Mecánica Estadística
Tarea 5: Teoría de Landau y Renormalización

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1. Teoría de Landau

1.1. Parámetro de orden

1. La derivada funcional de S es

$$\begin{aligned}\frac{\delta S}{\delta\phi(\mathbf{r})}[\phi] =&\frac{\delta H}{\delta\phi(\mathbf{r})}[\phi]-\frac{\delta}{\delta\phi(\mathbf{r})}\int \mathrm{d}^d\mathbf{r}'\,h(\mathbf{r}')\phi(\mathbf{r}')\\ =&\frac{\delta}{\delta\phi(\mathbf{r}')}\int \mathrm{d}^d\mathbf{r}'\left(\frac{1}{2}\|\boldsymbol{\nabla}\phi(\mathbf{r}')\|^2+A\phi(\mathbf{r}')^2+B\phi(\mathbf{r}')^4\right)\\ &-\int \mathrm{d}^d\mathbf{r}'\,h(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r})\\ =&\int \mathrm{d}^d\mathbf{r}'\,\boldsymbol{\nabla}\phi(\mathbf{r}')\cdot\boldsymbol{\nabla}\delta(\mathbf{r}'-\mathbf{r})\\ &+\int \mathrm{d}^d\mathbf{r}'\left(2A\phi(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r})+4B\phi(\mathbf{r}')^3\delta(\mathbf{r}'-\mathbf{r})\right)-h(\mathbf{r})\\ =&\int \mathrm{d}^d\mathbf{r}'\,\boldsymbol{\nabla}\phi(\mathbf{r}')\cdot\boldsymbol{\nabla}\delta(\mathbf{r}'-\mathbf{r})+2A\phi(\mathbf{r})+4B\phi(\mathbf{r})^3-h(\mathbf{r}).\end{aligned}\tag{1}$$

Note que según la definición de derivada de una distribución se tiene

$$\begin{aligned}\boldsymbol{\nabla}\phi(\mathbf{r}')\cdot\boldsymbol{\nabla}\delta(\mathbf{r}'-\mathbf{r})=&\sum_{i=1}^{d}\partial_i\phi(\mathbf{r}')\partial_i\delta(\mathbf{r}'-\mathbf{r})=-\sum_{i=1}^{d}\partial_i^2\phi(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r})\\ =&-\Delta\phi(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r}).\end{aligned}\tag{2}$$

Se concluye entonces que

$$\begin{aligned}\frac{\delta S}{\delta\phi(\mathbf{r})}[\phi]=&\int \mathrm{d}^d\mathbf{r}'\,\boldsymbol{\nabla}\phi(\mathbf{r}')\cdot\boldsymbol{\nabla}\delta(\mathbf{r}'-\mathbf{r})+2A\phi(\mathbf{r})+4B\phi(\mathbf{r})^3-h(\mathbf{r})\\ =&-\int \mathrm{d}^d\mathbf{r}'\,\Delta\phi(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r})+2A\phi(\mathbf{r})+4B\phi(\mathbf{r})^3-h(\mathbf{r})\\ =&-\Delta\phi(\mathbf{r})+2A\phi(\mathbf{r})+4B\phi(\mathbf{r})^3-h(\mathbf{r}).\end{aligned}\tag{3}$$

En efecto, la condición de minimización $0 = \frac{\delta S}{\delta \phi(\mathbf{r})}[\phi_0]$ toma la forma

$$h(\mathbf{r}) = 2A\phi_0(\mathbf{r}) + 4B\phi_0(\mathbf{r})^3 - \Delta\phi_0(\mathbf{r}). \tag{4}$$

2. En el caso $h = 0$ en el que ϕ_0 es constante, $\Delta\phi = 0$ y por lo tanto

$$0 = 2A\phi_0 + 4B\phi_0^3 = 2\phi_0(A + 2B\phi_0^2). \tag{5}$$

   Se obtienen las soluciones $\phi_0 = 0$ y, si A y B tienen signos opuestos, $\phi_0 = \pm\sqrt{-\frac{A}{2B}}$.

3. En este caso, si $T > T_c$, se tiene $A > 0$. Por lo tanto, la única solución posible es $\phi_0 = 0$. Por el otro lado, si $T < T_c$, tanto la solución $\phi_0 = 0$ como las soluciones

$$\phi_0 = \pm\sqrt{\frac{A_1(T_c - T)}{B}} \tag{6}$$

   son permitidas. Sin embargo, solo consideramos la solución (6) ya que esta minimiza la energía libre. En efecto, ya que $h = 0$, se tiene

$$\begin{aligned} F =& H[\phi_0] = \int \mathrm{d}^d\mathbf{r}\, \left(A\phi^2 + B\phi^4\right) = \int \mathrm{d}^d\mathbf{r}\, \left(A + B\phi^2\right)\phi^2 \\ =& \begin{cases} 0, & \phi_0 = 0 \\ -\int \mathrm{d}^d\mathbf{r}\, (A - B\frac{A}{2B})\frac{A}{2B} = -\int \mathrm{d}^d\mathbf{r}\, \frac{A^2}{4B} < 0, & \phi_0 = \pm\sqrt{-\frac{A}{2B}}. \end{cases} \end{aligned} \tag{7}$$

