{id}_{\mathcal{H}}) &= 2c_k c_k^\dagger - \text{id}_{\mathcal{H}} - 4c_j^\dagger c_j c_k c_k^\dagger + 2c_j^\dagger c_j \\
&= 2c_k c_k^\dagger - \text{id}_{\mathcal{H}} - 4c_k^\dagger c_k c_j c_j^\dagger + 2c_j^\dagger c_j \\
&= (2c_k c_k^\dagger - \text{id}_{\mathcal{H}})(\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j).
\end{aligned} \tag{22}$$

Entonces podemos reorganizar de manera que

$$\left(\prod_{j=1}^{n-1} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) c_n = \left(\prod_{j=1}^{n-1} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j)(2c_j c_j^\dagger - \text{id}_{\mathcal{H}}) \right) S_n^+. \tag{23}$$

Ahora bien, si $j \in \{1, \dots, N\}$ se tiene

$$\begin{aligned} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j)(2c_j c_j^\dagger - \text{id}_{\mathcal{H}}) &= 2c_j c_j^\dagger - \text{id}_{\mathcal{H}} - 4c_j^\dagger c_j c_j c_j^\dagger + 2c_j^\dagger c_j = 2\{c_j^\dagger, c_j\} - \text{id}_{\mathcal{H}} \\ &= \text{id}_{\mathcal{H}} \end{aligned} \quad (24)$$

donde se utilizó que $c_j c_j = c_j^\dagger c_j^\dagger = 0$ consecuencia de las relaciones de anticonmutación canónicas. Se confirma que

$$\left(\prod_{j=1}^{n-1} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) c_n = S_n^+. \quad (25)$$

De manera análoga tenemos

$$\begin{aligned} \left(\prod_{j=1}^{n-1} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) c_n^\dagger &= \left(\prod_{j=1}^{n-1} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) S_n^- \left(\prod_{k=1}^{n-1} (2c_k c_k^\dagger - \text{id}_{\mathcal{H}}) \right) \\ &= S_n^- \left(\prod_{j=1}^{n-1} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) \left(\prod_{k=1}^{n-1} (2c_k c_k^\dagger - \text{id}_{\mathcal{H}}) \right) \\ &= S_n^- \left(\prod_{j=1}^{n-1} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j)(2c_j c_j^\dagger - \text{id}_{\mathcal{H}}) \right) \\ &= S_n^- \left(\prod_{j=1}^{n-1} \text{id}_{\mathcal{H}} \right) = S_n^-. \end{aligned} \quad (26)$$

8. Reemplazando (??) y (??) en (??), notando que $c_j^\dagger c_j c_n^\dagger = (-1)^2 c_n^\dagger c_j^\dagger c_j = c_n^\dagger c_j^\dagger c_j$ y $c_n c_j^\dagger c_j = (-1)^2 c_j^\dagger c_j c_n = c_j^\dagger c_j c_n$ para todo $n, j \in \{0, \dots, N\}$ distintos, y calculando

$$\begin{aligned} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j)^2 &= \text{id}_{\mathcal{H}} - 4c_j^\dagger c_j + 4c_j^\dagger c_j c_j^\dagger c_j \\ &= \text{id}_{\mathcal{H}} - 4c_j^\dagger c_j + 4c_j^\dagger c_j (\{c_j^\dagger, c_j\} - c_j c_j^\dagger) \\ &= \text{id}_{\mathcal{H}} - 4c_j^\dagger c_j + 4c_j^\dagger c_j - 4c_j^\dagger c_j c_j c_j^\dagger = \text{id}_{\mathcal{H}} \end{aligned} \quad (27)$$

