Tarea 3: Aplicaciones de mecánica estadística cuántica

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1. El modelo XY cuántico en una dimensión

1.1. El modelo y preliminares

De la forma explicita de los operadores de spin vemos que podemos tomar como espacio de Hilbert de una partícula $\mathbb{C}^2$. El espacio de Hilbert total es $\mathcal{H} = \bigotimes_{n=1}^{N} \mathbb{C}^2$ y los operadores de spin son

$$S_n^i = \mathrm{id}_{\mathbb{C}^2} \otimes \cdots \otimes \mathrm{id}_{\mathbb{C}^2} \otimes \underbrace{S^i}_{\text{n-ésimo termino}} \otimes \mathrm{id}_{\mathbb{C}^2} \otimes \cdots \otimes \mathrm{id}_{\mathbb{C}}^2 \tag{1}$$

para todo $n \in \{1, \ldots, N\}$ e $i \in \{x, y, z, +, -\}$. Entonces podemos escoger $|+\rangle = (1, 0)$ y $|-\rangle = (0, 1)$. Denotaremos para $s_1, \ldots, s_N \in \{+, -\}$

$$|s_1 \cdots s_N\rangle = |s_1\rangle \otimes \cdots \otimes |s_N\rangle . \tag{2}$$

1. Calculando se obtiene

$$\begin{aligned}
S^+ |-\rangle =&(S^x + iS^y) |-\rangle = \frac{1}{2}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
=&\frac{1}{2}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |+\rangle \\
S^- |-\rangle =&(S^x - iS^y) |-\rangle = \frac{1}{2}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
=&\frac{1}{2}\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \\
S^+ |+\rangle =&\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \\
S^- |+\rangle =&\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |-\rangle .
\end{aligned} \tag{3}$$

1.2. Desarrollo

1. Se tiene

$$\begin{aligned}\{S^+, S^-\} \left|+\right\rangle =& (S^+S^- + S^-S^+) \left|+\right\rangle = \left|+\right\rangle + 0 = \left|+\right\rangle \\ \{S^+, S^-\} \left|-\right\rangle =& (S^+S^- + S^-S^+) \left|-\right\rangle = 0 + \left|-\right\rangle = \left|-\right\rangle ,\end{aligned} \tag{4}$$

es decir, $\{S^+, S^-\} = \mathrm{id}_{\mathbb{C}^2}$. Entonces para todo $n \in \{1, \dots, N\}$ se tiene

$$\begin{aligned}\{S_n^+, S_n^-\} =& (S_n^+S_n^- + S_n^-S_n^+) \\ =& \mathrm{id}_{\mathbb{C}^2} \otimes \cdots \otimes \mathrm{id}_{\mathbb{C}^2} \otimes \underbrace{S^+S^- + S^-S^+}_{\text{n-ésimo termino}} \otimes \mathrm{id}_{\mathbb{C}^2} \otimes \cdots \otimes \mathrm{id}_{\mathbb{C}}^2 \\ =& \mathrm{id}_{\mathcal{H}} .\end{aligned} \tag{5}$$

2. Se tiene

$$2S^+S^- - \mathrm{id}_{\mathbb{C}^2} = 2S^+S^- - \{S^+, S^-\} = [S^+, S^-] \tag{6}$$

y

$$\begin{aligned}[S^+, S^-] \left|+\right\rangle =& (S^+S^- - S^-S^+) \left|+\right\rangle = \left|+\right\rangle \\ [S^+, S^-] \left|-\right\rangle =& (S^+S^- - S^-S^+) \left|-\right\rangle = - \left|-\right\rangle .\end{aligned} \tag{7}$$

