Electrodynamics: Homework 5

Iván Mauricio Burbano Aldana

April 29, 2018

The Green function for the *massive photon* equation is given by

$$(\Box + m^2)G(x - x') = \delta^{(4)}(x - x'). \tag{1}$$

Consider the ansatz

$$G(x) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\, e^{-ikx}\tilde{G}(k). \tag{2}$$

By using the representation

$$\delta^{(4)}(x) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\, e^{-ikx} \tag{3}$$

on (1) with $x' = 0$ we obtain

$$\begin{aligned}
\frac{1}{(2\pi)^4}\int \mathrm{d}^4k\, e^{-ikx} =&(\Box + m^2)G(x) = (\Box + m^2)\frac{1}{(2\pi)^4}\int \mathrm{d}^4k\, e^{-ikx}\tilde{G}(k)\\
=&\frac{1}{(2\pi)^4}\int \mathrm{d}^4k\,(\Box + m^2)e^{-ikx}\tilde{G}(k)\\
=&\frac{1}{(2\pi)^4}\int \mathrm{d}^4k\,(\partial^\mu\partial_\mu + m^2)e^{-ikx}\tilde{G}(k)\\
=&\frac{1}{(2\pi)^4}\int \mathrm{d}^4k\,((-ik^\mu)(-ik_\mu) + m^2)e^{-ikx}\tilde{G}(k)\\
=&\frac{1}{(2\pi)^4}\int \mathrm{d}^4k\,(-k^2 + m^2)e^{-ikx}\tilde{G}(k).
\end{aligned} \tag{4}$$

This in turn tells us that

$$\tilde{G}(k) = \frac{1}{m^2 - k^2}. \tag{5}$$

Thus, the Green function is given by

$$\begin{aligned}
G(x) =&\frac{1}{(2\pi)^4}\int \mathrm{d}^4k\,\frac{e^{-ikx}}{m^2 - k^2} = \frac{1}{(2\pi)^4}\int \mathrm{d}^3\mathbf{k}\int \mathrm{d}k^0\,\frac{e^{-i(k^0x^0 - \mathbf{k}\cdot\mathbf{x})}}{m^2 - (k^0)^2 + \mathbf{k}^2}\\
=&-\frac{1}{(2\pi)^4}\int \mathrm{d}^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\int \mathrm{d}k^0\,\frac{e^{-ik^0x^0}}{(k^0)^2 - (m^2 + \mathbf{k}^2)}\\
=&-\frac{1}{(2\pi)^4}\int \mathrm{d}^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\int \mathrm{d}k^0\,\frac{e^{-ik^0x^0}}{(k^0)^2 - E_{\mathbf{k}}^2}.
\end{aligned} \tag{6}$$

where

$$E_{\mathbf{k}} := \sqrt{m^2 + \mathbf{k}^2}. \tag{7}$$

The procedure is now precisely the one followed in class. We consider the paths

$$\begin{aligned}
L_{R,\epsilon} :=& \{x - i\epsilon | x \in [-R, R]\}, \\
U_{R,\epsilon} :=& \{x + i\epsilon | x \in [-R, R]\}, \\
S_R^+ :=& \{Re^{i\theta} | \theta \in [0, \pi]\}, \\
S_R^- :=& \{Re^{i\theta} | \theta \in [-\pi, 0]\}, \\
\mathcal{C}_{R,\epsilon}^+ :=& L_{R,\epsilon} \cup \{\pm R - iy | y \in [0, \epsilon]\} \cup S_R^+, \\
\mathcal{D}_{R,\epsilon}^+ :=& U_{R,\epsilon} \cup S_{\sqrt{R^2+\epsilon^2}}^+ \setminus \{x + iy \in \mathbb{C} | y < \epsilon\}, \\
\mathcal{C}_{R,\epsilon}^- :=& U_{R,\epsilon} \cup \{\pm R + iy | y \in [0, \epsilon]\} \cup S_R^-, \\
\mathcal{D}_{R,\epsilon}^- :=& L_{R,\epsilon} \cup S_{\sqrt{R^2+\epsilon^2}}^- \setminus \{x + iy \in \mathbb{C} | y > -\epsilon\},
\end{aligned} \tag{8}$$

