

# Knowledge Representation and Processing

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# Administrative Information

# Format

## Room

- ▶ lectures and exercises in person — if we have a room
- ▶ interaction strongly encouraged      We don't want to lecture —  
we want to have a conversation during which you learn
- ▶ possibly make use of zoom

## Recordings

- ▶ maybe prerecorded video lectures or recorded zoom meeting
- ▶ to be decided along the way

# Background

## Instructor

- ▶ PD Dr. Florian Rabe
- ▶ Prof. Dr. Michael Kohlhase

Professor of Knowledge Representation and Processing

## Course

- ▶ This course is given for the fourth time
- ▶ Still a bit experimental **polishing and revising the materials this year**
- ▶ Signature course of our research group **same name!**

# Prerequisites

## Required

- ▶ basic knowledge about formal languages, context-free grammars  
but we'll do a quick revision here

## Helpful

- ▶ Algorithms and Data Structures mostly as a contrast to this lecture
- ▶ Basic logic we'll revise it slightly differently here
- ▶ all other courses as examples of how knowledge pervades all of CS

## General

- ▶ Curiosity this course is a bit unusual
- ▶ Interest in big picture  
this course touches on lots of things from all over CS

# Examination and Grading

## Suggestion

- ▶ grade determined by single exam
- ▶ probably written, 90 minutes
- ▶ exercises indirectly graded through conversation during exam

unless we agree on something else by the second week

## Exam-relevant

- ▶ anything mentioned in notes
- ▶ anything discussed in lectures
- ▶ anything done in exercises

none is a superset of another!

# Materials and Exam-Relevance

## Textbook

- ▶ does not exist
- ▶ normal for research-near specialization courses

## Notes

- ▶ textbook-style but not as comprehensive
- ▶ developed along the way
- ▶ systematic write-up, not necessarily in lecture order

## Slides

- ▶ not comprehensive
- ▶ used as visual aid, conversation starters

# Communication

## Open for questions

- ▶ open door policy in our offices
- ▶ always room for questions during lectures
- ▶ for personal questions, contact me during/after lecture
- ▶ matrix channel <https://matrix.to/#/#wuv:fau.de>
- ▶ studon forum not used

## Materials

- ▶ notes and slides are at: <https://github.com/florian-rabe/Teaching/tree/master/WuV>
- ▶ currently last year's version, will change throughout the semester  
you can read ahead, but maybe don't print everything right away
- ▶ pull requests and issues welcome



# Exercises

## Learning Goals

- ▶ Get acquainted with state of the art of practice
- ▶ Try out real tools

## Homeworks

- ▶ one major project as running example
- ▶ homeworks building on each other

build one large knowledge-based system  
details on later slides

# Overview and Essential Concepts

# Representation and Processing

Common pairs of concepts:

Representation	Processing
Static	Dynamic
Situation	Change
Be	Become
Data Structures	Algorithms
Set	Function
State	Transition
Space	Time

## Data and Knowledge

$2 \times 2$  key concepts

Syntax	Data
Semantics	Knowledge

- ▶ Data: any object that can be stored in a computer  
Example:  $((49.5739143, 11.0264941), "2020 - 04 - 21 T16 : 15 : 00 CEST")$
- ▶ Syntax: a system of rules that describes which data is **well-formed**  
Example: "a pair of (a pair of two IEEE double precision floating point numbers) and a string encoding of a time stamp"
- ▶ Semantics: system of rules that determines the meaning of well-formed data
- ▶ Knowledge: combination of some data with its syntax and semantics

# Knowledge is Elusive

## Representation of key concepts

- ▶ Data: using primitive objects  
implemented as bits, bytes, strings, records, arrays, ...
- ▶ Syntax: (context-free) grammars, (context-sensitive) type systems  
implemented as inductive data structures
- ▶ Semantics: functions for evaluation, interpretation, of well-formed data  
implemented as recursive algorithms on the syntax
- ▶ Knowledge: elusive  
emerges from applying and interacting with the semantics

## Semantics as Translation

- ▶ Knowledge can be captured by a higher layer of syntax
- ▶ Then semantics is translation into syntax

Data syntax	Semantics function	Knowledge syntax
SPARQL query	evaluation	result set
SQL query	evaluation	result table
program	compiler	binary code
program expression	interpreter	result value
logical formula	interpretation in a model	mathematical object
HTML document	rendering	graphics context

# Heterogeneity of Data and Knowledge

- ▶ Capturing knowledge is difficult
- ▶ Many different approaches to semantics
  - ▶ fundamental formal and methodological differences
  - ▶ often captured in different fields, conferences, courses, languages, tools
- ▶ Data formats equally heterogeneous
  - ▶ ontologies
  - ▶ programs
  - ▶ logical proofs
  - ▶ databases
  - ▶ documents

# Challenges of Heterogeneity

## Challenges

- ▶ collaboration across communities
- ▶ translation across languages
- ▶ conversion between data formats
- ▶ interoperability across tools

## Sources of problems

- ▶ interoperability across formats/tools major source of
  - ▶ complexity
  - ▶ bugs
- ▶ friction in project team due to differing preferences, expertise
- ▶ difficult choice between languages/tools with competing advantages
  - ▶ reverting choices difficult, costly
  - ▶ maintaining legacy choices increases complexity



## Aspects of Knowledge

- ▶ Tetrapod model of knowledge      **active research by our group**
- ▶ classifies approaches to knowledge into five aspects

Aspect	KRLs (examples)
ontologization	ontology languages (OWL), description logics (ALC)
concretization	relational databases (SQL, JSON)
computation	programming languages (C)
deduction	logics (HOL)
narration	document languages (HTML, LaTeX)

## Relations between the Aspects

Ontology is distinguished: capture the knowledge that the other four aspects share



## Complementary Advantages of the Aspects

Aspect	objects	characteristic		
		advantage	joint advantage of the other aspects	application
ded. comp.	formal proofs programs	correctness efficiency	ease of use well-definedness	verification execution
concr. narr.	concrete objects texts	queriability flexibility	abstraction formal semantics	storage/retrieval human understanding

Aspect pair	characteristic advantage
ded./comp. narr./concr.	rich meta-theory simple languages
ded./narr. comp./concr.	theorems and proofs normalization
ded./concr. comp./narr.	decidable well-definedness Turing completeness

# Structure of the Course

## Aspect-independent parts

- ▶ shared characteristics
- ▶ general methods

## Aspects-specific parts

- ▶ one part for each aspect
- ▶ high-level overview of state of the art
- ▶ focus on comparison/evaluation of the aspect-specific results

## Structure of the Exercises

### One major project

- ▶ representative for a project that a CS graduate might be put in charge of
- ▶ challenging heterogeneous data and knowledge
- ▶ requires integrating/combining different languages, tools

unique opportunity in this course because knowledge is everywhere

### Concrete project

- ▶ develop a campo/studnon-style KRP system for a university
- ▶ lots of heterogeneous knowledge
  - ▶ course and program descriptions
  - ▶ legal texts
  - ▶ websites
  - ▶ grade tables
  - ▶ code to generate diplomas
- ▶ build a functional system applying the lessons of the course  
we'll see how far we get — priority is learning

# Ontological Knowledge

## Components of an Ontology

8 main declarations

- ▶ **individual** — concrete objects that exist in the real world, e.g., "Florian Rabe" or "WuV"
- ▶ **concept** — abstract groups of individuals, e.g., "instructor" or "course"
- ▶ **relation** — binary relations between two individuals, e.g., "teaches"
- ▶ **properties** — binary relations between an individuals and a concrete value (a number, a date, etc.), e.g., "has-credits"
- ▶ **concept assertions** — the statement that a particular individual is an instance of a particular concept
- ▶ **relation assertions** — the statement that a particular relation holds about two individuals
- ▶ **property assertions** — the statement that a particular individual has a particular value for a particular property
- ▶ **axioms** — statements about relations between concepts, e.g., "instructor"  $\sqsubseteq$  "person"

# Divisions of an Ontology

## Abstract vs. concrete

- ▶ TBox: concepts, relations, properties, axioms  
everything that does not use individuals
- ▶ ABox: individuals and assertions

## Named vs. unnamed

- ▶ Signature: individuals, concepts, relations, properties  
together called entities or resources
- ▶ Theory: assertions, axioms



# Comparison of Terminology

Here	OWL	Description logics	ER model	UML	semantics via logics
individual	instance	individual	entity	object, instance	constant
concept	class	concept	entity-type	class	unary predicate
relation	object property	role	role	association	binary predicate
property	data property	(not common)	attribute	field of base type	binary predicate
		domain	individual	concept	
		type theory, logic	constant, term	type	
		set theory	element	set	
		database	row	table	
		philosophy <sup>1</sup>	object	property	
		grammar	proper noun	common noun	

<sup>1</sup>as in <https://plato.stanford.edu/entries/object/>

# Ontologies as Sets of Triples

General idea:

- ▶ Turn everything into a relation/property assertion
- ▶ Represent ontologies as sets of subject-predicate-object triples
- ▶ Obtain efficient representation of ontologies using RDF and RDFS

Assertion	Triple		
	Subject	Predicate	Object
entities	recover from what's mentioned in assertions		
concept assertion	"Florian Rabe"	is-a	"instructor"
relation assertion	"Florian Rabe"	"teaches"	"WuV"
property assertion	"WuV"	"has credits"	7.5
axiom	only some special cases work, e.g.,		
subconcept axiom	"instructor"	subClassOf	"person"

## Special Entities

RDF and RDFS define special entities for use in ontologies:

- ▶ "rdfs:Resource": concept of which all individuals are an instance and thus of which every concept is a subconcept
- ▶ "rdf:type": relates an entity to its type:
  - ▶ an individual to its concept (corresponding to is-a above)
  - ▶ other entities to their special type (see below)
- ▶ "rdfs:Class": special class for the type of classes
- ▶ "rdf:Property": special class for the type of properties
- ▶ "rdfs:subClassOf": a special relation that relates a subconcept to a superconcept
- ▶ "rdfs:domain": a special relation that relates a relation to the concepts of its subjects
- ▶ "rdfs:range": a special relation that relates a relation/property to the concept/type of its objects

Goal/effect: capture as many parts as possible as RDF triples.

# Declarations as Triples using Special Entities

Assertion	Triple		
	Subject	Predicate	Object
individual	individual	"rdf:type"	"rdfs:Resource"
concept	concept	"rdf:type"	"rdf:Class"
relation	relation	"rdf:type"	"rdf:Property"
property	property	"rdf:type"	"rdf:Property"
concept assertion	individual	"rdf:type"	concept
relation assertion	individual	relation	individual
property assertion	individual	property	value
for special forms of axioms			
$c \sqsubseteq d$	$c$	"rdfs:subClassOf"	$d$
$\text{dom } r \equiv c$	$r$	"rdfs:domain"	$c$
$\text{rng } r \equiv c$	$r$	"rdfs:range"	$c$

## An Example Ontology Language

see syntax of BOL in the lecture notes

## A Real-Life Ontology Language

See online resources for OWL.

Some specialties:

- ▶ Slightly different names than in BOL
- ▶ No strict distinction between individuals, concepts, relations - just resources
- ▶ Some special axioms, e.g., to make relations transitive
- ▶ Multiple sublanguages with varying expressivity/implementability: Lite, DL, Full

BOL vs. OWL:

- ▶ BOL is simpler, more systematically structured  
good for teaching, prototypes
- ▶ OWL is the standard  
the one to use for better or worse

## Exercise 1

As a team, build an ontology for a university.

Using git, OWL, and WebProtege are good ways to start.

(In WebProtege, set "suffix" to "user supplied name" in "New Entity Settings". Otherwise, it'll get messy when you share your ontology.)

## Example: The Common Sense Ontology

### Situation

- ▶ society uses one ontology for common sense knowledge
- ▶ changes over time

content relative to ontology: laws, regulations, etc.

### Special aspects

- ▶ unwritten
- ▶ not actually fully agreed upon
- ▶ sometimes subject to political debate
- ▶ no formal ontology language good enough to capture practical nuances
- ▶ many society members not comfortable with formal languages

but still always exists implicitly

Idea: see political proposals as ontology evolution



# Ontology Morphisms

# Idea

## Intuition of morphism $m$

- ▶ connects two ontologies, written  $m : V \rightarrow W$
- ▶ maps  $V$ -symbols to  $W$ -expressions
- ▶ extends homomorphically to map  $V$ -expressions to  $W$ -expressions
  - replace every symbol with its assignment
  - like substitutions for contexts

## Purpose

- ▶ extend  $V$  with entirely new declarations
  - special case of  $W = V, E$ , and identity morphism  $V \rightarrow W$
- ▶ extend the vocabulary with definitions
  - special case  $m : V, E \rightarrow V$ , and  $m$  maps new symbols to definitions
- ▶ ontology evolution:  $V$  is old ontology,  $W$  new,  $m$  interprets  $V$  in  $W$
- ▶ transfer legacy content from old to new ontology

## Exercise 2

We write ontologies for sex and gender.

Write two ontologies for

- ▶ cis-normative world view with just men and women
- ▶ trans-inclusive world view that accommodates sex and gender

and relate them with ontology morphisms.

## Side Note: Knowledge Representation is Apolitical

Knowledge representation makes no judgment about which ontologies or morphisms are fair, moral, politically correct, etc.

It can only judge practicality, e.g.,

- ▶ well-formedness and consistency
- ▶ decidability, efficiency of querying
- ▶ simplicity, e.g., measured by
  - ▶ number of declarations or the size of expressions
  - ▶ number of axioms about each symbol
- ▶ existence and simplicity of morphisms

Languages must allow for expressing whichever knowledge/opinion the user has.

Only users can judge if an ontology is correct.

## BOL Morphisms Formally

Syntax: Extend grammar with vocabulary morphisms

$M ::= A^* : O \rightarrow O$	morphisms
$A ::= i \mapsto I$	individual assignment
$\quad   \quad c \mapsto C$	concept assignment
$\quad   \quad r \mapsto R$	relation assignment
$\quad   \quad p \mapsto P$	property assignment

Well-formedness for  $M : O \rightarrow O'$ :

- ▶ one assignment  $ID \mapsto E$  for each declaration ID of  $O$
- ▶  $E$  must be an  $O'$ -expression of the right kind
  - ▶ individual symbols to individual expressions
  - ▶ concept symbols to concept expressions
  - ▶ relation symbols to relation expressions
  - ▶ property symbols of type  $V$  to property expressions of type  $V$
  - ▶ what about assertions and axioms? see below

## Homomorphic Extension

Given morphism  $m : O \rightarrow O'$ , define

- ▶ mapping  $\bar{m}$  from  $O$ -expressions  $E$  to  $O'$ -expressions  $\bar{m}(E)$  by
- ▶ replacing every  $O$ -symbol  $s$  in  $E$   
with the expression  $s \mapsto E$  provided by  $m$ .

Notation:  $m(E)$  instead of  $\bar{m}(E)$

Well-defined mapping because morphisms must contain exactly one assignment for every  $O$ -symbol.

## BOL Morphisms: What about Axioms?

A morphism  $m : O \rightarrow O'$  is well-formed if

- ▶ for every axiom/assertion  $F$  in  $O$ ,
- ▶ we have that  $m(F)$  is a theorem of  $O'$ .

Theorem: Morphisms preserve truth

- ▶ if  $\vdash_O E : E'$  then  $\vdash_{O'} m(E) : m(E')$
- ▶ if  $\vdash_O F$  then  $\vdash_{O'} m(F)$

Mapping axioms works best if

- ▶ every axiom/assertion has a name
- ▶ new expression kind for proofs
  - given by derivations of some absolute deductive semantics
  - axioms = proof symbols = atomic proofs
- ▶ morphisms contain assignments  $a \mapsto P$  of axiom  $a$  to proof  $P$

## Example/Exercise 2

Assume *CisNormative* is BOL vocabulary containing

- ▶ concepts `man`, `woman`
- ▶ axioms `man`  $\sqcup$  `woman`  $\equiv \top$  and `man`  $\sqcap$  `woman`  $\equiv \perp$

simplified cis-normative world view

and *TransFriendly* contains one way to accommodate transgender people

- ▶ concepts `sexmale`, `sexfemale`, `cis`, `trans`
- ▶ appropriate axioms

Now have ontology morphism *CisNormative*  $\rightarrow$  *TransFriendly*

- ▶ morphism `gendermatters`
  - ▶ `man`  $\mapsto$  (`sexmale`  $\sqcap$  `cis`)  $\sqcup$  (`sexfemale`  $\sqcap$  `trans`)
  - ▶ `woman`  $\mapsto$  (`sexfemale`  $\sqcap$  `cis`)  $\sqcup$  (`sexmale`  $\sqcap$  `trans`)
- ▶ alternative morphism `sexmatters`
  - ▶ `man`  $\mapsto$  `sexmale`
  - ▶ `woman`  $\mapsto$  `sexfemale`



# Prevalence of Morphisms

## Deduction

- ▶ algebraic hierarchy, e.g.,  $Monoid \rightarrow Group$
- ▶ theory  $\rightarrow$  model, e.g.,  $Group \rightarrow Integer$

## Computation

- ▶ class extension
- ▶ interface implementation
- ▶ type class instances
- ▶ functor
- ▶ API adapters

## Concrete data

- ▶ between tables: database views
- ▶ between schemas: database migration

General: module systems for building large vocabularies

# Language Layers

# Layers of Language Design

Layer	Specified by	Implemented by
Syntax		
Context-Free	grammar	AST+parser+printer
Context-Sensitive	inference system	type checker
Semantics	inference system, interpretation, or translation	
Pragmatics	human preferences	human judgment

KRP = syntax + semantics

# Layered Processing

Data is processed in phases

1. data representation format, e.g., string, JSON, XML, binary
2. parsed — well-formed context-free syntax tree
3. context-sensitive check by traversal of the syntax tree — well-typed syntax tree
4. computation by traversal of well-typed AST — semantics

## Possible Errors

Layer	Error
CFS	not derivable from grammar
CSS	symbols not used as declared, other conditions
Sem.	ambiguous/undefined semantics
Pragmatics	not useful

## Typical Errors by Layer

In a programming language:

Layer	Expression	Issue	Explanation
CFS	1/	syntax error	argument missing
CSS	1/" 2"	typing error	wrong type
Sem.	1/0	run-time error	undefined semantics
Pragm.	1/1	code review	unnecessarily complex

## Typical Errors by Layer

In a logic:

Layer	Expression	Issue	Explanation
CFS	$\forall x$	not well-formed	body missing
CSS	$\forall x.P(y)$	not well-typed	$y$ not declared
Sem.	the $x \in \mathbb{N}$ with $x < 0$	not well-defined	no such $x$ exists
Pragm.	$\exists x.x \neq x$	not useful	no model exists

# Context-Free Grammars



# The Chomsky Hierarchy

- ▶ CH-0, regular grammars:
  - ▶ equivalent to regular expressions and finite automata
  - ▶ not used much as grammars
- ▶ CH-1, context-free grammars (CFGs) our focus
- ▶ CH-2, context-sensitive grammars
  - ▶ important as languages, but awkward as grammars
  - ▶ instead: type system determines subset of context-free language
- ▶ CH-3, unrestricted grammars
  - ▶ Turing-complete, theoretically important
  - ▶ not used much as grammars

## Definitions

- ▶ An alphabet is a set of symbols.
- ▶ A word is a list of symbols from the alphabet.
- ▶ A production is pair of words.
  - ▶ A production is written  $lhs ::= rhs$ .
  - ▶ Multiple productions for the same left-hand side are abbreviated  $lhs ::= rhs_1 \mid \dots \mid rhs_n$ .
  - ▶ Right-hand side may also use regular expressions like  $*$  for repetition and  $[]$  for optional parts.
- ▶ A CFG is a set of productions where  $lhs$  is a single symbol.
  - ▶ If there is a production  $N ::= rhs$ ,  $N$  is called non-terminal, otherwise terminal.
  - ▶ If a word contains non-terminal symbols, it is called non-terminal, otherwise terminal.
- ▶ A syntax tree is a tree whose nodes are labeled with productions  $N ::= rhs$  where the non-terminals in  $rhs$  are exactly the  $lhs$ 's of the children.
- ▶ The word produced by a syntax tree is read off by exhaustively replacing every  $lhs$  with the respective  $rhs$ .

