Mecánica Cuántica Avanzada: Tarea 4

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3.3

Se tiene que

$$
\begin{aligned}
\partial_i\phi(t,\mathbf{y}) =&\partial_i\int\frac{\mathrm{d}^3\mathbf{p}}{\sqrt{(2\pi)^3 2E_p}}\big(a(p)e^{-ip\cdot x}+a^\dagger(p)e^{ip\cdot x}\big)\\
=&\int\frac{\mathrm{d}^3\mathbf{p}}{\sqrt{(2\pi)^3 2E_p}}\big(a(p)\partial_i e^{-ip\cdot x}+a^\dagger(p)\partial_i e^{ip\cdot x}\big)\\
=&\int\frac{\mathrm{d}^3\mathbf{p}}{\sqrt{(2\pi)^3 2E_p}}\big(a(p)(-ip_i)e^{-ip\cdot x}+a^\dagger(p)ip_i e^{ip\cdot x}\big)\\
=&i\int\frac{\mathrm{d}^3\mathbf{p}}{\sqrt{(2\pi)^3 2E_p}}p_i\big(-a(p)e^{-ip\cdot x}+a^\dagger(p)e^{ip\cdot x}\big).
\end{aligned}
\tag{1}
$$

Por lo tanto

$$
\begin{aligned}
\phi_i(t,\mathbf{x})\partial_i\phi(t,\mathbf{y}) =&\int\frac{\mathrm{d}^3\mathbf{p}}{\sqrt{(2\pi)^3 2E_p}}\big(a(p)e^{-ip\cdot x}+a^\dagger(p)e^{ip\cdot x}\big)\times\\
&i\int\frac{\mathrm{d}^3\mathbf{p}}{\sqrt{(2\pi)^3 2E_p}}p_i\big(-a(p)e^{-ip\cdot y}+a^\dagger(p)e^{ip\cdot y}\big)\\
=&i\int\frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2E_p}p_i'\times\\
&\big(a(p)e^{-ip\cdot x}+a^\dagger(p)e^{ip\cdot x}\big)\Big(-a(p')e^{-ip'\cdot y}+a^\dagger(p')e^{ip'\cdot y}\Big)\\
=&i\int\frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2E_p}p_i'\times\\
&\Big(-a(p)a(p')e^{-ip\cdot x}e^{-ip'\cdot y}+a(p)a^\dagger(p')e^{-ip\cdot x}e^{ip'\cdot y}\\
&-a^\dagger(p)a(p')e^{ip\cdot x}e^{-ip'\cdot y}+a^\dagger(p)a^\dagger(p')e^{ip\cdot x}e^{ip'\cdot y}\Big).
\end{aligned}
\tag{2}
$$

De manera similar

$$\begin{aligned}
\partial_i\phi(t,\mathbf{y})\phi_i(t,\mathbf{x}) =& i\int \frac{\mathrm{d}^3\mathbf{p}}{\sqrt{(2\pi)^3 2E_p}} p_i\big(-a(p)e^{-ip\cdot y} + a^\dagger(p)e^{ip\cdot y}\big)\times \\
& \int \frac{\mathrm{d}^3\mathbf{p}}{\sqrt{(2\pi)^3 2E_p}}\big(a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}\big) \\
=& i\int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p'}}{(2\pi)^3 2E_p} p'_i\times \\
& \Big(-a(p')e^{-ip'\cdot y} + a^\dagger(p')e^{ip'\cdot y}\Big)\big(a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}\big) \\
=& i\int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p'}}{(2\pi)^3 2E_p} p'_i\times \\
& \Big(-a(p')a(p)e^{-ip'\cdot y}e^{-ip\cdot x} - a(p')a^\dagger(p)e^{-ip'\cdot y}e^{ip\cdot x} \\
& +a^\dagger(p')a(p)e^{ip'\cdot y}e^{-ip\cdot x} + a^\dagger(p')a^\dagger(p)e^{ip'\cdot y}e^{ip\cdot x}\Big).
\end{aligned} \tag{3}$$

Entonces podemos calcular el conmutador

$$\begin{aligned}
[\phi_i(t,\mathbf{x}), \partial_i\phi(t,\mathbf{y})]_- =& \phi_i(t,\mathbf{x})\partial_i\phi(t,\mathbf{y}) - \partial_i\phi(t,\mathbf{y})\phi_i(t,\mathbf{x}) \\
=& i\int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p'}}{(2\pi)^3 2E_p} p'_i\times \\
& \Big(-a(p)a(p')e^{-ip\cdot x}e^{-ip'\cdot y} + a(p)a^\dagger(p')e^{-ip\cdot x}e^{ip'\cdot y} \\
& -a^\dagger(p)a(p')e^{ip\cdot x}e^{-ip'\cdot y} + a^\dagger(p)a^\dagger(p')e^{ip\cdot x}e^{ip'\cdot y} \\
& a(p')a(p)e^{-ip'\cdot y}e^{-ip\cdot x} + a(p')a^\dagger(p)e^{-ip'\cdot y}e^{ip\cdot x} \\
& -a^\dagger(p')a(p)e^{ip'\cdot y}e^{-ip\cdot x} - a^\dagger(p')a^\dagger(p)e^{ip'\cdot y}e^{ip\cdot x}\Big). \\
& = i\int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p'}}{(2\pi)^3 2E_p} p'_i\times \\
& \Big(e^{-ip\cdot x}e^{-ip'\cdot y}(-a(p)a(p') + a(p')a(p)) \\
& +e^{-ip\cdot x}e^{ip'\cdot y}(a(p)a^\dagger(p') - a^\dagger(p')a(p)) \\
& +e^{ip\cdot x}e^{-ip'\cdot y}(-a^\dagger(p)a(p') + a(p')a^\dagger(p)) \\
& +e^{ip\cdot x}e^{ip'\cdot y}(a^\dagger(p)a^\dagger(p') - a^\dagger(p')a^\dagger(p))\Big) \\
& = i\int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p'}}{(2\pi)^3 2E_p} p'_i\times \\
& \Big(-e^{-ip\cdot x}e^{-ip'\cdot y}[a(p), a(p')]_- + \\
& e^{-ip\cdot x}e^{ip'\cdot y}[a(p), a^\dagger(p')]_- - e^{ip\cdot x}e^{-ip'\cdot y}[a(p'), a^\dagger(p)] \\
& e^{ip\cdot x}e^{ip'\cdot y}[a^\dagger(p), a^\dagger(p')]\Big)
\end{aligned} \tag{4}$$

