7 Polarization: The Hidden Variable

We call polarization the hidden variable because it is a property of light not readily observed with the unaided eye. Everyone is aware of variations in color and brightness, so in teaching about these properties of light we can appeal to observations that everyone has made or can make with little effort. Alas, this is not so with the polarization of light, which to observe requires a bit of effort and, more important, a few simple tools. Once you have fully grasped polarized light, however, you sometimes can observe its manifestations with nothing but your eyes. We recommend that you have polarizing sunglasses or a polarizing filter near to hand as you read this chapter so that you can observe for yourselves some of the consequences of polarization we discuss.

Treatments of polarization in elementary physics textbooks sometimes are misleading, usually incomplete, and sometimes wrong or incomprehensible. And as one moves further from physics, into chemistry, biology, geology, and meteorology, what is bad becomes progressively worse. We once surveyed geology textbooks and reference works for discussions of the colored patterns seen when transparent crystals are interposed between crossed polarizing filters (see Sec. 7.1.6). What we found was mostly confusing and often erroneous.

In previous chapters we noted in passing that electromagnetic waves are vector waves but were able to sidestep this and make physical arguments based on scalar waves. For example, the simple phase difference arguments in Chapter 3, so helpful for understanding scattering by particles, are essentially independent of the vector nature of electromagnetic waves. To understand polarization, however, requires us to face this head on.

7.1 The Nature of Polarized Light

The more times you see an explanation of a physical phenomenon or a statement about physical reality, especially in the form of an invariable mantra, especially in a textbook unaccompanied by any qualifications, the more certain you can be that it is wrong. Stated more succinctly, repetition increases the probability of incorrectness. This is a law of almost universal validity. One example is the assertion that the electric and magnetic fields of light waves are *always* perpendicular to each other and to the direction of propagation. Because this assertion has been made so often, without qualification, you can be certain it is wrong. And indeed it is. Ask any electrical engineer who knows something about near fields.

In Section 4.1 we noted that the Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{7.1}$$

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specifies the magnitude and direction of energy transport by any electromagnetic field at any point. Both the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ are necessarily perpendicular to $\mathbf{S}$, although they are not necessarily perpendicular to each other or to the direction of propagation (if by which is meant the normal to a surface of constant phase). For example, the fields within an illuminated sphere (indeed, any particle) are not perpendicular to each other, and the concept of a surface of constant phase is meaningless. The fields scattered by a sphere also are not perpendicular to each other except approximately at sufficiently large (compared with the wavelength) distances; and the concept of a surface of constant phase has its limitations. All we can be certain of is that the electric and magnetic fields lie in a plane to which the Poynting vector is perpendicular. It has become the custom to specify the polarization properties of electromagnetic waves by the *electric* field, although the magnetic field would serve just as well, and you occasionally come across works (especially by British authors) in which polarization is based on the magnetic field.

7.1.1 Vibration Ellipse and Ellipsometric Parameters

The only assumption we make at this point is that the electric field is time-harmonic:

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}(\mathbf{x}) \exp(-i\omega t), \tag{7.2}$$

where $\mathbf{E}$ is the complex representation of the electric field (see Sec. 2.5). To specify the polarization state of $\mathbf{E}$, however, we need the real field. Because $\mathbf{E}$ lies in a plane perpendicular to $\mathbf{S}$, only two components are needed. We denote two orthogonal unit vectors as $\mathbf{e}_\perp$ and $\mathbf{e}_\|$ chosen such that $\mathbf{e}_\perp \times \mathbf{e}_\|$ is in the direction of the Poynting vector. It will become apparent when we discuss applications why the coordinate axes are denoted as perpendicular ($\perp$) and parallel ($\|$). The field components are the real parts of

$$E_\perp = a_\perp \exp\{-i(\vartheta_\perp + \omega t)\}, \; E_\| = a_\| \exp\{-i(\vartheta_\| + \omega t)\}, \tag{7.3}$$

where the amplitudes a and phases ϑ are real functions that may depend on position but not time. Without loss of generality we may take the amplitudes as positive because the field components can be negative by virtue of the phases (i.e., $\cos\pi = -1$).

At a fixed point in space the tip of the electric vector (the point with coordinates given by the real parts of $E_\perp$ and $E_\|$) endlessly traces out a closed, bounded curve. When the two phases are equal, this curve is a straight line with slope equal to the ratio of amplitudes. When the phases differ by $\pi/2$, the curve is an ellipse with principal axes aligned along the coordinate axes, where the lengths of the two semi-axes are $a_\perp$ and $a_\|$. A circle results when these two amplitudes are equal.

In general, Eq. (7.3) describes an arbitrarily oriented ellipse of arbitrary *ellipticity* (not to be confused with eccentricity), defined as the ratio of the minor to major axis lengths (Fig. 7.1). The *azimuth* of this *vibration ellipse* is the angle between the major axis and a reference axis (e.g., one of the coordinate axes). One more *ellipsometric parameter* of the vibration ellipse is its *handedness*, the rotational sense in which it is traced out in time. There is no universal convention for what is meant by right- and left-handed rotation. Moreover, investigators in a particular field often assume that everyone knows what their convention is so feel no need to state it. We adopt the convention of calling a field right-handed if the vibration ellipse is traced

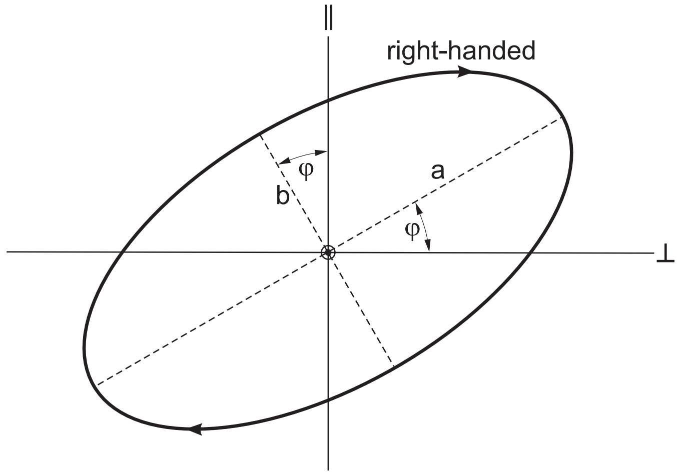

Figure 7.1: A time-harmonic electric field traces out an ellipse specified by its handedness, azimuth φ, and ellipticity b/a.

out clockwise as imagined to be seen looking into the Poynting vector. With this convention the helix traced out in space by the tip of the electric field vector is what all the world calls right-handed.

The electric field described by Eq. (7.3) is 100% or completely polarized in that it has a definite and fixed vibration ellipse. The general state of complete polarization is elliptical, special cases being linear and circular. But some textbooks, and, even more so, books on popular science, convey the notion that by polarization is meant linear polarization, no other kind being conceivable. To make matters worse, linearly polarized light is sometimes called plane polarized, especially in older works. This is a poor choice of terminology on several grounds. If a plane electromagnetic wave [Eq. (7.4)] is linearly polarized we would have the awkward designation plane-polarized plane wave (and polarized parallel or perpendicular to yet another plane). The first plane is defined by the electric field vector and the direction of propagation (equivalently, the plane surface traced out by the field as it propagates), the second plane is a surface of constant phase. To be consistent we would have to describe elliptically polarized light as elliptical-helicoidally polarized light because its electric field traces out an elliptical helicoid in space. Yet this is unnecessary because the polarization state of a plane wave (indeed, any wave) is specified by the ellipsometric parameters, which have nothing essential to do with surfaces.

Our experience has been that people who were taught at an impressionable age that light is plane polarized then find it difficult to understand elliptically polarized light and even more difficult to understand *partially polarized* light. Indeed, they sometimes confuse unpolarized light with circularly polarized light. And yet partially polarized light is readily understood beginning with a firm grasp of completely polarized light. The essential property of such light is *complete* correlation between two orthogonal components of the electric field. They may fluctuate in time, but if they do so synchronously (i.e., the ratio of amplitudes is constant as is the phase difference), the vibration ellipse has a definite and fixed form. Partially polarized light results when there is *partial* correlation between the two orthogonal components; unpolarized light results when there is *no* correlation.