obtiene

$$\begin{aligned}
H &= \frac{J}{2} \sum_{n=1}^{N-1} \left(\left(\prod_{j=1}^{n-1} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) c_n \left(\prod_{j=1}^n (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) c_{n+1}^\dagger \right. \\
&\quad \left. + \left(\prod_{j=1}^{n-1} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) c_n^\dagger \left(\prod_{j=1}^n (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) c_{n+1} \right) - h \sum_{n=1}^N (2c_n c_n^\dagger - \text{id}_{\mathcal{H}}) \\
&= \frac{J}{2} \sum_{n=1}^{N-1} \left(c_n \left(\prod_{j=1}^{n-1} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) \left(\prod_{j=1}^n (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) c_{n+1}^\dagger \right. \\
&\quad \left. + c_n^\dagger \left(\prod_{j=1}^{n-1} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) \left(\prod_{j=1}^n (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) c_{n+1} \right) - h \sum_{n=1}^N (2c_n c_n^\dagger - \text{id}_{\mathcal{H}}) \\
&= \frac{J}{2} \sum_{n=1}^{N-1} \left(c_n \left(\prod_{j=1}^{n-1} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) (\text{id}_{\mathcal{H}} - 2c_n^\dagger c_n) c_{n+1}^\dagger \right. \\
&\quad \left. + c_n^\dagger \left(\prod_{j=1}^{n-1} (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) (\text{id}_{\mathcal{H}} - 2c_j^\dagger c_j) \right) (\text{id}_{\mathcal{H}} - 2c_n^\dagger c_n) c_{n+1} \right) \\
&\quad - h \sum_{n=1}^N (2c_n c_n^\dagger - \text{id}_{\mathcal{H}}) \\
&= \frac{J}{2} \sum_{n=1}^{N-1} \left(c_n (\text{id}_{\mathcal{H}} - 2c_n^\dagger c_n) c_{n+1}^\dagger + c_n^\dagger (\text{id}_{\mathcal{H}} - 2c_n^\dagger c_n) c_{n+1} \right) \\
&\quad - 2h \sum_{n=1}^N c_n c_n^\dagger - hN \text{id}_{\mathcal{H}} \\
&= \frac{J}{2} \sum_{n=1}^{N-1} \left(c_n c_{n+1}^\dagger - 2c_n c_n^\dagger c_n c_{n+1}^\dagger + c_n^\dagger c_{n+1} - 2c_n^\dagger c_n^\dagger c_n c_{n+1} \right) \\
&\quad - 2h \sum_{n=1}^N c_n c_n^\dagger - hN \text{id}_{\mathcal{H}} \\
&= \frac{J}{2} \sum_{n=1}^{N-1} \left(c_n c_{n+1}^\dagger - 2(\text{id}_{\mathcal{H}} - c_n^\dagger c_n) c_n c_{n+1}^\dagger + c_n^\dagger c_{n+1} \right) - 2h \sum_{n=1}^N c_n c_n^\dagger - hN \text{id}_{\mathcal{H}} \\
&= \frac{J}{2} \sum_{n=1}^{N-1} \left(-c_n c_{n+1}^\dagger + c_n^\dagger c_{n+1} \right) - 2h \sum_{n=1}^N c_n c_n^\dagger - hN \text{id}_{\mathcal{H}} \\
&= \frac{J}{2} \sum_{n=1}^{N-1} \left(c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1} \right) - 2h \sum_{n=1}^N c_n c_n^\dagger - hN \text{id}_{\mathcal{H}}.
\end{aligned} \tag{28}$$