Utilizando las relaciones de conmutación

$$\begin{aligned}[a(p), a(p')]_- =&0 = [a^\dagger(p), a^\dagger(p')]_- \\ [a(p), a^\dagger(p')]_- =&\delta^3(\mathbf{p}-\mathbf{p}')\end{aligned} \tag{5}$$

se tiene

$$\begin{aligned}[\phi_i(t,\mathbf{x}), \partial_i\phi(t,\mathbf{y})]_- =& i\int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2E_p} p'_i \times \\ &\Big(e^{-ip\cdot x}e^{ip'\cdot y}\delta^3(\mathbf{p}-\mathbf{p}') - e^{ip\cdot x}e^{-ip'\cdot y}\delta^3(\mathbf{p}'-\mathbf{p})\Big) \\ =& i\int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2E_p} p'_i \times \\ &\Big(e^{-ip\cdot x}e^{ip'\cdot y} - e^{ip\cdot x}e^{-ip'\cdot y}\Big)\delta^3(\mathbf{p}'-\mathbf{p}) \\ =& i\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_p} p_i\big(e^{-ip\cdot x}e^{ip\cdot y} - e^{ip\cdot x}e^{-ip\cdot y}\big) \\ =& i\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_p} p_i\Big(e^{ip\cdot(y-x)} - e^{-ip\cdot(y-x)}\Big).\end{aligned} \tag{6}$$

Ya que el conmutador se toma en tiempos iguales se tiene que

$$p\cdot(y-x) = p_0(t-t) - \mathbf{p}\cdot(\mathbf{y}-\mathbf{x}) = -\mathbf{p}\cdot(\mathbf{y}-\mathbf{x}) = \mathbf{p}\cdot(\mathbf{x}-\mathbf{y}). \tag{7}$$

Por lo tanto

$$\begin{aligned}[\phi_i(t,\mathbf{x}), \partial_i\phi(t,\mathbf{y})]_- =& i\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_p} p_i\Big(e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\Big) \\ =& i\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_p} p_i 2\cos(\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})) \\ =& -i\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 E_p} p^i\cos(\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})).\end{aligned} \tag{8}$$

Note que la función

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R} \\ \mathbf{p} &\mapsto p^i\cos(\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}))\end{aligned} \tag{9}$$

es impar, es decir, $f(-\mathbf{p}) = -f(\mathbf{p})$. Se concluye entonces que su integral se debe anular y por lo tanto

$$[\phi_i(t,\mathbf{x}), \partial_i\phi(t,\mathbf{y})]_- = 0. \tag{10}$$

Debo aclarar que estos cálculos solo cobran sentido en el contexto de integrales oscilatorias.

3.4

Para invertir la relación de Fourier recuerde

$$\int \frac{\mathrm{d}^3x}{(2\pi)^3} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} = \delta^3(\mathbf{p}-\mathbf{p}') = \delta^3(\mathbf{p}'-\mathbf{p}) = \int \frac{\mathrm{d}^3x}{(2\pi)^3} e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}. \tag{11}$$

Por lo tanto

$$\begin{aligned}
\int \mathrm{d}^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}}\phi(x) &= \int \frac{\mathrm{d}^3x\, \mathrm{d}^3p'}{\sqrt{(2\pi)^3 2E_{p'}}} e^{-i\mathbf{p}\cdot\mathbf{x}}(a(p')e^{-ip'\cdot x} + a^\dagger(p')e^{ip'\cdot x}) \\
&= \int \frac{\mathrm{d}^3x\, \mathrm{d}^3p'}{\sqrt{(2\pi)^3 2E_{p'}}} (2\pi)^3 \Bigg(a(p')e^{-iE_{p'}t}\frac{e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}}{(2\pi)^3} \\
&\quad + a^\dagger(p')e^{iE_{p'}t}\frac{e^{-i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}}}{(2\pi)^3}\Bigg) \\
&= \int \mathrm{d}^3p' \sqrt{\frac{(2\pi)^3}{2E_{p'}}}(a(p')e^{-iE_{p'}t}\delta^3(\mathbf{p}-\mathbf{p}') \\
&\quad + a^\dagger(p')e^{iE_{p'}t}\delta^3(\mathbf{p}+\mathbf{p}')) \\
&= \sqrt{\frac{(2\pi)^3}{2E_p}}(a(p)e^{-iE_pt} + a^\dagger(\tilde{p})e^{iE_pt})
\end{aligned} \tag{12}$$

