# Knowledge Representation and Processing

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2020–2024

# Administrative Information

# Format

## Room

- lectures and exercises in person — if we have a room
- interaction strongly encouraged

We don't want to lecture —
we want to have a conversation during which you learn

- possibly make use of zoom

## Recordings

- maybe prerecorded video lectures or recorded zoom meeting
- to be decided along the way

# Background

## Instructor

- PD Dr. Florian Rabe
- Prof. Dr. Michael Kohlhase

  Professor of Knowledge Representation and Processing

## Course

- This course is given for the fourth time
- Still a bit experimental polishing and revising the materials this year
- Signature course of our research group same name!

# Prerequisites

## Required

- basic knowledge about formal languages, context-free grammars

  but we'll do a quick revision here

## Helpful

- Algorithms and Data Structures mostly as a contrast to this lecture
- Basic logic we'll revise it slightly differently here
- all other courses as examples of how knowledge pervades all of CS

## General

- Curiosity this course is a bit unusual
- Interest in big picture

  this course touches on lots of things from all over CS

# Examination and Grading

## Suggestion

- grade determined by single exam
- probably written, 90 minutes
- exercises indirectly graded through conversation during exam

unless we agree on something else by the second week

## Exam-relevant

- anything mentioned in notes
- anything discussed in lectures
- anything done in exercises

none is a superset of another!

# Materials and Exam-Relevance

## Textbook

- does not exist
- normal for research-near specialization courses

## Notes

- textbook-style but not as comprehensive
- developed along the way
- systematic write-up, not necessarily in lecture order

## Slides

- not comprehensive
- used as visual aid, conversation starters

# Communication

## Open for questions

- open door policy in our offices
- always room for questions during lectures
- for personal questions, contact me during/after lecture
- matrix channel `https://matrix.to/#/#wuv:fau.de`
- studon forum not used

## Materials

- notes and slides are at: `https://github.com/florian-rabe/Teaching/tree/master/WuV`
- currently last year's version, will change throughout the semester
  you can read ahead, but maybe don't print everything right away
- pull requests and issues welcome

# Exercises

## Learning Goals

- Get acquainted with state of the art of practice
- Try out real tools

## Homeworks

- one major project as running example
- homeworks building on each other

build one large knowledge-based system

details on later slides

# Overview and Essential Concepts

# Representation and Processing

Common pairs of concepts:

| Representation | Processing |
|---|---|
| Static | Dynamic |
| Situation | Change |
| Be | Become |
| Data Structures | Algorithms |
| Set | Function |
| State | Transition |
| Space | Time |

# Data and Knowledge

2 × 2 key concepts

| | |
|---|---|
| Syntax | Data |
| Semantics | Knowledge |

- Data: any object that can be stored in a computer
  Example: $((49.5739143, 11.0264941), "2020-04-21T16:15:00CEST")$
- Syntax: a system of rules that describes which data is **well-formed**
  Example: "a pair of (a pair of two IEEE double precision floating point numbers) and a string encoding of a time stamp"
- Semantics: system of rules that determines the meaning of well-formed data
- Knowledge: combination of some data with its syntax and semantics

# Knowledge is Elusive

Representation of key concepts

- Data: using primitive objects
  implemented as bits, bytes, strings, records, arrays, ...
- Syntax: (context-free) grammars, (context-sensitive) type systems
  implemented as inductive data structures
- Semantics: functions for evaluation, interpretation, of well-formed data
  implemented as recursive algorithms on the syntax
- Knowledge: elusive
  emerges from applying and interacting with the semantics

# Semantics as Translation

- Knowledge can be captured by a higher layer of syntax
- Then semantics is translation into syntax

| Data syntax | Semantics function | Knowledge syntax |
|---|---|---|
| SPARQL query | evaluation | result set |
| SQL query | evaluation | result table |
| program | compiler | binary code |
| program expression | interpreter | result value |
| logical formula | interpretation in a model | mathematical object |
| HTML document | rendering | graphics context |

# Heterogeneity of Data and Knowledge

- Capturing knowledge is difficult
- Many different approaches to semantics
  - fundamental formal and methodological differences
  - often captured in different fields, conferences, courses, languages, tools
- Data formats equally heterogeneous
  - ontologies
  - programs
  - logical proofs
  - databases
  - documents

# Challenges of Heterogeneity

## Challenges

- collaboration across communities
- translation across languages
- conversion between data formats
- interoperability across tools

## Sources of problems

- interoperability across formats/tools major source of
  - complexity
  - bugs
- friction in project team due to differing preferences, expertise
- difficult choice between languages/tools with competing advantages
  - reverting choices difficult, costly
  - maintaining legacy choices increases complexity

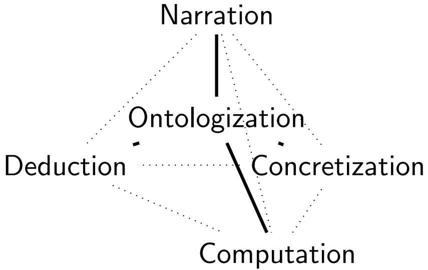
# Aspects of Knowledge

- Tetrapod model of knowledge   active research by our group
- classifies approaches to knowledge into five aspects

| Aspect | KRLs (examples) |
|---|---|
| ontologization | ontology languages (OWL), description logics (ALC) |
| concretization | relational databases (SQL, JSON) |
| computation | programming languages (C) |
| deduction | logics (HOL) |
| narration | document languages (HTML, LaTeX) |

# Relations between the Aspects

Ontology is distinguished: capture the knowledge that the other four aspects share



# Complementary Advantages of the Aspects

| Aspect | objects | characteristic | | |
|---|---|---|---|---|
| | | advantage | joint advantage of the other aspects | application |
| ded. | formal proofs | correctness | ease of use | verification |
| comp. | programs | efficiency | well-definedness | execution |
| concr. | concrete objects | queriability | abstraction | storage/retrieval |
| narr. | texts | flexibility | formal semantics | human understanding |

| Aspect pair | characteristic advantage |
|---|---|
| ded./comp. | rich meta-theory |
| narr./conc. | simple languages |
| ded./narr. | theorems and proofs |
| comp./conc. | normalization |
| ded./conc. | decidable well-definedness |
| comp./narr. | Turing completeness |

# Structure of the Course

## Aspect-independent parts

- shared characteristics
- general methods

## Aspects-specific parts

- one part for each aspect
- high-level overview of state of the art
- focus on comparison/evaluation of the aspect-specific results

# Structure of the Exercises

## One major project

- representative for a project that a CS graduate might be put in charge of
- challenging heterogeneous data and knowledge
- requires integrating/combining different languages, tools

unique opportunity in this course because knowledge is everywhere

## Concrete project

- develop a campo/studnon-style KRP system for a university
- lots of heterogeneous knowledge
  - course and program descriptions
  - legal texts
  - websites
  - grade tables
  - code to generate diplomas
- build a functional system applying the lessons of the course

we'll see how far we get — priority is learning

# Ontological Knowledge

# Components of an Ontology

8 main declarations

- **individual** — concrete objects that exist in the real world, e.g., "Florian Rabe" or "WuV"
- **concept** — abstract groups of individuals, e.g., "instructor" or "course"
- **relation** — binary relations between two individuals, e.g., "teaches"
- **properties** — binary relations between an individuals and a concrete value (a number, a date, etc.), e.g., "has-credits"
- **concept assertions** — the statement that a particular individual is an instance of a particular concept
- **relation assertions** — the statement that a particular relation holds about two individuals
- **property assertions** — the statement that a particular individual has a particular value for a particular property
- **axioms** — statements about relations between concepts, e.g., "instructor" $\sqsubseteq$ "person"

# Divisions of an Ontology

### Abstract vs. concrete

- TBox: concepts, relations, properties, axioms
  everything that does not use individuals
- ABox: individuals and assertions

### Named vs. unnamed

- Signature: individuals, concepts, relations, properties
  together called entities or resources
- Theory: assertions, axioms

# Comparison of Terminology

| Here | OWL | Description logics | ER model | UML | semantics via logics |
|---|---|---|---|---|---|
| individual | instance | individual | entity | object, instance | constant |
| concept | class | concept | entity-type | class | unary predicate |
| relation | object property | role | role | association | binary predicate |
| property | data property | (not common) | attribute | field of base type | binary predicate |

| domain | individual | concept |
|---|---|---|
| type theory, logic | constant, term | type |
| set theory | element | set |
| database | row | table |
| philosophy[1] | object | property |
| grammar | proper noun | common noun |

[1] as in https://plato.stanford.edu/entries/object/

# Ontologies as Sets of Triples

General idea:

- Turn everything into a relation/property assertion
- Represent ontologies as sets of subject-predicate-object triples
- Obtain efficient representation of ontologies using RDF and RDFS

| Assertion | Triple | | |
|---|---|---|---|
| | Subject | Predicate | Object |
| entities | recover from what's mentioned in assertions | | |
| concept assertion | "Florian Rabe" | `is-a` | "instructor" |
| relation assertion | "Florian Rabe" | "teaches" | "WuV" |
| property assertion | "WuV" | "has credits" | 7.5 |
| axiom | only some special cases work, e.g., | | |
| subconcept axiom | "instructor" | `subClassOf` | "person" |

# Special Entities

RDF and RDFS define special entities for use in ontologies:

- "rdfs:Resource": concept of which all individuals are an instance and thus of which every concept is a subconcept
- "rdf:type": relates an entity to its type:
  - an individual to its concept (corresponding to `is-a` above)
  - other entities to their special type (see below)
- "rdfs:Class": special class for the type of classes
- "rdf:Property": special class for the type of properties
- "rdfs:subClassOf": a special relation that relates a subconcept to a superconcept
- "rdfs:domain": a special relation that relates a relation to the concepts of its subjects
- "rdfs:range": a special relation that relates a relation/property to the concept/type of its objects

Goal/effect: capture as many parts as possible as RDF triples.

# Declarations as Triples using Special Entities

| Assertion | Triple | | |
|---|---|---|---|
| | Subject | Predicate | Object |
| individual | individual | "rdf:type" | "rdfs:Resource" |
| concept | concept | "rdf:type" | "rdf:Class" |
| relation | relation | "rdf:type" | "rdf:Property" |
| property | property | "rdf:type" | "rdf:Property" |
| concept assertion | individual | "rdf:type" | concept |
| relation assertion | individual | relation | individual |
| property assertion | individual | property | value |
| for special forms of axioms | | | |
| $c \sqsubseteq d$ | $c$ | "rdfs:subClassOf" | $d$ |
| $\mathrm{dom}\, r \equiv c$ | $r$ | "rdfs:domain" | $c$ |
| $\mathrm{rng}\, r \equiv c$ | $r$ | "rdfs:range" | $c$ |

# An Example Ontology Language

see syntax of BOL in the lecture notes

# A Real-Life Ontology Language

See online resources for OWL.

Some specialties:

- Slightly different names than in BOL
- No strict distinction between individuals, concepts, relations - just resources
- Some special axioms, e.g., to make relations transitive
- Multiple sublanguages with varying expressivity/implementability: Lite, DL, Full

BOL vs. OWL:

- BOL is simpler, more systematically structured
  good for teaching, prototypes
- OWL is the standard    the one to use for better or worse

# Exercise 1

As a team, build an ontology for a university.
Using git, OWL, and WebProtege are good ways to start.

(In WebProtege, set "suffix" to "user supplied name" in "New Entity Settings". Otherwise, it'll get messy when you share your ontology.)

# Example: The Common Sense Ontology

## Situation

- society uses one ontology for common sense knowledge
- changes over time

content relative to ontology: laws, regulations, etc.

## Special aspects

- unwritten
- not actually fully agreed upon
- sometimes subject to political debate
- no formal ontology language good enough to capture practical nuances
- many society members not comfortable with formal languages

but still always exists implicitly

Idea: see political proposals as ontology evolution

# Ontology Morphisms

# Idea

## Intuition of morphism $m$

- connects two ontologies, written $m : V \to W$
- maps $V$-symbols to $W$-expressions
- extends homomorphically to ma $V$-expressions to $W$-expressions

  replace every symbol with its assignment
  like substitutions for contexts

## Purpose

- extend $V$ with entirely new declarations
  special case of $W = V, E$, and identity morphism $V \to W$
- extend the vocabulary with definitions
  special case $m : V, E \to V$, and $m$ maps new symbols to definitions
- ontology evolution: $V$ is old ontology, $W$ new, $m$ interprets $V$ in $W$
- transfer legacy content from old to new ontology

# Exercise 2

We write ontologies for sex and gender.

Write two ontologies for

- cis-normative world view with just men and women
- trans-inclusive world view that accommodates sex and gender

and relate them with ontology morphisms.

# Side Note: Knowledge Representation is Apolitical

Knowledge representation makes no judgment about which ontologies or morphisms are fair, moral, politically correct, etc.

It can only judge practicality, e.g.,

- well-formedness and consistency
- decidability, efficiency of querying
- simplicity, e.g., measured by
  - number of declarations or the size of expressions
  - number of axioms about each symbol
- existence and simplicity of morphisms

Languages must allow for expressing whichever knowledge/opinion the user has.

Only users can judge if an ontology is correct.

# BOL Morphisms Formally

Syntax: Extend grammar with vocabulary morphisms

$$
\begin{array}{lcll}
M & ::= & A^* : O \to O & \text{morphisms} \\
A & ::= & i \mapsto I & \text{individual assignment} \\
 & \mid & c \mapsto C & \text{concept assignment} \\
 & \mid & r \mapsto R & \text{relation assignment} \\
 & \mid & p \mapsto P & \text{property assignment}
\end{array}
$$

Well-formedness for $M : O \to O'$:

- one assignment $\mathtt{ID} \mapsto E$ for each declaration ID of $O$
- $E$ must be an $O'$-expression of the right kind
  - individual symbols to individual expressions
  - concept symbols to concept expressions
  - relation symbols to relation expressions
  - property symbols of type $V$ to property expressions of type $V$
  - what about assertions and axioms? see below

# Homomorphic Extension

Given morphism $m : O \to O'$, define

- mapping $\overline{m}$ from $O$-expressions $E$ to $O'$-expressions $\overline{m}(E)$ by
- replacing every $O$-symbol $s$ in $E$
  with the expression $s \mapsto E$ provided by $m$.

Notation: $m(E)$ instead of $\overline{m}(E)$

Well-defined mapping because morphisms must contain exactly one assignment for every $O$-symbol.

# BOL Morphisms: What about Axioms?

A morphism $m : O \to O'$ is well-formed if

- for every axiom/assertion $F$ in $O$,
- we have that $m(F)$ is a theorem of $O'$.

Theorem: Morphisms preserve truth

- if $\vdash_O E : E'$ then $\vdash_{O'} m(E) : m(E')$
- if $\vdash_O F$ then $\vdash_{O'} m(F)$

Mapping axioms works best if

- every axiom/assertion has a name
- new expression kind for proofs

  given by derivations of some absolute deductive semantics

  axioms = proof symbols = atomic proofs
- morphisms contain assignments $a \mapsto P$ of axiom $a$ to proof $P$

# Example/Exercise 2

## Assume *CisNormative* is BOL vocabulary containing

- concepts `man`, `woman`
- axioms $\mathtt{man} \sqcup \mathtt{woman} \equiv \top$ and $\mathtt{man} \sqcap \mathtt{woman} \equiv \bot$

simplified cis-normative world view

## and *TransFriendly* contains

one way to accommodate transgender people

- concepts `sexmale`, `sexfemale`, `cis`, `trans`
- appropriate axioms

## Now have ontology morphism *CisNormative* $\to$ *TransFriendly*

- morphism `gendermatters`
  - $\mathtt{man} \mapsto (\mathtt{sexmale} \sqcap \mathtt{cis}) \sqcup (\mathtt{sexfemale} \sqcap \mathtt{trans})$
  - $\mathtt{woman} \mapsto (\mathtt{sexfemale} \sqcap \mathtt{cis}) \sqcup (\mathtt{sexmale} \sqcap \mathtt{trans})$
- alternative morphism `sexmatters`
  - $\mathtt{man} \mapsto \mathtt{sexmale}$
  - $\mathtt{woman} \mapsto \mathtt{sexfemale}$

# Prevalence of Morphisms

Deduction

- algebraic hierarchy, e.g., *Monoid* $\to$ *Group*
- theory $\to$ model, e.g., *Group* $\to$ *Integer*

Computation

- class extension
- interface implementation
- type class instances
- functor
- API adapters

Concrete data

- between tables: database views
- between schemas: database migration

General: module systems for building large vocabularies

# Language Layers

# Layers of Language Design

| Layer | Specified by | Implemented by |
|---|---|---|
| Syntax | | |
| Context-Free | grammar | AST+parser+printer |
| Context-Sensitive | inference system | type checker |
| Semantics | inference system, interpretation, or translation | |
| Pragmatics | human preferences | human judgment |

KRP = syntax + semantics

# Layered Processing

Data is processed in phases

1. data representation format, e.g., string, JSON, XML, binary
2. parsed — well-formed context-free syntax tree
3. context-sensitive check by traversal of the syntax tree — well-typed syntax tree
4. computation by traversal of well-typed AST — semantics

# Possible Errors

| Layer | Error |
|---|---|
| CFS | not derivable from grammar |
| CSS | symbols not used as declared, other conditions |
| Sem. | ambiguous/undefined semantics |
| Pragmatics | not useful |

# Typical Errors by Layer

In a programming language:

| Layer | Expression | Issue | Explanation |
|---|---|---|---|
| CFS | 1/ | syntax error | argument missing |
| CSS | 1/"2" | typing error | wrong type |
| Sem. | 1/0 | run-time error | undefined semantics |
| Pragm. | 1/1 | code review | unnecessarily complex |

## Typical Errors by Layer

In a logic:

| Layer | Expression | Issue | Explanation |
|---|---|---|---|
| CFS | $\forall x$ | not well-formed | body missing |
| CSS | $\forall x.P(y)$ | not well-typed | $y$ not declared |
| Sem. | the $x \in \mathbb{N}$ with $x < 0$ | not well-defined | no such $x$ exists |
| Pragm. | $\exists x.x \neq x$ | not useful | no model exists |

# Context-Free Grammars

# The Chomsky Hierarchy

- CH-0, regular grammars:
  - equivalent to regular expressions and finite automata
  - not used much as grammars
- CH-1, context-free grammars (CFGs) our focus
- CH-2, context-sensitive grammars
  - important as languages, but awkward as grammars
  - instead: type system determines subset of context-free language
- CH-3, unrestricted grammars
  - Turing-complete, theoretically important
  - not used much as grammars

# Definitions

- An alphabet is a set of symbols.
- A word is a list of symbols from the alphabet.
- A production is pair of words.
  - A production is written $lhs ::= rhs$.
  - Multiple productions for the same left-hand side are abbreviated $lhs ::= rhs_1 \mid \ldots \mid rhs_n$.
  - Right-hand side may also use regular expressions like $^*$ for repetition and [] for optional parts.
- A CFG is a set of productions where *lhs* is a single symbol.
  - If there is a production $N ::= rhs$, $N$ is called non-terminal, otherwise terminal.
  - If a word contains non-terminal symbols, it is called non-terminal, otherwise terminal.
- A syntax tree is a tree whose nodes are labeled with productions $N ::= rhs$ where the non-terminals in rhs are exactly the lhs's of the children.
- The word produced by a syntax tree is read off by exhaustively replacing every lhs with the respective rhs.

