Mecánica Cuántica Avanzada
Tarea 5

Iván Mauricio Burbano Aldana

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4.8 Multiplicando por izquierda $\overline{u}_r(\mathbf{p}')\gamma^\mu$ a la ecuación (4.46) de [1] se obtiene

$$0 = \overline{u}_r(\mathbf{p}')\gamma^\mu(\not{p} - m)u_s(\mathbf{p}) = \overline{u}_r(\mathbf{p}')\gamma^\mu\gamma^\nu p_\nu u_s(\mathbf{p}) - \overline{u}_r(\mathbf{p}')\gamma^\mu m u_s(\mathbf{p}). \quad (1)$$

Multiplicando por la derecha $\gamma^\mu u_s(\mathbf{p})$ a la ecuación (4.48) de [1] se obtiene

$$0 = \overline{u}_r(\mathbf{p}')(\not{p}' - m)\gamma^\mu u_s(\mathbf{p}) = \overline{u}_r(\mathbf{p}')\gamma^\nu\gamma^\mu p'_\nu u_s(\mathbf{p}) - \overline{u}_r(\mathbf{p}')\gamma^\mu m u_s(\mathbf{p}). \quad (2)$$

Sumandolas se concluye que

$$\begin{aligned} 0 =&\overline{u}_r(\mathbf{p}')\gamma^\mu\gamma^\nu p_\nu u_s(\mathbf{p}) - \overline{u}_r(\mathbf{p}')\gamma^\mu m u_s(\mathbf{p}) \\ &+\overline{u}_r(\mathbf{p}')\gamma^\nu\gamma^\mu p'_\nu u_s(\mathbf{p}) - \overline{u}_r(\mathbf{p}')\gamma^\mu m u_s(\mathbf{p}) \\ &=\overline{u}_r(\mathbf{p}')(\gamma^\mu\gamma^\nu p_\nu + \gamma^\nu\gamma^\mu p'_\nu)u_s(\mathbf{p}) - 2\overline{u}_r(\mathbf{p}')\gamma^\mu m u_s(\mathbf{p}). \end{aligned} \quad (3)$$

En particular, si $\mathbf{p} = \mathbf{p}'$

$$0 = \overline{u}_r(\mathbf{p})2g^{\mu\nu}p_\nu u_s(\mathbf{p}) - 2m\overline{u}_r(\mathbf{p})\gamma^\mu u_s(\mathbf{p}). \quad (4)$$

Dividiendo por 2 y subiendo el índice del momento entonces es claro que

$$\overline{u}_r(\mathbf{p})\gamma^\mu m u_s(\mathbf{p}) = \overline{u}_r(\mathbf{p})p^\mu u_s(\mathbf{p}). \quad (5)$$

Repitiendo este proceso al pie de la letra se tiene

$$0 = \overline{v}_r(\mathbf{p}')\gamma^\mu(\not{p} + m)v_s(\mathbf{p}) = \overline{v}_r(\mathbf{p}')\gamma^\mu\gamma^\nu p_\nu v_s(\mathbf{p}) + \overline{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}), \quad (6)$$

$$0 = \overline{v}_r(\mathbf{p}')(\not{p}' + m)\gamma^\mu v_s(\mathbf{p}) = \overline{v}_r(\mathbf{p}')\gamma^\nu\gamma^\mu p'_\nu v_s(\mathbf{p}) + \overline{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}), \quad (7)$$

$$\begin{aligned} 0 =&\overline{v}_r(\mathbf{p}')\gamma^\mu\gamma^\nu p_\nu v_s(\mathbf{p}) + \overline{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}) \\ &+\overline{v}_r(\mathbf{p}')\gamma^\nu\gamma^\mu p'_\nu v_s(\mathbf{p}) + \overline{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}) \\ &=\overline{v}_r(\mathbf{p}')(\gamma^\mu\gamma^\nu p_\nu + \gamma^\nu\gamma^\mu p'_\nu)v_s(\mathbf{p}) + 2\overline{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}) \\ &=\overline{v}_r(\mathbf{p}')2g^{\mu\nu}p_\nu v_s(\mathbf{p}) + 2m\overline{v}_r(\mathbf{p}')\gamma^\mu v_s(\mathbf{p}) \end{aligned} \quad (8)$$

en el caso $\mathbf{p} = \mathbf{p}'$ y

$$\overline{v}_r(\mathbf{p}')\gamma^\mu m v_s(\mathbf{p}) = -\overline{v}_r(\mathbf{p}')p^\mu v_s(\mathbf{p}). \quad (9)$$

Poniendo $\mu = 0$ en (5) se tiene haciendo uso de la normalización (4.49) de [1]

$$\begin{aligned}2E_p\delta_{rs}m =& mu_r^\dagger(\mathbf{p})u_s(\mathbf{p}) = mu_r^\dagger(\mathbf{p})\gamma^0\gamma^0u_s(\mathbf{p}) = m\overline{u}_r(\mathbf{p})\gamma^0u_s(\mathbf{p})\\ =& p^0\overline{u}_r(\mathbf{p})u(\mathbf{p}) = E_p\overline{u}_r(\mathbf{p})u_s(\mathbf{p}).\end{aligned} \tag{10}$$

Repitiendo con (9)

$$\begin{aligned}2E_p\delta_{rs}m =& mv_r^\dagger(\mathbf{p})v_s(\mathbf{p}) = mv_r^\dagger(\mathbf{p})\gamma^0\gamma^0v_s(\mathbf{p}) = m\overline{v}_r(\mathbf{p})\gamma^0v_s(\mathbf{p})\\ =& -p^0\overline{v}_r(\mathbf{p})v(\mathbf{p}) = -E_p\overline{v}_r(\mathbf{p})v_s(\mathbf{p}).\end{aligned} \tag{11}$$