y

$$\begin{aligned}
\int \mathrm{d}^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}}\Pi(x) &= \int \mathrm{d}^3x\, \mathrm{d}^3p'\, i\sqrt{\frac{E_{p'}}{(2\pi)^3 2}} e^{-i\mathbf{p}\cdot\mathbf{x}}(-a(p')e^{-ip'\cdot x} \\
&\quad + a^\dagger(p')e^{ip'\cdot x}) \\
&= \int \mathrm{d}^3x\, \mathrm{d}^3p'\, i\sqrt{\frac{E_{p'}}{(2\pi)^3 2}}(2\pi)^3 \\
&\quad \Bigg(-a(p')e^{-iE_{p'}t}\frac{e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}}{(2\pi)^3} \\
&\quad + a^\dagger(p')e^{iE_{p'}t}\frac{e^{-i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}}}{(2\pi)^3}\Bigg) \\
&= \int \mathrm{d}^3p'\, i\sqrt{\frac{(2\pi)^3E_{p'}}{2}}(-a(p')e^{-iE_{p'}t}\delta^3(\mathbf{p}-\mathbf{p}') \\
&\quad + a^\dagger(p')e^{iE_{p'}t}\delta^3(\mathbf{p}+\mathbf{p}')) \\
&= i\sqrt{\frac{(2\pi)^3E_p}{2}}(-a(p)e^{-iE_pt} + a^\dagger(\tilde{p})e^{iE_pt})
\end{aligned} \tag{13}$$

donde donde $\tilde{p} = (E_p, -\mathbf{p})$. Por lo tanto

$$\begin{aligned} a(p) =& \frac{1}{\sqrt{(2\pi)^3 2E_p}} \int \mathrm{d}^3x \,(E_p\phi(x)e^{-i\mathbf{p}\cdot\mathbf{x}}e^{iE_pt} + i\Pi(x)e^{-i\mathbf{p}\cdot\mathbf{x}}e^{iE_pt}) \\ =& \frac{1}{\sqrt{(2\pi)^3 2E_p}} \int \mathrm{d}^3x \, e^{ip\cdot x}(E_p\phi(x) + i\Pi(x)) \end{aligned} \tag{14}$$

y

$$a^\dagger(p) = \frac{1}{\sqrt{(2\pi)^3 2E_p}} \int \mathrm{d}^3x \, e^{-ip\cdot x}(E_p\phi(x) - i\Pi(x)). \tag{15}$$

Se concluye entonces

$$\begin{aligned} [a(p), a(p')]_- =& \frac{1}{(2\pi)^3 2\sqrt{E_pE_{p'}}} \int \mathrm{d}^3x \, \mathrm{d}^3y \, e^{i(p\cdot x + p'\cdot y)}[E_p\phi(x) + i\Pi(x) \\ & , E_{p'}\phi(y) + i\Pi(y)] \\ =& \frac{1}{(2\pi)^3 2\sqrt{E_pE_{p'}}} \int \mathrm{d}^3x \, \mathrm{d}^3y \, e^{i(p\cdot x + p'\cdot y)} \\ & (E_pE_{p'}[\phi(x), \phi(y)]_- + iE_p[\phi(x), \Pi(y)]_- \\ & + iE_{p'}[\Pi(x), \phi(y)]_- - [\Pi(x), \Pi(y)]_-), \end{aligned} \tag{16}$$

$$\begin{aligned} [a^\dagger(p), a^\dagger(p')]_- =& \frac{1}{(2\pi)^3 2\sqrt{E_pE_{p'}}} \int \mathrm{d}^3x \, \mathrm{d}^3y \, e^{-i(p\cdot x + p'\cdot y)}[E_p\phi(x) - i\Pi(x) \\ & , E_{p'}\phi(y) - i\Pi(y)] \\ =& \frac{1}{(2\pi)^3 2\sqrt{E_pE_{p'}}} \int \mathrm{d}^3x \, \mathrm{d}^3y \, e^{-i(p\cdot x + p'\cdot y)} \\ & (E_pE_{p'}[\phi(x), \phi(y)]_- - iE_p[\phi(x), \Pi(y)]_- \\ & - iE_{p'}[\Pi(x), \phi(y)]_- - [\Pi(x), \Pi(y)]_-) \end{aligned} \tag{17}$$

y

$$\begin{aligned} [a(p), a^\dagger(p')]_- =& \frac{1}{(2\pi)^3 2\sqrt{E_pE_{p'}}} \int \mathrm{d}^3x \, \mathrm{d}^3y \, e^{i(p\cdot x - p'\cdot y)}[E_p\phi(x) + i\Pi(x) \\ & , E_{p'}\phi(y) - i\Pi(y)] \\ =& \frac{1}{(2\pi)^3 2\sqrt{E_pE_{p'}}} \int \mathrm{d}^3x \, \mathrm{d}^3y \, e^{i(p\cdot x - p'\cdot y)} \\ & (E_pE_{p'}[\phi(x), \phi(y)]_- - iE_p[\phi(x), \Pi(y)]_- \\ & + iE_{p'}[\Pi(x), \phi(y)]_- + [\Pi(x), \Pi(y)]_-). \end{aligned} \tag{18}$$

Para evaluar estos conmutadores podemos utilizar las relaciones en tiempos simultaneos. Esto se debe a que los operadores de creación y aniquilación son constantes. En efecto, utilizando la ecuación de Klein-Gordon tenemos

$$\begin{aligned}\frac{\mathrm{d}a(p)}{\mathrm{d}t} =&\frac{1}{\sqrt{(2\pi)^3 2E_p}}\int \mathrm{d}^3x\, iE_p e^{ip\cdot x}(E_p\phi(x)+i\Pi(x))\\ &+\frac{1}{\sqrt{(2\pi)^3 2E_p}}\int \mathrm{d}^3x\, e^{ip\cdot x}(E_p\dot{\phi}(x)+i\dot{\Pi}(x))\\ =&\frac{1}{\sqrt{(2\pi)^3 2E_p}}\int \mathrm{d}^3x\, E_p e^{ip\cdot x}(iE_p\phi(x)-\dot{\phi}(x)+\dot{\phi(x)}+\frac{i}{E_p}\ddot{\phi}(x))\\ =&\frac{1}{\sqrt{(2\pi)^3 2E_p}}\int \mathrm{d}^3x\, ie^{ip\cdot x}(E_p^2\phi(x)+\ddot{\phi}(x))\\ =&\frac{1}{\sqrt{(2\pi)^3 2E_p}}\int \mathrm{d}^3x\, ie^{ip\cdot x}(E_p^2\phi(x)+\Delta\phi(x)-m^2\phi(x)).\end{aligned} \tag{19}$$