# Example: Syntax of Arithmetic Language

$\boxed{\text{Numbers}}$

| | | |
|---|---|---|
| $N ::= 0 \mid 1$ | | literals |
| $\mid$ | $N + N$ | sum |
| $\mid$ | $N * N$ | product |

$\boxed{\text{Formulas}}$

| | | |
|---|---|---|
| $F ::=$ | $N \doteq N$ | equality |
| $\mid$ | $N \leq N$ | ordering by size |

# Implementing CFGs via Inductive Data Types

## Correspondence

| CFG | IDT |
|---|---|
| non-terminal | type |
| production | constructor |
| non-terminal on left of production | return type of constructor |
| non-terminals on right of production | arguments types of constructor |
| terminals on right of production | notation of constructor |
| words derived from non-terminal $N$ | expressions of type $N$ |

# Classes of Languages

Functional languages:

- pure: ML, Haskell
- with OO: F#, Scala

inductive types are primitive

OO-languages:

- C#, Java, C++

inductive types simulated via classes

Untyped languages:

- Python, Javascript

inductive types simulated ad hoc

# Implementing the Example

Done interactively. See the examples in the repository. See also the notes.

# Exercise 3

Individually, using any programming language, implement the AST for the BOL language. Implement a printer for BOL by context-free traversal of the syntax tree.

Remarks:

- To simplify, you can drop most productions for concepts and relations.
- To simplify, you can drop properties altogether. But it would be nice to have them and allow for integers and strings as basic types.

# Context-Sensitive Syntax

# Vocabularies and Declarations

Generic structure of a context-sensitive language

- a vocabulary is a list of declarations
  - named: type/function/predicate symbol etc.
  - unnamed: axioms etc.
  - structural: namespaces/package, inclusion/import
- named declarations introduce atomic objects of various kinds
- for each kind, a non-terminal for complex expressions of that kind
- references to names introduced by declarations are base cases of expressions

# Example: Typed Expressions

$\boxed{\text{Vocabularies}}$

| | |
|---|---|
| $Voc \ ::= Decl^*$ | list of declarations |

$\boxed{\text{Declarations}}$

| | |
|---|---|
| $Decl \ ::= id : Type^* \to Type$ | typed function symbols |
| $\quad\quad \mid \ id : Type^* \to FORM$ | typed predicate symbols |

$\boxed{\text{Types}}$

| | |
|---|---|
| $Type ::= Nat \mid String$ | base types |

$\boxed{\text{Expressions}}$

| | |
|---|---|
| $Expr \ ::= 0 \mid 1 \mid Expr + Expr \mid Expr * Expr$ | as before |
| $\quad\quad \mid \ id(Expr^*)$ | application of a function symbol |

$\boxed{\text{Formulas}}$

| | |
|---|---|
| $Form ::= Expr \doteq Expr \mid Expr \leq Expr$ | as before |
| $\quad\quad \mid \ id(Expr^*)$ | application of a predicate symbol |

# Example: Vocabularies and Expressions

Example vocabulary $V$ containing the following declarations:

- $\mathit{fib} : \mathit{Nat} \to \mathit{Nat}$
- $\mathit{length} : \mathit{String} \to \mathit{Nat}$
- $\mathit{mod} : \mathit{Nat}\,\mathit{Nat} \to \mathit{Nat}$
- $\mathit{prime} : \mathit{Nat} \to \mathit{FORM}$

Example expressions relative to $V$

- expressions: $\mathit{fib}(0)$, $\mathit{mod}(\mathit{fib}(\mathit{fib}(1)), 1+1)$
- formulas: $\mathit{fib}(0) = 0$, $\mathit{prime}(\mathit{fib}(1))$

# Primitive vs. Declared

## Primitive

- built into the language
- assumed to exist a priori — fundamentals of nature
- fixed semantics (usually interpreted by identity function)

| | primitive | declared |
|---|---|---|
| introduced by | language designer | user |
| introduced in | grammar | vocabulary $V$ |
| visible in | all vocabularies | $V$ only |
| semantics given | explicitly | implicitly |
| ... by | translation function | axioms |

more expressive declarations → fewer primitives needed

paradoxical: more complex language may have simpler grammar

# Quasi-Primitive = Declared in standard library

## Standard library: a vocabulary *StdLib*

- present in every language — empty vocabulary by default
- one fixed vocabulary
  - implicitly included into every other vocabulary
  - implicitly fixed by any translation between vocabularies

Combination of advantages

- from the user's perspective: like a primitive
- from the theory's/system's perspective: no special treatment

## Examples

- sufficiently expressive languages
  - push many primitive objects to standard library — never all
  - simplifies language, especially when defining operations

  strings in C, BigInteger in Java, inductive type for $\mathbb{N}$
- inexpressive languages
  - many primitives — SQL, spreadsheet software
  - few (quasi)-primitives — few operations available in OWL

# Example: Removing Built-in Operations

Grammar without built-in operations

| | | |
|---|---|---|
| $Voc$ | $::= Decl^*$ | list of declarations |
| $Decl$ | $::= id : Type^* \to Type$ | typed function symbols |
| | $\mid\ id : Type^* \to FORM$ | typed predicate symbols |
| $Type$ | $::= Nat \mid String$ | base types |
| $Expr$ | $::= id(Expr^*)$ | application of a function symbol |
| $Form$ | $::= id(Expr^*)$ | application of a predicate symbol |

Standard library:

- $0 : Nat$, $1 : Nat$, $sum : Nat\ Nat \to Nat$, $product : Nat\ Nat \to Nat$,
- $equals : Nat\ Nat \to FORM$, $lesseq : Nat\ Nat \to FORM$

# Example: Removing more Primitive Operations

If we add type declarations, we can remove *Nat* as well

| | | | |
|---|---|---|---|
| $Voc$ | $::=$ | $Decl^*$ | list of declarations |
| $Decl$ | $::=$ | $id : Type^* \to Type$ | typed function symbols |
| | $\mid$ | $id : Type^* \to FORM$ | typed predicate symbols |
| | $\mid$ | $id : TYPE$ | type symbols |
| $Type$ | $::=$ | $id$ | reference to a type symbol |
| $Expr$ | $::=$ | $id(Expr^*)$ | application of a function symbol |
| $Form$ | $::=$ | $id(Expr^*)$ | application of a predicate symbol |

Note: *Type* and *Form* are non-terminals, *TYPE* and *FORM* are not

Add to default vocabulary: $Nat : TYPE$, $String : TYPE$

# Context-Sensitivity

A reference to a declared name must respect the way in which it was declared in the vocabulary examples below relative to $V$ above

- occur in a position where an expression of the right kind is expected example error: $\mathit{prime}(1) = 1$
- be applied to the right number of arguments example error: $\mathit{prime}(1, 1)$
- if a type system is used
  - arguments must have the right types $\mathit{length}(1)$
  - return type must match what is expected $\mathit{fib}(1)$ if a string is expected

# Syntax Traversal

Context-free traversal — mutually recursive functions

- one function for each non-terminal/inductive type
- for each such function, one case for each production/constructor
- for each such case, one recursive call for each non-terminal on the rhs/constructor argument
- Examples
  - printer/serializer (return, e.g., string)
  - test if a feature is used (return boolean)
  - find set of used identifiers (return set of names)

Context-sensitive traversal: as above but

- functions take extra argument for vocabulary
- cases for identifier references look up declaration in vocabulary
- Examples: anything that looks up identifier declarations
  - substitution of identifiers with new expressions
  - typical syntax transformations, e.g., in compilers (return same expression kind)
  - semantics by translation

# Type Checker

Syntax checker

- context-sensitive traversal where all functions return booleans
- case for vocabulary checks each declaration relative to the preceding vocabulary
- cases for identifier references check correct use of identifier

Type checker: as above but additionally

- functions for typed expression additionally take expected type as argument
- cases for identifiers references
  - check each argument against declared input type
  - compare output to expected type

# Exercise 4

Implement a type-checker for BOL.

The type-checker must check that all identifier references are used according to their declaration:

- concept identifiers as concepts
- relation identifiers as relations
- individual identifiers as individuals
- properties identifiers of type $Y$ as properties for values of type $Y$individual identifiers as individuals

# Natural Language

# Languages

We give an overview of narrative languages as one corner of the Tetrapod.

# Formal Languages for Natural Language

## Formal Languages

- Controlled natural languages: GF, . . . ,
- Word processors: Microsoft Word, Libre Office Write, . . .
- Web-oriented languages: Markdown, HTML, . . .

## Features

- Unrestricted or barely restricted natural language
- Usually: visual presentation, e.g., fonts, colors
- Possibly: formal structure, e.g.,
  - sections
  - lists, enumerations
  - cross-references
- Rarely: explicit representation of ontological knowledge, e.g., definitions, references

# Semantics by Translation

## Relative Semantics

- define semantics of a language $I$ relative to a language $L$ whose semantics is already fixed
- by translation of $I$ into $L$

## Compositional Translation

- translation is given as context-free traversal of checked syntax tree
- one function per non-terminal $N$
- one case per production $N ::= R$
- one recursive call per non-terminal in $R$

# A Narrative Semantics of BOL

We give a semantics of BOL by translation into English according to Section 6.6 of the notes.

# LaTeX

# The sTeX Dialect of LaTeX

LaTeX is a language for natural language documents.

sTeX enriches LaTeX to allow describing the ontology in addition to the narrative document. In particular: definitions and references to them:

## sTeX primitives

- \begin{smodule}{<name>} starts a module (groups definitions)
- \symdecl*{<sym>} delcares/reserves a symbol (name)
- \begin{ sdefinition } starts a definition
- \definiendum{<sym>}{<print>} makes a definiendum for <sym> in a definition
- \sr{<sym>}{<print>} references <sym> (e.g. as a link on <print>

# Exercise 5

Implement a semantics by translation for BOL that translates BOL to sTeX.

Concretely, your translation should take an ontology and a name N. And it should return a file N.tex containing

- for every named declaration: one English sentence with an sTeX declaration
- for every axiom: one English sentence in which all identifier references carry a hyperlink to the corresponding sTeX declaration

Run this on your university ontology to produce university.tex and university.pdf. Write a second sTeX document containing a mission statement for the university in such a way that every mention of a concept carries a hyperlink to the respective declaration in university.pdf.

# Typed First-Order Logic

# Syntax

We give an overview of first-order logic as the main example of the deductive corner of the Tetrapod. The grammar is given in Section 6.3 of the notes.

# Type Systems

# Overview

## General Goal

- subdivide expressions into groups (called sorts, types, kinds, etc.)
- written $\vdash_V E : T$ for expression $E$ of type $T$ and vocabulary $V$

## Basic Type System

- expressions are always subdivided by their non-terminals
- the types are the non-terminals
- type of expression immediately clear from expression
- context only needed to check well-formedness of expression, not to infer its type

## Refined Type System

- for some non-terminals, the expressions are additionally subdivided
- the types are other expressions

  $E$ and $T$ can be from same or different non-terminals
- relation between $E$ and $T$ entirely up to the language

# Local Variables

## Type Systems with Variables

- $\Gamma \vdash E : T$ for expression $E$ of type $T$
- $\Gamma = x_1 : Y_1, \ldots, x_n : Y_n$ declares free variables that may occur in $E$ and $T$
- confusingly: $\Gamma$ usually also called context

## Choice of $\Gamma$

- for BOL: nothing (BOL has no variable binding)
- for SFOL: variables introduced by $\forall$ and $\exists$

# Algorithms for Type Systems

Judgment $\Gamma \vdash_V E : T$

## Type Checking

- input: $V$, $\Gamma$, $E$, $T$
- output: boolean

## Type Inference

- input: $\Gamma$, $V$, $E$
- output: $T$

in practice: also return error messages, e.g., as exceptions

## Advanced Variants

- as above but additionally return $E'$ and $T'$
- $E$ and $T$ have gaps that are filled by the algorithms resulting in $E'$ and $T'$

# Implementing a Type-Checker

## Structure of Syntax

- structural level: vocabularies (and morphisms), declarations
- expressions: some non-terminals are designated as expressions
  - usually at least one per declaration kind
  - usually includes (or can be extended to include) formulas

## Structure of Type-Checker

- function check-$N$ for each non-terminal $N$
  - takes context ($V$ and $\Gamma$) and $N$-word
  - returns error message(s)
- whenever $N$-words are typed by $Y$-words, instead
  - function check-$N$ takes $V$, $\Gamma$, $N$-word, and $Y$-word (expected type)
  - function infer-$N$ takes $V$, $\Gamma$, $N$-word; returns $Y$-word (inferred type)

# Exercise 6

Implement the syntax (with printer but not necessarily with parser) and type-checking of SFOL.

# Theory Morphisms

# Syntax

## Syntax like for BOL: morphism $m : V \to W$ maps

- type symbols to type expressions

$$y := Y$$

- function symbols to function expressions

$$f := [x_1, \ldots, x_n]T$$

$n$-ary function expression = term with $n$ free variables

- predicate symbols to predicate expressions

$$p := [x_1, \ldots, x_n]F$$

$n$-ary predicate expression = formula with $n$ free variables

# Example

$V$:

```
type person
type int
fun age: person → int
pred sibling ⊆ person person
```

$W$:

```
type human
type int
fun minus: int int → int
fun birthYear: person → int
fun currentYear: → int
pred parent ⊆ person person
```

One possible morphism $m : V \to W$:

```
person := human
int    := int
age    := [p] minus(currentYear, birthYear(p))
sibling:= [x,y] ∀ p:human. parent(p,x) ↔ parent(p,y)
```

# Type-Checking (1)

### The easy cases — like for BOL

- for $V$-type symbol $y := Y$ mapped by $y := Y$
  well-typed if
  $$\vdash_W Y : \mathtt{type}$$
  i.e., if $Y$ is a type-expression
- $V$-axiom asserting $F$: mapping of named symbols must be such that $m(F)$ is a theorem

# Type-Checking (2)

## Conditions for function/predicate symbols slightly technical

- $V$-function symbol $f : Y_1 \dots Y_n \to Y$ mapped by $f := [x_1, \dots, x_n]T$

$$x_1 : m(Y_1), \dots, x_n : m(Y_n) \vdash_W f' : m(Y)$$

  i.e., well-typed if $T$ is a well-typed term with
  - free variables whose types correspond to the arguments of $f$
  - output corresponding to the output type of $f$
- $V$-predicate symbol $p \subseteq Y_1 \dots Y_n$ mapped by $p := [x_1, \dots, x_n]F$

$$x_1 : m(Y_1), \dots, x_n : m(Y_n) \vdash_W F : \mathtt{form}$$

  i.e., well-typed if $F$ is a well-typed formula with free variables whose types correspond to the arguments of $p$

  like for function symbols but no output type

# Homomorphic Extension

## Given morphism $m : V \to W$

- In principle like for BOL: map $V$-expression $E$ to $W$-expression $m(E)$ by replacing every symbol reference with the expression provided by $m$
- But one subtlety for function/predicate application
  - if $m$ contains $f := [x_1, \ldots, x_n]T$:

    $$m(f(t_1, \ldots, t_n)) = T[x_1/m(t_1), \ldots, x_n/m(t_n)]$$

    (term $T$ with each variable $x_i$ substituted with $t_i$)
  - if $m$ contains $p := [x_1, \ldots, x_n]F$:

    $$m(p(t_1, \ldots, t_n)) = F[x_1/m(t_1), \ldots, x_n/m(t_n)]$$

    (formula $F$ with each variable $x_i$ substituted with $t_i$)

# Theory Morphisms Preserve Theorems

## Given

- well-typed theories $V$, $W$
- well-typed theory morphism $m : V \to W$
- well-typed $V$-formula $F$ that is a theorem

i.e., consequence of the $V$-axioms

Then:

$$m(F) \quad \text{is a } W\text{-theorem}$$

## Enables Big Picture Applications

- reuse theorems across theories
- show equivalence of theories
- find structural connection between seemingly large theories
- build module system using inheritance and functors
- build large theories from small systematically

# Semantics

# Theorems

## Semantics for an SFOL theory $V$

- theorem: a formula that is implied (must be true) by the axioms
- contradiction: a formula that must be false due to the axioms

  if there is negation: $F$ is contradiction iff $\neg F$ is theorem
- $V$ is inconsistent iff all formulas are theorems
  (equivalently: iff all formulas are contradictions)

more details on semantics later

## Kinds of Formulas

- ill-formed: cannot be represented in AST, parse error
- ill-typed: can be represented in AST, but rejected by type-checker
- well-typed: accepted by type-checker and one of
  - theorem
  - contradiction
  - other

  usually, most formulas are neither theorem not contradiction

# Implementing Semantics

## Automated Theorem Prover (ATP)

- input:
  - an SFOL-theory $V$
  - a well-typed $V$-formula $F$ (called the conjecture)
- output: $F$ is theorem/$F$ is contradiction/timeout

## Theorem status undecidable for most logics

- theorem provers try proving/refuting until interrupted
- typical: proof/refutation in a few seconds/minutes or never

many ATPs for SFOL: Vampire, E, Spass, ...

# TPTP: A Standard Concrete Syntax

## SFOL Syntax Standardization

- basically only one choice for abstract syntax of SFOL
- but lots of options for concrete syntax

  recall: concrete syntax = grammar with all the terminal symbols

## TPTP Concrete Syntax

- designed by Sutcliffe for CASC competition

  annual competition of SFOL ATPs
- gradually became de facto standard syntax for SFOL
- today: every SFOL ATP supports TPTP as input syntax
- various extensions of TPTP for other logics

Online interface for SFOL ATPs at `http://www.tptp.org`

# TPTP Syntax: Declarations

## Theory

- text file containing one line per declaration/axiom/conjecture

## Declarations

- named: line tff (decl_N, decl, N: TYPE). where
  - N is the name
  - TYPE is
    - $tType for a type declaration
    - Y1∗...∗Yn>Y for a function declaration
    - Y1∗...∗Yn>$o for a predicate declaration
- axiom: line tff (N, axiom, F). where
  - N is some name
  - F is the formula

# TPTP Syntax: Expressions

## Expressions

- ![X:TYPE]:F and ?[X:TYPE]:F for quantifiers
- F & G, F | G, F => G, ~F for connectives
- s(T1 ,..., Tn) for function/predicate applications

## Identifiers

- variables must start with upper-case letter
- type/function/predicate symbols must start with a lower-case letter
- remaining characters alphanumeric

## Conjecture (the formula to prove)

- like an axiom
- but with conjecture instead of axiom

# Exercise 7

Extend your implementation of SFOL as follows:

- Add a printer for printing SFOL theories and expressions that produces strings in TPTP syntax.
- Add a function that takes a theory and a conjecture and produces the corresponding TPTP input. Use an ATP to (try to) prove the conjecture.
- Optionally:
  - Add theory morphisms to the AST.
  - Implement the homomorphic extension — a function that takes a morphism $m : V \to W$ and a $V$-expression $E$ and returns the $W$-expression $m(E)$.

    actually one function each for terms, types, formulas
  - Add a check method for morphisms that calls the homomorphic extension, your type checker, and a theorem prover as needed.
  - Check an example theory morphism.

