

Mecánica Cuántica Avanzada

Tarea 5

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21 de abril de 2018

4.8 Multiplicando por izquierda $\bar{u}_r(\mathbf{p}')\gamma^\mu$ a la ecuación (4.46) de [1] se obtiene

$$0 = \bar{u}_r(\mathbf{p}')\gamma^\mu(\not{p} - m)u_s(\mathbf{p}) = \bar{u}_r(\mathbf{p}')\gamma^\mu\gamma^\nu p_\nu u_s(\mathbf{p}) - \bar{u}_r(\mathbf{p}')\gamma^\mu m u_s(\mathbf{p}). \quad (1)$$

Multiplicando por la derecha $\gamma^\mu u_s(\mathbf{p})$ a la ecuación (4.48) de [1] se obtiene

$$0 = \bar{u}_r(\mathbf{p}')(\not{p}' - m)\gamma^\mu u_s(\mathbf{p}) = \bar{u}_r(\mathbf{p}')\gamma^\nu\gamma^\mu p'_\nu u_s(\mathbf{p}) - \bar{u}_r(\mathbf{p}')\gamma^\mu m u_s(\mathbf{p}). \quad (2)$$

Sumandolas se concluye que

$$\begin{aligned} 0 &= \bar{u}_r(\mathbf{p}')\gamma^\mu\gamma^\nu p_\nu u_s(\mathbf{p}) - \bar{u}_r(\mathbf{p}')\gamma^\mu m u_s(\mathbf{p}) \\ &\quad + \bar{u}_r(\mathbf{p}')\gamma^\nu\gamma^\mu p'_\nu u_s(\mathbf{p}) - \bar{u}_r(\mathbf{p}')\gamma^\mu m u_s(\mathbf{p}) \\ &= \bar{u}_r(\mathbf{p}')(\gamma^\mu\gamma^\nu p_\nu + \gamma^\nu\gamma^\mu p'_\nu)u_s(\mathbf{p}) - 2\bar{u}_r(\mathbf{p}')\gamma^\mu m u_s(\mathbf{p}). \end{aligned} \quad (3)$$

En particular, si $\mathbf{p} = \mathbf{p}'$

$$0 = \bar{u}_r(\mathbf{p})2g^{\mu\nu}p_\nu u_s(\mathbf{p}) - 2m\bar{u}_r(\mathbf{p})\gamma^\mu u_s(\mathbf{p}). \quad (4)$$

Dividiendo por 2 y subiendo el índice del momento entonces es claro que

$$\bar{u}_r(\mathbf{p})\gamma^\mu m u_s(\mathbf{p}) = \bar{u}_r(\mathbf{p})p^\mu u_s(\mathbf{p}). \quad (5)$$

Repitiendo este proceso al pie de la letra se tiene

$$0 = \bar{v}_r(\mathbf{p}')\gamma^\mu(\not{p} + m)v_s(\mathbf{p}) = \bar{v}_r(\mathbf{p}')\gamma^\mu\gamma^\nu p_\nu v_s(\mathbf{p}) + \bar{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}), \quad (6)$$

$$0 = \bar{v}_r(\mathbf{p}')(\not{p}' + m)\gamma^\mu v_s(\mathbf{p}) = \bar{v}_r(\mathbf{p}')\gamma^\nu\gamma^\mu p'_\nu v_s(\mathbf{p}) + \bar{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}), \quad (7)$$

$$\begin{aligned} 0 &= \bar{v}_r(\mathbf{p}')\gamma^\mu\gamma^\nu p_\nu v_s(\mathbf{p}) + \bar{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}) \\ &\quad + \bar{v}_r(\mathbf{p}')\gamma^\nu\gamma^\mu p'_\nu v_s(\mathbf{p}) + \bar{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}) \\ &= \bar{v}_r(\mathbf{p}')(\gamma^\mu\gamma^\nu p_\nu + \gamma^\nu\gamma^\mu p'_\nu)v_s(\mathbf{p}) + 2\bar{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}) \\ &= \bar{v}_r(\mathbf{p}')2g^{\mu\nu}p_\nu v_s(\mathbf{p}) + 2m\bar{v}_r(\mathbf{p}')\gamma^\mu v_s(\mathbf{p}) \end{aligned} \quad (8)$$

en el caso $\mathbf{p} = \mathbf{p}'$ y

$$\bar{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}) = -\bar{v}_r(\mathbf{p}')p^\mu v_s(\mathbf{p}). \quad (9)$$