Por lo tanto, asumiendo que $E_p \neq 0$, se tiene

$$\overline{u}_r(\mathbf{p})u_s(\mathbf{p}) = -\overline{v}_r(\mathbf{p})v_s(\mathbf{p}) = 2m\delta_{rs}. \tag{12}$$

4.9 Note que haciendo uso de la relaciones de conmutación de las matrices de Dirac se tiene

$$\begin{aligned}&(p+p')^\mu - i\sigma^{\mu\nu}q_\nu\\ =& p^\mu + p'^\mu - i\frac{i}{2}[\gamma^\mu,\gamma^\nu]_-(p_\nu - p'_\nu)\\ =& p^\mu + p'^\mu + \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p_\nu - p'_\nu)\\ =& p^\mu + p'^\mu + \frac{1}{2}(\gamma^\mu\gamma^\nu p_\nu - \gamma^\mu\gamma^\nu p'_\nu - \gamma^\nu\gamma^\mu p_\nu + \gamma^\nu\gamma^\mu p'_\nu)\\ =& p^\mu + p'^\mu\\ &+\frac{1}{2}(\gamma^\mu\gamma^\nu p_\nu - 2g^{\mu\nu}p'_\nu + \gamma^\nu\gamma^\mu p'_\nu - 2g^{\nu\mu}p_\nu + \gamma^\mu\gamma^\nu p_\nu + \gamma^\nu\gamma^\mu p'_\nu)\\ =& p^\mu + p'^\mu + \frac{1}{2}(2\gamma^\mu\gamma^\nu p_\nu - 2p'^\mu + 2\gamma^\nu\gamma^\mu p'_\nu - 2p^\mu)\\ =& \gamma^\mu\gamma^\nu p_\nu + \gamma^\nu\gamma^\mu p'_\nu.\end{aligned} \tag{13}$$

Por lo tanto, comparando con las ecuaciones (3) y (8) se obtiene

$$\begin{aligned}0 =& \overline{u}_r(\mathbf{p}')((p+p')^\mu - i\sigma^{\mu\nu}q_\nu)u_s(\mathbf{p}) - 2m\overline{u}_r(\mathbf{p}')\gamma^\mu u_s(\mathbf{p})\\ 0 =& \overline{v}_r(\mathbf{p}')((p+p')^\mu - i\sigma^{\mu\nu}q_\nu)v_s(\mathbf{p}) + 2m\overline{v}_r(\mathbf{p}')\gamma^\mu v_s(\mathbf{p}).\end{aligned} \tag{14}$$

Dividiendo por $2m$ se obtienen las identidades de Gordon

$$\begin{aligned}\overline{u}_r(\mathbf{p}')\gamma^\mu u_s(\mathbf{p}) =& \frac{1}{2m}\overline{u}_r(\mathbf{p}')((p+p')^\mu - i\sigma^{\mu\nu}q_\nu)u_s(\mathbf{p})\\ \overline{v}_r(\mathbf{p}')\gamma^\mu v_s(\mathbf{p}) =& -\frac{1}{2m}\overline{v}_r(\mathbf{p}')((p+p')^\mu - i\sigma^{\mu\nu}q_\nu)v_s(\mathbf{p}).\end{aligned} \tag{15}$$

4.10 Note que debido a la ecuación de Dirac (4.46) de [1]

$$\begin{aligned}(\not p + m)u_r(\mathbf{p}) =& (\not p - m + 2m)u_r(\mathbf{p}) = 2mu_r(\mathbf{p})\\ (\not p + m)v_r(\mathbf{p}) =& 0\\ (\not p - m)u_r(\mathbf{p}) =& 0\\ (\not p - m)v_r(\mathbf{p}) =& (\not p + m - 2m)v_r(\mathbf{p}) = -2mv_r(\mathbf{p}).\end{aligned} \tag{16}$$

Por el otro lado, haciendo uso de las relaciones de normalización halladas en el ejercicio 4.8 y el hecho de que $u_r(\mathbf{p})$ y $v_s(\mathbf{p})$ son ortogonales pues son vectores propios de la energía con valores propios asociados distintos

$$\begin{aligned}
\sum_s u_s(\mathbf{p})\overline{u}_s(\mathbf{p})u_r(\mathbf{p}) =& \sum_s u_s(\mathbf{p})2m\delta_{sr} = 2mu_r(\mathbf{p}) \\
\sum_s u_s(\mathbf{p})\overline{u}_s(\mathbf{p})v_r(\mathbf{p}) =& 0 \\
\sum_s v_s(\mathbf{p})\overline{v}_s(\mathbf{p})u_r(\mathbf{p}) =& 0 \\
\sum_s v_s(\mathbf{p})\overline{v}_s(\mathbf{p})v_r(\mathbf{p}) =& \sum_s v_s(\mathbf{p})(-2m\delta_{sr}) = -2mv_r(\mathbf{p}).
\end{aligned} \tag{17}$$

Ya que las matrices coinciden en una base, por extensión lineal deben ser iguales

$$\begin{aligned}
\sum_s u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) =& \not{p} + m \\
\sum_s v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) =& \not{p} - m.
\end{aligned} \tag{18}$$

Referencias

[1] A. Lahiri and P. B. Pal, *A First Book of Quantum Field Theory.* 2005.