Appendix E: Properties of the Legendre Polynomials and Addition Theorem

As indicated in Eqs. (5.2.42) and (5.2.43), the solution of the second-order differential equation

$$(1-\mu^2)\frac{d^2y}{d\mu^2} - 2\mu\frac{dy}{d\mu} + \left[\ell(\ell+1) - \frac{m^2}{1-\mu^2}\right]y = 0 \tag{E.1}$$

is given by

$$y(\mu) = P_\ell^m(\mu) = \frac{(1-\mu^2)^{m/2}}{2^\ell \ell!}\frac{d^{m+\ell}}{d\mu^{m+\ell}}(\mu^2-1), \tag{E.2}$$

where $\mu = \cos\theta$. When $m = 0$, $P_\ell^0(\mu) = P_\ell(\mu)$ are the Legendre polynomials. From Eq. (E.2) we have

$$P_\ell^m(\mu) = (1-\mu^2)^{m/2}\frac{d^m P_\ell(\mu)}{d\mu^m}. \tag{E.3}$$

The associated Legendre polynomials satisfy the orthogonal properties

$$\int_{-1}^{1} P_\ell^m(\mu)P_k^m(\mu)\,d\mu = \begin{cases} 0, & \ell \neq k, \\ \dfrac{2}{2\ell+1}\dfrac{(\ell+m)!}{(\ell-m)!}, & \ell = k, \end{cases} \tag{E.4}$$

$$\int_{-1}^{1} P_\ell^m(\mu)P_\ell^n(\mu)\frac{d\mu}{1-\mu^2} = \begin{cases} 0, & m \neq n, \\ \dfrac{1}{m}\dfrac{(\ell+m)!}{(\ell-m)!}, & m = n. \end{cases} \tag{E.5}$$

Some useful recurrence relations in conjunction with light scattering and radiative transfer are

$$\frac{dP_\ell^m}{d\theta} = -\sqrt{1-\mu^2}\,\frac{dP_\ell^m}{d\mu} = \frac{1}{2}\left[(\ell-m+1)(\ell+m)P_\ell^m - P_\ell^{m+1}\right], \tag{E.6}$$

$$(2\ell+1)\mu P_\ell^m = (\ell+m)P_{\ell-1}^m + (\ell-m+1)P_{\ell+1}^m, \tag{E.7}$$

$$(2\ell+1)(1-\mu^2)^{1/2}P_\ell^m = \left(P_{\ell+1}^{m+1} - P_{\ell-1}^{m+1}\right). \tag{E.8}$$

A number of low-order associated Legendre and Legendre polynomials are

$$\begin{aligned} &P_1^1(\mu) = (1-\mu^2)^{1/2}, && P_2^1(\mu) = 3\mu(1-\mu^2)^{1/2}, \\ &P_3^1(\mu) = \tfrac{3}{2}(5\mu^2-1)(1-\mu^2)^{1/2}, && P_2^2(\mu) = 3(1-\mu^2), \\ &P_3^2(\mu) = 15\mu(1-\mu^2), && P_3^3(\mu) = 15(1-\mu^2)^{3/2}, \end{aligned} \tag{E.9}$$

$$\begin{aligned} &P_0(\mu) = 1, && P_1(\mu) = \mu, \\ &P_2(\mu) = \tfrac{1}{2}(3\mu^2-1), && P_3(\mu) = \tfrac{1}{2}(5\mu^3-3\mu), \\ &P_4(\mu) = \tfrac{1}{8}(35\mu^4-30\mu^2+3). \end{aligned} \tag{E.10}$$

Below we present the addition theorem for the Legendre polynomials. Let $g(\mu, \phi)$ be an arbitrary function on the surface of a sphere where this function and all of its first and second derivatives are continuous. Then $g(\mu, \phi)$ may be represented by an absolutely convergent series of surface harmonics as follows:

$$g(\mu, \phi) = \sum_{\ell=0}^{\infty} \left[a_{\ell 0} P_\ell(\mu) + \sum_{m=1}^{\ell} (a_{\ell m} \cos m\phi + b_{\ell m} \sin m\phi) P_\ell^m(\mu) \right]. \tag{E.11}$$

The coefficients can be determined by

$$a_{\ell 0} = \frac{2\ell+1}{4\pi} \int_0^{2\pi} \int_{-1}^{1} g(\mu, \phi) P_\ell(\mu)\, d\mu\, d\phi, \tag{E.12}$$

$$a_{\ell m} = \frac{(2\ell+1)(\ell-m)!}{2\pi(\ell+m)!} \int_0^{2\pi} \int_{-1}^{1} g(\mu, \phi) P_\ell^m(\mu) \cos m\phi\, d\mu\, d\phi, \tag{E.13}$$

$$b_{\ell m} = \frac{(2\ell+1)(\ell-m)!}{2\pi(\ell+m)!} \int_0^{2\pi} \int_{-1}^{1} g(\mu, \phi) P_\ell^m(\mu) \sin m\phi\, d\mu\, d\phi. \tag{E.14}$$

