Mecánica Cuántica Avanzada
Tarea 5

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4.8 Multiplicando por izquierda $\overline{u}_r(\mathbf{p}')\gamma^\mu$ a la ecuación (4.46) de [1] se obtiene

$$0 = \overline{u}_r(\mathbf{p}')\gamma^\mu(\not{p} - m)u_s(\mathbf{p}) = \overline{u}_r(\mathbf{p}')\gamma^\mu\gamma^\nu p_\nu u_s(\mathbf{p}) - \overline{u}_r(\mathbf{p}')\gamma^\mu m u_s(\mathbf{p}). \quad (1)$$

Multiplicando por la derecha $\gamma^\mu u_s(\mathbf{p})$ a la ecuación (4.48) de [1] se obtiene

$$0 = \overline{u}_r(\mathbf{p}')(\not{p}' - m)\gamma^\mu u_s(\mathbf{p}) = \overline{u}_r(\mathbf{p}')\gamma^\nu\gamma^\mu p'_\nu u_s(\mathbf{p}) - \overline{u}_r(\mathbf{p}')\gamma^\mu m u_s(\mathbf{p}). \quad (2)$$

Sumandolas se concluye que

$$\begin{aligned} 0 =& \overline{u}_r(\mathbf{p}')\gamma^\mu\gamma^\nu p_\nu u_s(\mathbf{p}) - \overline{u}_r(\mathbf{p}')\gamma^\mu m u_s(\mathbf{p}) \\ &+\overline{u}_r(\mathbf{p}')\gamma^\nu\gamma^\mu p'_\nu u_s(\mathbf{p}) - \overline{u}_r(\mathbf{p}')\gamma^\mu m u_s(\mathbf{p}) \\ =& \overline{u}_r(\mathbf{p}')(\gamma^\mu\gamma^\nu p_\nu + \gamma^\nu\gamma^\mu p'_\nu)u_s(\mathbf{p}) - 2\overline{u}_r(\mathbf{p}')\gamma^\mu m u_s(\mathbf{p}). \end{aligned} \quad (3)$$

En particular, si $\mathbf{p} = \mathbf{p}'$

$$0 = \overline{u}_r(\mathbf{p})2g^{\mu\nu}p_\nu u_s(\mathbf{p}) - 2m\overline{u}_r(\mathbf{p})\gamma^\mu u_s(\mathbf{p}). \quad (4)$$

Dividiendo por 2 y subiendo el índice del momento entonces es claro que

$$\overline{u}_r(\mathbf{p})\gamma^\mu m u_s(\mathbf{p}) = \overline{u}_r(\mathbf{p})p^\mu u_s(\mathbf{p}). \quad (5)$$

Repitiendo este proceso al pie de la letra se tiene

$$0 = \overline{v}_r(\mathbf{p}')\gamma^\mu(\not{p} + m)v_s(\mathbf{p}) = \overline{v}_r(\mathbf{p}')\gamma^\mu\gamma^\nu p_\nu v_s(\mathbf{p}) + \overline{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}), \quad (6)$$

$$0 = \overline{v}_r(\mathbf{p}')(\not{p}' + m)\gamma^\mu v_s(\mathbf{p}) = \overline{v}_r(\mathbf{p}')\gamma^\nu\gamma^\mu p'_\nu v_s(\mathbf{p}) + \overline{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}), \quad (7)$$

$$\begin{aligned} 0 =& \overline{v}_r(\mathbf{p}')\gamma^\mu\gamma^\nu p_\nu v_s(\mathbf{p}) + \overline{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}) \\ &+\overline{v}_r(\mathbf{p}')\gamma^\nu\gamma^\mu p'_\nu v_s(\mathbf{p}) + \overline{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}) \\ =& \overline{v}_r(\mathbf{p}')(\gamma^\mu\gamma^\nu p_\nu + \gamma^\nu\gamma^\mu p'_\nu)v_s(\mathbf{p}) + 2\overline{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}) \\ =& \overline{v}_r(\mathbf{p}')2g^{\mu\nu}p_\nu v_s(\mathbf{p}) + 2m\overline{v}_r(\mathbf{p}')\gamma^\mu v_s(\mathbf{p}) \end{aligned} \quad (8)$$

en el caso $\mathbf{p} = \mathbf{p}'$ y

$$\overline{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}) = -\overline{v}_r(\mathbf{p}')p^\mu v_s(\mathbf{p}). \quad (9)$$