Note que si los campos decaen en el infinito

$$\begin{aligned}\int \mathrm{d}x^i\, e^{\pm ip\cdot x}\partial_i^2\phi(x) =& -\int \mathrm{d}x^i\,(\pm)ip_i e^{\pm ip\cdot x}\partial_i\phi(x)\\ =&\int \mathrm{d}x^i\,(ip_i)^2 e^{\pm ip\cdot x}\phi(x).\end{aligned} \tag{20}$$

Por lo tanto

$$\frac{\mathrm{d}a(p)}{\mathrm{d}t} =\frac{1}{\sqrt{(2\pi)^3 2E_p}}\int \mathrm{d}^3x\, ie^{ip\cdot x}(E_p^2-\mathbf{p}^2-m^2)\phi(x)=0. \tag{21}$$

Además,

$$\frac{\mathrm{d}a^\dagger(p)}{\mathrm{d}t}=\left(\frac{\mathrm{d}a(p)}{\mathrm{d}t}\right)^\dagger=0. \tag{22}$$

Sabiendo esto, concluimos

$$\begin{aligned}[a(p),a(p')]_- =&\frac{1}{(2\pi)^3 2\sqrt{E_pE_{p'}}}\int \mathrm{d}^3x\,\mathrm{d}^3y\, e^{i(p\cdot x+p'\cdot y)}\\ &(-E_p\delta^3(\mathbf{x}-\mathbf{y})+E_{p'}\delta^3(\mathbf{x}-\mathbf{y}))\\ =&\frac{1}{(2\pi)^3 2\sqrt{E_pE_{p'}}}\int \mathrm{d}^3x\, e^{i(p+p')\cdot x}(E_{p'}-E_p)\\ =&\frac{1}{2\sqrt{E_pE_{p'}}}\delta^3(\mathbf{p}+\mathbf{p}')(E_{p'}-E_p)\\ =&\frac{1}{2\sqrt{E_pE_{p'}}}\delta^3(\mathbf{p}+\mathbf{p}')(E_p-E_p)=0,\end{aligned} \tag{23}$$

$$
\begin{aligned}
[a^\dagger(p), a^\dagger(p')]_- =& \frac{1}{(2\pi)^3 2\sqrt{E_p E_{p'}}} \int \mathrm{d}^3x\, \mathrm{d}^3y\, e^{-i(p\cdot x + p'\cdot y)} \\
& (E_p \delta^3(\mathbf{x}-\mathbf{y}) - E_{p'}\delta^3(\mathbf{x}-\mathbf{y})) \\
=& \frac{1}{(2\pi)^3 2\sqrt{E_p E_{p'}}} \int \mathrm{d}^3x\, e^{-i(p+p')\cdot x}(E_p - E_{p'}) \\
=& \frac{1}{2\sqrt{E_p E_{p'}}}\delta^3(\mathbf{p}+\mathbf{p}')(E_p - E_{p'}) \\
=& \frac{1}{2\sqrt{E_p E_{p'}}}\delta^3(\mathbf{p}+\mathbf{p}')(E_p - E_p) = 0
\end{aligned} \tag{24}
$$

y

$$
\begin{aligned}
[a(p), a^\dagger(p')]_- =& \frac{1}{(2\pi)^3 2\sqrt{E_p E_{p'}}} \int \mathrm{d}^3x\, \mathrm{d}^3y\, e^{i(p\cdot x - p'\cdot y)} \\
& (E_p \delta^3(\mathbf{x}-\mathbf{y}) + E_{p'}\delta^3(\mathbf{x}-\mathbf{y})) \\
=& \frac{1}{(2\pi)^3 2\sqrt{E_p E_{p'}}} \int \mathrm{d}^3x\, e^{i(p-p')\cdot x}(E_p + E_{p'} \\
=& \frac{1}{2\sqrt{E_p E_{p'}}}\delta^3(\mathbf{p}-\mathbf{p}')(E_p + E_{p'}) \\
=& \frac{2E_p}{2E_p}\delta^3(\mathbf{p}-\mathbf{p}') = \delta^3(\mathbf{p}-\mathbf{p}').
\end{aligned} \tag{25}
$$

3.5

a)

En primer lugar necesitamos construir el tensor de energía-momento. Para el campo real masivo el Lagrangiano es

$$
\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2. \tag{26}
$$

Por lo tanto, el tensor de energía-momento es

$$
\begin{aligned}
T^{\mu\nu} =& \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L} \\
=& (\partial^\mu\phi)(\partial^\nu\phi) - \frac{1}{2}g^{\mu\nu}(\partial_\sigma\phi)(\partial^\sigma\phi) + \frac{1}{2}g^{\mu\nu}m^2\phi^2.
\end{aligned} \tag{27}
$$

De acá concluimos que el momento del campo es

$$
\begin{aligned}
P^\mu =& \int \mathrm{d}^3x\, T^{0\mu}(x) \\
=& \int \mathrm{d}^3x \left(\dot\phi(x)\partial^\nu\phi(x) - \frac{1}{2}g^{0\nu}(\partial_\sigma\phi)(\partial^\sigma\phi) + \frac{1}{2}g^{0\nu}m^2\phi(x)^2\right) \\
=& \int \mathrm{d}^3x \left(\Pi(x)\partial^\nu\phi(x) - \frac{1}{2}g^{0\nu}(\partial_\sigma\phi)(\partial^\sigma\phi) + \frac{1}{2}g^{0\nu}m^2\phi(x)^2\right).
\end{aligned} \tag{28}
$$

