

## Mathematical analysis II

### Collection of problems

Not for handing in

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#### Exercise 1 (Metrics).

a) Let

$$d(x, y) = \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right|.$$

Show that  $(\mathbb{R}, d)$  is a metric space. Are the metrics  $d$  and  $d_1(x, y) := |x - y|$  equivalent? Show or disprove.

b) Let  $(X, d)$  be a metric space. Find all  $k \in \mathbb{R}$  such that

$$d_1(x, y) := (k - 1)(k - 3)d(x, y)$$

is a metric on  $X$ .

c) Let  $(X, d)$  be a metric space and define like in HW 1

$$\delta(x, y) = \min\{d(x, y), 1\}.$$

Are, in general, the metrics  $d$  and  $\delta$  equivalent? Show or disprove.

#### Exercise 2 (Open and closed sets).

a) Give an example of a set  $A \subset \mathbb{R}$  (with the euclidean metric) that is neither closed nor open. Can you also find such a set  $A \subset \mathbb{R}^2$  or even  $A \subset \mathbb{R}^n$ ?

b) Give an example of metric spaces  $(X, d)$  and  $(Y, e)$ , a function  $f : X \rightarrow Y$ , and a closed subset  $B \subset Y$  such that  $f^{-1}[B]$  is not closed. Give also an example where you switch every “closed” with “open”.

c) Find a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , a closed set  $A \subset \mathbb{R}$  and an open set  $B \subset \mathbb{R}$  such that  $f[A]$  and  $f[B]$  are neither open nor closed.

d) A space  $X$  is called *connected*, if the only sets that are both closed and open (clopen) are just  $\emptyset$  and  $X$ .

1) Give an example  $X \subset \mathbb{R}$  that is not connected. Determine all clopen subsets of  $X$ .

2) Let  $(X, d)$  and  $(Y, e)$  be metric spaces and

$$f : X \rightarrow Y$$

be continuous and onto (surjective). Show that if  $X$  is connected, then  $Y$  is as well.

### Exercise 3 (Continuity).

- a) Determine whether or not the following functions  $f_i : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$  are continuous. Can we extend them in such a way that the functions are continuous on the whole of  $\mathbb{R}^2$ ?

1)  $f_1(x, y) = \frac{x+y}{\sqrt{x^2+y^2}}$

2)  $f_2(x, y) = \frac{x^2 y^2}{x^2 + y^2}$

3)  $f_3(x, y) = \frac{x^2 + y^2}{|x| + |y|}$

- b) Let  $(X, d), (Y, e)$  be metric spaces, where  $X = A \cup B$  is the union of open or closed subsets of  $X$ . Let moreover  $f_A : A \rightarrow Y$  and  $f_B : B \rightarrow Y$  be continuous with  $f_A = f_B$  on  $A \cap B$ . Show that the function

$$f : X \rightarrow Y, \quad f(x) = \begin{cases} f_A(x) & \text{if } x \in A, \\ f_B(x) & \text{if } x \in B \end{cases}$$

is continuous.

### Exercise 4 (Differentiability, extrema, tangential planes).

- a) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be an affine mapping, that is, there is some linear mapping  $L : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $f(y) - f(x) = L(y - x)$ . Show that  $f$  is totally differentiable on  $\mathbb{R}^n$ . How the total differential looks like?
- b) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable,  $x_0 \in \mathbb{R}^n$  and  $c = f(x_0)$ . Show that the gradient  $\nabla f(x_0)$  is perpendicular to the level set

$$N_f(c) = \{x \in \mathbb{R}^n : f(x) = c\},$$

i.e., the following holds: If  $\varepsilon > 0$  and  $\phi : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$  is a differentiable curve with  $\phi(0) = x_0$  and  $\phi[(-\varepsilon, \varepsilon)] \subset N_f(c)$ , then

$$\langle \phi'(0), \nabla f(x_0) \rangle = 0.$$

(Hint: Consider the function  $g := f \circ \phi$ .)

- c) Calculate the partial derivatives up to order 2 of  $f_1$  and  $f_2$  from Exercise 3.
- d) Determine position and kind of all extrema to the function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y) = (x^2 - 1)^2 + y^4.$$

Additionally, calculate the tangent plane at the point  $x_0 = (2, 3)$ . What happens to the extrema if we consider  $g(x, y) = (x^2 - 1)^2 + y^3$  instead?

- e) Let

$$f(t) = (1 + t, t^2, 1 - t), \quad g(x, y, z) = 1 + x + xyz.$$

Calculate once with and once without the help of the chain rule  $D(g \circ f)(0)$ .

