Electrodynamics: Homework 4

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1. We have
$$T^{i0}=F^i_{\gamma}F^{0\gamma}-\frac{1}{4}\eta^{i0}F_{\gamma\delta}F^{\gamma\delta}. \tag{1}$$
Given that $\eta^{\mu\nu}=0$ if $\mu\neq\nu$ and $i\in\{1,2,3\}$ we conclude
$$T^{i0}=F^i_{\gamma}F^{0\gamma}=\eta_{\rho\gamma}F^{i\rho}F^{0\gamma} \tag{2}$$
Remembering that $F^{\mu\nu}$ is antisymmetric and thus that $F^{00}=0$, we notice that
$$T^{i0}=\eta_{\rho j}F^{i\rho}F^{0j}=-\sum_{j=1}^{3}F^{ij}F^{0j}. \tag{3}$$
Recalling that $F^{0j}=E_j$ and $F^{ij}=\sum_{k=1}^3\epsilon_{ijk}B_k$ we have
$$T^{i0}=-\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}B_kE_j=-(\mathbf{E}\times\mathbf{B})_i \tag{4}$$

2. (i) Indeed, $\mathcal{L}$ is real. This is because $\mathcal{L}_\phi$, $\mathcal{L}_A$ and $\mathcal{L}_{\text{int}}$ are. We have
$$\begin{aligned}\mathcal{L}_\phi^* =&(\partial_\mu\phi)(\partial^\mu\phi)^*-m^2\phi\phi^*=(\partial_\mu\phi)(g^{\mu\nu}\partial_\nu\phi)^*-m^2\phi^*\phi\\ =&(g^{\mu\nu}\partial_\mu\phi)(\partial_\nu\phi)^*-m^2\phi^*\phi=(\partial^\nu\phi)(\partial_\nu\phi)^*-m^2\phi^*\phi\\ =&(\partial_\mu\phi)^*(\partial^\mu\phi)-m^2\phi^*\phi=\mathcal{L}_\phi,\\ \mathcal{L}_A^* =&\frac{1}{4}F_{\mu\nu}^*F^{\mu\nu *}=\frac{1}{4}F_{\mu\nu}F^{\mu\nu}=\mathcal{L}_A \quad \text{y}\\ \mathcal{L}_{\text{int}}^* =&ieA_\mu(\phi(\partial^\mu\phi)^*-(\partial^\mu\phi)\phi^*)+e^2A_\mu A^\mu\phi^*\phi\\ =&-ieA_\mu((\partial^\mu\phi)\phi^*-\phi(\partial^\mu\phi)^*)+e^2A_\mu A^\mu\phi^*\phi\\ =&\mathcal{L}_{\text{int}}\end{aligned} \tag{5}$$

   (ii) It is clear that $\mathcal{L}_A$ is invariant under gauge transformations since the electromagnetic tensor $F_{\mu\nu}$ is. This is because under a gauge transformation the commutation of partial derivatives tells us that
$$\begin{aligned}F_{\mu\nu}\mapsto F'_{\mu\nu}=&\partial_\mu A'_\nu-\partial_\nu A'_\mu=\partial_\mu A_\nu+\partial_\mu\partial_\nu\alpha-\partial_\nu A_\mu-\partial_\nu\partial_\mu\alpha\\ =&\partial_\mu A_\nu-\partial_\nu A_\mu=F_{\mu\nu}.\end{aligned} \tag{6}$$

For the other two terms in the Lagrangian we have the following behavior under gauge transformations

