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Mathematical analysis II

Collection of problems

Not for handing in

Exercise 1 (Metrics).

a) Let

$$d(x,y) = \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right|.$$

Show that $(\mathbb{R}, d)$ is a metric space. Are the metrics d and $d_1(x,y) := |x-y|$ equivalent? Show or disprove.

b) Let (X,d) be a metric space. Find all $k \in \mathbb{R}$ such that

$$d_1(x,y) := (k-1)(k-3)d(x,y)$$

is a metric on X.

c) Let (X,d) be a metric space and define like in HW 1

$$\delta(x,y) = \min\{d(x,y), 1\}.$$

Are, in general, the metrics d and δ equivalent? Show or disprove.

Exercise 2 (Open and closed sets).

a) Give an example of a set $A \subset \mathbb{R}$ (with the euclidean metric) that is neither closed nor open. Can you also find such a set $A \subset \mathbb{R}^2$ or even $A \subset \mathbb{R}^n$?

b) Give an example of metric spaces (X,d) and (Y,e), a function $f : X \to Y$, and a closed subset $B \subset Y$ such that $f^{-1}[B]$ is not closed. Give also an example where you switch every "closed" with "open".

c) Find a function $f : \mathbb{R} \to \mathbb{R}$, a closed set $A \subset \mathbb{R}$ and an open set $B \subset \mathbb{R}$ such that $f[A]$ and $f[B]$ are neither open nor closed.

d) A space X is called *connected*, if the only sets that are both closed an open (clopen) are just $\emptyset$ and X.

   1) Give an example $X \subset \mathbb{R}$ that is not connected. Determine all clopen subsets of X.
   2) Let (X,d) and (Y,e) be metric spaces and

   $$f : X \to Y$$

   be continuous and onto (surjective). Show that if X is connected, then Y is as well.

Exercise 3 (Continuity).

a) Determine whether or not the following functions $f_i : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ are continuous. Can we extend them in such a way that the functions are continuous on the whole of $\mathbb{R}^2$?

   1) $f_1(x,y) = \frac{x+y}{\sqrt{x^2+y^2}}$
   2) $f_2(x,y) = \frac{x^2y^2}{x^2+y^2}$
   3) $f_3(x,y) = \frac{x^2+y^2}{|x|+|y|}$

b) Let (X,d), (Y,e) be metric spaces, where $X = A \cup B$ is the union of open or closed subsets of X. Let moreover $f_A : A \to Y$ and $f_B : B \to Y$ be continuous with $f_A = f_B$ on $A \cap B$. Show that the function

$$f : X \to Y, \qquad f(x) = \begin{cases} f_A(x) & \text{if } x \in A, \\ f_B(x) & \text{if } x \in B \end{cases}$$

is continuous.

Exercise 4 (Differentiability, extrema, tangential planes).

a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be an affine mapping, that is, there is some linear mapping $L : \mathbb{R}^n \to \mathbb{R}$ such that $f(y) - f(x) = L(y-x)$. Show that f is totally differentiable on $\mathbb{R}^n$. How the total differential looks like?

b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, $x_0 \in \mathbb{R}^n$ and $c = f(x_0)$. Show that the gradient $\nabla f(x_0)$ is perpendicular to the level set

$$N_f(c) = \{x \in \mathbb{R}^n : f(x) = c\},$$

i.e., the following holds: If $\varepsilon > 0$ and $\phi : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is a differentiable curve with $\phi(0) = x_0$ and $\phi[(-\varepsilon,\varepsilon)] \subset N_f(c)$, then

$$\langle \phi'(0), \nabla f(x_0) \rangle = 0.$$

(Hint: Consider the function $g := f \circ \phi$.)

c) Calculate the partial derivatives up to order 2 of f_1 and f_2 from Exercise 3.

d) Determine position and kind of all extrema to the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) = (x^2-1)^2 + y^4.$$

Additionally, calculate the tangent plane at the point $x_0 = (2,3)$. What happens to the extrema if we consider $g(x,y) = (x^2-1)^2 + y^3$ instead?

e) Let

$$f(t) = (1+t, t^2, 1-t), \qquad g(x,y,z) = 1 + x + xyz.$$

Calculate once with and once without the help of the chain rule $D(g \circ f)(0)$.

Exercise 5 (Mean value theorem, second order derivatives).

a) Show that the mean value theorem fails for functions $f : \mathbb{R}^n \to \mathbb{R}^m$ with $m \geq 2$. More precisely, given $a = 0$, $h = 2\pi$, and

$$f : \mathbb{R} \to \mathbb{R}^2, \qquad f(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

show that there does not exist a $\theta \in (0, 1)$ such that $f(a + h) - f(a) = f'(a + \theta h)h$.

b) Show that for the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{else,} \end{cases}$$

we have $\partial_x \partial_y f(0, 0) \neq \partial_y \partial_x f(0, 0)$. Why this is not a contradiction to Schwarz' theorem?

Exercise 6 (Compactness, completeness).

a) Let (X, d) be a metric space. Show or disprove: if X is compact, then it is complete.

b) Let (X, d) be a metric space. Show or disprove: if X is complete, then it is compact.

c) Let $f : \mathbb{R} \to \mathbb{R}$ be a function, $K \subset \mathbb{R}$ and $V \subset \mathbb{R}$. Find examples for the following situations:

   1) K is compact, but $f[K]$ is not.
   2) K is compact, but $f^{-1}[K]$ is not.
   3) V is complete, but $f[V]$ is not.
   4) V is complete, but $f^{-1}[V]$ is not.

d) Show or disprove: if (X, d) and (Y, e) are metric spaces, $f : X \to Y$ is continuous and X is complete, then $f[X]$ is complete.

e) Let (X, d) be a metric space and $A \subset X$. For $x \in X$ we define the distance from x to A by $\operatorname{dist}(x, A) = \inf\{d(x, y) : y \in A\}$.

   1) Show that $\operatorname{dist}(\cdot, A) : X \to \mathbb{R}$ is continuous.
   2) Let $A \subset X$ be compact and $x \notin A$. Show that $\operatorname{dist}(x, A) > 0$. Does this also hold if we replace "compact" by "closed"?

f) (a bit harder task, don't waste too much time with that one) Let $A \subset \mathbb{R}^n$, I be any (countable or uncountable) index set and $U_i \subset \mathbb{R}^n$ be open for any $i \in I$. We call $(U_i)_{i \in I}$ an *(open) covering* of A if $A \subset \bigcup_{i \in I} U_i$. Show that A is compact if and only if for any open covering $(U_i)_{i \in I}$ there is a finite subcovering, i.e., there are finitely many $U_{i_1}, \ldots, U_{i_n}$ with $i_1, \ldots, i_n \in I$ such that $A \subset \bigcup_{j=1}^{n} U_{i_j}$. (In this sense, "compactness" is a generalization of "finiteness").

g) Give for $(0, 1)$ and $[0, \infty)$ open coverings that do not posses a finite subcovering.

Exercise 7 (A mixed problem).

Let $X = (0, 1)$ and

$$d(x, y) = \begin{cases} |x - y| + \frac{1}{x} + \frac{1}{y} + \frac{1}{1-x} + \frac{1}{1-y} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

a) Show that d is a metric on X.

b) Show that the space $(X, |x - y|)$ is not complete, but (X, d) is complete.

c) Are the metrics $|x - y|$ a d$d(x, y)$ equivalent?

d) Is $(0, 1)$ bounded/closed/open/compact in the metric d?