## Example: Syntax of Arithmetic Language

### Numbers

$N ::= 0 \mid 1$	literals
$\mid N + N$	sum
$\mid N * N$	product

### Formulas

$F ::= N \doteq N$	equality
$\mid N \leq N$	ordering by size

## Implementing CFGs via Inductive Data Types

## Correspondence

CFG	IDT
non-terminal	type
production	constructor
non-terminal on left of production	return type of constructor
non-terminals on right of production	arguments types of constructor
terminals on right of production	notation of constructor
words derived from non-terminal $N$	expressions of type $N$

# Classes of Languages

Functional languages:

- ▶ pure: ML, Haskell
- ▶ with OO: F#, Scala

inductive types are primitive

OO-languages:

- ▶ C#, Java, C++

inductive types simulated via classes

Untyped languages:

- ▶ Python, Javascript

inductive types simulated ad hoc

## Implementing the Example

Done interactively. See the examples in the repository. See also the notes.

## Exercise 3

Individually, using any programming language, implement the AST for the BOL language. Implement a printer for BOL by context-free traversal of the syntax tree.

Remarks:

- ▶ To simplify, you can drop most productions for concepts and relations.
- ▶ To simplify, you can drop properties altogether. But it would be nice to have them and allow for integers and strings as basic types.



# Context-Sensitive Syntax

# Vocabularies and Declarations

Generic structure of a context-sensitive language

- ▶ a vocabulary is a list of declarations
  - ▶ named: type/function/predicate symbol etc.
  - ▶ unnamed: axioms etc.
  - ▶ structural: namespaces/package, inclusion/import
- ▶ named declarations introduce atomic objects of various kinds
- ▶ for each kind, a non-terminal for complex expressions of that kind
- ▶ references to names introduced by declarations are base cases of expressions

## Example: Typed Expressions

### Vocabularies

$Voc ::= Decl^*$

list of declarations

### Declarations

$Decl ::= id : Type^* \rightarrow Type$   
 $\quad \quad \quad | \quad id : Type^* \rightarrow FORM$

typed function symbols  
 typed predicate symbols

### Types

$Type ::= Nat \mid String$

base types

### Expressions

$Expr ::= 0 \mid 1 \mid Expr + Expr \mid Expr * Expr$   
 $\quad \quad \quad | \quad id(Expr^*)$

as before  
 application of a function symbol

### Formulas

$Form ::= Expr \doteq Expr \mid Expr \leq Expr$   
 $\quad \quad \quad | \quad id(Expr^*)$

as before  
 application of a predicate symbol

## Example: Vocabularies and Expressions

Example vocabulary  $V$  containing the following declarations:

- ▶  $fib : Nat \rightarrow Nat$
- ▶  $length : String \rightarrow Nat$
- ▶  $mod : Nat\ Nat \rightarrow Nat$
- ▶  $prime : Nat \rightarrow FORM$

Example expressions relative to  $V$

- ▶ expressions:  $fib(0)$ ,  $mod(fib(fib(1)), 1 + 1)$
- ▶ formulas:  $fib(0) = 0$ ,  $prime(fib(1))$

## Primitive vs. Declared

### Primitive

- ▶ built into the language
- ▶ assumed to exist a priori fundamentals of nature
- ▶ fixed semantics (usually interpreted by identity function)

	primitive	declared
introduced by	language designer	user
introduced in	grammar	vocabulary $V$
visible in	all vocabularies	$V$ only
semantics given	explicitly	implicitly
... by	translation function	axioms

more expressive declarations  $\rightarrow$  fewer primitives needed

paradoxical: more complex language may have simpler grammar

## Quasi-Primitive = Declared in standard library

### Standard library: a vocabulary *StdLib*

- ▶ present in every language empty vocabulary by default
- ▶ one fixed vocabulary
  - ▶ implicitly included into every other vocabulary
  - ▶ implicitly fixed by any translation between vocabularies

#### Combination of advantages

- ▶ from the user's perspective: like a primitive
- ▶ from the theory's/system's perspective: no special treatment

### Examples

- ▶ sufficiently expressive languages
  - ▶ push many primitive objects to standard library never all
  - ▶ simplifies language, especially when defining operations  
strings in C, BigInteger in Java, inductive type for  $\mathbb{N}$
- ▶ inexpressive languages
  - ▶ many primitives SQL, spreadsheet software
  - ▶ few (quasi)-primitives few operations available in OWL

## Example: Removing Built-in Operations

Grammar without built-in operations

$Voc ::= Decl^*$	list of declarations
$Decl ::= id : Type^* \rightarrow Type$	typed function symbols
$\quad \mid id : Type^* \rightarrow FORM$	typed predicate symbols
$Type ::= Nat \mid String$	base types
$Expr ::= id(Expr^*)$	application of a function symbol
$Form ::= id(Expr^*)$	application of a predicate symbol

Standard library:

- ▶  $0 : Nat, 1 : Nat, sum : Nat\ Nat \rightarrow Nat, product : Nat\ Nat \rightarrow Nat,$
- ▶  $equals : Nat\ Nat \rightarrow FORM, lesseq : Nat\ Nat \rightarrow FORM$

## Example: Removing more Primitive Operations

If we add type declarations, we can remove *Nat* as well

$Voc ::= Decl^*$	list of declarations
$Decl ::= id : Type^* \rightarrow Type$	typed function symbols
$id : Type^* \rightarrow FORM$	typed predicate symbols
$id : TYPE$	type symbols
$Type ::= id$	reference to a type symbol
$Expr ::= id(Expr^*)$	application of a function symbol
$Form ::= id(Expr^*)$	application of a predicate symbol

Note: *Type* and *Form* are non-terminals, *TYPE* and *FORM* are not  
 Add to default vocabulary: *Nat* : *TYPE*, *String* : *TYPE*



## Context-Sensitivity

A reference to a declared name must respect the way in which it was declared in the vocabulary examples below relative to  $V$  above

- ▶ occur in a position where an expression of the right kind is expected  
example error:  $\text{prime}(1) = 1$
- ▶ be applied to the right number of arguments  
example error:  $\text{prime}(1, 1)$
- ▶ if a type system is used
  - ▶ arguments must have the right types  
 $\text{length}(1)$
  - ▶ return type must match what is expected  
 $\text{fib}(1)$  if a string is expected

# Syntax Traversal

## Context-free traversal

mutually recursive functions

- ▶ one function for each non-terminal/inductive type
- ▶ for each such function, one case for each production/constructor
- ▶ for each such case, one recursive call for each non-terminal on the rhs/constructor argument
- ▶ Examples
  - ▶ printer/serializer (return, e.g., string)
  - ▶ test if a feature is used (return boolean)
  - ▶ find set of used identifiers (return set of names)

## Context-sensitive traversal: as above but

- ▶ functions take extra argument for vocabulary
- ▶ cases for identifier references look up declaration in vocabulary
- ▶ Examples: anything that looks up identifier declarations
  - ▶ substitution of identifiers with new expressions
  - ▶ typical syntax transformations, e.g., in compilers (return same expression kind)
  - ▶ semantics by translation

# Type Checker

## Syntax checker

- ▶ context-sensitive traversal where all functions return booleans
- ▶ case for vocabulary checks each declaration relative to the preceding vocabulary
- ▶ cases for identifier references check correct use of identifier

## Type checker: as above but additionally

- ▶ functions for typed expression additionally take expected type as argument
- ▶ cases for identifiers references
  - ▶ check each argument against declared input type
  - ▶ compare output to expected type

## Exercise 4

Implement a type-checker for BOL.

The type-checker must check that all identifier references are used according to their declaration:

- ▶ concept identifiers as concepts
- ▶ relation identifiers as relations
- ▶ individual identifiers as individuals
- ▶ properties identifiers of type  $Y$  as properties for values of type  $Y$   
individual identifiers as individuals

# Natural Language

# Languages

We give an overview of narrative languages as one corner of the Tetrapod.

# Formal Languages for Natural Language

## Formal Languages

- ▶ Controlled natural languages: GF, ... ,
- ▶ Word processors: Microsoft Word, Libre Office Write, ...
- ▶ Web-oriented languages: Markdown, HTML, ...

## Features

- ▶ Unrestricted or barely restricted natural language
- ▶ Usually: visual presentation, e.g., fonts, colors
- ▶ Possibly: formal structure, e.g.,
  - ▶ sections
  - ▶ lists, enumerations
  - ▶ cross-references
- ▶ Rarely: explicit representation of ontological knowledge, e.g., definitions, references

# Semantics by Translation

## Relative Semantics

- ▶ define semantics of a language  $I$  relative to a language  $L$  whose semantics is already fixed
- ▶ by translation of  $I$  into  $L$

## Compositional Translation

- ▶ translation is given as context-free traversal of checked syntax tree
- ▶ one function per non-terminal  $N$
- ▶ one case per production  $N ::= R$
- ▶ one recursive call per non-terminal in  $R$



## A Narrative Semantics of BOL

We give a semantics of BOL by translation into English according to Section 6.6 of the notes.

LaTeX

## The sTeX Dialect of LaTeX

LaTeX is a language for natural language documents.

sTeX enriches LaTeX to allow describing the ontology in addition to the narrative document. In particular: definitions and references to them

## Exercise 5

Implement a semantics by translation for BOL that translates BOL to sTeX.

Concretely, your translation should take an ontology and a name  $N$ . And it should return a file  $N.tex$  containing

- ▶ for every named declaration: one English sentence with an sTeX declaration
- ▶ for every axiom: one English sentence in which all identifier references carry a hyperlink to the corresponding sTeX declaration

Run this on your university ontology to produce `university.tex` and `university.pdf`. Write a second sTeX document containing a mission statement for the university in such a way that every mention of a concept carries a hyperlink to the respective declaration in `university.pdf`.

# Typed First-Order Logic

# Syntax

We give an overview of first-order logic as the main example of the deductive corner of the Tetrapod. The grammar is given in Section 6.3 of the notes.

## Exercise 6

Implement the syntax (with printer but not necessarily with parser) and type-checking of SFOL.

# Type Systems



# Overview

## General Goal

- ▶ subdivide expressions into groups (called sorts, types, kinds, etc.)
- ▶ written  $\vdash_V E : T$  for expression  $E$  of type  $T$  and vocabulary  $V$

## Basic Type System

- ▶ expressions are always subdivided by their non-terminals
- ▶ the types are the non-terminals
- ▶ type of expression immediately clear from expression
- ▶ context only needed to **check** well-formedness of expression, not to **infer** its type

## Refined Type System

- ▶ for some non-terminals, the expressions are additionally subdivided
- ▶ the types are other expressions  
 $E$  and  $T$  can be from same or different non-terminals
- ▶ relation between  $E$  and  $T$  entirely up to the language

# Local Variables

## Type Systems with Variables

- ▶  $\Gamma \vdash E : T$  for expression  $E$  of type  $T$
- ▶  $\Gamma = x_1 : Y_1, \dots, x_n : Y_n$  declares free variables that may occur in  $E$  and  $T$
- ▶ confusingly:  $\Gamma$  usually also called **context**

## Choice of $\Gamma$

- ▶ for BOL: nothing (BOL has no variable binding)
- ▶ for SFOL: variables introduced by  $\forall$  and  $\exists$

# Algorithms for Type Systems

Judgment  $\Gamma \vdash_V E : T$

## Type Checking

- ▶ input:  $V, \Gamma, E, T$
- ▶ output: boolean

## Type Inference

- ▶ input:  $\Gamma, V, E$
- ▶ output:  $T$

in practice: also return error messages, e.g., as exceptions

## Advanced Variants

- ▶ as above but additionally return  $E'$  and  $T'$
- ▶  $E$  and  $T$  have gaps that are filled by the algorithms resulting in  $E'$  and  $T'$

# Implementing a Type-Checker

## Structure of Syntax

- ▶ structural level: vocabularies (and morphisms), declarations
- ▶ expressions: some non-terminals are designated as expressions
  - ▶ usually at least one per declaration kind
  - ▶ usually includes (or can be extended to include) formulas

## Structure of Type-Checker

- ▶ function  $\text{check-}N$  for each non-terminal  $N$ 
  - ▶ takes context ( $V$  and  $\Gamma$ ) and  $N$ -word
  - ▶ returns error message(s)
- ▶ whenever  $N$ -words are typed by  $Y$ -words, instead
  - ▶ function  $\text{check-}N$  takes  $V$ ,  $\Gamma$ ,  $N$ -word, **and**  $Y$ -word (expected type)
  - ▶ function  $\text{infer-}N$  takes  $V$ ,  $\Gamma$ ,  $N$ -word; returns  $Y$ -word (inferred type)

# Typed Ontologies and Database Schememas

# Motivation

## Main ideas

- ▶ Ontology abstractly describes concepts and relations
- ▶ Tool maintains concrete data set
- ▶ Focus on efficiently
  - ▶ identifying (i.e., assign names)
  - ▶ representing
  - ▶ processing
  - ▶ querying

large sets of concrete data

## Recall: TBox-ABox distinction

- ▶ TBox: general parts, abstract, fixed  
main challenge: correct modeling of domain
- ▶ ABox: concrete individuals and assertions about them, growing  
main challenge: aggregate them all

# Concrete Data

## Concrete is

- ▶ Base values: integers, strings, booleans, etc.
- ▶ Collections: sets, multisets, lists (always finite)
- ▶ Aggregations: tuples, records (always finite)
- ▶ User-defined concrete data: enumerations, inductive types
- ▶ Advanced objects: finite maps, graphs, etc.

## Concrete is not

- ▶ Free symbols to be interpreted by a model  
exception: foreign function interfaces
  - ▶ Variables (free or bound)  
 $\lambda$ -abstraction, quantification
  - ▶ Symbolic expressions  
formulas, algorithms
- Exceptions:
- ▶ expressions of inductive type
  - ▶ application of built-in functions
  - ▶ queries that return concrete data

## Breakout question

What is the difference between

- ▶ an OWL ontology
- ▶ an SQL database



## Two Approaches

### Based on **untyped** (Curry-typed) ontology languages

- ▶ Representation based on **knowledge graph**
- ▶ Ontology written in BOL-like language
- ▶ Data maintained as **set of triples** tool = triple store
- ▶ Typical language/tool design
  - ▶ ontology and query language **separate** e.g., OWL, SPARQL
  - ▶ triple store and query engine integrated e.g., Virtuoso tool

### Based on **typed** (Church-typed) ontology languages

- ▶ Representation based on **abstract data types**
- ▶ Ontology written as database schema
- ▶ Data maintained as **tables** tool = (relational) database
- ▶ Typical language/tool design
  - ▶ ontology and query language **integrated** e.g., SQL
  - ▶ table store and query engine integrated e.g., SQLite tool

## Evolution of Approaches

### Our usage is non-standard

- ▶ Common
  - ▶ ontologies = untyped approach, OWL, triples, SPARQL
  - ▶ databases = typed approach, tables, SQL
- ▶ Our understanding: two approaches evolved from same idea
  - ▶ triple store = untyped database
  - ▶ SQL schema = typed ontology

### Evolution

- ▶ Typed-untyped distinction minor technical difference
- ▶ Optimization of respective advantages causes speciation
- ▶ Today segregation into different
  - ▶ jargons
  - ▶ languages, tools
  - ▶ communities, conferences
  - ▶ courses

## Curry-typed concrete data

### Central data structure = knowledge graph

- ▶ nodes = individuals  $i$ 
  - ▶ identifier
  - ▶ sets of concepts of  $i$
  - ▶ key-value sets of properties of  $i$
- ▶ edges = relation assertions
  - ▶ from subject to object
  - ▶ labeled with name of relation

### Processing strengths

- ▶ store: as triple set
- ▶ edit: Protege-style or graph-based
- ▶ visualize: as graph different colors for concepts, relations
- ▶ query: match, traverse graph structure
- ▶ untyped data simplifies integration, migration

## Church-typed concrete data

### Central data structure = relational database

- ▶ tables = abstract data type
- ▶ rows = objects of that type
- ▶ columns = fields of ADT
- ▶ cells = values of fields

### Processing strengths

- ▶ store: as CSV text files, or similar
- ▶ edit: SQL commands or table editors
- ▶ visualize: as table view
- ▶ query: relational algebra
- ▶ typed data simplifies selecting, sorting, aggregating

# Identifiers

## Curry-Typed Knowledge graph

- ▶ concept, relation, property names given in TBox
- ▶ individual names attached to nodes

## Church-Typed Database

- ▶ table, column names given in schema
- ▶ row identified by distinguished column (= key)  
options
  - ▶ preexistent characteristic column
  - ▶ added upon insertion
    - ▶ UUID string
    - ▶ incremental integers
    - ▶ concatenation of characteristic list of columns
- ▶ column/row identifiers formed by qualifying with table name

# Axioms

## Curry-Typed Knowledge Graph

- ▶ traditionally very expressive axioms
- ▶ yields inferred assertions
- ▶ triple store must do consequence closure to return correct query results
- ▶ not all axioms supported by every triple store

## Church-Typed Database

- ▶ typically no axioms
- ▶ instead consistency constraints, triggers
- ▶ allows limited support for axioms without calling it that way
- ▶ stronger need for users to program the consequence closure manually

## Open/Closed World

- ▶ Question: is the data complete?
  - ▶ closed world: yes
  - ▶ open world: not necessarily
- ▶ Dimensions of openness
  - ▶ existence of individual objects
  - ▶ assertions about them
- ▶ Sources of openness
  - ▶ more exists but has not yet been added
  - ▶ more could be created later
- ▶ Orthogonal to typed/untyped distinction, but in practice
  - ▶ knowledge graphs use open world
  - ▶ databases use closed world

Open world is natural state, closing adds knowledge

# Closing the World

## Derivable consequences

- ▶ induction: prove universal property by proving for each object
- ▶ negation by failure: atomic property false if not provable
- ▶ term-generation constraint: only nameable objects exist

## Enabled operations

- ▶ universal set: all objects
- ▶ complement of concept/type
- ▶ defaults: assume default value for property if not otherwise asserted

## Monotonicity problem

- ▶ monotone operation: bigger world = more results
- ▶ examples: union, intersection,  $\exists R.C$ , join, IN conditions
- ▶ counter-examples: complement,  $\forall R.C$ , NOT IN conditions

technically, non-monotone operations in open world dubious



## Summary

	semantic web	relational databases
ontology aspect	TBox of ontology	SQL schema
conceptual model	knowledge graph	set of tables
concrete data aspect	ABox of ontology	SQL database
concrete data storage	set of triples	set of rows of the tables
concrete data formats	RDF	CSV
concrete data tool	triple store	database implementation
typing	soft/Curry	hard/Church
query language	SPARQL	SQL SELECT query
openness of world	tends to be open	tends to be closed

## Exercise 2

Extend your ontology with an ABox and axioms. Export it in RDF format to a triple store like Virtuoso and run a concrete query in SPARQL.

