

Mathematical analysis II

Collection of problems

Not for handing in

Exercise 1 (Metrics).

a) Let

$$d(x, y) = \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right|.$$

Show that (\mathbb{R}, d) is a metric space. Are the metrics d and $d_1(x, y) := |x - y|$ equivalent?
Show or disprove.

b) Let (X, d) be a metric space. Find all $k \in \mathbb{R}$ such that

$$d_1(x, y) := (k-1)(k-3)d(x, y)$$

is a metric on X .

c) Let (X, d) be a metric space and define like in HW 1

$$\delta(x, y) = \min\{d(x, y), 1\}.$$

Are, in general, the metrics d and δ equivalent? Show or disprove.

Exercise 2 (Open and closed sets).

- a) Give an example of a set $A \subset \mathbb{R}$ (with the euclidean metric) that is neither closed nor open.
Can you also find such a set $A \subset \mathbb{R}^2$ or even $A \subset \mathbb{R}^n$?
- b) Give an example of metric spaces (X, d) and (Y, e) , a function $f : X \rightarrow Y$, and a closed subset $B \subset Y$ such that $f^{-1}[B]$ is not closed. Give also an example where you switch every “closed” with “open”.
- c) Find a function $f : \mathbb{R} \rightarrow \mathbb{R}$, a closed set $A \subset \mathbb{R}$ and an open set $B \subset \mathbb{R}$ such that $f[A]$ and $f[B]$ are neither open nor closed.
- d) A space X is called *connected*, if the only sets that are both closed and open (clopen) are just \emptyset and X .
 - 1) Give an example $X \subset \mathbb{R}$ that is not connected. Determine all clopen subsets of X .
 - 2) Let (X, d) and (Y, e) be metric spaces and

$$f : X \rightarrow Y$$

be continuous and onto (surjective). Show that if X is connected, then Y is as well.

Exercise 3 (Continuity).

- a) Determine whether or not the following functions $f_i : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ are continuous. Can we extend them in such a way that the functions are continuous on the whole of \mathbb{R}^2 ?

$$1) f_1(x, y) = \frac{x+y}{\sqrt{x^2+y^2}}$$

$$2) f_2(x, y) = \frac{x^2y^2}{x^2+y^2}$$

$$3) f_3(x, y) = \frac{x^2+y^2}{|x|+|y|}$$

- b) Let (X, d) , (Y, e) be metric spaces, where $X = A \cup B$ is the union of open or closed subsets of X . Let moreover $f_A : A \rightarrow Y$ and $f_B : B \rightarrow Y$ be continuous with $f_A = f_B$ on $A \cap B$. Show that the function

$$f : X \rightarrow Y, \quad f(x) = \begin{cases} f_A(x) & \text{if } x \in A, \\ f_B(x) & \text{if } x \in B \end{cases}$$

is continuous.

Exercise 4 (Differentiability, extrema, tangential planes).

- a) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be an affine mapping, that is, there is some linear mapping $L : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f(y) - f(x) = L(y - x)$. Show that f is totally differentiable on \mathbb{R}^n . How the total differential looks like?

- b) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuously differentiable, $x_0 \in \mathbb{R}^n$ and $c = f(x_0)$. Show that the gradient $\nabla f(x_0)$ is perpendicular to the level set

$$N_f(c) = \{x \in \mathbb{R}^n : f(x) = c\},$$

i.e., the following holds: If $\varepsilon > 0$ and $\phi : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$ is a differentiable curve with $\phi(0) = x_0$ and $\phi[(-\varepsilon, \varepsilon)] \subset N_f(c)$, then

$$\langle \phi'(0), \nabla f(x_0) \rangle = 0.$$

(Hint: Consider the function $g := f \circ \phi$.)

- c) Calculate the partial derivatives up to order 2 of f_1 and f_2 from Exercise 3.

- d) Determine position and kind of all extrema to the function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y) = (x^2 - 1)^2 + y^4.$$

Additionally, calculate the tangent plane at the point $x_0 = (2, 3)$. What happens to the extrema if we consider $g(x, y) = (x^2 - 1)^2 + y^3$ instead?

- e) Let

$$f(t) = (1+t, t^2, 1-t), \quad g(x, y, z) = 1+x+xyz.$$

Calculate once with and once without the help of the chain rule $D(g \circ f)(0)$.