# Kinds of Semantics

# Recall

Recall:

| Syntax | Data |
|---|---|
| Semantics | Knowledge |

Representing

- syntax = formal language
  - grammar — context-free part
  - type system — context-sensitive well-formedness
- data = words in the syntax
  - set of vocabularies
  - set of typed expressions for each vocabulary
- semantics = ???
- knowledge = emergent property of having well-formed words with semantics

# Semantics as Querying

## General Idea

- Semantics answers questions about the syntax
- Based on the intended meaning of the syntax
- Specifies the meaning by giving the answers

## Special Cases

- Deductive semantics: answers the question whether a formula is a theorem
- Computational semantics: answers the question what the result of a program is
- Narrative semantics: allows answering any question if we understand natural language
- Concrete data semantics: answers the question what objects with certain properties exist

# Semantics as Imperfect Modeling

- Actual meaning of real-world difficult to model
  - practical argument: any practically interesting system has too many rules

    cf. physics, e.g., three-body problem already chaotic
  - theoretical argument: no language can fully model itself

    cf. Gödel's incompleteness theorems
- Practical semantics is approximation of ideal semantics
- Use practical purpose as guide for defining semantics
  - a set of questions that can be asked e.g., judgments or queries
  - a definition of what the answers are

restrict questions according to practical needs

# Aspects of Semantics

- Documentation: answers given as text
  - pro: easy to read for humans, critical to build intuitions
  - con: often ambiguous, contradictory, or incomplete
- Specification: correct answers defined by rule system

  also called calculus or inference system
  - pro: good stepping stone between the other two levels
  - con: accomplishes the pros of neither of them
- Implementation: answers computed by algorithm
  - pro: easy to automate, critical for efficiency and scale
  - con: essentially impossible to understand or analyze
- Unit testing: set of query/answer pairs
  - pro: easy to write, automate
  - con: does not cover the whole semantics

Rule system is sweet spot to connect human- and machine-friendly definitions.

# Relative Semantics by Translation

Components:

- Two syntaxes
  - object-language $l$ — e.g., BOL
  - meta-language $L$ — e.g., SFOL, Scala, SQL, English
- Semantics of $L$ assumed fixed — captures what we already know
- Semantics of $l$ by translation into $L$

semantics of $l$ relative to to existing semantics of $L$

Problem: just kicking the can?

# Discussion of Semantics by Translation

## Advantages

- a few meta-languages yield semantics for many languages
- easy to develop new languages
- good connection between syntax and semantics via compositionality, substitution theorem

## Disadvantages

- does not solve the problem once and for all
- impractical without implementation of semantics of meta-language
- meta-languages typically much more expressive than needed for object-languages
- translations can be difficult, error-prone

Also needed: absolute semantics

# Absolute vs. Relative Semantics

Absolute = self-contained, no use of meta-language $L$

## Get off the ground

- semantics for a few important meta-languages
  e.g., FOL, assembly language, set theory
- relative semantics for all other languages, e.g.,
  - model theory: logic $\to$ set theory
  - compilation: Scala $\to$ JVM $\to$ assembly

## Redundant semantics

- common to give
  - relative and absolute semantics for same syntax
  - multiple relative semantics    translations to different aspects
  - sometimes even maybe multiple absolute ones
- Allows understanding syntax from multiple perspectives
- Allows cross-checking    show equivalence of two semantics

# Example: Recall Syntax of Arithmetic Language

Syntax: represented as formal grammar

$\boxed{\text{Numbers}}$

$$
\begin{array}{llll}
N & ::= & 0 \mid 1 & \text{literals} \\
 & \mid & N + N & \text{sum} \\
 & \mid & N * N & \text{product}
\end{array}
$$

$\boxed{\text{Formulas}}$

$$
\begin{array}{llll}
F & ::= & N \doteq N & \text{equality} \\
 & \mid & N \leq N & \text{ordering by size}
\end{array}
$$

Implementation as inductive data type

# Example: Absolute Semantics

## Represented as judgments defined by sets of rules

- unclear what judgments to use
- here: computation $\vdash N \rightsquigarrow N$ and truth $\vdash F$

For numbers $n$: Rules to normalize numbers into values

$$\frac{}{\vdash N+0 \rightsquigarrow N} \qquad \frac{}{\vdash N*0 \rightsquigarrow 0} \qquad \frac{}{\vdash N*1 \rightsquigarrow N}$$

$$\frac{}{\vdash N*(R+S) \rightsquigarrow N*R+N*S}$$

and their commutative variants as well as

$$\frac{}{\vdash L+(M+N) \rightsquigarrow (L+M)+N}$$

For formulas $f$: rules to determine true formulas

$$\frac{}{\vdash N \doteq N} \qquad \frac{}{\vdash 0 \leq N} \qquad \frac{\vdash L \leq M}{\vdash L+N \leq M+N}$$

# Example: Absolute Semantics (2)

Checking if an absolute semantics works as intended is hard.

Here: number rules allow

1. eliminating all cases where arguments of $*$ are 0, 1, or $+$; thus, no more $*$
2. eliminating all cases where arguments of $+$ are 0
3. shift brackets of nested $+$ to the left
4. left: 0 or $(\ldots(1+1)\ldots+1)$ — isomorphic to natural numbers

formula rules allow

1. concluding equality if identical normal forms
2. reducing $M+1 \leq N+1$ to $M \leq N$, repeat until $0 \leq N$

# Example: Relative Semantics

Semantics: represented as translation into known language

Problem: Need to choose a known language first
Here: unary numbers represented as strings
Built-in data (strings and booleans):

$$
\begin{array}{rcll}
S & ::= & \texttt{""} & \text{empty} \\
 & | & \texttt{(Unicode)} & \text{character sequence} \\
B & ::= & \texttt{true} & \text{truth} \\
 & | & \texttt{false} & \text{falsity}
\end{array}
$$

Built-in operations to work on the data:

- concatenation of strings $S ::= \texttt{conc}(S, S)$
- replacing all occurrences of $c$ in $S_1$ with $S_2$
  $S ::= \texttt{replace}(S_1, c, S_2)$
- equality test: $B ::= S_1 == S_2$
- prefix test: $B ::= \texttt{startsWith}(S_1, S_2)$

# Example: Relative Semantics

## Represented as function from syntax to semantics

- mutually recursive, inductive functions for each non-terminal symbol
- compositional: recursive call on immediate subterms of argument

For numbers $n$: semantics $[\![n]\!]$ is a string

- $[\![0]\!] = ""$
- $[\![1]\!] = "|"$
- $[\![m+n]\!] = \texttt{conc}([\![m]\!], [\![n]\!])$
- $[\![m*n]\!] = \texttt{replace}([\![m]\!], "|", [\![n]\!])$

For formulas $f$: semantics $[\![f]\!]$ is a boolean

- $[\![m \doteq n]\!] = [\![m]\!] == [\![n]\!]$
- $[\![m \leq n]\!] = \texttt{startsWith}([\![n]\!], [\![m]\!])$

# Example: Equivalence of Semantics

## For formulas

- if $\vdash F$, then $[\![F]\!] = \mathit{true}$ usually called soundness
- if $[\![F]\!] = \mathit{true}$, then $\vdash F$ usually called completeness

## For numbers

- $\vdash N \rightsquigarrow 0$ iff $[\![N]\!] = ""$
- $\vdash N \rightsquigarrow (\ldots(1+1)\ldots+1)$ iff $[\![N]\!] = "|\ldots|"$

# Relative Semantics for BOL

# Semantics of BOL

| Aspect | kind of semantic language | semantic language |
|---|---|---|
| deduction | logic | SFOL |
| concretization | database language | SQL |
| computation | programming language | Scala |
| narration | natural language | English |

# Narrative Semantics of BOL in English

We discussed earlier

- the rough design of how natural languages can be seen as a formal systems
  - grammar book = syntax
  - sentences = formulas
  - dictionary + common sense statements = standard library
  - domain-specific dictionary + sentences = vocabulary

  , ,
- the translation from BOL to it
- the non-compositional aspects of natural language

see details in the lecture notes

# Deductive Semantics of BOL in SFOL

We discuss

- the grammar of SFOL
- context-sensitive languages with variable binding (of which SFOL is an example)
- an implementation of SFOL in Scala
- the translation from BOL to SFOL
- compositionality of the translation
- the issue of
  - non-compositionality
  - the need for a semantic prefix

see details in the lecture notes

# Relative Semantics by Translation

# General Definition

A semantics by translation consists of

- syntax: a formal system $I$
- semantic language: a formal system $L$

  different or same aspect as $I$
- semantic prefix: a vocabulary $P$ in $L$

  formalizes fundamentals that are needed to represent $I$-objects
- interpretation: translates every $I$-vocabulary $T$ to an $L$-vocabulary $P, [\![T]\!]$

# Common Principles

Properties shared by all semantics by translation

not part of formal definition, but best practices

- $l$-declaration translated to $L$-declaration for the same name
- vocabularies translated declaration-wise
- one inductive function for every kind of complex $l$-expression
  - individuals, concepts, relations, properties, formulas
  - maps $l$-expressions to $L$-expressions
- atomic cases (base cases): $l$-identifier translated to $L$-identifier of the same name — or something very similar
- complex cases (step cases): compositional

# Compositionality

Case for operator $*$ in translation function compositional iff interpretation of $*(e_1, \ldots, e_n)$ only depends on on the interpretation of the $e_i$

$$[\![*(e_1, \ldots, e_n)]\!] = [\![*]\!]([\![e_1]\!], \ldots, [\![e_n]\!])$$

for some function $[\![*]\!]$

Example: ;-operator of BOL in translation to FOL

- translation: $[\![R_1; R_2]\!] = \exists m : \iota.[\![R_1]\!](x, m) \wedge [\![R_2]\!](m, y)$
- special case of the above via
  - $* =;$
  - $n = 2$
  - $[\![;]\!] = (p_1, p_2) \mapsto \exists m : \iota.p_1(x, m) \wedge p_2(m, y)$
- Indeed, we have $[\![R_1; R_2]\!] = [\![;]\!]([\![R_1]\!], [\![R_2]\!])$

# Compositionality (2)

Translation compositional iff

- one translation function for each non-terminal all written $[\![-]\!]$
- each defined by one induction on syntax

i.e., one case for production
mutually recursive

- all cases compositional

Substitution theorem: a compositional translation satisfies

$$[\![E(e_1, \ldots, e_n)]\!] = [\![E]\!]([\![e_1]\!], \ldots, [\![e_n]\!])$$

for

- every expression $E(N_1, \ldots, N_n)$ with non-terminals $N_i$
- some function $[\![E]\!]$ that only depends on $E$

# Compositionality (3)

$$[\![E(e_1,\ldots,e_n)]\!] = [\![E]\!]([\![e_1]\!],\ldots,[\![e_n]\!])$$

for every expression $E(N_1,\ldots,N_n)$ with non-terminals $N_i$

Now think of

- variable $x_i$ of type $N_i$ instead of non-terminal $N_i$
- $E(x_1,\ldots,x_n)$ as expression with free variables $x_i$ of type $N_i$
- expressions $e$ derived from $N$ as expressions of type $N$
- $E(e_1,\ldots,e_n)$ as result of substituting $e_i$ for $x_i$
- $[\![E]\!](x_1,\ldots,x_n)$ as (semantic) expression with free variables $x_i$

Then both sides of equations act on $E(x_1,\ldots,x_n)$:

- left side yields $[\![E(e_1,\ldots,e_n)]\!]$ by
  - first substitution $e_i$ for $x_i$
  - then semantics $[\![-]\!]$ of the whole
- right side yields $[\![E]\!]([\![e_1]\!],\ldots,[\![e_n]\!])$ by
  - first semantics $[\![-]\!]$ of all parts
  - then substitution $[\![e_i]\!]$ for $x_i$

semantics commutes with substitution

# Non-Compositionality

## Examples

- deduction: cut elimination, translation from natural deduction to Hilbert calculus
- computation: optimizing compiler, e.g., loop unrolling
- concretization: query optimization, e.g., turning a WHERE of a join into a join of WHEREs,
- narration: ambiguous words are translated based on context

## Typical sources

- subcases in a case of translation function
  - based on inspecting the arguments, e.g., subinduction
  - based on context
- custom-built semantic prefix

# Translation vs. Embedding

## Translation

- as above, $l$ and $L$ are at the same level
- $l$-declarations represented as $L$-declarations

also called shallow embedding

## Embedding

- $L$ is used as meta-language to represent $l$

  e.g., $L$ is programming language to implement $l$
- $l$-declarations represented as $L$-objects using an inductive type

  also called deep embedding

# Exercise 8

Implement the translation from BOL to SFOL. Translate your university ontology.

Formulate a BOL-conjecture and prove it by sending the translated version to an SFOL theorem prover.

# Denotational Semantics

# Relative Semantics: Denotation vs. Translation

## Translation semantics

- vocabularies mapped to vocabularies, expressions to expressions
- symbols in input vocabulary yield symbols in output vocabulary
- examples:
  - BOL to SFOL
  - compiling a programming language to another language

## Denotational semantics

- symbols in vocabulary given concrete value

  value/meaning/denotation/interpretation
- model: maps every symbols to its concrete value
- expressions interpreted relative to fixed situation
- examples
  - interpreting a program

    situation = input+run-time environment
  - interpreting logical formulas in a model situation = model

# Interpreted vs. Uninterpreted Symbols

## Interpreted = symbols with fixed semantics

- base types and their operations — integer, etc.
- concrete data types — enumerations, inductive types

semantics fixed by language

## Uninterpreted = semantics open to interpretation

- e.g, $a : \mathtt{type}$, $f : a \to a$, etc.
- axioms/definitions constrain/specify possible interpretations

semantics constrained by vocabulary, fixed by situation

## Relative to situation

- semantics of all symbols fixed
- semantics of every expression can be determined

often but not necessarily computable

# Example

## Vocabulary: rules of the world

- type $a$
- operation $f : a \to a$
- relations $r : a \to \texttt{prop}$
- axioms $F$ about $f, r$

## Situation: one concrete world $S$

- specific set $a^S$
- specific function $f^S$ from $a^S$ to $a^S$
- specific subset $r^S$ of $a^S$
- proof that $F$ holds about $f^S$, $r^S$

| Aspect | Abstract Vocabulary | Situation |
|---|---|---|
| ontology | TBox | initial situation through ABox |
| data | schema | database |
| deduction | theory | model |

# Terminology

"vocabulary" and "situation" are not standard names. They are introduced here to unify the different kinds of languages.

The standard names vary by knowledge aspect:

| Aspect | Vocabulary | Situation | Vocabulary+Situation |
|---|---|---|---|
| ontology | TBox | ABox | ontology |
| data | schema | database | SQL dump |
| deduction | theory | model | concrete theory |
| computation | program | environment | execution |
| narration | dictionary | technical jargon | |

# Approaches to Uninterpreted Symbols

## Usually only one extreme

- Computation: typically mostly interpreted symbols, situation only provides input/environment — programs can be run directly
- Deduction: typically only uninterpreted symbols — focus on studying the possible models

## Some attempts at combining

- reasoning about programs — e.g., functions with pre-/postconditions
- logics with built-in base types — e.g., SMT solving

# Initial Semantics

## Some situations can be captured in vocabulary

- ABox part of ontology
- schema and table entries part of SQL syntax
- trickier when vocabulary symbols represent abstract sets/functions
  usually meta-language needed to define concrete semantics

## Any vocabulary induces default situation

- inhabitants of a type are exactly the terms of that type
- functions map exactly as given by axioms
- every situation must be an extension
- called initial situation, or initial semantics
- examples:
  - ABox, database tables if part of vocabulary
  - Herbrand model of logical theory

# Absolute Semantics for BOL

# Judgments

Goal: Answer the question whether a formula is a theorem.

Use deduction judgment:

$$\Gamma \vdash_V^{BOL} F$$

for formula $F$

Notation:

- We drop the superscript $^{BOL}$ whenever clear.
- We drop the subscript $_V$ whenever clear.
- We drop the context $\Gamma$ if it is empty.

# Lookup Rules

The main rules that need to access the vocabulary:

$$\frac{f \text{ in } V}{\vdash_V f}$$

for assertions or axioms f

Assumptions in the context are looked up accordingly:

$$\frac{x : f \text{ in } \Gamma}{\Gamma \vdash f}$$

# Rules for Subsumption and Equality

Subsumption is an order with respect to equality:

$$\frac{}{\vdash c \sqsubseteq c}$$

$$\frac{\vdash c \sqsubseteq d \qquad \vdash d \sqsubseteq e}{\vdash c \sqsubseteq e}$$

$$\frac{\vdash c \sqsubseteq d \qquad \vdash d \sqsubseteq c}{\vdash c \equiv d}$$

Equal concepts can be substituted for each other:

$$\frac{\vdash c \equiv d \qquad x : C \vdash f(x) : \mathtt{prop} \qquad \vdash f(c)}{\vdash f(d)}$$

This completely defines equality.

