Morphism Axioms

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Abstract

We introduce a new concept in the area of formal logic: axioms for model morphisms.

We work in the setting of specification languages that define the semantics of a theory as a category of models. While it is routine to use axioms to specify the class of models of a theory, there has so far been no analogue to systematically specify the morphisms between these models. This leads to subtle problems where it is difficult to give a theory that specifies the intended model category, or where seemingly isomorphic theories actually have non-isomorphic model categories. Our morphism axioms remedy this by providing new syntax for axiomatizing and reasoning about the properties of model morphisms.

Additionally, our system resolves a subtle incompatibility between theory morphisms and model morphisms: the semantics that maps theories to model categories is functorial. While this result is standard in principle, previous formulations had to restrict the allowed theory morphisms or the allowed model morphisms. Our system allows establishing the result in full generality.

Keywords logic, specification, institution, theory morphism, model morphism, functorial

1 Introduction

Motivation One of the most important techniques in formal logic, especially in specification, is the use of axioms to restrict the class of admissible models: the (model-theoretical) semantics of a theory Θ is given by the class $\mathbf{Mod}(\Theta)$ of models. The present paper explores two deep technical problems in this context.

Firstly, a frequent interest is to specify $\mathbf{Mod}(\Theta)$ not only as a class of models but as a category of models and model morphisms. Indeed, many interesting properties of models can be studied via the properties of the category, including initial models, product models, submodels, and quotient models. For example, we want the theory `Group` to give rise to the category $\mathbf{Mod}(\texttt{Group})$ of groups and

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group homomorphisms, and **Mod**(Top) should be the category of topological spaces and continuous functions. For almost every logic, there is a canonical way to define **Mod** in such a way that **Mod**(Θ) is indeed a useful category.

However, typically the author of Θ has only indirect and limited control over the choice of morphisms in **Mod**(Θ). Moreover, subtle variations of Θ—even if they do not change the class of models—may yield very different model morphisms and thus different model categories. This is because theories usually provide no syntax for fine-tuning specifically the morphisms in **Mod**(Θ): While theory authors can add axioms to Θ to change the class of models, they have no direct influence on the morphisms.

For example, there are many choices for the theory Top whose models are exactly the topological spaces. But different choices can yield very different model morphisms, which may or may not be the continuous functions.

Secondly, formal logic can use theory morphisms $\vartheta : \Theta \to \Theta'$ to translate between theories. This method—developed most deeply in the field of algebraic specification mostly through the concept of institutions [GB92]—yields a category **Th** of theories and theory morphisms. The main properties of theory morphisms are that

- ϑ extends homomorphically to a mapping of Θ-formulas to Θ'-formulas, which is guaranteed to map Θ-theorems to Θ'-theorems.
- ϑ induces a model reduction functor **Mod**(ϑ) : **Mod**(Θ') $\to$ **Mod**(Θ).

This dual role of translating both syntax and semantics[1] has made theory morphisms an extremely valuable tool for structuring and relating large theories [SW83, FGT92].

However, not every theory morphism is well-behaved with respect to model morphisms. While **Mod**(ϑ) always reduces Θ'-models to Θ-models, not every Θ'-model morphism can be reduced to a Θ-model morphism. Thus, **Mod**(ϑ) is not always a functor, and consequently **Mod** is not always a functor from **Th** to $\mathcal{CAT}^{op}$.

Combining both of the above problems, we can find isomorphic theories $\Theta \leftrightarrow \Theta'$, whose model categories are not isomorphic. Thus, when using theories to formally specify model categories, small changes in the syntax that appear inconsequential because they are justified by a theory isomorphism may significantly change the semantics. This is not a contrived problem—in fact, we will see below that it happens all the time, even for elementary examples such as the theory of monoids.

Related Work The two problems described above have not received much attention in the literature so far. We can distinguish two fields that have avoided the practical consequences in two different ways.

On the one hand, algebraic specification languages of the OBJ tradition such as OBJ [GWM+93] or CASL [CoF04] avoid the problem by restricting the allowed theory morphisms: they require that theory morphisms map symbols to

[1] Note that syntax (formulas and proofs) and semantics (models) are translated in opposite directions. We follow the convention of algebraic specification that the $\to$ in $\Theta \to \Theta'$ indicates the direction of *syntax* translation. Readers that are used to working with model translations (e.g., forgetful functors or ML-style functors) may prefer flipping these arrows.

symbols rather than to arbitrary *expressions*. In that special case, $\mathbf{Mod}(\vartheta)$ always yields a functor. Here by symbol-to-symbol maps, we mean that a theory morphism $\vartheta : \Theta \to \Theta'$ must map, e.g., every binary Θ-function symbol f to a binary Θ'-function *symbol*, whereas symbol-to-expression maps allow mapping f to a binary *function*, e.g., a λ-expression of the right type. An intermediate option was recently explored in [Dia16]: Here theory morphisms map function symbols to expressions and predicate symbols to atomic expressions, and $\mathbf{Mod}(\vartheta)$ is proved to be a functor.

Interestingly, the restriction to symbol-to-symbol maps does not seem to have been motivated by the problems we described above. Algebraic specification languages tend to be based on first-order logic, where it is anyway more convenient to work only with symbol-to-symbol maps. Therefore, individual researchers in the field[2] may falsely believe that algebraic specification languages can be easily extended to allow symbol-to-expression maps.

Different choices of model morphisms in institutions are usually considered only by switching to a different institution. For example, [Dia08] discusses the method of diagrams for a few institutions that differ only in the choice of model morphisms.

On the other hand, in type theory, theory morphisms (if they are used at all) routinely allow symbol-to-expression maps. This is because type theories tend to be based on λ-calculi, where symbol-to-expression maps are much more elegant. This is the case, for example, for the proof assistants Isabelle [Pau94] and Coq [Coq15] and the logical framework Twelf [RS09].

These languages do not encounter the problems described above because they do not consider model morphisms in the first place. There are two reasons for this. Firstly, the respective communities tend to be less interested in model theory to begin with. Secondly, for λ-calculi, model morphisms do not work very well at all, and logical relations going back to [Rey74] usually have to be used instead.

Contribution We introduce a general formalism for specifying the properties of not only the models but also of the model morphisms in $\mathbf{Mod}(\Theta)$. The key innovation is to allow theories to contain what we call *maxioms*[3] in analogy to axioms: Just like axioms specify the intended models, the maxioms specify the intended model morphisms.

Our approach provides an elegant solution of both of our motivating problems: It allows users to fine-tune the morphisms when specifying model categories, and it guarantees that every theory morphism ϑ yields a functor $\mathbf{Mod}(\vartheta)$.

Overview We introduce a general framework for developing first-order style logics in Sect. 2. This allows us to later instantiate our results for different logics. It also has the added benefit to help us understand the limitations of our results by asking to which logics they do not apply.

Then Sect. 2 develops the syntax, proof theory, and model theory of our logics excluding model morphisms. Building on this, Sect. 3 discusses the technical problems regarding model morphisms in more detail. This discussion leads to

[2] including until recently the author of this paper

[3] We will systematically form new words by prepending the letter m in order to emphasize the symmetry between existing and new concepts.

our solution in Sect. 4, which presents our main results. Specifically, we instantiate our framework with plain, typed, and polymorphic first-order logic. Sect. 5 discusses further generalizations for subtypes and partial functions.

2 A Framework for First-Order Logics

We give a systematic definition of the syntax, proof theory, and model theory of first-order logic in a way that abstracts from specific language features such as the details of the type and proof systems. Our presentation stays as close to first-order logic as possible while enjoying the advantages of higher-order formalisms, especially logical frameworks such as LF [HHP93].[4]

2.1 Syntax

2.1.1 Theories and Their Expressions

The syntax of our logic distinguishes four ontological concepts: **types** A, **terms** T, **formulas** F, and **proofs** P, which are collectively called **expressions**. **Theories** Θ can contain four different kinds of declarations, one for each concept: type symbols a, function symbols t, predicate symbols φ, and axioms a, which are collectively called **symbols**. Similarly, **contexts** Γ declare **variables** of each concept. The following table gives an overview:

Symbol Declaration		Expressions	Judgment
type	$a : \{\Gamma\}\mathtt{tp}$	types A	$\Gamma \vdash_\Theta A : \mathtt{tp}$
function	$t : \{\Gamma\}A$	terms T of type A	$\Gamma \vdash_\Theta T : A$
predicate	$\varphi : \{\Gamma\}\mathtt{form}$	formulas F	$\Gamma \vdash_\Theta F : \mathtt{form}$
axiom	$a : \{\Gamma\}F$	proofs P of formula F	$\Gamma \vdash_\Theta P : F$
generic case to unify all of the above			
symbol	$c : \{\Gamma\}C$	expressions E	$\Gamma \vdash_\Theta E : C$

Before giving the precise definition, we introduce our running example to explain our notations:

Example 2.1. The theory `SemiGroup` consists of

- a type declaration: $u : \{\}\mathtt{tp}$
- a binary function symbol taking inputs x and y of type u and returning a term of type u
$$\circ : \{x : u, y : u\}u$$
- an axiom
$$\mathtt{assoc} : \{\}\ \forall x : u.\forall y : u.\forall z : u.x \circ (y \circ z) \doteq_u (x \circ y) \circ z$$
where we write $\circ$ as infix for readability.

The context Γ declares the arguments of each symbol: $\circ$ takes two arguments whereas u and `assoc` take none. Types with arguments allow for type operators, and axioms with arguments allow for axiom schemata.

In the sequel, we may omit $\{\Gamma\}$ if Γ is empty, i.e., if a symbol takes no arguments.

[4] Readers familiar with λ-calculi should read the notations $\{x : A\}B$ and $[x : A]T$, which we introduce below, as $\Pi x : A.B$ and $\lambda x : A.T$, respectively.

Expressions are well-formed relative to a theory Θ and a context Γ. It is convenient to write all well-formedness judgments as typing judgments. Therefore, we use the special symbols `tp` for the universe of all types and `form` for the universe of all formulas. Terms are typed by types, and proofs are typed by the formulas that they prove. We will write c and E for arbitrary symbols and expressions (i.e., type, term, formula, or proof), and C stands for any type A, any formula F, or either of the universes `tp` and `form`. Then all well-formedness judgments are of the form $\Gamma \vdash_\Theta E : C$.

The formal definition of the above is as follows:

Definition 2.2 (Theories)**.** A **theory** is a list of declarations $D_1, \dots, D_m$ where each D_i is of the form $c : \{\Gamma\}C$ such that

- c is a unique name of the declared symbol,
- Γ is a context over the theory $D_1, \dots, D_{i-1}$,
- C is one of the four cases described below.

Γ declares the arguments of c, and C describes what c returns.

A **context** Γ over a theory Θ is a list of declarations $x_1 : C_1, \dots, x_n : C_n$ such that Θ, Γ is a theory. The x_i are called **variables**.

Based on the shape of C, we distinguish four kinds of **declarations** $c : \{\Gamma\}C$ in a theory or context:

- $C = \texttt{tp}$: c is a **type symbol** and returns a new type.
- $C = A$ for some $\Gamma \vdash_\Theta A : \texttt{tp}$: c is a **function symbol** and returns a term of type A.
- $C = \texttt{form}$: c is a **predicate symbol** and returns a formula.
- $C = F$ for some $\Gamma \vdash_\Theta F : \texttt{form}$: c is an **axiom**[5] and returns a proof of the formula F.

Symbol declarations in theories may take parameters (as declared by Γ), whereas variables may not. This gives our logic its first-order flavor: For example, we can declare function symbols in theories, but we cannot quantify over them because we cannot declare them as variables.

Expressions are formed inductively from the declared symbols and variables:

Definition 2.3 (Expressions)**.** Complex types, terms, formulas, and proofs are formed according to the following grammar

A	::=	$x \mid a(E_1, \dots, E_n)$	atomic types
T	::=	$x \mid t(E_1, \dots, E_n)$	atomic terms
F	::=	$x \mid \varphi(E_1, \dots, E_n)$	atomic formulas
	$\mid$	$F \wedge F' \mid F \vee F' \mid F \Rightarrow F' \mid \neg F \mid F \Leftrightarrow F'$	propositional logic
	$\mid$	$T \doteq_A T$	typed equality
	$\mid$	$\forall x : A.F(x) \mid \exists x : A.F(x)$	typed quantification
P	::=	$x \mid p(E_1, \dots, E_n)$	atomic proofs
	$\mid$	as in Fig. 1	natural deduction proofs

In examples, we will omit the type arguments of equality and quantifiers if they can be inferred.

