Appendix D: Complex Index of Refraction, Dispersion of Light, and Lorentz–Lorenz Formula

Within a dielectric, positive and negative charges are impelled to move in opposite directions by an applied electric field. As a result, electric dipoles are generated. The product of charges and the separation distance of positive and negative charges is called the dipole moment, which, when divided by the unit volume, is referred to as polarization **P**. The displacement vector **D** (charge per area) within a dielectric is defined by

$$\mathbf{D} = \varepsilon \mathbf{E} = \mathbf{E} + 4\pi \mathbf{P}, \tag{D.1}$$

where ε is the permittivity of the medium. Thus,

$$\varepsilon = 1 + 4\pi \mathbf{P} \cdot \mathbf{E}/E^2. \tag{D.2}$$

The velocity of light in terms of ε and the permeability μ is given by

$$c = \sqrt{\frac{1}{\mu\varepsilon}}. \tag{D.3}$$

The permeability μ in air or water is nearly equal to the permeability μ_0 in vacuum, i.e., $\mu \approx \mu_0$. The index of refraction is defined as the ratio of the velocity of light in vacuum to that in the medium and may be expressed by

$$m = \frac{c_0}{c} \approx \sqrt{\varepsilon} = \sqrt{1 + \frac{4\pi \mathbf{P} \cdot \mathbf{E}}{E^2}}. \tag{D.4}$$

But the polarization vector for N dipoles is [see Eq. (3.3.1)]

$$\mathbf{P} = N\alpha \mathbf{E}. \tag{D.5}$$

Inserting Eq. (D.5) into Eq. (D.4) leads to

$$m^2 = 1 + 4\pi N\alpha. \tag{D.6}$$

Now, we have to find the polarizibility in terms of frequency. On the basis of the definition of a polarization vector, we have

$$\mathbf{P} = Ne\mathbf{r}, \tag{D.7}$$

where e is the charge of an electron, and $\mathbf{r}$ represents the vector distance. Combining Eqs. (D.5) and (D.7), we find

$$\alpha \mathbf{E} = e\mathbf{r}. \tag{D.8}$$

Further, from the Lorentz force equation, the force generated by the electric and magnetic fields is given by

$$\mathbf{F} = e[\mathbf{E} + (\mu/c)\mathbf{v} \times \mathbf{H}], \tag{D.9}$$

where $\mathbf{v}$ denotes the velocity of an electron, which is very small compared to the velocity of light. Hence, the force produced by the magnetic field may be neglected. The force in the vibrating system in terms of the displacement r is due to (1) the acceleration of the electron; (2) the damping force, which carries away energy when the vibrating electrons emit electromagnetic waves, and which is proportional to the velocity of the electrons; and (3) the restoring force of the vibration, which is proportional to the distance r. From Newton's second law, we find

$$\frac{\mathbf{F}}{m_e} = \frac{e\mathbf{E}}{m_e} = \frac{d^2\mathbf{r}}{dt^2} + \gamma \frac{d\mathbf{r}}{dt} + \xi \mathbf{r}, \tag{D.10}$$

where γ and ξ are the damping and restoring coefficients, respectively, and m_e is the mass of the electron. In scalar form, we write

$$\frac{d^2 r}{dt^2} + \gamma \frac{dr}{dt} + \xi r = \frac{eE}{m_e}. \tag{D.11}$$