We note that

$$\int_{-1}^{1} P_\ell^m(\mu) P_k^m(\mu)\, d\mu = \begin{cases} 0, & \ell \neq k \\ \dfrac{2(\ell+m)!}{(2\ell+1)(\ell-m)!}, & \ell = k, \end{cases} \tag{E.15}$$

$$\int_0^{2\pi} \cos m\phi \cos n\phi\, d\phi = \begin{cases} 0, & m \neq n \\ \pi, & m = n, \end{cases} \tag{E.16}$$

and also that $P_\ell(1) = 1$, and $P_\ell^m(1) = 0$. Thus, we write

$$[g(\mu, \phi)]_{\mu=1} = \sum_{\ell=0}^{\infty} a_{\ell 0} = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell+1) \int_0^{2\pi} \int_{-1}^{1} g(\mu, \phi) P_\ell(\mu)\, d\mu\, d\phi. \tag{E.17}$$

We may now define the surface harmonic function in the form

$$Y_\ell(\mu, \phi) = \sum_{m=0}^{\ell} (a_{\ell m} \cos m\phi + b_{\ell m} \sin m\phi) P_\ell^m(\mu). \tag{E.18}$$

Let $Y_\ell(\mu, \phi)$ of order ℓ be $g(\mu, \phi)$, and by virtue of Eq. (E.17), we find

$$[Y_\ell(\mu, \phi)]_{\mu=1} = \frac{2\ell + 1}{4\pi} \int_0^{2\pi} \int_{-1}^{1} Y_\ell(\mu, \phi) P_\ell(\mu)\, d\mu\, d\phi. \tag{E.19}$$

From the scattering geometry, we have

$$\cos\Theta = \mu\mu' + (1 - \mu^2)^{1/2}(1 - \mu'^2)^{1/2} \cos(\phi - \phi'). \tag{E.20}$$

Thus, we may let

$$\begin{aligned} P_\ell(\cos\Theta) &= \sum_{m=0}^{\ell} (c_m \cos m\phi + d_m \sin m\phi) P_\ell^m(\mu) \\ &= \frac{c_0}{2} P_\ell(\mu) + \sum_{m=1}^{\ell} (c_m \cos m\phi + d_m \sin m\phi) P_\ell^m(\mu). \end{aligned} \tag{E.21}$$

Using the orthogonal properties denoted in Eqs. (E.15) and (E.16), we find

$$\int_0^{2\pi} \int_{-1}^{1} P_\ell(\cos\Theta) P_\ell^m(\mu) \cos m\phi\, d\mu\, d\phi = \frac{2\pi(\ell + m)!}{(2\ell + 1)(\ell - m)!} c_m. \tag{E.22}$$

By letting $P_\ell^m(\mu) \cos m\phi = Y_\ell(\mu, \phi)$, and using Eq. (E.19), Eq. (E.22) becomes

$$\begin{aligned} \int_0^{2\pi} \int_{-1}^{1} P_\ell(\cos\Theta) \left[P_\ell^m(\mu) \cos m\phi\right] d\mu\, d\phi &= \frac{4\pi}{2\ell + 1} \left[P_\ell^m(\mu) \cos m\phi\right]_{\cos\Theta = 1} \\ &= \frac{4\pi}{2\ell + 1} P_\ell^m(\mu') \cos m\phi'. \end{aligned} \tag{E.23}$$

Note that $\cos\Theta = 1$ and $\Theta = 0$, so we have $\mu = \mu'$, and $\phi = \phi'$. It follows from Eq. (E.21) that

$$c_m = \frac{2(\ell - m)!}{(\ell + m)!} P_\ell^m(\mu') \cos m\phi'. \tag{E.24}$$

In a similar manner, we find

$$d_m = \frac{2(\ell - m)!}{(\ell + m)!} P_\ell^m(\mu') \sin m\phi'. \tag{E.25}$$

Thus, from Eqs. (E.24), (E.25), and (E.21), we obtain

$$P_\ell(\cos\Theta) = P_\ell(\mu) P_\ell(\mu') + 2 \sum_{m=1}^{\ell} \frac{(\ell - m)!}{(\ell + m)!} P_\ell^m(\mu) P_\ell^m(\mu') \cos m(\phi' - \phi). \tag{E.26}$$