# Rules relating Instancehood and Subsumption

$$\frac{\vdash i\ \mathtt{is\text{-}a}\ c \qquad \vdash c \sqsubseteq d}{\vdash i\ \mathtt{is\text{-}a}\ d}$$

Read:

- if
  - $i\ \mathtt{is\text{-}a}\ c$
  - $c \sqsubseteq d$
- then $i\ \mathtt{is\text{-}a}\ d$

$$\frac{x : I, x\ \mathtt{is\text{-}a}\ c \vdash x\ \mathtt{is\text{-}a}\ d}{\vdash c \sqsubseteq d}$$

Read:

- if
  - assuming an individual $x$ and $x\ \mathtt{is\text{-}a}\ c$, then $x\ \mathtt{is\text{-}a}\ d$
- then $c \sqsubseteq d$

# Induction

Consider from before

$$\frac{x : I,\ x\ \mathtt{is\text{-}a}\ c \vdash x\ \mathtt{is\text{-}a}\ d}{\vdash c \sqsubseteq d}$$

Question: Do we allow proving the hypothesis by checking for each individual $x$? induction

# Induction

Consider from before

$$\frac{x : I,\ x\ \mathtt{is\text{-}a}\ c \vdash x\ \mathtt{is\text{-}a}\ d}{\vdash c \sqsubseteq d}$$

Question: Do we allow proving the hypothesis by checking for each individual $x$? induction

- Open world: no

# Induction

Consider from before

$$\frac{x : I,\ x\ \texttt{is-a}\ c \vdash x\ \texttt{is-a}\ d}{\vdash c \sqsubseteq d}$$

Question: Do we allow proving the hypothesis by checking for each individual $x$? induction

- Open world: no
- Closed world: yes

$$\frac{\Gamma[x = i] \vdash f[x = i] \ \text{ for every individual } i}{\Gamma, x : I \vdash f(x)}$$

effectively applicable if only finitely many individuals

# Rules for Union and Intersection of Concepts

Union as the least upper bound:

$$\frac{}{\vdash c \sqsubseteq c \sqcup d} \qquad \frac{}{\vdash d \sqsubseteq c \sqcup d}$$

$$\frac{\vdash c \sqsubseteq h \qquad \vdash d \sqsubseteq h}{\vdash c \sqcup d \sqsubseteq h}$$

Dually, intersection as the greatest lower bound:

$$\frac{}{\vdash c \sqcap d \sqsubseteq c} \qquad \frac{}{\vdash c \sqcap d \sqsubseteq d}$$

$$\frac{\vdash h \sqsubseteq c \qquad \vdash h \sqsubseteq d}{\vdash h \sqsubseteq c \sqcap d}$$

# Rules for Existential and Universal

Easy rules:

- Existential

$$\frac{\vdash i\ r\ j \qquad \vdash j\ \texttt{is-a}\ c}{\vdash i\ \texttt{is-a}\ \exists r.c}$$

- Universal

$$\frac{\vdash i\ \texttt{is-a}\ \forall r.c \qquad \vdash i\ r\ j}{\vdash j\ \texttt{is-a}\ c}$$

Other directions are trickier:

- Existential

$$\frac{\vdash i\ \texttt{is-a}\ \exists r.c \qquad j : I,\ i\ r\ j,\ j\ \texttt{is-a}\ c \vdash f}{\vdash f}$$

- Universal

$$\frac{j : I,\ i\ r\ j \vdash j\ \texttt{is-a}\ c}{\vdash i\ \texttt{is-a}\ \forall r.c}$$

## Selected Rules for Relations

Inverse:

$$\frac{\vdash i\, r\, j}{\vdash j\, r^{-1}\, i}$$

Composition:

$$\frac{\vdash i\, r\, j \qquad \vdash j\, s\, k}{\vdash i\,(r;s)\,k}$$

Transitive closure:

$$\frac{}{\vdash i\, r^{*}\, i} \qquad \frac{\vdash i\, r\, j \qquad \vdash j\, r^{*} k}{\vdash i\, r^{*} k}$$

Identity at concept $c$:

$$\frac{\vdash i\, \texttt{is-a}\, c}{\vdash i\, \Delta_c i}$$

# Kinds of Typing: Extrinsic and Intrinsic

## Breakout Question

Is this an improvement over BOL?

$\boxed{\text{Declarations}}$

| | | | |
|---|---|---|---|
| $D$ | $::=$ | **individual** $i : C$ | typed atomic individual |
| | $\mid$ | **concept** $c$ | atomic concept |
| | $\mid$ | **relation** $r \subseteq C \times C$ | typed atomic relation |
| | $\mid$ | **property** $p \subseteq C \times T$ | typed atomic property |

rest as before

# Actually, when is a language an improvement?

orthogonal, often mutually exclusive criteria

## Trade-off for syntax design

- expressivity: easy to express knowledge
  e.g., big grammar, complex type system
- simplicity: easy to implement/interpret
  e.g., few, carefully chosen productions, types

## Semantics

- specification, implementation, documentation

## Intended users

- skill level
- prior experience with related languages
- amount of training needed
- innovation height, differential evaluation against existing languages

# Actually, when is a language an improvement? (2)

## Support software ecosystem

- optional tool support: IDEs, debuggers, heuristic checkers, alternative implementations, interpreter/REPL
- many/large well-crafted vocabularies and package managers to find them
- integrations with other languages: translations, common run-time platforms, foreign function interface

## Long-term plans: re-answer the above question but now

- maintainability: syntax was changed, everything to be redone
- backwards compatibility: support for legacy input
- scalability: expressed knowledge content has reached huge sizes

# General Idea

## A **type system** for a syntax consists of

- some non-terminals $\mathcal{E}$, whose words are called $\mathcal{E}$-**expressions**,

  coarse, context-free, classification into disjoint sets
- for some symbols $\mathcal{E}$
  - set of types: $\mathcal{T}$-expressions for a non-terminal $\mathcal{T}$
  - typing relation $\Gamma \vdash^L_V e : T$ between $\mathcal{E}$-expressions $e$ and $\mathcal{T}$-expressions $T$

  fine, context-sensitive, classification into disjoint or overlapping sets

## Examples

- BOL: non-terminals $\mathcal{E}$ for expressions are $C, I, R, P, F$
  $I$-expressions typed by $C$-expressions

  overlapping, types undecidable/difficult to check
- SFOL: non-terminals $\mathcal{E}$ for expressions are $Y, T, F$
  $T$-expressions typed by $Y$-expressions disjoint, types easy to infer

# Church vs. Curry Typing

| | intrinsic | extrinsic |
|---|---|---|
| $\lambda$-calculus by | Church | Curry |
| type is | carried by object | given by environment |
| typing is a | function objects $\rightarrow$ types | relation objects $\times$ types |
| objects have | unique type | any number of types |
| types interpreted as | disjoint sets | unary predicates |
| type given by | part of declaration | additional axiom |
| example | **individual** "WuV":"course" | **individual** "Wuv", "WuV" `is-a` "course" |
| examples | SFOL, SQL | OWL, Scala, English |
| | most logics, functional PLs | ontology, OO, natural languages |
| | many type theories | set theories |

# Type Checking

| | intrinsic | extrinsic |
|---|---|---|
| type is | carried by object | given by environment |
| typing is a | function objects $\to$ types | relation objects $\times$ types |
| objects have | unique type | any number of types |
| type given by | part of declaration | additional axiom |
| example | **individual** WuV: course | **individual** Wuv,<br>WuV `is-a` course |
| type inference for $x$ | uniquely infer $A$ from $x$ | find minimal $A$ with $x : A$ |
| type checking | inferred=expected | prove $x : A$ |
| subtyping $A <: B$ | cast from $A$ to $B$ | $x : A$ implies $x : B$ |
| typing decidable | yes unless too expressive | no unless restricted |
| typing errors | static (compile-time) | dynamic (run-time) |
| advantages | easy<br>unique type inference | flexible<br>allows subtyping |

# Examples: Curry-Typing in BOL

| Semantics | objects | types | typing relation |
|---|---|---|---|
| absolute | individuals $I$ | concepts $C$ | $i$ `is-a` $C$ |
| SFOL | terms of type $\iota$ | predicates $C \subseteq \iota$ | $c(i)$ |
| English | proper nouns | common nouns | "$i$ is a $C$" |

# Examples

| System | typing | objects | types | typing relation |
|---|---|---|---|---|
| any | Church | expressions | non-terminals | derived from |
| BOL | Curry | individuals | concepts | `is-a` |
| SFOL | Church | terms | types | : |
| set theory | Curry | sets | sets | $\in$ |
| OO | Curry | instances | classes | `isInstanceOf` |

# Subtyping

Subtyping works best with Curry Typing

- explicit subtyping as in $\mathbb{N} <: \mathbb{Z}$
- comprehension/refinement as in $\{x : \mathbb{N} | x \neq 0\}$
- operations like union and intersection on types
- inheritance between classes, in which case subclass = subtype
- anonymous record types as in $\{x : \mathbb{N}, y : \mathbb{Z}\} <: \{x : \mathbb{N}\}$

# Ontologies vs. Databases

# Recall: Ontologies

## Main ideas

- Ontology abstractly describes concepts and relations
- Tool maintains concrete data set
- Focus on efficiently
  - identifying (i.e., assign names)
  - representing
  - processing
  - querying

  large sets of concrete data

## Recall: TBox-ABox distinction

- TBox: general parts, abstract, fixed

  main challenge: correct modeling of domain
- ABox: concrete individuals and assertions about them, growing

  main challenge: aggregate them all

# Concrete Data

## Concrete is

- Base values: integers, strings, booleans, etc.
- Collections: sets, multisets, lists (always finite)
- Aggregations: tuples, records (always finite)
- User-defined concrete data: enumerations, inductive types
- Advanced objects: finite maps, graphs, etc.

## Concrete is not

- Uninterpreted symbols
- Variables (free or bound) $\lambda$-abstraction, quantification
- Symbolic expressions formulas, algorithms

# Two Approaches to Representing Concrete Data

## Curry-typed ontology languages (e.g., BOL, OWL)

- Representation based on knowledge graph
- Ontology written in BOL-like language
- Data maintained as set of triples — tool = triple store
- Typical language/tool design
  - ontology and query language separate — e.g., OWL, SPARQL
  - triple store and query engine integrated — e.g., Virtuoso tool

## Church-typed languages (e.g., SQL, UML)

- Representation based on abstract data types
- Ontology written as set of related ADTs — SQL database schema
- Data maintained as tables — tool = (relational) database
- Typical language/tool design
  - ontology and query language integrated — e.g., SQL
  - table store and query engine integrated — e.g., SQLite tool

# Evolution of Approaches

## Our usage is non-standard

- Common
  - ontologies = untyped approach, OWL, triples, SPARQL
  - relational databases = typed approach, tables, SQL
- Our understanding: two approaches evolved from same idea
  - ontolgoies = Curry-typed ontology + data store
  - relational database = Church-typed ontology + data store

## Evolution

- Typed-untyped distinction minor technical difference
- Optimization of respective advantages causes speciation
- Today segregation into different
  - jargons
  - languages, tools
  - communities, conferences
  - courses

# Data structures for Curry-typed concrete data

## Central data structure = knowledge graph

- nodes = individuals *i*
  - identifier
  - sets of concepts of *i*
  - key-value sets of properties of *i*
- edges = relation assertions
  - from subject to object
  - labeled with name of relation

## Processing strengths

- store: as triple set
- edit: Protege-style or graph-based
- visualize: as graph different colors for concepts, relations
- query: match, traverse graph structure
- untyped data simplifies integration, migration

# Data structures for Church-typed concrete data

## Central data structure = relational database

- tables = abstract data type
- rows = objects of that type
- columns = fields of ADT
- cells = values of fields

## Processing strengths

- store: as CSV text files, or similar
- edit: SQL commands or table editors
- visualize: as table view
- query: relational algebra
- typed data simplifies selecting, sorting, aggregating

# Identifiers

## Curry-Typed Knowledge graph

- concept, relation, property names given in TBox
- individual names attached to nodes

## Church-Typed Database

- table, column names given in schema
- row identified by distinguished column (= key)
  options
  - preexistent characteristic column
  - added upon insertion
    - UUID string
    - incremental integers
    - concatenation of characteristic list of columns
- column/row identifiers formed by qualifying with table name

# Axioms

## Curry-Typed Knowledge Graph

- traditionally very expressive axioms
- yields inferred assertions
- triple store must do consequence closure to return correct query results
- not all axioms supported by every triple store

## Church-Typed Database

- typically no axioms
- instead consistency constraints, triggers
- allows limited support for axioms without calling it that way
- stronger need for users to program the consequence closure manually

# Open/Closed World

- Question: is the data complete?
  - closed world: yes
  - open world: not necessarily
- Dimensions of openness
  - existence of individual objects
  - assertions about them
- Sources of openness
  - more exists but has not yet been added
  - more could be created later
- Orthogonal to typed/untyped distinction, but in practice
  - knowledge graphs use open world
  - databases use closed world

Open world is natural state, closing adds knowledge

# Closing the World

## Derivable consequences

- induction: prove universal property by proving for each object
- negation by failure: atomic property false if not provable
- term-generation constraint: only nameable objects exist

## Enabled operations

- universal set: all objects
- complement of concept/type
- defaults: assume default value for property if not otherwise asserted

## Monotonicity problem

- monotone operation: bigger world = more results
- examples: union, intersection, $\exists R.C$, join, IN conditions
- counter-examples: complement, $\forall R.C$, NOT IN conditions

technically, non-monotone operations in open world dubious

# Summary

| | semantic web | relational databases |
|---|---|---|
| ontology aspect | TBox of ontology | SQL schema |
| conceptual model | knowledge graph | set of tables |
| concrete data aspect | ABox of ontology | SQL database |
| concrete data storage | set of triples | set of rows of the tables |
| concrete data formats | RDF | CSV |
| concrete data tool | triple store | database implementation |
| typing | soft/Curry | hard/Church |
| query language | SPARQL | SQL SELECT query |
| openness of world | tends to be open | tends to be closed |

# Exercise 9

Absolute semantics:

- Via concrete data: Export your ontology to a triple store like Virtuoso and run a concrete query in SPARQL.
- Via deduction: Export your ontology in OWL format to a reasoner like FaCT++ and prove a theorem. Potentially, do this by installing a plugin for a reasoner in your ontology IDE.

Relative semantics:

- Via deduction: finish exercise 8 and prove a theorem via an SFOL theorem prover.

# Kinds of Types

# Question

What kind of types are there?

# Abstract vs. Concrete Types

**Concrete** type: values are

- given by their internal form,
- defined along with the type, typically built from already-known pieces.

product types, enumeration types, collection types

main example: concrete (inductive/algebraic) data types

**Abstract** type: values are

- given by their externally visible properties,
- defined in any environment that understands the type definition.

structures, records, classes, aggregation types

main example: abstract data types

# Non-Recursive vs. Recursive Types

**Non-Recursive** types

- given by some expressions
- can be anonymous
- values given directly

integers, lists of strings, ...

**Recursive** types

- definition of the type must refer to the type itself
  so type must have a name
- type typically defines other named operations
- values obtained by fixed-point constructions

optional property of concrete and abstract data types

# Atomic vs. Complex Types

**Atomic** type

- given by its name
- values are a set

integers, strings, booleans, ...

**Complex** types

- arise by applying type symbol to arguments
- separate set of values for each tuple of arguments

two kinds of complex types (next slide)

# Type operators vs. Dependent type Families

Both are complex: take arguments and return a type

**Type operators** take only type arguments, e.g.,

- type operator $\times$
- takes two types $A, B$
- returns type $A \times B$

**Dependent types** take also value arguments, e.g.,

- dependent type operator *vector*
- takes natural number $n$, type $A$
- returns type $A^n$ of $n$-tuples over $A$

dependent types much more complicated, less uniformly used

harder to starndardize

# Built-in vs. User-Defined Types

**Built-in** types

- syntax and semantics fixed by language designer
- part of grammar, implementation, etc.
- usually concrete, atomic, non-recursive

typical: integers, strings, lists

sometimes also called primitive or basic types

**User-Defined** types

- declared by users in vocabulary
- standard syntax prescribed by grammar, possibly customizable
- semantics given by operations and their axioms

anything the language can axiomatize

usually difficult to axiomatize recursive properties

# Non-Recursive Data Types

# Common Built-in Types

## Typical (quasi-)primitive types

- natural numbers ($= \mathbb{N}$)
- arbitrary precision integers ($= \mathbb{Z}$)
- fixed precision integers (32 bit, 64 bit, ...)
- floating point (float, double, ...)
- Booleans
- characters (ASCII, Unicode)
- strings

Observation:

- essentially the same in every language
  including whatever language used for semantics
- semantics by translation trivial

# Less Common Types

## Problem

- quickly encounter primitive types not supported by common languages
- need to encode them using existing types

typically as strings, ints, or prodcuts/lists thereof

## Examples

- date, time, color, location on earth
- graph, function
- picture, audio, video
- physical quantities ($1m$, $1in$, etc.)
- gene, person

# Specifying a Type

## Components

- the type — eg: 'int'
- values of the type — eg: 0, 1, -1, ...
- string encoding — injective function from values to strings
- operations on type — eg: addition, multiplication, ...

## Examples

- IEEE floating point numbers
- ISO 8601 for date/time

# Type Operators

As for types, but taking $n$ type argument.

## Component

- the type eg: 'list'
- arity $n$ eg: 1
- for argument types $A_1, \ldots, A_n$ with string encodings $E_1, \ldots, E_n$
  - values of the type eg: $[a_1, \ldots, a_n]$ for $a_i$ values of $A_1$
  - string encoding eg: "[" $E(a_1)$ "," ..."," $E_1(a_n)$ "]"
  - operations on the type eg: concatenation, element access, ...

No good standards, but part of most languages

# Basic Atomic Types

## typical in IT systems

- fixed precision integers (32 bit, 64 bit, ...)
- IEEE float, double
- Booleans
- Unicode characters
- strings could be list of characters but usually bad idea

## typical in math

- natural numbers ($= \mathbb{N}$)
- arbitrary precision integers ($= \mathbb{Z}$)
- rational, real, complex numbers
- graphs, trees

clear: language must be modular, extensible

# Advanced Atomic Types

### general purpose

- date, time, color, location on earth
- picture, audio, video

### domain-specific

- physical quantities ($1m$, $1in$, etc.)
- gene, person
- semester, course id, ...

clear: language must be modular, extensible

# Collection Data Types

## Homogeneous Collection Types

- sets
- multisets (= bags)
- lists all unary type operators, e.g. *list A* is type of lists over *A*
- fixed-length lists (= Cartesian power, vector *n*-tuple)

dependent type operator

## Heterogeneous Collection Types

- lists
- fixed-length lists (= Cartesian power, *n*-tuple)
- sets
- multisets (= bags)

all atomic types, e.g., *list* is type of lists over any objects

# Aggregation Data Types

## Products

- Cartesian product of some types $A \times B$
  values are pairs $(x, y)$ numbered projections $_1$, $_2$ — order relevant
- labeled Cartesian product (= record) $\{a : A, b : B\}$
  values are records $\{a = x, b = y\}$
  named projections $a$, $b$ — order irrelevant

## Disjoint Unions

- disjoint union of some types $A \uplus B$
  values are $inj_1(x)$, $inj_2(y)$ numbered injections $_1$, $_2$ — order relevant
- labeled disjoint union $a(A)|b(B)$
  values are constructor applications $a(x)$, $b(y)$
  named injections $a$, $b$ — order irrelevant

labeled disjoint unions uncommon

but recursive labeled disjoint union = inductive data type

# A Basic Language for Data Types

Let BDL be given by

$\boxed{\text{Types}}$

| | | |
|---|---|---|
| $T ::= \mathit{int} \mid \mathit{float} \mid \mathit{string} \mid \mathit{bool}$ | | base types |
| $\mid$ | $\texttt{list}\ T$ | homogeneous lists |
| $\mid$ | $(\texttt{ID} : T)^*$ | record types |
| $\mid$ | $\ldots$ | additional types |

$\boxed{\text{Data}}$

| | | |
|---|---|---|
| $D ::=$ | $(\mathit{64\ bit\ integers})$ | |
| $\mid$ | $(\mathit{IEEE\ double})$ | |
| $\mid$ | $"(\mathit{Unicode\ strings})"$ | |
| $\mid$ | $\texttt{true} \mid \texttt{false}$ | |
| $\mid$ | $D^*$ | lists |
| $\mid$ | $(\texttt{ID} = D)^*$ | records |
| $\mid$ | $\ldots$ | constructors for additional types |

# Recursive Concrete Data Types

# Motivation

Idea

- describe infinite type in finite way
- describe words derived from grammars
- exploit inductive structure to catch all values

Name: usually called **inductive** data type, especially when recursive

## Examples

Natural numbers $Nat$ given by

- $zero$
- $succ(n)$ for every $n : Nat$

Lists $list\ A$ over type $A$ given by

- empty list $nil$
- $cons(a, l)$ for every $a : A$ and $l : list\ A$

Arithmetic expressions $E$ given by

- natural number $literal(n)$ for $n : Nat$
- sum $plus(e, f)$ for every $e, f : E$
- product $times(e, f)$ for every $e, f : E$

# Rigorous Definition

Let $T$ be the set of types that are known in the current context.