We are aware of the existence of polarization only because two beams, identical in all respects except in one or more ellipsometric parameters (ellipticity, azimuth, handedness) can interact with matter in observably different ways. Were it not for this, the polarization state of time-harmonic fields would be a kind of non-functional adornment, like the fins on 1959 Cadillacs. Only two things can happen to a field when it interacts with matter: its amplitude or phase (or both) are changed. If two orthogonal components of the field are changed *differently*, the polarization state is changed. By "changed" here is meant that the polarization state of an incident (or exciting) wave is different from the polarization state of waves it gives rise to.

7.1.2 Orthogonally Polarized Waves do not Interfere

The general plane harmonic (complex) electric wave has the form

$$\mathbf{E} = \mathbf{E}_0 \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t), \tag{7.4}$$

where $\mathbf{E}_0$ is constant in space and time and the wave vector $\mathbf{k}$ may be complex; the magnetic field $\mathbf{H}$ is given by a similar expression. These fields must satisfy

$$\mathbf{k}\cdot\mathbf{E} = 0,\ \mathbf{H} = C\mathbf{k}\times\mathbf{E},\ \mathbf{k}\cdot\mathbf{H} = 0, \tag{7.5}$$

where C is a frequency-dependent parameter (possibly complex if the propagation medium is absorbing) characteristic of the medium in which the wave propagates; its value is of no consequence here. Keep in mind that the real and imaginary parts of $\mathbf{k}$ need not be parallel to each other; that is, the surfaces of constant phase and the surfaces of constant amplitude need not coincide (see Prob. 7.52). If they do, the wave is said to be *homogeneous*; if not, it is *inhomogeneous*. Inhomogeneous waves are not the product of an unbridled imagination. They can be produced readily by illuminating an absorbing medium at oblique incidence. Only if a wave is homogeneous, that is, its wave vector has the form $\mathbf{k} = k\,\mathbf{e}$, where k may be complex but $\mathbf{e}$ is a real unit vector, are the (real) electric and magnetic fields perpendicular to $\mathbf{e}$, the direction of propagation. And only if k is real are the (real) electric and magnetic fields perpendicular to each other. From now on we assume, unless stated otherwise, that the waves of interest are homogeneous ($\mathbf{k} = k\,\mathbf{e}$) and the medium in which they propagate is nonabsorbing (k is real). Absolutely plane waves and nonabsorbing media do not exist. We can get away with assuming they do because measurements of polarization are almost always made in negligibly absorbing media (e.g., air) and at sufficiently large distances (compared with the wavelength) from finite sources (e.g., bounded scatterers) that the fields from them are approximately planar over the detector. But if we were to inquire about polarization of waves in absorbing media or close to sources, much of the following analysis would not be strictly applicable.

Given the assumptions in the previous paragraph, the Poynting vector is

$$\mathbf{S} = C\mathbf{k}(\mathbf{E}\cdot\mathbf{E}), \tag{7.6}$$

which follows from Eqs. (7.1) and (7.5) and the identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}). \tag{7.7}$$

Equation (7.6) is a generalization to vector waves of a result in Section 3.3.2, namely, energy propagation by a scalar plane harmonic wave (on a string) is proportional to the square of a wave function. To avoid cluttered notation we do not use different symbols for fields and their complex representations, trusting that context indicates which is meant.

Two plane harmonic waves are said to be *orthogonally polarized* if they are opposite in handedness and the azimuths of their vibration ellipses are perpendicular. Orthogonally polarized waves do not interfere in that the Poynting vector of their sum is the sum of their Poynting vectors. To prove this, consider two such waves:

$$\mathbf{E}_1 = a_1 \cos\omega t \, \mathbf{e}_\parallel + b_1 \sin\omega t \, \mathbf{e}_\perp, \; \mathbf{E}_2 = b_2 \sin\omega t \, \mathbf{e}_\parallel + a_2 \cos\omega t \, \mathbf{e}_\perp, \tag{7.8}$$

where a_j and b_j are positive but otherwise arbitrary. The Poynting vector corresponding to the sum of these two waves is

$$\mathbf{S} = C\mathbf{k}[(a_1^2 + a_2^2)\cos^2\omega t + (b_1^2 + b_2^2)\sin^2\omega t + 2(a_1 b_2 + a_2 b_1)\sin\omega t \cos\omega t]. \tag{7.9}$$

If the two fields are orthogonal in the restricted sense that $\mathbf{E}_1 \cdot \mathbf{E}_2 = 0$ then $a_1 b_2 + a_2 b_1 = 0$ and the waves do not interfere at *any* instant. Regardless of their state of orthogonal polarization the *time-averaged* Poynting vectors are additive because $\langle \sin\omega t \cos\omega t \rangle = 0$:

$$\langle \mathbf{S} \rangle = \langle \mathbf{S}_1 \rangle + \langle \mathbf{S}_2 \rangle. \tag{7.10}$$

We are usually interested in Poynting vectors averaged over times large compared with the period (inverse frequency) of waves. Although the fields in Eq. (7.1) must be real, we can determine time-averaged Poynting vectors directly from the complex representations of fields:

$$\langle \mathbf{S} \rangle = \frac{1}{2}\Re\{\mathbf{E} \times \mathbf{H}^*\}. \tag{7.11}$$

This equation is valid for any time-harmonic electromagnetic field. A more restricted version, applicable only to plane homogeneous waves in nonabsorbing media, is

$$\langle \mathbf{S} \rangle = \frac{1}{2} C\mathbf{k}(\mathbf{E} \cdot \mathbf{E}^*). \tag{7.12}$$

This equation is at the heart of what follows.

7.1.3 Stokes Parameters and the Ellipsometric Parameters

Although we can imagine watching the tip of an electric vector rotating at, say, 10^{15} Hz, to think that we could actually do so is pure fantasy. All that we can measure, usually, is time-averaged irradiances. Such measurements of the magnitude of the Poynting vector must therefore be the route to ellipsometric parameters, and given that Eq. (7.3) is the equation of an ellipse, they must depend only on the amplitudes $a_\parallel$ and $a_\perp$and the phases $\vartheta_\parallel$ and $\vartheta_\perp$.

From Eqs. (7.3) and (7.12) the time-averaged irradiance of a beam, denoted here by I, is the sum of squares of amplitudes

$$I = |\langle \mathbf{S} \rangle| = E_{\|} E_{\|}^{*} + E_{\perp} E_{\perp}^{*} = a_{\|}^{2} + a_{\perp}^{2}. \tag{7.13}$$

Missing from this equation is a constant factor, which we ignore here because absolute measurements are not needed to determine ellipsometric parameters. To obtain the separate amplitudes we need the help of an *ideal linear polarizer* (or *linear polarizing filter*). Such a filter completely transmits light linearly polarized in a particular direction but does not transmit light linearly polarized in the orthogonal direction. As its name implies, an ideal linear polarizer does not exist but we can come close, at least over a restricted range of wavelengths. An example is the sheet polarizers used in polarizing sunglasses or in polarizing filters for cameras (the function of which is explained in Sec. 7.4). Absorption by such a sheet polarizer is asymmetric in that $\kappa d \ll 1$, where κ is the absorption coefficient and d the thickness of the sheet, for light linearly polarized along the *transmission axis*, whereas $\kappa d \gg 1$ for light linearly polarized perpendicular to this axis. At visible and near-visible wavelengths this difference in absorption coefficients is a consequence of anisotropy of the sheet material on a molecular scale. We cannot see the transmission axis, although we might be able to see that of a polarizing filter for microwave radiation. A medium with different absorption coefficients for different orthogonal linear states of polarization is said to be *linearly dichroic*.