Tenemos que

$$\partial_\mu \phi(x) = \int \frac{\mathrm{d}^3 p}{\sqrt{(2\pi)^3 2E_p}} \big(-ip_\mu a(p) e^{-ip\cdot x} + ip_\mu a^\dagger(p) e^{ip\cdot x}\big), \tag{29}$$

es decir,

$$\partial^\nu \phi(x) = g^{\mu\nu}\partial_\mu \phi(x) = \int \frac{\mathrm{d}^3 p}{\sqrt{(2\pi)^3 2E_p}} ip^\nu \big(-a(p) e^{-ip\cdot x} + a^\dagger(p) e^{ip\cdot x}\big). \tag{30}$$

Por lo tanto,

$$\begin{aligned}
\Pi(x)\partial^\mu \phi(x) = &\int \frac{\mathrm{d}^3 p\, \mathrm{d}^3 p'}{\sqrt{(2\pi)^3 2E_{p'}}} i\sqrt{\frac{E_p}{2(2\pi)^3}} ip'^\mu \big(-a(p)e^{-ip\cdot x} + a^\dagger(p) e^{ip\cdot x}\big)\times \\
&\Big(-a(p')e^{-ip'\cdot x} + a^\dagger(p')e^{ip'\cdot x}\Big) \\
= &\int \frac{\mathrm{d}^3 p\, \mathrm{d}^3 p'}{(2\pi)^3 2}\sqrt{\frac{E_p}{E_{p'}}} p'^\mu \Big(-a(p)a(p')e^{-i(p+p')\cdot x} \\
&+ a(p)a^\dagger(p')e^{-i(p-p')\cdot x} + a^\dagger(p)a(p')e^{-i(p'-p)\cdot x} \\
&-a^\dagger(p)a^\dagger(p')e^{-i(-p-p')\cdot x}\Big) \\
= &\int \frac{\mathrm{d}^3 p\, \mathrm{d}^3 p'}{(2\pi)^3 2}\sqrt{\frac{E_p}{E_{p'}}} p'^\mu \Big(-a(p)a(p')e^{-i(E_p+E_{p'})t}e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \\
&+ a(p)a^\dagger(p')e^{-i(E_p-E_{p'})t}e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \\
&+ a^\dagger(p)a(p')e^{-i(E_{p'}-E_p)t}e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}} \\
&-a^\dagger(p)a^\dagger(p')e^{-i(-E_p-E_{p'})t}e^{i(-\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\Big).
\end{aligned} \tag{31}$$

Recordando (**??**) y notando que $E_p = E_p'$ si $\mathbf{p} = -\mathbf{p}'$, concluimos

$$\begin{aligned}
\int \mathrm{d}^3 x\, \Pi(x)\partial^\mu\phi(x) = &\int \frac{\mathrm{d}^3 p\, \mathrm{d}^3 p'}{2}\sqrt{\frac{E_p}{E_{p'}}} p'^\mu \\
&\Big(-a(p)a(p')e^{-i(E_p+E_{p'})t}\delta^3(\mathbf{p}+\mathbf{p}') \\
&+ a(p)a^\dagger(p')e^{-i(E_p-E_{p'})t}\delta^3(\mathbf{p}-\mathbf{p}') \\
&+ a^\dagger(p)a(p')e^{-i(E_{p'}-E_p)t}\delta^3(\mathbf{p}-\mathbf{p}') \\
&-a^\dagger(p)a^\dagger(p')e^{-i(-E_p-E_{p'})t}\delta^3(\mathbf{p}+\mathbf{p}')\Big) \\
= &\int \mathrm{d}^3 p\, \frac{1}{2}\big(-\tilde{p}^\mu a(p)a(\tilde{p})e^{-i2E_p t} \\
&+p^\mu a(p)a^\dagger(p) + p^\mu a^\dagger(p)a(p) - \tilde{p}^\mu a^\dagger(p)a^\dagger(\tilde{p})e^{i2E_p t}\big)
\end{aligned} \tag{32}$$

donde $\tilde{p} = (E_p, -\mathbf{p})$. De manera análoga tenemos que

$$
\begin{aligned}
\partial_\sigma \phi(x) \partial^\sigma \phi(x) = & \int \frac{\mathrm{d}^3 p \, \mathrm{d}^3 p'}{(2\pi)^3 2} \frac{1}{\sqrt{E_p E_{p'}}} (ip_\sigma)(ip'^\sigma) \times \\
& \left(-a(p) e^{-ip\cdot x} + a^\dagger(p) e^{ip\cdot x}\right) \times \\
& \left(-a(p') e^{-ip'\cdot x} + a^\dagger(p') e^{ip'\cdot x}\right) \\
= & \int \frac{\mathrm{d}^3 p \, \mathrm{d}^3 p'}{(2\pi)^3 2} \frac{1}{\sqrt{E_p E_{p'}}} p_\sigma p'^\sigma \\
& \Big(-a(p) a(p') e^{-i(E_p + E_{p'})t} e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \\
& + a(p) a^\dagger(p') e^{-i(E_p - E_{p'})t} e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \\
& + a^\dagger(p) a(p') e^{-i(E_{p'} - E_p)t} e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}} \\
& - a^\dagger(p) a^\dagger(p') e^{-i(-E_p - E_{p'})t} e^{i(-\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\Big)
\end{aligned}
\tag{33}
$$