Se concluye entonces que $2S^+S^- - \mathrm{id}_{\mathbb{C}^2} = [S^+, S^-] = 2S^z$. Tensorizando la ecuación a ambos lados con identidades es claro que

$$2S_n^+S_n^- - \mathrm{id}_{\mathcal{H}} = [S_n^+, S_n^-] = 2S_n^z. \tag{8}$$

3. Note que para todo $n \in \{1, \dots, N-1\}$

$$\begin{aligned}& S_n^+S_{n+1}^- + S_n^-S_{n+1}^+ \\ &= (S_n^x + iS_n^y)(S_{n+1}^x - iS_{n+1}^y) + (S_n^x - iS_n^y)(S_{n+1}^x + iS_{n+1}^y) \\ &= S_n^xS_{n+1}^x - iS_n^xS_{n+1}^y + iS_n^yS_{n+1}^x + S_n^yS_{n+1}^y \\ &+ S_n^xS_{n+1}^x + iS_n^xS_{n+1}^y - iS_n^yS_{n+1}^x + S_n^yS_{n+1}^y \\ &= 2S_n^xS_{n+1}^x + 2S_n^yS_{n+1}^y.\end{aligned} \tag{9}$$

Entonces, en vista de (8), es claro que

$$H = \frac{J}{2} \sum_{i=1}^{N-1} \left(S_n^+S_{n+1}^- + S_n^-S_{n+1}^+\right) - h \sum_{n=1}^{N} \left[S_n^+, S_n^-\right]. \tag{10}$$

4. Note que para todo $n \in \{1, \dots, N\}$ el operador ϵ_n está bien definido ya que en vista de (8) los terminos en la productoria son $2S_1^z, \dots, 2S_{n-1}^z$ y conmutan. Entonces se tiene para $n \in \{1, \dots, N\}$ y $s_1, \dots, s_N \in \{+, -\}$

$$\begin{aligned}\epsilon_n \left|s_1 \cdots s_N\right\rangle = \prod_{j=1}^{n-1} S_j^z \left|s_1 \cdots s_N\right\rangle = \prod_{j=1}^{n-1} \begin{Bmatrix} 1 & s_j = + \\ -1 & s_j = - \end{Bmatrix} \left|s_1 \cdots s_N\right\rangle \\ = (-1)^{\left|\{j \in \{1, \dots, n-1\} | s_j = -\}\right|} \left|s_1 \cdots s_N\right\rangle = (-1)^A \left|s_1 \cdots s_N\right\rangle .\end{aligned} \tag{11}$$

5. Se tiene que para todo $n \in \{1, \ldots, N\}$ y $s_1, \ldots, s_N \in \{+, -\}$

$$\epsilon_n^2 |s_1 \cdots s_N\rangle = (-1)^{2A} |s_1 \cdots s_N\rangle = |s_1 \cdots s_N\rangle . \tag{12}$$

Ya que $\{|s_1 \cdots s_N\rangle \, | s_1, \ldots, s_N \in \{+, -\}\}$ genera $\mathcal{H}$ se concluye que $\epsilon_n^2 = \mathrm{id}_{\mathcal{H}}$. Por otra parte, se tiene que

$$\begin{aligned} 2S^z 2S^z |+\rangle &= |+\rangle \\ 2S^z 2S^z |-\rangle &= (-1)^2 |-\rangle = |-\rangle . \end{aligned} \tag{13}$$

Entonces es claro que $(2S^z)^2 = \mathrm{id}_{\mathbb{C}^2}$ y por lo tanto $(2S_j^+ S_j^- - \mathrm{id}_{\mathcal{H}})^2 = (2S_j^z)^2 = \mathrm{id}_{\mathcal{H}}$ para todo $j \in \{1, \ldots, N\}$.

6. Note que

$$\begin{aligned} (S^+)^2 |-\rangle &= S^+ |+\rangle = 0 \\ (S^-)^2 |-\rangle &= 0 \\ (S^+)^2 |+\rangle &= 0 \\ (S^-)^2 |+\rangle &= S^- |-\rangle = 0, \end{aligned} \tag{14}$$

es decir, $(S^+)^2 = (S^-)^2 = 0$. Por otra parte

$$\begin{aligned} \{2S^z, S^+\} |+\rangle &= (2S^z S^+ + 2S^+ S^z) |+\rangle = 0 \\ \{2S^z, S^+\} |-\rangle &= (2S^z S^+ + 2S^+ S^z) |-\rangle = |+\rangle - |+\rangle = 0 \\ \{2S^z, S^-\} |+\rangle &= (2S^z S^- + 2S^- S^z) |+\rangle = -|-\rangle + |-\rangle = 0 \\ \{2S^z, S^-\} |-\rangle &= (2S^z S^- + 2S^- S^z) |-\rangle = 0, \end{aligned} \tag{15}$$

es decir, para todo $n \in \{1, \ldots, N\}$ se tiene $\{2S_n^z, S_n^\pm\} = 0$.