for $\epsilon > 0$ and $R > E_{\mathbf{k}}$. We orient $\mathcal{C}_{R,\epsilon}^+$ and $\mathcal{D}_{R,\epsilon}^+$ counterclockwise, and $\mathcal{C}_{R,\epsilon}^-$ and $\mathcal{D}_{R,\epsilon}^-$ clockwise. If

$$\begin{aligned}
f : \mathbb{C} &\to \mathbb{C} \\
z &\mapsto \frac{e^{-izx^0}}{z^2 - E_{\mathbf{k}}^2} = \frac{e^{-izx^0}}{(z - E_{\mathbf{k}})(z + E_{\mathbf{k}})}
\end{aligned} \tag{9}$$

by the residue theorem

$$\begin{aligned}
\int_{\mathcal{C}_{R,\epsilon}^+} \mathrm{d}z\, f(z) =& 2\pi i \left(\frac{e^{-iE_{\mathbf{k}}x^0}}{2E_{\mathbf{k}}} - \frac{e^{iE_{\mathbf{k}}x^0}}{2E_{\mathbf{k}}} \right) = 2\pi \frac{\sin\left(E_{\mathbf{k}}x^0\right)}{E_{\mathbf{k}}} \\
\int_{\mathcal{C}_{R,\epsilon}^-} \mathrm{d}z\, f(z) =& -2\pi \left(\frac{e^{-iE_{\mathbf{k}}x^0}}{2E_{\mathbf{k}}} - \frac{e^{iE_{\mathbf{k}}x^0}}{2E_{\mathbf{k}}} \right) = -2\pi \frac{\sin\left(E_{\mathbf{k}}x^0\right)}{E_{\mathbf{k}}} \\
\int_{\mathcal{D}_{R,\epsilon}^+} \mathrm{d}z\, f(z) =& 0 \\
\int_{\mathcal{D}_{R,\epsilon}^-} \mathrm{d}z\, f(z) =& 0
\end{aligned} \tag{10}$$

Take $x^0 < 0$. For $\theta \in [0, \pi]$, $\sin(\theta)x^0 < 0$ and thus

$$
\begin{aligned}
|f(Re^{i\theta})|^2 = & \left| \frac{e^{-iRe^{i\theta}x^0}}{R^2 e^{i2\theta} - E_{\mathbf{k}}^2} \right|^2 = \frac{\left| e^{-iR\cos(\theta)x^0} e^{R\sin(\theta)x^0} \right|^2}{(R^2 e^{i2\theta} - E_{\mathbf{k}}^2)(R^2 e^{-i2\theta} - E_{\mathbf{k}}^2)} \\
= & \frac{\left| e^{R\sin(\theta)x^0} \right|^2}{R^4 + E_{\mathbf{k}}^4 - R^2 E_{\mathbf{k}}^2 (e^{i2\theta} + e^{-i2\theta})} = \frac{e^{2R\sin(\theta)x^0}}{R^4 + E_{\mathbf{k}}^4 - 2R^2 E_{\mathbf{k}}^2 \cos(2\theta)} \\
\leq & \frac{1}{R^4 + E_{\mathbf{k}}^4 - 2R^2 E_{\mathbf{k}}^2 \cos(2\theta)} \leq \frac{1}{R^4 + E_{\mathbf{k}}^4 - 2R^2 E_{\mathbf{k}}^2} \\
= & \frac{1}{(R^2 - E_{\mathbf{k}}^2)^2}.
\end{aligned} \tag{11}
$$