An **inductive data type** is given by

- a name $n$, called the **type**,
- a set of **constructors** each consisting of
  - a name
  - a list of elements of $T \cup \{n\}$, called the **argument** types

Notation:

```
inductive n = c(A,...) | ...

inductive Nat = zero | succ(Nat)
inductive E = Number(Nat) | sum(E,E) | times(E,E)
```

# Induction Principle

The values of the inductive type are exactly the ones that can be built by the constructors.

- No junk: the constructors are jointly-surjective
  - no other values but union of their images
  - closed world
- No confusion: the constructors are jointly-injective in the following sense
  - each constructor is an injective function
  - images of the constructor are pairwise disjoint

Inductive definition: define function out of inductive type by giving one case per constructor pattern matching

- total because jointly-surjective
- well-defined because jointly-injective (no overlap between cases)

# Special case: No recursion

A concrete data types without recursive constructor arguments are called a **labeled union**. They are isomorphic to the union of the products of the constructor arguments.

Example:

```
inductive Value = Number(Nat) | true | false
inductive Product(A,B) = Pair(A,B)
```

A concrete data types without any constructor arguments is called an **enumeration**. They have exactly one element per constructor.

Example:

```
inductive Boolean = true | false
inductive Color = red | blue | green
```

# Generalization: Mutual Induction

Multiple inductive types whose definitions refer to each other.

Example:

```
inductive E = literal(Nat) | sum(E,E) | times(E,E)
inductive F = equal(E,E) | less(E,E)
```

# BDL Extended with Named Inductive Types

| | | | |
|---|---|---|---|
| $V$ | $::=$ | $Decl^*$ | Vocabularies |
| $Decl$ | $::=$ | $\textbf{inductive}\ t\ \{(\mathtt{ID} : T^* \to t)^*\}$ | type definitions |

$\boxed{\text{Types}}$

| | | | |
|---|---|---|---|
| $T$ | $::=$ | $\ldots$ | as before |
| | $\mid$ | $t$ | reference to a named type |

$\boxed{\text{Data}}$

| | | | |
|---|---|---|---|
| $D$ | $::=$ | $\ldots$ | as before |
| | $\mid$ | $\mathtt{ID}(D^*)$ | constructor application |

# Recursive Abstract Data Types

## Breakout Question

What do the following have in common?

- Java class
- SQL schema for a table
- logical theory (e.g., Monoid)

# Breakout Question

What do the following have in common?

- Java class
- SQL schema for a table
- logical theory (e.g., Monoid)

all are (essentially) abstract data types

## Motivation

Recall subject-centered representation of assertion triples:

```
individual "FlorianRabe"
  is-a "instructor" "male"
  "teach" "WuV" "KRMT"
  "age" 40
  "office" "11.137"
```

Can we use types to force certain assertions to occur together?

- Every instructor should teach a list of courses.
- Every instructor should have an office.

# Motivation

Inspires **subject-centered types**, e.g.,

```
concept instructor
  teach course*
  age: int
  office: string

individual "FlorianRabe": "instructor"
  is-a "male"
  teach "WuV" "KRMT"
  age 40
  office "11.137"
```

Incidental benefits:

- no need to declare relations/properties separately
- reuse relation/property names
  distinguish via qualified names: `instructor.age`

# Motivation

Natural next step: inheritance

```
concept person
  age: int

concept male <: person

concept instructor <: person
  teach course*
  office: string

individual "FlorianRabe": "instructor" ⊓ "male"
  "teach" "WuV" "KRMT"
  "age" 40
  "office" "11.137"
```

our language quickly gets a very different flavor

# Examples

Prevalence of abstract data types:

| aspect | language | abstract data type |
|---|---|---|
| ontologization | UML | class |
| concretization | SQL | table schema |
| computation | Scala | class, interface |
| deduction | various | theory, specification, module, locale |
| | | if recursive: coinductive type |
| narration | various | emergent feature |

same idea, but may look very different across languages

# Examples

| aspect | type | values |
|---|---|---|
| computation | abstract class | instances of implementing classes |
| concretization | table schema | table rows |
| deduction | theory | models |

Values depend on the environment in which the type is used:

- class defined in one specification language (e.g., UML), implementations in programing languages Java, Scala, etc.
  available values may depend on run-time state
- theory defined in logic,
  models defined in set theories, type theories, programming languages
  available values may depend on philosophical position

# Definition

Given some type system, an **abstract data type** (ADT) is defined by

$$\textbf{class}\, a = \{c_1 : T_1[= t_1], \ldots, c_n : T_n[= t_n]\}$$

where

- $c_i$ are distinct names
- $T_i$ are types
- $t_i$ are optional definitions; if given, $t_i : T_i$ required

## Recursion

- general case: $a$ may occur in the $T_i$
- if non-recursive: called a record type

# BDL Extended with Named ADTs

| | | | |
|---|---|---|---|
| $V$ | $::=$ | $\mathit{Decl}^*$ | Vocabularies |
| $\mathit{Decl}$ | $::=$ | $\textbf{class}\ t\ \{\mathtt{ID} : T^*\}$ | ADT definitions |

$\boxed{\text{Types}}$

| | | | |
|---|---|---|---|
| $T$ | $::=$ | $\ldots$ | as before |
| | $\mid$ | $t$ | reference to a named ADT |

$\boxed{\text{Data}}$

| | | | |
|---|---|---|---|
| $D$ | $::=$ | $\ldots$ | as before |
| | $\mid$ | $t\{(\mathtt{ID} = D)^*\}$ | ADT elements |

# Inheritance

Generalized ADT definition:

$$
\begin{array}{l}
\texttt{abstract class } a \texttt{ extends } a_1, \ldots, a_m \ \{ \\
\quad c_1 : \ T_1 \\
\quad \vdots \\
\quad c_n : \ T_n \\
\}
\end{array}
$$

Terminology:

- $a$ **inherits** from $a_i$
- $a_i$ are **super**-X or **parent**-X of $a$ where $X$ is whatever the language calls its ADTs (e.g., X=class)

# Flattening

Given ADT with inheritance as above, define **flattening** by

$$a^\flat = a_1^\flat \cup a_m^\flat \cup \{c_1 : T_1, \ldots, c_n : T_n\}$$

where duplicate field names are handled as follows

- same name, same type, same or omitted definition: merge

  details may be much more difficult

- otherwise: ill-formed

## Subtleties

We gloss over several major issues:

- How exactly do we merge duplicate field names? Does it always work? implement abstract methods, override, overload
- Is recursion allowed, i.e., can I define an ADT $a = A$ where $a$ occurs in $A$?
  common in OO-languages: use $a$ in the types of its fields
- What about ADTs with type arguments?
  e.g., generics in Java, square-brackets in Scala
- Is mutual recursion between fields in a flat type allowed?
  common in OO-languages
- Is * commutative? What about dependencies between fields?

no unique answers

incarnations of ADTs subtly different across languages

# Breakout question

When using typed concrete data,
how to fully realize abstract data types

- nesting: ADTs occurring as field types
- inheritance between ADTs
- mixins

# ADTs in Databases

### Nesting: field $a : A$ in ADT $B$

- field types must be base types, $a : A$ not allowed
- allow $ID$ as additional base type
- use field $a : ID$ in table $B$
- store value of $b$ in table $A$

### Inheritance: $B$ inherits from $A$

- add field $parent_A$ to table $B$
- store values of inherited fields of $B$ in table $A$

general principle: all objects of type $A$ stored in same table

### Mixin: $A * B$

- essentially join of tables $A$ and $B$ on common fields
- some subtleties depending on ADT flattening

# Subtyping

# Subtyping

- relatively easy if all data types disjoint
- better with subtyping    open problem how to do it nicely

## Subtyping Atomic Types

- $\mathbb{N} <: \mathbb{Z}$
- ASCII $<:$ Unicode

## Subtyping Complex Types

- covariance subtyping (= vertical subtyping) same for disjoint unions

$$A <: A' \Rightarrow list\ A <: list\ A'$$

$$A_i <: A_i' \Rightarrow \{\ldots, a_i : A_i, \ldots\} <: \{\ldots, a_i : A_i', \ldots\}$$

- structural subtyping (= horizontal subtyping)

$$\{a : A, b : B\} :> \{a : A, b : B, c : C\}$$

$$a(A)|b(B) <: a(A)|b(B)|c(C)$$

# Exercise 10

As a single group, write a joint document that specifies a choice of built-in datatypes for BOL and SFOL. They should be helpful in general and for the university ontology in particular.

For each new type, you have to give

- name
- values
- string encoding of each value (for printing, import/export)

Add the types and their values to your BOL and SFOL implementations and adjust your translation to translates values to themselves. (Note: Not all types will be supported by theorem provers; so your TPTP printer may have to be partial.)

# Codecs

# Data Interoperability

## Situation

- languages focus on different aspects frequent need to exchange data
- generally, lots of aspect/language-specific objects
  proofs, programs, tables, sentences
- but same/similar primitive data types used across systems
  should be easy to exchange

## Problem

- crossing system barriers usually requires interchange language
  serialize as string and reparse
- interchange languages typically untyped XML, JSON, YAML, ...

## Solution

- standardize basic data types
- standardize encoding in interchange languages

# Failures of Encodings

## Y2K bug

- date encoded as tuple of integers, using 2 digits for year
- needed fixing in year 2000
- estimated $300 billion spent to change software
- possible repeat: in 2038, number of seconds since 1970-01-01 (used by Unix to encode time as integer) overflows 32-bit integers

## Genes in Excel

- 2016 study found errors in 20% of spreadsheets accompanying genomics journal papers
- gene names encoded as strings but auto-converted to other types by Excel
  - "SEPT2" (Septin 2) converted to September 02
  - REKIN identifiers, e.g., "2310009E13", converted to float $2.31E+1$

https://genomebiology.biomedcentral.com/articles/10.1186/s13

# Failures of Encodings (2)

## Mars Climate Orbiter

- two components exchanged physical quantity
- specification required encoding as number using unit Newton seconds
- one component used wrong encoding (with pound seconds as unit)
- led to false trajectory and loss of $300 million device

## Shellshock

- bash allowed gaining root access from 1998 to 2014
- function definitions were encoded as source code
- not decoded at all; instead, code simply run (as root)
- allowed appending "; ..." to function definitions

SQL injection similar: complex data encoded as string, no decoding

# General Definition

Throughout this section, we fix a data representation language $L$.

$L$-words called codes

Given a data type $T$, a codec for $T$ consists

- coding function: $c : T \to L$
- partial decoding function: $d : L \to^? T$
- such that

$$d(c(x)) = x$$

# Codec Operators

Given a data type operator $T$ taking $n$ type arguments, a codec operator $C$ for $T$

- takes $n$ codecs $C_i$ for $T_i$
- returns a codec $C(C_1, \ldots, C_n)$ for $T(T_1, \ldots, T_n)$

# Codecs for Base Types

We define codecs for the base types using strings as the data representation language $L$.

Easy cases:

- StandardFloat: as specified in IEEE floating point standard
- StandardString: as themselves, quoted
- StandardBool: as *true* or *false*
- StandardInt (64-bit): decimal digit-sequences as usual

# Breakout Question

How to encode unlimited precision integers?

# Codecs for Unlimited Precision Integers

Encode $z \in \mathbb{Z}$

- $L$ is strings: decimal digit sequence as usual
- $L$ is JSON:
  - IntAsInt: decimal digit sequence as usual

    JSON does not specify precision
    but target systems may get in trouble
  - IntAsString: string containing decimal digit sequence

    safe but awkward
  - IntAsDecList: list of decimal digits — safe but awkward
  - IntAsList1: as list of digits for base $2^{64}$

    OK, but we can do better
  - IntAsList2: as list of
    - integer for the number of digits, sign indicate sign of $z$
    - list of digits of $|z|$ for base $2^{64}$

    Question: Why is this smart?

# Codecs for Unlimited Precision Integers

Encode $z \in \mathbb{Z}$

- $L$ is strings: decimal digit sequence as usual
- $L$ is JSON:
  - IntAsInt: decimal digit sequence as usual
    JSON does not specify precision
    but target systems may get in trouble
  - IntAsString: string containing decimal digit sequence
    safe but awkward
  - IntAsDecList: list of decimal digits safe but awkward
  - IntAsList1: as list of digits for base $2^{64}$
    OK, but we can do better
  - IntAsList2: as list of
    - integer for the number of digits, sign indicate sign of $z$
    - list of digits of $|z|$ for base $2^{64}$

    Question: Why is this smart?
    Can use lexicographic ordering for size comparison

# Codecs for Lists

Encode list $x$ of elements of type $T$

- $L$ is strings: e.g., comma-separated list of $T$-encoded elements of $x$
- $L$ is JSON:
  - ListAsString: like for strings above
  - ListAsArray: lists JSON array of $T$-encoded elements of $x$

# Additional Types

Examples: semester

Extend BDL:

$\boxed{\text{Types}}$

$T ::= Sem$ semester

$\boxed{\text{Data}}$

$D ::= sem(int, bool)$ i.e., year + summer?

Define standard codec:

$$sem(y, true) \leadsto "SSY"$$

$$sem(y, false) \leadsto "WSY"$$

where $Y$ is encoding of $y$

# Additional Types (2)

Examples: timestamps

Extend BDL:

$\boxed{\text{Types}}$

$T ::= \mathit{timestamp}$

$\boxed{\text{Data}}$

$D ::=$ (productions for dates, times, etc.)

Standard codec: encode as string as defined in ISO 8601

# Data Interchange

# Research Goal for Aspect-Independent Data in Tetrapod

## Standardization of Common Data Types

- Ontology language optimized for declaring types, values, operations
  semantics must exist but can be extra-linguistic
- Vocabulary declaring such objects
  should be standardized, modular, extensible

## Standardization of Codecs

- Fixed small set of primitive objects
  should be (quasi-)primitive in every language
  not too expressive, possibly untyped
- Standard codecs for translating common types to interchange languages

## Codec for type $A$ and int. lang. $L$

- coding function $A$-values $\rightarrow$ $L$-objects
- partial decoding function $L$-objects $\rightarrow$ $A$-values
- inverse to each other in some sense

# Design

1. Specify types
   - types
   - constructors
   - operations

   can be done in appropriate type theory

2. Pick data representation language $L$
3. Specify codecs for type system and $L$
   - at least one codec per base type
   - at least one codec operator per type operator

   on paper

4. Every system implements
   - type system (as they like) typically aspect-specific constraints
   - codecs as specified
   - function mapping types to codecs
5. Systems can exchange data by encoding-decoding

   type-safe because codecs chosen by type

# BDL-Mediated Interoperability

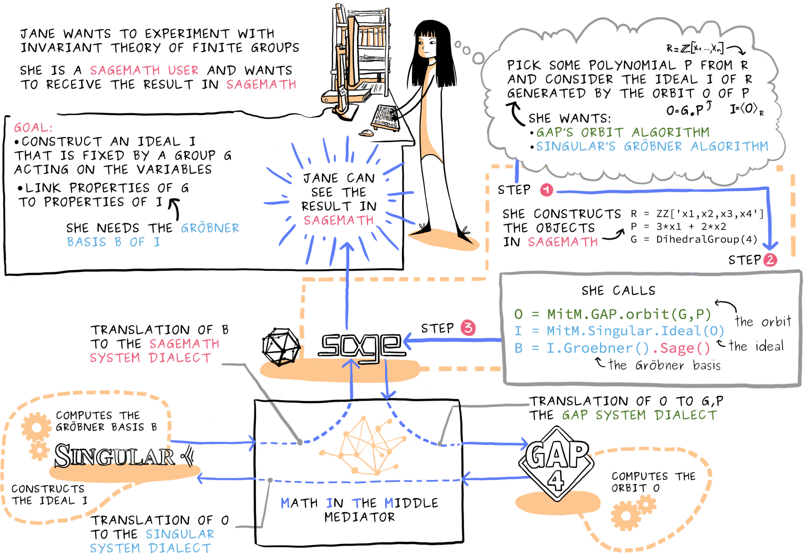
Idea

- define data types in BDL   or similar typed ontology language
- use ADTs
- generate corresponding
  - class definitions for programming languages PL
    one class per ADT
  - table definitions in SQL   one table per ADT
- use codecs to convert automatically when interchanging data between PL and SQL

Open research problem

no shiny solution yet that can be presented in lectures

# Example Application: OpenDreamKit research project



# Codecs in ADT Definitions

**SQL table schema = list of fields where field is**

- name
- type — only types of database supported

**BDL semantic table schema = list of fields where field is**

- name
- type $T$ of type system — independent of database
- codec for $T$ using primitive objects of database as codes
  see research paper `https://kwarc.info/people/frabe/Research/WKR_`

Codec could be chosen automatically, but we want to allow multiple users a choice of codecs for the same type.

# Example

Ontology based on BDL-ADTs with additional codec information:

```
schema Instructor
  name:     string        codec StandardString
  age:      int           codec StandardInt
  courses:  list Course   codec CommaSeparatedList CourseAsName
schema Course
  name:      string       codec StandardString
  credits:   float        codec StandardFloat
  semester:  Semester     codec SemesterAsString
```

Generated SQL tables:

```
CREATE TABLE Instructor
  (name string, age int, courses string)
CREATE TABLE Course
  (name string, credits float, semester string)
```

# Open Problem: Non-Compositionality

**Sometimes optimal translation is non-compositional**

- example translate *list*-type in ADT to comma-separated string in DB
- better break up *list* $B$ fields in type $A$ into separate table with columns for $A$ and $B$

Similar problems

- a pair type in an ADT could be translated to two separate columns
- an option type in an ADT could translated to a normal column using SQL's NULL value

# Open Problem: Querying

- General setup
  - write SQL-style queries using at the BDL level
  - automatically encode values when writing to database from PL
  - automatically decode query results when reading from DB
- But queries using semantic operations cannot always be translated to DB
  - operation $IsSummer : Semester \to bool$ in BDL
  - query SELECT $*$ FROM course WHERE$IsSummer$(semester)
  - how to map $IsSummer$ to SQL?
- Ontology operations need commuting operations on codes
  - given $f : A \to B$ in BDL, codecs $C, D$ for $A$ and $B$
  - SQL function $f'$ commutes with $f$ iff

$$B.decode(f'(C.encode\, a)) = f(a)$$

  for all $a : A$

# Concrete Data Representation Languages

# Overview

## General Properties

- general purpose or domain-specific
- typed or untyped
  typical: Church-typed but no type operators, quasi untyped
- text or binary serialization
- libraries for many programming languages
  - data structures
  - serialization (data structure to string)
  - parsing (string to data structure, partial)

## Candidates

- XML: standard on the web, notoriously verbose
- JSON: JavaScript objects, more human-friendly text syntax
  older than XML, probably better choice than XML in retrospect
- YAML: line/indentation-based

# Breakout Question

What is the difference between JSON, YAML, XML?