Dado que operadores en distintos sitios conmutan, los operadores S_n^i conmutan con ϵ_m para $n, m \in \{1, \ldots, N\}$, $m \geq n$ e $i \in \{x, y, z, +, -\}$. Se tiene entonces $c_n^2 = \epsilon_n^2 (S_n^+)^2 = 0 = (S_n^-)^2 \epsilon_n^2 = (c_n^\dagger)^2$, $c_n^\dagger c_n = S_n^- S_n^+ \epsilon_n^2 = S_n^- S_n^+$, $c_n c_n^\dagger = S_n^+ S_n^- \epsilon_n^2 = S_n^+ S_n^-$ y $\{c_n^\dagger, c_n\} = \{S_n^-, S_n^+\} = \mathrm{id}_{\mathcal{H}}$ para todo $n \in \{1, \ldots, N\}$.

Tome $n, m \in \{1, \ldots, N\}$ y suponga que $n > m$. Luego

$$\begin{aligned}
\{c_n^\dagger, c_m\} =& S_n^- \epsilon_n \epsilon_m S_m^+ + \epsilon_m S_m^+ S_n^- \epsilon_n \\
=& S_n^- \left(\prod_{j=1}^{n-1} 2S_j^z \right) S_m^+ \epsilon_m + \epsilon_m S_n^- S_m^+ \left(\prod_{j=1}^{n-1} 2S_j^z \right) \\
=& S_n^- \left(\prod_{\substack{j=1 \\ j\neq m}}^{n-1} 2S_j^z \right) 2S_m^z S_m^+ \epsilon_m + \epsilon_m S_n^- S_m^+ 2S_m^z \left(\prod_{\substack{j=1 \\ j\neq m}}^{n-1} 2S_j^z \right) \\
=& S_n^- \left(\prod_{\substack{j=1 \\ j\neq m}}^{n-1} 2S_j^z \right) \epsilon_m 2S_m^z S_m^+ + S_n^- \left(\prod_{\substack{j=1 \\ j\neq m}}^{n-1} 2S_j^z \right) \epsilon_m S_m^+ 2S_m^z \\
=& S_n^- \left(\prod_{\substack{j=1 \\ j\neq m}}^{n-1} 2S_j^z \right) \epsilon_m \{2S_m^z, S_m^+\} = 0
\end{aligned} \tag{16}$$

$$\begin{aligned}
\{c_n^\dagger, c_m^\dagger\} =& S_n^- \epsilon_n S_m^- \epsilon_m + S_m^- \epsilon_m S_n^- \epsilon_n \\
=& S_n^- \left(\prod_{j=1}^{n-1} 2S_j^z \right) S_m^- \epsilon_m + \epsilon_m S_n^- S_m^- \left(\prod_{j=1}^{n-1} 2S_j^z \right) \\
=& S_n^- \left(\prod_{\substack{j=1 \\ j\neq m}}^{n-1} 2S_j^z \right) 2S_m^z S_m^- \epsilon_m + \epsilon_m S_n^- S_m^- 2S_m^z \left(\prod_{\substack{j=1 \\ j\neq m}}^{n-1} 2S_j^z \right) \\
=& S_n^- \left(\prod_{\substack{j=1 \\ j\neq m}}^{n-1} 2S_j^z \right) \epsilon_m 2S_m^z S_m^- + S_n^- \left(\prod_{\substack{j=1 \\ j\neq m}}^{n-1} 2S_j^z \right) \epsilon_m S_m^- 2S_m^z \\
=& S_n^- \left(\prod_{\substack{j=1 \\ j\neq m}}^{n-1} 2S_j^z \right) \epsilon_m \{2S_m^z, S_m^-\} = 0
\end{aligned} \tag{17}$$