y

$$
\begin{aligned}
\int \mathrm{d}^3 x \, \frac{1}{2} g^{0\mu} \partial_\sigma \phi(x) \partial^\sigma \phi(x) = & \int \frac{\mathrm{d}^3 p \, \mathrm{d}^3 p'}{4} g^{0\mu} \frac{1}{\sqrt{E_p E_{p'}}} p_\sigma p'^\sigma \\
& \Big(-a(p) a(p') e^{-i(E_p + E_{p'})t} \delta^3(\mathbf{p}+\mathbf{p}') \\
& + a(p) a^\dagger(p') e^{-i(E_p - E_{p'})t} \delta^3(\mathbf{p}-\mathbf{p}') \\
& + a^\dagger(p) a(p') e^{-i(E_{p'} - E_p)t} \delta^3(\mathbf{p}-\mathbf{p}') \\
& - a^\dagger(p) a^\dagger(p') e^{-i(-E_p - E_{p'})t} \delta^3(\mathbf{p}+\mathbf{p}')\Big) \\
= & \int \mathrm{d}^3 p \, \frac{g^{0\mu}}{4E_p} \left(-p_\sigma \tilde{p}^\sigma a(p) a(\tilde{p}) e^{-i2E_p t}\right. \\
& + m^2 a(p) a^\dagger(p) + m^2 a^\dagger(p) a(p) \\
& \left. - p_\sigma \tilde{p}^\sigma a^\dagger(p) a^\dagger(\tilde{p}) e^{i2E_p t}\right).
\end{aligned}
\tag{34}
$$

Finalmente

$$
\begin{aligned}
\phi(x)^2 = &\int \frac{\mathrm{d}^3p\,\mathrm{d}^3p'}{\sqrt{(2\pi)^3 2E_p}\sqrt{(2\pi)^3 2E_{p'}}}\left(a(p)e^{-ip\cdot x}+a^\dagger(p)e^{ip\cdot x}\right)\times \\
&\left(a(p')e^{-ip'\cdot x}+a^\dagger(p')e^{ip'\cdot x}\right) \\
=&\int \frac{\mathrm{d}^3p\,\mathrm{d}^3p'}{(2\pi)^3 2}\frac{1}{\sqrt{E_pE_{p'}}}\Big(a(p)a(p')e^{-i(p+p')\cdot x} \\
&+a(p)a^\dagger(p')e^{-i(p-p')\cdot x}+a^\dagger(p)a(p')e^{-i(p'-p)\cdot x} \\
&a^\dagger(p)a^\dagger(p')e^{-i(-p-p')\cdot x}\Big) \\
=&\int \frac{\mathrm{d}^3p\,\mathrm{d}^3p'}{(2\pi)^3 2}\frac{1}{\sqrt{E_pE_{p'}}}\Big(a(p)a(p')e^{-i(E_p+E_{p'})t}e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \\
&+a(p)a^\dagger(p')e^{-i(E_p-E_{p'})t}e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \\
&+a^\dagger(p)a(p')e^{-i(E_{p'}-E_p)t}e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}} \\
&a^\dagger(p)a^\dagger(p')e^{-i(-E_p-E_{p'})t}e^{i(-\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\Big)
\end{aligned}
\tag{35}
$$

y por lo tanto

$$
\begin{aligned}
\int \mathrm{d}^3x\,\frac{1}{2}g^{0\mu}m^2\phi(x)^2 = &\int \frac{\mathrm{d}^3p\,\mathrm{d}^3p'}{4}\frac{m^2}{\sqrt{E_pE_{p'}}} \\
&\Big(a(p)a(p')e^{-i(E_p+E_{p'})t}\delta^3(\mathbf{p}+\mathbf{p}') \\
&+a(p)a^\dagger(p')e^{-i(E_p-E_{p'})t}\delta^3(\mathbf{p}-\mathbf{p}') \\
&+a^\dagger(p)a(p')e^{-i(E_{p'}-E_p)t}\delta^3(\mathbf{p}-\mathbf{p}') \\
&a^\dagger(p)a^\dagger(p')e^{-i(-E_p-E_{p'})t}\delta^3(\mathbf{p}+\mathbf{p}')\Big) \\
=&\int \mathrm{d}^3p\,\frac{g^{0\mu}m^2}{4E_p}\left(a(p)a(\tilde{p})e^{-i2E_pt}\right. \\
&\left.+a(p)a^\dagger(p)+a^\dagger(p)a(p)+a^\dagger(p)a^\dagger(\tilde{p})e^{i2E_pt}\right).
\end{aligned}
\tag{36}
$$

Entonces llegamos a que

$$
\begin{aligned}
P^\mu = \int \mathrm{d}^3 p \Bigg(& \frac{1}{2} \big(-\tilde{p}^\mu a(p) a(\tilde{p}) e^{-i2E_p t} \\
& + p^\mu a(p) a^\dagger(p) + p^\mu a^\dagger(p) a(p) - \tilde{p}^\mu a^\dagger(p) a^\dagger(\tilde{p}) e^{i2E_p t}\big) \\
& - \frac{g^{0\mu}}{4E_p} \big(-p_\sigma \tilde{p}^\sigma a(p) a(\tilde{p}) e^{-i2E_p t} \\
& + m^2 a(p) a^\dagger(p) + m^2 a^\dagger(p) a(p) - p_\sigma \tilde{p}^\sigma a^\dagger(p) a^\dagger(\tilde{p}) e^{i2E_p t}\big) \\
& + \frac{g^{0\mu} m^2}{4E_p} \big(a(p) a(\tilde{p}) e^{-i2E_p t} \\
& + a(p) a^\dagger(p) + a^\dagger(p) a(p) + a^\dagger(p) a^\dagger(\tilde{p}) e^{i2E_p t}\big)\big) \\
= \int \mathrm{d}^3 p \Bigg(& \frac{1}{2} \big(-\tilde{p}^\mu a(p) a(\tilde{p}) e^{-i2E_p t} \\
& + p^\mu a(p) a^\dagger(p) + p^\mu a^\dagger(p) a(p) - \tilde{p}^\mu a^\dagger(p) a^\dagger(\tilde{p}) e^{i2E_p t}\big) \\
& + \frac{g^{0\mu}}{4E_p} \big((p_\sigma \tilde{p}^\sigma + m^2) a(p) a(\tilde{p}) e^{-i2E_p t} \\
& + (p_\sigma \tilde{p}^\sigma + m^2) a^\dagger(p) a^\dagger(\tilde{p}) e^{i2E_p t}\big)\big).
\end{aligned}
\tag{37}
$$

