Electrodynamics: Homework 2

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1. In this exercise I will consider spacetime to be an affine space. Given a chart x and a vector v (which may be considered equivalently as an element of the tangent bundle or of the vector space after which spacetime is modeled), v_x^μ will be the components of v in the coordinate induced basis. We set $\gamma = \left(1-\beta^2\right)^{-1/2}$ and $\beta = \frac{v}{c}$.

1.1. Let us consider two events p and q which correspond to the endpoints of the bar at equal times according to observer S'. Let x be the coordinates according to observer S and y the coordinates according to S'. Assume that the coordinates are taken such that the rod is in the 12-plane, that is, $(p-q)_y^3 = 0$. The lengths can be calculated according to the formulae

$$\begin{aligned} L &= \sqrt{\sum_{i=1}^{3}((p-q)_x^i)^2} \\ (p-q)_y^1 &= L'\cos(\theta') \\ (p-q)_y^2 &= L'\sin(\theta') \end{aligned} \tag{1}$$

Using the Lorentz transformation

$$\frac{\partial x^\mu}{\partial y^\nu} = \Lambda^\mu_\nu = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

we have

$$\begin{aligned} L &= \sqrt{\sum_{i=1}^{3}(\Lambda^i_\nu (p-q)_y^\nu)^2} \\ &= \sqrt{(\gamma\beta(p-q)_y^0 + \gamma(p-q)_y^1)^2 + ((p-q)_y^2)^2 + ((p-q)_y^3)^2} \\ &= \sqrt{\gamma^2 L'^2\cos^2(\theta') + L'^2\sin^2(\theta')} \\ &= L'\sqrt{\gamma^2\cos^2(\theta') + \sin^2(\theta')}. \end{aligned} \tag{3}$$

1.2 As in the previous point we have

$$\tan(\theta) = \frac{(p-q)_x^2}{(p-q)_x^1} = \frac{(p-q)_y^2}{\gamma\beta(p-q)_y^0 + \gamma(p-q)_y^1} = \frac{1}{\gamma}\frac{(p-q)_y^2}{(p-q)_y^1} = \frac{1}{\gamma}\tan(\theta') \tag{4}$$

2. We have by the definition of a Lorentz transformation that they preserve the metric, which in matrix notation reads $g_{\mu\nu}\Lambda^\mu_\sigma\Lambda^\nu_\lambda = g_{\sigma\lambda}$. Therefore

$$g_{\mu\nu}A'^\mu B'^\nu = g_{\mu\nu}\Lambda^\mu_\sigma A^\sigma \Lambda^\nu_\lambda B^\lambda = g_{\sigma\lambda}A^\sigma B^\lambda = g_{\mu\nu}A^\mu B^\nu. \tag{5}$$

3.

(i) We set $c = 1$. Through integration we find

$$\mathbf{p}(t) = \mathbf{p}(0) + \int_0^t \mathbf{p}'(u)du = \mathbf{p}_0 + \int_0^t q(\mathcal{E},0,0)du = \frac{m\mathbf{v}_0}{\sqrt{1-\mathbf{v}_0^2}} + q\mathcal{E}t(1,0,0). \tag{6}$$

(ii) The energy of the particle is then

$$\begin{aligned} E(t) = \sqrt{\mathbf{p}(t)^2 + m^2} &= \sqrt{\left(\frac{m\mathbf{v}_0}{\sqrt{1-\mathbf{v}_0^2}} + q\mathcal{E}t(1,0,0)\right)^2 + m^2} \\ &= \sqrt{\frac{m^2\mathbf{v}_0^2}{1-\mathbf{v}_0^2} + \frac{2mq\mathcal{E}t(\mathbf{v}_0)_x}{\sqrt{1-\mathbf{v}_0^2}} + q^2\mathcal{E}^2t^2 + m^2} \\ &= \sqrt{\frac{m^2\mathbf{v}_0^2 + 2mq\mathcal{E}t(\mathbf{v}_0)_x\sqrt{1-\mathbf{v}_0^2} + (q^2\mathcal{E}^2t^2+m^2)(1-\mathbf{v}_0^2)}{1-\mathbf{v}_0^2}} \\ &= \sqrt{\frac{m^2 + 2mq\mathcal{E}t(\mathbf{v}_0)_x\sqrt{1-\mathbf{v}_0^2} + q^2\mathcal{E}^2t^2(1-\mathbf{v}_0^2)}{1-\mathbf{v}_0^2}}. \end{aligned} \tag{7}$$

(iii) We thus obtain for the velocity

$$\mathbf{v}(t) = \frac{\mathbf{p}(t)}{E(t)} = \frac{m\mathbf{v}_0 + q\mathcal{E}t\sqrt{1-\mathbf{v}_0^2}}{\sqrt{m^2 + 2mq\mathcal{E}t(\mathbf{v}_0)_x\sqrt{1-\mathbf{v}_0^2} + q^2\mathcal{E}^2t^2(1-\mathbf{v}_0^2)}} \tag{8}$$

which we can directly integrate for the position. It is the nevertheless easier to begin with a less developed form where we have more control for the time dependence.