Export it in OWL format to a reasoner like FaCT++ and run a deductive query. Potentially, do this by installing a plugin for a reasoner in your ontology IDE.

## Extrinsic and Intrinsic Typing

## Breakout Question

Is this an improvement over BOL?

Declarations
--------------

$D ::=$	<b>individual</b> $i : C$	typed atomic individual
	<b>concept</b> $c$	atomic concept
	<b>relation</b> $r \subseteq C \times C$	typed atomic relation
	<b>property</b> $p \subseteq C \times T$	typed atomic property

rest as before

## Actually, when is a language an improvement?

Criteria: orthogonal, often mutually exclusive

- ▶ syntax design trade-off
  - ▶ expressivity: easy to express knowledge  
e.g., big grammar, complex type system
  - ▶ simplicity: easy to implement/interpret  
e.g., few, carefully chosen productions, types
- ▶ semantics: specify, implement, documents
- ▶ intended users
  - ▶ skill level
  - ▶ prior experience with related languages
  - ▶ amount of training needed
  - ▶ innovation height, differential evaluation against existing languages

## Actually, when is a language an improvement? (2)

More criteria:

- ▶ long-term plans: re-answer the above question but now
  - ▶ maintainability: syntax was changed, everything to be redone
  - ▶ backwards compatibility: support for legacy input
  - ▶ scalability: expressed knowledge content has reached huge sizes
- ▶ support software ecosystem
  - ▶ optional tool support: IDEs, debuggers, heuristic checkers, alternative implementations, interpreter/REPL
  - ▶ many/large well-crafted vocabularies and package managers to find them
  - ▶ integrations with other languages: translations, common run-time platforms, foreign function interface

# Church vs. Curry Typing

	intrinsic	extrinsic
$\lambda$ -calculus by type is typing is a objects have types interpreted as	Church carried by object function objects $\rightarrow$ types unique type disjoint sets	Curry given by environment relation objects $\times$ types any number of types unary predicates
type given by example	part of declaration <b>individual</b> "WuV" : "course"	additional axiom <b>individual</b> "Wuv", "WuV" is-a "course"
examples	SFOL, SQL most logics, functional PLs  many type theories	OWL, Scala, English ontology, OO, natural languages set theories

# Type Checking

	intrinsic	extrinsic
type is typing is a objects have	carried by object function objects $\rightarrow$ types unique type	given by environment relation objects $\times$ types any number of types
type given by example	part of declaration <b>individual</b> WuV: course	additional axiom <b>individual</b> WuV, WuV is-a course
type inference for $x$ type checking subtyping $A <: B$ typing decidable typing errors	uniquely infer $A$ from $x$ inferred=expected cast from $A$ to $B$ yes unless too expressive static (compile-time)	find minimal $A$ with $x : A$ prove $x : A$ $x : A$ implies $x : B$ no unless restricted dynamic (run-time)
advantages	easy unique type inference	flexible allows subtyping



## Examples: Curry Typing in BOL Semantics

language	objects	types	typing relation
Syntax	individuals	concepts	$i$ is-a $c$
Semantics in			
FOL	type $\iota$	predicates $c \subseteq \iota$	$c(i)$ true
SQL	table Individuals	tables containing ids	id of $i$ in table $c$
Scala	String	hash sets of strings	$c.\text{contains}(i)$
English	proper nouns	common nouns	" $i$ is a $c$ " is true

# Subtyping

Subtyping works best with Curry Typing

- ▶ explicit subtyping as in  $\mathbb{N} <: \mathbb{Z}$
- ▶ comprehension/refinement as in  $\{x : \mathbb{N} \mid x \neq 0\}$
- ▶ operations like union and intersection on types
- ▶ inheritance between classes, in which case subclass = subtype
- ▶ anonymous record types as in  $\{x : \mathbb{N}, y : \mathbb{Z}\} <: \{x : \mathbb{N}\}$

## A General Definition of a Type System

A **type system** for a syntax consists of

- ▶ some non-terminals  $\mathcal{E}$ , whose words are called  $\mathcal{E}$ -**expressions**,  
coarse, easy to check classification into disjoint sets
- ▶ for some (possibly no) intrinsic types  $\mathcal{E}$ 
  - ▶ a non-terminal  $\mathcal{T}$
  - ▶ a typing relation between  $\mathcal{E}$ -expressions and  $\mathcal{T}$ -expressions,  
called the **extrinsic typing** relation for  $\mathcal{E}$   
fine-granular subclassification of the expressions

## Examples

System	intrinsic types	extrinsic types	extr. typing
pure Church	one per type	none	none
pure Curry	one for all expressions	types $T$	:
FOL	one per type	none	none
OO	primitive types, Object	classes	<code>isInstanceOf</code>
BOL	<i>Ind</i> , <i>Conc</i> , ...	concepts	<i>is-a</i>
set theory	<i>Set</i> , <i>Prop</i>	sets	$\in$

## Breakout Questions

What kind of extrinsic types could we choose?

## Abstract vs. Concrete Types

**Concrete** type: values are

- ▶ given by their internal form,
- ▶ defined along with the type, typically built from already-known pieces.

product types, enumeration types, collection types

main example: concrete (inductive/algebraic) data types

**Abstract** type: values are

- ▶ given by their externally visible properties,
- ▶ defined in any environment that understands the type definition.

structures, records, classes, aggregation types

main example: abstract data types

## Plain vs. Recursive Types

### **Plain** types

- ▶ given by some expressions
- ▶ can be anonymous
- ▶ values given directly

integers, lists of strings, ...

### **Recursive** types

- ▶ definition of the type must refer to the type itself  
so type must have a name
- ▶ type typically defines other named operations
- ▶ values obtained by fixed-point constructions

optional property of concrete and abstract data types

## Atomic vs. Complex Types

### **Atomic** type

- ▶ given by its name
- ▶ values are a set

integers, strings, booleans, ...

### **Complex** types

- ▶ arise by applying type symbol to arguments
- ▶ separate set of values for each tuple of arguments

two kinds of complex types (next slide)



## Type operators vs. Dependent type Families

Both are symbols that take arguments and return a type.

**Type operators** take **only type arguments**, e.g.,

- ▶ type operator  $\times$
- ▶ takes two types  $A, B$
- ▶ returns type  $A \times B$

**Dependent types** take **also value arguments**, e.g.,

- ▶ dependent type operator *vector*
- ▶ takes natural number  $n$ , type  $A$
- ▶ returns type  $A^n$  of  $n$ -tuples over  $A$

dependent types much more complicated, less uniformly used  
harder to standardize

## Non-Recursive Data Types

## Basic Atomic Types

### typical in IT systems

- ▶ fixed precision integers (32 bit, 64 bit, ...)
- ▶ IEEE float, double
- ▶ Booleans
- ▶ Unicode characters
- ▶ strings                      could be list of characters but usually bad idea

### typical in math

- ▶ natural numbers ( $= \mathbb{N}$ )
- ▶ arbitrary precision integers ( $= \mathbb{Z}$ )
- ▶ rational, real, complex numbers
- ▶ graphs, trees

clear: language must be modular, extensible

# Advanced Atomic Types

## general purpose

- ▶ date, time, color, location on earth
- ▶ picture, audio, video

## domain-specific

- ▶ physical quantities (*1m*, *1in*, etc.)
- ▶ gene, person
- ▶ semester, course id, ...

clear: language must be modular, extensible

# Collection Data Types

## Homogeneous Collection Types

- ▶ sets
- ▶ multisets (= bags)
- ▶ lists      all unary type operators, e.g. *list A* is type of lists over *A*
- ▶ fixed-length lists (= Cartesian power, vector *n*-tuple)  
dependent type operator

## Heterogeneous Collection Types

- ▶ lists
- ▶ fixed-length lists (= Cartesian power, *n*-tuple)
- ▶ sets
- ▶ multisets (= bags)  
all atomic types, e.g., *list* is type of lists over any objects

# Aggregation Data Types

## Products

- ▶ Cartesian product of some types  $A \times B$   
values are pairs  $(x, y)$     numbered projections  $_1, _2$  — order relevant
- ▶ labeled Cartesian product (= record)  $\{a : A, b : B\}$   
values are records  $\{a = x, b = y\}$   
named projections  $a, b$  — order irrelevant

## Disjoint Unions

- ▶ disjoint union of some types  $A \uplus B$   
values are  $inj_1(x), inj_2(y)$  numbered injections  $_1, _2$  — order relevant
- ▶ labeled disjoint union  $a(A) | b(B)$   
values are constructor applications  $a(x), b(y)$   
named injections  $a, b$  — order irrelevant

labeled disjoint unions uncommon  
but recursive labeled disjoint union = inductive data type

## Subtyping

- ▶ relatively easy if all data types disjoint
- ▶ better with subtyping      open problem how to do it nicely

### Subtyping Atomic Types

- ▶  $\mathbb{N} <: \mathbb{Z}$
- ▶ ASCII <: Unicode

### Subtyping Complex Types

- ▶ covariance subtyping (= vertical subtyping) same for disjoint unions

$$A <: A' \Rightarrow \text{list } A <: \text{list } A'$$

$$A_i <: A'_i \Rightarrow \{\dots, a_i : A_i, \dots\} <: \{\dots, a_i : A'_i, \dots\}$$

- ▶ structural subtyping (= horizontal subtyping)

$$\{a : A, b : B\} :> \{a : A, b : B, c : C\}$$

$$a(A)|b(B) <: a(A)|b(B)|c(C)$$

# Concrete Data Types



# Motivation

## Idea

- ▶ describe infinite type in finite way
- ▶ exploit inductive structure to catch all values

Name: usually called **inductive** data type, especially when recursive

## Examples

Natural numbers  $Nat$  given by

- ▶ *zero*
- ▶ *succ*( $n$ ) for every  $n : Nat$

Lists *list*  $A$  over type  $A$  given by

- ▶ empty list *nil*
- ▶ *cons*( $a, l$ ) for every  $a : A$  and  $l : list\ A$

Arithmetic expressions  $E$  given by

- ▶ natural number *literal*( $n$ ) for  $n : Nat$
- ▶ sum *plus*( $e, f$ ) for every  $e, f : E$
- ▶ product *times*( $e, f$ ) for every  $e, f : E$

## Rigorous Definition

Let  $T$  be the set of types that are known in the current context.

An **inductive data type** is given by

- ▶ a name  $n$ , called the **type**,
- ▶ a set of **constructors** each consisting of
  - ▶ a name
  - ▶ a list of elements of  $T \cup \{n\}$ , called the **argument** types

Notation:

`inductive n = c(A, ...) | ...`

`inductive Nat = zero | succ(Nat)`

`inductive E = Number(Nat) | sum(E,E) | times(E,E)`

## Induction Principle

The values of the inductive type are exactly the ones that can be built by the constructors.

- ▶ No junk: the constructors are jointly-surjective
  - ▶ no other values but union of their images
  - ▶ closed world
- ▶ No confusion: the constructors are jointly-injective in the following sense
  - ▶ each constructor is an injective function
  - ▶ images of the constructor are pairwise disjoint

Inductive definition: define function out of inductive type by giving one case per constructor pattern matching

- ▶ total because jointly-surjective
- ▶ well-defined because jointly-injective (no overlap between cases)

## Special case: No recursion

A concrete data types without recursive constructor arguments are called a **labeled union**. They are isomorphic to the union of the products of the constructor arguments.

Example:

```
inductive Value = Number(Nat) | true | false
inductive Product(A,B) = Pair(A,B)
```

A concrete data types without any constructor arguments is called an **enumeration**. They have exactly one element per constructor.

Example:

```
inductive Boolean = true | false
inductive Color = red | blue | green
```

## Generalization: Mutual Induction

Multiple inductive types whose definitions refer to each other.

Example:

```
inductive E = literal(Nat) | sum(E,E) | times(E,E)
inductive F = equal(E,E) | less(E,E)
```

# Abstract Data Types

## Breakout Question

What do the following have in common?

- ▶ Java class
- ▶ SQL schema for a table
- ▶ logical theory (e.g., Monoid)



## Breakout Question

What do the following have in common?

- ▶ Java class
- ▶ SQL schema for a table
- ▶ logical theory (e.g., Monoid)

all are (essentially) abstract data types

## Motivation

Recall subject-centered representation of assertion triples:

```
individual "FlorianRabe"  
  is-a "instructor" "male"  
  "teach" "WuV" "KRMT"  
  "age" 40  
  "office" "11.137"
```

Can we use types to force certain assertions to occur together?

- ▶ Every instructor should teach a list of courses.
- ▶ Every instructor should have an office.

## Motivation

Inspires **subject-centered types**, e.g.,

```
concept instructor
  teach course*
  age: int
  office: string
```

```
individual "FlorianRabe": "instructor"
  is-a "male"
  teach "WuV" "KRMT"
  age 40
  office "11.137"
```

Incidental benefits:

- ▶ no need to declare relations/properties separately
- ▶ reuse relation/property names  
distinguish via qualified names: `instructor .age`

## Motivation

Natural next step: inheritance

```
concept person  
  age: int
```

```
concept male <: person
```

```
concept instructor <: person  
  teach course*  
  office: string
```

```
individual "FlorianRabe": "instructor" ⊑ "male"  
  "teach" "WuV" "KRMT"  
  "age" 40  
  "office" "11.137"
```

our language quickly gets a very different flavor

## Examples

Prevalence of abstract data types:

aspect	language	abstract data type
ontologization	UML	class
concretization	SQL	table schema
computation	Scala	class, interface
deduction	various	theory, specification, module, locale
narration	various	emergent feature

same idea, but may look very different across languages

## Examples

aspect	type	values
computation	abstract class	instances of implementing classes
concretization	table schema	table rows
deduction	theory	models

Values depend on the environment in which the type is used:

- ▶ class defined in one specification language (e.g., UML),  
implementations in programming languages Java, Scala, etc.  
available values may depend on run-time state
- ▶ theory defined in logic,  
models defined in set theories, type theories, programming  
languages  
available values may depend on philosophical position

## Definition

Given some type system, an **abstract data type** (ADT) is

- ▶ a **flat** type

$$\{c_1 : T_1 [= t_1], \dots, c_n : T_n [= t_n]\}$$

where

- ▶  $c_i$  are distinct names
  - ▶  $T_i$  are types
  - ▶  $t_i$  are optional definitions; if given,  $t_i : T_i$  required
- ▶ or a **mixin** type

$$A_1 * \dots * A_n$$

for ADTs  $A_i$ .

Languages may or may not make ADTs additional types of the type system

## Class Definitions

A class definition in OO:

```
abstract class  $a$  extends  $a_1$  with ... with  $a_m$  {  
   $c_1 : T_1$   
   $\vdots$   
   $c_n : T_n$   
}
```

Corresponding ADT definition:

$$a = a_1 * \dots * a_m * \{c_1 : T_1, \dots, c_n : T_n\}$$

The usual terminology:

- ▶  $a$  **inherits** from  $a_i$
- ▶  $a_i$  are **super-X** or **parent-X** of  $a$  where  $X$  is whatever the language calls its ADTs (e.g.,  $X=\text{class}$ )



# Flattening

The **flattening**  $A^b$  of an ADT  $A$  is

- ▶ if  $A$  is flat:  $A^b = A$
- ▶  $(A_1 * \dots * A_n)^b$  is union of all  $A_i^b$   
where duplicate field names are handled as follows
  - ▶ same name, same type, same or omitted definition: merge  
details may be much more difficult
  - ▶ otherwise: ill-formed

## Subtleties

We gloss over several major issues:

- ▶ How exactly do we merge duplicate field names? Does it always work? **implement abstract methods, override, overload**
- ▶ Is recursion allowed, i.e., can I define an ADT  $a = A$  where  $a$  occurs in  $A$ ?  
**common in OO-languages: use  $a$  in the types of its fields**
- ▶ What about ADTs with type arguments?  
**e.g., generics in Java, square-brackets in Scala**
- ▶ Is mutual recursion between fields in a flat type allowed?  
**common in OO-languages**
- ▶ Is  $*$  commutative? What about dependencies between fields?  
**no unique answers**  
**incarnations of ADTs subtly different across languages**

## Breakout question

When using typed concrete data,  
how to fully realize abstract data types

- ▶ nesting: ADTs occurring as field types
- ▶ inheritance between ADTs
- ▶ mixins

## ADTs in Typed Concrete Data

### Nesting: field $a : A$ in ADT $B$

- ▶ field types must be base types,  $a : A$  not allowed
- ▶ allow  $ID$  as additional base type
- ▶ use field  $a : ID$  in table  $B$
- ▶ store value of  $b$  in table  $A$

### Inheritance: $B$ inherits from $A$

- ▶ add field  $parent_A$  to table  $B$
- ▶ store values of inherited fields of  $B$  in table  $A$

general principle: all objects of type  $A$  stored in same table

### Mixin: $A * B$

- ▶ essentially join of tables  $A$  and  $B$  on common fields
- ▶ some subtleties depending on ADT flattening

# Motivation

# Data Interoperability

## Situation

- ▶ languages focus on different aspects frequent need to exchange data
- ▶ generally, lots of aspect/language-specific objects  
proofs, programs, tables, sentences
- ▶ but same/similar primitive data types used across systems  
should be easy to exchange

## Problem

- ▶ crossing system barriers usually requires interchange language  
serialize as string and reparse
- ▶ interchange languages typically untyped XML, JSON, YAML, ...

## Solution

- ▶ standardize primitive data types
- ▶ standardize encoding in interchange languages

# Treatment in this Course

## BOL syntax and semantics so far

- ▶ primitive types omitted in syntax
- ▶ assumed reasonable collection available
- ▶ assumed same (quasi-)primitive objects in semantic languages  
irrelevant if interpreting primitive objects as primitive or quasi-primitive

largely justified by practice in common languages

No universal set of types standardized, but some types much more common than others.

