# Appendix C: Spherical Geometry

In reference to Fig. C.1, we define

$$\overline{CD} = \overline{CO}\tan\theta', \qquad \overline{OD} = \overline{CO}\sec\theta',$$
$$\overline{CE} = \overline{CO}\tan\theta, \qquad \overline{OE} = \overline{CO}\sec\theta, \tag{C.1}$$

where $\overline{CD}$ and $\overline{CE}$ are the tangent lines of the arcs $CA$ and $CB$, respectively. For the triangle $\Delta CDE$, we find

$$\overline{DE}^2 = \overline{CD}^2 + \overline{CE}^2 - 2\overline{CE}\,\overline{CD}\cos DCE. \tag{C.2}$$

For the triangle $\Delta ODE$, we have

$$\overline{DE}^2 = \overline{OD}^2 + \overline{OE}^2 - 2\overline{OD}\,\overline{OE}\cos DOE. \tag{C.3}$$

Upon substituting Eq. (C.1) into Eqs. (C.2) and (C.3), we obtain

$$\overline{DE}^2 = \overline{CO}^2[\tan^2\theta' + \tan^2\theta - 2\tan\theta'\tan\theta\cos(\phi - \phi')], \tag{C.4}$$
$$\overline{DE}^2 = \overline{CO}^2[\sec^2\theta' + \sec^2\theta - 2\sec\theta'\sec\theta\cos\Theta]. \tag{C.5}$$

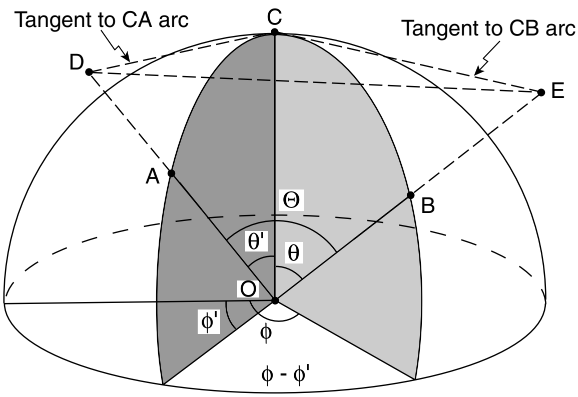


**Figure C.1** Relationship between the scattering angle, zenith angle, and azimuthal angle in spherical coordinates.

It follows that

$$\begin{aligned}
&\tan^2\theta' + \tan^2\theta - 2\tan\theta'\tan\theta\cos(\phi - \phi') \\
&\quad = \sec^2\theta' + \sec^2\theta - 2\sec\theta'\sec\theta\cos\Theta.
\end{aligned} \tag{C.6}$$

But $\sec^2\theta - \tan^2\theta = 1$, so Eq. (C.6) becomes

$$2 - 2\sec\theta'\sec\theta\cos\Theta = -2\tan\theta'\tan\theta\cos(\phi - \phi'). \tag{C.7}$$

Thus, we have

$$\begin{aligned}
\cos\Theta &= \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi') \\
&= \mu\mu' + (1 - \mu^2)^{1/2}(1 - \mu'^2)^{1/2}\cos(\phi - \phi'),
\end{aligned} \tag{C.8}$$

where $\mu = \cos\theta$ and $\mu' = \cos\theta'$.