The homogeneous solution of this second-order differential equation is given by

$$r = r_0 e^{-i\omega t} = r_0 e^{-i2\pi\tilde{\nu}t}. \tag{D.12}$$

Substituting Eq. (D.12) into Eq. (D.11), we obtain

$$[(\xi - 4\pi^2\tilde{\nu}^2) - i2\pi\tilde{\nu}\gamma]r = eE/m_e. \tag{D.13}$$

The natural (or resonant) frequency is defined by $\tilde{\nu}_0 = \sqrt{\xi}/2\pi$. Thus, we find

$$\begin{aligned} \alpha = \frac{er}{E} &= \frac{e^2}{m_e} \frac{1}{4\pi^2\left(\tilde{\nu}_0^2 - \tilde{\nu}^2\right) - i2\pi\gamma\tilde{\nu}} \\ &= \frac{e^2}{m_e} \left[\frac{\tilde{\nu}_0^2 - \tilde{\nu}^2}{4\pi^2\left(\tilde{\nu}_0^2 - \tilde{\nu}^2\right)^2 + \gamma^2\tilde{\nu}^2} + \frac{i}{2\pi} \frac{\gamma\tilde{\nu}}{4\pi^2\left(\tilde{\nu}_0^2 - \tilde{\nu}^2\right)^2 + \gamma^2\tilde{\nu}^2} \right]. \end{aligned} \tag{D.14}$$

Let the real and imaginary parts of the index of refraction be m_r and m_i, respectively, so that the index of refraction is defined by

$$m = m_r + im_i. \tag{D.15}$$

From Eq. (D.6), we can then show that

$$m_r^2 - m_i^2 = 1 + \frac{4\pi N e^2}{m_e} \frac{\tilde{\nu}_0^2 - \tilde{\nu}^2}{4\pi^2\left(\tilde{\nu}_0^2 - \tilde{\nu}^2\right)^2 + \gamma^2\tilde{\nu}^2}, \tag{D.16a}$$

$$2m_r m_i = \frac{2Ne^2}{m_e} \frac{\gamma\tilde{\nu}}{4\pi^2\left(\tilde{\nu}_0^2 - \tilde{\nu}^2\right)^2 + \gamma^2\tilde{\nu}^2}. \tag{D.16b}$$

For air, $m_r \approx 1$ and $m_i \ll (m_r - 1)$. Also, in the neighborhood of the resonant frequency $(\tilde{\nu}^2 - \tilde{\nu}_0^2) = (\tilde{\nu}_0 + \tilde{\nu}) \cdot (\tilde{\nu} - \tilde{\nu}_0) \cong 2\tilde{\nu}_0(\tilde{\nu} - \tilde{\nu}_0)$. Further, the half-width of the natural broadening depends on the damping and is given in the form $\alpha_N = \gamma/4\pi$, while the line strength S is $\pi N e^2/(m_e c)$. Thus, we obtain the real part

$$m_r - 1 = -\frac{Ne^2}{4\pi m_e \tilde{\nu}_0} \frac{\tilde{\nu} - \tilde{\nu}_0}{(\tilde{\nu} - \tilde{\nu}_0)^2 + \alpha_N^2}, \tag{D.17}$$

and the absorption coefficient (Born and Wolf, 1975)

$$k_{\tilde{\nu}} = \frac{4\pi\tilde{\nu}_0 m_i}{c} = \frac{S}{\pi} \frac{\alpha_N}{(\tilde{\nu} - \tilde{\nu}_0)^2 + \alpha_N^2}. \tag{D.18}$$

Equation (D.18) is the Lorentz profile discussed in Section 1.3.

Shown in Fig. D.1 is the dependence of $(m_r - 1)$ and $k_{\tilde{\nu}}$ on the frequency. The value of $(m_r - 1)$ increases as the frequency increases when $(\tilde{\nu}_0 - \alpha_N) > \tilde{\nu}$. This mode is