9. El Hamiltoniano (??) se puede reescribir con la cotransformada de Fourier

$$\begin{aligned}
H &= -hN \text{id}_{\mathcal{H}} + \frac{J}{2} \sum_{n=1}^N \left(\frac{1}{N} \sum_{p=1}^N e^{ik_p(n+1)} \hat{c}_{k_p}^\dagger \sum_{p'=1}^N e^{-ik_{p'}n} \hat{c}_{k_{p'}} \right. \\
&\quad \left. + \frac{1}{N} \sum_{p=1}^N e^{ik_p n} \hat{c}_{k_p}^\dagger \sum_{p'=1}^N e^{-ik_{p'}(n+1)} \hat{c}_{k_{p'}} \right) \\
&\quad + 2h \sum_{n=1}^N \frac{1}{N} \sum_{p=1}^N e^{ik_p n} \hat{c}_{k_p}^\dagger \sum_{p'=1}^N e^{-ik_{p'}n} \hat{c}_{k_{p'}} \\
&= -hN \text{id}_{\mathcal{H}} + \frac{1}{N} \sum_{n=1}^N \sum_{p=1}^N \sum_{p'=1}^N \left(\frac{J}{2} \left(e^{ik_p(n+1)} \hat{c}_{k_p}^\dagger e^{-ik_{p'}n} \hat{c}_{k_{p'}} \right. \right. \\
&\quad \left. \left. + e^{ik_p n} \hat{c}_{k_p}^\dagger e^{-ik_{p'}(n+1)} \hat{c}_{k_{p'}} \right) + 2h e^{ik_p n} \hat{c}_{k_p}^\dagger e^{-ik_{p'}n} \hat{c}_{k_{p'}} \right) \\
&= -hN \text{id}_{\mathcal{H}} + \frac{1}{N} \sum_{n=1}^N \sum_{p=1}^N \sum_{p'=1}^N \left(\frac{J}{2} \left(e^{ik_p} e^{i(k_p-k_{p'})n} \right. \right. \\
&\quad \left. \left. + e^{i(k_p-k_{p'})n} e^{-ik_{p'}} \right) + 2h e^{i(k_p-k_{p'})n} \right) \hat{c}_{k_p}^\dagger \hat{c}_{k_{p'}} \\
&= -hN \text{id}_{\mathcal{H}} + \frac{1}{N} \sum_{n=1}^N \sum_{p=1}^N \sum_{p'=1}^N e^{i(k_p-k_{p'})n} \left(\frac{J}{2} (e^{ik_p} + e^{-ik_{p'}}) + 2h \right) \hat{c}_{k_p}^\dagger \hat{c}_{k_{p'}} \\
&= -hN \text{id}_{\mathcal{H}} + \frac{1}{N} \sum_{p=1}^N \sum_{p'=1}^N N \delta_{k_p k_{p'}} (2h + J \cos(k_p)) \hat{c}_{k_p}^\dagger \hat{c}_{k_{p'}} \\
&= -hN \text{id}_{\mathcal{H}} + \sum_{p=1}^N (2h + J \cos(k_p)) \hat{c}_{k_p}^\dagger \hat{c}_{k_p}
\end{aligned} \tag{29}$$

donde se utilizó $k_p := 2\pi p/N$ para todo $p \in \{1, \dots, N\}$.

10. Note que para $p \in \{0, \dots, N\}$

$$\begin{aligned}
H\hat{c}_{k_p}^\dagger |0\rangle &= -hN\hat{c}_{k_p}^\dagger |0\rangle + \sum_{p'=1}^N (2h + J \cos(k_{p'})) \hat{c}_{k_{p'}}^\dagger \hat{c}_{k_{p'}} \hat{c}_{k_p}^\dagger |0\rangle \\
&= -hN\hat{c}_{k_p}^\dagger |0\rangle + \sum_{\substack{p'=1 \\ p' \neq p}}^N (2h + J \cos(k_{p'})) (-1)^2 \hat{c}_{k_p}^\dagger \hat{c}_{k_{p'}} \hat{c}_{k_{p'}}^\dagger |0\rangle \\
&\quad + (2h + J \cos(k_p)) \hat{c}_{k_p}^\dagger \hat{c}_{k_p} \hat{c}_{k_p}^\dagger |0\rangle \\
&= -hN\hat{c}_{k_p}^\dagger |0\rangle + (2h + J \cos(k_p)) \hat{c}_{k_p}^\dagger (1 - \hat{c}_{k_p}^\dagger \hat{c}_{k_p}) |0\rangle \\
&= -hN\hat{c}_{k_p}^\dagger |0\rangle + (2h + J \cos(k_p)) \hat{c}_{k_p}^\dagger |0\rangle \\
&= (-hN + 2h + J \cos(k_p)) \hat{c}_{k_p}^\dagger |0\rangle.
\end{aligned} \tag{30}$$

Luego la energía propia correspondiente al número de onda k_p es $2h + J \cos(k_p)$.

11. Los vectores propios del operador número de ocupación son aquellos de la forma $(\hat{c}_{k_1}^\dagger)^{n_1} \dots (\hat{c}_{k_N}^\dagger)^{n_N} |0\rangle$ para $n_1, \dots, n_N \in \mathbb{N}$. Más aún, las relaciones canónicas de anticonmutación aseguran que $n_1, \dots, n_N \in \{0, 1\}$ ya que $(\hat{c}_{k_p}^\dagger)^2 = 0$ para todo $p \in \{1, \dots, N\}$. Para $p \in \{1, \dots, N\}$ se tiene