Now we imagine inserting an ideal linear polarizing filter in the beam and measuring transmitted irradiances, first for the transmission axis along $\mathbf{e}_{\|}$, then along $\mathbf{e}_{\perp}$, and then subtracting these two irradiances:

$$Q = E_{\|} E_{\|}^{*} - E_{\perp} E_{\perp}^{*} = a_{\|}^{2} - a_{\perp}^{2}. \tag{7.14}$$

We now have done enough to obtain the amplitudes:

$$a_{\|}^{2} = \frac{1}{2}(I + Q), \; a_{\perp}^{2} = \frac{1}{2}(I - Q). \tag{7.15}$$

What about the phases? From Eqs. (7.3) and (7.12) it would seem that to obtain phases we must transmit a bit of both orthogonal components of an electric field. For example, if we align a linear polarizing filter with its transmission axis at 45° to $\mathbf{e}_{\|}$, the transmitted amplitude is

$$\frac{1}{\sqrt{2}}(E_{\|} + E_{\perp}). \tag{7.16}$$

Rotate the filter by 90° and the transmitted amplitude is

$$\frac{1}{\sqrt{2}}(E_{\perp} - E_{\|}). \tag{7.17}$$

The difference in the irradiances corresponding to Eqs. (7.16) and (7.17) is

$$U = E_{\|} E_{\perp}^{*} + E_{\perp} E_{\|}^{*} = 2 a_{\|} a_{\perp} \cos\delta, \tag{7.18}$$

where $\delta = \vartheta_{\|} - \vartheta_{\perp}$.

Measurement of I, Q, and U is sufficient to obtain $\cos\delta$, but because $\cos\delta = \cos(-\delta)$ is not sufficient to determine the handedness of the wave. Given $\cos\delta$ we cannot say if δ is positive or negative, which determines handedness. To find this quantity requires the help of *ideal circular polarizers* (or *circular polarizing filters*), devices that completely transmit circularly polarized light of one handedness but do not transmit circularly polarized light of the opposite handedness. Such circular polarizers are much more difficult to find than linear polarizers. Media with different absorption coefficients for different states of circularly polarized light, said to be *circularly dichroic*, exist. For example, our bodies and all organic matter are chock full of helical molecules (e.g., the double helix of DNA), and helices are not superposable on their mirror images: the reflection of a right-handed helix is a left-handed helix. Because of this mirror asymmetry we expect absorption by such molecules to be different for different states of circular polarization. And indeed this is so, but the difference is usually greatest at ultraviolet frequencies and, moreover, media with greatly different absorption coefficients are difficult to find. Nevertheless, we can imagine thought experiments with ideal circular polarizers.

To discuss circularly polarized light it is convenient to introduce a set of complex basis vectors:

$$\mathbf{e}_{\mathrm{R}} = \frac{1}{\sqrt{2}}(\mathbf{e}_{\parallel} + i\mathbf{e}_{\perp}),\ \mathbf{e}_{\mathrm{L}} = \frac{1}{\sqrt{2}}(\mathbf{e}_{\parallel} - i\mathbf{e}_{\perp}), \tag{7.19}$$

which are orthonormal in that

$$\mathbf{e}_{\mathrm{R}} \cdot \mathbf{e}_{\mathrm{R}}^* = 1,\ \mathbf{e}_{\mathrm{L}} \cdot \mathbf{e}_{\mathrm{L}}^* = 1,\ \mathbf{e}_{\mathrm{R}} \cdot \mathbf{e}_{\mathrm{L}}^* = 0. \tag{7.20}$$

$\mathbf{e}_{\mathrm{R}}$ corresponds to a right-circularly polarized wave of unit amplitude, $\mathbf{e}_{\mathrm{L}}$ to a left-circularly polarized wave. An arbitrary electric field thus can be written

$$\mathbf{E} = E_{\mathrm{R}}\mathbf{e}_{\mathrm{R}} + E_{\mathrm{L}}\mathbf{e}_{\mathrm{L}}, \tag{7.21}$$

where the circularly polarized (complex) amplitudes are related to the linearly polarized amplitudes by

$$E_{\mathrm{R}} = \frac{1}{\sqrt{2}}(E_{\parallel} - iE_{\perp}),\ E_{\mathrm{L}} = \frac{1}{\sqrt{2}}(E_{\parallel} + iE_{\perp}). \tag{7.22}$$

Now imagine that an ideal right-circular polarizer is inserted in the beam and the transmitted irradiance $E_{\mathrm{R}}E_{\mathrm{R}}^*$ is measured, then the irradiance $E_{\mathrm{L}}E_{\mathrm{L}}^*$ transmitted by an ideal left-circular polarizer is measured, and the second irradiance subtracted from the first:

$$V = E_{\mathrm{R}}E_{\mathrm{R}}^* - E_{\mathrm{L}}E_{\mathrm{L}}^* = i(E_{\parallel}E_{\perp}^* - E_{\perp}E_{\parallel}^*) = 2\Im\{E_{\perp}E_{\parallel}^*\} = 2a_{\parallel}a_{\perp}\sin\delta. \tag{7.23}$$

Knowing both $\sin\delta$ and $\cos\delta$ we can determine the sign of δ and hence the handedness of the wave.

The four quantities $\{I, Q, U, V\}$, which are no more than sums and differences of irradiances, are called the *Stokes parameters*, first set down by Sir George Gabriel Stokes in 1852. Even more than 150 years later his paper "On the composition and resolution of streams of

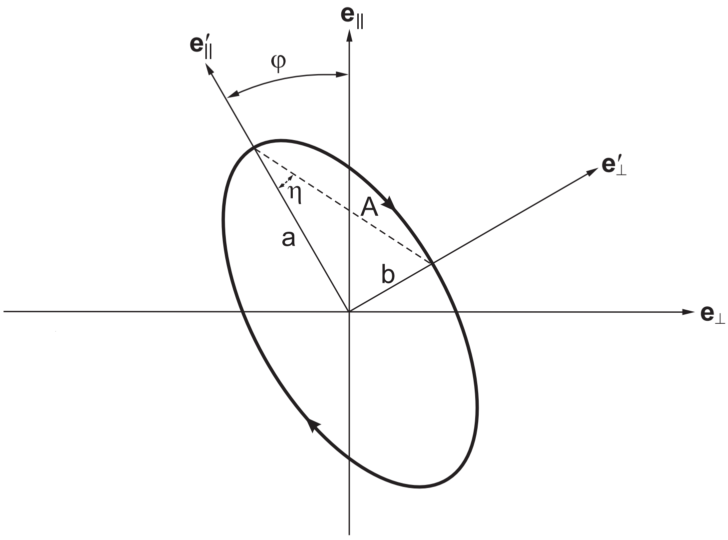

Figure 7.2: The unprimed coordinate system is rotated relative to the primed coordinate system, the axes of which are along the minor axis b and major axis a of the vibration ellipse.

polarized light from different sources" is still worth reading. You are likely to encounter different symbols for the Stokes parameters (he used A, B, C, and D), and linear combinations of Stokes parameters are also valid Stokes parameters. They sometimes are written compactly as a column matrix

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} E_{\|}E_{\|}^{*} + E_{\perp}E_{\perp}^{*} \\ E_{\|}E_{\|}^{*} - E_{\perp}E_{\perp}^{*} \\ E_{\|}E_{\perp}^{*} + E_{\perp}E_{\|}^{*} \\ i(E_{\|}E_{\perp}^{*} - E_{\perp}E_{\|}^{*}) \end{pmatrix} = \begin{pmatrix} a_{\|}^2 + a_{\perp}^2 \\ a_{\|}^2 - a_{\perp}^2 \\ 2a_{\|}a_{\perp}\cos\delta \\ 2a_{\|}a_{\perp}\sin\delta \end{pmatrix}, \tag{7.24}$$

called the *Stokes vector*, although it does not have the same rights and privileges as proper vectors. For example, the Stokes parameters are not independent in that

$$I^2 = Q^2 + U^2 + V^2. \tag{7.25}$$

Stokes's paper appeared a dozen years before the publication of Maxwell's famous electromagnetic theory of light and 32 years before the publication of Poynting's work. Much was known about the properties of light waves even before they were fully grounded in an adequate theory.