[5] Because we allow parameters, we should technically speak of an *axiom schema*, but we will use the word *axiom* for brevity.

The rules for well-formed proofs are given in Fig. 1. There we use the usual notation $\Gamma \vdash_\Theta F$ in proof rules, i.e., we omit the proof expression P in the judgment $\Gamma \vdash_\Theta P : F$. It is straightforward to form proof expressions from the names of the proof rules.

Definition 2.4 (Substitution)**.** We write $E[x_1/E_1, \dots, x_n/E_n]$ for the result of (capture-avoiding) **substitution** of E_i for x_i in E.

If the free variables in E are clear from the context, we also write this as $E(E_1, \dots, E_n)$.

All four kinds of expressions share the construction of atomic expressions. They also share the well-formedness rules for them:

$$\frac{x : C \text{ in } \Gamma}{\Gamma \vdash_\Theta x : C}$$

$$\frac{c : \{x_1 : C_1, \dots, x_n : C_n\}C \text{ in } \Theta \qquad \Gamma \vdash_\Theta E_i : C_i(\vec{E}) \text{ for } i = 1, \dots, n}{\Gamma \vdash_\Theta c(E_1, \dots, E_n) : C(\vec{E})}$$

Here each x_i may occur in $C_{i+1}, \dots, C_n, C$, and we write $C(\vec{E})$ for the result of substituting each x_i with E_i.

We omit the straightforward well-formedness rules for formulas.

Examples and Remarks We can obtain several logics as special cases:

Example 2.5 (Variant Logics). Untyped intuitionistic first-order logic (FOL) arises as the special case where

- there is a single fixed type symbol $u : \texttt{tp}$ and other type declarations are not allowed,
- contexts may declare only term variables.

Thus, all FOL-contexts are of the form $x_1 : u, \dots, x_n : u$. `SemiGroup` from Ex. 2.1 is an example of a FOL-theory.

Typed first-order logic (TFOL) arises as the special case where

- type declarations must be of the form $a : \texttt{tp}$ (i.e., types may not take parameters),
- contexts may declare only term variables.

Thus, all TFOL-contexts are of the form $x_1 : a_1, \dots, x_n : a_n$.

Typed first-order logic with polymorphism (PFOL), which has recently received more systematic attention [BP13], arises as the special case where

- type declarations must be of the form $a : \{x_1 : \texttt{tp}, \dots, x_n : \texttt{tp}\}\texttt{tp}$ (i.e., n-ary type operators),
- contexts may only declare type and term variables.

Thus, up to reordering, all PFOL-contexts are of the form $x_1 : \texttt{tp}, \dots, x_m : \texttt{tp}, y_1 : A_1, \dots, y_n : A_n$.

All of the above logics can be classical or intuitionistic. The classical variant arises by adding the axiom $\texttt{classical} : \{x : \texttt{form}\}x \vee \neg x$.

	Introduction	Elimination
$\wedge$	$\frac{\Gamma\vdash_\Theta F \quad \Gamma\vdash_\Theta G}{\Gamma\vdash_\Theta F\wedge G}\texttt{conjI}$	$\frac{\Gamma\vdash_\Theta F\wedge G}{\Gamma\vdash_\Theta F}\texttt{conjEl} \quad \frac{\Gamma\vdash_\Theta F\wedge G}{\Gamma\vdash_\Theta G}\texttt{conjEr}$
$\vee$	$\frac{\Gamma\vdash_\Theta F}{\Gamma\vdash_\Theta F\vee G}\texttt{disjIl} \quad \frac{\Gamma\vdash_\Theta G}{\Gamma\vdash_\Theta F\vee G}\texttt{disjIr}$	$\frac{\Gamma\vdash_\Theta F\vee G \quad \Gamma,p{:}F\vdash_\Theta H \quad \Gamma,p{:}G\vdash_\Theta H}{\Gamma\vdash_\Theta H}\texttt{disjE}$
$\to$	$\frac{\Gamma,p{:}F\vdash_\Theta G}{\Gamma\vdash_\Theta F\to G}\texttt{implI}$	$\frac{\Gamma\vdash_\Theta F\to G \quad \Gamma\vdash_\Theta F}{\Gamma\vdash_\Theta G}\texttt{implE}$
$\Leftrightarrow$	$\frac{\Gamma,p{:}F\vdash_\Theta G \quad \Gamma,p{:}G\vdash_\Theta F}{\Gamma\vdash_\Theta F\Leftrightarrow G}\texttt{equivI}$	$\frac{\Gamma\vdash_\Theta F\Leftrightarrow G \quad \Gamma\vdash_\Theta F}{\Gamma\vdash_\Theta G}\texttt{equivEl}$ $\frac{\Gamma\vdash_\Theta F\Leftrightarrow G \quad \Gamma\vdash_\Theta G}{\Gamma\vdash_\Theta F}\texttt{equivEr}$
$\neg$	$\frac{\Gamma,p{:}F,f{:}\texttt{form}\vdash_\Theta f}{\Gamma\vdash_\Theta \neg F}\texttt{negI}$	$\frac{\Gamma\vdash_\Theta \neg F \quad \Gamma\vdash_\Theta F \quad \Gamma\vdash_\Theta H{:}\texttt{form}}{\Gamma\vdash_\Theta H}\texttt{negE}$
$\texttt{true}$	$\frac{}{\Gamma\vdash_\Theta \texttt{true}}\texttt{trueI}$	
$\texttt{false}$		$\frac{\Gamma\vdash_\Theta \texttt{false} \quad \Gamma\vdash_\Theta H{:}\texttt{form}}{\Gamma\vdash_\Theta H}\texttt{falseE}$
$\forall$	$\frac{\Gamma,x{:}A\vdash_\Theta F \quad x\notin\Gamma}{\Gamma\vdash_\Theta \forall x{:}A.\ F}\texttt{forallI}$	$\frac{\Gamma\vdash_\Theta \forall x{:}A.\ F \quad \Gamma\vdash_\Theta T{:}A}{\Gamma\vdash_\Theta F[x/T]}\texttt{forallE}$
$\exists$	$\frac{\Gamma\vdash_\Theta F[x/T] \quad \Gamma\vdash_\Theta T{:}A}{\Gamma\vdash_\Theta \exists x{:}A.\ F}\texttt{existsI}$	$\frac{\Gamma\vdash_\Theta \exists x{:}A.\ F \quad \Gamma,x{:}A,p{:}F\vdash_\Theta H \quad x\notin\Gamma,H}{\Gamma\vdash_\Theta H}\texttt{existsI}$
$\doteq$	$\frac{\Gamma\vdash_\Theta T{:}A}{\Gamma\vdash_\Theta T\doteq_A T}\texttt{equalI}$	$\frac{\Gamma\vdash_\Theta E(S){:}C(S) \quad \Gamma\vdash_\Theta S\doteq_A T}{\Gamma\vdash_\Theta E(T){:}C(T)}\texttt{equalE}$

Figure 1: Natural Deduction Rules

Example 2.6 (Continuing Ex. 2.1). There are two equivalent ways to extend the theory `SemiGroup` to the theory of monoids. Let $\texttt{isunit}(x)$ abbreviate $\forall y : u.\, y \circ x \doteq_u y \wedge x \circ y \doteq_u y$. The theory `MonoidCon` uses a constant for the unit

$$e : u,\ \texttt{neut} : \texttt{isunit}(e)$$

The theory `MonoidAx` uses an axiom for the existence of a unit

$$\texttt{unit} : \exists x : u.\, \texttt{isunit}(x)$$

It is well-known that these two theories induce isomorphic model *classes*. However, as we will see in Ex. 2.11, they do not induce isomorphic model *categories* in standard first-order logic.

Remark 2.7 (Empty Types). Our logic allows types to be empty. This is important for applications in mathematics, where many important theories naturally have empty models, e.g., the theory of sets or the theory of orders. This breaks with the convention of standard first-order logic that all types are non-empty, i.e., that $\exists x : A.\texttt{true}$ is a theorem.

For readers not familiar with this effect, this may be surprising because our

natural deduction rules appear to be exactly the ones of standard first-order logic. But our proof rules subtly deviate from standard first-order logic by using the assumption $\Gamma \vdash_\Theta T : A$ in the rules `forallE` and `existsI`. If we dropped these assumptions, we could instantiate the rules with variables not in Γ. That is equivalent to assuming that all types are non-empty.

If we want to recover the standard convention, we can add an axiom `nonempty` : $\{y : \texttt{tp}\}\exists x : y.\,\texttt{true}$.

Remark 2.8 (Separating Formulas from Terms and Types). Our logic uses a layered approach where types and terms are separated from formulas and proofs. Many logics use a simpler ontology where formulas are special cases of terms or types.

For example, higher-order logic assumes $\vdash \texttt{form} : \texttt{tp}$, i.e., `form` is a special type and all formulas are special cases of terms. Similarly, type theories following the propositions-as-types paradigm assume $\texttt{form} = \texttt{tp}$, i.e., formulas and proofs are the same, respectively, as types and terms.

All statements of Sect. 2 easily specialize to those simpler ontologies. However, these simpler ontologies do not interact well with model morphisms. The reasons are subtle, and we will come back to this in Sect. 3. Essentially, when working with model morphisms, we have to treat formulas differently than terms and types. Therefore, we strictly segregate them from the beginning.

2.1.2 Theory Morphisms and Their Action on Expressions

Our logic allows for a very general definition of theory morphisms $\Theta \to \Theta'$. These are compositional translations, which map

- Θ-contexts to Θ'-contexts,
- Θ-expressions to Θ'-expressions.

Definition 2.9 (Theory Morphisms)**.** Consider two theories Θ and Θ'.

A **theory morphism** $\vartheta : \Theta \to \Theta'$ contains
for every declaration $c : \{\Gamma\}C$ in Θ
exactly one assignment $c \mapsto [\vartheta(\Gamma)]E$
such that $\vartheta(\Gamma) \vdash_{\Theta'} E : \vartheta(C)$.

The **homomorphic extension** $\vartheta(-)$ is defined as follows: To obtain $\vartheta(X)$
replace every occurrence of an atomic expression $c(E_1, \ldots, E_n)$ in X
where $c \mapsto [\vartheta(\Gamma)]E$ in ϑ
with $E(E_1, \ldots, E_n)$.

The key property of theory morphisms is that they preserve all judgments:

Theorem 2.10 (Judgment Preservation)**.** *Given a theory morphism* $\vartheta : \Theta \to \Theta'$*, then*

$$\Gamma \vdash_\Theta E : C \qquad \text{implies} \qquad \vartheta(\Gamma) \vdash_{\Theta'} \vartheta(E) : \vartheta(C)$$

Note that Thm. 2.10 contains as a special case the preservation of theorems: If F is provable in Θ, then $\vartheta(F)$ is provable in Θ'. Judgment preservation holds for a wide variety of formal systems—a proof for a very general case that subsumes the logic we use here can be found in [Rab14].

Example 2.11 (Continuing Ex. 2.6). We can give a theory morphism $\texttt{axCon} : \texttt{MonoidAx} \to \texttt{MonoidCon}$. It maps all symbols to themselves except for the axiom `unit`. It maps that symbol to a `MonoidCon`-proof by $\texttt{unit} \mapsto \texttt{existsI}(e, \texttt{neut})$.

`axCon` is essentially an isomorphism in the category of theories and morphisms. The only caveat is that first-order logic is not expressive enough to write down the inverse morphism. If we extend first-order logic with a description operator ι, we can map e to the uniquely determined unit element:

$$\texttt{e} \mapsto \iota x : u.\, \texttt{isunit}(x)$$

In the presence of such a description operator, the two theories are indeed isomorphic, i.e., they compose to the identity morphism (where equality of theory morphisms is up to provable equality in the logic).

We introduce one more running example:

Example 2.12. Consider the theories

$$\texttt{Special} = u : \texttt{tp},\ \texttt{sp} : \{x : u\}\texttt{form}$$

of a unary predicate on a type u and

$$\texttt{Order} = u : \texttt{tp},\ \texttt{leq} : \{x : u, y : u\}\texttt{form}, \ldots$$

of orders on a type u (where we omit the axioms).

