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Mathematical analysis II

Homework 4

To be handed in by Wednesday, 05.11.25, 23:59 h via OWL

Exercise 1 (Compactness). *(2+2=4 points)*

a) Let X be a non-empty set and

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{else.} \end{cases}$$

Show that a set $A \subset X$ is compact if and only if it is finite. (Hint: you might prove one direction directly, the other one by contradiction.)

b) Show that the union of finitely many compact sets is again compact. Show via an example that for general (infinite) unions, this fails to be true.

Exercise 2 (Completeness). *(1+2+3=6 points)*

a) Give an example of a metric space (X,d) different from part c) that is not complete.

b) Prove or disprove: There is a non-empty set M such that for any metric d, the metric space (M,d) is not complete.

c) We already know from the lecture that the metric space $(\mathbb{R}, |x-y|)$ is a complete metric space. The aim of this task is to emphasize that the property of being complete depends on the chosen metric:
Let

$$d(x,y) = |\arctan(x) - \arctan(y)|.$$

You can use without proof that this function defines a metric on $\mathbb{R}$. Show that the space $(\mathbb{R}, d)$ is not complete. Here $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is the inverse function of tan. (Hint: investigate the sequence $x_n = n$ for each $n \in \mathbb{N}$.)