Now we have to show that the Stokes parameters determine the ellipsometric parameters. In what follows fields are real. The real parts of the fields in Eq. (7.3) can be expanded using the identity for the cosine of the sum of angles and written in matrix form as

$$\begin{pmatrix} E_{\|} \\ E_{\perp} \end{pmatrix} = \begin{pmatrix} a_{\|}\cos\vartheta_{\|} & -a_{\|}\sin\vartheta_{\|} \\ a_{\perp}\cos\vartheta_{\perp} & -a_{\perp}\sin\vartheta_{\perp} \end{pmatrix} \begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix}. \tag{7.26}$$

In a coordinate system for which the field components are

$$E'_{\parallel} = a\cos\omega t,\ E'_{\perp} = b\sin\omega t, \tag{7.27}$$

the field traces out a right-handed ellipse with minor axis b, major axis a (if $a > b$), which follows from solving Eq. (7.27) for $\cos\omega t$ and $\sin\omega t$, then squaring and adding them. The transformation from the original coordinate system to the primed coordinate system is (Fig. 7.2)

$$\begin{aligned} \begin{pmatrix} E_{\parallel} \\ E_{\perp} \end{pmatrix} &= \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} E'_{\parallel} \\ E'_{\perp} \end{pmatrix} \\ &= \begin{pmatrix} a\cos\varphi & b\sin\varphi \\ -a\sin\varphi & b\cos\varphi \end{pmatrix} \begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix}. \end{aligned} \tag{7.28}$$

Equality of Eqs. (7.26) and (7.28) requires that

$$a_{\parallel}\cos\vartheta_{\parallel} = a\cos\varphi, \tag{7.29}$$

$$-a_{\parallel}\sin\vartheta_{\parallel} = b\sin\varphi, \tag{7.30}$$

$$a_{\perp}\cos\vartheta_{\perp} = -a\sin\varphi, \tag{7.31}$$

$$-a_{\perp}\sin\vartheta_{\perp} = b\cos\varphi. \tag{7.32}$$

Square and add the left sides of these equations and set the result equal to the sum of the squares of the right side:

$$a_{\parallel}^2 + a_{\perp}^2 = a^2 + b^2 = I. \tag{7.33}$$

We may write

$$a = A\cos\eta,\ b = A\sin\eta, \tag{7.34}$$

where $0 \le \eta \le \pi/4$ and $\tan\eta = b/a$. Now multiply Eq. (7.29) by Eq. (7.30), Eq. (7.31) by Eq. (7.32), add the result, and use the identities for the sine and cosine of the sum of angles:

$$U = 2a_{\parallel}a_{\perp}\cos\delta = (b^2 - a^2)\sin 2\varphi = -A^2\cos 2\eta\sin 2\varphi. \tag{7.35}$$

Now square Eqs. (7.32) and (7.30), take their difference, square Eqs. (7.29) and (7.31), take their difference, and finally take the difference of the result:

$$a_{\perp}^2 - a_{\parallel}^2 = (b^2 - a^2)\cos 2\varphi, \tag{7.36}$$

which yields

$$Q = a_{\parallel}^2 - a_{\perp}^2 = A^2\cos 2\eta\cos 2\varphi. \tag{7.37}$$

Finally, to obtain V, multiply Eqs. (7.29) and (7.32), Eqs. (7.30) and (7.31), and add to obtain

$$a_{\parallel}a_{\perp}\sin\delta = ab, \tag{7.38}$$

which from Eq. (7.34) yields

$$V = 2a_{\|}a_{\perp}\sin\delta = A^2\sin 2\eta. \tag{7.39}$$

If we go through the same steps for a left-circularly polarized wave, I, Q, and U are unchanged whereas V becomes

$$V = -A^2\sin 2\eta. \tag{7.40}$$

Instead of having separate sets of equations for left- and right-circularly polarized light we can combine them by allowing η to lie in the range $(-\pi/4, \pi/4)$, where negative angles correspond to left-circularly polarized light, positive angles to right-circularly polarized light. To recapitulate:

$$I = A^2,\ Q = A^2\cos 2\eta\cos 2\varphi,\ U = -A^2\cos 2\eta\sin 2\varphi,\ V = A^2\sin 2\eta, \tag{7.41}$$

$$\frac{U}{Q} = -\tan 2\varphi,\ \frac{V}{\sqrt{Q^2+U^2}} = \tan 2\eta,\ \frac{a}{b} = |\tan\eta|, \tag{7.42}$$

where $0 \leq \varphi \leq \pi$ and $-\pi/4 \leq \eta \leq \pi/4$. Because $\tan 2\varphi$ does not uniquely determine φ we need additional information: if $U < 0$, $0 < \varphi < \pi/2$, whereas if $U > 0$, $\pi/2 < \varphi < \pi$. From Eq. (7.41) it follows that I and V do not depend on the coordinate system (i.e., on φ), Q and U do, but the sum of their squares does not.

The surfaces of constant phase and amplitude for the plane wave Eq. (7.4) are infinite in extent. Thus the electric field of this wave occupies all space, which, of course, is physically unrealistic. To apply the previous analysis to real beams finite in lateral extent their properties have to be more or less laterally uniform. The Stokes parameters [Eq. (7.24)] were obtained by way of thought experiments that are easy to state but not all of them readily done in practice. Nevertheless, once we know the form of these parameters we can devise feasible ways of measuring them with readily available linear retarders and polarizing filters (see Prob. 7.36 at the end of this chapter).

7.1.4 Unpolarized and Partially Polarized Light

An electric wave described by Eq. (7.2) or Eq. (7.3) is necessarily completely polarized in that its vibration ellipse is traced out with monotonous regularity from the beginning until the end of time (actually, this time interval need not span eternity, just much longer than the period of the wave). Radiation from a microwave or radio antenna might closely fit this description because an antenna is a coherent object, its parts fixed relative to each other (on the scale of the wavelength), driven by electric currents that are more or less time-harmonic. It would take some ingenuity to make a microwave or radio antenna that *did not* radiate completely polarized waves. Radiation at much shorter wavelengths, however, often originates from vast arrays of tiny antennas (molecules) emitting more or less independently of each other, and hence we would not expect the same degree of regularity of the radiation from such sources. The extreme example of irregularity is unpolarized light whereas the extreme example of regularity is completely polarized light, both idealizations never strictly realized in nature. But what is unpolarized light?

Perhaps the simplest way to define such light is operationally, subject to previous caveats about ideal linear and circular polarizers. What kind of experimental tests can we devise to determine if a beam is unpolarized? Suppose that we transmit it through an ideal linear polarizer and discover that regardless of the orientation of its transmission axis, the transmitted irradiance is the same. This implies that there is no preferred direction of the electric field, for if there were the irradiance would vary. According to our operational definition of the Stokes parameters, $Q = U = 0$ for this beam. But wait! A circularly-polarized beam would yield the same result. So we now have to determine if the beam exhibits a preferential handedness. First transmit the beam through an ideal left-circular polarizer, then through a right-circular polarizer. If the two transmitted irradiances are equal, $V = 0$, and the electric field of the beam exhibits no preference for left-handed over right-handed rotation. Thus our operational definition of unpolarized light is that for which $Q = U = V = 0$. The Stokes parameters of *partially polarized* light also do not satisfy Eq. (7.25) but Q, U, V are not all zero.

