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Mathematical analysis II

Homework 1

To be handed in by Wednesday, 15.10.25, 23:59 h via OWL

Exercise 1 (Metrics). *(3+1=4)*

a) Let $X = [1, \infty)$. Show that the function defined via

$$d(x,y) = \left| \frac{1}{x} - \frac{1}{y} \right|$$

is a metric.

b) Let $X = \mathbb{R}$. Is the function defined by

$$d(x,y) = |x|^{|y|}$$

a metric? Show or disprove!

Solution. a) Symmetry and $d(x,y) \geq 0$ is obvious since the absolute value $|\cdot|$ fulfils this. If $d(x,y) = 0$, then

$$\left| \frac{1}{x} - \frac{1}{y} \right| = 0 \Leftrightarrow \frac{1}{x} - \frac{1}{y} = 0 \Leftrightarrow \frac{1}{x} = \frac{1}{y} \Leftrightarrow x = y,$$

where the first equivalence uses that $|a| = 0$ if and only if $a = 0$. Finally, the triangle inequality follows by

$$d(x,y) = \left| \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{1}{x} - \frac{1}{z} + \frac{1}{z} - \frac{1}{y} \right| \leq \left| \frac{1}{x} - \frac{1}{z} \right| + \left| \frac{1}{z} - \frac{1}{y} \right| = d(x,z) + d(z,y),$$

and the inequality comes from the triangle inequality fulfilled by $|\cdot|$. *(Note that inserting $\pm 1/z$ corresponds to adding a zero, something that every mathematician should be able to do.)* Note also that even if X for us is an infinite axis, the so-called *diameter* of this space X under d is

$$\operatorname{diam}(X) := \sup\{d(x,y) : x, y \in X\} = 1$$

such that X is *bounded* and equal to B_1^d, its ball of radius 1.

Remark: Many of you did this wrongly, so let me emphasize that definiteness means $d(x,y) = 0 \Rightarrow x = y$. That $d(x,x) = 0$ is usually trivial to see.

b) It is not a metric since it violates (almost) everything: first, it is not symmetric by setting, e.g., $x = 1$ and $y = 2$. Second, it is not definite since for any $y \neq 0$, we have $d(0,y) = |0|^{|y|} = 0$. Third, triangle inequality fails, set $x = 3,\ y = 2,\ z = 1$.

Exercise 2 (Open and closed sets). *(3+3=6)*

In here, we assume $X = \mathbb{R}$.

a) Let $d : X \times X \to [0, \infty)$ be a metric, and define

$$\delta(x, y) = \min\{d(x, y),\ 1\}.$$

Show that δ is really a metric, and that d and δ define the same open and closed sets.

b) Let

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Show that under this metric, every set is both open and closed.

Solution. a) Obviously $\delta \geq 0$ since both $d(x, y) \geq 0$ and $1 \geq 0$. Moreover

$$\delta(x, y) = 0 \Leftrightarrow \min\{d(x, y), 1\} = 0 \Leftrightarrow d(x, y) = 0 \Leftrightarrow x = y$$

since d is a metric. Symmetry is trivial as well since $d(x, y)$ is symmetric as it is a metric. Triangle inequality follows from

$$\delta(x, y) = \min\{d(x, y), 1\} \leq \min\{d(x, z) + d(z, y), 1\} \leq \min\{d(x, z), 1\} + \min\{d(z, y), 1\},$$

where the first inequality uses that d is a metric and that $\min\{a, c\} \leq \min\{b, c\}$ for any $a \leq b$ and any c, and the last equality uses subadditivity of min for non-negative arguments (for understanding: find an example where $\min\{a + b, 1\} > \min\{a, 1\} + \min\{b, 1\}$).

Let's come to open sets. I will denote

$$B^d_\varepsilon(a) = \{x \in X : d(x, a) < \varepsilon\}, \qquad B^\delta_\varepsilon(a) = \{x \in X : \delta(x, a) < \varepsilon\}.$$

We have to show 2 directions:

1) Let $A \subset X$ be open wrt. δ, that is,

$$\forall a \in A \exists \varepsilon > 0 : B^\delta_\varepsilon(a) \subset A.$$

Now, we need to find for any $a \in A$ some $\varepsilon_1 > 0$ such that $B^d_{\varepsilon_1}(a) \subset A$. This is achieved by $\varepsilon_1 = \varepsilon$, since by definition, always $\delta(x, y) \leq d(x, y)$ for any $x, y \in X$, hence $B^d_\varepsilon(a) \subset B^\delta_\varepsilon(a)$ for any a (notice the reversed subset relation). In other words, if $B^\delta_\varepsilon(a) \subset A$, then also $B^d_\varepsilon(a) \subset B^\delta_\varepsilon(a) \subset A$, and hence A is open wrt. d.

2) Let $A \subset X$ be open wrt. d, that is,

$$\forall a \in A \exists \varepsilon > 0 : B^d_\varepsilon(a) \subset A.$$

Again, we need to find for any $a \in A$ some $\varepsilon_1 > 0$ such that $B^\delta_{\varepsilon_1}(a) \subset A$. This is achieved by $\varepsilon_1 = \min\{\varepsilon, 1\}$. Indeed, if $x \in B^\delta_{\varepsilon_1}(a)$, then $\delta(x, a) < \varepsilon_1 = \min\{\varepsilon, 1\} \leq 1$, which means that $\delta(x, a) = d(x, a)$ (otherwise we would have $\delta(x, a) = 1$, which contradicts the fact that $\delta(x, a) < 1$). But then $d(x, a) = \delta(x, a) < \min\{\varepsilon, 1\} \leq \varepsilon$, i.e., $d(x, a) < \varepsilon$, giving $x \in B^d_\varepsilon(a)$ and so $x \in A$ as A is open wrt. d. This shows that for this ε_1, we have $B^\delta_{\varepsilon_1}(a) \subset A$ and thus A is open wrt. δ.

b) Let $A \subset X$. First we show that A is open, that means for any $a \in A$, we need to find $\varepsilon > 0$ such that $B_\varepsilon(a) \subset A$. We choose (amazingly independently of the set A!) $\varepsilon = \frac{1}{2}$, then, by definition of d, we have $B_\varepsilon(a) = \{a\}$, a single point. Since $a \in A$, this means $B_\varepsilon(a) = \{a\} \subset A$ such that A is open.
Next, A is also closed: by definition (or one of the equivalent definitions), A is closed if and only if $X \setminus A$ is open. Since obviously $X \setminus A \subset X$ is a subset, and we just showed that any subset of X is open, also $X \setminus A$ is open. According to the definition, this means that A is closed. (Here, we could also argue with sequences, since under d any convergent sequence will be constant from some point on. Try to formalize this!)