Note que

$$
p_\sigma \tilde{p}^\sigma + m^2 = p_\sigma(\tilde{p}^\sigma + p^\sigma) = E_p(E_p + E_p) + 0 = 2E_p^2. \tag{38}
$$

Entonces el momento toma la forma

$$
\begin{aligned}
P^\mu = \int \mathrm{d}^3 p \Bigg(& \frac{1}{2} \big(-\tilde{p}^\mu a(p) a(\tilde{p}) e^{-i2E_p t} \\
& + p^\mu a(p) a^\dagger(p) + p^\mu a^\dagger(p) a(p) - \tilde{p}^\mu a^\dagger(p) a^\dagger(\tilde{p}) e^{i2E_p t}\big) \\
& + \frac{g^{0\mu} E_p}{2} \big(a(p) a(\tilde{p}) e^{-i2E_p t} + a^\dagger(p) a^\dagger(\tilde{p}) e^{i2E_p t}\big)\Bigg).
\end{aligned}
\tag{39}
$$

En particular,

$$
\begin{aligned}
P^i = \int \mathrm{d}^3 p \, & \frac{1}{2} \big(-\tilde{p}^i a(p) a(\tilde{p}) e^{-i2E_p t} \\
& + p^i a(p) a^\dagger(p) + p^i a^\dagger(p) a(p) - \tilde{p}^i a^\dagger(p) a^\dagger(\tilde{p}) e^{i2E_p t}\big) \\
& \int \mathrm{d}^3 p \, \frac{p^i}{2} \big(a(p) a(\tilde{p}) e^{-i2E_p t} \\
& + a(p) a^\dagger(p) + a^\dagger(p) a(p) + a^\dagger(p) a^\dagger(\tilde{p}) e^{i2E_p t}\big)
\end{aligned}
\tag{40}
$$

Note que bajo la transformación $p^i \mapsto -p^i$ se tiene que $p \mapsto \tilde{p}$. Como $[a(p), a(p')]_- = 0 = [a(p), a(p')]_-$ se tiene

$$
\begin{aligned}
p^i a(p) a(\tilde{p}) e^{-i2E_p t} &\mapsto -p^i a(\tilde{p}) a(p) e^{-i2E_{\tilde{p}} t} = -p^i a(p) a(\tilde{p}) e^{-i2E_p t} \\
p^i a^\dagger(p) a^\dagger(\tilde{p}) e^{i2E_p t} &\mapsto -p^i a^\dagger(\tilde{p}) a^\dagger(p) e^{i2E_{\tilde{p}} t} = -p^i a^\dagger(p) a^\dagger(\tilde{p}) e^{i2E_p t}.
\end{aligned}
\tag{41}
$$

Entonces por paridad se concluye que

$$P^i = \int \mathrm{d}^3p\, \frac{p^i}{2}\big(a(p)a^\dagger(p) + a^\dagger(p)a(p)\big). \tag{42}$$

No es necesaria la prescripción de orden normal ya que el momento se puede poner en la forma

$$P^i = \int \mathrm{d}^3p\, p^i \left(a^\dagger(p)a(p) + \frac{1}{2}\delta^3(0)\right). \tag{43}$$

El termino $p^i\delta^3(0)$ es impar y por lo tanto su integral se anula[1]. Entonces

$$P^i = \int \mathrm{d}^3p\, p^i a^\dagger(p)a(p). \tag{44}$$

b)

Note que

$$\begin{aligned} P^0 = &\int \mathrm{d}^3p \left(\frac{E_p}{2}\big(-a(p)a(\tilde{p})e^{-i2E_pt} \right. \\ &+a(p)a^\dagger(p) + a^\dagger(p)a(p) - a^\dagger(p)a^\dagger(\tilde{p})e^{i2E_pt}\big) \\ &\left. +\frac{E_p}{2}\big(a(p)a(\tilde{p})e^{-i2E_pt} + a^\dagger(p)a^\dagger(\tilde{p})e^{i2E_pt}\big)\right) \\ = &\int \mathrm{d}^3p\, \frac{E_p}{2}\big(a(p)a^\dagger(p) + a^\dagger(p)a(p)\big) \end{aligned} \tag{45}$$

Por lo tanto, concluimos que

$$P^\mu = \int \mathrm{d}^3p\, \frac{p^\mu}{2}\big(a(p)a^\dagger(p) + a^\dagger(p)a(p)\big). \tag{46}$$

Entonces tenemos

$$\begin{aligned} [\phi(x), P_\mu]_- = &\int \frac{\mathrm{d}^3p\,\mathrm{d}^3p'}{\sqrt{(2\pi)^3 2E_p}}\frac{p'_\mu}{2}[a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x} \\ &, a(p')a^\dagger(p') + a^\dagger(p')a(p')]_-. \end{aligned} \tag{47}$$

[1]Este tipo de manipulaciones son comunes en la teoría cuántica de campos. Se pueden hacer más rigurosos mediante cercas técnicas. Sin embargo, siguiendo la tradición y teniendo en cuenta que esta parte de la clase es de carácter introductorio, me permitiré cometer esta y otras infracciones. Lo siento. Me gustaría poder responder de mejor manera a esta pregunta.

