

Mathematical analysis II

Homework 9

To be handed in by Wednesday, 10.12.25, 23:59 h via OWL

Exercise 1 (Integration by parts).
(2+3=5 points)

- a) Show that for two continuously differentiable functions $f, g : [a, b] \rightarrow \mathbb{R}$, it holds

$$\int_a^b f(x)g'(x) dx = f(x)g(x) \Big|_a^b - \int_a^b f'(x)g(x) dx.$$

(Hint: consider the function $F(x) = f(x)g(x)$.)

- b) Let $f : [a, b] \rightarrow \mathbb{R}$ be continuously differentiable and set for any $k \in \mathbb{N}$

$$a_k := \int_a^b f(x) \sin(kx) dx.$$

Show that $\lim_{k \rightarrow \infty} a_k = 0$.

Exercise 2 (Riemann integral in nD).
(3+2=5 points)

The words “interval” and “brick” are used synonymously here.

- a) Show *via definition* that for any compact interval (brick) $J \subset \mathbb{R}^n$ and any constant $c \in \mathbb{R}$, the Riemann integral $\int_J c dx$ exists and that it holds $\int_J c dx = c \cdot \text{vol}(J)$.
- b) Let $J = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ be a compact interval. A function $\phi : J \rightarrow \mathbb{R}$ is called a *step function* if there are constants $c_1, \dots, c_k \in \mathbb{R}$ and pairwise disjoint intervals (bricks) $I_1, \dots, I_k \subset J$ such that¹

$$\phi(x) = \sum_{i=1}^k c_i \chi_{I_i}(x), \quad \text{where} \quad \chi_{I_i}(x) = \begin{cases} 1 & \text{if } x \in I_i, \\ 0 & \text{else.} \end{cases}$$

Show that for any step function, it holds

$$\int_J \phi(x) dx = \sum_{i=1}^k c_i \cdot \text{vol}(I_i).$$

(You can use without proof that for intervals $I \in \{(a, b), [a, b), (a, b], [a, b]\}$ it holds $\text{vol}(I) = b - a$, and that the function χ_{I_i} is integrable on the brick I_i . A sketch might be helpful, say, for $k = 2$, $J = [0, 1]$, $I_1 = [0, \frac{1}{2}]$, $I_2 = (\frac{1}{2}, 1]$, and $c_1 = 1$, $c_2 = 2$.)

¹Although our definition of the integral uses it, the intervals I_i need not to be closed here.