1.2. Correlaciones

1. Tomando la derivada funcional y utilizando la relación vista en clase $G(\mathbf{r}', \mathbf{r}) = \frac{\delta}{\delta h(\mathbf{r})}\phi_0(\mathbf{r}')$ se tiene

$$\begin{aligned} \delta(\mathbf{r}' - \mathbf{r}) =& \frac{\delta}{\delta h(\mathbf{r})} h(\mathbf{r}') = \frac{\delta}{\delta h(\mathbf{r})}\big(2A\phi_0(\mathbf{r}') + 4B\phi_0(\mathbf{r}')^3 - \Delta\phi_0(\mathbf{r}')\big) \\ =& 2A\frac{\delta}{\delta h(\mathbf{r})}\phi_0(\mathbf{r}') + 12B\phi_0(\mathbf{r}')^2\frac{\delta}{\delta h(\mathbf{r})}\phi_0(\mathbf{r}') - \Delta\frac{\delta}{\delta h(\mathbf{r})}\phi_0(\mathbf{r}') \\ =& 2AG(\mathbf{r}', \mathbf{r}) + 12B\phi_0(\mathbf{r}')^2 G(\mathbf{r}', \mathbf{r}) - \Delta_{\mathbf{r}'}G(\mathbf{r}', \mathbf{r}). \end{aligned} \tag{8}$$

   Si $h = 0$ y el sistema es homogeneo, ϕ_0 es constante y $G(\mathbf{r}', \mathbf{r}) = G(\mathbf{r}' - \mathbf{r})$. Es claro entonces por la regla de la cadena que

$$\Delta_{\mathbf{r}'}G(\mathbf{r}', \mathbf{r}) = \Delta_{\mathbf{r}'}G(\mathbf{r}' - \mathbf{r}) = \Delta G(\mathbf{r}' - \mathbf{r}). \tag{9}$$

   Tomando

$$\xi^{-2} = 2A + 12B\phi_0^2 \tag{10}$$

   y haciendo el cambio de variable $\mathbf{r}' - \mathbf{r} \mapsto \mathbf{r}$ se concluye

$$\delta(\mathbf{r}) = (-\Delta + \xi^{-2})G(\mathbf{r}). \tag{11}$$

Podemos determinar

$$
\begin{aligned}
\xi =& \frac{1}{\sqrt{2A + 12B\phi^2}} \\
=& \begin{cases} \frac{1}{\sqrt{2A-12B\frac{A}{2B}}} = \frac{1}{2\sqrt{-A}} = \frac{1}{2\sqrt{A_1(T_c - T)}}, & T < T_c \\ \frac{1}{\sqrt{2A}} = \frac{1}{\sqrt{2A_1(T-T_c)}}, & T > T_c. \end{cases}
\end{aligned}
\tag{12}
$$

2. Utilizando la forma integral de la distribución delta e insertando la transformada de Fourier en (11) se tiene

$$
\begin{aligned}
\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{r}} =& \delta(\mathbf{r}) = (-\Delta + \xi^{-2}) \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{r}} \tilde{G}(\mathbf{k}) \\
=& \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} (-\Delta + \xi^{-2}) e^{i\mathbf{k}\cdot\mathbf{r}} \tilde{G}(\mathbf{k}) \\
=& \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} (\mathbf{k}^2 + \xi^{-2}) e^{i\mathbf{k}\cdot\mathbf{r}} \tilde{G}(\mathbf{k})
\end{aligned}
\tag{13}
$$

en vista de que $\Delta e^{i\mathbf{k}\cdot\mathbf{r}} = i\mathbf{k} \cdot i\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} = -\mathbf{k}^2 e^{i\mathbf{k}\cdot\mathbf{r}}$. Comparando se obtiene que

$$
\tilde{G}(\mathbf{k}) = \frac{1}{\mathbf{k}^2 + \xi^{-2}}. \tag{14}
$$

3. Utilizando el resultado anterior y restringiendo a $d = 3$ tenemos

$$
\begin{aligned}
G(\mathbf{r}) =& \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2 + \xi^{-2}} \\
=& \frac{1}{(2\pi)^3} \int_0^\infty \mathrm{d}u \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \, u^2 \sin(\theta) \frac{e^{iu\|\mathbf{r}\|\cos(\theta)}}{u^2 + \xi^{-2}}.
\end{aligned}
\tag{15}
$$