# Common Primitive Types

## Typical (quasi-)primitive types

- ▶ natural numbers ( $= \mathbb{N}$ )
- ▶ arbitrary precision integers ( $= \mathbb{Z}$ )
- ▶ fixed precision integers (32 bit, 64 bit, ...)
- ▶ floating point (float, double, ...)
- ▶ Booleans
- ▶ characters (ASCII, Unicode)
- ▶ strings

## Observation:

- ▶ essentially the same in every language  
including whatever language used for semantics
- ▶ semantics by translation trivial



# Less Common Primitive Types

## Problem

- ▶ quickly encounter primitive types not supported by common languages
- ▶ need to encode them using existing types  
typically as strings, ints, or products/lists thereof

## Examples

- ▶ date, time, color, location on earth
- ▶ graph, function
- ▶ picture, audio, video
- ▶ physical quantities ( $1m$ ,  $1in$ , etc.)
- ▶ gene, person

# Types and their Operations

Types usually come with additional operations:

## Triple Structure: 3 kinds of named objects

- ▶ the type eg: 'int'
- ▶ values of the type eg: 0, 1, -1, ...
- ▶ operations on type eg: addition, multiplication, ...

# Specifying a Primitive Type

## Components

- ▶ name
- ▶ set of values
- ▶ string encoding

injective function from values to strings

## Examples

- ▶ IEEE floating point numbers
- ▶ ISO 8601 for date/time

## Exercise 5

As a group, define an XML schema for BOL using the RelaxNG compact syntax. See

<https://relaxng.org/compact-tutorial-20030326.html>.

Individually, extend your implementations of BOL with XML importers and exporters relative to that schema. The exporter is a context-free traverser. For the importer, use an XML library for your chosen programming language to parse the input. Then write a context-free traverser over the XML data structures that translates the XML into your BOL data structures.

After importing a vocabulary, type-check it.

Write some example vocabularies for BOL and export-import them to each other.

## Exercise 6

As a group, specify additional primitive datatypes for BOL that are helpful the university knowledge ontology.

For each new type, you have to give

- ▶ name
- ▶ values
- ▶ string encoding of each value

The name is added by an additional BOL-production for non-terminal, the values by additional productions for V. The string encodings are used by the import/export functions.

# Failures of Encodings

## Y2K bug

- ▶ date encoded as tuple of integers, using 2 digits for year
- ▶ needed fixing in year 2000
- ▶ estimated \$300 billion spent to change software
- ▶ possible repeat: in 2038, number of seconds since 1970-01-01 (used by Unix to encode time as integer) overflows 32-bit integers

## Genes in Excel

- ▶ 2016 study found errors in 20% of spreadsheets accompanying genomics journal papers
- ▶ gene names encoded as strings but auto-converted to other types by Excel
  - ▶ "SEPT2" (Septin 2) converted to September 02
  - ▶ REKIN identifiers, e.g., "2310009E13", converted to float  $2.31E + 1$

## Failures of Encodings (2)

### Mars Climate Orbiter

- ▶ two components exchanged physical quantity
- ▶ specification required encoding as number using unit Newton seconds
- ▶ one component used wrong encoding (with pound seconds as unit)
- ▶ led to false trajectory and loss of \$300 million device

### Shellshock

- ▶ bash allowed gaining root access from 1998 to 2014
- ▶ function definitions were encoded as source code
- ▶ not decoded at all; instead, code simply run (as root)
- ▶ allowed appending ";" ... to function definitions

SQL injection similar: complex data encoded as string, no decoding

# Research Goal for Aspect-Independent Data in Tetrapod

## Standardization of Common Data Types

- ▶ Ontology language optimized for declaring types, values, operations  
semantics must exist but can be extra-linguistic
- ▶ Vocabulary declaring such objects  
should be standardized, modular, extensible

## Standardization of Codecs

- ▶ Fixed small set of primitive objects  
should be (quasi-)primitive in every language  
not too expressive, possibly untyped
- ▶ Standard codecs for translating common types to interchange languages

## Codec for type $A$ and int. lang. $L$

- ▶ coding function  $A$ -values  $\rightarrow L$ -objects
  - ▶ partial decoding function  $L$ -objects  $\rightarrow A$ -values
  - ▶ inverse to each other
- in some sense



# Overview

## Next steps

1. Data interchange languages
2. Data types
3. Codecs

# Data Representation Languages

# Overview

## General Properties

- ▶ general purpose or domain-specific
- ▶ typed or untyped
  - typical: Church-typed but no type operators, quasi untyped
- ▶ text or binary serialization
- ▶ libraries for many programming languages
  - ▶ data structures
  - ▶ serialization (data structure to string)
  - ▶ parsing (string to data structure, partial)

## Candidates

- ▶ XML: standard on the web, notoriously verbose
- ▶ JSON: JavaScript objects, more human-friendly text syntax
  - older than XML, probably better choice than XML in retrospect
- ▶ YAML: line/indentation-based

## Breakout Question

What is the difference between JSON, YAML, XML?

# Typical Data Representation Languages

XML, JSON, YAML essentially the same

except for concrete syntax

## Atomic Types

- ▶ integer, float, boolean, string
- ▶ need to read fine-print on precision

## (Not Very) Complex Types

- ▶ heterogeneous lists
- ▶ records

a single type for all lists

a single type for all records

## Example: JSON

JSON:

```
{  
  "individual" : "FlorianRabe",  
  "age" : 40,  
  "concepts" : ["instructor", "male"],  
  "teach" : [  
    {"name" : "Wuv", "credits" : 7.5},  
    {"name" : "KRMT", "credits" : 5}  
  ]  
}
```

Weirdnesses:

- ▶ atomic/list/record = basic/array/object
- ▶ record field names are arbitrary strings, must be quoted
- ▶ records use : instead of =

## Example: YAML

inline syntax: same as JSON but without quoted field names

alternative: indentation-sensitive syntax

```
individual : "FlorianRabe"  
age : 40  
concepts :  
  - "instructor"  
  - "male"  
teach :  
  - name : "WuV"  
    credits : 7.5  
  - name : "KRMT"  
    credits : 5
```

Weirdnesses:

- ▶ atomic/list/record = scalar/collection/structure
- ▶ records use : instead of =

## Example: XML

Weird structure but very similar

- ▶ elements both record (= attributes) and list (= children)
- ▶ elements carry name of type (= tag)

```
<Person individual="Florian Rabe" age="40">  
  <concepts>  
    <Concept>instructor </Concept>  
    <Concept>male</Concept>  
  </concepts>  
  <teach>  
    <Course name="WuV" credits="7.5" />  
    <Course name="KRMT" credits="5" />  
  </teach>  
</Person>
```

- ▶ Good: Person, Course, Concept give type of object  
easier to decode
- ▶ Bad: value of record field must be string  
concepts cannot be given in attribute  
integers, Booleans, whitespace-separated lists coded as strings



# Structure Sharing

## Problem

- ▶ Large objects are often redundant specially when machine-produced
- ▶ Same string, URL, mathematical objects occurs in multiple places
- ▶ Handled in memory via pointers
- ▶ Size of serialization can explode

## Solution 1: in language

- ▶ Add definitions to language common part of most languages anyway
  - ▶ Users should introduce name whenever object used twice
  - ▶ Problem: only works if
    - ▶ duplication anticipated
    - ▶ users introduced definition
    - ▶ duplication within same context
- structure-sharing most powerful if across contexts

## Structure Sharing (2)

### Solution 2: in tool

- ▶ Use factory methods instead of constructors
- ▶ Keep huge hash set of all objects
- ▶ Reuse existing object if already in hash set
- ▶ Advantages
  - ▶ allows optimization
  - ▶ transparent to users
- ▶ Problem: only works if
  - ▶ for immutable data structures
  - ▶ if no occurrence-specific metadata e.g., source reference

### In data representation language

- ▶ Allow any subobject to carry identifier
- ▶ Allow identifier references as subobjects
  - allows preserving structure-sharing in serialization

supported by XML, YAML

# Data Types

## A Basic Language for Typed Data

Let BDL be given by

### Types

$T ::=$	$int \mid float \mid string \mid bool$	base types
	$list\ T$	homogeneous lists
	$(ID : T)^*$	record types
	$\dots$	additional types

### Data

$D ::=$	$(64\ bit\ integers)$	
	$(IEEE\ double)$	
	$"(Unicode\ strings)"$	
	$true \mid false$	
	$D^*$	lists
	$(ID = D)^*$	records
	$\dots$	constructors for additional types

## BDL Extended with Named ADTs

$V$	$::= Decl^*$	Vocabularies
$Decl$	$::= \mathbf{adt} \ t \ \{ID : T^*\}$	ADT definitions
	$\quad   \quad \mathbf{datum} \ d : T = D$	data definitions

## Types

$T$	$::= \dots$	as before
	$\quad   \quad t$	reference to a named ADT

## Data

$D$	$::= \dots$	as before
	$\quad   \quad d$	reference to a named datum
	$\quad   \quad t\{(ID = D)^*\}$	ADT elements

# Codecs

## General Definition

Throughout this section, we fix a data representation language  $L$ .

$L$ -words called codes

Given a data type  $T$ , a codec for  $T$  consists

- ▶ coding function:  $c : T \rightarrow L$
- ▶ partial decoding function:  $d : L \rightarrow^? T$
- ▶ such that

$$d(c(x)) = x$$

## Codec Operators

Given a data type operator  $T$  taking  $n$  type arguments,  
a codec operator  $C$  for  $T$

- ▶ takes  $n$  codecs  $C_i$  for  $T_i$
- ▶ returns a codec  $C(C_1, \dots, C_n)$  for  $T(T_1, \dots, T_n)$



## Codecs for Base Types

We define codecs for the base types using strings as the data representation language  $L$ .

Easy cases:

- ▶ StandardFloat: as specified in IEEE floating point standard
- ▶ StandardString: as themselves, quoted
- ▶ StandardBool: as *true* or *false*
- ▶ StandardInt (64-bit): decimal digit-sequences as usual

## Breakout Question

How to encode unlimited precision integers?

## Codecs for Unlimited Precision Integers

Encode  $z \in \mathbb{Z}$

- ▶  $L$  is strings: decimal digit sequence as usual
- ▶  $L$  is JSON:
  - ▶ IntAsInt: decimal digit sequence as usual  
JSON does not specify precision  
but target systems may get in trouble
  - ▶ IntAsString: string containing decimal digit sequence  
safe but awkward
  - ▶ IntAsDecList: list of decimal digits  
safe but awkward
  - ▶ IntAsList1: as list of digits for base  $2^{64}$   
OK, but we can do better
  - ▶ IntAsList2: as list of
    - ▶ integer for the number of digits, sign indicate sign of  $z$
    - ▶ list of digits of  $|z|$  for base  $2^{64}$

Question: Why is this smart?

## Codecs for Unlimited Precision Integers

Encode  $z \in \mathbb{Z}$

- ▶  $L$  is strings: decimal digit sequence as usual
- ▶  $L$  is JSON:
  - ▶ IntAsInt: decimal digit sequence as usual  
JSON does not specify precision  
but target systems may get in trouble
  - ▶ IntAsString: string containing decimal digit sequence  
safe but awkward
  - ▶ IntAsDecList: list of decimal digits  
safe but awkward
  - ▶ IntAsList1: as list of digits for base  $2^{64}$   
OK, but we can do better
  - ▶ IntAsList2: as list of
    - ▶ integer for the number of digits, sign indicate sign of  $z$
    - ▶ list of digits of  $|z|$  for base  $2^{64}$

Question: Why is this smart?

Can use lexicographic ordering for size comparison

## Codecs for Lists

Encode list  $x$  of elements of type  $T$

- ▶  $L$  is strings: e.g., comma-separated list of  $T$ -encoded elements of  $x$
- ▶  $L$  is JSON:
  - ▶ ListAsString: like for strings above
  - ▶ ListFromArray: lists JSON array of  $T$ -encoded elements of  $x$

## Additional Types

Examples: semester

Extend BDL:

Types

$T ::= \text{Sem}$                       semester

Data

$D ::= \text{sem}(\text{int}, \text{bool})$     i.e., year + summer?

Define standard codec:

$\text{sem}(y, \text{true}) \rightsquigarrow \text{"SSY"}$

$\text{sem}(y, \text{false}) \rightsquigarrow \text{"WSY"}$

where  $Y$  is encoding of  $y$

## Additional Types (2)

Examples: timestamps

Extend BDL:

Types

$T ::= \text{timestamp}$

Data

$D ::= (\text{productions for dates, times, etc.})$

Standard codec: encode as string as defined in ISO 8601

# Data Interchange



# Design

## 1. Specify types

- ▶ types
- ▶ constructors
- ▶ operations

can be done in appropriate type theory

## 2. Pick data representation language $L$

## 3. Specify codecs for type system and $L$

- ▶ at least one codec per base type
- ▶ at least one codec operator per type operator

on paper

## 4. Every system implements

- ▶ type system (as they like) typically aspect-specific constraints
- ▶ codecs as specified
- ▶ function mapping types to codecs

## 5. Systems can exchange data by encoding-decoding

type-safe because codecs chosen by type

## BDL-Mediated Interoperability

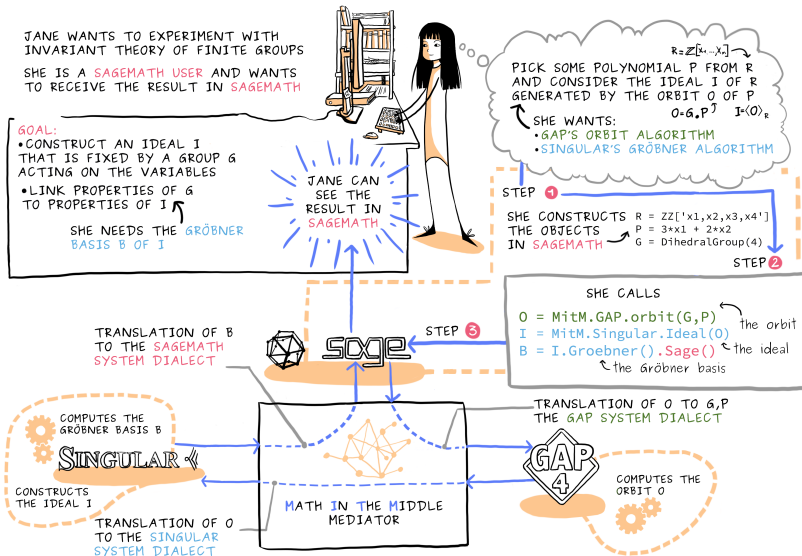
### Idea

- ▶ define data types in BDL or similar typed ontology language
- ▶ use ADTs
- ▶ generate corresponding
  - ▶ class definitions for programming languages PL  
one class per ADT
  - ▶ table definitions in SQL  
one table per ADT
- ▶ use codecs to convert automatically when interchanging data between PL and SQL

### Open research problem

no shiny solution yet that can be presented in lectures

# Example Application: OpenDreamKit research project



## Codecs in ADT Definitions

SQL table schema = list of fields where field is

- ▶ name
- ▶ type only types of database supported

BDL semantic table schema = list of fields where field is

- ▶ name
- ▶ type  $T$  of **type system** independent of database
- ▶ codec for  $T$  using primitive objects of database as codes  
see research paper [https://kwarc.info/people/frabe/Research/WKR\\_](https://kwarc.info/people/frabe/Research/WKR_)

Codec could be chosen automatically, but we want to allow multiple users a choice of codecs for the same type.

## Example

Ontology based on BDL-ADTs with additional codec information:

```
schema Instructor
  name:      string      codec StandardString
  age:       int         codec StandardInt
  courses:   list Course codec CommaSeparatedList CourseAsName
schema Course
  name:      string      codec StandardString
  credits:   float       codec StandardFloat
  semester:  Semester    codec SemesterAsString
```

Generated SQL tables:

```
CREATE TABLE Instructor
  (name string , age int , courses string)
CREATE TABLE Course
  (name string , credits float , semester string)
```

## Open Problem: Non-Compositionality

### Sometimes optimal translation is non-compositional

- ▶ example translate *list*-type in ADT to comma-separated string in DB
- ▶ better break up *list B* fields in type *A* into separate table with columns for *A* and *B*

### Similar problems

- ▶ a pair type in an ADT could be translated to two separate columns
- ▶ an option type in an ADT could be translated to a normal column using SQL's NULL value

## Open Problem: Querying

- ▶ General setup
  - ▶ write SQL-style queries using at the BDL level
  - ▶ automatically encode values when writing to database from PL
  - ▶ automatically decode query results when reading from DB
- ▶ But queries using semantic operations cannot always be translated to DB
  - ▶ operation  $IsSummer : Semester \rightarrow bool$  in BDL
  - ▶ query `SELECT * FROM course WHERE  $IsSummer$ (semester)`
  - ▶ how to map  $IsSummer$  to SQL?
- ▶ Ontology operations need commuting operations on codes
  - ▶ given  $f : A \rightarrow B$  in BDL, codecs  $C, D$  for  $A$  and  $B$
  - ▶ SQL function  $f'$  commutes with  $f$  iff

$$B.decode(f'(C.encode\ a)) = f(a)$$

for all  $a : A$

# Overview



## General Ideas

- ▶ Recall
  - ▶ syntax = context-free grammar
  - ▶ semantics = translation to another language
- ▶ Example: BOL translated to SQL, SFOL, Scala, English
- ▶ Querying = use semantics to answer questions about syntax

Note:

- ▶ Not the standard definition of querying
- ▶ Design of a new Tetrapod-level notion of querying  
ongoing research
- ▶ Subsumes concepts of different names from the various aspects

# Propositions

syntax with propositions =  
designated non-terminals for propositions

Examples:

aspect	basic propositions
ontology language	assertions, concept equality/subsumption
programming language	equality for some types
database language	equality for base types
logic	equality for all types
natural language	sentences

Aspects vary critically in how propositions can be formed

- ▶ any program in computation
- ▶ quantifiers in deductions
- ▶  $\exists$  in databases

undecidable

# Propositions as Queries

Propositions allow defining queries

	Query	Result
deduction	proposition	yes/no
concretization	proposition with free variables	true ground instances
computation	term	value
narration	question	answer

# Semantics of Propositions

syntax with propositions =  
designated non-terminals for propositions

needed to ask queries

semantics with theorems =  
designates some propositions as theorems or contradictions

needed to answer queries

Note:

- ▶ A propositions may be neither theorem nor contradiction.
- ▶ We say that language has negation if:  
 $F$  theorem iff  $\neg F$  contradiction and vice versa.

We write  $\vdash F$  if  $F$  is theorem.

