

## Mathematical analysis II

### Collection of problems

Not for handing in

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#### Exercise 1 (Metrics).

a) Let

$$d(x, y) = \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right|.$$

Show that  $(\mathbb{R}, d)$  is a metric space. Are the metrics  $d$  and  $d_1(x, y) := |x - y|$  equivalent? Show or disprove.

b) Let  $(X, d)$  be a metric space. Find all  $k \in \mathbb{R}$  such that

$$d_1(x, y) := (k - 1)(k - 3)d(x, y)$$

is a metric on  $X$ .

c) Let  $(X, d)$  be a metric space and define like in HW 1

$$\delta(x, y) = \min\{d(x, y), 1\}.$$

Are the metrics  $d$  and  $\delta$  equivalent? Show or disprove.

#### Exercise 2 (Open and closed sets).

A space  $X$  is called *connected*, if the only sets that are both closed and open (clopen) are just  $\emptyset$  and  $X$ .

a) Give an example  $X \subset \mathbb{R}$  that is not connected. Determine all clopen subsets of  $X$ .

b) Let  $(X, d)$  and  $(Y, e)$  be metric spaces and

$$f : X \rightarrow Y$$

be continuous and onto (surjective). Show that if  $X$  is connected, then  $Y$  is as well.

#### Exercise 3 (Continuity).

a) Determine whether or not the following functions  $f_i : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$  are continuous. Can we extend them in such a way that the functions are continuous on the whole of  $\mathbb{R}^2$ ?

1)  $f_1(x, y) = \frac{x+y}{\sqrt{x^2+y^2}}$

2)  $f_2(x, y) = \frac{x^2 y^2}{x^2 + y^2}$

$$3) f_3(x, y) = \frac{x^2 + y^2}{|x| + |y|}$$

- b) Let  $(X, d), (Y, e)$  be metric spaces, where  $X = A \cup B$  is the union of open or closed subsets of  $X$ . Let moreover  $f_A : A \rightarrow Y$  and  $f_B : B \rightarrow Y$  be continuous with  $f_A = f_B$  on  $A \cap B$ . Show that the function

$$f : X \rightarrow Y, \quad f(x) = \begin{cases} f_A(x) & \text{if } x \in A, \\ f_B(x) & \text{if } x \in B \end{cases}$$

is continuous.

#### Exercise 4 (Differentiability, extrema, tangential planes).

- a) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be an affine mapping, that is, there is some linear mapping  $L : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $f(y) - f(x) = L(y - x)$ . Show that  $f$  is totally differentiable on  $\mathbb{R}^n$ . How the total differential looks like?
- b) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable,  $x_0 \in \mathbb{R}^n$  and  $c = f(x_0)$ . Show that the gradient  $\nabla f(x_0)$  is perpendicular to the level set

$$N_f(c) = \{x \in \mathbb{R}^n : f(x) = c\},$$

i.e., the following holds: If  $\varepsilon > 0$  and  $\phi : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$  is a differentiable curve with  $\phi(0) = x_0$  and  $\phi[(-\varepsilon, \varepsilon)] \subset N_f(c)$ , then

$$\langle \phi'(0), \nabla f(x_0) \rangle = 0.$$

(Hint: Consider the function  $g := f \circ \phi$ .)

- c) Calculate the partial derivatives up to order 2 of  $f_1$  and  $f_2$  from Exercise 3.
- d) Determine position and kind of all extrema to the function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y) = (x^2 - 1)^2 + y^4.$$

Additionally, calculate the tangent plane at the point  $x_0 = (2, 3)$ .

#### Exercise 5 (Mean value theorem, second order derivatives).

- a) Show that the mean value theorem fails for functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  with  $m \geq 2$ . More precisely, given  $a = 0$ ,  $h = 2\pi$ , and

$$f : \mathbb{R} \rightarrow \mathbb{R}^2, \quad f(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

show that there does not exist a  $\theta \in (0, 1)$  such that  $f(a + h) - f(a) = f'(a + \theta h)h$ .

- b) Show that for the function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{else,} \end{cases}$$

we have  $\partial_x \partial_y f(0, 0) \neq \partial_y \partial_x f(0, 0)$ . Why this is not a contradiction to Schwarz' theorem?