Poniendo $\mu = 0$ en (5) se tiene haciendo uso de la normalización (4.49) de [1]

$$\begin{aligned}2E_p\delta_{rs}m =&mu_r^\dagger(\mathbf{p})u_s(\mathbf{p}) = mu_r^\dagger(\mathbf{p})\gamma^0\gamma^0u_s(\mathbf{p}) = m\overline{u}_r(\mathbf{p})\gamma^0u_s(\mathbf{p})\\ =&p^0\overline{u}_r(\mathbf{p})u(\mathbf{p}) = E_p\overline{u}_r(\mathbf{p})u_s(\mathbf{p}).\end{aligned} \tag{10}$$

Repitiendo con (9)

$$\begin{aligned}2E_p\delta_{rs}m =&mv_r^\dagger(\mathbf{p})v_s(\mathbf{p}) = mv_r^\dagger(\mathbf{p})\gamma^0\gamma^0v_s(\mathbf{p}) = m\overline{v}_r(\mathbf{p})\gamma^0v_s(\mathbf{p})\\ =&-p^0\overline{v}_r(\mathbf{p})v(\mathbf{p}) = -E_p\overline{v}_r(\mathbf{p})v_s(\mathbf{p}).\end{aligned} \tag{11}$$

Por lo tanto, asumiendo que $E_p \neq 0$, se tiene

$$\overline{u}_r(\mathbf{p})u_s(\mathbf{p}) = -\overline{v}_r(\mathbf{p})v_s(\mathbf{p}) = 2m\delta_{rs}. \tag{12}$$

4.9 Note que haciendo uso de la relaciones de conmutación de las matrices de Dirac se tiene

$$\begin{aligned}&(p+p')^\mu - i\sigma^{\mu\nu}q_\nu\\ =&p^\mu + p'^\mu - i\frac{i}{2}[\gamma^\mu,\gamma^\nu]_-(p_\nu - p'_\nu)\\ =&p^\mu + p'^\mu + \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p_\nu - p'_\nu)\\ =&p^\mu + p'^\mu + \frac{1}{2}(\gamma^\mu\gamma^\nu p_\nu - \gamma^\mu\gamma^\nu p'_\nu - \gamma^\nu\gamma^\mu p_\nu + \gamma^\nu\gamma^\mu p'_\nu)\\ =&p^\mu + p'^\mu\\ &+\frac{1}{2}(\gamma^\mu\gamma^\nu p_\nu - 2g^{\mu\nu}p'_\nu + \gamma^\nu\gamma^\mu p'_\nu - 2g^{\nu\mu}p_\nu + \gamma^\mu\gamma^\nu p_\nu + \gamma^\nu\gamma^\mu p'_\nu)\\ =&p^\mu + p'^\mu + \frac{1}{2}(2\gamma^\mu\gamma^\nu p_\nu - 2p'^\mu + 2\gamma^\nu\gamma^\mu p'_\nu - 2p^\mu)\\ =&\gamma^\mu\gamma^\nu p_\nu + \gamma^\nu\gamma^\mu p'_\nu.\end{aligned} \tag{13}$$

Por lo tanto, comparando con las ecuaciones (3) y (8) se obtiene

$$\begin{aligned}0 =&\overline{u}_r(\mathbf{p}')((p+p')^\mu - i\sigma^{\mu\nu}q_\nu)u_s(\mathbf{p}) - 2m\overline{u}_r(\mathbf{p}')\gamma^\mu u_s(\mathbf{p})\\ 0 =&\overline{v}_r(\mathbf{p}')((p+p')^\mu - i\sigma^{\mu\nu}q_\nu)v_s(\mathbf{p}) + 2m\overline{v}_r(\mathbf{p}')\gamma^\mu v_s(\mathbf{p}).\end{aligned} \tag{14}$$

Dividiendo por $2m$ se obtienen las identidades de Gordon

$$\begin{aligned}\overline{u}_r(\mathbf{p}')\gamma^\mu u_s(\mathbf{p}) =&\frac{1}{2m}\overline{u}_r(\mathbf{p}')((p+p')^\mu - i\sigma^{\mu\nu}q_\nu)u_s(\mathbf{p})\\ \overline{v}_r(\mathbf{p}')\gamma^\mu v_s(\mathbf{p}) =&-\frac{1}{2m}\overline{v}_r(\mathbf{p}')((p+p')^\mu - i\sigma^{\mu\nu}q_\nu)v_s(\mathbf{p}).\end{aligned} \tag{15}$$