# Typical Data Representation Languages

XML, JSON, YAML essentially the same

except for concrete syntax

## Atomic Types

- integer, float, boolean, string
- need to read fine-print on precision

## (Not Very) Complex Types

- heterogeneous lists — a single type for all lists
- records — a single type for all records

# Example: JSON

JSON:

$$
\begin{array}{l}
\{ \\
\quad "individual" : "FlorianRabe", \\
\quad "age" : 40, \\
\quad "concepts" : ["instructor", "male"], \\
\quad "teach" : [ \\
\quad\quad \{"name" : "Wuv", credits : 7.5\}, \\
\quad\quad \{"name" : "KRMT", credits : 5\} \\
\quad ] \\
\}
\end{array}
$$

Weirdnesses:

- atomic/list/record = basic/array/object
- record field names are arbitrary strings, must be quoted
- records use : instead of =

# Example: YAML

inline syntax: same as JSON but without quoted field names

alternative: indentation-sensitive syntax

```
individual : "FlorianRabe"
age : 40
concepts :
    – "instructor"
    – "male"
teach :
    – name : "WuV"
      credits : 7.5
    – name : "KRMT"
      credits : 5
```

Weirdnesses:

- atomic/list/record = scalar/collection/structure
- records use : instead of =

# Example: XML

Weird structure but very similar

- elements both record (= attributes) and list (= children)
- elements carry name of type (= tag)

```
<Person individual="Florian Rabe" age="40">
 <concepts>
   <Concept>instructor</Concept/>
   <Concept>male</Concept/>
 </concepts>
 <teach>
   <Course name="WuV" credits="7.5"/>
   <Course name="KRMT" credits="5"/>
 </teach>
</Person>
```

- Good: Person, Course, Concept give type of object

  easier to decode

- Bad: value of record field must be string

  concepts cannot be given in attribute

  integers, Booleans, whitespace-separated lists coded as strings

# Structure Sharing

## Problem

- Large objects are often redundant specially when machine-produced
- Same string, URL, mathematical objects occurs in multiple places
- Handled in memory via pointers
- Size of serialization can explode

## Solution 1: in language

- Add definitions to language common part of most languages anyway
- Users should introduce name whenever object used twice
- Problem: only works if
  - duplication anticipated
  - users introduced definition
  - duplication within same context

structure-sharing most powerful if across contexts

# Structure Sharing (2)

## Solution 2: in tool

- Use factory methods instead of constructors
- Keep huge hash set of all objects
- Reuse existing object if already in hash set
- Advantages
  - allows optimization
  - transparent to users
- Problem: only works if
  - for immutable data structures
  - if no occurrence-specific metadata e.g., source reference

## In data representation language

- Allow any subobject to carry identifier
- Allow identifier references as subobjects

allows preserving structure-sharing in serialization

supported by XML, YAML

# Syntax Trees as Data

## Two kinds of data in syntax

- Basic data that occurs literally in the source
  - in BOL, SFOL: strings for names, basic types/values
  - typically leaves of the syntax tree
  - often needs a codec
- The syntax tree structure
  - names of the productions/constructors
  - names of the children/constructor arguments
  - nesting of nodes

# Representing a Syntax Tree in XML

Context-free traverser of the AST (printer)

- For basic data, use the type as the tag and give the string encoding in an attribute:

$$< \text{identifier value=''NAME''} / >$$

$$< \text{int value=''ENCODED\_VALUE''} / >$$

- For node $N = c(e_1, \ldots, e_n)$ where
  - $c$ is the constructor/production name
  - $e_i$ are the subtrees

$$XML(N) = < c > XML(e_1) \ldots XML(e_n) < /c >$$

  optionally: wrap every $XML(e_i)$ in another tag that gives the name of the constructor argument, e.g., for $C \sqcup D$

$$<union><left>XML(C)</left><right>XML(D)</right></union>$$

$$<union>XML(C)XML(D)</union>$$

# Exercise 11

As a group, define an XML schema for BOL using the RelaxNG compact syntax. See `https://relaxng.org/compact-tutorial-20030326.html`.

Use names for all productions and constructor arguments, to obtain the tag names for the tree structure.

Use the string encodings for your agreed-upon primitive types to encode values of primitive types.

Individually, extend your implementations of BOL with XML importers and exporters relative to that schema.

The exporter is a context-free traverser. For the importer, use an XML library for your chosen programming language to parse the input. Then write a context-free traverser over the XML data structures that translates the XML into your BOL data structures.

Exchange your vocabularies with each other. Type-check each other's vocabularies after importing.

# Overview

## General Ideas

- Recall
  - syntax = context-free grammar
  - semantics = translation to another language
- Example: BOL translated to SQL, SFOL, Scala, English
- Querying = use semantics to answer questions about syntax

Note:

- Not the standard definition of querying
- Design of a new Tetrapod-level notion of querying

  ongoing research
- Subsumes concepts of different names from the various aspects

# Propositions

syntax with propositions $=$
designated non-terminals for propositions

Examples:

| aspect | basic propositions |
|---|---|
| ontology language | assertions, concept equality/subsumption |
| programming language | equality for some types |
| database language | equality for base types |
| logic | equality for all types |
| natural language | sentences |

Aspects vary critically in how propositions can be formed

- any program in computation
- quantifiers in deductions undecidable
- *IN* in databases

# Propositions as Queries

Propositions allow defining queries

| | Query | Result |
|---|---|---|
| deduction | proposition | yes/no |
| concretization | proposition with free variables | true ground instances |
| computation | term | value |
| narration | question | answer |

# Semantics of Propositions

syntax with propositions =
designated non-terminals for propositions

needed to ask queries

semantics with theorems =
designates some propositions as theorems or contradictions

needed to answer queries

Note:

- A propositions may be neither theorem nor contradiction.
- We say that language has negation if:
  $F$ theorem iff $\neg F$ contradiction and vice versa.

We write $\vdash F$ if $F$ is theorem.

# Deductive Queries

# Definition

We assume

- a semantics $[\![-]\!]$ from $I$ to $L$
- $I$ has propositions
- there is an operation $\mathtt{True}$ that maps translations of $I$-propositions to $L$-propositions called truth lifting
- $L$ has semantics with propositions

We define

- a deductive query is an $I$-proposition $p$
- the result is
  - yes if $\mathtt{True}[\![p]\!]$ is a theorem of $L$
  - no if $\mathtt{True}[\![p]\!]$ is a contradiction in $L$

# The Truth-Lift operator

## Problem with type-preserving translation $[\![-]\!]$

- must translate $F : \mathtt{prop}^I$ to $[\![F]\!] : [\![\mathtt{prop}^I]\!]$
- but need not satisfy $[\![\mathtt{prop}^I]\!] = \mathtt{prop}^L$

  $I$-propositions maybe not translated to $L$-propositions
- If not, $\vdash^L [\![F]\!]$ cannot be used to answer query $\vdash^I F$

## Solution: $L$-operator $\mathtt{True} : [\![\mathtt{prop}^I]\!] \to \mathtt{prop}^L$

- maps translations of $I$-propositions to $L$-propositions
- then use $\vdash^L \mathtt{True}[\![F]\!]$ to answer $\vdash^I F$

## Example: probabilistic semantics with cutoff

- $\mathtt{prop}^I$: propositions; $\mathtt{prop}^L$: truth values
- probabilistic semantics: $[\![\mathtt{prop}^I]\!] = [0;1]$
- $\mathtt{True}$ maps $p \in [0;1]$ to truth value, e.g., via cutoff:

  $\mathtt{True}\,p = p \geq 0.75$

# Situational Semantics as Translation

## Example: standard semantics of SFOL in set theory

- set theory propositions: truth values $\{0, 1\}$
- situations (= models): provide interpretations for all vocabulary symbols
- semantics is model-specific interpretation function: given vocabulary $V$, $V$-model $M$, and $V$-expression $E$, write $[\![E]\!]_M$ for semantics of $E$ in $M$

## Reformulate as semantics by translation

- vocabulary $V$ translated to set-theoretical vocabulary containing required components of model
- expression $E$ translated to mapping $[\![E]\!] : M \mapsto [\![E]\!]_M$

## Truth-Lift operator defines theorems

- $[\![\mathtt{prop}^I]\!]$ is set of mappings $\mu$ from models to truth values

  standard notation for $[\![F]\!](M) = 1$: $[\![F]\!]_M = 1$ or $M \models F$
- $\mathtt{True}\,\mu = 1$ iff $\mu(M) = 1$ for all models $M$ (theorem = true in all models)

# Breakout question

What can go wrong?

## Problem: Inconsistency

In general, (in)consistency of semantics

- Some propositions may be both a theorem and a contradiction.
- In that case, queries do not have a result.

In practice, however:

- If this holds for some propositions, it typically holds for all of them.
- In that, we call $L$ inconsistent.
- We usually assume $L$ to be consistent.

# Problem: Incompleteness

In general, (in)completeness of semantics

- We cannot in general assume that every proposition in $L$ is either a theorem or a contradiction.
- In fact, most propositions are neither.
- So, queries do not necessarily have a result.
- We speak of incompleteness.

Note: not the same as the usual (in)completeness of logic

In practice, however:

- It may be that $L$ is complete for all propositions in the image of $\mathtt{True}[\![-]\!]$.
- This is the case if $I$ is simple enough

typical for ontology languages

# Problem: Undecidability

In general, (un)decidability of semantics:

- We cannot in general assume that it is decidable whether a proposition in $L$ is a theorem or a contradiction.
- In fact, it usually isn't.
- So, we cannot necessarily compute the result of a query.
- However: If we have completeness, decidability is likely.

  run provers for $F$ and $\neg F$ in parallel

In practice, however:

- It may be that $L$ is decidable for all propositions in the image of $\mathtt{True}[\![-]\!]$.
- This is the case if $I$ is simple enough

  typical for ontology languages

# Problem: Inefficiency

In general, (in)efficiency of semantics:

- Answering deductive queries is very slow.
- Even if we are complete and decidable.

In practice, however:

- Decision procedures for the image of $\texttt{True}[\![-]\!]$ may be quite efficient.
- Dedicated implementations for specific fragments.
- This is the case if $I$ is simple enough

typical for ontology languages

# Contexts and Free Variables

# Concepts

Recall the analogy between grammars and typing:

| grammars | typing |
|---|---|
| non-terminal | type |
| production | constructor |
| non-terminal on left of production | return type of constructor |
| non-terminals on right of production | arguments types of constructor |
| terminals on right of production | notation of constructor |
| words derived from non-terminal $N$ | expressions of type $N$ |

We will now add contexts and substitutions.

# Contexts

Independent of whether $I$ already has contexts/variables, we can define:

- A context $\Gamma$ is of the form $x_1 : N_1, \ldots, x_n : N_n$ where the
  - $x_i$ are names
  - $N_i$ are non-terminals

  We write this as $\vdash_I \Gamma$.
- A substitution for $\Gamma$ is of the form $x_1 := w_1, \ldots, x_n := w_n$ where the
  - $x_i$ are as in $\Gamma$
  - $w_i$ derived from the corresponding $N_i$

  We write this as $\vdash_I \gamma : \Gamma$.
- An expression in context $\Gamma$ of type $N$ is a word $w$ derived from $N$ using additionally the productions $N_i ::= x_i$.
  We write this as $\Gamma \vdash_I w : N$.
- Given $\Gamma \vdash w : N$ and $\vdash \gamma : \Gamma$ as above, the substitution of $\gamma$ in $w$ is obtained by replacing every $x_i$ in $w$ with $w_i$. We write this as $w[\gamma]$.

## Contexts under Compositional Translation

Consider a compositional semantics $[\![-]\!]$ from $I$ to $L$ between context-free languages.

- Every $\vdash_I w : N$ is translated to some $\vdash_L [\![w]\!] : N'$ for some $N'$.
- Compositionality ensures that $N'$ is the same for all $w$ derived from $N$.
- We write $[\![N]\!]$ for that $N'$.
- Then we have

$$\vdash_I w : N \qquad \text{implies} \qquad \vdash_L [\![w]\!] : [\![N]\!]$$

Now we translate contexts, substitutions, and variables as well:

$$[\![x_1 : N_1, \ldots, x_n : N_n]\!] := x_1 : [\![N_1]\!], \ldots, x_n : [\![N_n]\!]$$
$$[\![x_1 := w_1, \ldots, x_n := w_n]\!] := x_1 := [\![w_1]\!], \ldots, x_n := [\![w_n]\!]$$
$$[\![x]\!] := x$$

Then we have

$$\Gamma \vdash_I w : N \qquad \text{implies} \qquad [\![\Gamma]\!] \vdash_L [\![w]\!] : [\![N]\!]$$

# Substitution under Compositional Translation

From previous slide:

$$[\![x_1 : N_1, \ldots, x_n : N_n]\!] := x_1 : [\![N_1]\!], \ldots, x_n : [\![N_n]\!]$$

$$[\![x_1 := w_1, \ldots, x_n := w_n]\!] := x_1 := [\![w_1]\!], \ldots, x_n := [\![w_n]\!]$$

$$[\![x]\!] := x$$

$$\Gamma \vdash_I w : N \qquad \text{implies} \qquad [\![\Gamma]\!] \vdash_L [\![w]\!] : [\![N]\!]$$

We can now restate the substitution theorem as follows:

$$[\![E[\gamma]]\!] = [\![E]\!][[\![\gamma]\!]]$$

# Concretized Queries

# Definition

We assume

- as for deductive queries
- semantics must be compositional

We define

- a concretized query is an $l$-proposition $p$ in context $\Gamma$
- a single result is a
  - a substitution $\vdash_l \gamma : \Gamma$
  - such that $\vdash_L \mathtt{True}[\![p[\gamma]]\!]$
- the result set is the set of all results

# Example

1. BOL ontology:

$$\textit{concept male},\ \textit{concept person},\ \textit{axiom male} \sqsubseteq \textit{person},$$
$$\textit{individual FlorianRabe},\ \textit{assertion FlorianRabe isa male}$$

2. Query $x : \textit{individual} \vdash_{BOL} x \text{ isa } \textit{person}$
3. Translation to SFOL: $x : \iota \vdash_{SFOL} \textit{person}(x)$
4. SFOL calculus yields theorem $\vdash_{SFOL} \textit{person}(\textit{FlorianRabe})$
5. Query result $[\![\gamma]\!] = x := \textit{FlorianRabe}$
6. Back-translating the result to BOL: $\gamma = x := \textit{FlorianRabe}$

back translation is deceptively simple:
translates SFOL-constant to BOL-individual of same name

# Yes/No vs. Wh-Questions

### Queries about $\vdash F$ are yes/no questions

- specialty of deductive semantics
- but maybe only because everything else is ever harder to do deductively

### Queries about ground instances of $\Gamma \vdash F$ are Wh questions

- specialty of concrete databases
- for the special case of retrieving finite results sets from a fixed concrete store
- only situation where Wh questions are easy

# Breakout question

What can go wrong?

# Problem: Open World

In general, semantics uses open world:

- open world: result contains all known results

  same query might yield more results later
- closed world: result set contains all results

always relative to concrete database for $L$

In practice, however,

- system explicitly assumes closed world typical for databases
- users aware of open world and able to process results correctly

# Problem: Infinity of Results

In general, there may be infinitely many results:

- e.g., query for all $x$ such that $\vdash x$,

In practice, however,

- systems pull results from finite database e.g., SQL, SPARQL
- systems enumerate results, require user to explicitly ask for more e.g., Prolog

## Problem: Back-Translation of Results

In general, $[\![-]\!]$ may be non-trivial to invert

- easy to obtain $[\![p]\!]$ in context $[\![\Gamma]\!]$ just apply semantics
- possible to find substitutions

$$\vdash_L \delta : [\![\Gamma]\!] \qquad \text{where} \qquad [\![\Gamma]\!] \vdash_L \mathtt{True}[\![p]\!][\delta]$$

easiest case: just look them up in database

- but how to translate $\delta$ to $I$-substitutions $\gamma$ with

$$\vdash_I \gamma : \Gamma \qquad \text{where} \qquad [\![\Gamma]\!] \vdash_L \mathtt{True}[\![p[\gamma]]\!]$$

substitution theorem: pick such that $[\![\gamma]\!] = \delta$
the more $[\![-]\!]$ does, the harder to invert

In practice, however:

- often only interested in concrete substitutions
- translation of concrete data usually identity

But: practice restricted to what works even if more is needed

# Computational Queries

# Definition

We assume

- the same as for deductive queries
- semantics has equality/equivalence $\doteq$

We define

- a computational query is an $l$-expression $e$
- the result is an $l$-expression $e'$ so that $\vdash_L [\![e]\!] \doteq [\![e']\!]$

intuition: $e'$ is the result of evaluating $e$

If semantics is compositional, $e$ may contain free variables

evaluate to themselves

# Problem: Back-Translation of Results

In general, $[\![-]\!]$ may be non-trivial to invert

- easy to obtain $E := [\![e]\!]$
- possible to find $E'$ with $\vdash_L E' \doteq E$ by working in the semantics
- non-obvious how to obtain $e'$ such that $[\![e']\!] = E'$

In practice, however:

- evaluation meant to simplify, i.e., only useful if $E'$ very simple
- simple $E'$ usually in the image of $[\![-]\!]$
- typical case: $E'$ is concrete data and $e' = E'$ called a value

# Problem: Non-Termination

In general, computation of $E'$ from $E$ might not terminate

- while-loops
- recursion
- $(\lambda x.x\,x)\,(\lambda x.x\,x)$ with $\beta$-rule
- simplification rule $x \cdot y \rightsquigarrow y \cdot x$

similar: distributivity, associativity

In practice, however:

- image of $[\![-]\!]$ part of terminating fragment

But: if $I$ is Turing-complete or undecidable, general termination not possible

# Problem: Lack of Confluence

In general, there may be multiple $E'$ that are simpler than $E$

- there may be multiple rules that apply to $E$
- e.g., $f(g(x))$
  - call-by-value: first simplify $g(x) \rightsquigarrow y$, then $f(y) \rightsquigarrow z$
  - call-by-name: first plug $g(x)$ into definition of $f$, then simplify
- Normal vs. canonical form
  - normal: $\vdash_L E \doteq E'$
  - canonical: normal and $\vdash_L E_1 \doteq E_2$ iff $E_1' = E_2'$

equivalent expressions have identical evaluation

allows deciding equality

In practice, however:

- image of $[\![-]\!]$ part of confluent fragment
- typical: evaluation to a value is canonical form

works for BDL-types but not for, e.g., function types

# Narrative Queries

# Definition

We assume

- semantics into natural language

We define

- a narrative query is an $L$-question about some $l$-expressions
- the result is the answer to the question

## Problem: Unimplementable

very expressive = very difficult to implement

- Natural language understanding
  - no implementable syntax of natural language
    needs restriction to controlled natural language
  - specifying semantics hard even when controlled
- Knowledge base for question answering needed
  - very large must include all common sense
  - might be inconsistent common sense often is
  - finding answers still very hard

In practice, however:

- accept unreliability attach probability measures to answers
- implement special cases
  e.g., lookup in databases like Wikidata
- search knowledge base for related statements Google, Watson

# Syntactic Querying

# Search

- "search" not systematically separated from "querying"
- often interchangeable
- querying tends to imply formal languages for queries with well-specified semantics e.g., SQL
- search tends to imply less targeted process e.g., Google

we will not distinguish between the two

# Syntactic vs. Semantic Querying

## Semantic querying

- Query results specified by vocabulary $V$ but (usually) not contained in it
- Query answered using semantics of language
- Challenge: apply semantics to find results
  - deductive query $\vdash f : \mathtt{prop}$ requires theorem prover
  - computation query $\vdash e : E$ requires evaluator
  - concrete query $\Gamma \vdash f : \mathtt{prop}$ requires enumerating all substitutions, running theorem prover/evaluator on all of them

what we've looked at so far

## Syntactic querying

- Query is an expression $e$
- Result is set of occurrences of $e$ in $V$
- Independent of semantics
- Much easier to realize

# Challenges for Syntactic Search

Easier to realize $\rightarrow$ scale until new challenges arise

- large vocabularies
  - narrative: all text documents in a domain
    e.g., all websites, all math papers
  - deductive: large repositories of formalization in proof assistants
    $10^5$ theorems
  - computational: package managers for all programming languages
  - concrete: all databases in a domain TBs routine
- incremental indexing: reindex only new/changed parts
- incremental search to handle large result sets pagination
- sophisticated techniques for
  - indexing: to allow for fast retrieval
  - similarity: to select likely results
  - quality: to rank selected results
- integration of some semantic parts

## Overview

- Deduction
  - semantic: theorem proving called search
  - syntactic: text search
- Concretization
  - semantic: complex query languages (nestable queries)
    SQL, SPARQL
  - syntactic: search by identifier (linked data)
- Computation
  - semantic: interpreters called execution
  - mixed: IDEs search for occurrences, dependencies
  - syntactic: search in IDE, package manager
- Narration:
  - semantic: very difficult
  - syntactic: bag of words search

# Abstract Definition: Document

**Document** =

- file or similar resource that contains vocabularies
- often with comments, metadata
- different names per aspect
  - deduction: formalization, theory, article
  - computation: source files
  - concretization: database, ontology ABox
  - narrative: document, web site

**Library** =

- collection of documents
- usually structured into folders, files or similar
- often grouped by user access — e.g., git repository
- vocabularies interrelated within and across libraries

# Abstract Definition: Document Fragment

**Fragment** = subdivision of documents into nested semantic units

Examples

- deductive: theory, section, theorem, definition, proof step, etc.
- computational: class, function, command, etc.
- concrete: table, row, cell
- narrative: section, paragraph, etc.

Assign unique **fragment URI**, e.g., LIB/DOC?FRAG where

- LIB: base URI of library e.g., repository URL
- DOC: path to document within library
  e.g., folder structure, file name
- FRAG: id of fragment within document
  e.g., class name/method name

# Abstract Definition: Index(er)

**Indexer** consists of

- data structure $O$ for indexable objects
  specific to aspect, index design
  e.g., words, syntax trees
- function that maps library to index  the indexing

**Index entry** consists of

- object that occurred in the library
- URI of the containing fragment
- information on where in the fragment it was found

**Index** $=$ set of index entries

# Abstract Definition: Query and Result

Given

- indexer $I$ with data structure $O$
- set of libraries
- union of their indexes computed once, queried often

**Query** = object $\Gamma \vdash^I q : O$

**Result** consists of

- index entry with object $o$
- substitution for $\Gamma$ such that $q$ matches $o$
  definition of "match" index-specific, e.g., $q[\gamma] = o$

**Result set** = set of all results in the index

## Bag of Words Search

Definition:

- Index data structure = sequences of words (n-grams) up to a certain length
- Query = bag of words   bag = multiset
- Match: (most) words in query occur in same n-gram or n-grams near each other

Example implementations

- internet search engines for websites
- Elasticsearch: open source engine for custom vocabularies

Mostly used for narrative documents

- can treat concrete values as words   e.g., numbers
- could treat other expressions as words   works badly

## Symbolic Search

Definition:

- Index data structure = syntax tree (of any grammar) of expressions $o$ with free/bound variables
- Query = expression $q$ with free (meta-)variables
- Match: $q[\gamma] =_\alpha o$, i.e., up to variable renaming

Example implementation

- MathWebSearch

see separate slides on MathWebSearch in the repository

Mostly used for formal documents

- deductive
- computational

# Knowledge Graph Search

Definition:

- Index data structure = assertion forming node/edge in a knowledge graph
- Index = big knowledge graph $G$
- Query = knowledge graph $g$ with free variables
- Match: $g[\gamma]$ is part of $G$

Example implementations

- SPARQL engines without consequence closure

  i.e., the most common case in practice
- graph databases

Mainly used for ABoxes of untyped ontologies

# Value Search

Definition:

- Index data structure $=$ BDL values $v$
- Query $=$ BDL expression $q$ with free variables
- Match: $q[\gamma] = v$

Example implementations

- no systematic implementation yet
- special cases part of most database systems

Could be used for values occurring in any document

- all aspects
- may need to decode/encode before putting in index

# Cross-Aspect Occurrences

Observation

- libraries are written in one primary aspect
- indexer focuses on one aspect and kind of object
- but documents may contain indexable objects of any index

# Cross-Aspect Occurrences: Examples

- Any library can contain
  - metadata on fragments
    - relation assertions induce knowledge graph structure between fragments
    - property assertions contain values narrative, symbolic objects, or values
  - cross-references to fragments of any other library
  - narrative comments
- Narrative text may contain symbolic expressions

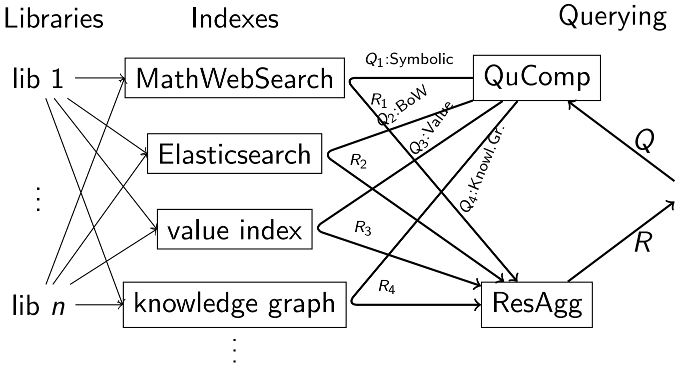
  STEM documents
- Database table may have columns containing
  - text
  - encoded BDL values
  - symbolic expression (often as strings)
- Symbolic fragments may contain database references

  e.g., when using database for persistent memoization

# A New Indexing Design

recent paper `https://kwarc.info/people/frabe/Research/BKR_mdql_20.pdf`

with K. Bercic



# A New Indexing Design (2)

Tricky question: What is the query language that allows combining queries for each index?

Easy:

- query = conjunction of atomic queries
- each atom queries one index
- QuComp splits into atoms
- ResAgg take intersection of results

Better: allow variables to be shared across atoms

open research question

# A New Indexing Design: Example

Consider

- table of graphs with human-recognizable names and arc-transitivity property
  indexed into
  - value index for the graph `sparse6 codec`
  - Boolean computed property for the arc-transitivity in knowledge graph
  - text index for name
- papers from arXiv in narrative index
  indexed into
  - narrative index for text
  - MathWebSearch for formulas
  - knowledge graph for metadata

Query goal: find arc-transitive graphs mentioned by name in articles with h-index greater than 50

# Integrating Semantic Querying

Word search

- find multi-meaning words for only one meaning

  "normal" in math
- special treatment of certain queries e.g., "weather" in Google

Symbolic search

- match query $e \doteq e'$ against occurrence $e' \doteq e$
- similarly: associativity, commutativity, etc.
- slippery slope to deductive queries

Value search

- match query 1.5 against interval $1.4 \pm 0.2$
- match query $5 \cdot x$ against 25
- slippery slope to computational queries

frontiers of research — in our group: for STEM documents

# Formal Systems

# Typical Structure of a Formal System

Vocabularies

- vocabularies $V$ = lists of declarations
- vocabulary morphisms $m : V \to W$ = lists of definitions of $V$-identifiers in terms of $W$-expressions

Declarations

- named
- at least one for each expression kind
- may contain other expressions e.g., type, definition
- may contain nested declarations e.g., fields in an ADT

Expressions

- inductive data type
- relative to vocabulary names occur as base cases
- formulas as special case

Morphisms $m$ map $V$-expressions to $W$-expressions (homomorphic extension)

# Example: Vocabularies and Expressions

| Aspect | vocabulary $\Theta$ | expression kinds |
|---|---|---|
| Ontologization | ontology | individual, concept, relation, property, formula |
| Concretization | database schema | cell, row, table, formula |
| Computation | program | term, type, object, class, ... |
| Logic | signature, theory | term, type, formula, ... |
| Narration | dictionary | phrases, sentences, texts |

# Components and Well-Formedness

## Components of formal system $I$

- context-free syntax
- distinguished non-terminal symbols $\mathcal{V}$ and $\mathcal{M}$
  words called **vocabularies** and **morphisms**
- some distinguished non-terminal symbols words called **expressions**
- some expressions may act as types for other expressions

## Intuition

- context-free syntax generates more than needed
- context-sensitive well-formedness defines the exact subset

Question: How do we define the well-formed subsets?

use a typing via context-sensitive inference rules

# Inference System

## Judgments

- contexts $\Gamma$ of the form $x_1 : E_1, \ldots, x_n : E_n$ for expressions $E_i$
- types: either non-terminals (default coarse typing) or certain expressions (fine-granular typing)
- a set of judgments including
  - a judgment $\vdash^I V$ on vocabularies $V$

    $V$ is well-formed vocabulary
  - a judgment $\Gamma \vdash^I_V e : E$ between expressions $e, E$

    $e$ is well-formed with type $E$ over $V$
- inference system = a set of rules for the judgments, each one of the form

$$\frac{J_1 \qquad \ldots \qquad J_n}{J}$$

  where the $J$'s are judgments

  called type system or proof system depending on focus

conventions: leave out superscript $I$, subscript $V$ if clear

leave out $\Gamma$ if empty

# Derivations in an Inference System

## For an inference system, define

- derivation: tree of judgments such that for every node $J$ with children $J_1, \ldots, J_n$, there is a rule

$$\frac{J_1 \qquad \ldots \qquad J_n}{J}$$

- derivation of $J$: a derivation with root $J$
- $J$ holds: there is a derivation of $J$
- piece of syntax is well-formed: if it is part of a derivable judgment

# Special Cases of Formal Systems

## Formal system with propositions

- syntax additionally has a distinguished type `prop`
- $F$ is proposition if $\vdash_V F : \mathtt{prop}$
- inference system typically has judgment $\vdash_V F$ for $F$ being theorem

## Formal system with equality

- syntax additionally has a distinguished proposition $e_1 \doteq_E e_2$ whenever $\vdash e_i : E$
- $e_1, e_2$ are equal at type $E$ if $\vdash e_1 \doteq_E e_2$
- inference system has rules to make $\doteq$ behave like equality

# Example: BOL

## Syntax

- non-terminal for vocabularies: $V$ (ontologies)
- expressions kinds: concepts $C$, individuals $I$, relations $R$, properties $P$, formulas $F$
- propositions: `prop` is $F$
- equality: only for concepts, $\doteq_{\mathrm{Concept}}$ is the operator $\leftrightarrow$

## Absolute Semantics via Inference System

- trivial church type system with 5 types for the 5 kinds of expressions
- proof system for judgment $\vdash F$
- Curry type systems with formula $I$ `is-a` $C$ using concepts as types for individuals

  no need for typing rules

  $I$ `is-a` $C$ is proposition, and typing is special case of $\vdash F$

# Example: SFOL

## Syntax

- non-terminal for vocabularies: $V$ (theories)
- expressions kinds: type $Y$, term $T$, formula $F$
- proposition: `prop` is the non-terminal for formulas
- equality
  - $\doteq_Y$ for terms
  - $\Leftrightarrow_{\text{Formula}}$ behaves like equality of formulas

## Absolute Semantics via Inference System

- trivial church type system with 3 types for the 3 kinds of expressions
- Church type system with types $Y$ typing terms $T$

  decidable, typing rules given/implemented independent of $\vdash F$
- proof system for judgment $\vdash F$

# Morphisms

| | vocabulary | morphism |
|---|---|---|
| identifiers | declarations of identifiers $e$ | assignments $e := E$ |
| expressions | cont.-free ind. type | ind. def. of hom. ext. |
| typing | cont.-sens. inference rules | ind. type preservation proof |

cont.=context, ind.=inductive, def.=definition

Vocabulary $V$

- $V$ introduces list of identifiers (of various kinds)
- $V$-expressions are syntax trees with $V$-identifiers as leaves
- well-formed $V$-expressions defined by rules

Homomorphisms $m : V \to W$

- $m$ maps $V$-identifiers to $W$-expressions (of the same kind)
- inductive definition of hom. ext. to map every $V$-expression to a $W$-expression
- inductive proof that $\overline{m}$ preserves well-formedness

# An Even More Abstract Definition

fully abstract — no appeal to grammar or inference system

## Formal system $I$ consists of

- set $\mathrm{Voc}^I$ — elements called vocabularies
- for any two vocabularies $V, W$, a set $\mathrm{Mor}^I(V, W)$ — elements called morphisms
- for any vocabulary $V$, a set $\mathrm{Exp}^I(V)$ — elements called expressions
  with a binary relation $:_V$ on the expressions
- for any morphism $m \in \mathrm{Mor}^I(V, W)$, a map
  $\mathrm{Exp}^I(m) : \mathrm{Exp}^I(V) \to \mathrm{Exp}^I(W)$ — called homomorphic extension
  such that $:$ is preserved:

$$e :_V E \quad \text{implies} \quad \mathrm{Exp}^I(e) :_W \mathrm{Exp}^I(E)$$

## ... with propositions

- special expression $\mathtt{prop} \in \mathrm{Exp}^I(V)$ for every $V$
  expressions $F$ with $F :_V \mathtt{prop}$ called propositions

# Deductive Semantics

# Deductive Semantics

## Definition

- subset $\mathrm{Thm}^I(V) \subseteq \{F \in \mathrm{Exp}^I(V) | F :_V \mathtt{prop}\}$
  elements called theorems
- special judgment $\vdash^I_V F$ for $F \in \mathrm{Thm}^I(V)$

## Terminology

- Calculus = proof system = inference system for $\vdash^I_V F$
- Logic: formal system with deductive semantics
- Theorem prover: implementation of deductive semantics
- Decision procedure: theorem prover for decidable $\vdash F$

## Examples

- Natural deduction for first-order logic
- Axiomatic set theory for (most of) mathematics

# Examples

## Absolute Deductive Semantics

- Natural deduction for first-order logic
- Axiomatic set theory for (most of) mathematics

## Relative Deductive Semantics

- given translation $[\![-]\!] : I \to L$ with truth lifting $\mathtt{True}$
- define $\vdash^{I}_{V} F$ iff $\vdash^{L}_{[\![V]\!]} \mathtt{True}[\![F]\!]$

# Redundant Deductive Semantics

## Multiple deductive semantics for the same syntax, e.g.,

- Proof system: absolute semantics
- Model theory: relative semantics via translation to set theory $L$

write $\models F$ for $\vdash_L \mathtt{True}[\![F]\!]$

## Equivalence Theorems

- Soundness: $\vdash F$ implies $\models F$
- Completeness: $\models F$ implies $\vdash F$

accordingly for other translations

# Example: Redundant Semantics of BOL

## Are these two BOL semantics deductively equivalent

- absolute deductive semantics via proof system
- relative deductive semantics via translation $\llbracket - \rrbracket$ to SFOL

## Soundness: $\vdash^{BOL}_{V} f$ implies $\vdash^{SFOL}_{\llbracket V \rrbracket} \llbracket f \rrbracket$

- induction on derivations of $\vdash^{BOL}_{V} f$
- one case per rule — induction rule from above not sound
- several pages of work but straightforward and relatively easy

# Example: Redundant Semantics of BOL

## Are these two BOL semantics deductively equivalent

- absolute deductive semantics via proof system
- relative deductive semantics via translation $[\![-]\!]$ to SFOL

## Completeness: $\vdash^{BOL}_{V} f$ implied by $\vdash^{SFOL}_{[\![V]\!]} [\![f]\!]$

- induction on SFOL derivations does not work
  - SFOL more expressive than BOL
  - $[\![-]\!]$ not surjective
- instead show that $[\![-]\!]$ preserves consistency of vocabularies

  no universal recipe how to do that
- then a typical proof uses $V$ extended with $\neg f$
  - if $V$ inconsistent, $\vdash_V f$ for all $f$, done
  - if $V$ consistent and $V + \neg f$ inconsistent, then $\vdash_V f$, done
  - if $V + \neg f$ consistent, so is $[\![V + \neg f]\!]$, which contradicts $\vdash^{SFOL}_{[\![V]\!]} [\![f]\!]$

# Side Note: Proof system as special case of type system

### Propositions-as-Types, Proofs-as-Terms

- extend grammar so that there are expressions for proofs
- use propositions as types for proofs:
  $\vdash P : F$ means $P$ is proof of $F$
- define: $\vdash F$ iff there is $P$ such that $\vdash P : F$

often called Curry-Howard representation

### Then: given an absolute semantics and a semantics by translation

- type preservation of translation = soundness
- conservativity of translation = completeness

# Computational Semantics

# Computational Semantics

## Definition

- for every $V$, a function $\mathrm{Eval}^I_V : \mathrm{Exp}^I_V \to \mathrm{Exp}^I_V$
- special judgment $\vdash^I_V e \rightsquigarrow e'$ for $e' = \mathrm{Eval}^I_V(e)$

determines how expressions evaluate to values

## Terminology

- Programming language: language plus computational semantics
- Operational semantics: rules defining computational semantics
- Interpreter: implementation of absolute computational semantics
- Compiler: implementation of relative computational semantics
- Machine language: microchip for absolute computational semantics

# Caveats

Evaluation $\vdash^{I}_{V} e \rightsquigarrow e'$ insufficient in general

## Actually more complex

- side effects:
  - IO channels
  - object creation/destruction
  - mutable variables

  needs environment, heap, stack, references, ...
- non-termination: no $e'$ exists
- non-determinism: multiple $e'$ exist

  $e \rightsquigarrow e'$ must be relation, not function

# Redundant Computational Semantics

## Multiple computational semantics

- Specification: absolute as rules on paper
- Interpreter: absolute as implementation
- Compiler: relative via translation to assembly $L$

  write $\models E \rightsquigarrow V$ for $\vdash_L [\![E]\!] \rightsquigarrow [\![V]\!]$
- Cross-compilation: relative via translation into other languages

  Church-Turing thesis: always possible

## Equivalence Theorems

- e.g., correctness of compiler: $\vdash E \rightsquigarrow V$ iff $\models E \rightsquigarrow V$

  accordingly for other translations

# Relationships to other judgments

## Big Step vs. Small Step

- big step: $\vdash^{I}_{V} e \rightsquigarrow e'$ is the entire evaluation
- small step: $\vdash^{I}_{V} e \rightsquigarrow e'$ is just one step and semantics requires exhaustive chaining of steps

## Typing

- subject reduction: if $\vdash e : E$, then $\vdash \mathrm{Eval}(e) : E$

## Deductive semantics with equality

- normal forms:
  - $\mathrm{Eval}^{I}_{V}$ idempotent, i.e., $\mathrm{Eval}^{I}_{V}(x) = x$ if $x$ already a value
  - $\vdash^{I}_{V} e \doteq_{E} \mathrm{Eval}^{I}_{V}(e)$
- canonical forms: $\vdash^{I}_{V} e_1 \doteq_{E} e_2$ iff $\mathrm{Eval}^{I}_{V}(e_1) = \mathrm{Eval}^{I}_{V}(e_2)$

# Interdefinability

**Given a computational semantics, define a deductive one:**

- distinguished expression $\vdash true : \mathtt{prop}$,
- $\vdash F$ iff $\mathrm{Eval}(F) = true$

  implies decidability, so usually only possible for some $F$

**Given a deductive semantics, define computational one:**

- $\mathrm{Eval}(e)$ is some $e'$ such that $\vdash e \doteq e'$

  trivially normal, but usually not canonical

Both kinds of semantics add different value. We usually want both.