Note que

$$
\begin{aligned}
[a(p), a(p')a^\dagger(p')]_- =&[a(p), a(p')]_- a^\dagger(p') + a(p')[a(p), a^\dagger(p')]_- \\
=&a(p')\delta^3(\mathbf{p}-\mathbf{p}') \\
[a(p), a^\dagger(p')a(p')]_- =&[a(p), a^\dagger(p')]_- a(p') + a^\dagger(p')[a(p), a(p')]_- \\
=&a(p')\delta^3(\mathbf{p}-\mathbf{p}') \\
[a^\dagger(p), a(p')a^\dagger(p')]_- =&[a^\dagger(p), a(p')]_- a^\dagger(p') + a(p')[a^\dagger(p), a^\dagger(p')]_- \\
=&-a^\dagger(p')\delta^3(\mathbf{p}-\mathbf{p}') \\
[a^\dagger(p), a^\dagger(p')a(p')]_- =&[a^\dagger(p), a^\dagger(p')]_- a(p') + a^\dagger(p')[a^\dagger(p), a(p')]_- \\
=&-a^\dagger(p')\delta^3(\mathbf{p}-\mathbf{p}'),
\end{aligned}
\tag{48}
$$

donde se hizo uso de

$$
\begin{aligned}
[A, BC]_- =&ABC - BCA = ABC - BAC + BAC - BCA \\
=&[A, B]_- C + B[A, C]_- .
\end{aligned}
\tag{49}
$$

Por lo tanto el conmutador se reduce a

$$
\begin{aligned}
[\phi(x), P_\mu]_- =&\int \frac{\mathrm{d}^3p\,\mathrm{d}^3p'}{\sqrt{(2\pi)^3 2E_p}} \frac{p'_\mu}{2}(e^{-ip\cdot x}a(p')\delta^3(\mathbf{p}-\mathbf{p}') \\
&+ e^{-ip\cdot x}a(p')\delta^3(\mathbf{p}-\mathbf{p}') - e^{ip\cdot x}a^\dagger(p')\delta^3(\mathbf{p}-\mathbf{p}') \\
&- e^{ip\cdot x}a^\dagger(p')\delta^3(\mathbf{p}-\mathbf{p}')) \\
=&\int \frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2E_p}} \frac{p_\mu}{2}(e^{-ip\cdot x}a(p) \\
&+ e^{-ip\cdot x}a(p) - e^{ip\cdot x}a^\dagger(p) - e^{ip\cdot x}a^\dagger(p)) \\
=&\int \frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2E_p}} p_\mu(e^{-ip\cdot x}a(p) - e^{ip\cdot x}a^\dagger(p)) \\
=&i\int \frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2E_p}} ip_\mu(-e^{-ip\cdot x}a(p) + e^{ip\cdot x}a^\dagger(p)) \\
=&i\partial_\mu\phi(x).
\end{aligned}
\tag{50}
$$

3.6

Haciendo uso de (48) se tiene

$$
\begin{aligned}
[\mathcal{N}, a^\dagger(k)]_- =&\int \mathrm{d}^3p\,[a^\dagger(p)a(p), a^\dagger(k)]_- = \int \mathrm{d}^3p\,a^\dagger(p)\delta^3(\mathbf{p}-\mathbf{k}) = a^\dagger(\mathbf{k}) \\
[\mathcal{N}, a(k)]_- =&\int \mathrm{d}^3p\,[a^\dagger(p)a(p), a(k)]_- = -\int \mathrm{d}^3p\,a(p)\delta^3(\mathbf{p}-\mathbf{k}) \\
=&-a(\mathbf{k}).
\end{aligned}
\tag{51}
$$

Por lo tanto

$$\begin{aligned}\mathcal{N}\left|p_1,\ldots,p_N\right\rangle =&\mathcal{N}a^\dagger(p_1)\cdots a^\dagger(p_N)\left|0\right\rangle = ([\mathcal{N},a^\dagger(p_1)\cdots a^\dagger(p_N)]_- \\ &+ a^\dagger(p_1)\cdots a^\dagger(p_N)\mathcal{N})\left|0\right\rangle .\end{aligned} \tag{52}$$

Es claro que

$$\mathcal{N}\left|0\right\rangle = \int \mathrm{d}^3p\, a^\dagger(p)a(p)\left|0\right\rangle = 0 \tag{53}$$

y mediante una generalización de (49)

$$\begin{aligned}&[\mathcal{N},a^\dagger(p_1)\cdots a^\dagger(p_N)]_- = \\ &[\mathcal{N},a^\dagger(p_1)]_- a^\dagger(p_2)\cdots a^\dagger(p_N) + a^\dagger(p_1)[\mathcal{N},a^\dagger(p_2)\cdots a^\dagger(p_N)]_- = \\ &a^\dagger(p_1)a^\dagger(p_2)\cdots a^\dagger(p_N) + a^\dagger(p_1)[\mathcal{N},a^\dagger(p_2)\cdots a^\dagger(p_N)]_- = \cdots = \\ &\underbrace{a^\dagger(p_1)a^\dagger(p_2)\cdots a^\dagger(p_N) + \cdots + a^\dagger(p_1)a^\dagger(p_2)\cdots a^\dagger(p_N)}_{\text{N veces}} \\ &= N a^\dagger(p_1)\cdots a^\dagger(p_N).\end{aligned} \tag{54}$$

Se concluye que

$$\mathcal{N}\left|p_1,\ldots,p_N\right\rangle = Na^\dagger(p_1)\cdots a^\dagger(p_N)\left|0\right\rangle = N\left|p_1,\ldots,p_N\right\rangle . \tag{55}$$

Este cálculo no depende de que los momentos sean distintos. Por lo tanto

$$\begin{aligned}\mathcal{N}\left|p(n)\right\rangle =&\frac{1}{\sqrt{n!}}(\mathcal{N}a^\dagger(p))^n\left|0\right\rangle = \frac{1}{\sqrt{n!}}((a^\dagger(p))^n\mathcal{N} + [\mathcal{N},(a^\dagger(p))^n]_-)\left|0\right\rangle \\ =&0 + \frac{1}{\sqrt{n!}}n(a^\dagger(p))^n\left|0\right\rangle = n\left|p(n)\right\rangle .\end{aligned} \tag{56}$$

Se concluye que $\mathcal{N}$ cuenta el número de partículas.