4.10 Como en el ejercicio 4.8, multiplicando a izquierda por $\overline{u}_r(\mathbf{p})\gamma^\mu$ a la ecuación (4.46) de [1] y por la derecha por $\gamma^\mu v_s(\mathbf{p})$ a (4.48) de [1] se obtiene

$$\begin{aligned}\overline{u}_r(\mathbf{p})\gamma^\mu(\not{p}+m)v_s(\mathbf{p}) =&0\\ \overline{u}_r(\mathbf{p})(\not{p}-m)\gamma^\mu v_s(\mathbf{p}) =&0.\end{aligned} \tag{16}$$

Al sumar estas ecuaciones se concluye

$$\begin{aligned}0 =&\overline{u}_r(\mathbf{p})\gamma^\mu(\not{p}+m)v_s(\mathbf{p})+\overline{u}_r(\mathbf{p})(\not{p}-m)\gamma^\mu v_s(\mathbf{p})\\ =&\overline{u}_r(\mathbf{p})(\gamma^\mu\not{p}+\not{p}\gamma^\mu)v_s(\mathbf{p})=\overline{u}_r(\mathbf{p})(\gamma^\mu\gamma^\nu p_\nu+\gamma^\nu\gamma^\mu p_\nu)v_s(\mathbf{p})\\ =&\overline{u}_r(\mathbf{p})(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)p_\nu v_s(\mathbf{p})=\overline{u}_r(\mathbf{p})2g^{\mu\nu}p_\nu v_s(\mathbf{p})\\ =&2p^\mu\overline{u}_r(\mathbf{p})v_s(\mathbf{p}),\end{aligned}\tag{17}$$

es decir,

$$0=\overline{u}_r(\mathbf{p})v_s(\mathbf{p}).\tag{18}$$

Conjugando se tiene

$$\begin{aligned}0 =&(\overline{u}_r(\mathbf{p})v_s(\mathbf{p}))^\dagger=(u_r^\dagger(\mathbf{p})\gamma^0 v_s(\mathbf{p}))^\dagger=v_s(\mathbf{p})^\dagger\gamma^0 u_r(\mathbf{p})\\ =&\overline{v}_s(\mathbf{p})u_r(\mathbf{p}).\end{aligned}\tag{19}$$

Note que debido a la ecuación de Dirac (4.46) de [1]

$$\begin{aligned}(\not{p}+m)u_r(\mathbf{p}) =&(\not{p}-m+2m)u_r(\mathbf{p})=2mu_r(\mathbf{p})\\ (\not{p}+m)v_r(\mathbf{p}) =&0\\ (\not{p}-m)u_r(\mathbf{p}) =&0\\ (\not{p}-m)v_r(\mathbf{p}) =&(\not{p}+m-2m)v_r(\mathbf{p})=-2mv_r(\mathbf{p}).\end{aligned}\tag{20}$$

Por el otro lado, haciendo uso de las relaciones de normalización halladas en el ejercicio 4.8 y las relaciones (18) y (19)

$$\begin{aligned}\sum_s u_s(\mathbf{p})\overline{u}_s(\mathbf{p})u_r(\mathbf{p}) =&\sum_s u_s(\mathbf{p})2m\delta_{sr}=2mu_r(\mathbf{p})\\ \sum_s u_s(\mathbf{p})\overline{u}_s(\mathbf{p})v_r(\mathbf{p}) =&0\\ \sum_s v_s(\mathbf{p})\overline{v}_s(\mathbf{p})u_r(\mathbf{p}) =&0\\ \sum_s v_s(\mathbf{p})\overline{v}_s(\mathbf{p})v_r(\mathbf{p}) =&\sum_s v_s(\mathbf{p})(-2m\delta_{sr})=-2mv_r(\mathbf{p}).\end{aligned}\tag{21}$$

Ya que las matrices coinciden en una base, por extensión lineal deben ser iguales

$$\begin{aligned}\sum_s u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) =&\not{p}+m\\ \sum_s v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) =&\not{p}-m.\end{aligned}\tag{22}$$