We define a theory morphism $\texttt{least} : \texttt{Special} \to \texttt{Order}$ by

$$\texttt{least} = u \mapsto u,\ \texttt{sp} \mapsto [x : u] \forall y : u.\, \texttt{leq}(x, y)$$

It interprets the predicate of being special as the property of being the least element.

We will see in Ex. 3.8 that the standard definition of first-order does not yield a model reduction functor for this theory morphism.

2.2 Semantics

2.2.1 Models and Their Interpretation Functions

Before concisely stating our general definition of models, it is instructive to recall the general structure of how models are defined for the simple case of typed first-order logic (TFOL):

Remark 2.13 (Standard Definition of Models). For an TFOL-theory Θ, a model M

1. (a) provides a set a^M for each Θ-type symbol $a : \texttt{tp}$,
   (b) induces (by induction on types A) a set A^M for every type A (which is trivial for TFOL because the type symbols are the only types),
2. (a) provides an interpretation for each Θ-function symbol $t : \{x_1 : A_1, \ldots, x_n : A_n\}a$ as a mapping

$$t^M : (A_1^M \times \ldots \times A_n^M) \to A^M$$

(b) induces (by induction on terms T) the interpretation $T^{M,\alpha} \in A^M$ under assignment α for every term T of type A,

3. (a) provides an interpretation for each Θ-predicate symbol $\varphi : \{x_1 : A_1, \ldots, x_n : A_n\}\texttt{form}$ as a mapping

$$\varphi^M : (A_1^M \times \ldots \times A_n^M) \to \{0,1\}$$

(b) induces (by induction on formulas F) the interpretation $F^{M,\alpha} \in \{0,1\}$ under assignment α for every formula F,

4. (a) must satisfy each Θ-axiom symbol $a : \{\Gamma\}F$, i.e., $F^{M,\alpha} = 1$ for all assignments α

(b) is shown to satisfy (by induction on proofs P) every Θ-theorem F with proof P.

Our definition of model follows quite straightforwardly from applying these principles. The only subtlety is that we use a particular treatment of assignments that allows for a very concise definition:

Remark 2.14 (Treatment of Assignments). We interpret every context as the set of assignments for its variables. Thus, $(x_1 : A_1, \ldots, x_n : A_n)^M$ is the set of assignments $\alpha = (\alpha_1, \ldots, \alpha_n)$ that assign α_i to x_i.

We will also write $\alpha.e$ for the tuple $(\alpha_1, \ldots, \alpha_n, e)$.

Then we are ready to state our definitions of models and interpretation function:

Definition 2.15 (Models)**.** Given a theory Θ, the class $\mathbf{Mod}(\Theta)$ of **models** contains tuples $(\ldots, c^M, \ldots)$ providing for every Θ-symbol $c : \{\Gamma\}C$ a denotation c^M that maps $\alpha \in \Gamma^M$ to $c^M(\alpha) \in C^{M,\alpha}$.

Every such model induces an **interpretation function** that maps

- every Θ-context Γ to the set Γ^M of assignments for Γ,
- for every assignment $\alpha \in \Gamma^M$, every expression $\Gamma \vdash_\Theta E : C$ to its denotation $E^{M,\alpha} \in C^{M,\alpha}$

and is defined as follows:

- for contexts $\Gamma = x_1 : C_1, \ldots, x_n : C_n$, assignments α to M are n-tuples and
$$(\alpha_1, \ldots, \alpha_n) \in \Gamma^M \quad \text{iff} \quad \alpha_i \in C_i^{M,(\alpha_1,\ldots,\alpha_{i-1})}$$
- for the universes
$$\texttt{tp}^{M,\alpha} = \mathcal{SET}$$
$$\texttt{form}^{M,\alpha} = \{0,1\}$$
- for variables x_i declared in the context
$$x_i^{M,\alpha} = \alpha_i$$
- for atomic expressions $E = c(E_1, \ldots, E_n)$
$$E^{M,\alpha} = c^M(E_1^{M,\alpha}, \ldots, E_n^{M,\alpha})$$
- for complex expressions see Def. 2.18.

Definition 2.16 (Satisfaction)**.** As usual, we write $M \models F$ if $F^{M,\alpha} = 1$ for all assignments α to the context of F and say that M **satisfies** F.

Remark 2.17 (Interpretation of Proofs). Proofs of F are interpreted as elements of the interpretation of F. This corresponds to the *formulas as types* interpretation where every formula serves as the type of its proofs.

A formula is interpreted as either 0 or 1, and we assume the usual definitions of $0 := \varnothing$ and $1 := \{0\}$, i.e., the interpretation of F is either empty or a singleton. Consequently, the interpretation of a proof $\vdash_\Theta P : F$ is either impossible or uniquely determined. Thus, well-definedness of the interpretation function subsumes the soundness of the proof calculus: There may only ever be proofs of F if F is satisfied by every model M.

This corresponds to the *proof irrelevance* interpretation: It matters only that a proof can be interpreted, but its interpretation is irrelevant (because it is uniquely determined). In particular, even though a model M must provide an interpretation p^M of every Θ-axiom p, only the existence of p^M is relevant and two models can never differ in the choice of p^M. Therefore, when giving concrete models, we can drop the interpretations p^M from the notation.

Def. 2.15 does not define the interpretation of complex expressions. These can be supplied separately depending on the choice of complex expressions. For all expressions from Def. 2.3, the interpretation is straightforward:

Definition 2.18 (Interpretation Function)**.** We define the interpretation of complex expressions in the usual way

$$\forall x : A.F(x)^{M,\alpha} = \begin{cases} 1 & \text{if } F^{M,\alpha.e} = 1 \text{ for all } e \in A^{M,\alpha} \\ 0 & \text{otherwise} \end{cases}$$

$$S \doteq_A T^{M,\alpha} = \begin{cases} 1 & \text{if } S^{M,\alpha} = T^{M,\alpha} \\ 0 & \text{otherwise} \end{cases}$$

and accordingly for the other complex formulas. The interpretation of all complex proofs is uniquely determined: All proofs are interpreted as the element of $\{0\}$.

2.2.2 The Action of Theory Morphisms on Models

The following definition provides the semantic analogue to Def. 2.9. Just like theory morphisms map expressions in one direction, they map models in the opposite direction:

Definition 2.19 (Model Reduction)**.** Consider a theory morphism $\vartheta : \Theta \to \Theta'$.

We define $\mathbf{Mod}(\vartheta) : \mathbf{Mod}(\Theta') \to \mathbf{Mod}(\Theta)$ as follows. We consider $M' \in \mathbf{Mod}(\Theta')$ and define $M = \mathbf{Mod}(\vartheta)(M') \in \mathbf{Mod}(\Theta)$ by:

For every declaration $c : \{\Gamma\}C$ in Θ,
if $c \mapsto [\vartheta(\Gamma)]\vartheta(E)$ is the corresponding assignment in ϑ,
then $c^M(\alpha) = E^{M',\alpha}$ for every $\alpha \in \Gamma^M$.

The well-definedness of Def. 2.19 is proved together with the main theorem about model reduction, which forms the semantic analogue to Thm. 2.10:

Theorem 2.20 (Denotation Condition)**.** *Given a theory morphism $\vartheta : \Theta \to \Theta'$ and an expression $\Gamma \vdash_\Theta E : C$, we have for every model $M' \in \mathbf{Mod}(\Theta')$ and every assignment α from $\vartheta(\Gamma)$ to M':*

$$\vartheta(\Gamma)^{M'} = \Gamma^{\mathbf{Mod}(\vartheta)(M')}$$

$$\vartheta(E)^{M',\alpha} = E^{\mathbf{Mod}(\vartheta)(M'),\alpha}.$$

Proof. The straightforward proof proceeds by induction on expressions. Proofs of this form are well-known from the study of institutions [GB92] (whose satisfaction condition is a special case of our denotation condition). □

Thus, the reduced model $\mathbf{Mod}(\vartheta)(M')$ is essentially the same mathematical structure as M'. It only differs by being framed as a model of Θ according to ϑ.

Example 2.21 (Continuing Ex. 2.12). A model $N' \in \mathbf{Mod}(\texttt{Order})$ is given by $u^{N'} = \mathbb{N}$, $\texttt{leq}^{N'} = \leq$ (and $p^{N'} = 0$ for every axiom p). Model reduction yields $N = \mathbf{Mod}(\texttt{least})(N')$ with $u^N = \mathbb{N}$ and $\texttt{sp}^N : n \mapsto \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$

Another model $Z' \in \mathbf{Mod}(\texttt{Order})$ is given by $u^{Z'} = \mathbb{Z}$, $\texttt{leq}^{Z'} = \leq$. Model reduction yields $Z = \mathbf{Mod}(\texttt{least})(Z')$ with $u^Z = \mathbb{Z}$ and $\texttt{sp}^Z : n \mapsto 0$.

Thus, **Mod** is a functor from the category of theories and theory morphisms to the category of classes and mappings. Our work is motivated by the observation that functoriality fails if we naively add model morphisms to $\mathbf{Mod}(\Theta)$.

3 Motivating Considerations

3.1 Limitations of Standard Model Morphisms

Before stating our definition of model morphism, let us recall the standard definition for the special case of TFOL in analogy to Rem. 2.13:

Remark 3.1 (Standard Definition of Model Morphisms). For a TFOL-theory Θ, a model morphism $m : M \to M'$ between Θ-models

1. (a) provides a function $a^m : a^M \to a^{M'}$ for each Θ-type symbol a,
   (b) induces (by induction on types A) a function $A^m : A^m \to A^{M'}$ for every type A (which is trivial for TFOL because the type symbols are the only types),
2. (a) must commute with the interpretation of each Θ-function symbol $t : \{x_1 : A_1, \ldots, x_n : A_n\}a$, i.e.,

$$a^m\big(t(x_1, \ldots, x_n)^{M,\alpha}\big) = t(x_1, \ldots, x_n)^{M',m(\alpha)}$$

   (b) is shown (by induction on terms T) to commute with the interpretation of every Θ-term T,

3. (a) must preserve the interpretation of each Θ-predicate symbol $\varphi : \{x_1 : A_1, \ldots, x_n : A_n\}\texttt{form}$, i.e.,

$$a^m\big(\varphi(x_1, \ldots, x_n)^{M,\alpha}\big) \leq \varphi(x_1, \ldots, x_n)^{M',m(\alpha)}$$

   (b) cannot be shown by induction on formulas F to preserve the interpretation of all Θ-formulas,
4. (a) does nothing with Θ-axioms,
   (b) has no relation with Θ-theorems.

To understand the above notations, we have to explain how model morphisms treat assignments in analogy to Rem. 3.1:

Remark 3.2 (Treatment of Assignments). A model morphism m maps every M-assignments α for Γ component-wise to an M'-assignment for Γ. Formally, we define a function $\Gamma^m : \Gamma^M \to \Gamma^{M'}$ as follows: Let $\Gamma = \ldots, x_i : A_i, \ldots$. Then $\Gamma^m(\ldots, \alpha_i, \ldots) = (\ldots, A_i^m(\alpha_i), \ldots)$,

We abbreviate $\Gamma^m(\alpha)$ as $m(\alpha)$ if Γ is clear,

Rem. 3.1 indicates a broken symmetry between models and morphisms: The cases (3b) and (4) do not extend to model morphisms. To understand this better, we discuss the relation between model morphisms and formulas.

The ideal but unrealistic invariants of model morphisms are concisely captured by the following table:

Expression	Interpretation by	
	model M	morphism $m : M \to M'$
Type A	set $A^{M,\alpha}$	function $A^{m,\alpha} : A^{M,\alpha} \to A^{M',\alpha}$
Term T of A	elements $T^{M,\alpha} \in A^{M,\alpha}$	commutativity $T^{M,\alpha} = T^{M',m(\alpha)}$
Formula F	truth value $F^{M,\alpha} \in \{0,1\}$	preservation $F^{M,\alpha} \leq F^{M',m(\alpha)}$
Proof P of F	uniquely determined	commutativity trivial

The upper half of the table works well. Models interpret types as sets and terms as elements of these sets. Model morphisms interpret types as functions between these sets that commute with the interpretation of terms.