# Deductive Queries

## Definition

We assume

- ▶ a semantics  $\llbracket - \rrbracket$  from  $I$  to  $L$
- ▶  $I$  has propositions
- ▶ there is an operation  $\text{True}$  that maps translations of  $I$ -propositions to  $L$ -propositions called truth lifting
- ▶  $L$  has semantics with propositions

We define

- ▶ a deductive query is an  $I$ -proposition  $p$
- ▶ the result is
  - ▶ yes if  $\text{True}[\llbracket p \rrbracket]$  is a theorem of  $L$
  - ▶ no if  $\text{True}[\llbracket p \rrbracket]$  is a contradiction in  $L$

## The Truth-Lift operator

## Problem with type-preserving translation $\llbracket - \rrbracket$

- ▶ must translate  $F : \text{prop}'$  to  $\llbracket F \rrbracket : \llbracket \text{prop}' \rrbracket$
- ▶ but need not satisfy  $\llbracket \text{prop}' \rrbracket = \text{prop}^L$   
 $I$ -propositions maybe not translated to  $L$ -propositions
- ▶ If not,  $\vdash^L \llbracket F \rrbracket$  cannot be used to answer query  $\vdash' F$

Solution:  $L$ -operator True :  $\llbracket \text{prop}' \rrbracket \rightarrow \text{prop}^L$

- ▶ maps translations of  $I$ -propositions to  $L$ -propositions
- ▶ then use  $\vdash^L \text{True} \llbracket F \rrbracket$  to answer  $\vdash^I F$

## Example: probabilistic semantics with cutoff

- ▶  $\text{prop}^I$ : propositions;  $\text{prop}^L$ : truth values
- ▶ probabilistic semantics:  $\llbracket \text{prop}^I \rrbracket = [0; 1]$
- ▶ True maps  $p \in [0; 1]$  to truth value, e.g., via cutoff:  
 $\text{True}p = p \geq 0.75$

# Situational Semantics as Translation

## Example: standard semantics of SFOL in set theory

- ▶ set theory propositions: truth values  $\{0, 1\}$
- ▶ situations (= models): provide interpretations for all vocabulary symbols
- ▶ semantics is model-specific interpretation function: given vocabulary  $V$ ,  $V$ -model  $M$ , and  $V$ -expression  $E$ , write  $\llbracket E \rrbracket_M$  for semantics of  $E$  in  $M$

## Reformulate as semantics by translation

- ▶ vocabulary  $V$  translated to set-theoretical vocabulary containing required components of model
- ▶ expression  $E$  translated to mapping  $\llbracket E \rrbracket : M \mapsto \llbracket E \rrbracket_M$

## Truth-Lift operator defines theorems

- ▶  $\llbracket \text{prop}' \rrbracket$  is set of mappings  $\mu$  from models to truth values  
 standard notation for  $\llbracket F \rrbracket(M) = 1$ :  $\llbracket F \rrbracket_M = 1$  or  $M \models F$
- ▶  $\text{True}_\mu = 1$  iff  $\mu(M) = 1$  for all models  $M$  (theorem = true in all models)



## Breakout question

What can go wrong?

## Problem: Inconsistency

In general, (in)consistency of semantics

- ▶ Some propositions may be both a theorem and a contradiction.
- ▶ In that case, queries do not have a result.

In practice, however:

- ▶ If this holds for some propositions, it typically holds for all of them.
- ▶ In that, we call  $L$  inconsistent.
- ▶ We usually assume  $L$  to be consistent.

## Problem: Incompleteness

In general, (in)completeness of semantics

- ▶ We cannot in general assume that every proposition in  $L$  is either a theorem or a contradiction.
- ▶ In fact, most propositions are neither.
- ▶ So, queries do not necessarily have a result.
- ▶ We speak of incompleteness.

Note: not the same as the usual (in)completeness of logic

In practice, however:

- ▶ It may be that  $L$  is complete for all propositions in the image of  $\text{True}[\![ - ]\!]$ .
- ▶ This is the case if  $L$  is simple enough  
typical for ontology languages

## Problem: Undecidability

In general, (un)decidability of semantics:

- ▶ We cannot in general assume that it is decidable whether a proposition in  $L$  is a theorem or a contradiction.
- ▶ In fact, it usually isn't.
- ▶ So, we cannot necessarily compute the result of a query.
- ▶ However: If we have completeness, decidability is likely.

run provers for  $F$  and  $\neg F$  in parallel

In practice, however:

- ▶ It may be that  $L$  is decidable for all propositions in the image of  $\text{True}[\![ - ]\!]$ .
- ▶ This is the case if  $I$  is simple enough

typical for ontology languages

## Problem: Inefficiency

In general, (in)efficiency of semantics:

- ▶ Answering deductive queries is very slow.
- ▶ Even if we are complete and decidable.

In practice, however:

- ▶ Decision procedures for the image of  $\text{True}[\![ - ]\!]$  may be quite efficient.
- ▶ Dedicated implementations for specific fragments.
- ▶ This is the case if  $I$  is simple enough  
typical for ontology languages

## Contexts and Free Variables

## Concepts

Recall the analogy between grammars and typing:

grammars	typing
non-terminal	type
production	constructor
non-terminal on left of production	return type of constructor
non-terminals on right of production	arguments types of constructor
terminals on right of production	notation of constructor
words derived from non-terminal $N$	expressions of type $N$

We will now add contexts and substitutions.

## Contexts

Independent of whether  $I$  already has contexts/variables, we can define:

- ▶ A **context**  $\Gamma$  is of the form  $x_1 : N_1, \dots, x_n : N_n$  where the
  - ▶  $x_i$  are names
  - ▶  $N_i$  are non-terminals

We write this as  $\vdash_I \Gamma$ .

- ▶ A **substitution** for  $\Gamma$  is of the form  $x_1 := w_1, \dots, x_n := w_n$  where the
  - ▶  $x_i$  are as in  $\Gamma$
  - ▶  $w_i$  derived from the corresponding  $N_i$

We write this as  $\vdash_I \gamma : \Gamma$ .

- ▶ An **expression in context**  $\Gamma$  of type  $N$  is a word  $w$  derived from  $N$  using additionally the productions  $N_i ::= x_i$ .

We write this as  $\Gamma \vdash_I w : N$ .

- ▶ Given  $\Gamma \vdash w : N$  and  $\vdash \gamma : \Gamma$  as above, the **substitution of**  $\gamma$  in  $w$  is obtained by replacing every  $x_i$  in  $w$  with  $w_i$ . We write this as  $w[\gamma]$ .



## Contexts under Compositional Translation

Consider a compositional semantics  $\llbracket - \rrbracket$  from  $I$  to  $L$  between context-free languages.

- ▶ Every  $\vdash_I w : N$  is translated to some  $\vdash_L \llbracket w \rrbracket : N'$  for some  $N'$ .
- ▶ Compositionality ensures that  $N'$  is the same for all  $w$  derived from  $N$ .
- ▶ We write  $\llbracket N \rrbracket$  for that  $N'$ .
- ▶ Then we have

$$\vdash_I w : N \quad \text{implies} \quad \vdash_L \llbracket w \rrbracket : \llbracket N \rrbracket$$

Now we translate contexts, substitutions, and variables as well:

$$\llbracket x_1 : N_1, \dots, x_n : N_n \rrbracket := x_1 : \llbracket N_1 \rrbracket, \dots, x_n : \llbracket N_n \rrbracket$$

$$\llbracket x_1 := w_1, \dots, x_n := w_n \rrbracket := x_1 := \llbracket w_1 \rrbracket, \dots, x_n := \llbracket w_n \rrbracket$$

$$\llbracket x \rrbracket := x$$

Then we have

$$\Gamma \vdash_I w : N \quad \text{implies} \quad \llbracket \Gamma \rrbracket \vdash_L \llbracket w \rrbracket : \llbracket N \rrbracket$$

# Substitution under Compositional Translation

From previous slide:

$$\llbracket x_1 : N_1, \dots, x_n : N_n \rrbracket := x_1 : \llbracket N_1 \rrbracket, \dots, x_n : \llbracket N_n \rrbracket$$

$$\llbracket x_1 := w_1, \dots, x_n := w_n \rrbracket := x_1 := \llbracket w_1 \rrbracket, \dots, x_n := \llbracket w_n \rrbracket$$

$$\llbracket x \rrbracket := x$$

$$\Gamma \vdash_I w : N \quad \text{implies} \quad \llbracket \Gamma \rrbracket \vdash_L \llbracket w \rrbracket : \llbracket N \rrbracket$$

We can now restate the substitution theorem as follows:

$$\llbracket E[\gamma] \rrbracket = \llbracket E \rrbracket \llbracket \llbracket \gamma \rrbracket \rrbracket$$

# Concretized Queries

## Definition

We assume

- ▶ as for deductive queries
- ▶ semantics must be compositional

We define

- ▶ a concretized query is an  $l$ -proposition  $p$  in context  $\Gamma$
- ▶ a **single** result is a
  - ▶ a substitution  $\vdash_I \gamma : \Gamma$
  - ▶ such that  $\vdash_L \text{True}[\![p[\gamma]]\!]$
- ▶ the **result set** is the set of all results

## Example

1. BOL ontology:

*concept male, concept person, axiom male  $\sqsubseteq$  person,*  
*individual FlorianRabe, assertion FlorianRabe isa male*

2. Query  $x : \textit{individual} \vdash_{BOL} x \textit{ isa person}$
3. Translation to SFOL:  $x : \iota \vdash_{SFOL} \textit{person}(x)$
4. SFOL calculus yields theorem  $\vdash_{SFOL} \textit{person}(\textit{FlorianRabe})$
5. Query result  $\llbracket \gamma \rrbracket = x := \textit{FlorianRabe}$
6. Back-translating the result to BOL:  $\gamma = x := \textit{FlorianRabe}$   
back translation is deceptively simple:  
translates SFOL-constant to BOL-individual of same name

## Breakout question

What can go wrong?

## Problem: Open World

In general, semantics uses open world:

- ▶ open world: result contains **all known** results  
same query might yield more results later
- ▶ closed world: result set contains **all** results

always relative to concrete database for  $L$

In practice, however,

- ▶ system explicitly assumes closed world    typical for databases
- ▶ users aware of open world and able to process results correctly

## Problem: Infinity of Results

In general, there may be infinitely many results:

- ▶ e.g., query for all  $x$  such that  $\vdash x$ ,

In practice, however,

- ▶ systems pull results from finite database e.g., SQL, SPARQL
- ▶ systems enumerate results, require user to explicitly ask for more e.g., Prolog



## Problem: Back-Translation of Results

In general,  $\llbracket - \rrbracket$  may be non-trivial to invert

- ▶ easy to obtain  $\llbracket p \rrbracket$  in context  $\llbracket \Gamma \rrbracket$       just apply semantics
- ▶ possible to find substitutions

$$\vdash_L \delta : \llbracket \Gamma \rrbracket \quad \text{where} \quad \llbracket \Gamma \rrbracket \vdash_L \text{True}[\llbracket p \rrbracket][\delta]$$

easiest case: just look them up in database

- ▶ but how to translate  $\delta$  to  $l$ -substitutions  $\gamma$  with

$$\vdash_l \gamma : \Gamma \quad \text{where} \quad \llbracket \Gamma \rrbracket \vdash_L \text{True}[\llbracket p[\gamma] \rrbracket]$$

substitution theorem: pick such that  $\llbracket \gamma \rrbracket = \delta$

the more  $\llbracket - \rrbracket$  does, the harder to invert

In practice, however:

- ▶ often only interested in concrete substitutions
- ▶ translation of concrete data usually identity

But: practice restricted to what works even if more is needed

# Computational Queries

## Definition

We assume

- ▶ the same as for deductive queries
- ▶ semantics has equality/equivalence  $\doteq$

We define

- ▶ a computational query is an  $l$ -expression  $e$
- ▶ the result is an  $l$ -expression  $e'$  so that  $\vdash_L \llbracket e \rrbracket \doteq \llbracket e' \rrbracket$

intuition:  $e'$  is the result of evaluating  $e$

If semantics is compositional,  $e$  may contain free variables

evaluate to themselves

## Problem: Back-Translation of Results

In general,  $\llbracket - \rrbracket$  may be non-trivial to invert

- ▶ easy to obtain  $E := \llbracket e \rrbracket$
- ▶ possible to find  $E'$  with  $\vdash_L E' \doteq E$  by working in the semantics
- ▶ non-obvious how to obtain  $e'$  such that  $\llbracket e' \rrbracket = E'$

In practice, however:

- ▶ evaluation meant to simplify, i.e., only useful if  $E'$  very simple
- ▶ simple  $E'$  usually in the image of  $\llbracket - \rrbracket$
- ▶ typical case:  $E'$  is concrete data and  $e' = E'$       called a value

## Problem: Non-Termination

In general, computation of  $E'$  from  $E$  might not terminate

- ▶ while-loops
- ▶ recursion
- ▶  $(\lambda x.x x) (\lambda x.x x)$  with  $\beta$ -rule
- ▶ simplification rule  $x \cdot y \rightsquigarrow y \cdot x$

similar: distributivity, associativity

In practice, however:

- ▶ image of  $\llbracket - \rrbracket$  part of terminating fragment

But: if  $I$  is Turing-complete or undecidable, general termination not possible

## Problem: Lack of Confluence

In general, there may be multiple  $E'$  that are simpler than  $E$

- ▶ there may be multiple rules that apply to  $E$
- ▶ e.g.,  $f(g(x))$ 
  - ▶ call-by-value: first simplify  $g(x) \rightsquigarrow y$ , then  $f(y) \rightsquigarrow z$
  - ▶ call-by-name: first plug  $g(x)$  into definition of  $f$ , then simplify
- ▶ Normal vs. canonical form
  - ▶ normal:  $\vdash_L E \doteq E'$
  - ▶ canonical: normal and  $\vdash_L E_1 \doteq E_2$  iff  $E'_1 = E'_2$ 
    - equivalent expressions have identical evaluation
    - allows deciding equality

In practice, however:

- ▶ image of  $\llbracket - \rrbracket$  part of confluent fragment
- ▶ typical: evaluation to a value is canonical form
  - works for BDL-types but not for, e.g., function types

# Narrative Queries

## Definition

We assume

- ▶ semantics into natural language

We define

- ▶ a narrative query is an  $L$ -question about some  $I$ -expressions
- ▶ the result is the answer to the question



## Problem: Unimplementable

very expressive = very difficult to implement

- ▶ Natural language understanding
  - ▶ no implementable syntax of natural language  
needs restriction to controlled natural language
  - ▶ specifying semantics hard even when controlled
- ▶ Knowledge base for question answering needed
  - ▶ very large  
must include all common sense
  - ▶ might be inconsistent  
common sense often is
  - ▶ finding answers still very hard

In practice, however:

- ▶ accept unreliability  
attach probability measures to answers
- ▶ implement special cases  
e.g., lookup in databases like Wikidata
- ▶ search knowledge base for related statements  
Google, Watson

## Syntactic Querying

# Search

- ▶ “search” not systematically separated from “querying”
  - ▶ often interchangeable
  - ▶ querying tends to imply formal languages for queries with well-specified semantics e.g., SQL
  - ▶ search tends to imply less targeted process e.g., Google
- we will not distinguish between the two

# Syntactic vs. Semantic Querying

## Semantic querying

- ▶ Query results specified by vocabulary  $V$  but (usually) not contained in it
- ▶ Query answered using semantics of language
- ▶ Challenge: apply semantics to find results
  - ▶ deductive query  $\vdash f : \text{prop}$  requires theorem prover
  - ▶ computation query  $\vdash e : E$  requires evaluator
  - ▶ concrete query  $\Gamma \vdash f : \text{prop}$  requires enumerating all substitutions, running theorem prover/evaluator on all of them

what we've looked at so far

## Syntactic querying

- ▶ Query is an expression  $e$
- ▶ Result is set of occurrences of  $e$  in  $V$
- ▶ Independent of semantics
- ▶ Much easier to realize

## Challenges for Syntactic Search

Easier to realize → scale until new challenges arise

- ▶ large vocabularies
  - ▶ narrative: all text documents in a domain  
e.g., all websites, all math papers
  - ▶ deductive: large repositories of formalization in proof assistants  
10<sup>5</sup> theorems
  - ▶ computational: package managers for all programming languages
  - ▶ concrete: all databases in a domain  
TBs routine
- ▶ incremental indexing: reindex only new/changed parts
- ▶ incremental search to handle large result sets  
pagination
- ▶ sophisticated techniques for
  - ▶ indexing: to allow for fast retrieval
  - ▶ similarity: to select likely results
  - ▶ quality: to rank selected results
- ▶ integration of some semantic parts

# Overview

- ▶ Deduction
  - ▶ semantic: theorem proving called search
  - ▶ syntactic: text search
- ▶ Concretization
  - ▶ semantic: complex query languages (nestable queries)  
SQL, SPARQL
  - ▶ syntactic: search by identifier (linked data)
- ▶ Computation
  - ▶ semantic: interpreters called execution
  - ▶ mixed: IDEs search for occurrences, dependencies
  - ▶ syntactic: search in IDE, package manager
- ▶ Narration:
  - ▶ semantic: very difficult
  - ▶ syntactic: bag of words search

## Abstract Definition: Document

### **Document** =

- ▶ file or similar resource that contains vocabularies
- ▶ often with comments, metadata
- ▶ different names per aspect
  - ▶ deduction: formalization, theory, article
  - ▶ computation: source files
  - ▶ concretization: database, ontology ABox
  - ▶ narrative: document, web site

### **Library** =

- ▶ collection of documents
- ▶ usually structured into folders, files or similar
- ▶ often grouped by user access e.g., git repository
- ▶ vocabularies interrelated within and across libraries

## Abstract Definition: Document Fragment

**Fragment** = subdivision of documents into nested semantic units

Examples

- ▶ deductive: theory, section, theorem, definition, proof step, etc.
- ▶ computational: class, function, command, etc.
- ▶ concrete: table, row, cell
- ▶ narrative: section, paragraph, etc.

