Electrodynamics
Third Exam
2018-I

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May 3, 2018

(i) As we have seen in class the potential A^μ is given by

$$A^\mu(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\,\frac{j^\mu(\mathbf{r}',t-\|\mathbf{r}-\mathbf{r}'\|/c)}{\|\mathbf{r}-\mathbf{r}'\|} \tag{1}$$

ignoring all background waves, that is, solutions of $\Box A^\mu = 0$. Therefore, its temporal Fourier transform, if we assume exists, is given by

$$\begin{aligned}A^\mu(\mathbf{r},\omega) &= \int \mathrm{d}t\, e^{i\omega t}A^\mu(\mathbf{r},t)\\ &= \frac{\mu_0}{4\pi}\int \mathrm{d}t\, e^{i\omega t}\int \mathrm{d}^3\mathbf{r}'\,\frac{j^\mu(\mathbf{r}',t-\|\mathbf{r}-\mathbf{r}'\|/c)}{\|\mathbf{r}-\mathbf{r}'\|}\\ &= \frac{\mu_0}{4\pi}\int \mathrm{d}t\int \mathrm{d}^3\mathbf{r}'\,\frac{1}{\|\mathbf{r}-\mathbf{r}'\|}e^{i\omega t}j^\mu(\mathbf{r}',t-\|\mathbf{r}-\mathbf{r}'\|/c).\end{aligned} \tag{2}$$

Assuming the proper conditions are met to apply Fubini's theorem we have

$$\begin{aligned}A^\mu(\mathbf{r},\omega) &= \frac{\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\int \mathrm{d}t\,\frac{1}{\|\mathbf{r}-\mathbf{r}'\|}e^{i\omega t}j^\mu(\mathbf{r}',t-\|\mathbf{r}-\mathbf{r}'\|/c)\\ &= \frac{\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\,\frac{1}{\|\mathbf{r}-\mathbf{r}'\|}\int \mathrm{d}t\, e^{i\omega t}j^\mu(\mathbf{r}',t-\|\mathbf{r}-\mathbf{r}'\|/c)\end{aligned} \tag{3}$$

Through the change of variables

$$\begin{aligned}\mathbb{R}&\to\mathbb{R}\\ t&\mapsto t+\|\mathbf{r}-\mathbf{r}'\|/c\end{aligned} \tag{4}$$

which keeps the domain $\mathbb{R}$ invariant we obtain

$$\begin{aligned}A^\mu(\mathbf{r},\omega) &= \frac{\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\,\frac{1}{\|\mathbf{r}-\mathbf{r}'\|}\int \mathrm{d}t\, e^{i\omega(t+\|\mathbf{r}-\mathbf{r}'\|/c)}j^\mu(\mathbf{r}',t)\\ &= \frac{\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\,\frac{1}{\|\mathbf{r}-\mathbf{r}'\|}\int \mathrm{d}t\, e^{i\omega\|\mathbf{r}-\mathbf{r}'\|/c}e^{i\omega t}j^\mu(\mathbf{r}',t)\\ &= \frac{\mu_0}{4\pi}\int \mathrm{d}^3\mathbf{r}'\,\frac{e^{iK\|\mathbf{r}-\mathbf{r}'\|}}{\|\mathbf{r}-\mathbf{r}'\|}\int \mathrm{d}t\, e^{i\omega t}j^\mu(\mathbf{r}',t).\end{aligned} \tag{5}$$

We now recognize the Fourier transform

$$j^\mu(\mathbf{r},\omega) = \int \mathrm{d}t\, e^{i\omega t} j^\mu(\mathbf{r},t) \tag{6}$$

concluding

$$A^\mu(\mathbf{r},\omega) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3\mathbf{r'}\, \frac{e^{iK\|\mathbf{r}-\mathbf{r'}\|}}{\|\mathbf{r}-\mathbf{r'}\|} j^\mu(\mathbf{r'},\omega). \tag{7}$$

Separating this equation by remembering that $A^\mu = (\phi/c, \mathbf{A})$, $j^\mu = (c\rho, \mathbf{J})$, and $c^2\mu_0 = \frac{\mu_0}{\mu_0\epsilon_0} = \frac{1}{\epsilon_0}$ we obtain

$$\begin{aligned}
\phi(\mathbf{r},\omega) =& \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3\mathbf{r'}\, \frac{e^{iK\|\mathbf{r}-\mathbf{r'}\|}}{\|\mathbf{r}-\mathbf{r'}\|} \rho(\mathbf{r'},\omega), \\
\mathbf{A}(\mathbf{r},\omega) =& \frac{\mu_0}{4\pi} \int \mathrm{d}^3\mathbf{r'}\, \frac{e^{iK\|\mathbf{r}-\mathbf{r'}\|}}{\|\mathbf{r}-\mathbf{r'}\|} \mathbf{J}(\mathbf{r'},\omega).
\end{aligned} \tag{8}$$