Example 3.3 (First-Order Logic). Recall FOL from Ex. 2.5, which has a single type u. Thus, a model morphism m must provide a single function $u^m : u^M \to u^{M'}$. For a function symbol $t : \{x_1 : u, \ldots, x_n : u\}u$, the commutativity conditions becomes the well-known $u^m(t^M(e_1, \ldots, e_n)) = t^{M'}(u^m(e_1), \ldots, u^m(e_n))$.

For example, for $n = 2$ and $t = \circ$, this yields the condition for semigroup homomorphisms: $u^m(e_1 \circ^M e_2) = u^m(e_1) \circ^{M'} u^m(e_2)$.

We might now try to generalize these results to the lower half of the table. Ideally, a model morphism would provide a function $F^{M,\alpha} \to F^{M',m(\alpha)}$ for every formula F. Using $0 = \varnothing$ and $1 = \{0\}$, we see that such a function exists uniquely or not at all. It exists iff the morphism preserves F, i.e., if $F^{M,\alpha} = 1$ implies $F^{M',m(\alpha)} = 1$. We can write this concisely as $F^{M,\alpha} \leq F^{M',m(\alpha)}$. In that case, the commutativity condition for proofs is trivial because the interpretation of proofs is determined uniquely anyway. However, if F is

not preserved, morphisms cannot map proofs at all, and the commutativity condition cannot even be stated anymore.

Example 3.4 (First-Order Logic). For a predicate symbol $\varphi : \{x_1 : u, \dots, x_n : u\}\texttt{form}$, the preservation of atomic formulas becomes the well-known "if $\varphi^M(e_1, \dots, e_n)$ then $\varphi^{M'}(u^m(e_1), \dots, u^m(e_n))$".

For example, for $n = 2$ and $\varphi = \leq$, this yields the condition for monotonous maps between orders: if $e_1 \leq^M e_2$, then $u^m(e_1) \leq^{M'} u^m(e_2)$.

Thus, it is natural to require that all formulas be preserved by model morphisms. However—while elegant and consistent—this is impractical: It is a very strong condition that is not satisfied by many interesting maps between models. For example, the preservation of the formula $\neg x \doteq_u y$ by a model morphism m is already equivalent to the injectivity of u^m. Similarly, universally quantified formulas can usually only be preserved by surjective morphisms. But model morphisms are used extensively to describe submodels and quotient models—which are non-trivial exactly for non-surjective and non-injective morphisms, respectively.

Therefore, the preservation condition on formulas must be relaxed to be practical for mathematical applications. This leads to the standard definition described in Rem. 3.1: require preservation only for the *atomic* formulas. This choice is made, for example, in the CASL standard for first-order logic [CoF04].

Remark 3.5 (Proving Preservation Inductively). Starting with the knowledge that all atomic formulas are preserved, one might try to prove the preservation of all formulas F by induction on F. This is not possible. The induction step succeeds for equality, truth, falsity, disjunction, conjunction, and existential quantification. But it fails for negation, implication, and universal quantification.

Remark 3.6 (Segregating Formulas (following up on Rem. 2.8)). We can now explain why our logic does not treat formulas as a special case of terms: Formulas generally do not commute with model morphisms, i.e., we do not expect $F^{M,\alpha} = F^{M',m(\alpha)}$ to hold in general.

We might require this stronger property in order to allow treating terms and formulas uniformly, but that would be impractically strong. At best, we would have to restrict it to atomic formulas—the resulting model morphisms are called *closed* in [Dia08]. But many interesting model morphisms are not closed. For example, order homomorphisms are the ones that preserve (but do not necessarily commute with) atomic formulas.

The standard choice of requiring the preservation of atomic formulas is often but not necessarily the best choice. Ultimately, it is the reason for the two motivating problems described in Sect. 1: It fixes one set of morphisms when other choices might be interesting as well, and it prevents the functoriality of $\mathbf{Mod}(\vartheta)$ in the general case.

Example 3.7 (Monoid Morphisms (Continuing Ex. 2.11). Applying the standard definition of model morphisms to `MonoidCon` yields the usual monoid

morphisms. But because the standard definition ignores axioms entirely, the theory `MonoidAx` induces the same model morphisms as the theory `SemiGroup`.

Thus, **Mod**(`MonoidAx`) also allows semigroup morphisms between monoids that are not monoid morphisms. An example is the inclusion from $(\mathbb{N} \setminus \{0\}, \max)$ to $(\mathbb{N}, \max)$ (where max returns the greater one of two numbers): Both are monoids, and the uniquely determined units are 1 and 0, respectively. The inclusion commutes with the binary operation but not with the unit. Thus, it is a **Mod**(`MonoidAx`)-morphism but not a **Mod**(`MonoidCon`)-morphism.

Therefore, the categories **Mod**(`MonoidAx`) and **Mod**(`MonoidCon`) are not isomorphic, and **Mod**(`axCon`) is not an isomorphic functor between them.

If we want to use `MonoidAx` instead of `MonoidCon`, we have no way to fine-tune the `MonoidAx`-model morphisms to exclude such morphisms. Or rather, the only way to fine-tune the morphisms is to add a constant for the unit, in which case we end up with the theory `MonoidCon`.

Example 3.8 (Continuing Ex. 2.21). We can give a model morphism $i' : N' \to Z'$ by the inclusion mapping $u^{i'} : n \mapsto n$. It preserves the ordering and thus conforms to the standard definition: If $\mathtt{leq}(x,y)^{N',(m,n)} = 1$ then $\mathtt{leq}(x,y)^{Z',i(m,n)} = 1$.

The standard definition of **Mod**(ϑ) reduces i' to i with $u^i = u^{i'}$. i does not preserve the predicate `sp` and thus is not a standard model morphism: $\mathtt{sp}(x)^{N,(0)} = 1$ but $\mathtt{sp}(x)^{Z,i(0)} = 0$.

Thus, **Mod**(`least`) is not a functor if we use the standard definition of **Mod**($-$).

3.2 Interesting Properties of Model Morphisms

To understand better what kind of properties we might require of model morphisms, we discuss some properties they may or may not have.

Consider two models M and M' and a model morphism $m : M \to M'$. For the purposes of this section, it does not matter how m is defined in detail. All we need is that m induces the translation function $m(-)$ that maps M-assignment to M'-assignments.

Definition 3.9 (Preserving Formulas)**.** We say that m **preserves** $\Gamma \vdash_\Theta F : \mathtt{form}$ if $F^{M,\alpha} \leq F^{M',m(\alpha)}$ for all $\alpha \in \Gamma^M$.

Definition 3.10 (Reflecting Formulas)**.** We say that m **reflects** F if $A^{M,\alpha} \geq A^{M',m(\alpha)}$ for all $\alpha \in \Gamma^M$.

Definition 3.11 (Commuting with Expressions)**.** We say that m **commutes** with an expression $\Gamma \vdash_\Theta E : C$ if $E^{M,\alpha} = E^{M',m(\alpha)}$ for all $\alpha \in \Gamma^M$.

We have the following relations betweens these notions:

Theorem 3.12 (Terms)**.** *Consider a term T of type A. A model morphism commutes with T iff it preserves $y \doteq_A T$ for a fresh variable $y : A$.*

Proof. This is shown by a straightforward unfolding of the definitions. □

Theorem 3.13 (Formulas)**.** *Consider a formula F. A model morphism*

- *reflects F iff it preserves $\neg F$,*
- *preserves F iff it reflects $\neg F$,*
- *commutes with F iff it commutes with $\neg F$,*
- *commutes with F iff it preserves and reflects F.*

Proof. All proofs are straightforward. □

Theorem 3.14 (Proofs)**.** *Consider a proof P of F. Every model morphism commutes with P.*

Proof. This holds trivially because all models interpret proofs in the same way. □

Remark 3.15 (Types). Consider a type A. Interesting model morphisms usually do not commute with A. This is because the whole point of a model morphism is to relate two different models, i.e., two models for which A^M and $A^{M'}$ are different.

In some situations, it may be useful to speak about whether a morphism commutes with a type, but we will not consider this further here.

Considering Thm. 3.13, Thm. 3.14, and Rem. 3.15, we see that the commutation property is most interesting for terms and can be ignored for types, formulas, and proofs. Moreover, considering Thm. 3.12 and Thm. 3.13, we see that many interesting properties can be expressed in terms of preservation alone.

Example 3.16 (Equality and Injectivity). All model morphisms m preserve the formula $x \doteq_a y$ in context $x : a, y : a$. This expresses the fact that two equal values cannot be teased apart by a model morphism.

Conversely, m preserves $\neg x \doteq_a y$ iff a^m is injective.

Example 3.17 (Preserving All Formulas and Surjectivity). Model isomorphisms preserve and reflect all formulas.

More generally, elementary equivalences are defined as those model morphisms that preserve all formulas. For example the inclusion model morphism $i : (\mathbb{Q}, <) \hookrightarrow (\mathbb{R}, <)$ preserves all formulas over the theory of dense linear orders without end points.

This example also shows that we cannot specify the surjectivity of model morphisms in terms of preservation: i preserves all formulas but is not surjective. This is reminiscent of the result that we cannot specify the class of all term-generated models using any set of first-order formulas.

Example 3.18 (Continuing Ex. 3.7). We can now state the problem of Ex. 2.11 concisely. We want to specify the category containing only those `MonoidAx`-model morphisms that preserve the formula $\texttt{isunit}(x)$ in context $x : u$.

Example 3.19 (Continuing Ex. 3.8). We can now state the problem of Ex. 3.8 concisely.

The model morphism i' preserves the formula $x : u, y : u \vdash_{\texttt{Order}} \texttt{leq}(x, y) : \texttt{form}$ but not the formula $x : u \vdash_{\texttt{Order}} \forall y : u.\, \texttt{leq}(x, y) : \texttt{form}$.

Thus, the theory morphism `least` maps `sp` to a non-preserved formula. Therefore, i does not preserve $x : u \vdash_{\texttt{Special}} \texttt{sp}(x) : \texttt{form}$.

Ex. 3.18 and 3.19 indicate our solution. Just like theories may declare axioms that specify the desired properties of models, theories should be allowed to make declarations that specify the desired properties of model morphisms. We will develop this solution in Sect. 4.

Remark 3.20 (Intuitionistic Logic). Thm. 3.13 holds because our models are always classical, i.e., bivalent. But we could easily generalize our definition of model to intuitionistic logic, e.g., by using an arbitrary Heyting-Algebra $\texttt{form}^M$ instead of the fixed boolean algebra $\{0, 1\}$.

If both models use the same Heyting algebra, Def. 3.9 and 3.10 make sense without changes. In that case, Thm. 3.13 has to be weakened as follows: The right-to-left implications of the items involving negation do not hold. (The left-to-right implications still hold.)

We can also give a definition of model morphism between models based on different Heyting algebras. Then the results above can be generalized accordingly.

4 A Logic With Model Morphisms

In this section, we extend the basic logic defined in Sect. 2.1. Our main goal is to define model morphisms in a way that introduces flexibility while guaranteeing that every theory morphism induces a model reduction functor.

Our central idea is to add a new kind of declaration in theories: *maxiom* declarations postulate a property for all model morphisms in the same way in which axiom declarations postulate a property for all models. Just like formulas are properties of models asserted by axioms, we introduce *mormulas* as properties of model morphisms that are asserted by maxioms. Just like theorems are formulas that are implied by the axioms, we introduce *meorems* as the mormulas implied by the maxioms. Finally, just like proofs are the expressions that establish theorems, we introduce *moofs* as the expressions that establish meorems.

The following tables summarize the new concepts for model morphisms and their analogy to the existing concepts for models:

Declared symbols	Expressions	Judgment
-	mormulas V	$\Gamma \vdash_\Theta V : \texttt{morm}$
maxioms $q : V$	moofs Q of mormula V	$\Gamma \vdash_\Theta Q : V$

	Models	Model morphisms
property	formula F	mormula V
postulated property	axiom $p : F$	maxiom $q : V$
established property	theorem F	meorem V
establishing evidence	proof $\vdash_\Theta P : F$	moof $\vdash_\Theta Q : V$

The empty cell in the first table raises the question whether there should also be declarations of mormula constructors. These would be analogous to

predicate symbols. Our logic could easily accommodate such declarations. But we omit them here because we have so far not found a use for such declarations.

