

Mathematical analysis II

Collection of problems

Not for handing in

Exercise 1 (Metrics).

a) Let

$$d(x, y) = \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right|.$$

Show that (\mathbb{R}, d) is a metric space. Are the metrics d and $d_1(x, y) := |x - y|$ equivalent? Show or disprove.

b) Let (X, d) be a metric space. Find all $k \in \mathbb{R}$ such that

$$d_1(x, y) := (k - 1)(k - 3)d(x, y)$$

is a metric on X .

c) Let (X, d) be a metric space and define like in HW 1

$$\delta(x, y) = \min\{d(x, y), 1\}.$$

Are the metrics d and δ equivalent? Show or disprove.

Exercise 2 (Open and closed sets).

A space X is called *connected*, if the only sets that are both closed and open (clopen) are just \emptyset and X .

a) Give an example $X \subset \mathbb{R}$ that is not connected. Determine all clopen subsets of X .

b) Let (X, d) and (Y, e) be metric spaces and

$$f : X \rightarrow Y$$

be continuous and onto (surjective). Show that if X is connected, then Y is as well.

Exercise 3 (Continuity).

a) Determine whether or not the following functions $f_i : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ are continuous. Can we extend them in such a way that the functions are continuous on the whole of \mathbb{R}^2 ?

1) $f_1(x, y) = \frac{x+y}{\sqrt{x^2+y^2}}$

2) $f_2(x, y) = \frac{x^2 y^2}{x^2 + y^2}$

3) $f_3(x, y) = \frac{x^2 + y^2}{|x| + |y|}$

- b) Let $(X, d), (Y, e)$ be metric spaces, where $X = A \cup B$ is the union of open or closed subsets of X . Let moreover $f_A : A \rightarrow Y$ and $f_B : B \rightarrow Y$ be continuous with $f_A = f_B$ on $A \cap B$. Show that the function

$$f : X \rightarrow Y, \quad f(x) = \begin{cases} f_A(x) & \text{if } x \in A, \\ f_B(x) & \text{if } x \in B \end{cases}$$

is continuous.

Exercise 4 (Differentiability, extrema, tangential planes).

- a) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be an affine mapping, that is, there is some linear mapping $L : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f(y) - f(x) = L(y - x)$. Show that f is totally differentiable on \mathbb{R}^n . How the total differential looks like?
- b) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuously differentiable, $x_0 \in \mathbb{R}^n$ and $c = f(x_0)$. Show that the gradient $\nabla f(x_0)$ is perpendicular to the level set

$$N_f(c) = \{x \in \mathbb{R}^n : f(x) = c\},$$

i.e., the following holds: If $\varepsilon > 0$ and $\phi : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$ is a differentiable curve with $\phi(0) = x_0$ and $\phi[(-\varepsilon, \varepsilon)] \subset N_f(c)$, then

$$\langle \phi'(0), \nabla f(x_0) \rangle = 0.$$

(Hint: Consider the function $g := f \circ \phi$.)

- c) Calculate the partial derivatives up to order 2 of f_1 and f_2 from Exercise 3.
- d) Determine position and kind of all extrema to the function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y) = (x^2 - 1)^2 + y^4.$$

Additionally, calculate the tangent plane at the point $x_0 = (2, 3)$.

- e) Let

$$f(t) = (1 + t, t^2, 1 - t), \quad g(x, y, z) = 1 + x + xyz.$$

Calculate once with and once without the help of the chain rule $D(g \circ f)(0)$.

Exercise 5 (Mean value theorem, second order derivatives).

- a) Show that the mean value theorem fails for functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $m \geq 2$. More precisely, given $a = 0$, $h = 2\pi$, and

$$f : \mathbb{R} \rightarrow \mathbb{R}^2, \quad f(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

show that there does not exist a $\theta \in (0, 1)$ such that $f(a + h) - f(a) = f'(a + \theta h)h$.

- b) Show that for the function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{else,} \end{cases}$$

we have $\partial_x \partial_y f(0, 0) \neq \partial_y \partial_x f(0, 0)$. Why this is not a contradiction to Schwarz' theorem?