Assign unique **fragment URI**, e.g., LIB/DOC?FRAG where

- ▶ LIB: base URI of library e.g., repository URL
- ▶ DOC: path to document within library e.g., folder structure, file name
- ▶ FRAG: id of fragment within document e.g., class name/method name



## Abstract Definition: Index(er)

**Indexer** consists of

- ▶ data structure  $O$  for indexable objects  
specific to aspect, index design  
e.g., words, syntax trees
- ▶ function that maps library to index  
the indexing

**Index entry** consists of

- ▶ object that occurred in the library
- ▶ URI of the containing fragment
- ▶ information on where in the fragment it was found

**Index** = set of index entries

## Abstract Definition: Query and Result

Given

- ▶ indexer  $I$  with data structure  $O$
- ▶ set of libraries
- ▶ union of their indexes computed once, queried often

**Query** = object  $\Gamma \vdash^I q : O$

**Result** consists of

- ▶ index entry with object  $o$
- ▶ substitution for  $\Gamma$  such that  $q$  matches  $o$   
definition of “match” index-specific, e.g.,  $q[\gamma] = o$

**Result set** = set of all results in the index

## Bag of Words Search

### Definition:

- ▶ Index data structure = sequences of words (n-grams) up to a certain length
- ▶ Query = bag of words bag = multiset
- ▶ Match: (most) words in query occur in same n-gram or n-grams near each other

### Example implementations

- ▶ internet search engines for websites
- ▶ Elasticsearch: open source engine for custom vocabularies

### Mostly used for narrative documents

- ▶ can treat concrete values as words e.g., numbers
- ▶ could treat other expressions as words works badly

# Symbolic Search

## Definition:

- ▶ Index data structure = syntax tree (of any grammar) of expressions  $o$  with free/bound variables
- ▶ Query = expression  $q$  with free (meta-)variables
- ▶ Match:  $q[\gamma] =_{\alpha} o$ , i.e., up to variable renaming

## Example implementation

- ▶ MathWebSearch  
see separate slides on MathWebSearch in the repository

## Mostly used for formal documents

- ▶ deductive
- ▶ computational

# Knowledge Graph Search

## Definition:

- ▶ Index data structure = assertion forming node/edge in a knowledge graph
- ▶ Index = big knowledge graph  $G$
- ▶ Query = knowledge graph  $g$  with free variables
- ▶ Match:  $g[\gamma]$  is part of  $G$

## Example implementations

- ▶ SPARQL engines without consequence closure  
i.e., the most common case in practice
- ▶ graph databases

Mainly used for ABoxes of untyped ontologies

## Value Search

Definition:

- ▶ Index data structure = BDL values  $v$
- ▶ Query = BDL expression  $q$  with free variables
- ▶ Match:  $q[\gamma] = v$

Example implementations

- ▶ no systematic implementation yet
- ▶ special cases part of most database systems

Could be used for values occurring in any document

- ▶ all aspects
- ▶ may need to decode/encode before putting in index

## Cross-Aspect Occurrences

### Observation

- ▶ libraries are written in one primary aspect
- ▶ indexer focuses on one aspect and kind of object
- ▶ but documents may contain indexable objects of any index

## Cross-Aspect Occurrences: Examples

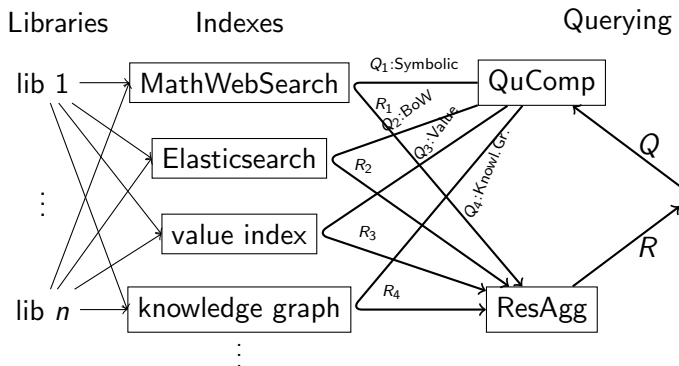
- ▶ Any library can contain
  - ▶ metadata on fragments
    - ▶ relation assertions induce knowledge graph structure between fragments
    - ▶ property assertions contain values narrative, symbolic objects, or values
  - ▶ cross-references to fragments of any other library
  - ▶ narrative comments
- ▶ Narrative text may contain symbolic expressions  
STEM documents
- ▶ Database table may have columns containing
  - ▶ text
  - ▶ encoded BDL values
  - ▶ symbolic expression (often as strings)
- ▶ Symbolic fragments may contain database references  
e.g., when using database for persistent memoization



## A New Indexing Design

recent paper [https://kwarc.info/people/frabe/Research/BKR\\_mdql\\_20.pdf](https://kwarc.info/people/frabe/Research/BKR_mdql_20.pdf)

with K. Bercic



## A New Indexing Design (2)

Tricky question: What is the query language that allows combining queries for each index?

Easy:

- ▶ query = conjunction of atomic queries
- ▶ each atom queries one index
- ▶ QuComp splits into atoms
- ▶ ResAgg take intersection of results

Better: allow variables to be shared across atoms

open research question

## A New Indexing Design: Example

Consider

- ▶ table of graphs with human-recognizable names and arc-transitivity property indexed into
  - ▶ value index for the graph sparse6 codec
  - ▶ Boolean computed property for the arc-transitivity in knowledge graph
  - ▶ text index for name
- ▶ papers from arXiv in narrative index indexed into
  - ▶ narrative index for text
  - ▶ MathWebSearch for formulas
  - ▶ knowledge graph for metadata

Query goal: find arc-transitive graphs mentioned by name in articles with h-index greater than 50

# Integrating Semantic Querying

## Word search

- ▶ find multi-meaning words for only one meaning  
“normal” in math
- ▶ special treatment of certain queries e.g., “weather” in Google

## Symbolic search

- ▶ match query  $e \doteq e'$  against occurrence  $e' \doteq e$
- ▶ similarly: associativity, commutativity, etc.
- ▶ slippery slope to deductive queries

## Value search

- ▶ match query 1.5 against interval  $1.4 \pm 0.2$
- ▶ match query  $5 \cdot x$  against 25
- ▶ slippery slope to computational queries

frontiers of research — in our group: for STEM documents

# Kinds of Semantics

# Recall

Recall:

Syntax	Data
Semantics	Knowledge

Representing

- ▶ syntax = formal language
  - ▶ grammar context-free part
  - ▶ type system context-sensitive well-formedness
- ▶ data = words in the syntax
  - ▶ set of vocabularies
  - ▶ set of typed expressions for each vocabulary
- ▶ semantics = ???
- ▶ knowledge = emergent property of having well-formed words with semantics

## Relative Semantics by Translation

Components:

- ▶ Two syntaxes
  - ▶ object-language  $I$  e.g., BOL
  - ▶ meta-language  $L$  e.g., SFOL, Scala, SQL, English
- ▶ Semantics of  $L$  assumed fixed captures what we already know
- ▶ Semantics of  $I$  by translation into  $L$   
semantics of  $I$  relative to existing semantics of  $L$

Problem: just kicking the can?

# Discussion of Semantics by Translation

## Advantages

- ▶ a few meta-languages yield semantics for many languages
- ▶ easy to develop new languages
- ▶ good connection between syntax and semantics via compositionality, substitution theorem

## Disadvantages

- ▶ does not solve the problem once and for all
- ▶ impractical without implementation of semantics of meta-language
- ▶ meta-languages typically much more expressive than needed for object-languages
- ▶ translations can be difficult, error-prone

Also needed: absolute semantics



## Absolute vs. Relative Semantics

Absolute = self-contained, no use of meta-language  $L$

### Get off the ground

- ▶ semantics for a few important meta-languages  
e.g., FOL, assembly language, set theory
- ▶ relative semantics for all other languages, e.g.,
  - ▶ model theory: logic  $\rightarrow$  set theory
  - ▶ compilation: Scala  $\rightarrow$  JVM  $\rightarrow$  assembly

### Redundant semantics

- ▶ common to give
  - ▶ relative and absolute semantics for same syntax
  - ▶ multiple relative semantics      translations to different aspects
  - ▶ sometimes even maybe multiple absolute ones
- ▶ Allows understanding syntax from multiple perspectives
- ▶ Allows cross-checking      show equivalence of two semantics

## Example: Recall Syntax of Arithmetic Language

Syntax: represented as formal grammar

### Numbers

$N ::= 0$		literals
$N + N$		sum
$N * N$		product

### Formulas

$F ::= N \doteq N$	equality
$N \leq N$	ordering by size

Implementation as inductive data type

## Example: Absolute Semantics

Represented as judgments defined by sets of rules

- unclear what judgments to use
- here: computation  $\vdash N \rightsquigarrow N$  and truth  $\vdash F$

For numbers  $n$ : Rules to normalize numbers into values

$$\overline{\vdash N + 0 \rightsquigarrow N} \quad \overline{\vdash N * 0 \rightsquigarrow 0} \quad \overline{\vdash N * 1 \rightsquigarrow N}$$

$$\overline{\vdash N * (R + S) \rightsquigarrow N * R + N * S}$$

and their commutative variants as well as

$$\overline{\vdash L + (M + N) \rightsquigarrow (L + M) + N}$$

For formulas  $f$ : rules to determine true formulas

$$\overline{\vdash N \doteq N} \quad \overline{\vdash 0 \leq N} \quad \overline{\vdash L + N \leq M + N} \quad \overline{\vdash L \leq M}$$

## Example: Absolute Semantics (2)

Checking if an absolute semantics works as intended is hard.

Here: number rules allow

1. eliminating all cases where arguments of  $*$  are 0, 1, or  $+$ ; thus, no more  $*$
2. eliminating all cases where arguments of  $+$  are 0
3. shift brackets of nested  $+$  to the left
4. left: 0 or  $(\dots(1+1)\dots+1)$  — isomorphic to natural numbers

formula rules allow

1. concluding equality if identical normal forms
2. reducing  $M + 1 \leq N + 1$  to  $M \leq N$ , repeat until  $0 \leq N$

## Example: Relative Semantics

Semantics: represented as translation into known language

Problem: Need to choose a known language first

Here: unary numbers represented as strings

Built-in data (strings and booleans):

$S ::= ""$	empty
(Unicode)	character sequence
$B ::= \text{true}$	truth
false	falsity

Built-in operations to work on the data:

- ▶ concatenation of strings  $S ::= \text{conc}(S, S)$
- ▶ replacing all occurrences of  $c$  in  $S_1$  with  $S_2$   
 $S ::= \text{replace}(S_1, c, S_2)$
- ▶ equality test:  $B ::= S_1 == S_2$
- ▶ prefix test:  $B ::= \text{startsWith}(S_1, S_2)$

## Example: Relative Semantics

Represented as function from syntax to semantics

- ▶ mutually recursive, inductive functions for each non-terminal symbol
- ▶ compositional: recursive call on immediate subterms of argument

For numbers  $n$ : semantics  $\llbracket n \rrbracket$  is a string

- ▶  $\llbracket 0 \rrbracket = ""$
- ▶  $\llbracket 1 \rrbracket = "|"$
- ▶  $\llbracket m + n \rrbracket = \text{conc}(\llbracket m \rrbracket, \llbracket n \rrbracket)$
- ▶  $\llbracket m * n \rrbracket = \text{replace}(\llbracket m \rrbracket, "|", \llbracket n \rrbracket)$

For formulas  $f$ : semantics  $\llbracket f \rrbracket$  is a boolean

- ▶  $\llbracket m \dot{=} n \rrbracket = \llbracket m \rrbracket == \llbracket n \rrbracket$
- ▶  $\llbracket m \leq n \rrbracket = \text{startsWith}(\llbracket n \rrbracket, \llbracket m \rrbracket)$

## Example: Equivalence of Semantics

### For formulas

- ▶ if  $\vdash F$ , then  $\llbracket F \rrbracket = \text{true}$
- ▶ if  $\llbracket F \rrbracket = \text{true}$ , then  $\vdash F$

usually called **soundness**

usually called **completeness**

### For numbers

- ▶  $\vdash N \rightsquigarrow 0$  iff  $\llbracket N \rrbracket = ""$
- ▶  $\vdash N \rightsquigarrow (\dots(1 + 1) \dots + 1)$  iff  $\llbracket N \rrbracket = " | \dots | "$

## Semantics as Imperfect Modeling

- ▶ Actual meaning of real-world difficult to model
  - ▶ practical argument: any practically interesting system has too many rules
    - cf. physics, e.g., three-body problem already chaotic
  - ▶ theoretical argument: no language can fully model itself
    - cf. Gödel's incompleteness theorems
- ▶ Practical semantics is approximation of ideal semantics
- ▶ Use practical purpose as guide for defining semantics
  - ▶ a set of questions that can be asked e.g., judgments or queries
  - ▶ a definition of what the answers are
    - restrict questions according to practical needs



## Levels of Semantics

- ▶ Documentation: answers given as text
  - ▶ pro: easy to read for humans, critical to build intuitions
  - ▶ con: often ambiguous, contradictory, or incomplete
- ▶ Specification: correct answers defined by rule system  
also called calculus or inference system
  - ▶ pro: good stepping stone between the other two levels
  - ▶ con: accomplishes the pros of neither of them
- ▶ Implementation: answers computed by algorithm
  - ▶ pro: easy to automate, critical for efficiency and scale
  - ▶ con: essentially impossible to understand or analyze
- ▶ Unit testing: set of query/answer pairs
  - ▶ pro: easy to write, automate
  - ▶ con: does not cover the whole semantics

Rule system is sweet spot to connect human- and machine-friendly definitions.

# Relative Semantics for BOL

## Semantics of BOL

Aspect	kind of semantic language	semantic language
deduction	logic	SFOL
concretization	database language	SQL
computation	programming language	Scala
narration	natural language	English

## Deductive Semantics of BOL in SFOL

We discuss

- ▶ the grammar of SFOL
- ▶ context-sensitive languages with variable binding (of which SFOL is an example)
- ▶ an implementation of SFOL in Scala
- ▶ the translation from BOL to SFOL
- ▶ compositionality of the translation
- ▶ the issue of
  - ▶ non-compositionality
  - ▶ the need for a semantic prefix

see details in the lecture notes

## Exercise 7

Extend your implementation of BOL with implementations of SFOL and the translation from BOL to SFOL.

## Exercise 8

Use an existing absolute SFOL semantics for BOL. Concretely this semantics is given in the form of theorem provers like Vampire, E, or Spass. These use TPTP as the input format.

So extend your implementation of SFOL with a printer to TPTP format. Compose your translation with that printer to translate your example ontology and an example query (a formula to be proved) to a theorem prover.

See <https://www.tptp.org/> for the TPTP resources, including the grammar and a web interface to theorem provers. The basic syntax is

```
fof(id , type , name : the type ).
fof(id , axiom , the formula ).
fof(id , conjecture , the formula ).
```

where formulas are formed using  $![X:TYPE]:F$ ,  $?[X:TYPE]:F$ ,  $F \& F$ ,  $F|F$ ,  $F \Rightarrow F$ ,  $\sim F$

## Concrete Data Semantics of BOL in SQL

We discuss

- ▶ the grammar of an SQL-like language
- ▶ the translation from BOL to it
- ▶ the limitations of the translation
  - ▶ partiality, e.g., for concept assertions where the concept is not atomic
  - ▶ lack of formal axioms in SQL

see details in the lecture notes

# Computational Semantics of BOL in Scala

We discuss

- ▶ the grammar of a Scala-like language
- ▶ the translation from BOL to it
- ▶ the limitations of the translation

see details in the lecture notes



## Concrete Data Semantics of BOL in English

We discuss

- ▶ the rough design of how natural languages can be seen as a formal systems
  - ▶ grammar book = syntax
  - ▶ sentences = formulas
  - ▶ dictionary + common sense statements = standard library
  - ▶ domain-specific dictionary + sentences = vocabulary
- , ,
- ▶ the translation from BOL to it
- ▶ the non-compositional aspects of natural language

see details in the lecture notes

## Relative Semantics by Translation

## General Definition

A semantics by translation consists of

- ▶ syntax: a formal system  $I$
- ▶ semantic language: a formal system  $L$   
different or same aspect as  $I$
- ▶ semantic prefix: a vocabulary  $P$  in  $L$   
formalizes fundamentals that are needed to represent  $I$ -objects
- ▶ interpretation: translates every  $I$ -vocabulary  $T$  to an  $L$ -vocabulary  $P, \llbracket T \rrbracket$

## Common Principles

Properties shared by all semantics by translation

not part of formal definition, but best practices

- ▶  $I$ -declaration translated to  $L$ -declaration for the same name
- ▶ vocabularies translated declaration-wise
- ▶ one inductive function for every kind of complex  $I$ -expression
  - ▶ individuals, concepts, relations, properties, formulas
  - ▶ maps  $I$ -expressions to  $L$ -expressions
- ▶ atomic cases (base cases):  $I$ -identifier translated to  $L$ -identifier of the same name or something very similar
- ▶ complex cases (step cases): compositional

## Compositionality

Case for operator  $*$  in translation function compositional iff interpretation of  $*(e_1, \dots, e_n)$  only depends on the interpretation of the  $e_i$

$$\llbracket *(e_1, \dots, e_n) \rrbracket = \llbracket * \rrbracket (\llbracket e_1 \rrbracket, \dots, \llbracket e_n \rrbracket)$$

for some function  $\llbracket * \rrbracket$

Example:  $;$ -operator of BOL in translation to FOL

- ▶ translation:  $\llbracket R_1; R_2 \rrbracket = \exists m : \iota. \llbracket R_1 \rrbracket(x, m) \wedge \llbracket R_2 \rrbracket(m, y)$
- ▶ special case of the above via
  - ▶  $* = ;$ ;
  - ▶  $n = 2$
  - ▶  $\llbracket ; \rrbracket = (p_1, p_2) \mapsto \exists m : \iota. p_1(x, m) \wedge p_2(m, y)$
- ▶ Indeed, we have  $\llbracket R_1; R_2 \rrbracket = \llbracket ; \rrbracket (\llbracket R_1 \rrbracket, \llbracket R_2 \rrbracket)$

## Compositionality (2)

Translation compositional iff

- ▶ one translation function for each non-terminal all written  $\llbracket - \rrbracket$
- ▶ each defined by one induction on syntax
  - i.e., one case for production
  - mutually recursive
- ▶ all cases compositional

Substitution theorem: a compositional translation satisfies

$$\llbracket E(e_1, \dots, e_n) \rrbracket = \llbracket E \rrbracket(\llbracket e_1 \rrbracket, \dots, \llbracket e_n \rrbracket)$$

for

- ▶ every expression  $E(N_1, \dots, N_n)$  with non-terminals  $N_i$
- ▶ some function  $\llbracket E \rrbracket$  that only depends on  $E$

## Compositionality (3)

$$\llbracket E(e_1, \dots, e_n) \rrbracket = \llbracket E \rrbracket(\llbracket e_1 \rrbracket, \dots, \llbracket e_n \rrbracket)$$

for every expression  $E(N_1, \dots, N_n)$  with non-terminals  $N_i$

Now think of

- ▶ variable  $x_i$  of type  $N_i$  instead of non-terminal  $N_i$
- ▶  $E(x_1, \dots, x_n)$  as expression with free variables  $x_i$  of type  $N_i$
- ▶ expressions  $e$  derived from  $N$  as expressions of type  $N$
- ▶  $E(e_1, \dots, e_n)$  as result of substituting  $e_i$  for  $x_i$
- ▶  $\llbracket E \rrbracket(x_1, \dots, x_n)$  as (semantic) expression with free variables  $x_i$

Then both sides of equations act on  $E(x_1, \dots, x_n)$ :

- ▶ left side yields  $\llbracket E(e_1, \dots, e_n) \rrbracket$  by
  - ▶ first substitution  $e_i$  for  $x_i$
  - ▶ then semantics  $\llbracket - \rrbracket$  of the whole
- ▶ right side yields  $\llbracket E \rrbracket(\llbracket e_1 \rrbracket, \dots, \llbracket e_n \rrbracket)$  by
  - ▶ first semantics  $\llbracket - \rrbracket$  of all parts
  - ▶ then substitution  $\llbracket e_i \rrbracket$  for  $x_i$

semantics commutes with substitution

# Non-Compositionality

## Examples

- ▶ deduction: cut elimination, translation from natural deduction to Hilbert calculus
- ▶ computation: optimizing compiler, e.g., loop unrolling
- ▶ concretization: query optimization, e.g., turning a WHERE of a join into a join of WHEREs,
- ▶ narration: ambiguous words are translated based on context

## Typical sources

- ▶ subcases in a case of translation function
  - ▶ based on inspecting the arguments, e.g., subinduction
  - ▶ based on context
- ▶ custom-built semantic prefix



# Translation vs. Embedding

## Translation

- ▶ as above,  $I$  and  $L$  are at the same level
- ▶  $I$ -declarations represented as  $L$ -declarations

also called shallow embedding

## Embedding

- ▶  $L$  is used as meta-language to represent  $I$   
e.g.,  $L$  is programming language to implement  $I$
- ▶  $I$ -declarations represented as  $L$ -objects using an inductive type  
also called deep embedding

## Exercise 9

Choose a parser generator for your programming language. Then extend your implementation with generated parser for BOL. Write your example ontology in BOL and parse it into BOL.