Rem. 3.1 pointed out a lack of symmetry between the definitions of models and model morphisms. Our logic will yield the following symmetry between models and model morphisms:

- Regarding types
  - A model interprets every type as a set.
  - A model morphism interprets every type as a map.
- Regarding terms
  - A model interprets every term as element of the set interpreting its type.
  - A model morphism may or may not commute with a term.
- Regarding formulas and mormulas
  - A model interprets every formula as a truth value.
  - A model morphism interprets every mormula as a truth value.
- Regarding proofs and moofs
  - A model must satisfy all axioms; proofs show that it also satisfies all theorems.
  - A model morphism must satisfy all maxioms; moofs show that it also satisfies all meorems.

4.1 Syntax: Mormulas and Maxioms

To allow maxiom declarations in theories, we amend Def. 2.2 as follows:

Definition 4.1 (Mormulas, Moofs, Maxioms)**.** In addition to `tp` and `form`, we have the universe `morm` of mormulas. Expressions $\Gamma \vdash_\Theta V : \texttt{morm}$ are called **mormulas**.

Expressions $\Gamma \vdash_\Theta Q : V$ are called **moofs** of Q.

In addition to the declarations of Def. 2.2, theories may contain declarations of the form

- **maxioms** $q : \{\Gamma\}V$ for $\Gamma \vdash_\Theta V : \texttt{morm}$

The key design choice to make now is what mormulas to allow. Inspired by Sect. 3.2, we choose the following:

Definition 4.2 (Complex Mormulas)**.** Mormulas are formed according to the following grammar

$$\begin{array}{rcll} V & ::= & x \mid q(E_1, \dots, E_n) & \text{atomic mormulas} \\ & \mid & \mathfrak{I}_\Gamma T & \text{commutation with } T \\ & \mid & \mathfrak{q}_\Gamma F & \text{preservation of } F \\ & \mid & \mathfrak{R}_\Gamma F & \text{reflection of } F \end{array}$$

Their typing rules are

$$\frac{\Delta, \Gamma \vdash_\Theta T : A}{\Delta \vdash_\Theta \mathfrak{I}_\Gamma T : \texttt{morm}} \qquad \frac{\Delta, \Gamma \vdash_\Theta F : \texttt{form}}{\Delta \vdash_\Theta \#_\Gamma F : \texttt{morm}} \text{ for } \# \in \{\mathfrak{q}, \mathfrak{R}\}$$

where each Γ may declare only term variables.

All mormulas have a similar syntax: They bind term variables in a formula. Thus, they behave similarly to $\forall$ and $\exists$, which inspired their notations as mirrored upper case letters.

However, contrary to formulas, mormulas are not nested. For example, we can only write $\mathfrak{I}_{x:a,y:b}T$ but not $\mathfrak{I}_{x:a}\mathfrak{I}_{y:b}T$. This is important because we will define the interpretation of $\mathfrak{I}_{x:a,y:b}T$ in one step rather than in two compositional steps.

We can immediately recover the standard definition of model morphisms in the sense of Rem. 3.1:

Definition 4.3 (Standard Maxioms)**.** For a theory Θ, the standard maxioms are:

- for each function symbol $t : \{\Gamma\}A$ in Θ:
$$t^{\text{std}} : \mathfrak{I}_\Gamma t(x_1, \dots, x_n)$$
- for each predicate symbol $\varphi : \{\Gamma\}\texttt{form}$ in Θ:
$$\varphi^{\text{std}} : \mathfrak{q}_\Gamma \varphi(x_1, \dots, x_n)$$

where in each case $x_1, \dots, x_n$ are the variables declared in Γ.

We do not require that the standard maxioms are present in every theory Θ. But it is good to have a name for them because they are present very often.

Example 4.4 (Continuing Ex. 3.18). We can now solve the problem of Ex. 3.7 concisely. The theory `MonoidCon` can use the standard maxioms. These are:

$$\circ^{\text{std}} : \mathfrak{I}_{x:u,y:u} x \circ y, \quad e^{\text{std}} : \mathfrak{I} e$$

But in the theory `MonoidAx`, the standard maxiom $\circ^{\text{std}}$ is not enough. We need the additional maxiom

$$\texttt{unit\_commute} : \mathfrak{q}_{x:u}\, \texttt{isunit}(x)$$

With these maxioms, both `MonoidCon` and `MonoidAx` will yield isomorphic model categories.

Remark 4.5 (Expressivity of the Standard Maxioms). One might argue that the standard maxioms are already sufficient because all maxioms can be seen as a special case of the standard maxiom for a fresh symbol. For example, we can replace the maxiom $q : \mathfrak{q}_\Gamma F$ where $\Gamma = \dots, x_i : A_i, \dots$ with the following declarations:

- a new predicate symbol $\varphi : \{\Gamma\}\texttt{form}$
- an axiom $p : \forall x_1 : A_1 \dots \forall x_n : A_n . \varphi(x_1, \dots, x_n) \Leftrightarrow F$

Now φ^{std} has the same effect as the maxiom q.

Our solution goes beyond this lightweight approach in several ways:

- It allows using fewer than the standard maxioms, e.g., when
  - a standard maxiom is not satisfied by the intended model morphisms as in Ex. 5.1,

  - a standard maxiom is redundant because it is a meorem, i.e., implied by the other maxioms, as in Ex. 4.6,
  - a theory uses auxiliary symbols that should be ignored by model morphisms, e.g., the base of a vector space or the size of a finite group.[6]
- It allows us to reason about meoremhood in theory morphisms as we will see in Sect. 4.2. This is the key step to show that $\mathbf{Mod}(\vartheta)$ is always functorial.

Example 4.6 (Continuing Ex. 4.4). To extend the theory `MonoidCon` to the theory `Group`, we have to add the symbols

$$\mathtt{i} : \{x : u\}u, \quad \mathtt{inverse} : \forall x : u.\, x \circ \mathtt{i}(x) \doteq e \wedge \mathtt{i}(x) \circ x \doteq e$$

We do not have to add a maxiom—it is well known that the category of groups is a full subcategory of the theory of monoids, i.e., every monoid morphism between groups is already a group morphism. In our terminology, that means that the mormula $\Im_{x:u}\, \mathtt{i}(x)$ is a meorem of the theory `Group`. Indeed, using the moof calculus from Sect. 4.2, we give a moof Q such that $\vdash_{\mathtt{Group}} Q : \Im_{x:u}\, \mathtt{i}(x)$ in Ex. 4.11.

In fact, groups are even a full subcategory of semigroups. Therefore, `Group` does not need the maxiom e^{std} either—that is also a meorem. However, monoids are not a full subcategory of semigroups. Therefore, the theory `MonoidCon` does need the maxiom e^{std}.

Example 4.7 (Elementary Equivalences). We can specify elementary equivalences by using the maxiom $\mathtt{ee} : \{x : \mathtt{form}\}\mathfrak{q}x$, which asserts the preservation of all formulas.

Remark 4.8 (Other Choices of Mormulas). In light of Sect. 3.2, we see that $\Im$ and $\mathfrak{R}$ are definable in terms of $\mathfrak{q}$. Thus, one might argue that they are redundant. We prefer to include them for several reasons.

Firstly, it follows the spirit of first-order logic to include important but redundant connectives. Secondly, the definition of $\Im$ in terms of $\mathfrak{q}$ would be awkward in practice, and using the dual connectives $\mathfrak{q}$ and $\mathfrak{R}$ together makes the moof calculus somewhat more elegant.

Finally, our choice is not meant to be final. Instead, it serves as a starting point that exemplifies the approach and indicates the extensibility of the language. There are a several candidate connectives that could be added to $\Im$, $\mathfrak{q}$, and $\mathfrak{R}$. A particularly interesting addition is $\mathtt{surj}\, A$, which would express the surjectivity of A^m.

Moreover, it is straightforward to add propositional connectives for mormulas. For example, if we add, e.g., conjunction $V \wedge V'$, then $(\mathfrak{q}_\Gamma F) \wedge (\mathfrak{q}_\Gamma F')$ is not definable in terms of $\mathfrak{q}$: In particular, it is stronger than $\mathfrak{q}_\Gamma(F \wedge F')$.

[6] These examples are taken from the category library of SageMath [S+13], which specifies model categories common in mathematics.

4.2 Proof Theory: Moofs and Syntactic Meorems

4.2.1 Theories

The deduction rules for moofs are given in Fig. 2. They can be verbalized as follows:

- `commvar`: Variables commute.
- `commsubs`: Substituting commuting terms into commuting terms yields commuting terms.
- `commpres`: Terms commute if their identity is preserved.
- `subs`: Substituting commuting terms into preserved (reflected) formulas, yields preserved (reflected) formulas.
- `equiv`: If F is preserved (reflected), so is every equivalent formula.
- `true` and `false`: `true` and `false` are preserved (reflected) by all morphisms.
- `conj` and `disj`: Conjunctions and disjunctions are preserved (reflected) if their arguments are.
- `neg`: $\neg F$ is preserved if F is reflected, and vice versa for reflection.
- `impl`: $F \Rightarrow G$ is preserved if F is reflected and G preserved, and vice versa for reflection.
- `equal`: Equalities between commuting terms are preserved.
- `exists`: Existential quantifications are preserved if their argument is.
- `forall`: Universal quantifications are reflected if their argument is.

Definition 4.9 (Meorems)**.** A mormula V is called a **syntactic**[7] **meorem** if there is a moof Q such that $\vdash_\Theta Q : V$, i.e., if $\vdash_\Theta V$.

Example 4.10 (Theorems and Contradictions are Meorems). Theorems and contradictions F have the same truth values in all models. Therefore, they are trivially preserved by all model morphisms and thus $\mathfrak{q}F$ and $\mathfrak{R}F$ are meorems.

Our calculus derives them using the rules for `equiv`, `true`, and `false`. This is possible because `true` $\Leftrightarrow F$ is a theorem if F is, and `false` $\Leftrightarrow F$ is a theorem if F is a contradiction.

Example 4.11 (Continuing Ex. 4.6). We show that $\mathfrak{O}e$ is a meorem of the theory `Group` using $\circ^{\text{std}}$ as the only maxiom:

$$\dfrac{\dfrac{\dfrac{}{\vdash_{\texttt{Group}} \mathfrak{O}_{y:u} y \circ y}\,\circ^{\text{std}} \qquad \dfrac{}{\vdash_{\texttt{Group}} \mathfrak{O}_{y:u} y}\,\texttt{commvar}}{\vdash_{\texttt{Group}} \mathfrak{q}_{y:u} y \circ y \doteq_u y}\,\texttt{equal} \qquad P}{\vdash_{\texttt{Group}} \mathfrak{q}_{y:u} y \doteq_u e}\,\texttt{equiv}}{\vdash_{\texttt{Group}} \mathfrak{O}e}\,\texttt{commpres}$$

where $y : u \vdash_{\texttt{Group}} P : y \circ y \doteq_u y \Leftrightarrow y \doteq_u e$ is a straightforward proof.

The moof of $\vdash_{\texttt{Group}} \mathfrak{O}_{x:u}\, \mathtt{i}(x)$ proceeds in essentially the same way. The key step is to apply `equiv` with the equivalence $y \doteq_u \mathtt{i}(x) \Leftrightarrow x \circ y \doteq_u e$ and then use $\circ^{\text{std}}$ and $\mathfrak{O}e$.

[7] We use the qualifier *syntactic* to distinguish from *semantic* meorems, which are defined model theoretically in Sect. 4.3. We discuss the soundness/completeness relation between the two notions in Sect. 4.4.