- f) Let  $n$  points  $(x_1, y_1), \dots, (x_n, y_n)$  be given. Find the equation of the line  $y = ax + b$ , for which the sum

$$f(a, b) = \sum_{i=1}^n (y_i - (ax_i + b))^2$$

becomes minimal.

**Exercise 5 (Mean value theorem, second order derivatives, chain rule).**

- a) Show that the mean value theorem fails for functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  with  $m \geq 2$ . More precisely, given  $a = 0$ ,  $h = 2\pi$ , and

$$f : \mathbb{R} \rightarrow \mathbb{R}^2, \quad f(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

show that there does not exist a  $\theta \in (0, 1)$  such that  $f(a + h) - f(a) = f'(a + \theta h)h$ .

- b) Show that for the function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{else,} \end{cases}$$

we have  $\partial_x \partial_y f(0, 0) \neq \partial_y \partial_x f(0, 0)$ . Why this is not a contradiction to Schwarz' theorem?

- c) A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is called *homogeneous of degree*  $k \in \mathbb{Z}$  if for any  $s \in \mathbb{R} \setminus \{0\}$ , we have  $f(sx_1, sx_2, \dots, sx_n) = s^k f(x_1, x_2, \dots, x_n)$ .

- 1) Give examples of functions  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  that are homogeneous of degree -1, 0, 1, 2, respectively.
- 2) Show: If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is partially differentiable and homogeneous of degree  $k$ , then

$$\sum_{i=1}^n x_i \partial_{x_i} f(x_1, \dots, x_n) = k f(x_1, \dots, x_n).$$

**Exercise 6 (Compactness, completeness).**

- a) Let  $(X, d)$  be a metric space. Show or disprove: if  $X$  is compact, then it is complete.
- b) Let  $(X, d)$  be a metric space. Show or disprove: if  $X$  is complete, then it is compact.
- c) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function,  $K \subset \mathbb{R}$  and  $V \subset \mathbb{R}$ . Find examples for the following situations:
  - 1)  $K$  is compact, but  $f[K]$  is not.
  - 2)  $K$  is compact, but  $f^{-1}[K]$  is not.
  - 3)  $V$  is complete, but  $f[V]$  is not.
  - 4)  $V$  is complete, but  $f^{-1}[V]$  is not.
- d) Show or disprove: if  $(X, d)$  and  $(Y, e)$  are metric spaces,  $f : X \rightarrow Y$  is continuous and  $X$  is complete, then  $f[X]$  is complete.
- e) Let  $(X, d)$  be a metric space and  $A \subset X$ . For  $x \in X$  we define the distance from  $x$  to  $A$  by  $\text{dist}(x, A) = \inf\{d(x, y) : y \in A\}$ .
  - 1) Show that  $\text{dist}(\cdot, A) : X \rightarrow \mathbb{R}$  is continuous.
  - 2) Let  $A \subset X$  be compact and  $x \notin A$ . Show that  $\text{dist}(x, A) > 0$ . Does this also hold if we replace “compact” by “closed”?
- f) (a bit harder task, don't waste too much time with that one) Let  $A \subset \mathbb{R}^n$ ,  $I$  be any (countable or uncountable) index set and  $U_i \subset \mathbb{R}^n$  be open for any  $i \in I$ . We call  $(U_i)_{i \in I}$  an *(open) covering* of  $A$  if  $A \subset \bigcup_{i \in I} U_i$ . Show that  $A$  is compact if and only if for any open covering  $(U_i)_{i \in I}$  there is a finite subcovering, i.e., there are finitely many  $U_{i_1}, \dots, U_{i_n}$  with  $i_1, \dots, i_n \in I$  such that  $A \subset \bigcup_{j=1}^n U_{i_j}$ . (In this sense, “compactness” is a generalization of “finiteness”).

g) Give for  $(0, 1)$  and  $[0, \infty)$  open coverings that do not possess a finite subcovering.

**Exercise 7 (A mixed problem).**

Let  $X = (0, 1)$  and

$$d(x, y) = \begin{cases} |x - y| + \frac{1}{x} + \frac{1}{y} + \frac{1}{1-x} + \frac{1}{1-y} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

- a) Show that  $d$  is a metric on  $X$ .
- b) Show that the space  $(X, |x - y|)$  is not complete, but  $(X, d)$  is complete.
- c) Are the metrics  $|x - y|$  and  $d(x, y)$  equivalent?
- d) Is  $(0, 1)$  bounded/closed/open/compact in the metric  $d$ ?

**Exercise 8 (Taylor polynomial, implicit function theorem, inverse function theorem).**

- a) Find the Taylor polynomials of degree two and three of the function

$$f(x, y) = \frac{x - y}{x + y}$$

in the point  $(1, 1)$ .