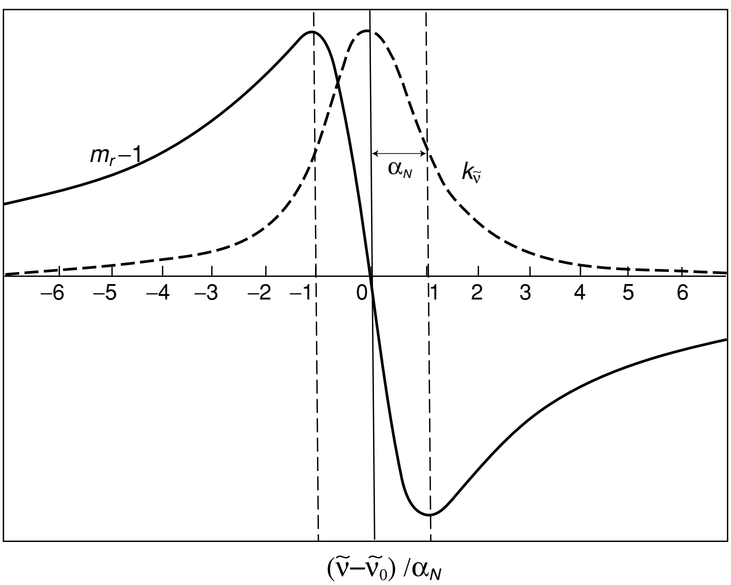

Figure D.1 Real and imaginary parts of the complex index of refraction as functions of the frequency.

referred to as *normal dispersion* under which light is dispersed by a prism into its component colors. For the region $(\tilde{\nu}_0 + \alpha_N) > \tilde{\nu} > (\tilde{\nu}_0 - \alpha_N)$, $(m_r - 1)$ decreases with increasing frequency and is called *anomalous dispersion*. For the range $\tilde{\nu} > (\tilde{\nu}_0 + \alpha_N)$, normal dispersion takes place again, but $(m_r - 1)$ is smaller than unity.

In this appendix we also wish to prove Eq. (3.3.16). We consider a dielectric placed between the plates of a parallel-plate condenser without the end effect. Moreover, we consider an individual molecule constituting this dielectric and draw a sphere with radius a about this molecule. The molecule is, therefore, affected by the fields caused by (1) the charges of the surfaces of the condenser plates; (2) the surface charge on the dielectric facing the condenser plates; (3) the surface charge on the spherical boundary of radius a; and (4) the charges of molecules (other than the one under consideration) contained within the sphere. For items (1) and (2), the electric field produced by these charges is

$$\mathbf{E}_1 + \mathbf{E}_2 = (\mathbf{E} + 4\pi\mathbf{P}) - 4\pi\mathbf{P} = \mathbf{E}. \tag{D.19}$$

For item (3), the electric field, which is produced by the polarization charge presented on the inside of the sphere, is given by

$$dE_3 = \frac{4\pi P \cos\theta \, dA}{4\pi a^2}, \tag{D.20}$$

where $P\cos\theta$ represents the component of the polarization vector in the direction of the electric field vector, and the differential area $dA = a^2 \sin\theta \, d\theta \, d\phi \times \cos\theta$. Thus,

$$E_3 = \int_0^{2\pi} \int_0^{\pi} \frac{4\pi P \cos\theta}{4\pi a^2} a^2 \sin\theta \cos\theta \, d\theta \, d\phi = \frac{4\pi P}{3}. \tag{D.21}$$

For item (4), it turns out that $E_4 = 0$. Thus, the effective electric field is

$$\mathbf{E}' = \mathbf{E} + 4\pi\mathbf{P}/3. \tag{D.22}$$

However, from Eq. (D.5), we have

$$\mathbf{P} = \alpha N \mathbf{E}' = \alpha N(\mathbf{E} + 4\pi\mathbf{P}/3). \tag{D.23}$$

It follows that

$$\mathbf{P} = \alpha N \mathbf{E}/(1 - 4\pi\alpha N/3). \tag{D.24}$$

Thus, from the definition of the index of refraction in Eq. (D.4), we find

$$m^2 = 1 + 4\pi\alpha N/(1 - 4\pi\alpha N/3). \tag{D.25}$$

Rearranging the terms, we obtain the Lorentz–Lorenz formula in the form

$$\alpha = \frac{3}{4\pi N} \frac{m^2 - 1}{m^2 + 2}. \tag{D.26}$$