- f) Let n points $(x_1, y_1), \dots, (x_n, y_n)$ be given. Find the equation of the line $y = ax + b$, for which the sum

$$f(a, b) = \sum_{i=1}^n (y_i - (ax_i + b))^2$$

becomes minimal.

Exercise 5 (Mean value theorem, second order derivatives, chain rule).

- a) Show that the mean value theorem fails for functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $m \geq 2$. More precisely, given $a = 0$, $h = 2\pi$, and

$$f : \mathbb{R} \rightarrow \mathbb{R}^2, \quad f(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

show that there does not exist a $\theta \in (0, 1)$ such that $f(a + h) - f(a) = f'(a + \theta h)h$.

- b) Show that for the function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{else,} \end{cases}$$

we have $\partial_x \partial_y f(0, 0) \neq \partial_y \partial_x f(0, 0)$. Why this is not a contradiction to Schwarz' theorem?

- c) A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called *homogeneous of degree $k \in \mathbb{Z}$* if for any $s \in \mathbb{R} \setminus \{0\}$, we have $f(sx_1, sx_2, \dots, sx_n) = s^k f(x_1, x_2, \dots, x_n)$.
- 1) Give examples of functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ that are homogeneous of degree -1, 0, 1, 2, respectively.
 - 2) Show: If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is partially differentiable and homogeneous of degree k , then

$$\sum_{i=1}^n x_i \partial_{x_i} f(x_1, \dots, x_n) = k f(x_1, \dots, x_n).$$

Exercise 6 (Compactness, completeness).

- a) Let (X, d) be a metric space. Show or disprove: if X is compact, then it is complete.
- b) Let (X, d) be a metric space. Show or disprove: if X is complete, then it is compact.
- c) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function, $K \subset \mathbb{R}$ and $V \subset \mathbb{R}$. Find examples for the following situations:
- 1) K is compact, but $f[K]$ is not.
 - 2) K is compact, but $f^{-1}[K]$ is not.
 - 3) V is complete, but $f[V]$ is not.
 - 4) V is complete, but $f^{-1}[V]$ is not.
- d) Show or disprove: if (X, d) and (Y, e) are metric spaces, $f : X \rightarrow Y$ is continuous and X is complete, then $f[X]$ is complete.
- e) Let (X, d) be a metric space and $A \subset X$. For $x \in X$ we define the distance from x to A by $\text{dist}(x, A) = \inf\{d(x, y) : y \in A\}$.
- 1) Show that $\text{dist}(\cdot, A) : X \rightarrow \mathbb{R}$ is continuous.
 - 2) Let $A \subset X$ be compact and $x \notin A$. Show that $\text{dist}(x, A) > 0$. Does this also hold if we replace “compact” by “closed”?
- f) (a bit harder task, don’t waste too much time with that one) Let $A \subset \mathbb{R}^n$, I be any (countable or uncountable) index set and $U_i \subset \mathbb{R}^n$ be open for any $i \in I$. We call $(U_i)_{i \in I}$ an (*open*) *covering* of A if $A \subset \bigcup_{i \in I} U_i$. Show that A is compact if and only if for any open covering $(U_i)_{i \in I}$ there is a finite subcovering, i.e., there are finitely many U_{i_1}, \dots, U_{i_n} with $i_1, \dots, i_n \in I$ such that $A \subset \bigcup_{j=1}^n U_{i_j}$. (In this sense, “compactness” is a generalization of “finiteness”).

- g) Give for $(0, 1)$ and $[0, \infty)$ open coverings that do not possess a finite subcovering.

Exercise 7 (A mixed problem).

Let $X = (0, 1)$ and

$$d(x, y) = \begin{cases} |x - y| + \frac{1}{x} + \frac{1}{y} + \frac{1}{1-x} + \frac{1}{1-y} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

- a) Show that d is a metric on X .
- b) Show that the space $(X, |x - y|)$ is not complete, but (X, d) is complete.
- c) Are the metrics $|x - y|$ and $d(x, y)$ equivalent?
- d) Is $(0, 1)$ bounded/closed/open/compact in the metric d ?

Exercise 8 (Taylor polynomial, implicit function theorem, inverse function theorem).

- a) Find the Taylor polynomials of degree two and three of the function

$$f(x, y) = \frac{x - y}{x + y}$$

in the point $(1, 1)$.