$$
\begin{aligned}
\mathcal{L}_\phi \mapsto \mathcal{L}'_\phi &= (\partial_\mu\phi')^*(\partial^\mu\phi') - m^2\phi'^*\phi' \\
=&(\partial_\mu(e^{-ie\alpha}\phi))^*(\partial^\mu(e^{-ie\alpha}\phi)) - m^2 e^{ie\alpha}\phi^* e^{-ie\alpha}\phi \\
=&(e^{-ie\alpha}\partial_\mu\phi - iee^{-ie\alpha}\phi\partial_\mu\alpha)^*(e^{-ie\alpha}\partial^\mu\phi - iee^{-ie\alpha}\phi\partial^\mu\alpha) \\
&- m^2\phi^*\phi \\
=&(e^{ie\alpha}(\partial_\mu\phi)^* + iee^{ie\alpha}\phi^*\partial_\mu\alpha)(e^{-ie\alpha}\partial^\mu\phi - iee^{-ie\alpha}\phi\partial^\mu\alpha) \\
&- m^2\phi^*\phi \\
=&(\partial_\mu\phi)^*\partial^\mu\phi + ie\phi^*(\partial^\mu\phi)(\partial_\mu\alpha) - ie\phi(\partial_\mu\phi)^*(\partial^\mu\alpha) \\
&+ e^2\phi^*\phi(\partial_\mu\alpha)(\partial^\mu\alpha) - m^2\phi^*\phi \\
=&\mathcal{L}_\phi + ie(\phi^*\partial^\mu\phi - \phi(\partial^\mu\phi)^*)\partial_\mu\alpha + e^2\phi^*\phi(\partial_\mu\alpha)(\partial^\mu\alpha)
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
\mathcal{L}_{\text{int}} \mapsto \mathcal{L}'_{\text{int}} &= -ieA'_\mu((\phi')^*\partial^\mu\phi' - (\partial^\mu\phi')^*\phi') + e^2A'_\mu A'^\mu\phi'^*\phi \\
=&-ie(A_\mu + \partial_\mu\alpha)(e^{ie\alpha}\phi^*\partial^\mu(e^{-ie\alpha}\phi) - (\partial^\mu(e^{-ie\alpha}\phi))^*e^{-ie\alpha}\phi) \\
&+ e^2(A_\mu + \partial_\mu\alpha)(A^\mu + \partial^\mu\alpha)e^{ie\alpha}\phi^*e^{-ie\alpha}\phi \\
=&-ie(A_\mu + \partial_\mu\alpha)(e^{ie\alpha}\phi^*(e^{-ie\alpha}\partial^\mu\phi - iee^{-ie\alpha}\phi\partial^\mu\alpha) \\
&- (e^{-ie\alpha}\partial^\mu\phi - iee^{-ie\alpha}\phi\partial^\mu\alpha)^*e^{-ie\alpha}\phi) \\
&+ e^2A_\mu A^\mu\phi^*\phi + 2e^2A_\mu\phi^*\phi\partial^\mu\alpha + e^2\phi^*\phi(\partial_\mu\alpha)(\partial^\mu\alpha) \\
=&-ie(A_\mu + \partial_\mu\alpha)(\phi^*\partial^\mu\phi - (\partial^\mu\phi)^*\phi - 2ie\phi^*\phi\partial^\mu\alpha) \\
&+ e^2A_\mu A^\mu\phi^*\phi + 2e^2A_\mu\phi^*\phi\partial^\mu\alpha + e^2\phi^*\phi(\partial_\mu\alpha)(\partial^\mu\alpha) \\
=&\mathcal{L}_{\text{int}} - 2e^2\phi^*\phi A_\mu\partial^\mu\alpha - ie(\phi^*\partial^\mu\phi - (\partial^\mu\phi)^*\phi)\partial_\mu\alpha \\
&- 2e^2\phi^*\phi(\partial_\mu\alpha)(\partial^\mu\alpha) + 2e^2A_\mu\phi^*\phi\partial^\mu\alpha + e^2\phi^*\phi(\partial_\mu\alpha)(\partial^\mu\alpha) \\
=&\mathcal{L}_{\text{int}} - ie(\phi^*\partial^\mu\phi - (\partial^\mu\phi)^*\phi)\partial_\mu\alpha - e^2\phi^*\phi(\partial_\mu\alpha)(\partial^\mu\alpha).
\end{aligned}
\tag{8}
$$

We thus conclude that the sum $\mathcal{L}_\phi + \mathcal{L}_{\text{int}}$ is gauge invariant and the full Lagrangian $\mathcal{L}$ is as well.

3. (i) Notice that the Lagrangian $\mathcal{L}'$ is a function of the 30 fields ϕ, ϕ^*, $\partial_\mu\phi$, $\partial_\mu\phi^*$, A_μ, and $\partial_\mu A^\nu$. We have the derivatives