Rules for commutation:

$$\frac{x : A \text{ in } \Gamma}{\triangle \vdash_\Theta \mathcal{O}_\Gamma x}\texttt{commvar}$$

$$\frac{\triangle \vdash_\Theta \mathcal{O}_{x_1:A_1,\ldots,x_n:A_n} T \quad \triangle \vdash_\Theta \mathcal{O}_\Gamma T_1 \quad \ldots \quad \triangle \vdash_\Theta \mathcal{O}_\Gamma T_n}{\triangle \vdash_\Theta \mathcal{O}_\Gamma T[\ldots, x_i/T_i, \ldots]}\texttt{commsubs}$$

$$\frac{\Gamma \vdash_\Theta T : A \quad \triangle \vdash_\Theta \mathcal{Q}_{\Gamma,y:A} y \doteq_A T}{\triangle \vdash_\Theta \mathcal{O}_\Gamma T}\texttt{commpres}$$

Common rules for preservation/reflection for $\# \in \{\mathcal{Q}, \mathcal{R}\}$:

$$\frac{\triangle \vdash_\Theta \#_{x_1:A_1,\ldots,x_n:A_n} F \quad \triangle \vdash_\Theta \mathcal{O}_\Gamma T_i \quad \ldots \quad \triangle \vdash_\Theta \mathcal{O}_\Gamma T_n}{\triangle \vdash_\Theta \#_\Gamma F[\ldots, x_i/T_i, \ldots]}\texttt{subs}$$

$$\frac{\triangle \vdash_\Theta \#_\Gamma F \quad \Gamma \vdash_\Theta F \Leftrightarrow G}{\triangle \vdash_\Theta \#_\Gamma G}\texttt{equiv} \qquad \frac{}{\triangle \vdash_\Theta \#_\Gamma \texttt{true}}\texttt{true} \qquad \frac{}{\triangle \vdash_\Theta \#_\Gamma \texttt{false}}\texttt{false}$$

$$\frac{\triangle \vdash_\Theta \#_\Gamma F \quad \triangle \vdash_\Theta \#_\Gamma G}{\triangle \vdash_\Theta \#_\Gamma F \wedge G}\texttt{conj} \qquad \frac{\triangle \vdash_\Theta \#_\Gamma F \quad \triangle \vdash_\Theta \#_\Gamma G}{\triangle \vdash_\Theta \#_\Gamma F \vee G}\texttt{disj}$$

Rules specific to preservation:

$$\frac{\triangle \vdash_\Theta \mathcal{R}_\Gamma F}{\triangle \vdash_\Theta \mathcal{Q}_\Gamma \neg F}\texttt{neg} \qquad \frac{\triangle \vdash_\Theta \mathcal{R}_\Gamma F \quad \triangle \vdash_\Theta \mathcal{Q}_\Gamma G}{\triangle \vdash_\Theta \mathcal{Q}_\Gamma F \Rightarrow G}\texttt{impl}$$

$$\frac{\triangle \vdash_\Theta \mathcal{Q}_{\Gamma,x:A} F}{\triangle \vdash_\Theta \mathcal{Q}_\Gamma \exists x : A.F}\texttt{exists} \qquad \frac{\triangle \vdash_\Theta \mathcal{O}_\Gamma T \quad \triangle \vdash_\Theta \mathcal{O}_\Gamma T'}{\triangle \vdash_\Theta \mathcal{Q}_\Gamma T \doteq_A T'}\texttt{equal}$$

Rules specific to reflection:

$$\frac{\triangle \vdash_\Theta \mathcal{Q}_\Gamma F}{\triangle \vdash_\Theta \mathcal{R}_\Gamma \neg F}\texttt{neg} \qquad \frac{\triangle \vdash_\Theta \mathcal{Q}_\Gamma F \quad \triangle \vdash_\Theta \mathcal{R}_\Gamma G}{\triangle \vdash_\Theta \mathcal{R}_\Gamma F \Rightarrow G}\texttt{impl}$$

$$\frac{\triangle \vdash_\Theta \mathcal{R}_{\Gamma,x:A} F}{\triangle \vdash_\Theta \mathcal{R}_\Gamma \forall x : A.F}\texttt{forall}$$

Figure 2: Moof Calculus

4.2.2 Theory Morphisms

Def. 2.9 for theory morphisms can be applied without change to theories that contain maxioms. In particular, a theory morphism $\vartheta : \Theta \to \Theta'$ must map every maxiom to a moof of the corresponding mormula. Concretely,

- the homomorphic extension maps $\vartheta(\mathtt{morm}) = \mathtt{morm}$ and $\vartheta(\#_\Gamma F) = \#_{\vartheta(\Gamma)}\vartheta(F)$ for $\# \in \{\mathfrak{O}, \mathfrak{q}, \mathfrak{R}\}$,
- ϑ must provide for every maxiom $q : \{\Gamma\}V$ in Θ, an assignment $q \mapsto [\vartheta(\Gamma)]Q$ such that $\vartheta(\Gamma) \vdash_{\Theta'} Q : \vartheta(V)$.

Similarly, Thm. 2.10 holds for maxioms without change:

Theorem 4.12 (Meorem Preservation)**.** *Let $\vartheta : \Theta \to \Theta'$ be a theory morphism. If V is a syntactic Θ-meorem, then $\vartheta(V)$ is a syntactic Θ'-meorem.*

Proof. As for Thm. 2.10. □

Example 4.13 (Continuing Ex. 4.4). We can now extend the theory morphism $\mathtt{axCon} : \mathtt{MonoidAx} \to \mathtt{MonoidCon}$ from Ex. 2.11.

Both theories contain the standard maxiom $\circ^{\mathrm{std}}$, and we can simply use the assignment

$$\circ^{\mathrm{std}} \mapsto \circ^{\mathrm{std}}$$

For the additional maxiom of `MonoidAx` we use

$$\mathtt{unit\_commute} \mapsto Q$$

where Q is the moof given by the following derivation:

$$\dfrac{\dfrac{}{\vdash_{\mathtt{MonoidCon}} \mathfrak{O}_{x:u} x}\,\mathtt{commvar} \qquad \dfrac{\dfrac{}{\vdash_{\mathtt{MonoidCon}} \mathfrak{O} e}\, e^{\mathrm{std}}}{\vdash_{\mathtt{MonoidCon}} \mathfrak{O}_{x:u} e}\,\mathtt{commsubs}}{\vdash_{\mathtt{MonoidCon}} \mathfrak{q}_{x:u} x \doteq_u e}\,\mathtt{equal} \qquad P}{\vdash_{\mathtt{MonoidCon}} \mathfrak{q}_{x:u}\, \mathtt{isunit}(x)}\,\mathtt{equiv}$$

for some proof $x : u \vdash_{\mathtt{MonoidCon}} P : x \doteq_u e \Leftrightarrow \mathtt{isunit}(x)$.

With the same caveat about a description operator as in Ex. 2.11, `axCon` is now essentially an isomorphism.

Example 4.14 (Continuing Ex. 3.19). We can now solve the problem of Ex. 3.8 concisely.

We can choose not to add any maxioms to the theory `Special`. Then we obtain the category of sets with special elements whose morphisms do not have to preserve specialty. `least` is a theory morphism.

Alternatively, we can add the standard maxiom $\mathtt{sp}^{\mathrm{std}}$, thus obtaining a different theory $\mathtt{Special}'$. This theory yields the category whose morphisms must preserve specialty. Consequently, we cannot extend `least` to a theory morphism out of $\mathtt{Special}'$ because there is no fitting moof in the theory `Order`.

4.3 Model Theory: Model Categories and Semantic Meorems

So far all definitions were generic, covering all variants of first-order logic allowed by our framework of Sect. 2. To state our main results, we now restrict attention to PFOL as defined in Ex. 2.5. The definition specializes to all fragments of PFOL, in particular FOL and TFOL.

We can generalize our results to more complex logics, but the definitions become increasingly complex. In fact, already the case for PFOL is so complex that it is instructive to consider the special case of TFOL first.

4.3.1 Model Categories for TFOL

We avoid the dilemmas described in Sect. 3.1 by using a weak definition of model morphism that only requires the maps a^m and imposes no commutativity or preservation conditions. Any properties of model morphisms are then imposed later by declaring maxioms. This is analogous to the standard definition of models, which uses a weak definition of model first and then allows imposing conditions on models by declaring axioms.

Definition 4.15 (Model Morphisms)**.** Consider a fixed TFOL-theory Θ and two models $M, M' \in \mathbf{Mod}(\Theta)$.

A **model morphism** $m : M \to M'$ is a tuple $(\ldots, a^m, \ldots)$ containing for every Θ-type symbol $a : \mathtt{tp}$ a function

$$a^m : a^M \to a^{M'}$$

such that for every Θ-maxiom $q : \{\Gamma\}V$, we have that $m \models V(E_1, \ldots, E_n)$ for every mormula $\vdash_\Theta V(E_1, \ldots, E_n) : \mathtt{morm}$ that arises by substituting for the free variables of V.

$m \models V$ will be defined in Def. 4.16 and Def. 4.17.

Every model morphism $m : M \to M'$ induces an interpretation function Γ^m. For contexts Γ, we have that $\Gamma^m : \Gamma^M \to \Gamma^{M'}$ translates M-assignments to M'-assignments. We will abbreviate $\Gamma^m(\alpha)$ it by $m(\alpha)$ if Γ is clear.

Similarly to formulas, a mormula $\vdash_\Theta V : \mathtt{morm}$ is interpreted as its truth value $V^m \in \{0, 1\}$. Accordingly, the interpretation of a moof is uniquely determined.

Definition 4.16 (Interpretation Function)**.** The **interpretation function** of a model morphism $m : M \to M'$ is defined as follows:

- for contexts $\Gamma = \ldots, x_i : a_i, \ldots$:

$$\Gamma^m : \Gamma^M \ni (\ldots, \alpha_i, \ldots) \mapsto (\ldots, a_i^m(\alpha_i), \ldots) \in \Gamma^{M'}$$

- for the universe

$$\mathtt{morm}^m = \{0, 1\}$$

- for complex mormulas:

$$(\mathfrak{O}_\Gamma T)^m = \begin{cases} 1 & \text{if } T^{M,\alpha} = T^{M',m(\alpha)} \text{ for all } \alpha \in \Gamma^M \\ 0 & \text{otherwise} \end{cases}$$

$$(\mathfrak{q}_\Gamma F)^m = \begin{cases} 1 & \text{if } F^{M,\alpha} \leq F^{M',m(\alpha)} \text{ for all } \alpha \in \Gamma^M \\ 0 & \text{otherwise} \end{cases}$$

$$(\mathfrak{R}_\Gamma F)^m = \begin{cases} 1 & \text{if } F^{M,\alpha} \geq F^{M',m(\alpha)} \text{ for all } \alpha \in \Gamma^M \\ 0 & \text{otherwise} \end{cases}$$

- for complex moofs: Q^m is uniquely determined

Definition 4.17 (Satisfaction)**.** we write $m \models V$ if $V^m = 1$.

It is straightforward to package models and model morphisms into model categories:

Definition 4.18 (Model Categories)**.** For every theory Θ, we define $\mathbf{Mod}(\Theta)$ to be the category of

- models as in Def. 2.15,
- model morphisms as in Def. 4.15,
- identity and composition of model morphisms in the obvious way.

We can now state the semantic analogue to Def. 4.9. Just like a (semantic) theorem is a formula that is satisfied by all models, a semantic meorem is a mormula that is satisfied by all model morphisms:

Definition 4.19 (Meorems)**.** A Θ-mormula V is called a **semantic meorem** if $m \models V$ for all Θ-model morphism m.

4.3.2 Model Categories for PFOL

To generalize Def. 4.15 to polymorphic theories, we need an auxiliary definitions first:

Definition 4.20. Let $\mathcal{FUN}$ be the class of triples $(A \in \mathcal{SET}, B \in \mathcal{SET}, f : A \to B)$. For a tuple $(k_1, \ldots, k_n) \in \mathcal{FUN}^n$ and $i = 1, 2$ we write $k^{(i)} \in \mathcal{SET}^n$ for the tuple $(k_{1i}, \ldots, k_{ni})$.

The intuition behind $\mathcal{FUN}$ is that for every type A, the triple $(A^M, A^{M'}, A^m) \in \mathcal{FUN}$ packages the interpretations of A by the models and by the model morphism $m : M \to M'$. Because PFOL-types may contain type variables, we have to carry an assignment α when interpreting them. Therefore, we will actually work with triples $(A^{M,\alpha}, A^{M',m(\alpha)}, A^{m,\alpha}) \in \mathcal{FUN}$.