We can put more theoretical flesh onto these bare bones by extending Eq. (7.3) to *quasi-monochromatic* radiation with (real) electric field components

$$E_{\parallel}(t) = a_{\parallel}(t)\cos\{\vartheta_{\parallel}(t) + \omega t\},\ E_{\perp}(t) = a_{\perp}(t)\cos\{\vartheta_{\perp}(t) + \omega t\}, \tag{7.43}$$

where the amplitudes and phases now vary with time but much more slowly than $\cos\omega t$. With this restriction the electric field Eq. (7.43) and its associated magnetic field *approximately* satisfy Eq. (7.5). The instantaneous Poynting vector corresponding to Eq. (7.43) is

$$\mathbf{S} = C\mathbf{k}(E_{\parallel}^2 + E_{\perp}^2), \tag{7.44}$$

the magnitude of which (within a constant factor) is

$$\begin{aligned} |\mathbf{S}| = \ & (a_{\parallel}^2\cos^2\vartheta_{\parallel} + a_{\perp}^2\cos^2\vartheta_{\perp})\cos^2\omega t \\ & +(a_{\parallel}^2\sin^2\vartheta_{\parallel} + a_{\perp}^2\sin^2\vartheta_{\perp})\sin^2\omega t \\ & -2(a_{\parallel}^2\cos\vartheta_{\parallel}\sin\vartheta_{\parallel} + a_{\perp}^2\cos\vartheta_{\perp}\sin\vartheta_{\perp})\sin\omega t\cos\omega t, \end{aligned} \tag{7.45}$$

where the amplitudes and phases may depend on time (not explicit to keep the notation uncluttered). To determine the time average of Eq. (7.45) requires evaluating integrals of the form

$$\frac{1}{\tau}\int_0^\tau f(t)\cos^2\omega t\,dt,\ \frac{1}{\tau}\int_0^\tau g(t)\sin^2\omega t\,dt,\ \frac{1}{\tau}\int_0^\tau h(t)\sin\omega t\cos\omega t\,dt. \tag{7.46}$$

We need consider only the first of these integrals because it sets the pattern for the other two.

Divide the range of integration into N equal intervals Δt:

$$\langle f\cos^2\omega t\rangle = \frac{1}{\tau}\int_0^\tau f(t)\cos^2\omega t\,dt = \frac{1}{N\Delta t}\sum_{i=1}^{N}\int_{t_i}^{t_i+\Delta t} f(t)\cos^2\omega t\,dt. \tag{7.47}$$

From the mean-value theorem of integral calculus

$$\int_{t_i}^{t_i+\Delta t} f(t)\cos^2\omega t\,dt = f(\overline{t_i})\int_{t_i}^{t_i+\Delta t}\cos^2\omega t\,dt, \tag{7.48}$$

where $t_i \le \overline{t_i} \le t_i + \Delta t$. By the definition of quasi-monochromatic light we can choose $\Delta t \gg 1/\omega$ such that $f(t)$ is approximately constant [call the value $f(\overline{t_i})$] over this time interval, and hence the integral of the cosine squared is approximately $\Delta t/2$ and the time average is approximately

$$\langle f \cos^2 \omega t\rangle \approx \frac{1}{2N}\sum_{i=1}^{N} f(\overline{t_i}). \tag{7.49}$$

Similarly,

$$\langle g \sin^2 \omega t\rangle \approx \frac{1}{2N}\sum_{i=1}^{N} g(\overline{t_i}),\ \ \langle h \sin\omega t\cos\omega t\rangle \approx 0. \tag{7.50}$$

From Eqs. (7.45), (7.49), and (7.50) it therefore follows that the time-averaged irradiance for quasi-monochromatic light is

$$I = \langle |\mathbf{S}|\rangle = \langle a_\parallel^2 + a_\perp^2\rangle = \langle E_\parallel E_\parallel^* + E_\perp E_\perp^*\rangle. \tag{7.51}$$

As previously, all common factors are omitted. Because the Stokes parameters are sums and differences of irradiances, the other three parameters for such light are given by similar expressions:

$$Q = \langle E_\parallel E_\parallel^* - E_\perp E_\perp^*\rangle = \langle a_\parallel^2 - a_\perp^2\rangle, \tag{7.52}$$

$$U = \langle E_\parallel E_\perp^* + E_\perp E_\parallel^*\rangle = \langle 2a_\parallel a_\perp \cos\delta\rangle, \tag{7.53}$$

$$V = \langle i(E_\parallel E_\perp^* - E_\perp E_\parallel^*)\rangle = \langle 2a_\parallel a_\perp \sin\delta\rangle. \tag{7.54}$$

An example of a quasi-monochromatic beam of light is collimated sunlight passed through an ordinary (as opposed to a polarizing) filter, a device that transmits light only over a band of frequencies. The spectral width of this transmitted light may be quite narrow but the amplitudes and phases of its orthogonal field components fluctuate over times large compared with the period and small compared with the response time of the detector.

According to Eq. (7.42) the ellipsometric parameters depend only on ratios of Stokes parameters, which in turn implies that they depend only on the *ratio* of the amplitudes $a_\parallel$ and $a_\perp$ and the *difference* in phases $\delta = \vartheta_\parallel - \vartheta_\perp$. Suppose that these amplitudes and phases fluctuate in time but do so synchronously, that is, they are *correlated*, the ratio of amplitudes and the difference in phases constant in time. It then follows from Eqs. (7.51)–(7.54) that $I^2 = Q^2 + U^2 + V^2$, and hence the light is completely polarized despite the fluctuations. Correlation is essential to understanding polarized light. Complete correlation corresponds to completely polarized light, no correlation to unpolarized light, and partial correlation to partially polarized light.

We may visualize Eq. (7.43) as follows. Over a time interval of several periods the electric field vector traces out a more or less definite vibration ellipse, but with the passage of time the vibration ellipse changes. If all vibration ellipses are traced out over the response time of the detector the light is unpolarized.

7.1.5 Degree of Polarization

Any beam with Stokes parameters I, Q, U, V may be considered the incoherent superposition of two beams, one unpolarized , one completely polarized:

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I_{\mathrm{u}} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} I_{\mathrm{p}} \\ Q \\ U \\ V \end{pmatrix}, \tag{7.55}$$

where

$$I_{\mathrm{p}}^2 = Q^2 + U^2 + V^2. \tag{7.56}$$

Because $I_{\mathrm{p}} \leq I$ it follows (but see Prob. 7.27) that

$$Q^2 + U^2 + V^2 \leq I^2, \tag{7.57}$$

equality holding for completely polarized light. We define the *degree of (elliptical) polarization* of this beam as the ratio of the irradiance of the polarized component to the total irradiance:

$$\frac{I_{\mathrm{p}}}{I_{\mathrm{p}} + I_{\mathrm{u}}} = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}, \tag{7.58}$$

often multiplied by 100 and expressed as a percentage ($\leq 100\%$).

We can go further and imagine the beam to be a superposition of three beams, one unpolarized, one linearly polarized, and one circularly polarized:

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I_{\mathrm{u}} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} I_{\mathrm{lp}} \\ Q \\ U \\ 0 \end{pmatrix} + \begin{pmatrix} I_{\mathrm{cp}} \\ 0 \\ 0 \\ V \end{pmatrix}, \tag{7.59}$$

where

$$I_{\mathrm{lp}} = \sqrt{Q^2 + U^2},\ I_{\mathrm{cp}} = |V|. \tag{7.60}$$

This naturally leads to definitions of the *degree of linear polarization*

$$\frac{I_{\mathrm{lp}}}{I} = \frac{\sqrt{Q^2 + U^2}}{I}, \tag{7.61}$$

and the *degree of circular polarization*

$$\frac{I_{\mathrm{cp}}}{I} = \frac{V}{I}, \tag{7.62}$$

a signed quantity: positive values correspond to right-circular polarization, negative to left-circular polarization.

