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Mathematical analysis II

Homework 5

To be handed in by Wednesday, 19.11.25, 23:59 h via OWL

Exercise 1 (Implicitness and more). *(3+1*+3+1=7+1* points)*

Justify all your answers in this task well (i.e., verify assumptions of Theorems you use, conclusions from them, etc.)!

a) Show that close to $x = 0$, there exist a $\delta > 0$ and a continuously differentiable function $g : (-\delta, \delta) \to \mathbb{R}$ such that

$$g(x) = [g(x)]^3 + 2e^{g(x)} \sin(x).$$

**Bonus:* How many such functions exist (for $\delta > 0$ but suitably small)?

b) For an interval $I \subseteq (-\delta, \delta)$, the graph of a function $f : I \to \mathbb{R}$ is defined as

$$\mathrm{gr}(f) = \{(x, f(x)) : x \in I\}.$$

Show that for any compact interval $I \subset (-\delta, \delta)$ and any continuous function $f : I \to \mathbb{R}$, the graph $\mathrm{gr}(f)$ is compact in $\mathbb{R}^2$. What can you say about the graphs $\mathrm{gr}(g)$ and $\mathrm{gr}(g')$ on such intervals I? (Hint: consider the function $\Phi : I \to \mathbb{R}^2$, $\Phi(x) = (x, f(x))$. Which properties Φ has?)

c) What can you say about the monotonicity of g in $x = 0$?

Exercise 2 (Taylor in higher dimensions). *(2+1=3 points)*

Sometimes, Taylor polynomials in higher dimensions can be easier obtained than just using the formula

$$T^n_{f;x_0}(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0) \cdot [H_f(x_0)(x - x_0)] + ...$$

a) Calculate the Taylor polynomials of $\log(1 + x)$ and $\cos(x)$ around $x = 0$ up to second order.

b) Conclude that for the function

$$f(x, y, z) = (x + 1) \log(y + 1) \cos(z),$$

the Taylor polynomial of second order around $x_0 = (0, 0, 0)$ is given by

$$T^2_{f;x_0}(x) = y + xy - \frac{1}{2}y^2.$$