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\begin{aligned}
\mathbf{r}(t) = \mathbf{r}(0) + \int_0^t du r\mathbf{r}'(u) &= \mathbf{r}_0 + \int_0^t du \frac{\mathbf{p}(u)}{\sqrt{\mathbf{p}(u)^2 + m^2}} \\
&= \mathbf{r}_0 + \int_0^t du \frac{\sum_{i=1}^3 \mathbf{p}(u)_i \hat{\mathbf{e}}_i}{\sqrt{\sum_{i=1}^3 \mathbf{p}(u)_i^2 + m^2}} \\
&= \mathbf{r}_0 + \int_0^t du \frac{\mathbf{p}(u)_1 \hat{\mathbf{e}}_1 + \sum_{i=2}^3 \mathbf{p}(0)_i \hat{\mathbf{e}}_i}{\sqrt{\mathbf{p}(u)_1^2 + \sum_{i=2}^3 \mathbf{p}(0)_i^2 + m^2}} \\
&= \mathbf{r}_0 + \int_{\mathbf{p}(0)_1}^{\mathbf{p}(t)_1} dv \frac{1}{\mathbf{p}_1'(\mathbf{p}_1^{-1}(v))} \frac{v \hat{\mathbf{e}}_1 + \sum_{i=2}^3 \mathbf{p}(0)_i \hat{\mathbf{e}}_i}{\sqrt{v^2 + \sum_{i=2}^3 \mathbf{p}(0)_i^2 + m^2}} \\
&= \mathbf{r}_0 + \frac{1}{q\mathcal{E}} \int_{\mathbf{p}(0)_1}^{\mathbf{p}(t)_1} dv \frac{v \hat{\mathbf{e}}_1 + \sum_{i=2}^3 \mathbf{p}(0)_i \hat{\mathbf{e}}_i}{\sqrt{v^2 + \sum_{i=2}^3 \mathbf{p}(0)_i^2 + m^2}} \\
&= \mathbf{r}_0 + \frac{1}{2q\mathcal{E}} \int_{\mathbf{p}(0)_1^2}^{\mathbf{p}(t)_1^2} dw \frac{\hat{\mathbf{e}}_1}{\sqrt{w + \sum_{i=2}^3 \mathbf{p}(0)_i^2 + m^2}} \\
&+ \frac{1}{q\mathcal{E}} \sum_{i=2}^3 \mathbf{p}(0)_i \hat{\mathbf{e}}_i \int_{\mathbf{p}(0)_1}^{\mathbf{p}(t)_1} dv \frac{1}{\sqrt{v^2 + \sum_{i=2}^3 \mathbf{p}(0)_i^2 + m^2}} \\
&= \mathbf{r}_0 + \frac{\hat{\mathbf{e}}_1}{q\mathcal{E}} \left(\sqrt{\mathbf{p}(t)_1^2 + \sum_{i=2}^3 \mathbf{p}(0)_i^2 + m^2} - \sqrt{\mathbf{p}(0)_1^2 + \sum_{i=2}^3 \mathbf{p}(0)_i^2 + m^2} \right) \\
&+ \frac{1}{q\mathcal{E}} \sum_{i=2}^3 \mathbf{p}(0)_i \hat{\mathbf{e}}_i \ln \left(\frac{\sqrt{\mathbf{p}(t)_1^2 + \sum_{i=2}^3 \mathbf{p}(0)_i^2 + m^2} + \mathbf{p}(t)_1}{\sqrt{\mathbf{p}(0)_1^2 + \sum_{i=2}^3 \mathbf{p}(0)_i^2 + m^2} + \mathbf{p}(0)_1} \right) \\
&= \mathbf{r}_0 + \frac{\hat{\mathbf{e}}_1}{q\mathcal{E}} \left(\sqrt{\mathbf{p}(t)^2 + m^2} - \sqrt{\mathbf{p}(0)^2 + m^2} \right) \\
&+ \frac{1}{q\mathcal{E}} \sum_{i=2}^3 \mathbf{p}(0)_i \hat{\mathbf{e}}_i \ln \left(\frac{\sqrt{\mathbf{p}(t)^2 + m^2} + \mathbf{p}(0)_1 + q\mathcal{E}t}{\sqrt{\mathbf{p}(0)^2 + m^2} + \mathbf{p}(0)_1} \right) \\
&= \mathbf{r}_0 + \frac{\hat{\mathbf{e}}_1}{q\mathcal{E}} (E(t) - E(0)) \\
&+ \frac{1}{q\mathcal{E}} \sum_{i=2}^3 \mathbf{p}(0)_i \hat{\mathbf{e}}_i \ln \left(\frac{E(t) + \mathbf{p}(0)_1 + q\mathcal{E}t}{E(0) + \mathbf{p}(0)_1} \right)
\end{aligned}
\tag{9}
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Conveniently expressed using our previous results.