Poniendo $\mu = 0$ en (5) se tiene haciendo uso de la normalización (4.49) de [1]

$$\begin{aligned} 2E_p \delta_{rs} m &= m u_r^\dagger(\mathbf{p}) u_s(\mathbf{p}) = m u_r^\dagger(\mathbf{p}) \gamma^0 \gamma^0 u_s(\mathbf{p}) = m \bar{u}_r(\mathbf{p}) \gamma^0 u_s(\mathbf{p}) \\ &= p^0 \bar{u}_r(\mathbf{p}) u(\mathbf{p}) = E_p \bar{u}_r(\mathbf{p}) u_s(\mathbf{p}). \end{aligned} \quad (10)$$

Repitiendo con (9)

$$\begin{aligned} 2E_p \delta_{rs} m &= m v_r^\dagger(\mathbf{p}) v_s(\mathbf{p}) = m v_r^\dagger(\mathbf{p}) \gamma^0 \gamma^0 v_s(\mathbf{p}) = m \bar{v}_r(\mathbf{p}) \gamma^0 v_s(\mathbf{p}) \\ &= -p^0 \bar{v}_r(\mathbf{p}) v(\mathbf{p}) = -E_p \bar{v}_r(\mathbf{p}) v_s(\mathbf{p}). \end{aligned} \quad (11)$$

Por lo tanto, asumiendo que $E_p \neq 0$, se tiene

$$\bar{u}_r(\mathbf{p}) u_s(\mathbf{p}) = -\bar{v}_r(\mathbf{p}) v_s(\mathbf{p}) = 2m \delta_{rs}. \quad (12)$$

4.9 Note que haciendo uso de la relaciones de conmutación de las matrices de Dirac se tiene

$$\begin{aligned} &(p + p')^\mu - i\sigma^{\mu\nu} q_\nu \\ &= p^\mu + p'^\mu - i\frac{1}{2}[\gamma^\mu, \gamma^\nu]_-(p_\nu - p'_\nu) \\ &= p^\mu + p'^\mu + \frac{1}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)(p_\nu - p'_\nu) \\ &= p^\mu + p'^\mu + \frac{1}{2}(\gamma^\mu \gamma^\nu p_\nu - \gamma^\mu \gamma^\nu p'_\nu - \gamma^\nu \gamma^\mu p_\nu + \gamma^\nu \gamma^\mu p'_\nu) \\ &= p^\mu + p'^\mu \\ &\quad + \frac{1}{2}(\gamma^\mu \gamma^\nu p_\nu - 2g^{\mu\nu} p'_\nu + \gamma^\nu \gamma^\mu p'_\nu - 2g^{\nu\mu} p_\nu + \gamma^\mu \gamma^\nu p_\nu + \gamma^\nu \gamma^\mu p'_\nu) \\ &= p^\mu + p'^\mu + \frac{1}{2}(2\gamma^\mu \gamma^\nu p_\nu - 2p'^\mu + 2\gamma^\nu \gamma^\mu p'_\nu - 2p^\mu) \\ &= \gamma^\mu \gamma^\nu p_\nu + \gamma^\nu \gamma^\mu p'_\nu. \end{aligned} \quad (13)$$

Por lo tanto, comparando con las ecuaciones (3) y (8) se obtiene

$$\begin{aligned} 0 &= \bar{u}_r(\mathbf{p}')((p + p')^\mu - i\sigma^{\mu\nu} q_\nu) u_s(\mathbf{p}) - 2m \bar{u}_r(\mathbf{p}') \gamma^\mu u_s(\mathbf{p}) \\ 0 &= \bar{v}_r(\mathbf{p}')((p + p')^\mu - i\sigma^{\mu\nu} q_\nu) v_s(\mathbf{p}) + 2m \bar{v}_r(\mathbf{p}') \gamma^\mu v_s(\mathbf{p}). \end{aligned} \quad (14)$$