$$
\begin{aligned}
\{c_n, c_m\} =&\epsilon_n S_n^+ \epsilon_m S_m^+ + \epsilon_m S_m^+ \epsilon_n S_n^+ \\
=&\left(\prod_{j=1}^{n-1} 2S_j^z\right) S_m^+ S_n^+ \epsilon_m + \epsilon_m S_m^+ \left(\prod_{j=1}^{n-1} 2S_j^z\right) S_n^+ \\
=&\left(\prod_{\substack{j=1\\ j\neq m}}^{n-1} 2S_j^z\right) 2S_m^z S_m^+ S_n^+ \epsilon_m + \epsilon_m S_m^+ 2S_m^z \left(\prod_{\substack{j=1\\ j\neq m}}^{n-1} 2S_j^z\right) S_n^+ \\
=&S_n^+ \left(\prod_{\substack{j=1\\ j\neq m}}^{n-1} 2S_j^z\right) \epsilon_m 2S_m^z S_m^+ + S_n^+ \left(\prod_{\substack{j=1\\ j\neq m}}^{n-1} 2S_j^z\right) \epsilon_m S_m^+ 2S_m^z \\
=&S_n^+ \left(\prod_{\substack{j=1\\ j\neq m}}^{n-1} 2S_j^z\right) \epsilon_m \{2S_m^z, S_m^+\} = 0.
\end{aligned} \tag{18}
$$

Las mismas relaciones son validas si $m > n$ pues

$$
\{c_n^\dagger, c_m\} = (c_n^\dagger c_m + c_m c_n^\dagger) = ((c_m^\dagger c_n)^\dagger + (c_n c_m^\dagger)^\dagger) = \{c_m^\dagger, c_n\}^\dagger = 0 \tag{19}
$$

y el anticonmutador es simétrico. Se concluye entonces que se satisfacen la relaciones de anticonmutación canónicas para $n, m \in \{1, \ldots, N\}$

$$
\begin{aligned}
\{c_n^\dagger, c_m\} =&\delta_{nm}\, \mathrm{id}_{\mathcal{H}} \\
\{c_n^\dagger, c_m^\dagger\} =&0 = \{c_n, c_m\}.
\end{aligned} \tag{20}
$$

7. En el punto anterior se demostró que para todo $n \in \{1, \ldots, N\}$ se tiene $c_n c_n^\dagger = S_n^+ S_n^-$ debido a que $S_n^\pm$ conmuta con ϵ_n. Entonces

$$
\left(\prod_{j=1}^{n-1} (1 - 2c_j^\dagger c_j)\right) c_n = \left(\prod_{j=1}^{n-1} (1 - 2c_j^\dagger c_j)\right) \left(\prod_{k=1}^{n-1} (2c_k c_k^\dagger - 1)\right) S_n^+. \tag{21}
$$

Note que para $j, k \in \{1, \ldots, N\}$ si $j \neq k$

$$
\begin{aligned}
(1 - 2c_j^\dagger c_j)(2c_k c_k^\dagger - 1) =&2c_k c_k^\dagger - 1 - 4c_j^\dagger c_j c_k c_k^\dagger + 2c_j^\dagger c_j \\
=&2c_k c_k^\dagger - 1 - 4c_k^\dagger c_k c_j c_j^\dagger + 2c_j^\dagger c_j \\
=&(2c_k c_k^\dagger - 1)(1 - 2c_j^\dagger c_j).
\end{aligned} \tag{22}
$$

Entonces podemos reorganizar de manera que

$$
\left(\prod_{j=1}^{n-1} (1 - 2c_j^\dagger c_j)\right) c_n = \left(\prod_{j=1}^{n-1} (1 - 2c_j^\dagger c_j)(2c_j c_j^\dagger - 1)\right) S_n^+. \tag{23}
$$