Definition 4.21 (Model Morphisms)**.** A PFOL-**model morphism** $m : M \to M'$ is defined as in Def. 4.15 except that the tuple $(\ldots, a^m, \ldots)$ contains for every Θ-type symbol $a : \{x_1 : \mathtt{tp}, \ldots, x_n : \mathtt{tp}\}\mathtt{tp}$ a dependent function

$$a^m : (k \in \mathcal{FUN}^n) \to a^M(k^{(1)}) \to a^{M'}(k^{(2)}).$$

Note that this definition contains Def. 4.15 as the special case where $n = 0$. Similarly, the interpretation of mormulas from Def. 4.16 remains essentially unchanged. We only have to extend the translation of assignments because PFOL-contexts may contain type variables:

Definition 4.22 (Assignment Translation)**.** A PFOL-model morphism $m : M \to M'$ translates assignments for a context $\Gamma = \ldots, x_i : C_i, \ldots$ as follows:

$$\Gamma^m : \Gamma^M \ni (\ldots, \alpha_i, \ldots) \mapsto (\ldots, C_i^{m,(\alpha_1,\ldots,\alpha_{i-1})}(\alpha_i), \ldots) \in \Gamma^{M'}$$

There are two cases for the C_i:

- If $C = \mathtt{tp}$, we need a function from $\mathtt{tp}^{M,\alpha} = \mathcal{SET}$ to $\mathtt{tp}^{M',m(\alpha)} = \mathcal{SET}$ and define $\mathtt{tp}^{m,\alpha} = id_{\mathcal{SET}}$. Thus m maps type assignments to themselves.
- If C is a type A, we define $A^{m,\alpha} : A^{M,\alpha} \to A^{M',m(\alpha)}$ by

$$x^{m,\alpha} = id_{\alpha(x)}$$

and

$$a(\ldots, A_i, \ldots)^{m,\alpha} : a^M(\ldots, A_i^{M,\alpha}, \ldots) \to a^{M'}(\ldots, A_i^{M',m(\alpha)}, \ldots)$$

$$a(\ldots, A_i, \ldots)^{m,\alpha} = a^m(\ldots, (A_i^{M,\alpha}, A_i^{M',m(\alpha)}, A_i^{m,\alpha}), \ldots)$$

With the assignment translation in place, all remaining definitions remain unchanged. In general, the assignment translation is the key part of the definition that has to be generalized when considering more general logics.

Example 4.23 (Lists). For a theory of lists, we use the declarations

$$\mathtt{list} : \{X : \mathtt{tp}\}\mathtt{tp}$$

$$\mathtt{nil} : \{X : \mathtt{tp}\}\,\mathtt{list}(X), \quad \mathtt{cons} : \{X : \mathtt{tp}, x : X, l : \mathtt{list}(X)\}\,\mathtt{list}(X)$$

along with the usual axioms for inductive types as well as the standard maxioms

$$\mathtt{nil}^{\mathrm{std}} : \{X : \mathtt{tp}\}\mathfrak{I}\,\mathtt{nil}(X), \quad \mathtt{cons}^{\mathrm{std}} : \{X : \mathtt{tp}\}\mathfrak{I}_{x:X,l:\mathtt{list}(X)}\,\mathtt{cons}(X, x, l)$$

Let M and M' be two models of this theory using two different interpretations of lists:

$$\mathtt{list}^M(S) = \{(n, l) | n \in \mathbb{N}, l : \{1, \ldots, n\} \to S\}$$

$$\mathtt{list}^{M'}(S) = \bigcup_{n \in \mathbb{N}} S^n$$

along with the obvious values for $\mathtt{nil}^M$, $\mathtt{nil}^{M'}$, $\mathtt{cons}^M$, and $\mathtt{cons}^{M'}$.

To give a model morphism $m : M \to M'$, we define $\mathtt{list}^m$ by

$$\mathtt{list}^m : ((S, S', f) \in \mathcal{FUN}) \to \mathtt{list}^M(S) \to \mathtt{list}^{M'}(S')$$

$$\mathtt{list}^m((S, S', f)) : (n, l) \mapsto (f(l(1)), \ldots, f(l(n)))$$

In fact, because the axioms make `list` an inductive type, the maxioms (which guarantee that morphisms commute with all constructors) already uniquely determine $\mathtt{list}^m$.

4.3.3 Model Reduction Functors

We can now establish our main result: For every PFOL-theory morphism ϑ, we obtain a model reduction functor $\mathbf{Mod}(\vartheta)$, and $\mathbf{Mod}$ is itself functorial.

Definition 4.24 (Model Reduction)**.** For every PFOL-theory morphism $\vartheta : \Theta \to \Theta'$, we define $\mathbf{Mod}(\vartheta) : \mathbf{Mod}(\Theta') \to \mathbf{Mod}(\Theta)$ to be the functor that maps

- models as in Def. 2.19,
- model morphisms $m' : M' \to N'$ to model morphisms $m : \mathbf{Mod}(\vartheta)(M') \to \mathbf{Mod}(\vartheta)(N')$ such that $a^m = \vartheta(a)^{m'}$ for all type symbols a declared in Θ.

Theorem 4.25 (Functorial Model Reduction)**.** $\mathbf{Mod}(\vartheta)$ *is a functor for every theory morphism* ϑ*.*

Proof. We only have to show that $\mathbf{Mod}(\vartheta)(m')$ from Def. 4.24 is indeed a model morphism and preserves all Θ-maxioms.

The former follows immediately from the definition. The latter follows from the fact that ϑ must contain an assignment for every Θ-maxiom and the soundness of the moof calculus (which we show in Thm. 4.29 below). □

Theorem 4.26 (Functorial Model Reduction)**.** $\mathbf{Mod}$ *is a functor from the category of theories to* $\mathcal{CAT}^{op}$*.*

Proof. Thm. 4.25 shows that $\mathbf{Mod}$ indeed maps theories and morphisms to categories and functors. We only have to verify the functor laws which is straightforward. □

Moreover, we have the following analogue of Thm. 2.20:

Theorem 4.27 (Preservation Condition)**.** *Consider a theory morphism* $\vartheta : \Theta' \to \Theta$*.*

For every Θ*-formula* F *and every* Θ'*-model* M'

$$M' \models \vartheta(F) \quad \text{iff} \quad \mathbf{Mod}(\vartheta)(M') \models F$$

For every Θ*-mormula* V *and every* Θ'*-model morphism* $m' : M' \to N'$

$$m' \models \vartheta(V) \quad \text{iff} \quad \mathbf{Mod}(\vartheta)(m') \models V$$

Proof. The first statement is a special case of Thm. 2.20, which remains true without change.

For the second statement, abbreviate $M = \mathbf{Mod}(\vartheta)(M')$, $N = \mathbf{Mod}(\vartheta)(N')$, and $m = \mathbf{Mod}(\vartheta)(m')$.

We prove the left-to-right direction for the case $V = \mathsf{q}_\Gamma F$. We abbreviate $\Gamma' = \vartheta(\Gamma)$ and $F' = \vartheta(F)$.

Assume the left property (1). To prove the right property, assume an assignment $\alpha \in \Gamma^M$ such that $F^{M,\alpha} = 1$ (2). We have to prove $F^{N,\Gamma^m(\alpha)} = 1$ $(*)$. Note that $\Gamma'^{M'} = \Gamma^M$ and $\Gamma'^{m'}(\alpha) = \Gamma^m(\alpha)$. Observe that we can treat

- Θ, Γ and $\Theta, \vartheta(\Gamma)$ as theories,
- ϑ, id_Γ as a theory morphism between them,
- M', α and $N', \Gamma'^{m'}(\alpha)$ as models of the latter theory,

- M, α and $N, \Gamma^m(\alpha)$ as the reducts of these models along ϑ, id_Γ.

Using this, we can apply the first statement to the theory morphism ϑ, id_Γ and (2). This yields $F'^{M',\alpha} = 1$ (3). Applying (1) to (3) yields $F'^{N',\Gamma'^{m'}(\alpha)} = 1$ (4). Applying the first statement to the theory morphism ϑ, id_Γ and (4) yields $(*)$.

The proof of the right-to-left direction proceeds accordingly.

The cases for $V = \mathfrak{R}_\Gamma F$ and $V = \mathfrak{I}_\Gamma T$ proceed accordingly. □

Finally, we can package the model theoretical properties of our logic into a single statement:

Theorem 4.28. *PFOL forms an institution in the sense of [GB92].*

Proof. This is proved in the usual way. The key step is provided by Thm. 4.26. □

The corresponding results hold for (classical or intuitionistic) fragments of PFOL, including FOL and TFOL.

4.4 Soundness and Completeness

It remains to relate syntactic and semantic meorems. Ideally, the notions should coincide, i.e., the moof calculus should be sound and complete. However, we have only been able to prove soundness at this point:

Theorem 4.29 (Soundness)**.** *The moof calculus is sound, i.e., if V is a syntactic meorem, then V is a semantic meorem.*

Proof. We proceed by induction on derivations. All cases are straightforward. □

We conjecture that the moof calculus for meorems is also complete. More precisely, we conjecture it is complete for any variant of our logic for which the proof calculus for theorems is complete. However, we do not have a proof at this point.

Our conjecture is motivated by an informal analysis of possible moof rules that one might add to the calculus. All rules we could conceive of were easily seen to be either derivable or unsound.

A promising proof idea uses that, in light of Ex. 4.10, any counter-example to completeness (i.e., a semantic meorem that is not a syntactic meorem) would have to be a Θ-formula F that is neither a theorem nor a contradiction. Consequently, both F and $\neg F$ would be consistent with Θ, and the completeness of the proof calculus would yield models M of $\Theta \cup \{F\}$ and M' of $\Theta \cup \{\neg F\}$. Completeness of the moof calculus would follow if we could exhibit an appropriate model morphism from M to M'.

5 Generalizations

5.1 Power Types

In this section, we extend PFOL with a built-in unary type operator for power types. That is necessary because power types cannot be axiomatized in first-order logic even if we use PFOL-type operators. We apply the extended logic to axiomatize the category of topological spaces and continuous functions.

We extend PFOL with a type constructor $\mathtt{Set}(A)$. Our guiding intuition is that $\Gamma \vdash_\Theta T : \mathtt{Set}(A)$ holds if T is a subtype of A.

Syntax We add the following productions to the grammar of Def. 2.3:

A	::=	$\mathtt{Set}(A)$	power type of A
T	::=	$\{x : A \mid F\}$	subtype of A given by F
F	::=	$T \in T'$	membership test for a subtype
P	::=	see below	
Q	::=	see below	

with the following typing rules

$$\frac{\Gamma \vdash_\Theta A : \mathtt{tp}}{\Gamma \vdash_\Theta \mathtt{Set}(A) : \mathtt{tp}}$$

$$\frac{\Gamma, x : A \vdash_\Theta F : \mathtt{form}}{\Gamma \vdash_\Theta \{x : A \mid F\} : \mathtt{Set}(A)} \qquad \frac{\Gamma \vdash_\Theta U : \mathtt{Set}(A) \quad \Gamma \vdash_\Theta T : A}{\Gamma \vdash_\Theta T \in U : \mathtt{form}}$$

We also add the following proof rules

$$\frac{\Gamma, x : A \vdash_\Theta F : \mathtt{form} \quad \Gamma \vdash_\Theta T : A}{\Gamma \vdash_\Theta \big(T \in \{x : A \mid F(x)\}\big) \Leftrightarrow F(T)}\mathtt{reduce}$$

$$\frac{}{\Gamma, X : \mathtt{tp}, y : \mathtt{Set}(X) \vdash_\Theta y \doteq_{\mathtt{Set}(X)} \{x : X \mid x \in y\}}\mathtt{expand}$$

Everything so far is straightforward: If we used higher-order logic[8], we could define $\mathtt{Set}(A) := A \to \mathtt{bool}$. Then $\{x : A \mid F\}$, $T \in U$, $\mathtt{reduce}$, and $\mathtt{expand}$, respectively, would become special cases of λ-abstraction, application, β-reduction, and η-expansion.

The key novelty now is that we can add the maxiom

$$\frac{}{X : \mathtt{tp} \vdash_\Theta \mathsf{q}_{x:X, y:\mathtt{Set}(X)} x \in y}\mathtt{preserve}$$

which guarantees that membership is preserved by model morphisms. Notably, we do not require that membership is reflected. Thus, model morphisms do not commute with all terms. As we will see below, that requirement would be impractically strong.