- b) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by $f(x, p) = x^2 + px - 1$. Aim of this exercise is to train implicit function theorem (IFT) with a known example.
 - 1) Show directly (completing the square) that the equation is solvable for x , i.e., there is a function g_1 such that $f(x, p) = 0 \Leftrightarrow x = g_1(p)$.
 - 2) Use IFT to show that there exists a function g_2 such that $f(x, p) = 0 \Leftrightarrow x = g_2(p)$ for all (x, p) in some neighborhood of $(1, 0)$.
 - 3) Differentiate the equation $f(g_2(p), p) = 0$ wrt. p using chain rule to obtain an equation for $g'_2(p)$.
 - 4) Check whether also g_1 fulfills this equation.
- c) Show that the system of equations

$$\begin{aligned} x^2 + y^2 &= 2uv, \\ x^3 + y^3 &= v^3 - u^3 \end{aligned}$$

defines in some neighborhood of the point $(x_0, y_0) = (-1, 1)$ implicitly a function $g(x, y) = (u(x, y), v(x, y))$ with $g(-1, 1) = (1, 1)$. Determine the Jacobi matrix of g in $(-1, 1)$.

- d) (A student's logarithmic rule) Show that there is a continuously differentiable function g defined in a neighborhood of $x = 1/e$ such that

$$\log(g(x) - x) = \frac{\log g(x)}{\log x}.$$

Determine first a “good” value y_0 such that $g(1/e) = y_0$. (For “generalizers”: Take $1/e^r$ with $r \geq 1$ instead of $1/e$.)

e) Find for the function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad f(x, y) = (e^x \sin y, e^x \cos y)$$

an open set $U \subset \mathbb{R}^2$ such that the restriction $f|U$ is injective. Is f globally invertible?

f) Let $f(x, y, z) = (x + y + z, xy + xz + yz, xyz)$. Show that f is continuously differentiable and determine the Jacobi matrix. Decide also whether or not the inverse function theorem is applicable in the points $(1, 1, 0)$ and $(1, -1, 0)$.

g) Let

$$f(x) = \begin{cases} x + 2x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is differentiable and that the derivative is bounded on the interval $(-1, 1)$. Show moreover that in no neighborhood of $x = 0$, the function is injective. Why this isn't a contradiction with the inverse function theorem?

Exercise 9 (Directional derivatives, extrema under constraints).

a) Calculate the directional derivative of $f(x, y, z) = e^{xyz}$ in the point $x_0 = (1, 1, 1)$ with direction $v = (1, 2, -1)$. Show that indeed $D_v f(x_0) = \nabla f(x_0) \cdot v$.

b) Let

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is discontinuous in $(x, y) = (0, 0)$ (in particular, it is not totally differentiable there), but all directional derivatives exist there.

c) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be totally differentiable in a point $x_0 \in \mathbb{R}^2$, and let this total differential be nonzero. Let $v, w \in \mathbb{R}^2$ such that $D_v f(x_0) = D_w f(x_0) = 0$. Show that v and w are linearly dependent. Does this also hold when replacing \mathbb{R}^2 with \mathbb{R}^3 (or even \mathbb{R}^n)?

d) Define

$$M = \{(x, y) \in \mathbb{R}^2 : x = y \text{ and } x \neq 0\},$$

and let

$$f(x, y) = \begin{cases} e^x - 1 & \text{if } (x, y) \in M, \\ 0 & \text{else.} \end{cases}$$

1) Show that f is partially differentiable in $(x, y) \in \mathbb{R}^2$ if and only if $(x, y) \notin M$.

2) The directional derivative $D_v f(0)$ exists for any $v \in \mathbb{R}^2$.

3) There is some $v \in \mathbb{R}^2$ with $|v| = 1$ such that $D_v f(0) \neq \nabla f(0) \cdot v$.

e) Let $f(x, y) = 4x^2 - 3xy$. Find all extrema of f in the disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Argue why such extrema do exist. (How to: first find the extrema inside D in the usual way, then find extrema on the boundary using constraints.)

Exercise 10 (Uniform continuity, Riemann integral 1D).