Ahora bien, si $j \in \{1, \dots, N\}$ se tiene

$$\begin{aligned}(1-2c_j^\dagger c_j)(2c_jc_j^\dagger - 1) =& 2c_jc_j^\dagger - 1 - 4c_j^\dagger c_jc_jc_j^\dagger + 2c_j^\dagger c_j = 2\{c_j^\dagger, c_j\} - \mathrm{id}_{\mathcal{H}} \\ =& \mathrm{id}_{\mathcal{H}}\end{aligned} \tag{24}$$

donde se utilizó que $c_jc_j = c_j^\dagger c_j^\dagger = 0$ consecuancia de las relaciones de anticonmutación canónicas. Se confirma que

$$\left(\prod_{j=1}^{n-1}(1-2c_j^\dagger c_j)\right)c_n = S_n^+. \tag{25}$$

De manera análoga tenemos

$$\begin{aligned}\left(\prod_{j=1}^{n-1}(1-2c_j^\dagger c_j)\right)c_n^\dagger =& \left(\prod_{j=1}^{n-1}(1-2c_j^\dagger c_j)\right)S_n^-\left(\prod_{k=1}^{n-1}(2c_kc_k^\dagger - 1)\right) \\ =& S_n^-\left(\prod_{j=1}^{n-1}(1-2c_j^\dagger c_j)\right)\left(\prod_{k=1}^{n-1}(2c_kc_k^\dagger - 1)\right) \\ =& S_n^-\left(\prod_{j=1}^{n-1}(1-2c_j^\dagger c_j)(2c_jc_j^\dagger - 1)\right) \\ =& S_n^-\left(\prod_{j=1}^{n-1}\mathrm{id}_{\mathcal{H}}\right) = S_n^-.\end{aligned} \tag{26}$$

8. Reemplazando (25) y (26) en (10) y notando que $c_j^\dagger c_jc_n^\dagger = (-1)^2c_n^\dagger c_j^\dagger c_j = c_n^\dagger c_j^\dagger c_j$ y $c_nc_j^\dagger c_j = (-1)^2c_j^\dagger c_jc_n = c_j^\dagger c_jc_n$ para todo $n, j \in \{0, \dots, N\}$ distintos,

se obtiene

$$
\begin{aligned}
H =&\frac{J}{2}\sum_{n=1}^{N-1}\left(\left(\prod_{j=1}^{n-1}(1-2c_j^\dagger c_j)\right)c_n\left(\prod_{j=1}^{n}(1-2c_j^\dagger c_j)\right)c_{n+1}^\dagger\right.\\
&+\left.\left(\prod_{j=1}^{n-1}(1-2c_j^\dagger c_j)\right)c_n^\dagger\left(\prod_{j=1}^{n}(1-2c_j^\dagger c_j)\right)c_{n+1}\right)-h\sum_{n=1}^{N}(2c_nc_n^\dagger-1)\\
=&\frac{J}{2}\sum_{n=1}^{N-1}\left(c_n\left(\prod_{j=1}^{n-1}(1-2c_j^\dagger c_j)\right)\left(\prod_{j=1}^{n}(1-2c_j^\dagger c_j)\right)c_{n+1}^\dagger\right.\\
&+\left.c_n^\dagger\left(\prod_{j=1}^{n-1}(1-2c_j^\dagger c_j)\right)\left(\prod_{j=1}^{n}(1-2c_j^\dagger c_j)\right)c_{n+1}\right)-h\sum_{n=1}^{N}(2c_nc_n^\dagger-1)\\
=&\frac{J}{2}\sum_{n=1}^{N-1}\left(c_n\left(\prod_{j=1}^{n-1}(1-2c_j^\dagger c_j)(1-2c_j^\dagger c_j)\right)(1-2c_n^\dagger c_n)c_{n+1}^\dagger\right.\\
&+\left.c_n^\dagger\left(\prod_{j=1}^{n-1}(1-2c_j^\dagger c_j)(1-2c_j^\dagger c_j)\right)(1-2c_n^\dagger c_n)c_{n+1}\right)\\
&-h\sum_{n=1}^{N}(2c_nc_n^\dagger-1)\\
=&\frac{J}{2}\sum_{n=1}^{N-1}\left(c_n(1-2c_n^\dagger c_n)c_{n+1}^\dagger+c_n^\dagger(1-2c_n^\dagger c_n)c_{n+1}\right)\\
&-2h\sum_{n=1}^{N}c_nc_n^\dagger-hN\\
=&\frac{J}{2}\sum_{n=1}^{N-1}\left(c_nc_{n+1}^\dagger-2c_nc_n^\dagger c_nc_{n+1}^\dagger+c_n^\dagger c_{n+1}-2c_n^\dagger c_n^\dagger c_nc_{n+1}\right)\\
&-2h\sum_{n=1}^{N}c_nc_n^\dagger-hN\\
=&\frac{J}{2}\sum_{n=1}^{N-1}\left(c_nc_{n+1}^\dagger-2(1-c_n^\dagger c_n)c_nc_{n+1}^\dagger+c_n^\dagger c_{n+1}\right)-2h\sum_{n=1}^{N}c_nc_n^\dagger-hN\\
=&\frac{J}{2}\sum_{n=1}^{N-1}\left(-c_nc_{n+1}^\dagger+c_n^\dagger c_{n+1}\right)-2h\sum_{n=1}^{N}c_nc_n^\dagger-hN\\
=&\frac{J}{2}\sum_{n=1}^{N-1}\left(c_{n+1}^\dagger c_n+c_n^\dagger c_{n+1}\right)-2h\sum_{n=1}^{N}c_nc_n^\dagger-hN.
\end{aligned}
\tag{27}
$$