$$
\begin{aligned}
\frac{\partial\mathcal{L}'}{\partial(\partial_\mu\phi^*)} =&\frac{\partial\mathcal{L}_\phi}{\partial(\partial_\mu\phi^*)} + \frac{\partial\mathcal{L}_{\text{int}}}{\partial(\partial_\mu\phi^*)} \\
=&\frac{\partial(\partial_\nu\phi)^*(\partial^\nu\phi)}{\partial(\partial_\mu\phi)} - ie\frac{\partial A_\nu(\phi^*\partial^\nu\phi - (\partial^\nu\phi^*)\phi)}{\partial(\partial_\mu\phi)} \\
=&(\partial^\nu\phi)\frac{\partial(\partial_\nu\phi^*)}{\partial(\partial_\mu\phi)^*} + ieA^\nu\phi\frac{\partial(\partial_\nu\phi^*)}{\partial(\partial_\mu\phi^*)} \\
=&(\partial^\nu\phi)\delta^\mu_\nu + ieA^\nu\phi\delta^\mu_\nu = \partial^\mu\phi + ieA^\mu\phi,
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}'}{\partial \phi^*} =& \frac{\partial \mathcal{L}_\phi}{\partial \phi^*} + \frac{\partial \mathcal{L}_{\text{int}}}{\partial \phi^*} + \frac{\partial V}{\partial \phi^*} \\
=& - m^2\phi - ieA_\mu \partial^\mu \phi + e^2 A_\mu A^\mu \phi + 2\lambda(\phi^*\phi - \phi_0^2)\phi,
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}'}{\partial(\partial_\mu A_\nu)} =& \frac{\partial \mathcal{L}_A}{\partial(\partial_\mu A_\nu)} = \frac{1}{4}\frac{\partial(F^{\sigma\rho}F_{\sigma\rho})}{\partial(\partial_\mu A_\nu)} = \frac{1}{4} g^{\sigma\lambda} g^{\rho\xi} \frac{\partial(F_{\lambda\xi}F_{\sigma\rho})}{\partial(\partial_\mu A_\nu)} \\
=& \frac{1}{4} g^{\sigma\lambda} g^{\rho\xi} \left(F_{\sigma\rho} \frac{\partial F_{\lambda\xi}}{\partial(\partial_\mu A_\nu)} + F_{\lambda\xi} \frac{\partial F_{\sigma\rho}}{\partial(\partial_\mu A_\nu)} \right) \\
=& \frac{1}{4} g^{\sigma\lambda} g^{\rho\xi} \left(F_{\sigma\rho} \left(\delta^\mu_\lambda \delta^\nu_\xi - \delta^\mu_\xi \delta^\nu_\lambda \right) + F_{\lambda\xi} \left(\delta^\mu_\sigma \delta^\nu_\rho - \delta^\mu_\rho \delta^\nu_\sigma \right) \right) \\
=& \frac{1}{4} \left(F^{\lambda\xi} \left(\delta^\mu_\lambda \delta^\nu_\xi - \delta^\mu_\xi \delta^\nu_\lambda \right) + F^{\sigma\rho} \left(\delta^\mu_\sigma \delta^\nu_\rho - \delta^\mu_\rho \delta^\nu_\sigma \right) \right) \\
=& \frac{1}{4} (F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu}) = F^{\mu\nu},
\end{aligned} \tag{11}
$$

and

$$
\begin{aligned}
\frac{\partial \mathcal{L}'}{\partial A_\nu} =& \frac{\partial \mathcal{L}_{\text{int}}}{\partial A_\nu} = -ie((\partial^\mu \phi)\phi^* - \phi(\partial^\mu \phi)^*)\delta^\nu_\mu + e^2 \phi^* \phi \frac{\partial A^\mu A_\mu}{\partial A_\nu} \\
=& - ie((\partial^\nu \phi)\phi^* - \phi(\partial^\nu \phi)^*) + e^2 \phi^* \phi g^{\sigma\mu} \frac{\partial A_\sigma A_\mu}{\partial A_\nu} \\
=& - ie((\partial^\nu \phi)\phi^* - \phi(\partial^\nu \phi)^*) + e^2 \phi^* \phi g^{\sigma\mu} (A_\sigma \delta^\nu_\mu + A_\mu \delta^\nu_\sigma) \\
=& - ie((\partial^\nu \phi)\phi^* - \phi(\partial^\nu \phi)^*) + 2e^2 \phi^* \phi A^\nu.
\end{aligned} \tag{12}
$$

Therefore, the Euler-Lagrange equations read for the ϕ field

$$
\begin{aligned}
0 =& \partial_\mu \frac{\partial \mathcal{L}'}{\partial(\partial_\mu \phi^*)} - \frac{\partial \mathcal{L}'}{\partial \phi^*} \\
=& \partial^\mu \partial_\mu \phi + ie\partial_\mu (A^\mu \phi) + m^2 \phi + ieA_\mu \partial^\mu \phi - e^2 A_\mu A^\mu \phi \\
& - 2\lambda(\phi^*\phi - \phi_0^2)\phi.
\end{aligned} \tag{13}
$$

Familiarity with the Klein-Gordon equation compels us to write this result in the form

$$
(\Box + m^2)\phi = 2\lambda(\phi^*\phi - \phi_0^2)\phi + e^2 A^\mu A_\mu - ie(\partial_\mu(A^\mu \phi) + A^\mu \partial_\mu \phi) \tag{14}
$$

where $\Box = \partial^\mu \partial_\mu$. On the other hand, the Euler-Lagrange equations for the A_μ field read

$$
\begin{aligned}
0 =& \partial_\mu \frac{\partial \mathcal{L}'}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}'}{\partial A_\nu} \\
=& \partial_\mu F^{\mu\nu} + ie((\partial^\nu \phi)\phi^* - \phi(\partial^\nu \phi)^*) - 2e^2 \phi^* \phi A^\nu.
\end{aligned} \tag{15}
$$

Once again, familiarity with Maxwell's equations suggests that we write this equation in the form

$$
\partial_\mu F^{\mu\nu} = 2e^2 \phi^* \phi A^\nu - ie((\partial^\nu \phi)\phi^* - \phi(\partial^\nu \phi)^*). \tag{16}
$$

(ii) Comparing (16) with Maxwell's equations $\partial_\mu F^{\mu\nu} = j^\nu$ we conclude that the electromagnetic current density is

$$j^\mu = 2e^2\phi^*\phi A^\mu - ie((\partial^\mu\phi)\phi^* - \phi(\partial^\mu\phi)^*). \qquad (17)$$