4.24 Se tiene con la ecuación (A.32) de [1] que

$$\begin{aligned}
W^\mu W_\mu =& \frac{1}{4}\epsilon^{\mu\nu\lambda\rho}\epsilon_{\mu\nu'\lambda'\rho'}P_\nu J_{\lambda\rho}P^{\nu'}J^{\lambda'\rho'} \\
=& -\frac{1}{4}(\delta^\nu_{\nu'}\delta^\lambda_{\lambda'}\delta^\rho_{\rho'} + \delta^\nu_{\lambda'}\delta^\lambda_{\rho'}\delta^\rho_{\nu'} + \delta^\nu_{\rho'}\delta^\lambda_{\nu'}\delta^\rho_{\lambda'} \\
& - \delta^\nu_{\lambda'}\delta^\lambda_{\nu'}\delta^\rho_{\rho'} - \delta^\nu_{\rho'}\delta^\lambda_{\lambda'}\delta^\rho_{\nu'} - \delta^\nu_{\nu'}\delta^\lambda_{\rho'}\delta^\rho_{\lambda'})P_\nu J_{\lambda\rho}P^{\nu'}J^{\lambda'\rho'} \\
=& -\frac{1}{4}P_\nu J_{\lambda\rho}(P^\nu J^{\lambda\rho} + P^\rho J^{\nu\lambda} + P^\lambda J^{\rho\nu} \\
& - P^\lambda J^{\nu\rho} - P^\rho J^{\lambda\nu} - P^\nu J^{\rho\lambda}) \\
=& -\frac{1}{4}P_\nu J_{\lambda\rho}(P^\nu(J^{\lambda\rho} - J^{\rho\lambda}) + P^\rho(J^{\nu\lambda} - J^{\lambda\nu}) \\
& + P^\lambda(J^{\rho\nu} - J^{\nu\rho}))
\end{aligned} \tag{23}$$

Note las siguientes propiedades de antisimetría

$$\begin{aligned}
\sigma_{\mu\nu} =& \frac{i}{2}[\gamma^\mu, \gamma^\nu]_- = -\frac{i}{2}[\gamma^\nu, \gamma^\mu]_- = -\sigma^{\nu\mu} \\
J_{\mu\nu} =& i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \frac{1}{2}\sigma_{\mu\nu} = -i(x_\nu\partial_\mu - x_\mu\partial_\nu) - \frac{1}{2}\sigma_{\nu\mu} = -J_{\nu\mu}.
\end{aligned} \tag{24}$$

Por lo tanto

$$W^\mu W_\mu = -\frac{1}{2}P_\nu J_{\lambda\rho}(P^\nu J^{\lambda\rho} + P^\rho J^{\nu\lambda} + P^\lambda J^{\rho\nu}) \tag{25}$$

Ahora bien, note que $W^\mu W_\mu$ es un escalar y por lo tanto no depende del sistema de referencia en el que se evalue. Ya que la fórmula (4.95) de [1] solo tiene sentido para partículas masivas, asumimos que nuestra partícula tiene $m \neq 0$. Por lo tanto en el sistema de reposo de la partícula las componentes espaciales del momento se anulan y

$$\begin{aligned}
W^\mu W_\mu =& -\frac{1}{2}P_0 J_{\lambda\rho}(P^0 J^{\lambda\rho} + P^\rho J^{0\lambda} + P^\lambda J^{\rho 0}) \\
=& -\frac{1}{2}(P_0 J_{\lambda\rho}P^0 J^{\lambda\rho} + P_0 J_{\lambda\rho}P^\rho J^{0\lambda} + P_0 J_{\lambda\rho}P^\lambda J^{\rho 0}) \\
=& -\frac{1}{2}(P_0 J_{\lambda\rho}P^0 J^{\lambda\rho} + P_0 J_{\lambda 0}P^0 J^{0\lambda} + P_0 J_{0\rho}P^0 J^{\rho 0}) \\
=& -\frac{1}{2}(P_0 J_{\lambda\rho}P^0 J^{\lambda\rho} + P_0 J_{\lambda 0}P^0 J^{0\lambda} + P_0 J_{\rho 0}P^0 J^{0\rho}) \\
=& -\frac{1}{2}(P_0 J_{\lambda\rho}P^0 J^{\lambda\rho} + 2P_0 J_{\lambda 0}P^0 J^{0\lambda}.)
\end{aligned} \tag{26}$$

Más aún, en el sistema de reposo el momento angular orbital es nulo y por lo tanto

$$J_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}. \tag{27}$$

Ya que actuan en espacios distintos se tiene

$$[P^\mu, \sigma^{\nu\lambda}]_- = 0. \tag{28}$$

Además, en el centro de masa el cuadrado de la energía es m^2, es decir $P^0 P_0 = m^2$. Se concluye

$$W^\mu W_\mu = -\frac{1}{8} m^2 (\sigma_{\lambda\rho}\sigma^{\lambda\rho} + 2\sigma_{\lambda 0}\sigma^{0\lambda}). \tag{29}$$

