Mathematical analysis II

Homework 8

To be handed in by Wednesday, 03.12.25, 23:59 h via OWL

Exercise 1 (Uniform continuity). *(1+3=4 points)*

Let (X, d) and (Y, e) be metric spaces. A function $f : X \to Y$ is said to be *Lipschitz continuous* (resp. just *Lipschitz*) if there exists some $L \geq 0$ such that

$$e(f(x), f(y)) \leq Ld(x, y) \quad \forall x, y \in X.$$

a) Show that any Lipschitz continuous function is uniformly continuous.

b) Show that $f : [0, \infty) \to [0, \infty)$, $f(x) = \sqrt{x}$ is uniformly continuous, but not Lipschitz. Here, we use $d(x, y) = e(x, y) = |x - y|$. (*Hint:* Split the nonnegative real line into $[0, 2]$ and $[1, \infty)$ and show uniform continuity on each of these intervals. Then argue why this implies uniform continuity everywhere on $[0, \infty)$.)

Solution. a) Let $\varepsilon > 0$. We need to find $\delta = \delta(\varepsilon)$ such that for any $x, y \in X$ with $d(x, y) < \delta$, we have $e(f(x), f(y)) < \varepsilon$. Let now $d(x, y) < \delta$, then

$$e(f(x), f(y)) \leq Ld(x, y) < L\delta;$$

hence, we can simply choose $\delta = \varepsilon/L$. If $L = 0$, the f is constant and hence we can choose any $\delta > 0$.

b) On $[0, 2]$, the function is uniformly continuous since it is continuous and $[0, 2]$ is compact. On $[1, \infty)$ we have

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{|x - y|}{2},$$

thus our function is even Lipschitz on $[1, \infty)$ with $L = \frac{1}{2}$ and thus by part a) also uniformly continuous. Here, we just used that $a^2 - b^2 = (a - b)(a + b)$ and that $x, y \geq 1$. It follows that f is uniformly continuous everywhere since the intervals $[0, 2]$ and $[1, \infty)$ overlap. More precisely, if $\varepsilon > 0$ and $\delta_1 = \delta_1(\varepsilon)$ is the δ from the definition for the interval $[0, 2]$, and $\delta_2 = \delta_2(\varepsilon)$ is the one for $[1, \infty)$, the we can choose $\delta = \min\{\delta_1, \delta_2\}$ that works for the whole semi-axis $[0, \infty)$. The function f is however not Lipschitz on $[0, \infty)$: assume it is and choose $y = 0$, then there is some $L \geq 0$ such that for any $x > 0$, we have

$$\sqrt{x} \leq Lx \Leftrightarrow L \geq 1/\sqrt{x};$$

but this is for arbitrarily small $x > 0$ impossible since the right hand side goes to ∞ if $x \to 0$. Thus, f is not Lipschitz.

A proof of uniform continuity that works without interval splitting is as follows: For each $x, y \geq 0$, it holds

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}.$$

The proof of this fact is simply squaring both sides (and assuming by symmetry without loss of generality that $x \geq y$). Then, for $|x-y| < \delta$,

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|} < \sqrt{\delta},$$

so it is enough to choose $\delta = \varepsilon^2$.

Exercise 2 (Riemann integral). *(2+2+2=6 points)*

Let $f : [a,b] \to \mathbb{R}$ be integrable (that is, $\int_a^b f(x)\,\mathrm{d}x$ exists and is finite), and set

$$F(x) := \int_a^x f(t)\,\mathrm{d}t.$$

Show:

a) The function f is bounded[1].

b) $F : [a,b] \to \mathbb{R}$ is Lipschitz continuous, i.e., there is some $L \geq 0$ such that

$$|F(x) - F(y)| \leq L|x-y| \quad \forall x, y \in [a,b].$$

c) If $f \geq 0$, then F is monotonically increasing.

Solution. a) Without loss of generality assume that $f \geq 0$, otherwise we consider separately $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$ such that $f = f_+ - f_-$, and use linearity of the Riemann integral. Assume for contradiction that f is unbounded. Then, there is a point $x_0 \in [a,b]$ such that for any $\varepsilon > 0$

$$\sup_{x \in (x_0 - \varepsilon, x_0 + \varepsilon) \cap [a,b]} f(x) = \infty.$$

Let $P = \{t_i : t_0 = a,\ t_n = b,\ t_i < t_{i+1}\}$ be any partition of the interval $[a,b]$. Then we have for the upper Riemann sum that

$$S(f,P) = \sum_{i=0}^{n-1} \Big[\sup_{x \in [t_i, t_{i+1}]} f(x) \Big] \cdot (t_{i+1} - t_i) \geq \Big[\sup_{x \in [t_j, t_{j+1}]} f(x) \Big] \cdot (t_{j+1} - t_j) = \infty,$$

where $[t_j, t_{j+1}]$ is the unique interval such that $x_0 \in [t_j, t_{j+1}]$. Note that the inequality holds since $f \geq 0$, and we may make ε even smaller such that $(x_0 - \varepsilon, x_0 + \varepsilon) \subset [t_j, t_{j+1}]$. Since the partition P was arbitrary, this yields

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \inf\{S(f,P) : P \text{ partition}\} = \infty;$$

but this is a contradiction since f was assumed to be integrable. Thus, f must be bounded.

b) We already know that f is bounded, that is, there is a number $L \geq 0$ such that

$$\sup_{t \in [a,b]} |f(t)| \leq L.$$

Without loss of generality let $y \geq x$. Then

$$|F(x) - F(y)| = \left| \int_a^x f(t)dt - \int_a^y f(t)dt \right| = \left| \int_x^y f(t)dt \right| \leq \int_x^y |f(t)|dt \leq L \int_x^y 1dt = L|x-y|.$$

[1]This holds generally: any Riemann integrable function is bounded.

c) Let $x, y \in [a, b]$ with $y \geq x$. Then

$$F(y) = \int_a^y f(t)dt = \int_a^x f(t)dt + \int_x^y f(t)dt \geq \int_a^x f(t)dt = F(x),$$

where we used that $f \geq 0$ on the interval $[x, y]$ and thus $\int_x^y f(t)dt \geq 0$.