9. El Hamiltoniano (27) se puede reescribir con la cotransformada de Fourier

$$
\begin{aligned}
H =& -hN + \frac{J}{2}\sum_{n=1}^{N}\Bigg(\frac{1}{N}\sum_{p=1}^{N} e^{ik_p(n+1)}\hat{c}^\dagger_{k_p}\sum_{p'=1}^{N} e^{-ik_{p'}n}\hat{c}_{k_{p'}}\\
&+\frac{1}{N}\sum_{p=1}^{N} e^{ik_p n}\hat{c}^\dagger_{k_p}\sum_{p'=1}^{N} e^{-ik_{p'}(n+1)}\hat{c}_{k_{p'}}\Bigg)\\
&+2h\sum_{n=1}^{N}\frac{1}{N}\sum_{p=1}^{N} e^{ik_p n}\hat{c}^\dagger_{k_p}\sum_{p'=1}^{N} e^{-ik_{p'}n}\hat{c}_{k_{p'}}\\
=& -hN + \frac{1}{N}\sum_{n=1}^{N}\sum_{p=1}^{N}\sum_{p'=1}^{N}\Bigg(\frac{J}{2}\Big(e^{ik_p(n+1)}\hat{c}^\dagger_{k_p}e^{-ik_{p'}n}\hat{c}_{k_{p'}}\\
&+e^{ik_p n}\hat{c}^\dagger_{k_p}e^{-ik_{p'}(n+1)}\hat{c}_{k_{p'}}\Big) + 2he^{ik_p n}\hat{c}^\dagger_{k_p}e^{-ik_{p'}n}\hat{c}_{k_{p'}}\Bigg)\\
=& -hN + \frac{1}{N}\sum_{n=1}^{N}\sum_{p=1}^{N}\sum_{p'=1}^{N}\Bigg(\frac{J}{2}\Big(e^{ik_p}e^{i(k_p-k_{p'})n}\\
&+e^{i(k_p-k_{p'})n}e^{-ik_{p'}}\Big) + 2he^{i(k_p-k_{p'})n}\Bigg)\hat{c}^\dagger_{k_p}\hat{c}_{k_{p'}}\\
=& -hN + \frac{1}{N}\sum_{n=1}^{N}\sum_{p=1}^{N}\sum_{p'=1}^{N} e^{i(k_p-k_{p'})n}\left(\frac{J}{2}\left(e^{ik_p}+e^{-ik_{p'}}\right)+2h\right)\hat{c}^\dagger_{k_p}\hat{c}_{k_{p'}}\\
=& -hN + \frac{1}{N}\sum_{p=1}^{N}\sum_{p'=1}^{N} N\delta_{k_p k_{p'}}\left(2h + J\cos(k_p)\right)\hat{c}^\dagger_{k_p}\hat{c}_{k_{p'}}\\
=& -hN + \sum_{p=1}^{N}\left(2h + J\cos(k_p)\right)\hat{c}^\dagger_{k_p}\hat{c}_{k_p}
\end{aligned}
\tag{28}
$$