Therefore

$$
\begin{aligned}
\left| \int_{S_R^+} \mathrm{d}z \, f(z) \right| \leq \int_{S_R^+} \mathrm{d}z \, |f(z)| \leq \pi R \sup \left\{ \left| f\left(Re^{i\theta}\right) \right| \middle| \theta \in [0, \pi] \right\} \\
\leq \frac{\pi R}{R^2 - E_{\mathbf{k}}^2} = \frac{\pi / R}{1 - E_{\mathbf{k}}^2 / R^2} \to 0
\end{aligned} \tag{12}
$$

as $R \to \infty$. Similarly for $x^0 > 0$ we have for $\theta \in [-\pi, 0]$ that

$$
\begin{aligned}
|f(Re^{i\theta})|^2 = & \left| \frac{e^{-iRe^{i\theta}x^0}}{R^2 e^{i2\theta} - E_{\mathbf{k}}^2} \right|^2 = \frac{\left| e^{-iR\cos(\theta)x^0} e^{R\sin(\theta)x^0} \right|^2}{(R^2 e^{i2\theta} - E_{\mathbf{k}}^2)(R^2 e^{-i2\theta} - E_{\mathbf{k}}^2)} \\
= & \frac{\left| e^{R\sin(\theta)x^0} \right|^2}{R^4 + E_{\mathbf{k}}^4 - R^2 E_{\mathbf{k}}^2 (e^{i2\theta} + e^{-i2\theta})} = \frac{e^{2R\sin(\theta)x^0}}{R^4 + E_{\mathbf{k}}^4 - 2R^2 E_{\mathbf{k}}^2 \cos(2\theta)} \\
\leq & \frac{1}{R^4 + E_{\mathbf{k}}^4 - 2R^2 E_{\mathbf{k}}^2 \cos(2\theta)} \leq \frac{1}{R^4 + E_{\mathbf{k}}^4 - 2R^2 E_{\mathbf{k}}^2} \\
= & \frac{1}{(R^2 - E_{\mathbf{k}}^2)^2},
\end{aligned} \tag{13}
$$

and thus

$$
\begin{aligned}
\left| \int_{S_R^-} \mathrm{d}z \, f(z) \right| \leq \int_{S_R^-} \mathrm{d}z \, |f(z)| \leq \pi R \sup \left\{ \left| f\left(Re^{i\theta}\right) \right| \middle| \theta \in [-\pi, 0] \right\} \\
\leq \frac{\pi R}{R^2 - E_{\mathbf{k}}^2} = \frac{\pi / R}{1 - E_{\mathbf{k}}^2 / R^2} \to 0
\end{aligned} \tag{14}
$$

as $R \to \infty$. We conclude for $x^0 < 0$ that

$$
\begin{aligned}
\int_{L_{R,\epsilon}} \mathrm{d}z\, f(z) =& \int_{\mathcal{C}^+_{R,\epsilon}} \mathrm{d}z\, f(z) - \int_{\{\pm R - iy | y\in[0,\epsilon]\}} \mathrm{d}z\, f(z) - \int_{S^+_R} \mathrm{d}z\, f(z) \\
=& 2\pi \frac{\sin\left(E_{\mathbf{k}} x^0\right)}{E_{\mathbf{k}}} - \int_{\{\pm R - iy | y\in[0,\epsilon]\}} \mathrm{d}z\, f(z) - \int_{S^+_R} \mathrm{d}z\, f(z) \\
\to& 2\pi \frac{\sin\left(E_{\mathbf{k}} x^0\right)}{E_{\mathbf{k}}} - \int_{\emptyset} \mathrm{d}z\, f(z) = 2\pi \frac{\sin\left(E_{\mathbf{k}} x^0\right)}{E_{\mathbf{k}}}, \\
\int_{U_{R,\epsilon}} \mathrm{d}z\, f(z) =& \int_{\mathcal{D}^+_{R,\epsilon}} \mathrm{d}z\, f(z) - \int_{S^+_{\sqrt{R^2+\epsilon^2}} \setminus \{x+iy\in\mathbb{C} | y<\epsilon\}} \mathrm{d}z\, f(z) \\
=& - \int_{S^+_{\sqrt{R^2+\epsilon^2}} \setminus \{x+iy\in\mathbb{C} | y<\epsilon\}} \mathrm{d}z\, f(z) \to \int_{S^+_R} \mathrm{d}z\, f(z) \to 0
\end{aligned}
\tag{15}
$$