- a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be uniformly continuous. Show that there is some $M \geq 0$ such that for any $x \in \mathbb{R}$, we have $|f(x)| \leq M(1 + |x|)$.
- b) Construct an example of a function $f : [0, 1] \rightarrow \mathbb{R}$ such that f is continuous but not uniformly continuous.
- c) Prove: Let (a, b) be a bounded open interval. A continuous function $f : (a, b) \rightarrow \mathbb{R}$ is uniformly continuous if and only if we can extend it to a function that is continuous on the closed interval $[a, b]$.
- d) Show that for any Riemann integrable function $f : [a, b] \rightarrow \mathbb{R}$, it holds that

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

- e) Let $x_0 \in (0, 1)$ and define

$$\chi_{x_0}(x) = \begin{cases} 1 & \text{if } x = x_0, \\ 0 & \text{else.} \end{cases}$$

Show via definition that $\int_0^1 \chi_{x_0}(x) dx = 0$.

Exercise 11 (Fubini's theorem in different variants, change of variables).

- a) Calculate the integral $\int_G f(x, y) d(x, y)$, where:
 - 1) $f(x, y) = 2xy + \cos(x + y)$, $G = [-\pi, 0] \times [0, 2]$
 - 2) $f(x, y) = x^2$, $G = \{x \in (-1, 2), 0 < y < x^2\}$
 - 3) $f(x, y) = x + y + xy$, where G is the area enclosed by the functions $g(x) = x^2 + 1$ and $h(x) = x + 3$
- b) Let $b > 0$ and $f : [0, b] \rightarrow \mathbb{R}$ be integrable. Show that the function $g(x) := \int_x^b t^{-1} f(t) dt$ is continuous on $(0, b]$ and integrable on $[0, b]$, and that $\int_0^b g(x) dx = \int_0^b f(t) dt$.
- c) A material with density $\varrho(x, y, z) = x + y + z$ is formed into the unit cube $C = [0, 1]^3$. Calculate the mass $m = \int_C \varrho dV$ of this cube. Here the volume element is given by $dV = dx dy dz$.
- d) Another material with density $\varrho(x, y, z) = x^2 + y^2$ is now formed into a cylinder $Z = \{x, y, z \in \mathbb{R} : x^2 + y^2 \leq 1, 0 \leq z \leq 2\}$. Calculate the mass of this cylinder. (It might be worth to use a change of coordinates. The volume element for cylindrical coordinates is given by $dV = r dr d\theta dz$.)
- e) Let $\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq R^2, x^2 + y^2 \leq z^2, z \geq 0\}$. Sketch the set Ω and calculate its volume (a usage of spherical coordinates $(x, y, z) = (r \sin(\theta) \cos(\phi), r \sin(\theta) \sin(\phi), r \cos(\theta))$ might be useful).
- f) Make a reasoned decision for which $\alpha \in \mathbb{R}$ the 3D integrals

$$\int_{B_1(0)} |x|^{-\alpha} dx \quad \text{and} \quad \int_{\mathbb{R}^3 \setminus \overline{B_1(0)}} |x|^{-\alpha} dx$$

converge.

- g) A cylinder $x^2 + y^2 \leq R^2$ cuts from the ball $x^2 + y^2 + z^2 \leq 4R^2$ a body B . Calculate the volume of B .

- h) Let $U \subset \mathbb{R}^3$ be a compact set filled with mass of (R-integrable) density ϱ , and let $M = \int_U \varrho dV$ be its mass. The *center of mass (COM)* is given by the coordinates

$$\bar{x} = \frac{1}{M} \int_U x\varrho(x, y, z) d(x, y, z), \quad \bar{y} = \frac{1}{M} \int_U y\varrho(x, y, z) d(x, y, z), \quad \bar{z} = \frac{1}{M} \int_U z\varrho(x, y, z) d(x, y, z),$$

resp. in vector form with $\underline{x} = (x, y, z)$ simply

$$\bar{\underline{x}} = \frac{1}{M} \int_U \underline{x}\varrho(\underline{x}) d\underline{x}.$$

Calculate the COM of

- 1) the octant $U = \{x^2 + y^2 + z^2 \leq R^2, x \geq 0, y \geq 0, z \geq 0\}$ with constant density $\varrho = 1$;
- 2) the half ball $U = \{x^2 + y^2 + z^2 \leq R^2, z \geq 0\}$ with constant density $\varrho = 1$;
- 3) the same octant and half ball, but with density $\varrho(x, y, z) = az$ for some constant $a > 0$.