Mecánica Cuantica Avanzada: Tarea 3

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2.1 En primer lugar tenemos gracias a que el tensor métrico es covariantemente conservado que

$$F_{\sigma\rho} = \partial_\sigma A_\rho - \partial_\rho A_\sigma = g_{\lambda\rho}\partial_\sigma A^\lambda - g_{\lambda\sigma}\partial_\rho A^\lambda. \tag{1}$$

Por lo tanto

$$\frac{\partial F_{\sigma\rho}}{\partial(\partial_\mu A^\nu)} = \delta^\mu_\sigma g_{\nu\rho} - \delta^\mu_\rho g_{\nu\sigma} \tag{2}$$

y

$$\begin{aligned}\frac{\partial F^{\lambda\gamma}}{\partial(\partial_\mu A^\nu)} =& \frac{\partial g^{\lambda\sigma} g^{\gamma\rho} F_{\sigma\rho}}{\partial(\partial_\mu A^\nu)} = g^{\lambda\sigma} g^{\gamma\rho}\frac{\partial F_{\sigma\rho}}{\partial(\partial_\mu A^\nu)} = g^{\lambda\sigma} g^{\gamma\rho}(\delta^\mu_\sigma g_{\nu\rho} - \delta^\mu_\rho g_{\nu\sigma}) \\ =& \delta^\mu_\sigma \delta^\gamma_\nu g^{\lambda\sigma} - \delta^\mu_\rho \delta^\lambda_\nu g^{\gamma\rho} = \delta^\gamma_\nu g^{\lambda\mu} - \delta^\lambda_\nu g^{\gamma\mu}.\end{aligned} \tag{3}$$

Entonces se tiene

$$\begin{aligned}\frac{\partial F^{\sigma\rho}F_{\sigma\rho}}{\partial(\partial_\mu A^\nu)} =& F^{\sigma\rho}\frac{\partial F_{\sigma\rho}}{\partial(\partial_\mu A^\nu)} + \frac{\partial F^{\sigma\rho}}{\partial(\partial_\mu A^\nu)}F_{\sigma\rho} \\ =& F^{\sigma\rho}(\delta^\mu_\sigma g_{\nu\rho} - \delta^\mu_\rho g_{\nu\sigma}) + (\delta^\rho_\nu g^{\sigma\mu} - \delta^\sigma_\nu g^{\rho\mu})F_{\sigma\rho} \\ =& F^{\mu\rho} g_{\nu\rho} - F^{\sigma\mu} g_{\nu\sigma} + F_{\sigma\nu} g^{\sigma\mu} - F_{\nu\rho} g^{\rho\mu}.\end{aligned} \tag{4}$$

Utilizando el hecho de que $F^{\mu\nu}$ es antisimétrico y cambiando el nombre del indice mudo

$$\sigma \to \rho \tag{5}$$

se tiene

$$\frac{\partial F^{\sigma\rho}F_{\sigma\rho}}{\partial(\partial_\mu A^\nu)} = 2(g_{\nu\rho}F^{\mu\rho} + g^{\rho\mu}F_{\rho\nu}). \tag{6}$$

Finalmente note que nombrando el indice mudo

$$\sigma \to \rho \tag{7}$$

en

$$g^{\rho\mu}F_{\rho\nu} = \delta^\lambda_\nu g^{\rho\mu}F_{\rho\lambda} = g_{\nu\sigma}g^{\sigma\lambda}g^{\rho\mu}F_{\rho\lambda} = g_{\nu\sigma}F^{\mu\sigma} \tag{8}$$

se obtiene que

$$\frac{\partial F^{\sigma\rho}F_{\sigma\rho}}{\partial(\partial_\mu A^\nu)} = 4g_{\nu\rho}F^{\mu\rho}. \tag{9}$$

Con esta información concluimos que la ecuación de Euler-Lagrange es

$$-j_\nu = \frac{\partial \mathcal{L}}{\partial A^\nu} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} \right) = -\frac{1}{4}\partial_\mu \left(\frac{\partial F^{\sigma\rho} F_{\sigma\rho}}{\partial(\partial_\mu A^\nu)} \right) = -g_{\nu\rho}\partial_\mu F^{\mu\rho}. \tag{10}$$

Multiplicando por $g^{\sigma\nu}$ se obtiene

$$j^\sigma = g^{\sigma\nu} j_\nu = g^{\sigma\nu} g_{\nu\rho} \partial_\mu F^{\mu\rho} = \delta^\sigma_\rho \partial_\mu F^{\mu\rho} = \partial_\mu F^{\mu\sigma}, \tag{11}$$

es decir, la ecuación de Maxwell no homogénea

$$\partial_\mu F^{\mu\nu} = j^\nu. \tag{12}$$

2.2 Note que

$$\partial^\mu \phi \partial_\mu \phi = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = (\partial_0 \phi)^2 - \sum_{i=1}^{3} (\partial_i \phi)^2. \tag{13}$$

Entonces

$$\frac{\partial \partial^\mu \phi \partial_\mu \phi}{\partial(\partial_\sigma \phi)} = 2 \begin{cases} \partial_0 \phi & \sigma = 0 \\ -\partial_i \phi & \sigma = i \in \{1,2,3\} \end{cases} = 2\partial^\mu \phi. \tag{14}$$