(ii) Recall that the law of charge current conservation is given by

$$\partial_\mu j^\mu(\mathbf{r},t) = 0. \tag{9}$$

Assuming that the Fourier transformation (6) is invertible, we must have

$$j^\mu(\mathbf{r},t) = \frac{1}{2\pi} \int \mathrm{d}\omega\, e^{-i\omega t} j^\mu(\mathbf{r},\omega) \tag{10}$$

and

$$0 = \frac{1}{2\pi} \partial_\mu \int \mathrm{d}\omega\, e^{-i\omega t} j^\mu(\mathbf{r},\omega). \tag{11}$$

Assuming that the right conditions are met to interchange differentiation with integration we have

$$\begin{aligned}
0 &= \int \mathrm{d}\omega\, \partial_\mu \big(e^{-i\omega t} j^\mu(\mathbf{r},\omega)\big) \\
&= \int \mathrm{d}\omega \left(\frac{1}{c}\frac{\partial e^{-i\omega t}}{\partial t} c\rho(\mathbf{r},\omega) + e^{-i\omega t}\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},\omega)\right) \\
&= \int \mathrm{d}\omega \left(-i\omega e^{-i\omega t}\rho(\mathbf{r},\omega) + e^{-i\omega t}\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},\omega)\right) \\
&= \int \mathrm{d}\omega\, e^{-i\omega t}(-i\omega\rho(\mathbf{r},\omega) + \boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},\omega)).
\end{aligned} \tag{12}$$

Under the right conditions the uniqueness theorem is valid and a function is null if and only if is Fourier transform is. We thus conclude that the law of charge current conservation can be expressed as

$$\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},\omega) = i\omega\rho(\mathbf{r},\omega). \tag{13}$$

(iii) We recall the definition of the electric dipole moment and magnetic dipole moment

$$\begin{aligned}\mathbf{p} =& \int \mathrm{d}^3\mathbf{r}\,\mathbf{r}\rho(\mathbf{r}),\\ \mathbf{m} =& \frac{1}{2}\int \mathrm{d}^3\mathbf{r}\,\mathbf{r}\times\mathbf{J}(\mathbf{r}).\end{aligned} \tag{14}$$

By using the product rule of the divergence

$$\begin{aligned}\int \mathrm{d}^3\mathbf{r}\,\mathbf{J}(\mathbf{r},\omega) =& \sum_{i=1}^{3}\hat{\mathbf{e}}_i \int \mathrm{d}^3\mathbf{r}\,\mathbf{J}(\mathbf{r},\omega)\cdot\hat{\mathbf{e}}_i\\ =& \sum_{i=1}^{3}\hat{\mathbf{e}}_i \int \mathrm{d}^3\mathbf{r}\,\mathbf{J}(\mathbf{r},\omega)\cdot\boldsymbol{\nabla}x^i\\ =& \sum_{i=1}^{3}\hat{\mathbf{e}}_i \int \mathrm{d}^3\mathbf{r}\,(\boldsymbol{\nabla}\cdot\big(x^i\mathbf{J}\big)(\mathbf{r},\omega) - x^i\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},\omega))\\ =& \sum_{i=1}^{3}\hat{\mathbf{e}}_i \bigg(\int \mathrm{d}^3\mathbf{r}\,\boldsymbol{\nabla}\cdot\big(x^i\mathbf{J}\big)(\mathbf{r},\omega) - \int \mathrm{d}^3\mathbf{r}\,x^i\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},\omega)\bigg).\end{aligned} \tag{15}$$

The first integral vanishes since it is a total differential on a region without a boundary, namely $\mathbb{R}^3$. Therefore, by using (13) we obtain

$$\begin{aligned}\int \mathrm{d}^3\mathbf{r}\,\mathbf{J}(\mathbf{r},\omega) =& -\sum_{i=1}^{3}\hat{\mathbf{e}}_i \int \mathrm{d}^3\mathbf{r}\,x^i\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},\omega)\\ =& -\int \mathrm{d}^3\mathbf{r}\sum_{i=1}^{3}\hat{\mathbf{e}}_i x^i i\omega\rho(\mathbf{r},\omega) = -\int \mathrm{d}^3\mathbf{r}\,\mathbf{r}i\omega\rho(\mathbf{r},\omega)\\ =& -i\omega\mathbf{p}.\end{aligned} \tag{16}$$