as $R \to \infty$ and $\epsilon \to 0$. On the other hand, for $x^0 > 0$ we have

$$
\begin{aligned}
\int_{U_{R,\epsilon}} \mathrm{d}z\, f(z) =& \int_{\mathcal{C}^-_{R,\epsilon}} \mathrm{d}z\, f(z) - \int_{\{\pm R + iy | y\in[0,\epsilon]\}} \mathrm{d}z\, f(z) - \int_{S^-_R} \mathrm{d}z\, f(z) \\
=& - 2\pi \frac{\sin\left(E_{\mathbf{k}} x^0\right)}{E_{\mathbf{k}}} - \int_{\{\pm R + iy | y\in[0,\epsilon]\}} \mathrm{d}z\, f(z) - \int_{S^-_R} \mathrm{d}z\, f(z) \\
\to& - 2\pi \frac{\sin\left(E_{\mathbf{k}} x^0\right)}{E_{\mathbf{k}}} - \int_{\emptyset} \mathrm{d}z\, f(z) = -2\pi \frac{\sin\left(E_{\mathbf{k}} x^0\right)}{E_{\mathbf{k}}}, \\
\int_{L_{R,\epsilon}} \mathrm{d}z\, f(z) =& \int_{\mathcal{D}^-_{R,\epsilon}} \mathrm{d}z\, f(z) - \int_{S^-_{\sqrt{R^2+\epsilon^2}} \setminus \{x+iy\in\mathbb{C} | y>-\epsilon\}} \mathrm{d}z\, f(z) \\
=& - \int_{S^-_{\sqrt{R^2+\epsilon^2}} \setminus \{x+iy\in\mathbb{C} | y>-\epsilon\}} \mathrm{d}z\, f(z) \to \int_{S^-_R} \mathrm{d}z\, f(z) \to 0
\end{aligned}
\tag{16}
$$

as $R \to \infty$ and $\epsilon \to 0$. These results may be summarized in

$$
\begin{aligned}
\int_{U_{R,\epsilon}} \mathrm{d}z\, f(z) \to& - 2\pi\theta(x^0) \frac{\sin\left(E_{\mathbf{k}} x^0\right)}{E_{\mathbf{k}}} \\
\int_{L_{R,\epsilon}} \mathrm{d}z\, f(z) \to& 2\pi\theta(-x^0) \frac{\sin\left(E_{\mathbf{k}} x^0\right)}{E_{\mathbf{k}}}
\end{aligned}
\tag{17}
$$

as $R \to \infty$ and $\epsilon \to 0$. In here θ is the Heaviside step function.

Both of these integrals are on equal footing to be regarded as the solutions of

$$
\int \mathrm{d}k^0\, \frac{e^{-ik^0 x^0}}{(k^0)^2 - E^2_{\mathbf{k}}}. \tag{18}
$$

Considering the first, we obtain

$$
G(x) = \frac{\theta(x^0)}{(2\pi)^3} \int \mathrm{d}^3\mathbf{k}\, \frac{e^{i\mathbf{k}\cdot\mathbf{x}} \sin\left(E_{\mathbf{k}} x^0\right)}{E_{\mathbf{k}}} \tag{19}
$$

On the other hand, the second yields

$$G(x) = -\frac{\theta(-x^0)}{(2\pi)^3} \int \mathrm{d}^3\mathbf{k}\, \frac{e^{i\mathbf{k}\cdot\mathbf{x}} \sin\bigl(E_{\mathbf{k}} x^0\bigr)}{E_{\mathbf{k}}}. \tag{20}$$

These results may be summarized by

$$G_{\pm}(x - x') = \pm \frac{\theta(\pm(x^0 - x'^0)}{(2\pi)^3} \int \mathrm{d}^3\mathbf{k}\, \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \sin\Bigl(\Bigl(x^0 - x'^0\Bigr)\sqrt{m^2 + \mathbf{k}^2}\Bigr)}{\sqrt{m^2 + \mathbf{k}^2}}.$$