Note las siguientes identidades

$$\begin{aligned}
\gamma^0\gamma_0 =& \gamma_0\gamma_0 = 1, \\
\gamma^\mu\gamma_\mu =& g^{\mu\nu}\gamma_\nu\gamma_\mu = \frac{1}{2}(g^{\mu\nu} + g^{\nu\mu})\gamma_\nu\gamma_\mu = \frac{1}{2}(g^{\mu\nu}\gamma_\nu\gamma_\mu + g^{\nu\mu}\gamma_\nu\gamma_\mu) \\
=& \frac{1}{2}(g^{\mu\nu}\gamma_\nu\gamma_\mu + g^{\mu\nu}\gamma_\mu\gamma_\nu) = \frac{1}{2}g^{\mu\nu}[\gamma_\mu, \gamma_\nu]_- = \frac{1}{2}g^{\mu\nu}2g_{\mu\nu} \\
=& \delta^\mu_\mu = 4, \\
\gamma^\mu\gamma^\nu\gamma_\mu =& (2g^{\mu\nu} - \gamma^\nu\gamma^\mu)\gamma_\mu = 2\gamma^\nu - 4\gamma^\nu = -2\gamma^\nu.
\end{aligned} \tag{30}$$

Por lo tanto se concluye que

$$\begin{aligned}
\sigma_{\lambda\rho}\sigma^{\lambda\rho} =& -\frac{1}{4}(\gamma_\lambda\gamma_\rho - \gamma_\rho\gamma_\lambda)(\gamma^\lambda\gamma^\rho - \gamma^\rho\gamma^\lambda) \\
=& -\frac{1}{4}(\gamma_\lambda\gamma_\rho\gamma^\lambda\gamma^\rho - \gamma_\lambda\gamma_\rho\gamma^\rho\gamma^\lambda - \gamma_\rho\gamma_\lambda\gamma^\lambda\gamma^\rho + \gamma_\rho\gamma_\lambda\gamma^\rho\gamma^\lambda) \\
=& -\frac{1}{4}(\gamma_\lambda(-2\gamma^\lambda) - 4\gamma_\lambda\gamma^\lambda - 4\gamma_\rho\gamma^\rho + \gamma_\rho(-2\gamma^\rho)) \\
=& -\frac{1}{4}(-8 - 16 - 16 - 8) = \frac{48}{4} = 12 \\
\sigma_{\lambda 0}\sigma^{0\lambda} =& -\frac{1}{4}(\gamma_\lambda\gamma_0 - \gamma_0\gamma_\lambda)(\gamma^0\gamma^\lambda - \gamma^\lambda\gamma^0) \\
=& -\frac{1}{4}(\gamma_\lambda\gamma_0\gamma^0\gamma^\lambda - \gamma_\lambda\gamma_0\gamma^\lambda\gamma^0 - \gamma_0\gamma_\lambda\gamma^0\gamma^\lambda + \gamma_0\gamma_\lambda\gamma^\lambda\gamma^0) \\
=& -\frac{1}{4}(\gamma_\lambda\gamma^\lambda - (-2\gamma_0)\gamma^0 - \gamma_0(-2\gamma^0) + 4\gamma_0\gamma^0) \\
=& -\frac{1}{4}(4 + 2 + 2 + 4) = -\frac{12}{4} = -3.
\end{aligned} \tag{31}$$

Este invariante entonces satisface

$$-m^2 s(s+1) = W^\mu W_\mu = -\frac{1}{8}m^2(12 - 2\times 3) = -\frac{6}{8}m^2 = -\frac{3}{4}m^2. \tag{32}$$

Esto nos lleva a la ecuación cuadrática $s^2 + s - \frac{3}{4} = 0$ cuyas soluciones son

$$s = \frac{-1 \pm \sqrt{1+3}}{2} = \frac{-1 \pm 2}{2} = \begin{cases} \frac{1}{2} \\ -\frac{3}{2}. \end{cases} \tag{33}$$

En particular, ya que el espín total es positivo, se concluye que

$$s = \frac{1}{2}. \tag{34}$$

Referencias

[1] A. Lahiri and P. B. Pal, *A First Book of Quantum Field Theory.* 2005.