Electrodynamics: Homework 3

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March 13, 2018

1. Vamos a asumir en este punto que el marco de referencia está alineado de manera que el primer eje corresponde a x, el segundo a y y el tercero a z. Ademas, los indices latinos perteneceran a $\{1,2,3\}$ mientras que los griegos a $\{0,1,2,3\}$. Se tiene entonces

$$\begin{aligned} E'_x =&E'_1 = F'^{01} = \Lambda^0_\sigma \Lambda^1_\rho F^{\sigma\rho} \\ =&\Lambda^0_0\Lambda^1_0 F^{00} + \Lambda^0_0\Lambda^1_i F^{0i} + \Lambda^0_i\Lambda^1_0 F^{i0} + \Lambda^0_i\Lambda^1_j F^{ij}. \end{aligned} \tag{1}$$

Por antisimetría $F^{00}=0$. Por otra parte, en vista de que $\mathbf{B}=0$ se tiene que $F^{ij}=0$ para todo $i,j\in\{1,2,3\}$. Entonces

$$\begin{aligned} E'_x =&\Lambda^0_0\Lambda^1_i F^{0i} + \Lambda^0_i\Lambda^1_0 F^{i0} = \gamma\left(\delta_{1i} + v_1 v_i \frac{\gamma-1}{\mathbf{v}^2}\right)E_i - \gamma^2 v_i v_1 F^{0i} \\ =&\gamma E_1 + \gamma v_1 v_i \frac{\gamma-1}{\mathbf{v}^2}E_i - \gamma^2 v_i v_1 E_i. \end{aligned} \tag{2}$$

En vista de que $v_1 = v$ y $v_2=v_3=0$ se tiene

$$\begin{aligned} E'_x =&\gamma E_1 + \left(\gamma\frac{\gamma-1}{v^2} - \gamma^2\right)v^2 E_1 = E_1\left(\gamma + \gamma^2 - \gamma - v^2\gamma^2\right) \\ =&E_1\gamma^2(1-v^2) = E_1 = E_x. \end{aligned} \tag{3}$$

2.

(i) Se tiene

$$
\begin{aligned}
F^i =& \frac{\mathrm{d}p^i}{\mathrm{d}t} = \frac{\mathrm{d}p^i}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = f^i\frac{\mathrm{d}\tau}{\mathrm{d}t} = e\frac{\mathrm{d}\tau}{\mathrm{d}t}\eta_{\beta\gamma}F^{i\beta}u^\gamma \\
=& e\frac{\mathrm{d}\tau}{\mathrm{d}t}\eta_{0\gamma}F^{i0}u^\gamma + e\frac{\mathrm{d}\tau}{\mathrm{d}t}\eta_{j\gamma}F^{ij}u^\gamma \\
=& -e\frac{\mathrm{d}\tau}{\mathrm{d}t}F^{i0}u^0 + e\frac{\mathrm{d}\tau}{\mathrm{d}t}F^{ij}u^j = e\frac{\mathrm{d}\tau}{\mathrm{d}t}E_i\frac{\mathrm{d}t}{\mathrm{d}\tau} + e\frac{\mathrm{d}\tau}{\mathrm{d}t}\epsilon_{pij}B_p\frac{\mathrm{d}x^j}{\mathrm{d}\tau} \\
=& e\frac{\mathrm{d}\tau}{\mathrm{d}t}E_i\frac{\mathrm{d}t}{\mathrm{d}\tau} + e\frac{\mathrm{d}\tau}{\mathrm{d}t}\epsilon_{pij}B_p\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = eE_i + e\frac{\mathrm{d}\tau}{\mathrm{d}t}\epsilon_{pij}B_pv_j\frac{\mathrm{d}t}{\mathrm{d}\tau} \\
=& eE_i + e\epsilon_{pij}B_pv_j = e(E_i + \epsilon_{ijp}v_jB_p) = e(\mathbf{E}_i + (\mathbf{v}\times\mathbf{B})_i) \\
=& e(\mathbf{E} + \mathbf{v}\times\mathbf{B})_i.
\end{aligned} \tag{4}
$$

Por lo tanto $\mathbf{F} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$.

(ii) De manera análoga

$$
\begin{aligned}
F^0 =& \frac{\mathrm{d}p^0}{\mathrm{d}t} = \frac{\mathrm{d}p^0}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = f^0\frac{\mathrm{d}\tau}{\mathrm{d}t} = e\frac{\mathrm{d}\tau}{\mathrm{d}t}\eta_{\beta\gamma}F^{0\beta}u^\gamma \\
=& e\frac{\mathrm{d}\tau}{\mathrm{d}t}\eta_{0\gamma}F^{00}u^\gamma + e\frac{\mathrm{d}\tau}{\mathrm{d}t}\eta_{j\gamma}F^{0j}u^\gamma = e\frac{\mathrm{d}\tau}{\mathrm{d}t}F^{0j}u^j \\
=& e\frac{\mathrm{d}\tau}{\mathrm{d}t}E_j\frac{\mathrm{d}x^j}{\mathrm{d}\tau} = e\frac{\mathrm{d}\tau}{\mathrm{d}t}E_j\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = eE_jv_j = e\mathbf{E}\cdot\mathbf{v}
\end{aligned} \tag{5}
$$

Por lo tanto $F^0 = e\mathbf{E}\cdot\mathbf{v}$.