Dividiendo por $2m$ se obtienen las identidades de Gordon

$$\begin{aligned} \bar{u}_r(\mathbf{p}') \gamma^\mu u_s(\mathbf{p}) &= \frac{1}{2m} \bar{u}_r(\mathbf{p}')((p + p')^\mu - i\sigma^{\mu\nu} q_\nu) u_s(\mathbf{p}) \\ \bar{v}_r(\mathbf{p}') \gamma^\mu v_s(\mathbf{p}) &= -\frac{1}{2m} \bar{v}_r(\mathbf{p}')((p + p')^\mu - i\sigma^{\mu\nu} q_\nu) v_s(\mathbf{p}). \end{aligned} \quad (15)$$

4.10 Como en el ejercicio 4.8, multiplicando a izquierda por $\bar{u}_r(\mathbf{p}) \gamma^\mu$ a la ecuación (4.46) de [1] y por la derecha por $\gamma^\mu v_s(\mathbf{p})$ a (4.48) de [1] se obtiene

$$\begin{aligned} \bar{u}_r(\mathbf{p}) \gamma^\mu (\not{p} + m) v_s(\mathbf{p}) &= 0 \\ \bar{u}_r(\mathbf{p}) (\not{p} - m) \gamma^\mu v_s(\mathbf{p}) &= 0. \end{aligned} \quad (16)$$

Al sumar estas ecuaciones se concluye

$$\begin{aligned}
0 &= \bar{u}_r(\mathbf{p})\gamma^\mu(\not{p} + m)v_s(\mathbf{p}) + \bar{u}_r(\mathbf{p})(\not{p} - m)\gamma^\mu v_s(\mathbf{p}) \\
&= \bar{u}_r(\mathbf{p})(\gamma^\mu \not{p} + \not{p}\gamma^\mu)v_s(\mathbf{p}) = \bar{u}_r(\mathbf{p})(\gamma^\mu \gamma^\nu p_\nu + \gamma^\nu \gamma^\mu p_\nu)v_s(\mathbf{p}) \\
&= \bar{u}_r(\mathbf{p})(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)p_\nu v_s(\mathbf{p}) = \bar{u}_r(\mathbf{p})2g^{\mu\nu}p_\nu v_s(\mathbf{p}) \\
&= 2p^\mu \bar{u}_r(\mathbf{p})v_s(\mathbf{p}),
\end{aligned} \tag{17}$$

es decir,

$$0 = \bar{u}_r(\mathbf{p})v_s(\mathbf{p}). \tag{18}$$

Conjugando se tiene

$$\begin{aligned}
0 &= (\bar{u}_r(\mathbf{p})v_s(\mathbf{p}))^\dagger = (u_r^\dagger(\mathbf{p})\gamma^0 v_s(\mathbf{p}))^\dagger = v_s(\mathbf{p})^\dagger \gamma^0 u_r(\mathbf{p}) \\
&= \bar{v}_s(\mathbf{p})u_r(\mathbf{p}).
\end{aligned} \tag{19}$$

Note que debido a la ecuación de Dirac (4.46) de [1]

$$\begin{aligned}
(\not{p} + m)u_r(\mathbf{p}) &= (\not{p} - m + 2m)u_r(\mathbf{p}) = 2mu_r(\mathbf{p}) \\
(\not{p} + m)v_r(\mathbf{p}) &= 0 \\
(\not{p} - m)u_r(\mathbf{p}) &= 0 \\
(\not{p} - m)v_r(\mathbf{p}) &= (\not{p} + m - 2m)v_r(\mathbf{p}) = -2mv_r(\mathbf{p}).
\end{aligned} \tag{20}$$