# Contexts

# Syntax with Contexts

## Syntax with contexts

- contexts: for every $V$, a set $\mathrm{Cont}^{l}_{V}$ write $\vdash_V \Gamma$
- substitutions: for $\Gamma, \Delta \in \mathrm{Cont}_V$, a set $\mathrm{Subs}_V(\Gamma, \Delta)$ write $\vdash_V \gamma : \Gamma \to \Delta$

## Expressions in context

- expressions: sets $\mathrm{Exp}_V(\Gamma)$
- substitution application: functions $\mathrm{Exp}(\gamma) : \mathrm{Exp}(\Gamma) \to \mathrm{Exp}(\Delta)$ for $\gamma \in \mathrm{Subs}(\Gamma, \Delta)$ write $\mathrm{Exp}(\gamma)(e)$ as $e[\gamma]$

## Typing in context

- expressions: sets $\mathrm{Exp}_V(\Gamma, E)$, written as $\Gamma \vdash_V e : E$
- substitution preserves types: if $\Gamma \vdash e : E$ and $\vdash \gamma : \Gamma \to \Delta$, then $\Delta \vdash e[\gamma] : E[\gamma]$

# Example: Contexts as Lists of Variables

Given formal systems, define

- contexts $\Gamma$: lists
$$x_1 : E_1, \ldots, x_n : E_n$$
where $E_i \in \mathrm{Exp}(x_1 : E_1, \ldots, x_{i-1} : E_{i-1})$
- for $\Gamma$ as above, substitutions $\Gamma \to \Delta$: lists
$$x_1 = e_1, \ldots, x_n = e_n$$
where $\Delta \vdash e_i : E_i[x_1 = e_1, \ldots, x_{i-1} = e_{i-1}]$

Type preservation of substitution must be proved individually for every formal system.
But if it does not hold, we can consider the formal system mis-designed.

# Semantics with Contexts

## Deductive semantics

- define: theorem sets $\mathrm{Thm}_V(\Gamma)$ write $F \in \mathrm{Thm}_V(\Gamma)$ as $\Gamma \vdash_V F$
- such that theorems are preserved by substitution:
  if $\Gamma \vdash_V F$ and $\vdash \gamma : \Gamma \to \Delta$, then $\Delta \vdash_V F[\gamma]$

## Computational semantics

- define: evaluation functions $\mathrm{Eval}_V(\Gamma) : \mathrm{Exp}_V(\Gamma) \to \mathrm{Exp}_V(\Gamma)$
  write $e' = \mathrm{Eval}_V(\Gamma)(e)$ as $\Gamma \vdash_V e \rightsquigarrow e'$
- extend to substitutions:
  $\mathrm{Eval}_V(\Delta)(\ldots, x = e, \ldots) \; = \; \ldots, x = \mathrm{Eval}_V(\Delta)(e), \ldots$
- require that evaluation is preserved by substitution $\vdash \gamma : \Gamma \to \Delta$
  $\mathrm{Eval}_V(\Delta)(e[\gamma]) = \mathrm{Eval}_V(\Delta)(e)[\mathrm{Eval}_V(\Delta)(\gamma)]$
  substitution theorem for $\mathrm{Eval}$ as a translation from $I$ to itself

# Terminology

- write $\cdot$ for empty context/substitution
- ground expression is expression in empty context

  also called closed; opposite is open
- ground substitution: $\vdash \gamma : \Gamma \to \cdot$

  no free variables after substitution
- true instance of $\Gamma \vdash F : \mathtt{prop}$: a ground substitution $\gamma$ such that $\vdash F[\gamma]$

# Concrete Semantics

# Concrete Semantics

## Definition

- for every $\Gamma \vdash^I_V F : \mathtt{prop}$, a set $\mathrm{Inst}^I_V(\Gamma, F)$ of ground instances

write $\vdash^I_V \gamma : \Gamma$ and $\vdash F[\gamma]$ for $\gamma \in \mathrm{Inst}_V(\Gamma, F)$

determines the true instances of propositions

## Terminology

- Query languages (in the usual, narrower sense than used here): languages plus concrete semantics
- Database: implementation of concrete semantics

  usually optimized for fast query answering

## Examples

- SQL for Church-typed ontologies with ADTs (relational databases)
- SPARQL for Curry-typed ontologies (triple stores)
- Prolog for first-order logic

# Redundant Concrete Semantics

## Multiple concrete semantics

- Specification: absolute as rules on paper
- Database: absolute by custom database
- Database: relative via translation to assembly $L$

## Equivalence Theorems

- typically: choose one, no redundancy, no equivalence theorems
- infinite results: easy on paper, hard in database
- open world: are all known ground instances in database?

# Interdefinability

## Given concrete semantics, define a deductive one

- for ground $F$, $\mathrm{Inst}(\cdot, F)$ is either $\{\cdot\}$ or $\{\}$
- $\vdash F$ iff $\mathrm{Inst}(\cdot, F) = \{\cdot\}$

  but concrete semantics usually cannot find all substitutions for all $F$

## Given concrete semantics, define a computational one

- $\vdash e \rightsquigarrow e'$ iff $(x = e') \in \mathrm{Inst}(x : E, e \doteq_E x)$

  but concrete semantics usually cannot find that substitution for all $e$

## Given deductive semantics, define a concrete one

- $\mathrm{Inst}(\Gamma, F) = \{\vdash \gamma : \Gamma \to \cdot \mid \vdash F[\gamma]\}$

  but deductive semantics usually does not allow computing that set

## Given computational semantics, define a concrete one

- $\mathrm{Inst}(\Gamma, F) = \{\mathrm{Eval}(\cdot, \gamma) \mid \vdash \gamma : \Gamma \to \cdot, \vdash F[\gamma] \rightsquigarrow \mathit{true}\}$
- allows restricting results to value substitutions

  composition of previous inter-definitions, inherits both problems

# Narrative Semantics

# Narrative Semantics

## Definition

- Describes how to answer (some) questions
- Implementations tend to be AI-complete, hypothetical
- In practice, information retrieval = find related documents

## More precisely?

- Not much theory, wide open research problem
- Some natural language document with interspersed definitions, formulas
- Maybe judgment: $\vdash Q?A$ for "$A$ is answer to $Q$"

## Examples

- "W3C Recommendation OWL 2" and Google
- "ISO/IEC 14882: 1998 Programming Language C++" and Stroustrup's book
- Mathematics textbooks and mathematicians

# Equivalence with respect to a semantics

## General Definition

So far: equivalence of two semantics wrt. all expressions

Related concept: equivalence of two expressions wrt. one semantics

- $F$, $G$ deductively equivalence:

$$\vdash F \quad \text{iff} \quad \vdash G$$

may be internalized by syntax as proposition $F \leftrightarrow G$

- $F, G$ concretely equivalent:

$$\vdash F[\gamma] \quad \text{iff} \quad \vdash G[\gamma]$$

for all ground substitutions $\gamma$ weaker than $\Gamma \vdash F \leftrightarrow G$

- closed $e, e'$ computationally equivalent:

$$\vdash e \rightsquigarrow v \quad \text{iff} \quad \vdash e' \rightsquigarrow v$$

may be internalized by syntax as proposition $e \doteq e'$

# Specific Variants

Interesting variants of computational semantics

- open $e, e'$ extensionally equivalent:

  $$\vdash e[\gamma] \leadsto v \qquad \text{iff} \qquad \vdash e'[\gamma] \leadsto v$$

  for all ground substitutions $\gamma$

  equal inputs produce equal outputs

  weaker then $\Gamma \vdash e \doteq e'$ — intensional equivalence

- machines $M, M'$ observationally equivalent:
  produce equal sequences of outputs for the same sequence of inputs e.g., automata, objects in OO-programming

choice of semantics defines legal optimizations in compiler

# Abstract Semantics with Morphisms

# Adjusted Definitions

If the formal system has morphisms $m \in \mathrm{Mor}^I(V, W)$, the homomorphic extension $\mathrm{Exp}^I_m : \mathrm{Exp}^I_V \to \mathrm{Exp}^I_W$ must preserve the semantics.

## Deductive semantics

- morphisms must preserve theorems
- for morphism $m \in \mathrm{Mor}^I(V, W)$

$$F \in \mathrm{Thm}^I(V) \quad \text{implies} \quad \mathrm{Exp}^I_m(F) \in \mathrm{Thm}^I(W)$$

## Computational semantics

- morphisms must preserve evaluation
- for morphism $m \in \mathrm{Mor}^I(V, W)$

$$\mathrm{Eval}^I_W(\mathrm{Exp}^I_m(e)) = \mathrm{Exp}^I_m(\mathrm{Eval}^I_V(e))$$

This must be proved individually for every formal system.
But if it does not hold, we can consider the formal system mis-designed.

# Translations

# Translations

Given: formal systems $I$ and $L$

## A translation $T$ from $I$ to $L$ consists

- function $\mathrm{Voc}^T : \mathrm{Voc}^I \to \mathrm{Voc}^L$ vocabulary translation
- family of functions $\mathrm{Exp}^T_V : \mathrm{Exp}^I_V \to \mathrm{Exp}^L_{\mathrm{Voc}^T(V)}$

expression translation

one function for each expression kind

## Example: BOL→SFOL

- $\mathrm{Voc}^T(V)$
  - semantic prefix to declare type $\iota$
  - for each BOL-declaration, one SFOL declaration of the same name
- $\mathrm{Exp}^T_V$
  - $\vdash^{BOL}_V C : \mathtt{Concept}$ translated to $x : \iota \vdash_{\mathrm{Voc}^T(V)} \mathrm{Exp}^T_V(C) : \mathtt{prop}$, etc.
  - identifiers mapped to themselves

# Properties of Translations

## Desirable properties

- Should satisfy type preservation:

$$\vdash^I_V e : E \quad \text{implies} \quad \vdash^L_{\mathrm{Voc}^T(V)} \mathrm{Exp}^T_V(e) : \mathrm{Exp}^T_V(E)$$

intuition: what we have, is preserved

- Should satisfy type reflection

$$\vdash^L_{\mathrm{Voc}^T(V)} \mathrm{Exp}^T_V(e) : \mathrm{Exp}^T_V(E) \quad \text{implies} \quad \vdash^I_V e : E$$

intuition: nothing new gets added

- Might satisfy conservativity for $V$-propositions $F$

$$\vdash^L_{\mathrm{Voc}^T(V)} \mathrm{Exp}^T_V(F) \quad \text{implies} \quad \vdash^I_V F$$

intuition: semantics is not changed

# Translation with Contexts

## Translations extend to contexts and substitutions

- $\mathrm{Cont}^T(\ldots, x : E, \ldots) \ = \ \ldots, x : \mathrm{Exp}^T(E), \ldots$
- $\mathrm{Subs}^T(\ldots, x = e, \ldots) \ = \ \ldots, x = \mathrm{Exp}^T(e), \ldots$
- $\mathrm{Exp}^T(x) = x$ for all variables

## Desirable properties

- Type preservation:

$$\Gamma \vdash^I_V e : E \qquad \text{implies} \qquad \mathrm{Cont}^T_V(\Gamma) \vdash^L_{\mathrm{Voc}^T(V)} \mathrm{Exp}^T_V(e) : \mathrm{Exp}^T_V(E)$$

- Type reflection

$$\mathrm{Cont}^T_V(\Gamma) \vdash^L_{\mathrm{Voc}^T(V)} \mathrm{Exp}^T_V(e) : \mathrm{Exp}^T_V(E) \qquad \text{implies} \qquad \Gamma \vdash^I_V e : E$$

- Conservativity:

$$\mathrm{Cont}^T_V(\Gamma) \vdash^L_{\mathrm{Voc}^T(V)} \mathrm{Exp}^T_V(F) \qquad \text{implies} \qquad \Gamma \vdash^I_V F$$

# Relative Semantics

## Given

- formal systems $I$ and $L$
- semantics for $L$
- translation $T$ from $I$ to $L$

## Define semantics for $I$ using $T$ and $L$

- deductive: define

$$\vdash^I_V F \qquad \text{iff} \qquad \vdash^L_{\mathrm{Voc}^T(V)} \mathrm{Exp}^T_V(F)$$

- computational: define

$$\vdash^I_V e \rightsquigarrow e' \qquad \text{iff} \qquad \vdash^L_{\mathrm{Voc}^T(V)} \mathrm{Exp}^T_V(e) \rightsquigarrow \mathrm{Exp}^T_V(e')$$

both work accordingly with a context $\Gamma$

- concrete: define

$$\Gamma \vdash^I_V \gamma : F \qquad \text{iff} \qquad \mathrm{Cont}^T_V(\Gamma) \vdash^L_{\mathrm{Voc}^T(V)} \mathrm{Subs}^T_V(\gamma) : \mathrm{Exp}^T_V(F)$$

# Compositionality

# 3 Layers of Syntax

| symbol | origin | owner | examples |
|---|---|---|---|
| logical, built-in | language | language creator | $\wedge$ in SFOL, $\sqcup$ in BOL |
| non-logical, user-defined | vocabulary | vocabulary author | `person`, `WuV` |
| variables, locally bound | context | containing declaration | $x, y, \ldots$ |

# 3 Layers of Translations

| level | translation | define by |
|---|---|---|
| language | translation | one case per production (= built-in) |
| vocabulary | morphism | one expression per symbol |
| context | substitution | one expressions per variable |

Morphisms and substitutions are always compositional by definition.
They keep the syntax tree structure unchanged and just replace symbols/variables with expressions.
Language translations should be compositional but might not.

# Substitution Theorem

Translations/morphisms/substitutions act independently parts of the syntax tree: productions/symbols/variables.

If they are compositional, they do not disturb each other, and their order can be exchanged.

This is called the substitution theorem.

# Substitution Theorem for translation/morphisms

Given:

- formal systems $I$, $L$ with morphisms
- a compositional translation $[\![-]\!] : I \to L$
- an $I$-morphism $m \in \mathrm{Mor}^I(V, W)$

$$[\![\overline{m}(E)]\!] \; = \; \overline{[\![m]\!]}([\![E]\!])$$

$$\begin{tikzcd}
\mathrm{Exp}^I(V) \arrow[r, "{[\![-]\!]}"] \arrow[d, "\overline{m}"'] & \mathrm{Exp}^L([\![V]\!]) \arrow[d, "\overline{[\![m]\!]}"] \\
\mathrm{Exp}^I(W) \arrow[r, "{[\![-]\!]}"] & \mathrm{Exp}^L([\![W]\!])
\end{tikzcd}$$

$\overline{m}$ is the homomorphic extension of $m$, also written $\mathrm{Exp}^I_m$

# Substitution Theorem for translation/substitutions

Given:

- formal systems $I$, $L$ with contexts and substitutions
- a compositional translation $[\![-]\!] : I \to L$
- an $I$-vocabulary $V$ and a $V$-substitution $\gamma : \Gamma \to \Delta$

$$[\![E[\gamma]]\!] \;=\; [\![E]\!][[\![\gamma]\!]]$$

$$\begin{tikzcd}
\mathrm{Exp}^{I}_{V}(\Gamma) \arrow[r, "{[\![-]\!]}"] \arrow[d, "{-[\gamma]}"'] & \mathrm{Exp}^{L}_{[\![V]\!]}([\![\Gamma]\!]) \arrow[d, "{-[[\![\gamma]\!]]}"] \\
\mathrm{Exp}^{I}_{V}(\Delta) \arrow[r, "{[\![-]\!]}"] & \mathrm{Exp}^{L}_{[\![V]\!]}([\![\Delta]\!])
\end{tikzcd}$$

# Substitution Theorem for morphisms/substitutions

Given:

- formal system $I$ with morphisms and contexts and substitutions
- a morphism $m : V \to W$
- a $V$-substitution $\gamma : \Gamma \to \Delta$

$$\overline{m}(E[\gamma]) \;=\; \overline{m}(E)[\overline{m}(\gamma)]$$

$$\begin{tikzcd}
\mathrm{Exp}^I_V(\Gamma) \arrow[r, "\overline{m}"] \arrow[d, "-[\gamma]"'] & \mathrm{Exp}^I_W(\overline{m}(\Gamma)) \arrow[d, "-[\overline{m}(\gamma)]"] \\
\mathrm{Exp}^I_V(\Delta) \arrow[r, "\overline{m}"] & \mathrm{Exp}^I_W(\overline{m}(\Delta))
\end{tikzcd}$$

# Substitution Theorem for the general case

The previous cases can be seen as faces of a cube where corners are triples of language, vocabulary, context.

In general, we have

- formal systems $I$, $L$ with morphisms and contexts and substitutions and a compositional translation $[\![-]\!] : I \to L$
- $I$-vocabularies $V$, $W$, and an $I$-morphism $m : V \to W$
- $V$-contexts $\Gamma$, $\Delta$, and a $V$-substitution $\gamma : \Gamma \to \Delta$

and all 6 orders of applying $[\![-]\!]$, $\overline{m}$, and $\gamma$ yield equal results.