donde se utilizó $k_p := 2\pi p/N$ para todo $p \in \{1, \ldots, N\}$.

10. Note que para $p \in \{0, \ldots, N\}$

$$
\begin{aligned}
H\hat{c}^\dagger_{k_p}|0\rangle =& -hN\hat{c}^\dagger_{k_p}|0\rangle + \sum_{p'=1}^{N}\left(2h + J\cos(k_{p'})\right)\hat{c}^\dagger_{k_{p'}}\hat{c}_{k_{p'}}\hat{c}^\dagger_{k_p}|0\rangle\\
=& -hN\hat{c}^\dagger_{k_p}|0\rangle + \sum_{\substack{p'=1\\p'\neq p}}^{N}\left(2h + J\cos(k_{p'})\right)(-1)^2\hat{c}^\dagger_{k_p}\hat{c}^\dagger_{k_{p'}}\hat{c}_{k_{p'}}|0\rangle\\
&+\left(2h + J\cos(k_p)\right)\hat{c}^\dagger_{k_p}\hat{c}_{k_p}\hat{c}^\dagger_{k_p}|0\rangle\\
=& -hN\hat{c}^\dagger_{k_p}|0\rangle + \left(2h + J\cos(k_p)\right)\hat{c}^\dagger_{k_p}(1-\hat{c}^\dagger_{k_p}\hat{c}_{k_p})|0\rangle\\
=& -hN\hat{c}^\dagger_{k_p}|0\rangle + \left(2h + J\cos(k_p)\right)\hat{c}^\dagger_{k_p}|0\rangle\\
=&(-hN + 2h + J\cos(k_p))\hat{c}^\dagger_{k_p}|0\rangle.
\end{aligned}
\tag{29}
$$

Luego la energía propia correspondiente al número de onda k_p es $2h + J\cos(k_p)$.

11. Los vectores propios del operador número de ocupación son aquellos de la forma $\left(\hat{c}^\dagger_{k_1}\right)^{n_1}\cdots\left(\hat{c}^\dagger_{k_N}\right)^{n_N}|0\rangle$ para $n_1,\ldots,n_N\in\mathbb{N}$. Más aún, las relaciones canónicas de anticonmutación aseguran que $n_1,\ldots,n_N\in\{0,1\}$ ya que $\left(\hat{c}^\dagger_{k_p}\right)^2 = 0$ para todo $p\in\{1,\ldots,N\}$. Para $p\in\{1,\ldots,N\}$ se tiene