Por lo tanto, las ecuaciones de Euler-Lagrange son

$$-m^2 \phi - V'(\phi) = \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \partial_\mu \partial^\mu \phi = \Box \phi, \tag{15}$$

o de manera más familiar,

$$(\Box + m^2)\phi = -V'(\phi). \tag{16}$$

2.4 Debemos realizar la derivada funcional

$$\begin{aligned} \Pi_\mu(t, \mathbf{x}) =& \frac{\delta L}{\delta \dot{A}^\mu(t, \mathbf{x})} = \frac{\delta}{\delta \dot{A}^\mu(t, \mathbf{x})} \int \mathrm{d}^3\mathbf{y}\, \mathcal{L}(A^\sigma(t, \mathbf{y}), \partial_\sigma A^\rho(t, \mathbf{y})) \\ =& \int \mathrm{d}^3 x \left(\frac{\partial \mathcal{L}(A^\lambda(t, \mathbf{y}), \partial_\gamma A^\kappa(t, \mathbf{y}))}{\partial A^\rho} \frac{\delta A^\rho(t, \mathbf{y})}{\delta \dot{A}^\mu(t, \mathbf{x})} \right. \\ & \left. + \frac{\partial \mathcal{L}(A^\lambda(t, \mathbf{y}), \partial_\gamma A^\kappa(t, \mathbf{y}))}{\partial(\partial_\sigma A^\rho)} \frac{\delta(\partial_\sigma A^\rho(t, \mathbf{y}))}{\delta \dot{A}^\mu(t, \mathbf{x})} \right). \end{aligned} \tag{17}$$

Se tiene que

$$\frac{\delta A^\rho(t, \mathbf{y})}{\delta \dot{A}^\mu(t, \mathbf{x})} = 0 = \frac{\delta(\partial_i A^\rho(t, \mathbf{y}))}{\delta \dot{A}^\mu(t, \mathbf{x})} \tag{18}$$

para todo $i \in \{1,2,3\}$. Por lo tanto

$$
\begin{aligned}
\Pi_\mu(t,\mathbf{x}) &= \int \mathrm{d}^3x \, \frac{\partial \mathcal{L}(A^\lambda(t,\mathbf{y}), \partial_\gamma A^\kappa(t,\mathbf{y}))}{\partial(\partial_\sigma A^\rho)} \frac{\delta(\partial_\sigma A^\rho(t,\mathbf{y}))}{\delta \dot{A}^\mu(t,\mathbf{x})} \\
&= \int \mathrm{d}^3x \left(\frac{\partial \mathcal{L}(A^\lambda(t,\mathbf{y}), \partial_\gamma A^\kappa(t,\mathbf{y}))}{\partial \dot{A}^\rho} \frac{\delta \dot{A}^\rho(t,\mathbf{y})}{\delta \dot{A}^\mu(t,\mathbf{x})} \right. \\
&\quad \left. \sum_{i=1}^{3} \frac{\partial \mathcal{L}(A^\lambda(t,\mathbf{y}), \partial_\gamma A^\kappa(t,\mathbf{y}))}{\partial(\partial_i A^\rho)} \frac{\delta(\partial_i A^\rho(t,\mathbf{y}))}{\delta \dot{A}^\mu(t,\mathbf{x})} \right) \\
&= \int \mathrm{d}^3x \, \frac{\partial \mathcal{L}(A^\lambda(t,\mathbf{y}), \partial_\gamma A^\kappa(t,\mathbf{y}))}{\partial \dot{A}^\rho} \frac{\delta \dot{A}^\rho(t,\mathbf{y})}{\delta \dot{A}^\mu(t,\mathbf{x})}.
\end{aligned}
\tag{19}
$$

Ya que

$$
\frac{\delta \dot{A}^\rho(t,\mathbf{y})}{\delta \dot{A}^\mu(t,\mathbf{x})} = \delta^\rho_\mu \delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{20}
$$

concluimos que

$$
\begin{aligned}
\Pi_\mu(t,\mathbf{x}) = & \int \mathrm{d}^3x \, \frac{\partial \mathcal{L}(A^\lambda(t,\mathbf{y}), \partial_\gamma A^\kappa(t,\mathbf{y}))}{\partial \dot{A}^\rho} \delta^\rho_\mu \delta^{(3)}(\mathbf{x}-\mathbf{y}) \\
= & \delta^\rho_\mu \frac{\partial \mathcal{L}(A^\lambda(t,\mathbf{x}), \partial_\gamma A^\kappa(t,\mathbf{x}))}{\partial \dot{A}^\rho} = \frac{\partial \mathcal{L}(A^\lambda(t,\mathbf{x}), \partial_\gamma A^\kappa(t,\mathbf{x}))}{\partial \dot{A}^\mu}.
\end{aligned}
\tag{21}
$$

Haciendo uso de la expresión (**??**) y la antisimetría de $F^{\mu\nu}$ concluimos que

$$
\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} = -\frac{1}{4} \frac{\partial F^{\sigma\rho} F_{\sigma\rho}}{\partial(\partial_0 A^\mu)} = -\frac{1}{4} 4 g_{\mu\rho} F^{0\rho} = g_{\mu\rho} F^{\rho 0}. \tag{22}
$$

Podemos expresar este resultado de manera más natural notando

$$
\Pi^\mu = g^{\mu\nu} \Pi_\nu = g^{\mu\nu} g_{\nu\rho} F^{\rho 0} = \delta^\mu_\rho F^{\rho 0} = F^{\mu 0}. \tag{23}
$$

Se concluye que el momento conjugado canónico es

$$
\Pi^\mu = F^{\mu 0}. \tag{24}
$$

Estas ecuaciones no se pueden invertir ya que $\Pi^0 = 0$. Es claro que sin más restricciones es imposible expresar el cuadrivector A^μ en términos de un Π^μ cuya primera componente se desvanece idénticamente.