- b) Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be given by  $f(x, p) = x^2 + px - 1$ . Aim of this exercise is to train implicit function theorem (IFT) with a known example.
  - 1) Show directly (completing the square) that the equation is solvable for  $x$ , i.e., there is a function  $g_1$  such that  $f(x, p) = 0 \Leftrightarrow x = g_1(p)$ .
  - 2) Use IFT to show that there exists a function  $g_2$  such that  $f(x, p) = 0 \Leftrightarrow x = g_2(p)$  for all  $(x, p)$  in some neighborhood of  $(1, 0)$ .
  - 3) Differentiate the equation  $f(g_2(p), p) = 0$  wrt.  $p$  using chain rule to obtain an equation for  $g_2'(p)$ .
  - 4) Check whether also  $g_1$  fulfils this equation.
- c) Show that the system of equations

$$\begin{aligned} x^2 + y^2 &= 2uv, \\ x^3 + y^3 &= v^3 - u^3 \end{aligned}$$

defines in some neighborhood of the point  $(x_0, y_0) = (-1, 1)$  implicitly a function  $g(x, y) = (u(x, y), v(x, y))$  with  $g(-1, 1) = (1, 1)$ . Determine the Jacobi matrix of  $g$  in  $(-1, 1)$ .

- d) (A student's logarithmic rule) Show that there is a continuously differentiable function  $g$  defined in a neighborhood of  $x = 1/e$  such that

$$\log(g(x) - x) = \frac{\log g(x)}{\log x}.$$

Determine first a "good" value  $y_0$  such that  $g(1/e) = y_0$ . (For "generalizers": Take  $1/e^r$  with  $r \geq 1$  instead of  $1/e$ .)

e) Find for the function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad f(x, y) = (e^x \sin y, e^x \cos y)$$

an open set  $U \subset \mathbb{R}^2$  such that the restriction  $f|_U$  is injective. Is  $f$  globally invertible?

f) Let  $f(x, y, z) = (x + y + z, xy + xz + yz, xyz)$ . Show that  $f$  is continuously differentiable and determine the Jacobi matrix. Decide also whether or not the inverse function theorem is applicable in the points  $(1, 1, 0)$  and  $(1, -1, 0)$ .

g) Let

$$f(x) = \begin{cases} x + 2x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that  $f$  is differentiable and that the derivative is bounded on the interval  $(-1, 1)$ . Show moreover that in no neighborhood of  $x = 0$ , the function is injective. Why this isn't a contradiction with the inverse function theorem?

### Exercise 9 (Directional derivatives, extrema under constraints).

a) Calculate the directional derivative of  $f(x, y, z) = e^{xyz}$  in the point  $x_0 = (1, 1, 1)$  with direction  $v = (1, 2, -1)$ . Show that indeed  $D_v f(x_0) = \nabla f(x_0) \cdot v$ .

b) Let

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that  $f$  is discontinuous in  $(x, y) = (0, 0)$  (in particular, it is not totally differentiable there), but all directional derivatives exist there.

c) Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be totally differentiable in a point  $x_0 \in \mathbb{R}^2$ , and let this total differential be nonzero. Let  $v, w \in \mathbb{R}^2$  such that  $D_v f(x_0) = D_w f(x_0) = 0$ . Show that  $v$  and  $w$  are linearly dependent. Does this also hold when replacing  $\mathbb{R}^2$  with  $\mathbb{R}^3$  (or even  $\mathbb{R}^n$ )?

d) Define

$$M = \{(x, y) \in \mathbb{R}^2 : x = y \text{ and } x \neq 0\},$$

and let

$$f(x, y) = \begin{cases} e^x - 1 & \text{if } (x, y) \in M, \\ 0 & \text{else.} \end{cases}$$

1) Show that  $f$  is partially differentiable in  $(x, y) \in \mathbb{R}^2$  if and only if  $(x, y) \notin M$ .

2) The directional derivative  $D_v f(0)$  exists for any  $v \in \mathbb{R}^2$ .

3) There is some  $v \in \mathbb{R}^2$  with  $|v| = 1$  such that  $D_v f(0) \neq \nabla f(0) \cdot v$ .

e) Let  $f(x, y) = 4x^2 - 3xy$ . Find all extrema of  $f$  in the disc  $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ . Argue why such extrema do exist. (How to: first find the extrema inside  $D$  in the usual way, then find extrema on the boundary using constraints.)