Por el otro lado, haciendo uso de las relaciones de normalización halladas en el ejercicio 4.8 y las relaciones (18) y (19)

$$\begin{aligned}
\sum_s u_s(\mathbf{p})\bar{u}_s(\mathbf{p})u_r(\mathbf{p}) &= \sum_s u_s(\mathbf{p})2m\delta_{sr} = 2mu_r(\mathbf{p}) \\
\sum_s u_s(\mathbf{p})\bar{u}_s(\mathbf{p})v_r(\mathbf{p}) &= 0 \\
\sum_s v_s(\mathbf{p})\bar{v}_s(\mathbf{p})u_r(\mathbf{p}) &= 0 \\
\sum_s v_s(\mathbf{p})\bar{v}_s(\mathbf{p})v_r(\mathbf{p}) &= \sum_s v_s(\mathbf{p})(-2m\delta_{sr}) = -2mv_r(\mathbf{p}).
\end{aligned} \tag{21}$$

Ya que las matrices coinciden en una base, por extensión lineal deben ser iguales

$$\begin{aligned}
\sum_s u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) &= \not{p} + m \\
\sum_s v_s(\mathbf{p})\bar{v}_s(\mathbf{p}) &= \not{p} - m.
\end{aligned} \tag{22}$$

4.24 Se tiene con la ecuación (A.32) de [1] que

$$\begin{aligned}
W^\mu W_\mu &= \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} \epsilon_{\mu\nu'\lambda'\rho'} P_\nu J_{\lambda\rho} P^{\nu'} J^{\lambda'\rho'} \\
&= -\frac{1}{4} (\delta_{\nu'}^\nu \delta_{\lambda'}^\lambda \delta_{\rho'}^\rho + \delta_{\lambda'}^\nu \delta_{\rho'}^\lambda \delta_{\nu'}^\rho + \delta_{\rho'}^\nu \delta_{\nu'}^\lambda \delta_{\lambda'}^\rho \\
&\quad - \delta_{\lambda'}^\nu \delta_{\nu'}^\lambda \delta_{\rho'}^\rho - \delta_{\rho'}^\nu \delta_{\lambda'}^\lambda \delta_{\nu'}^\rho - \delta_{\nu'}^\nu \delta_{\rho'}^\lambda \delta_{\lambda'}^\rho) P_\nu J_{\lambda\rho} P^{\nu'} J^{\lambda'\rho'} \\
&= -\frac{1}{4} P_\nu J_{\lambda\rho} (P^\nu J^{\lambda\rho} + P^\rho J^{\nu\lambda} + P^\lambda J^{\rho\nu} \\
&\quad - P^\lambda J^{\nu\rho} - P^\rho J^{\lambda\nu} - P^\nu J^{\rho\lambda}) \\
&= -\frac{1}{4} P_\nu J_{\lambda\rho} (P^\nu (J^{\lambda\rho} - J^{\rho\lambda}) + P^\rho (J^{\nu\lambda} - J^{\lambda\nu}) \\
&\quad + P^\lambda (J^{\rho\nu} - J^{\nu\rho}))
\end{aligned} \tag{23}$$

Note las siguientes propiedades de antisimetría

$$\begin{aligned}
\sigma_{\mu\nu} &= \frac{i}{2} [\gamma^\mu, \gamma^\nu]_- = -\frac{i}{2} [\gamma^\nu, \gamma^\mu]_- = -\sigma^{\nu\mu} \\
J_{\mu\nu} &= i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2} \sigma_{\mu\nu} = -i(x_\nu \partial_\mu - x_\mu \partial_\nu) - \frac{1}{2} \sigma_{\nu\mu} = -J_{\nu\mu}.
\end{aligned} \tag{24}$$

Por lo tanto

$$W^\mu W_\mu = -\frac{1}{2} P_\nu J_{\lambda\rho} (P^\nu J^{\lambda\rho} + P^\rho J^{\nu\lambda} + P^\lambda J^{\rho\nu}) \tag{25}$$

Referencias

- [1] A. Lahiri and P. B. Pal, *A First Book of Quantum Field Theory*. 2005.