To determine the degree of linear polarization of a beam, insert an ideal linear polarizer in it and measure the irradiance of the transmitted light. Suppose that the transmission axis of the polarizer makes an angle ξ (between 0 and π) with the $\mathbf{e}_\perp$ axis. The transmitted amplitude along this axis is

$$E_{\|\mathrm{i}} \sin\xi + E_{\perp\mathrm{i}} \cos\xi, \tag{7.63}$$

where the subscript i denotes components of the incident field. The components of this transmitted field are therefore

$$E_{\perp\mathrm{t}} = E_{\|\mathrm{i}} \sin\xi\cos\xi + E_{\perp\mathrm{i}} \cos^2\xi, \tag{7.64}$$

$$E_{\|\mathrm{t}} = E_{\|\mathrm{i}} \sin^2\xi + E_{\perp\mathrm{i}} \sin\xi\cos\xi. \tag{7.65}$$

From these equations and the definition of the Stokes parameters it follows that the transmitted irradiance is

$$I_\mathrm{t} = \frac{1}{2}(I_\mathrm{i} - Q_\mathrm{i}\cos 2\xi + U_\mathrm{i}\sin 2\xi). \tag{7.66}$$

The maximum and minimum of I_t occur for

$$\tan 2\xi = -\frac{U_\mathrm{i}}{Q_\mathrm{i}}. \tag{7.67}$$

The two solutions to Eq. (7.67) are separated by $\pi/2$. Without loss of generality we may take U_i and Q_i to be positive, in which instance the maximum occurs when the cosine is negative and the sine positive, whereas the minimum occurs when the cosine is positive and the sine negative:

$$I_\mathrm{max} = \frac{1}{2}(I_\mathrm{i} - Q_\mathrm{i}\cos 2\xi + U_\mathrm{i}\sin 2\xi), \tag{7.68}$$

$$I_\mathrm{min} = \frac{1}{2}(I_\mathrm{i} + Q_\mathrm{i}\cos 2\xi - U_\mathrm{i}\sin 2\xi), \tag{7.69}$$

where ξ is the solution to Eq. (7.67) for which the cosine is negative. Subtract these two equations, add them, and take their ratio:

$$\frac{I_\mathrm{max} - I_\mathrm{min}}{I_\mathrm{max} + I_\mathrm{min}} = \frac{-Q_\mathrm{i}\cos 2\xi + U_\mathrm{i}\sin 2\xi}{I_\mathrm{i}}. \tag{7.70}$$

Because of Eq. (7.67) we can write

$$Q_\mathrm{i} = -A\cos 2\xi,\ U_\mathrm{i} = A\sin 2\xi, \tag{7.71}$$

where

$$A = \sqrt{Q_\mathrm{i}^2 + U_\mathrm{i}^2}. \tag{7.72}$$

This then yields

$$\frac{\sqrt{Q_\mathrm{i}^2 + U_\mathrm{i}^2}}{I_\mathrm{i}} = \frac{I_\mathrm{max} - I_\mathrm{min}}{I_\mathrm{max} + I_\mathrm{min}}, \tag{7.73}$$

the degree of linear polarization Eq. (7.61). Now we have a procedure for measuring it: rotate a linear polarizing filter in the beam and measure the minimum and maximum irradiances. We don't need absolute irradiances because the degree of polarization is a ratio.

7.1.6 Linear Retarders and Birefringence

We noted previously that circular polarizing filters are rare. But this does not mean that we cannot readily transform unpolarized light into circularly polarized light. We can by using a sandwich composed of an ideal linear polarizing filter and an ideal *linear retarder*. A linear polarizing filter has different absorption coefficients for different orthogonal states of linear polarization. Stated another way, the imaginary parts of its refractive index are different (see Sec. 3.5.2). Thus it is hardly a stretch to imagine a medium for which the *real* parts are different, called *linear birefringence*.

We take the retarder to be a transparent (at the wavelength of interest) plate of uniform thickness illuminated at normal incidence. One direction in the plane of this plate is called the *slow axis*, the other the *fast axis* (for reasons that will become apparent). With the assumption that reflection by the plate is negligible, the electric field components in the plate along the fast and slow axes are

$$E_{\mathrm{f}} = a_{\mathrm{f}} \exp(ik_{\mathrm{f}} z - i\omega t),\ E_{\mathrm{s}} = a_{\mathrm{s}} \exp(ik_{\mathrm{s}} z - i\omega t). \tag{7.74}$$

The wavenumbers are

$$k_{\mathrm{f}} = \frac{2\pi}{\lambda} n_{\mathrm{f}},\ k_{\mathrm{s}} = \frac{2\pi}{\lambda} n_{\mathrm{s}}, \tag{7.75}$$

where n_{f} and n_{s} are the corresponding (real) refractive indices. Because of the inverse relation between phase speed and refractive index, the fast axis is so named because $n_{\mathrm{f}} < n_{\mathrm{s}}$. The phase difference of the two components after transmission a distance h is

$$\Delta\vartheta = \vartheta_{\mathrm{s}} - \vartheta_{\mathrm{f}} = \frac{2\pi h}{\lambda}(n_{\mathrm{s}} - n_{\mathrm{f}}), \tag{7.76}$$

whence the name retarder: the plate retards one phase relative to the other.

At visible and near-visible wavelengths the difference in refractive indices is a consequence of anisotropy at the molecular level, either because the medium is a crystal (with other than cubic symmetry), which gives *natural linear birefringence*, or because of non-uniform stresses in the medium, which gives *induced linear birefringence*. And, of course, birefringence could be both natural and induced (e.g., a stressed crystal). Ice is a naturally birefringent material; transparent adhesive tape is an induced birefringent material (stresses along the axis of the tape are different from those perpendicular to this axis). Although the difference in refractive indices may be, and usually is $\ll 1$, the phase difference Eq. (7.76) may be appreciable, $\pi/2$ or greater, if the plate is much thicker than the wavelength.

We now have set the stage for showing how to produce circularly polarized light. First an unpolarized beam is transmitted by an ideal linear polarizer, resulting in light linearly polarized along a direction we take to be the $\mathbf{e}_{\parallel}$ axis. Following this polarizer is an ideal linear retarder with its slow and fast axes oriented at $45°$ to the $\mathbf{e}_{\parallel}$ axis (Fig. 7.3). The real field components along the fast and slow axes transmitted by the retarder are

$$E_{\mathrm{ft}} = \frac{a}{\sqrt{2}} \cos(\vartheta_{\mathrm{f}} - \omega t),\ E_{\mathrm{st}} = \frac{a}{\sqrt{2}} \cos(\vartheta_{\mathrm{f}} - \omega t + \Delta\vartheta). \tag{7.77}$$

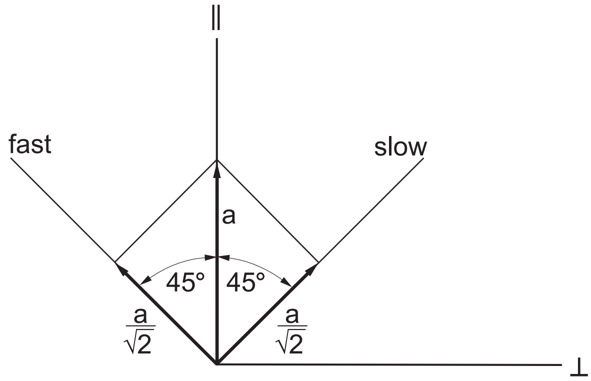

Figure 7.3: An electric field with amplitude a linearly polarized along the $\mathbf{e}_{\|}$ axis is incident on an ideal linear retarder with its slow and fast axes oriented at $45°$ to the $\mathbf{e}_{\|}$ axis. The equal-amplitude components of the incident field along the fast and slow axes undergo different phase shifts upon transmission.

If the thickness h of the retarder is such that $\Delta\vartheta = \pi/2$, these field components are

$$E_{\mathrm{ft}} = \frac{a}{\sqrt{2}}\cos(\vartheta_{\mathrm{f}} - \omega t), \; E_{\mathrm{st}} = -\frac{a}{\sqrt{2}}\sin(\vartheta_{\mathrm{f}} - \omega t), \tag{7.78}$$

which corresponds to a right-circularly polarized beam. Rotate the retarder by $90°$ around the axis defined by the beam and the transmitted light is left-circularly polarized. The requirement that the phase difference Eq. (7.76) be $\pi/2$ implies that

$$h(n_{\mathrm{s}} - n_{\mathrm{f}}) = \frac{\lambda}{4}, \tag{7.79}$$

and hence such a retarder is sometimes called a *quarter-wave plate*.