### Exercise 10 (Uniform continuity, Riemann integral 1D).

- a) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be uniformly continuous. Show that there is some  $M \geq 0$  such that for any  $x \in \mathbb{R}$ , we have  $|f(x)| \leq M(1 + |x|)$ .
- b) Construct an example of a function  $f : [0, 1) \rightarrow \mathbb{R}$  such that  $f$  is continuous but not uniformly continuous.
- c) Prove: Let  $(a, b)$  be a bounded open interval. A continuous function  $f : (a, b) \rightarrow \mathbb{R}$  is uniformly continuous if and only if we can extend it to a function that is continuous on the closed interval  $[a, b]$ .
- d) Show that for any Riemann integrable function  $f : [a, b] \rightarrow \mathbb{R}$ , it holds that

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

- e) Let  $x_0 \in (0, 1)$  and define

$$\chi_{x_0}(x) = \begin{cases} 1 & \text{if } x = x_0, \\ 0 & \text{else.} \end{cases}$$

Show via definition that  $\int_0^1 \chi_{x_0}(x) dx = 0$ .

### Exercise 11 (Fubini's theorem in different variants, change of variables).

- a) Calculate the integral  $\int_G f(x, y) d(x, y)$ , where:
- 1)  $f(x, y) = 2xy + \cos(x + y)$ ,  $G = [-\pi, 0] \times [0, 2]$
  - 2)  $f(x, y) = x^2$ ,  $G = \{x \in (-1, 2), 0 < y < x^2\}$
  - 3)  $f(x, y) = x + y + xy$ , where  $G$  is the area enclosed by the functions  $g(x) = x^2 + 1$  and  $h(x) = x + 3$
- b) Let  $b > 0$  and  $f : [0, b] \rightarrow \mathbb{R}$  be integrable. Show that the function  $g(x) := \int_x^b t^{-1} f(t) dt$  is continuous on  $(0, b]$  and integrable on  $[0, b]$ , and that  $\int_0^b g(x) dx = \int_0^b f(t) dt$ .
- c) A material with density  $\varrho(x, y, z) = x + y + z$  is formed into the unit cube  $C = [0, 1]^3$ . Calculate the mass  $m = \int_C \varrho dV$  of this cube. Here the volume element is given by  $dV = dx dy dz$ .
- d) Another material with density  $\varrho(x, y, z) = x^2 + y^2$  is now formed into a cylinder  $Z = \{x, y, z \in \mathbb{R} : x^2 + y^2 \leq 1, 0 \leq z \leq 2\}$ . Calculate the mass of this cylinder. (It might be worth to use a change of coordinates. The volume element for cylindrical coordinates is given by  $dV = r dr d\theta dz$ .)
- e) Let  $\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq R^2, x^2 + y^2 \leq z^2, z \geq 0\}$ . Sketch the set  $\Omega$  and calculate its volume (a usage of spherical coordinates  $(x, y, z) = (r \sin(\theta) \cos(\phi), r \sin(\theta) \sin(\phi), r \cos(\theta))$  might be useful).
- f) Make a reasoned decision for which  $\alpha \in \mathbb{R}$  the 3D integrals

$$\int_{B_1(0)} |x|^{-\alpha} dx \quad \text{and} \quad \int_{\mathbb{R}^3 \setminus \overline{B_1(0)}} |x|^{-\alpha} dx$$

converge.

- g) A cylinder  $x^2 + y^2 \leq R^2$  cuts from the ball  $x^2 + y^2 + z^2 \leq 4R^2$  a body  $B$ . Calculate the volume of  $B$ .

h) Let  $U \subset \mathbb{R}^3$  be a compact set filled with mass of (R-integrable) density  $\varrho$ , and let  $M = \int_U \varrho dV$  be its mass. The *center of mass (COM)* is given by the coordinates

$$\bar{x} = \frac{1}{M} \int_U x \varrho(x, y, z) d(x, y, z), \quad \bar{y} = \frac{1}{M} \int_U y \varrho(x, y, z) d(x, y, z), \quad \bar{z} = \frac{1}{M} \int_U z \varrho(x, y, z) d(x, y, z),$$

resp. in vector form with  $\underline{x} = (x, y, z)$  simply

$$\bar{\underline{x}} = \frac{1}{M} \int_U \underline{x} \varrho(\underline{x}) d\underline{x}.$$

Calculate the COM of

- 1) the octant  $U = \{x^2 + y^2 + z^2 \leq R^2, x \geq 0, y \geq 0, z \geq 0\}$  with constant density  $\varrho = 1$ ;
- 2) the half ball  $U = \{x^2 + y^2 + z^2 \leq R^2, z \geq 0\}$  with constant density  $\varrho = 1$ ;
- 3) the same octant and half ball, but with density  $\varrho(x, y, z) = az$  for some constant  $a > 0$ .