# Situational Relative Semantics

# Situational vs. Generic Relative Semantics

## Generic relative semantics

- ▶ vocabularies mapped to vocabularies, expressions to expressions
- ▶ abstract symbols in input vocabulary yield abstract symbols in output vocabulary
- ▶ examples:
  - ▶ all BOL translations so far
  - ▶ compiling a programming language to another language

## Situational relative semantics

- ▶ uninterpreted symbols in vocabulary given concrete semantics
- ▶ situation = maps every symbols to its concrete semantics
- ▶ expressions interpreted relative to fixed situation
- ▶ examples
  - ▶ interpreting a program  
situation = input+run-time environment
  - ▶ interpreting logical formulas  
situation = model

# Interpreted vs. Uninterpreted Symbols

Interpreted = symbols with fixed intended semantics

- ▶ base types and their operations integer, etc.
- ▶ concrete data types enumerations, inductive types
- ▶ defined symbols i.e., abbreviations

Uninterpreted = other symbols

- ▶ e.g,  $a : \text{type}$ ,  $f : a \rightarrow a$ , etc.
- ▶ semantics narrowed down by axioms possibly underspecified

## Approaches to Uninterpreted Symbols

### Broad spectrum

- ▶ Computation: typically only interpreted symbols  
programs can be run directly
- ▶ Deduction: typically only uninterpreted symbols  
focus on studying the possibly choices of interpretations

### Combinations

- ▶ reasoning about programs  
e.g., functions with pre-/postconditions
- ▶ logics with built-in base types

# Vocabulary vs. Situation

## Vocabulary: rules of the world

- ▶ type  $a$
- ▶ operation  $f : a \rightarrow a$
- ▶ relations  $r : a \rightarrow \text{prop}$
- ▶ axioms  $F$  about  $f, r$

## Situation: one concrete world $S$

- ▶ specific set  $S_a$
- ▶ specific function  $S_f$  from  $S_a$  to  $S_a$
- ▶ specific subset  $S_r$  of  $S_a$
- ▶ proof that  $F$  holds about  $S_f, S_r$

# Situations

## Structure

- ▶ provides interpretation to every abstract symbol in vocabulary
- ▶ abstract symbols in input vocabulary yield abstract symbols in output vocabulary
- ▶ examples:
  - ▶ all BOL translations so far
  - ▶ compiling a programming language to another language

Aspect	Abstract Vocabulary	Situation
ontology	TBox	initial situation through ABox
data	schema	database
deduction	theory	model
computation	program	environment
narration	dictionary	domain-specific definitions



## Initial Semantics

Default situation that is induced by vocabulary itself

- ▶ inhabitants of a type are exactly the terms of that type
- ▶ functions map exactly as given by axioms

Special case: ABox

- ▶ individuals exactly the ones that are declared
- ▶ concepts, relations hold exactly if assertion declared

# Absolute Semantics for BOL

## Judgments

Typing:

$$\Gamma \vdash_V^{BOL} e : E$$

Deduction:

$$\Gamma \vdash_V^{BOL} F$$

Propositions prop:

- ▶  $C \sqsubseteq D, C \equiv D$
- ▶ all three kinds of assertions

Notation:

- ▶ We drop the superscript  $BOL$  everywhere.
- ▶ We drop the subscript  $v$  unless we need to use  $V$ .
- ▶ We drop the context  $\Gamma$  unless we need to use/change  $\Gamma$ .

# Typing

Trivial intrinsic typing (Church)  $\vdash e :^{int} E$

- ▶  $E$  is a non-terminal
- ▶  $e$  an expression derived from  $E$

Refined by extrinsic typing (Curry)  $\vdash e :^{ext} E$

- ▶  $e$  is an individual, i.e.,  $\vdash e :^{int} I$
- ▶  $E$  is a concept, i.e.,  $\vdash E :^{int} C$   
where  $I$  and  $C$  are the non-terminals from the grammar
- ▶  $e$  has concept  $E$ , i.e.,  $\vdash e \text{ is-a } E$

## Propositions as Types

Say also  $\vdash p : f$  for proofs  $p$  of proposition  $f$   
in particular:  $x : f$  in contexts to make local assumptions

Notation:

$\Gamma, f$  instead of  $\Gamma, p : f$

sufficient if we only state the rules, not build proofs

## Lookup Rules

The main rules that need to access the vocabulary:

$$\frac{f \text{ in } V}{\vdash_V f}$$

for assertions or axioms  $f$

Assumptions in the context are looked up accordingly:

$$\frac{x : f \text{ in } \Gamma}{\Gamma \vdash f}$$

## Rules for Subsumption and Equality

Subsumption is an order with respect to equality:

$$\overline{\vdash c \sqsubseteq c}$$

$$\frac{\vdash c \sqsubseteq d \quad \vdash d \sqsubseteq e}{\vdash c \sqsubseteq e}$$

$$\frac{\vdash c \sqsubseteq d \quad \vdash d \sqsubseteq c}{\vdash c \equiv d}$$

Equal concepts can be substituted for each other:

$$\frac{\vdash c \equiv d \quad x : C \vdash f(x) : \text{prop} \quad \vdash f(c)}{\vdash f(d)}$$

This completely defines equality.

## Rules relating Instancehood and Subsumption

$$\frac{\vdash i \text{ is-a } c \quad \vdash c \sqsubseteq d}{\vdash i \text{ is-a } d}$$

Read:

- ▶ if
  - ▶  $i \text{ is-a } c$
  - ▶  $c \sqsubseteq d$
- ▶ then  $i \text{ is-a } d$

$$\frac{x : I, x \text{ is-a } c \vdash x \text{ is-a } d}{\vdash c \sqsubseteq d}$$

Read:

- ▶ if
  - ▶ assuming an individual  $x$  and  $x \text{ is-a } c$ , then  $x \text{ is-a } d$
- ▶ then  $c \sqsubseteq d$



## Induction

Consider from before

$$\frac{x : I, x \text{ is-a } c \vdash x \text{ is-a } d}{\vdash c \sqsubseteq d}$$

Question: Do we allow proving the hypothesis by checking for each individual  $x$ ? induction

## Induction

Consider from before

$$\frac{x : I, x \text{ is-a } c \vdash x \text{ is-a } d}{\vdash c \sqsubseteq d}$$

Question: Do we allow proving the hypothesis by checking for each individual  $x$ ? induction

- Open world: no

## Induction

Consider from before

$$\frac{x : I, x \text{ is-a } c \vdash x \text{ is-a } d}{\vdash c \sqsubseteq d}$$

Question: Do we allow proving the hypothesis by checking for each individual  $x$ ? induction

- ▶ Open world: no
- ▶ Closed world: yes

$$\frac{\Gamma[x = i] \vdash f[x = i] \text{ for every individual } i}{\Gamma, x : I \vdash f(x)}$$

effectively applicable if only finitely many individuals

## Rules for Union and Intersection of Concepts

Union as the least upper bound:

$$\begin{array}{c}
 \overline{\vdash c \sqsubseteq c \sqcup d} \qquad \overline{\vdash d \sqsubseteq c \sqcup d} \\
 \\
 \frac{\vdash c \sqsubseteq h \quad \vdash d \sqsubseteq h}{\vdash c \sqcup d \sqsubseteq h}
 \end{array}$$

Dually, intersection as the greatest lower bound:

$$\begin{array}{c}
 \overline{\vdash c \sqcap d \sqsubseteq c} \qquad \overline{\vdash c \sqcap d \sqsubseteq d} \\
 \\
 \frac{\vdash h \sqsubseteq c \quad \vdash h \sqsubseteq d}{\vdash h \sqsubseteq c \sqcap d}
 \end{array}$$

## Rules for Existential and Universal

Easy rules:

► Existential

$$\frac{\vdash irj \quad \vdash j \text{ is-a } c}{\vdash i \text{ is-a } \exists r.c}$$

► Universal

$$\frac{\vdash i \text{ is-a } \forall r.c \quad \vdash irj}{\vdash j \text{ is-a } c}$$

Other directions are trickier:

► Existential

$$\frac{\vdash i \text{ is-a } \exists r.c \quad j : I, irj, j \text{ is-a } c \vdash f}{\vdash f}$$

► Universal

$$\frac{j : I, irj \vdash j \text{ is-a } c}{\vdash i \text{ is-a } \forall r.c}$$

## Selected Rules for Relations

Inverse:

$$\frac{\vdash irj}{\vdash jr^{-1}i}$$

Composition:

$$\frac{\vdash irj \quad \vdash js k}{\vdash i(r;s)k}$$

Transitive closure:

$$\frac{}{\vdash ir^*i} \quad \frac{\vdash irj \quad \vdash jr^*k}{\vdash ir^*k}$$

Identity at concept  $c$ :

$$\frac{\vdash i \text{ is-a } c}{\vdash i \Delta_c i}$$

## Exercise 10

Implement an absolute semantics for BOL for the judgment  $\Gamma \vdash F$ . This is too difficult in general. But it becomes doable if we choose a sufficiently simple fragment of BOL, e.g., only

- ▶ formulas:  $I \text{ is } - aC, C \sqsubseteq C$
- ▶ concepts: identifiers,  $C \sqcup C, \exists R.C$
- ▶ relations: identifiers

## Exercise 11

Implement an absolute concrete semantics for BOL.



# Formal Systems

# Typical Structure of a Formal System

## Vocabularies

- ▶ vocabularies  $V$  = lists of declarations
- ▶ vocabulary morphisms  $m : V \rightarrow W$  = lists of definitions of  $V$ -identifiers in terms of  $W$ -expressions

## Declarations

- ▶ named
- ▶ at least one for each expression kind
- ▶ may contain other expressions e.g., type, definition
- ▶ may contain nested declarations e.g., fields in an ADT

## Expressions

- ▶ inductive data type
- ▶ relative to vocabulary names occur as base cases
- ▶ formulas as special case

Morphisms  $m$  map  $V$ -expressions to  $W$ -expressions (homomorphic extension)

## Example: Vocabularies and Expressions

Aspect	vocabulary $\Theta$	expression kinds
Ontologization	ontology	individual, concept, relation, property, formula
Concretization	database schema	cell, row, table, formula
Computation	program	term, type, object, class, ...
Logic	signature, theory	term, type, formula, ...
Narration	dictionary	phrases, sentences, texts

# Components and Well-Formedness

## Components of formal system /

- ▶ context-free syntax
- ▶ distinguished non-terminal symbols  $\mathcal{V}$  and  $\mathcal{M}$   
words called **vocabularies** and **morphisms**
- ▶ some distinguished non-terminal symbols    words called **expressions**
- ▶ unary predicate  $\text{wfv}(V)$  on vocabularies  $V$     well-formed vocabulary
- ▶ ternary predicate  $\text{wfm}_{V \rightarrow W}(m)$  on morphisms  $m$  from  $V$  to  $W$   
well-formed morphism
- ▶ unary predicates  $\text{wff}_V(e)$     well-formed expressions

## Intuition

- ▶ context-**free** syntax generates more than needed
- ▶ context-**sensitive** well-formedness defines the exact subset

Question: How do we define the well-formedness predicates?

use an inference system with context-sensitive rules

# Inference System

## Define well-formedness via type system

- ▶ contexts  $\Gamma$  of the form  $x_1 : E_1, \dots, x_n : E_n$  for expressions  $E_i$
- ▶ a set of judgments including
  - ▶ a judgment  $\vdash^I V$  on vocabularies  $V$   $\text{wfv}(V) \text{ iff } \vdash_V^I$
  - ▶ a judgment  $\Gamma \vdash_V^I e : E$  between expressions  $e, E$   $\text{wff}_V(e) \text{ iff } \vdash_V^I e : E \text{ for some } E$
- ▶ a set of rules for the judgments, each one of the form

$$\frac{J_1 \quad \dots \quad J_n}{J}$$

where the  $J$ 's are judgments

conventions: leave out superscript  $I$ , subscript  $V$  if clear  
 leave out  $\Gamma$  if empty

# Morphisms

	static	dynamic
introduce identifiers	vocabularies	morphisms
build context-free syntax	expressions	homomorphic extension
restrict context-sensitively	well-formed	well-formedness preservation

- ▶ Vocabulary  $V$  introduces list of identifiers (of various kinds)
- ▶  $V$ -expressions (of various kinds) are syntax trees using  $V$ -identifiers as leaves
- ▶ Well-formed  $V$  expressions satisfy additional context-sensitive typing conditions

Homomorphisms  $m : V \rightarrow W$

- ▶ map every  $V$ -identifier to a  $W$ -expression (of the same kind)
- ▶ map every  $V$ -expression to a  $W$ -expression (of the same kind) by homomorphic extensions
- ▶ preserve well-formedness

## Terminology

For an inference system, we define

- ▶ derivation: tree of judgments such that for every node  $J$  with children  $J_1, \dots, J_n$ , there is a rule

$$\frac{J_1 \quad \dots \quad J_n}{J}$$

- ▶ derivation of  $J$ : a derivation with root  $J$
- ▶  $J$  holds: there is a derivation of  $J$

$$\text{Voc}^I = \{V \mid \vdash^I V\}$$

sometimes written  $|I|$

$$\text{Mor}^I(V, W) = \{m \mid \vdash^I m : V \rightarrow W\}$$

sometimes written  $I(V, W)$

$$\text{Exp}_V^I(E) = \{e \mid \vdash_V^I e : E\} \quad \text{Exp}_V^I = \bigcup_E \text{Exp}_V^I(E)$$

$\text{Exp}_m^I(e) : \text{Exp}_V^I \rightarrow \text{Exp}_W^I$ , defined by homomorphic extension

## Special Cases

### A formal system with propositions

- ▶ additionally has a distinguished expression `prop`
- ▶ define  $F$  is proposition if  $\vdash_V F : \text{prop}$

### A formal system with equality

- ▶ additionally has a distinguished proposition  $e_1 \doteq_E e_2$  whenever  $\vdash e_i : E$

in the sequel: fix / as above



# Deductive Semantics

# Deductive Semantics

## Definition

- ▶ a system that determines which propositions are theorems
- ▶ for every  $V$ , a subset  $\text{Thm}'_V \subseteq \text{Exp}'_V(\text{prop})$  of theorems  
write  $\vdash'_V F$  for  $F \in \text{Thm}'_V$

## Terminology

- ▶ Logic: language plus deductive semantics
- ▶ Calculus: set of rules defining absolute deductive semantics
- ▶ Theorem prover: implementation of deductive semantics
- ▶ Decision procedure: special case of theorem prover when decidable

## Examples

- ▶ Natural deduction for first-order logic
- ▶ Axiomatic set theory for (most of) mathematics

# Redundant Deductive Semantics

## Multiple deductive semantics

- ▶ Proof theory: absolute
- ▶ Model theory: relative via translation to set theory  $L$   
write  $\models F$  for  $\vdash_L \text{True}[[F]]$
- ▶ Logic translation: relative via translation into standard logics, e.g., SFOL

## Equivalence Theorems

- ▶ Soundness:  $\vdash F$  implies  $\models F$
- ▶ Completeness:  $\models F$  implies  $\vdash F$     accordingly for other translations

## Relationship to Typing

Define deductive semantics as a special case of typing

- ▶ propositions as types
- ▶ proofs as expressions
- ▶ extend grammar so that there are expressions for proofs
- ▶ add typing rules such that  $\vdash P : F$  captures the statement “ $P$  is proof of  $F$ ”
- ▶ define:  $\vdash F$  iff there is  $P$  such that  $\vdash P : F$

often called Curry-Howard representation

# Computational Semantics

# Computational Semantics

## Definition

- ▶ determines how expressions evaluate to values
- ▶ for every  $V$ , a function  $\text{Eval}'_V : \text{Exp}'_V \rightarrow \text{Exp}'_V$   
write  $\vdash'_V e \rightsquigarrow e'$  for  $e' = \text{Eval}'_V(e)$

## Terminology

- ▶ Programming language: languages plus computational semantics
- ▶ Operational semantics: rules defining computational semantics
- ▶ Interpreter: implementation of absolute semantics
- ▶ Compiler: implementation of relative semantics

## Examples

- ▶ Any interpreted language Python, bash, ...
- ▶ Machine language interpretation rules built into microchips

## Caveat

Evaluation  $\vdash_V^I e \rightsquigarrow e'$  insufficient in general

special case of interpreter when evaluation pure and terminating

Actual programming languages more complex

- ▶ IO channels
- ▶ Object creation/destruction
- ▶ Mutable variables
- ▶ Non-termination (needed for Turing completeness)

Semantics requires environment, heap, stack, references, ...

