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Mathematical analysis II

Voluntary Homework 11

To be voluntarily handed in whenever via OWL

Transformers

Exercise 1 (Isodiametric inequality in 2D).

Let $g : \mathbb{R} \to (0, \infty)$ be continuous and π-periodic, that is, $g(\theta + \pi) = g(\theta)$ for any $\theta \in \mathbb{R}$. We define

$$\Omega = \{(x, y) = (r\cos(\theta), r\sin(\theta)) \in \mathbb{R}^2 : \theta \in [0, 2\pi],\ 0 \le r \le g(\theta)\},$$

and define the diameter of Ω as

$$\operatorname{diam}(\Omega) = \sup\{|x - y| : x, y \in \Omega\}.$$

a) Show that the area A of Ω is given by

$$A = \frac{1}{2}\int_0^{2\pi} [g(\theta)]^2 d\theta.$$

(*Hint*: write $A = \int_\Omega 1\, d(x, y)$, use a coordinate transformation to polar coordinates $(x, y) = (r\cos\phi, r\sin\phi)$, and use Fubini's theorem.)

b) Show that

$$A \le \pi \left(\frac{\operatorname{diam}(\Omega)}{2}\right)^2.$$

Give a geometrical interpretation of this inequality *for fixed diameter* $\operatorname{diam}(\Omega)$. To this end, compare an arbitrary function g with the special constant one $g_0 = \operatorname{diam}(\Omega)/2$. (*Hint*: what geometric property is included in the periodicity condition on g? Which relationship between $\operatorname{diam}(\Omega)$ and $\sup_{\theta\in\mathbb{R}} g(\theta)$ you can conclude from that? A sketch might be helpful.)

Exercise 2 (Christmas).

We define

$$h(z) = z - \frac{1}{2}\lfloor z\rfloor - \lfloor z/4\rfloor(z - 5/2), \quad z \in (0, 5).$$

Here, $\lfloor x\rfloor = \max\{k \in \mathbb{Z} : k \le x\}$ is the largest integer less than a real number $x \in \mathbb{R}$. Let further

$$CT = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \le h(z)^2\}.$$

a) Sketch h and the set CT (using colors if wished).

b) Calculate the volume of CT.

Merry Christmas and a Happy New Year 2026!