A sandwich composed of a linear polarizing filter and a quarter-wave plate oriented at $45°$ to the transmission axis of the polarizing filter is not a circular polarizing filter. And it is a one-way device: light incident from the polarizer side is transformed into circularly polarized light, whereas light incident from the retarder side is transformed into linearly polarized light.

What is the difference between birefringence and *double refraction*? All doubly refracting media are birefringent but the converse is not necessarily true. A doubly refracting medium is one with a difference in refractive indices so large that perceptible double images can be seen through it. Keep in mind that *all* solid objects made of amorphous materials (e.g., glass, plastic) always have some residual non-uniform stresses, and hence exhibit some induced birefringence. About 30 years ago one of the authors attempted to make a glass container free of birefringence. Even after many hours of annealing, the container still exhibited measurable birefringence, although to the eye it certainly was not doubly refracting.

Now consider another sandwich, a triple-decker composed of a polarizing filter, a linear retarder, and another polarizing filter with its transmission axis perpendicular to that of the first filter. Transmission of unpolarized light by the first filter results in light linearly polarized

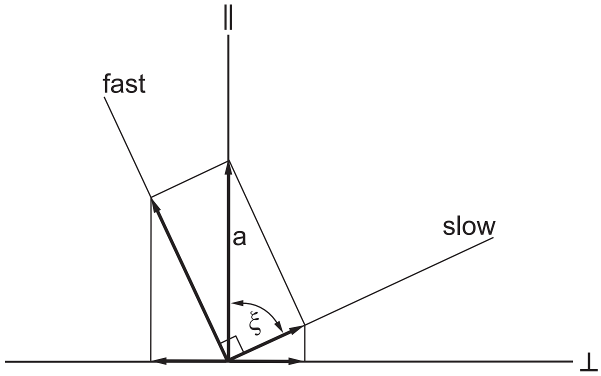

Figure 7.4: The orthogonal slow and fast axes of a linear retarder are rotated by an angle ξ relative to the reference coordinates axes $\perp$ and $\|$.

along the $\mathbf{e}_{\|}$ axis, say, with amplitude a. The field components along the slow and fast axes of the retarder are

$$a\cos\xi,\ \ a\sin\xi, \tag{7.80}$$

where ξ is the angle between the slow axis and $\mathbf{e}_{\|}$ (Fig. 7.4). After transmission by the retarder these components are

$$a\cos\xi\exp(ik_{\mathrm{s}}h),\ \ a\sin\xi\exp(ik_{\mathrm{f}}h). \tag{7.81}$$

Only the projections of these components onto the transmission axis of the final polarizing filter are transmitted, and hence the transmitted field is

$$E_{\mathrm{t}} = a\sin\xi\cos\xi\{\exp(ik_{\mathrm{s}}h) - \exp(ik_{\mathrm{f}}h)\}. \tag{7.82}$$

The transmitted irradiance is

$$I_{\mathrm{t}} = E_{\mathrm{t}}E_{\mathrm{t}}^{*} = 2a^2\sin^2\xi\cos^2\xi\{1 - \cos[(k_{\mathrm{s}} - k_{\mathrm{f}})h]\}. \tag{7.83}$$

By using the identity

$$1 - \cos x = 2\sin^2(x/2), \tag{7.84}$$

we can write Eq. (7.83) as

$$I_{\mathrm{t}} = 4a^2\sin^2\xi\cos^2\xi\sin^2\delta, \tag{7.85}$$

where the *retardance* is defined as

$$\delta = \frac{\pi h}{\lambda}(n_{\mathrm{s}} - n_{\mathrm{f}}). \tag{7.86}$$

For zero retardance, no light is transmitted by this triple-decker sandwich. Note that the retardance explicitly depends on wavelength by way of the factor $1/\lambda$ and implicitly by way of the wavelength dependence of the two refractive indices. Because of this wavelength dependence colored patterns can be seen when a birefringent medium is between crossed polarizing filters. Charles and Nancy Knight have photographed hailstones between crossed polarizing filters as a way of elucidating how hail is formed. Hailstones are agglomerations of many small crystals of different size and orientation, and hence from Eqs. (7.85) and (7.86) the wavelength dependence of transmission by each crystal is different. As a consequence, a hailstone seen through crossed polarizing filters displays a striking multi-colored mosaic, and the photographs taken by the Knights are as much art as science. Sun dogs (see Sec. 8.5.1) result from scattering of sunlight by ice crystals falling in air. Günther Können has noted that because of the birefringence of ice the (angular) position of a sun dog shifts slightly (a fraction of a degree) when observed through a polarizing filter while it is rotated.

Another observable consequence of birefringence in which the atmosphere plays a role is provided by airplane windows, which are made of tough plastic, several millimeters thick, and highly stressed. As we show in Section 7.3, skylight is partially polarized. Such light transmitted through an airplane window (i.e., retarder), then through a polarizing filter such as polarizing sunglasses or a polarizing filter for a camera, can result in striking colored patterns. You can observe them even without such a filter. As we show in the following section, a specular (mirror-like) reflecting interface is a kind of polarizing filter in that it reflects light of linear orthogonal polarization states differently. Thus a passenger in an airplane may see colored patterns in the reflected image of a window illuminated by skylight. Although we have called polarization the hidden variable, it is not completely hidden from those who know where to look.

7.2 Polarization upon Specular Reflection

As noted at the end of the previous section, specular reflection can change the polarization state of light. Consider two optically homogeneous, isotropic media separated by an optically smooth planar interface. Strictly speaking both media should be infinite for theory to be applicable, but this ideal (and unobtainable) condition is satisfied to good approximation if the dimensions of the media are much larger than the wavelength of the light of interest. A plane wave with wave vector

$$\mathbf{k}_{\mathrm{i}} = -k(\sin\vartheta_{\mathrm{i}}\mathbf{e}_{\mathrm{y}} + \cos\vartheta_{\mathrm{i}}\mathbf{e}_{\mathrm{z}}) \tag{7.87}$$

is incident on the interface from a negligibly absorbing medium (e.g., air) with wavenumber k. This wave gives rise to (i.e., excites) a reflected wave with wave vector $\mathbf{k}_{\mathrm{r}}$ and a transmitted wave with wave vector $\mathbf{k}_{\mathrm{t}}$:

$$\mathbf{k}_{\mathrm{r}} = -k(\sin\vartheta_{\mathrm{r}}\mathbf{e}_{\mathrm{y}} - \cos\vartheta_{\mathrm{r}}\mathbf{e}_{\mathrm{z}}),\ \mathbf{k}_{\mathrm{t}} = -k_{\mathrm{t}}(\sin\vartheta_{\mathrm{t}}\mathbf{e}_{\mathrm{y}} + \cos\vartheta_{\mathrm{t}}\mathbf{e}_{\mathrm{z}}), \tag{7.88}$$

where k_{t} is the wavenumber in the transmitting medium. The *plane of incidence*, which we take to be the yz-plane (Fig. 7.5), is determined by the normal to the interface and the incident