# Redundant Computational Semantics

## Multiple computational semantics

- ▶ Specification: absolute as rules on paper
- ▶ Interpreter: absolute as implementation
- ▶ Compiler: relative via translation to assembly  $L$

write  $\models E \rightsquigarrow V$  for  $\vdash_L \llbracket E \rrbracket \rightsquigarrow \llbracket V \rrbracket$

- ▶ Cross-compilation: relative via translation into other languages

Church-Turing thesis: always possible

## Equivalence Theorems

- ▶ Correctness of compiler:  $\vdash E \rightsquigarrow V$  iff  $\models E \rightsquigarrow V$

accordingly for other translations



## Relationships to other judgments

### Big Step vs. Small Step

- ▶ big step:  $\vdash_V^I e \rightsquigarrow e'$  is the entire evaluation
- ▶ small step:  $\vdash_V^I e \rightsquigarrow e'$  is just one step and semantics requires exhaustive chaining of steps

### Typing

- ▶ subject reduction: if  $\vdash e : E$ , then  $\vdash \text{Eval}(e) : E$

### Deductive semantics with equality

- ▶ normal forms:
  - ▶  $\text{Eval}_V^I$  idempotent, i.e.,  $\text{Eval}_V^I(x) = x$  if  $x$  value
  - ▶  $\vdash_V^I e \doteq_E \text{Eval}_V^I(e)$
- ▶ canonical forms:  $\vdash_V^I e_1 \doteq_E e_2$  iff  $\text{Eval}_V^I(e_1) = \text{Eval}_V^I(e_2)$

# Interdefinability

Given a computational semantics, define a deductive one:

- ▶ distinguished expression  $\vdash \text{true} : \text{prop}$ ,
- ▶  $\vdash F$  iff  $\text{Eval}(F) = \text{true}$   
implies decidability, so usually only possible for some  $F$

Given a deductive semantics, define computational one:

- ▶  $\text{Eval}(e)$  is some  $e'$  such that  $\vdash e \doteq e'$   
trivially normal, but usually not canonical

Both kinds of semantics add different value. We usually want both.

# Contexts and Substitutions

## Syntax with Contexts

If we want to talk about contexts, too, we need to expand all of the above.

### Syntax with contexts

- ▶ contexts: for every  $V$ , a set  $\text{Cont}'_V$  write  $\vdash_V \Gamma$
- ▶ substitutions: for  $\Gamma, \Delta \in \text{Cont}_V$ , a set  $\text{Subs}_V(\Gamma, \Delta)$   
write  $\vdash_V \gamma : \Gamma \rightarrow \Delta$

### Expressions in context

- ▶ expressions: sets  $\text{Exp}_V(\Gamma)$
- ▶ substitution application: functions  $\text{Exp}(\gamma) : \text{Exp}(\Gamma) \rightarrow \text{Exp}(\Delta)$  for  $\gamma \in \text{Subs}(\Gamma, \Delta)$   
write  $\text{Exp}(\gamma)(e)$  as  $e[\gamma]$

### Typing in context

- ▶ expressions: sets  $\text{Exp}_V(\Gamma, E)$ , written as  $\Gamma \vdash_V e : E$
- ▶ substitution preserves types: if  $\Gamma \vdash e : E$  and  $\vdash \gamma : \Gamma \rightarrow \Delta$ , then  $\Delta \vdash e[\gamma] : E[\gamma]$

## Contexts: General Definition

We can leave contexts abstract or spell out a concrete definition:

- ▶ contexts  $\Gamma$  are of the form

$$x_1 : E_1, \dots, x_n : E_n$$

where  $E_i \in \text{Exp}(x_1 : E_1, \dots, x_{i-1} : E_{i-1})$

- ▶ for  $\Gamma$  as above, substitutions  $\Gamma \rightarrow \Delta$  are of the form:

$$x_1 = e_1, \dots, x_n = e_n$$

where  $\Delta \vdash e_i : E_i[x_1 = e_1, \dots, x_{i-1} = e_{i-1}]$

This works uniformly for any formal system. But most formal systems are a bit more restrictive, e.g., by requiring that all  $E_i$  are types.

# Semantics with Contexts

## Deductive semantics

- ▶ define: theorem sets  $\text{Thm}_V(\Gamma)$       write  $F \in \text{Thm}_V(\Gamma)$  as  $\Gamma \vdash_V F$
- ▶ such that theorems are preserved by substitution:  
if  $\Gamma \vdash_V F$  and  $\vdash \gamma : \Gamma \rightarrow \Delta$ , then  $\Delta \vdash_V F[\gamma]$

## Computational semantics

- ▶ define: evaluation functions  $\text{Eval}_V(\Gamma) : \text{Exp}_V(\Gamma) \rightarrow \text{Exp}_V(\Gamma)$   
write  $e' = \text{Eval}_V(\Gamma)(e)$  as  $\Gamma \vdash_V e \rightsquigarrow e'$
- ▶ extend to substitutions:  
 $\text{Eval}_V(\Delta)(\dots, x = e, \dots) = \dots, x = \text{Eval}_V(\Delta)(e), \dots$
- ▶ require that evaluation is preserved by substitution  $\vdash \gamma : \Gamma \rightarrow \Delta$   
 $\text{Eval}_V(\Delta)(e[\gamma]) = \text{Eval}_V(\Delta)(e)[\text{Eval}_V(\Delta)(\gamma)]$   
substitution theorem for  $\text{Eval}$  as a translation from  $I$  to itself

## Definitions for Substitutions

- ▶ write  $\cdot$  for empty context/substitution
- ▶ ground expression is expression in empty context  
also called closed; then opposite is open
- ▶ ground substitution:  $\vdash \gamma : \Gamma \rightarrow \cdot$   
no free variables after substitution
- ▶ if we have computational semantics:  
value substitution is ground substitution where all expressions are values
- ▶ if we have deductive semantics:  
true instance of  $\Gamma \vdash F : \text{prop}$  is  $\gamma$  such that  $\vdash F[\gamma]$

# Concrete Semantics



# Concrete Semantics

## Definition

- ▶ determines the true instances of propositions
- ▶ for every  $\Gamma \vdash_V^I F : \text{prop}$ , a set  $\text{Inst}_V^I(\Gamma, F)$  of ground substitutions  
write  $\vdash_V^I \gamma : \Gamma$  and  $\vdash F[\gamma]$  for  $\gamma \in \text{Inst}_V^I(\Gamma, F)$

## Terminology

- ▶ Query languages (in the usual, narrower sense than used here):  
languages plus concrete semantics
- ▶ Database: implementation of concrete semantics  
usually optimized for fast query answering

## Examples

- ▶ SQL for Church-typed ontologies with ADTs (relational databases)
- ▶ SPARQL for Curry-typed ontologies (triple stores)
- ▶ Prolog for first-order logic

## Yes/No vs. Wh-Questions

Deductive/concrete semantics may be a bit of a misnomer

- ▶ Queries about  $\vdash F$  are yes/no questions
  - ▶ specialty of deductive semantics
  - ▶ but maybe only because everything else is ever harder to do deductively
- ▶ Queries about ground instances of  $\Gamma \vdash F$  are Wh questions
  - ▶ specialty of concrete databases
  - ▶ for the special case of retrieving finite results sets from a fixed concrete store
  - ▶ only situation where Wh questions are easy

But Yes/no and Wh questions exist in all aspects.

# Redundant Concrete Semantics

## Multiple concrete semantics

- ▶ Specification: absolute as rules on paper
- ▶ Database: absolute by custom database
- ▶ Database: relative via translation to assembly  $L$

## Equivalence Theorems

- ▶ typically: choose one, no redundancy, no equivalence theorems
- ▶ infinite results: easy on paper, hard in database
- ▶ open world: are all known ground instances in database?

## Interdefinability

Given concrete semantics, define a deductive one

- ▶ for ground  $F$ ,  $\text{Inst}(\cdot, F)$  is either  $\{\cdot\}$  or  $\{\}$
- ▶  $\vdash F$  iff  $\text{Inst}(\cdot, F) = \{\cdot\}$   
but concrete semantics usually cannot find all substitutions for all  $F$

Given concrete semantics, define a computational one

- ▶  $\vdash e \rightsquigarrow e'$  iff  $(x = e') \in \text{Inst}(x : E, e \doteq_E x)$   
but concrete semantics usually cannot find that substitution for all  $e$

Given deductive semantics, define a concrete one

- ▶  $\text{Inst}(\Gamma, F) = \{\vdash \gamma : \Gamma \rightarrow \cdot \mid \vdash F[\gamma]\}$   
but deductive semantics usually does not allow computing that set

Given computational semantics, define a concrete one

- ▶  $\text{Inst}(\Gamma, F) = \{\text{Eval}(\cdot, \gamma) \mid \vdash \gamma : \Gamma \rightarrow \cdot, \vdash F[\gamma] \rightsquigarrow \text{true}\}$
- ▶ allows restricting results to value substitutions  
composition of previous inter-definitions, inherits both problems

# Narrative Semantics

# Narrative Semantics

## Definition

- ▶ Describes how to answer (some) questions
- ▶ Implementations tend to be AI-complete, hypothetical
- ▶ In practice, information retrieval = find related documents

## More precisely?

- ▶ Not much theory, wide open research problem
- ▶ Some natural language document with interspersed definitions, formulas
- ▶ Maybe judgment:  $\vdash Q?A$  for “A is answer to Q”

## Examples

- ▶ “W3C Recommendation OWL 2” and Google
- ▶ “ISO/IEC 14882: 1998 Programming Language C++” and Stroustrup’s book
- ▶ Mathematics textbooks and mathematicians

# Relative Semantics

# Translations

A translation  $T$  from formal system  $I$  to formal system  $L$  consists of

- ▶ function  $\text{Voc}^T : \text{Voc}^I \rightarrow \text{Voc}^L$
- ▶ family of functions  $\text{Exp}_V^T : \text{Exp}_V^I \rightarrow \text{Exp}_{\text{Voc}^T(V)}^L$

## Desirable properties

- ▶ Should satisfy type preservation:

$$\vdash_V^I e : E \quad \text{implies} \quad \vdash_{\text{Voc}^T(V)}^L \text{Exp}_V^T(e) : \text{Exp}_V^T(E)$$

intuition: what we have, is preserved

- ▶ Might satisfy type reflection/conservativity:

$$\vdash_{\text{Voc}^T(V)}^L e' : \text{Exp}_V^T(E) \quad \text{implies} \quad \vdash_V^I e : E \text{ for some } e$$

intuition: nothing new is added



# Translation of Contexts

## Translations extend to contexts and substitutions

- ▶  $\text{Cont}^T(\dots, x : E, \dots) = \dots, x : \text{Exp}^T(E), \dots$
- ▶  $\text{Subs}^T(\dots, x = e, \dots) = \dots, x = \text{Exp}^T(e), \dots$
- ▶  $\text{Exp}^T(x) = x$  for all variables

## Desirable properties for arbitrary contexts

- ▶ Type preservation:

$$\Gamma \vdash_V^I e : E \quad \text{implies} \quad \text{Cont}_V^T(\Gamma) \vdash_{\text{Voc}^T(V)}^L \text{Exp}_V^T(e) : \text{Exp}_V^T(E)$$

- ▶ Conservativity:

$$\text{Cont}_V^T(\Gamma) \vdash_{\text{Voc}^T(V)}^L e' : \text{Exp}_V^T(E) \quad \text{implies} \quad \Gamma \vdash_V^I e : E \text{ for some } e$$

## Compositionality

Define: a translation is compositional iff we can show the substitution theorem for it

Given

$$\Gamma \vdash_V^I e : E \quad \vdash_V^I \gamma : \Gamma \rightarrow \Delta$$

we have that

$$\text{Cont}_V^T(\Delta) \vdash_{\text{Voc}^T(V)}^L \text{Exp}_V^T(e[\gamma]) \doteq_{\text{Exp}_V^T(E[\gamma])} \text{Exp}_V^T(e)[\text{Subs}_V^T(\gamma)]$$

Simplify: write  $T(-)$  for  $\text{Voc}^T(-)$ ,  $\text{Exp}_V^T(-)$ ,  $\text{Cont}_V^T(-)$ ,  $\text{Subs}_V^T(-)$

$$T(\Delta) \vdash_{T(V)}^L T(e[\gamma]) \doteq_{T(E[\gamma])} T(e)[T(\gamma)]$$

# Relative Semantics

Given

- ▶ formal systems  $I$  and  $L$
- ▶ semantics for  $L$
- ▶ translation  $T$  from  $I$  to  $L$

define semantics for  $I$

- ▶ deductive: define

$$\vdash_V^I F \quad \text{iff} \quad \vdash_{\text{Voc}^T(V)}^L \text{Exp}_V^T(F)$$

- ▶ computational: define

$$\vdash_V^I e \rightsquigarrow e' \quad \text{iff} \quad \vdash_{\text{Voc}^T(V)}^L \text{Exp}_V^T(e) \rightsquigarrow \text{Exp}_V^T(e')$$

both work accordingly with a context  $\Gamma$

- ▶ concrete: define

$$\Gamma \vdash_V^I \gamma : F \quad \text{iff} \quad \text{Cont}_V^T(\Gamma) \vdash_{\text{Voc}^T(V)}^L \text{Subs}_V^T(\gamma) : \text{Exp}_V^T(F)$$

# Equivalence of Semantics

## Definition

Two semantics  $\vdash^1$  and  $\vdash^2$  for  $I$  are equivalent if

- ▶ deductive:  $\vdash^1 F$  iff  $\vdash^2 F$
- ▶ computational:  $\vdash^1 e \rightsquigarrow e'$  iff  $\vdash^2 e \rightsquigarrow e'$
- ▶ concrete:  $\Gamma \vdash^1 \gamma : F$  iff  $\Gamma \vdash^2 \gamma : F$

Example for deductive semantics:

- ▶  $\vdash^1$  absolute semantics by calculus  
e.g., natural deduction for SFOL
- ▶  $\vdash^2$  relative semantics by translation e.g.,  $L$  is set theory,  $T$  is model theory of SFOL
- ▶ Assume proofs-as-expressions
- ▶ Then:
  - ▶ type preservation = soundness
  - ▶ conservativity = completeness

## Example: Relative Computational Semantics for BOL

Scala, SQL semantics evaluates

- ▶ concept  $c$  to
  - ▶ SQL: table of individuals result of running query  $\llbracket c \rrbracket$
  - ▶ Scala: hashset of individuals result of running program  $\llbracket c \rrbracket$
- ▶ propositions to booleans accordingly

Technically, results not in image of  $\llbracket - \rrbracket$

Fix: add productions for all values

$F ::= \text{true} \mid \text{false}$	truth values
$C ::= \{I, \dots, I\}$	finite concepts

## Equivalence with respect to Semantics

So far: equivalence of **two semantics** wrt **all queries**

Related concept: equivalence of **two queries** wrt **one semantics**

- $F, G$  deductively equivalence:

$$\vdash F \quad \text{iff} \quad \vdash G$$

may be internalized by syntax as proposition  $F \leftrightarrow G$

- $F, G$  concretely equivalent:

$$\vdash F[\gamma] \quad \text{iff} \quad \vdash G[\gamma]$$

for all ground substitutions  $\gamma$       weaker than  $\Gamma \vdash F \leftrightarrow G$

- closed  $e, e'$  computationally equivalent:

$$\vdash e \rightsquigarrow v \quad \text{iff} \quad \vdash e' \rightsquigarrow v$$

may be internalized by syntax as proposition  $e \doteq e'$

## Equivalence with respect to Semantics (2)

Interesting variants of computational semantics

- ▶ open  $e, e'$  extensionally equivalent:

$$\vdash e[\gamma] \rightsquigarrow v \quad \text{iff} \quad \vdash e'[\gamma] \rightsquigarrow v$$

for all ground substitutions  $\gamma$

equal inputs produce equal outputs

weaker than  $\Gamma \vdash e \doteq e'$  — intensional equivalence

- ▶ machines  $M, M'$  observationally equivalent:  
produce equal sequences of outputs for the same sequence of inputs  
e.g., automata, objects in OO-programming  
choice of semantics defines legal optimizations in compiler



## Equivalence of BOL Semantics

Now 5 semantics for BOL

- ▶ absolute deductive via calculus
- ▶ relative deductive via SFOL
- ▶ relative computational via Scala
- ▶ relative concrete via SQL
- ▶ relative narrative via English

Moreover, these are interdefinable.

e.g., Scala translation also induces deductive semantics

Can compare equivalence

- ▶ for every pair of semantics
- ▶ for every kind of equivalence (deductive, concrete, computational)

Question: Which of them hold?

## Questions

For example, consider:

- ▶ Are the absolute semantics and the Scala semantics deductively equivalent?
- ▶ Assuming BOL and SQL have the base types and values: Are the absolute semantics and the SQL semantics concretely equivalent?

## Deductive Semantics of BOL

Are these two BOL semantics deductively equivalent

- ▶ absolute deductive semantics
- ▶ relative deductive semantics via translation  $\llbracket - \rrbracket$  to SFOL

Soundness:  $\vdash_V^{BOL} f$  implies  $\vdash_{\llbracket V \rrbracket}^{SFOL} \llbracket f \rrbracket$

- ▶ induction on derivations of  $\vdash_V^{BOL} f$
- ▶ one case per rule                      induction rule from above not sound
- ▶ several pages of work but straightforward and relatively easy

## Deductive Semantics of BOL

Are these two BOL semantics deductively equivalent

- ▶ absolute deductive semantics
- ▶ relative deductive semantics via translation  $\llbracket - \rrbracket$  to SFOL

Completeness:  $\vdash_V^{BOL} f$  implied by  $\vdash_{\llbracket V \rrbracket}^{SFOL} \llbracket f \rrbracket$

works if we add missing rules

- ▶ induction on SFOL derivations does not work
  - ▶ SFOL more expressive than BOL
  - ▶  $\llbracket - \rrbracket$  not surjective
- ▶ instead show that  $\llbracket - \rrbracket$  preserves consistency of vocabularies
 

no universal recipe how to do that
- ▶ then a typical proof uses  $V$  extended with  $\neg f$ 
  - ▶ if  $V$  inconsistent,  $\vdash_V f$  for all  $f$ , done
  - ▶ if  $V$  consistent and  $V + \neg f$  inconsistent, then  $\vdash_V f$ , done
  - ▶ if  $V + \neg f$  consistent, so is  $\llbracket V + \neg f \rrbracket$ , which contradicts  $\vdash_{\llbracket V \rrbracket}^{SFOL} \llbracket f \rrbracket$

# Computational Semantics of BOL

Are these two BOL semantics deductively equivalent

- ▶ absolute deductive semantics
- ▶ relative deductive semantics via translation  $\llbracket - \rrbracket$  to Scala

Soundness:  $\vdash_V^{BOL} f$  implies  $\vdash_{\llbracket V \rrbracket}^{Scala} \llbracket f \rrbracket \rightsquigarrow true$

- ▶ Problem: Absolute semantics performs consequence closure, e.g.,
    - ▶ transitivity of  $\sqsubseteq$
    - ▶ relationship between  $\sqsubseteq$  and is-a
  - ▶ Scala semantics does so only if we explicitly implemented it
 

we didn't
- same problem for SQL semantics

# Computational Semantics of BOL

Are these two BOL semantics deductively equivalent

- ▶ absolute deductive semantics
- ▶ relative deductive semantics via translation  $\llbracket - \rrbracket$  to Scala

Completeness:  $\vdash_V^{BOL} f$  implied by  $\vdash_{\llbracket V \rrbracket}^{Scala} \llbracket f \rrbracket \rightsquigarrow true$

- ▶ absolute semantics leaves closed world optional
- ▶ Scala uses closed worlds
  - e.g., used to compute  $c \sqsubseteq d$  by checking all individuals
- ▶ complete only if we add induction rule