$$\begin{aligned}
&\hat{c}^\dagger_{k_p}\hat{c}_{k_p}\left(\hat{c}^\dagger_{k_1}\right)^{n_1}\cdots\left(\hat{c}^\dagger_{k_N}\right)^{n_N}|0\rangle\\
=&\left(\hat{c}^\dagger_{k_1}\right)^{n_1}\cdots\left(\hat{c}^\dagger_{k_{p-1}}\right)^{n_{p-1}}\hat{c}^\dagger_{k_p}\hat{c}_{k_p}\left(\hat{c}^\dagger_{k_p}\right)^{n_p}\left(\hat{c}^\dagger_{k_{p+1}}\right)^{n_{p+1}}\cdots\left(\hat{c}^\dagger_{k_N}\right)^{n_N}|0\rangle\\
=&\left(\hat{c}^\dagger_{k_1}\right)^{n_1}\cdots\left(\hat{c}^\dagger_{k_{p-1}}\right)^{n_{p-1}}\hat{c}^\dagger_{k_p}\\
&(1-\hat{c}^\dagger_{k_p}\hat{c}_{k_p})\left(\hat{c}^\dagger_{k_p}\right)^{n_p-1}\left(\hat{c}^\dagger_{k_{p+1}}\right)^{n_{p+1}}\cdots\left(\hat{c}^\dagger_{k_N}\right)^{n_N}|0\rangle\\
=&\left(\hat{c}^\dagger_{k_1}\right)^{n_1}\cdots\left(\hat{c}^\dagger_{k_{p-1}}\right)^{n_{p-1}}\hat{c}^\dagger_{k_p}\left(\hat{c}^\dagger_{k_p}\right)^{n_p-1}\left(\hat{c}^\dagger_{k_{p+1}}\right)^{n_{p+1}}\cdots\left(\hat{c}^\dagger_{k_N}\right)^{n_N}|0\rangle\\
=&\left(\hat{c}^\dagger_{k_1}\right)^{n_1}\cdots\left(\hat{c}^\dagger_{k_N}\right)^{n_N}|0\rangle
\end{aligned}\tag{30}$$

si $n_p = 1$ y

$$\begin{aligned}
&\hat{c}^\dagger_{k_p}\hat{c}_{k_p}\left(\hat{c}^\dagger_{k_1}\right)^{n_1}\cdots\left(\hat{c}^\dagger_{k_N}\right)^{n_N}|0\rangle\\
=&\left(\hat{c}^\dagger_{k_1}\right)^{n_1}\cdots\left(\hat{c}^\dagger_{k_{p-1}}\right)^{n_{p-1}}\hat{c}^\dagger_{k_p}\left(\hat{c}^\dagger_{k_{p+1}}\right)^{n_{p+1}}\cdots\left(\hat{c}^\dagger_{k_N}\right)^{n_N}\hat{c}_{k_p}|0\rangle = 0
\end{aligned}\tag{31}$$

si $n_p = 0$. Luego los valores propios del operador número de ocupación son 0 o 1 como es de esperarse para fermiones. Al notar que los vectores propios de el operador número de ocupación son los del Hamiltoniano por la expresión (28) se tiene

$$\begin{aligned}
Z(\beta) = \operatorname{tr}\left(e^{-\beta H}\right) &= \sum_{n_1=0}^{1}\cdots\sum_{n_N=0}^{1}\exp\left(\beta hN - \beta\sum_{p=1}^{N}(2h+J\cos k_p)n_p\right)\\
&= e^{\beta hN}\sum_{n_1=0}^{1}\cdots\sum_{n_N=0}^{1}\prod_{p=1}^{N}e^{-\beta(2h+J\cos k_p)n_p}\\
&= e^{\beta hN}\prod_{p=1}^{N}\sum_{n=0}^{1}e^{-\beta(2h+J\cos k_p)n} = e^{\beta hN}\prod_{p=1}^{N}\left(1+e^{-\beta(2h+J\cos k_p)}\right)
\end{aligned}\tag{32}$$

12. Se tiene que la energía libre por spin del sistema es

$$\begin{aligned}
f(\beta) :=& -\frac{1}{\beta N}\ln(Z(\beta)) = -\frac{1}{\beta N}\left(\beta hN + \sum_{p=1}^{N}\ln\left(1+e^{-\beta(2h+J\cos k_p)}\right)\right)\\
=& -h - \frac{1}{\beta N}\sum_{p=1}^{N}\ln\left(1+e^{-\beta(2h+J\cos k_p)}\right).
\end{aligned}\tag{33}$$

En el límite termodinámico se tiene la densidad de estados

$$\frac{1}{N}\sum_{p=1}^{N} \to \int \frac{dk}{2\pi}. \tag{34}$$

Se concluye entonces

$$f(\beta) = -h - k_B T \int \ln\Big(1 + e^{-\beta(2h + J\cos k)}\Big)\frac{dk}{2\pi}. \tag{35}$$