Electrodynamics: Homework 1

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In this homework I will use the notation $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ for the canonical basis of $\mathbb{R}^3$. I will also make use of Stokes' Theorem, the Divergence Theorem, and the cyclicity of the triple vector product without mention.

1.

$$\begin{aligned}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{G}) &= \epsilon_{ijk}\partial_j(\boldsymbol{\nabla} \times \mathbf{G})_k\hat{\mathbf{e}}_i = \epsilon_{ijk}\partial_j\epsilon_{klm}\partial_l G_m\hat{\mathbf{e}}_i \\ &= \epsilon_{kij}\epsilon_{klm}\partial_j\partial_l G_m\hat{\mathbf{e}}_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(\partial_j\partial_l G_m)\hat{\mathbf{e}}_i \\ &= (\partial_j\partial_i G_j - \partial_j\partial_j G_i)\hat{\mathbf{e}}_i = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{G}) - \nabla^2\mathbf{G}\end{aligned} \tag{1}$$

2.

$$\begin{aligned}&(\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G} + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} + \mathbf{F} \times (\boldsymbol{\nabla} \times \mathbf{G}) + \mathbf{G} \times (\boldsymbol{\nabla} \times F) \\ &= (F_j\partial_j G_i + G_j\partial_j F_i + \epsilon_{ijk}F_j\epsilon_{klm}\partial_l G_m + \epsilon_{ijk}G_j\epsilon_{klm}\partial_l F_m)\hat{\mathbf{e}}_i \\ &= (F_j\partial_j G_i + G_j\partial_j F_i + \epsilon_{kij}\epsilon_{klm}(F_j\partial_l G_m + G_j\partial_l F_m))\hat{\mathbf{e}}_i \\ &= (F_j\partial_j G_i + G_j\partial_j F_i + (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(F_j\partial_l G_m + G_j\partial_l F_m))\hat{\mathbf{e}}_i \\ &= (F_j\partial_j G_i + G_j\partial_j F_i + F_j\partial_i G_j + G_j\partial_i F_j - F_j\partial_j G_i - G_j\partial_j F_i)\hat{\mathbf{e}}_i \\ &= (F_j\partial_i G_j + G_j\partial_i F_j)\hat{\mathbf{e}}_i = \partial_i(F_j G_j)\hat{\mathbf{e}}_i = \boldsymbol{\nabla}(\mathbf{F} \cdot \mathbf{G})\end{aligned} \tag{2}$$

3.

$$\begin{aligned}\int_V d^3x \boldsymbol{\nabla} \times \mathbf{F} &= \int_V d^3x \epsilon_{ijk}\partial_j F_k\hat{\mathbf{e}}_i = \int_V d^3x\partial_j(\epsilon_{jki}F_k\hat{\mathbf{e}}_i) \\ &= \int_V d^3x\partial_j(\epsilon_{jki}F_k\delta_{il})\hat{\mathbf{e}}_l = \int_V d^3x\partial_j(\mathbf{F} \times \hat{\mathbf{e}}_l)_j\hat{\mathbf{e}}_l \\ &= \int_V d^3x\boldsymbol{\nabla} \cdot (\mathbf{F} \times \hat{\mathbf{e}}_l)\hat{\mathbf{e}}_l = \int_{\partial V} dA\hat{\mathbf{n}} \cdot (\mathbf{F} \times \hat{\mathbf{e}}_l)\hat{\mathbf{e}}_l \\ &= \int_{\partial V} dA\hat{\mathbf{e}}_l \cdot (\hat{\mathbf{n}} \times \mathbf{F})\hat{\mathbf{e}}_l = \int_{\partial V} dA\hat{\mathbf{n}} \times \mathbf{F}\end{aligned} \tag{3}$$

4.

$$
\begin{aligned}
\int_{\partial S} d\mathbf{r}\psi &= \int_{\partial S} d\mathbf{r}\cdot\hat{\mathbf{e}}_i\psi\hat{\mathbf{e}}_i = \int_{\partial S} d\mathbf{r}\cdot(\psi\hat{\mathbf{e}}_i)\hat{\mathbf{e}}_i \\
&= \int_S dA\hat{\mathbf{n}}\cdot(\boldsymbol{\nabla}\times(\psi\hat{\mathbf{e}}_i))\hat{\mathbf{e}}_i = \int_S dA\hat{\mathbf{n}}\cdot(\epsilon_{klm}\partial_l\psi\delta_{mi}\hat{\mathbf{e}}_k)\hat{\mathbf{e}}_i \\
&= \int_S dA\hat{\mathbf{n}}\cdot(\boldsymbol{\nabla}\psi\times\hat{\mathbf{e}}_i)\hat{\mathbf{e}}_i = \int_S dA\hat{\mathbf{e}}_i\cdot(\hat{\mathbf{n}}\times\boldsymbol{\nabla}\psi)\hat{\mathbf{e}}_i \\
&= \int_S dA\hat{\mathbf{n}}\times\boldsymbol{\nabla}\psi
\end{aligned} \tag{4}
$$

5.

$$
\begin{aligned}
-\int_V d^3x\mathbf{F}\cdot\boldsymbol{\nabla}f + \int_{\partial V} dAf\mathbf{F}\cdot\hat{\mathbf{n}} = \\
-\int_V d^3x\mathbf{F}\cdot\boldsymbol{\nabla}f + \int_V d^3x\boldsymbol{\nabla}\cdot(f\mathbf{F}) = \\
\int_V d^3x(-\mathbf{F}\cdot\boldsymbol{\nabla}f + \partial_i(fF_i)) = \\
\int_V d^3x(-F_i\partial_i f + F_i\partial_i f + f\partial_i F_i) = \int_V d^3xf\partial_i F_i = \\
\int_V d^3xf\boldsymbol{\nabla}\cdot\mathbf{F}
\end{aligned} \tag{5}
$$