In the far zone we have as hinted in the problem sheet

$$\frac{1}{\|\mathbf{r}-\mathbf{r}'\|} \approx \frac{1}{\|\mathbf{r}\| - \frac{\mathbf{r}\cdot\mathbf{r}'}{\|\mathbf{r}\|}} = \frac{1}{r - \frac{\mathbf{r}\cdot\mathbf{r}'}{r}} = \frac{1}{r}\frac{1}{1-\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}} \tag{17}$$

where $r := \|\mathbf{r}\|$ to lighten notation. Given that

$$\left|\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right| \leq \frac{rr'}{r^2} = \frac{r'}{r} \tag{18}$$

if we assume that the fields are local, that is, that $r' \ll r$ in the region where fields are relevant we may say that under the integral sign $|r'/r| \ll 1$. Therefore, we may employ the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{19}$$

to approximate

$$\frac{1}{\|\mathbf{r}-\mathbf{r}'\|} \approx \frac{1}{r}\left(1+\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right). \tag{20}$$

Similarly, we may stablish the estimate

$$e^{ik\|\mathbf{r}-\mathbf{r}'\|} \approx e^{ik\left(r-\frac{\mathbf{r}\cdot\mathbf{r}'}{r}\right)} = e^{ikr}e^{-ik\frac{\mathbf{r}\cdot\mathbf{r}'}{r}} = e^{ikr}\sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}\left(\frac{\mathbf{r}\cdot\mathbf{r}'}{r}\right)^n. \tag{21}$$

Thus,

$$\begin{aligned}\frac{e^{ik\|\mathbf{r}-\mathbf{r}'\|}}{\|\mathbf{r}-\mathbf{r}'\|} \approx& \frac{1}{r}\left(1+\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)e^{ikr}\sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}\left(\frac{\mathbf{r}\cdot\mathbf{r}'}{r}\right)^n\\ =&\left(1+\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)e^{ikr}\sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}\frac{1}{r}\left(\frac{\mathbf{r}\cdot\mathbf{r}'}{r}\right)^n.\end{aligned} \tag{22}$$

Due to estimate (18) we can truncate the series to obtain

$$\frac{e^{ik\|\mathbf{r}-\mathbf{r}'\|}}{\|\mathbf{r}-\mathbf{r}'\|} \approx \left(1+\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)e^{ikr}\left(\frac{1}{r}-ik\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right). \tag{23}$$

Thus far, all of our estimates have been consistent up to $\mathcal{O}(\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2})$. Therefore, we may estimate further

$$\frac{e^{ik\|\mathbf{r}-\mathbf{r}'\|}}{\|\mathbf{r}-\mathbf{r}'\|} \approx e^{ikr}\left(\frac{1}{r}-ik\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right) = \frac{e^{ikr}}{r}\left(1-ik\frac{\mathbf{r}\cdot\mathbf{r}'}{r}\right). \tag{24}$$

Pluggin this estimate in (8) we obtain

$$\mathbf{A}(\mathbf{r},\omega) = \frac{\mu_0 e^{ikr}}{4\pi r}\int \mathrm{d}^3\mathbf{r}'\left(1-ik\frac{\mathbf{r}\cdot\mathbf{r}'}{r}\right)\mathbf{J}(\mathbf{r}',\omega). \tag{25}$$

Remembering (16) and the hint in the exam sheet we obtain

$$\begin{aligned}\mathbf{A}(\mathbf{r},\omega) =&\frac{\mu_0 e^{ikr}}{4\pi r}\left(-i\omega\mathbf{p}-\frac{ik}{r}\int \mathrm{d}^3\mathbf{r}'\,(\mathbf{r}\cdot\mathbf{r}')\mathbf{J}(\mathbf{r}',\omega)\right)\\ =&-\frac{\mu_0 e^{ikr}}{4\pi r}\left(i\omega\mathbf{p}+\frac{ik}{2r}\int \mathrm{d}^3\mathbf{r}'\,(\mathbf{r}'\times\mathbf{J}(\mathbf{r}',\omega))\times\mathbf{r}\right)\\ =&-\frac{\mu_0 e^{ikr}}{4\pi r}\left(i\omega\mathbf{p}+\frac{ik}{r}\mathbf{m}\times\mathbf{r}\right).\end{aligned} \tag{26}$$

(iv) We can express the Fourier transform of the electric and magnetic field by

$$\mathbf{B}(\mathbf{r},\omega) = \int \mathrm{d}t\, e^{i\omega t}\mathbf{B}(\mathbf{r},t) = \int \mathrm{d}t\, e^{i\omega t}\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r},t). \tag{27}$$

Given that $e^{i\omega t}$ has no spatial dependence and assuming that the right conditions are met to exchange differentiation and integration we obtain

$$\mathbf{B}(\mathbf{r},\omega) = \boldsymbol{\nabla} \times \int \mathrm{d}t\, e^{i\omega t} \mathbf{A}(\mathbf{r},t) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r},\omega). \tag{28}$$

Making use of the product rule we have by inserting