$$\begin{aligned}
&\hat{c}_{k_p}^\dagger \hat{c}_{k_p} (\hat{c}_{k_1}^\dagger)^{n_1} \dots (\hat{c}_{k_N}^\dagger)^{n_N} |0\rangle \\
&= (\hat{c}_{k_1}^\dagger)^{n_1} \dots (\hat{c}_{k_{p-1}}^\dagger)^{n_{p-1}} \hat{c}_{k_p}^\dagger \hat{c}_{k_p} (\hat{c}_{k_p}^\dagger)^{n_p} (\hat{c}_{k_{p+1}}^\dagger)^{n_{p+1}} \dots (\hat{c}_{k_N}^\dagger)^{n_N} |0\rangle \\
&= (\hat{c}_{k_1}^\dagger)^{n_1} \dots (\hat{c}_{k_{p-1}}^\dagger)^{n_{p-1}} \hat{c}_{k_p}^\dagger \\
&\quad (1 - \hat{c}_{k_p}^\dagger \hat{c}_{k_p}) (\hat{c}_{k_p}^\dagger)^{n_p-1} (\hat{c}_{k_{p+1}}^\dagger)^{n_{p+1}} \dots (\hat{c}_{k_N}^\dagger)^{n_N} |0\rangle \\
&= (\hat{c}_{k_1}^\dagger)^{n_1} \dots (\hat{c}_{k_{p-1}}^\dagger)^{n_{p-1}} \hat{c}_{k_p}^\dagger (\hat{c}_{k_p}^\dagger)^{n_p-1} (\hat{c}_{k_{p+1}}^\dagger)^{n_{p+1}} \dots (\hat{c}_{k_N}^\dagger)^{n_N} |0\rangle \\
&= (\hat{c}_{k_1}^\dagger)^{n_1} \dots (\hat{c}_{k_N}^\dagger)^{n_N} |0\rangle
\end{aligned} \tag{31}$$

si $n_p = 1$ y

$$\begin{aligned}
&\hat{c}_{k_p}^\dagger \hat{c}_{k_p} (\hat{c}_{k_1}^\dagger)^{n_1} \dots (\hat{c}_{k_N}^\dagger)^{n_N} |0\rangle \\
&= (\hat{c}_{k_1}^\dagger)^{n_1} \dots (\hat{c}_{k_{p-1}}^\dagger)^{n_{p-1}} \hat{c}_{k_p}^\dagger (\hat{c}_{k_{p+1}}^\dagger)^{n_{p+1}} \dots (\hat{c}_{k_N}^\dagger)^{n_N} \hat{c}_{k_p} |0\rangle = 0
\end{aligned} \tag{32}$$

si $n_p = 0$. Luego los valores propios del operador número de ocupación son 0 o 1 como es de esperarse para fermiones. Al notar que los vectores propios de el operador número de ocupación son los del Hamiltoniano por la expresión (??)

se tiene

$$\begin{aligned}
Z(\beta) &= \text{tr}(e^{-\beta H}) = \sum_{n_1=0}^1 \cdots \sum_{n_N=0}^1 \exp\left(\beta h N - \beta \sum_{p=1}^N (2h + J \cos k_p) n_p\right) \\
&= e^{\beta h N} \sum_{n_1=0}^1 \cdots \sum_{n_N=0}^1 \prod_{p=1}^N e^{-\beta(2h + J \cos k_p) n_p} \\
&= e^{\beta h N} \prod_{p=1}^N \sum_{n=0}^1 e^{-\beta(2h + J \cos k_p) n} = e^{\beta h N} \prod_{p=1}^N (1 + e^{-\beta(2h + J \cos k_p)})
\end{aligned} \tag{33}$$

12. Se tiene que la energía libre por spin del sistema es

$$\begin{aligned}
f(\beta) &:= -\frac{1}{\beta N} \ln(Z(\beta)) = -\frac{1}{\beta N} \left(\beta h N + \sum_{p=1}^N \ln(1 + e^{-\beta(2h + J \cos k_p)}) \right) \\
&= -h - \frac{1}{\beta N} \sum_{p=1}^N \ln(1 + e^{-\beta(2h + J \cos k_p)}).
\end{aligned} \tag{34}$$

En el límite termodinámico se tiene la densidad de estados

$$\frac{1}{N} \sum_{p=1}^N \rightarrow \int \frac{dk}{2\pi}. \tag{35}$$

Se concluye entonces

$$f(\beta) = -h - k_B T \int \ln(1 + e^{-\beta(2h + J \cos k)}) \frac{dk}{2\pi}. \tag{36}$$