Abbreviations We can immediately define the usual lattice operations on the type $\mathtt{Set}(A)$:

$$\bot_A := \{x : A \mid \mathtt{false}\}$$
$$\top_A := \{x : A \mid \mathtt{true}\}$$
$$S \cap T := \{x : A \mid x \in S \wedge x \in T\}$$
$$S \cup T := \{x : A \mid x \in S \vee x \in T\}$$

[8] Recall that using higher-order logic in its entirety would make it difficult to define model morphisms.

$$\mathbb{C}S := \{x : A \mid \neg x \in S\}$$

$$S \subseteq T := \forall x : A.\, x \in S \Rightarrow x \in T$$

We can also define the big operators that map from $\texttt{Set}(\texttt{Set}(A))$ to $\texttt{Set}(A)$:

$$\bigcap K := \{x : A \mid \forall k : \texttt{Set}(A).k \in K \Rightarrow x \in k\}$$

$$\bigcup K := \{x : A \mid \exists k : \texttt{Set}(A).k \in K \wedge x \in k\}$$

Semantics We add the following cases to the interpretation of expressions in models in Def. 2.18:

$$\texttt{Set}(A)^{M,\alpha} = \mathcal{P}(A^{M,\alpha})$$

$$\{x : A \mid F\}^{M,\alpha} = \{e \in A^{M,\alpha} \mid F^{M,\alpha.e} = 1\}$$

$$(T \in U)^{M,\alpha} = \begin{cases} 1 & \text{if } T^{M,\alpha} \in U^{M,\alpha} \\ 0 & \text{otherwise} \end{cases}$$

This fixes the expected interpretation of power types as power sets.

Moreover, we add one case to the translation function of model morphisms in Def. 4.22:

$$\texttt{Set}(A)^{m,\alpha} : \mathcal{P}(A^{M,\alpha}) \to \mathcal{P}(A^{M',m(\alpha)})$$

$$\texttt{Set}(A)^{m,\alpha} : S \mapsto \{A^{m,\alpha}(s) : s \in S\}$$

This extends maps between sets to maps between power sets point-wise, i.e., m maps a set S of points $s \in A^{M,\alpha}$ to the set of points $A^{m,\alpha}(s) \in A^{M',m(\alpha)}$.

Extending the proof of soundness in Thm. 4.29 with cases for the new proof and moof rules is straightforward.

Non-Commutation of Terms Model morphisms do not have to commute with terms $\{x : A \mid F\}$. Using the moof rule we added, it is easy to show that for all terms $\Gamma \vdash_\Theta U : \texttt{Set}(A)$, we have $U^{M,\alpha} \subseteq U^{M',m(\alpha)}$. But equality may or may not hold.

For example, it is easy to show that model morphisms commute with $\bot_A$, $\cup$, $\cap$, and $\bigcup$ but not with $\top_A$, $\mathbb{C}$, and $\bigcap$. For example, if a^m is not surjective, the point-wise definition yields $(\top_a)^M \subsetneq (\top_a)^{M'}$.

Applications to Topology There are multiple conceptually different but isomorphic theories whose models are the topological spaces. But if the standard maxioms are used, these isomorphic theories yield very different model morphisms and thus do not yield isomorphic model categories.

The following table gives an overview:

Axiomatization is based on $\texttt{point} : \texttt{tp}$ and	Standard maxioms yield model morphisms that
open sets $\texttt{open} : \{s : \texttt{Set}(\texttt{point})\}\texttt{form}$	preserve openness
neighborhoods of a point $\texttt{neighborhood} : \{p : \texttt{point}, n : \texttt{Set}(\texttt{point})\}\texttt{form}$	preserve openness
closed sets $\texttt{closed} : \{s : \texttt{Set}(\texttt{point})\}\texttt{form}$	preserve closedness
closure operator $\texttt{closure} : \{s : \texttt{Set}(\texttt{point})\}\texttt{Set}(\texttt{point})$	preserve closedness
closeness relation between points and sets $\texttt{close} : \{p : \texttt{point}, s : \texttt{Set}(\texttt{point})\}\texttt{form}$	are continuous

Moreover, similar to Ex. 2.6, subtle variations can further influence the resulting model morphisms. For example, the standard maxioms yield different morphisms if the open sets are given by a set $\texttt{open}' : \texttt{Set}(\texttt{Set}(\texttt{point}))$ than if they are given by a unary predicate $\texttt{open} : \{s : \texttt{Set}(\texttt{point})\}\texttt{form}$.

The following example looks at two of the above in more detail:

Example 5.1 (Topological Spaces). The theory `Open` of topological spaces based on open sets contains the declarations

$$\texttt{point} : \texttt{tp}, \quad \texttt{open} : \{x : \texttt{Set}(\texttt{point})\}\texttt{form}$$

The theory `Closed` of topological spaces based on closed sets contains the declarations

$$\texttt{point} : \texttt{tp}, \quad \texttt{closed} : \{x : \texttt{Set}(\texttt{point})\}\texttt{form}$$

In both cases, we omit the well-known axioms.

It is straightforward to give theory isomorphisms between these theories. Both directions maps $\texttt{point} \mapsto \texttt{point}$. Additionally, the isomorphism $\texttt{oc} : \texttt{Open} \to \texttt{Closed}$ maps $\texttt{open} \mapsto [x : \texttt{point}]\, \texttt{closed}(\mathbb{C}x)$, and the isomorphism $\texttt{co} : \texttt{Closed} \to \texttt{Open}$ maps $\texttt{closed} \mapsto [x : \texttt{point}]\, \texttt{open}(\mathbb{C}x)$. We show that these morphisms are indeed isomorphisms by checking that they compose to the identity, which follows easily using $\mathbb{C}(\mathbb{C}x) \doteq_{\texttt{Set}(\texttt{point})} x$.

$\mathbf{Mod}(\texttt{Open})$ and $\mathbf{Mod}(\texttt{Closed})$ have bijective model classes, and $\mathbf{Mod}(\texttt{oc})$ and $\mathbf{Mod}(\texttt{co})$ are bijections between them.

But if we use the standard maxioms, the two categories have different model morphisms. $\mathbf{Mod}(\texttt{Open})$-morphisms preserve openness (i.e., map open sets to open sets), whereas $\mathbf{Mod}(\texttt{Closed})$-morphisms preserve closedness (i.e., map closed sets to closed sets). Neither of these two conditions implies the other, and neither is sufficient or necessary for being a continuous function.

If, instead, we do not use the standard maxioms, it becomes possible to give preservation maxioms that yield the continuous functions: To `Closed`, we add[9]

$$\texttt{continuous} : \mathsf{q}_{p:\texttt{point},s:\texttt{Set}(\texttt{point})} p \in \texttt{closure}(s)$$

where $\texttt{closure}(s) := \bigcap\{S : \texttt{Set}(\texttt{point}) \mid s \subseteq S \wedge \texttt{closed}(S)\}$ is the smallest closed superset of s.

The theory `Open` can be extended accordingly. It is now straightforward to extend the isomorphisms `oc` and `co`, which then imply that the resulting model categories are isomorphic.

5.2 Partial Functions

The theory of fields poses an awkward problem in first-order-based algebraic specification: It is an elementary theory in mathematics but the partiality of the division function cannot naturally be handled with plain first-order logic.

There are several solutions. For example we can use PFOL and introduce option types akin to Ex. 4.23. Among others, this would use the declarations

$$\texttt{option} : \{x : \texttt{tp}\}\texttt{tp},\quad u : \texttt{tp},\quad / : \{x : u, y : u\}\,\texttt{option}(u)$$

Another common solution is to extend FOL with a definedness predicate and partial function symbols.

Here we explore an alternative that uses guarded arguments. This example has the added benefit that it explores some advanced details of our treatment of model morphisms. We show how it can be used first and discuss the subtleties afterwards:

Example 5.2 (Fields). The theory of rings contains among others the declarations

$$u : \texttt{tp},\ 0 : u\ , 1 : u,\ + : \{x : u, y : u\}u,\ - : \{x : u, y : u\}u,\ * : \{x : u, y : u\}u$$

To obtain the theory `Field` of fields, we add a ternary function symbol for division, whose third argument is a guard that prevents division by zero:

$$\texttt{div} : \{x : u, y : u, p : \neg y \doteq_u 0\}u$$

`Field` only needs the standard maxioms $+^{\text{std}}$ and $*^{\text{std}}$. The other standard maxioms are meorems that we can establish in essentially the same way as in Ex. 4.11.

However, to make all this work, we have to extend our syntax to allow quantifying over proof variables. Firstly, we need it to state the axiom

$$\texttt{mult\_inv} : \forall x : u.\, \forall p : \neg x \doteq_u 0.\ \texttt{div}(x, x, p) \doteq_u 1$$

We cannot use an implication $\forall x : u.\, x \doteq_u 0 \Rightarrow \texttt{div}(x, x, ?) \doteq_u 1$ anymore because we need a name p for the assumption so that we can use it as the guard. Secondly, we need it in the mormula of the standard axiom for division (which is a meorem):

$$\texttt{div\_comm} : \mathfrak{I}_{x:u,y:u,p:\neg y\doteq_u 0}\,\texttt{div}(x, y, p)$$

Quantifying over proof variables is a technically major but intuitively straightforward generalization of TFOL.

With that in place, we can establish the meorem that non-zeroness is preserved:

$$\texttt{nz} : \mathfrak{q}_{x:u}\neg x \doteq_u 0$$

[9] Maybe surprisingly, the seemingly obvious maxiom $\mathfrak{R}_{x:\texttt{Set(point)}}\,\texttt{open}(x)$ is not strong enough to yield the continuous functions: It states "S is open if $f(S)$ is" whereas continuity of f is the subtly stronger "$f^{-1}(S)$ is open if S is".

The key part of the moof is

$$\dfrac{\dfrac{\dfrac{(\text{ omitted })}{\vdash_{\texttt{Field}} \mathsf{q}_{x:u,y:u} x * y \doteq 1}}{\vdash_{\texttt{Field}} \mathsf{q}_{x:u} \exists y : u.\, x * y \doteq_u 1}\,\texttt{exists} \qquad P}{\vdash_{\texttt{Field}} \mathsf{q}_{x:u} \neg x \doteq_u 0}\,\texttt{equiv}$$

where $x : u \vdash_{\texttt{Field}} P : \neg x \doteq_u 0 \Leftrightarrow \exists y : u.\, x * y \doteq_u 1$.

From this, we can prove an important meorem about field morphisms: that they are injective (i.e., preserve inequality). The moof is

$$\dfrac{\dfrac{\dfrac{}{\vdash_{\texttt{Field}} \mathsf{q}_{x:u} \neg x \doteq_u 0}\,\texttt{nz} \qquad \dfrac{(\text{ omitted })}{\vdash_{\texttt{Field}} \mathsf{J}_{x:u,y:u} x - y}}{\vdash_{\texttt{Field}} \mathsf{q}_{x:u,y:u} \neg x - y \doteq_u 0}\,\texttt{commsubs} \qquad P}{\vdash_{\texttt{Field}} \mathsf{q}_{x:u,y:u} \neg x \doteq_u y}\,\texttt{equiv}$$

where $x : u, y : u \vdash_{\texttt{Field}} P : \neg x - y \doteq_u 0 \Leftrightarrow \neg x \doteq_u y$.

Working out the extension of TFOL used in Ex. 5.2 leads to a subtle problem. The assignment translation function Γ^m is defined for contexts Γ that declare only term variables in Def. 4.16 and generalized to type variables in Def. 4.22. But it cannot easily be generalized to proof variables.

Consider a context $p : F$ for a closed formula F and a model morphism $m : M \to M'$. We want to define $(p : F)^m$. Note that $(p : F)^M$ is non-empty iff $F^M = 1$. Therefore, it is possible that no function $(p : F)^M \to (p : F)^{M'}$ exists. This happens if $F^M = 1$ but $F^{M'} = 0$, which is equivalent to $(\mathsf{q}F)^m = 0$.

Therefore, it is non-obvious how to define the needed extension of TFOL. A promising idea is to allow proof variable declarations $p : F$ only in contexts in which $\mathsf{q}F$ is a meorem. Indeed, the proof variables used in Ex. 5.2 have this property. We leave the details to future work.

6 Conclusion

We have introduced new syntactic concepts for specifying and reasoning about model morphisms. Our new concepts of mormulas, maxioms, moofs, and meorems are for model morphisms what the existing concepts of formulas, axioms, proofs, and theorems are for models.

This allows designing specification languages in which users have more fine-grained control over the intended semantics of a theory: they can now specify not only the models but also the model morphisms in the categories. The most important property of these languages is that the semantics of every theory morphism is a model reduction functor.

We developed a general framework for such specification languages and instantiated it with untyped, typed, and polymorphic first-order logic. Moreover, we discussed further generalizations to power types and partial functions.

Existing implementations of specification languages such as in the Hets system [MML07] can be easily augmented with our new concepts. Besides allowing for more precise specifications of categories, this enables tool support for reasoning about model morphisms.

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