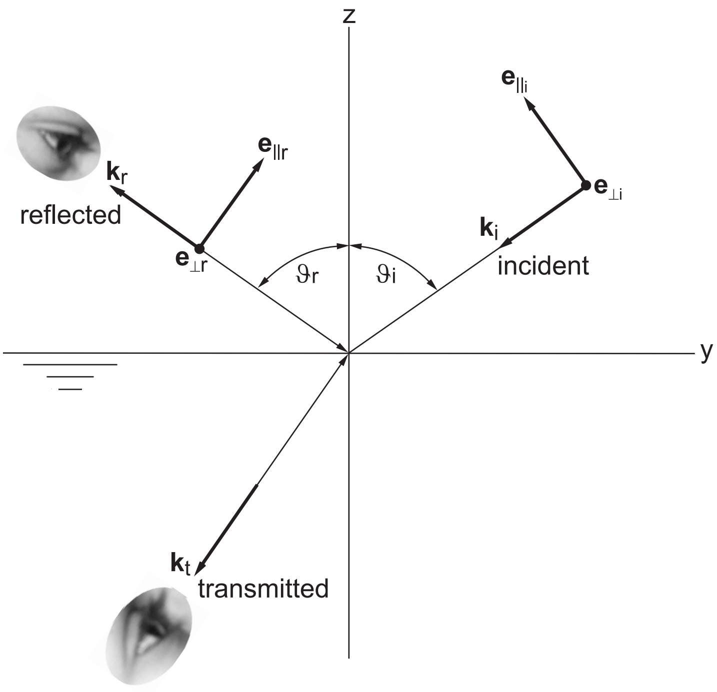

Figure 7.5: A plane wave illuminating the optically smooth interface between optical homogeneous (infinite) media gives rise to (specularly) reflected and transmitted (refracted) waves.

wave vector. All three wave vectors lie in the plane of incidence. Moreover, the angle of reflection equals the angle of incidence (law of specular reflection)

$$\vartheta_\mathrm{r} = \vartheta_\mathrm{i} \tag{7.89}$$

and ϑ_t is given by the law of refraction (Snel's law)

$$\sin\vartheta_\mathrm{i} = \frac{k_\mathrm{t}}{k}\sin\vartheta_\mathrm{t} = m\sin\vartheta_\mathrm{t}, \tag{7.90}$$

where m is the refractive index of the transmitting medium relative to that of the incident medium. Because m may be complex, so may be the angle of refraction, but the real part of this complex angle is *not* the angle of refraction defined as the angle between the real part of the complex wave vector and the normal to the interface. When m is complex, the transmitted wave is inhomogeneous (except for normal incidence, $\vartheta_\mathrm{i} = 0$): the real and imaginary parts of the complex wave vector are not parallel.

Despite what you see elsewhere, the spelling of Snel here is correct. Whatever fame Snel may deserve for discovering (empirically) the law of refraction, which has been disputed, he likely has the dubious honor of having his name misspelled more than that of any other scientist in history. Kirchhoff probably runs a close second. A resurrected Snel, upon seeing his name in hundreds of textbooks and thousands of papers, might exclaim, "What the 1!", whereas Kirchhoff's reaction might be, "Where the h?"

The incident and reflected electric field components are specified relative to orthogonal basis vectors parallel and perpendicular to the plane of incidence (Fig. 7.5), defined such that $\mathbf{e}_\perp \times \mathbf{e}_\parallel$ is in the direction of the wave vector. Note that the parallel basis vector for the incident field is *not* the same as that for the reflected field. The complex field components of the reflected field relative to those of the incident field are

$$\tilde{r}_\parallel = \frac{E_{\parallel \mathrm{r}}}{E_{\parallel \mathrm{i}}} = \frac{m\cos\vartheta_\mathrm{i} - \cos\vartheta_\mathrm{t}}{m\cos\vartheta_\mathrm{i} + \cos\vartheta_\mathrm{t}} = \frac{\tan(\vartheta_\mathrm{i} - \vartheta_\mathrm{t})}{\tan(\vartheta_\mathrm{i} + \vartheta_\mathrm{t})}, \tag{7.91}$$

$$\tilde{r}_\perp = \frac{E_{\perp \mathrm{r}}}{E_{\perp \mathrm{i}}} = \frac{\cos\vartheta_\mathrm{i} - m\cos\vartheta_\mathrm{t}}{\cos\vartheta_\mathrm{i} + m\cos\vartheta_\mathrm{t}} = \frac{\sin(\vartheta_\mathrm{i} - \vartheta_\mathrm{t})}{\sin(\vartheta_\mathrm{i} + \vartheta_\mathrm{t})}, \tag{7.92}$$

where subscripts i and r denote incident and reflected, respectively. Equations (7.91) and (7.92) are the *Fresnel coefficients*. Derived before the electromagnetic theory of light had been developed (Fresnel died in 1827), they specify the amplitude and phase of the reflected field for any angle of incidence and illuminated medium. The corresponding reflectivities for the two orthogonal polarization states are

$$R_\parallel = \left|\tilde{r}_\parallel\right|^2, \; R_\perp = \left|\tilde{r}_\perp\right|^2. \tag{7.93}$$

Underlying Eqs. (7.91) and (7.92) is the additional assumption that both media are nonmagnetic at the wavelength of interest. At normal incidence ($\vartheta_\mathrm{i} = 0°$) both reflectivities are equal,

$$R_\parallel(0°) = R_\perp(0°) = \left|\frac{m-1}{m+1}\right|^2, \tag{7.94}$$

and at glancing incidence ($\vartheta_\mathrm{i} = 90°$) both are 1 for arbitrary m. But for all intermediate angles of incidence the two are different, as, for example, those for an air–water interface illuminated by visible light (Fig. 7.6). In particular, $R_\parallel = 0$ if

$$\vartheta_\mathrm{i} + \vartheta_\mathrm{t} = \frac{\pi}{2}, \tag{7.95}$$

whereas $R_\perp$ does not vanish for any angle of incidence. From Eq. (7.90) it follows that Eq. (7.95) is equivalent to

$$\tan\vartheta_\mathrm{i} = m. \tag{7.96}$$

Equation (7.96) can be satisfied strictly only for m real.

Because $R_\parallel = 0$ for the angle of incidence satisfied by Eq. (7.96), unpolarized light reflected at this angle is 100% linearly polarized perpendicular to the plane of incidence. This angle is called the *polarizing angle* or the *Brewster angle* to honor Sir David Brewster, who first discovered it empirically. Brewster's paper, published in 1815, has aged well. Written with a clarity almost banned from scientific writing today, it ends with a touching homage to Etienne-Louis Malus, who discovered polarization upon reflection in 1809, coined the term

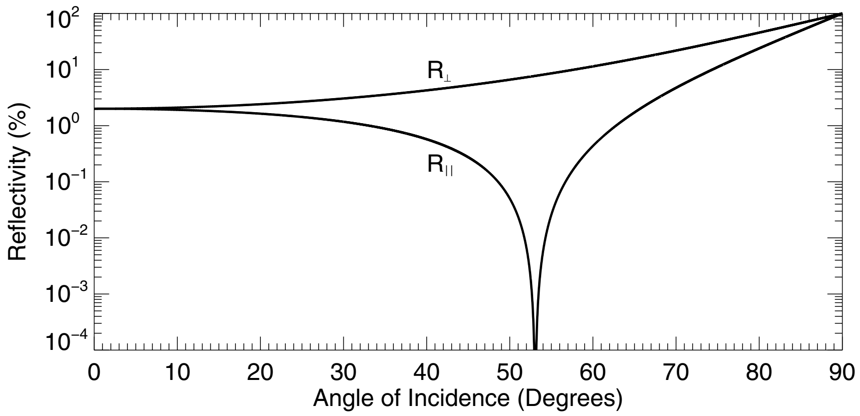

Figure 7.6: Reflectivities of water at visible wavelengths ($n = 1.33$) for incident light polarized parallel ($\|$) and perpendicular ($\perp$) to the plane of incidence.

polarization by way of a (faulty) analogy with magnetic poles, but failed to recognize the regular law later enunciated by Brewster: "*The index of refraction is the tangent of the angle of polarisation.*" Brewster measured the polarizing angle for more than a dozen transparent substances and compared its value calculated from the tangent equation, finding agreement, on average, to within $15'$. But Brewster knew that he hadn't derived this equation: "In these enquiries I have made use of no hypothetical assumptions... the language of theory has been occasionally employed, but the terms thus introduced are merely expressive of experimental results... When discoveries shall have accumulated a greater number of facts, and connected them together with general laws, we may then safely begin... to speculate respecting the cause of those wonderful phenomena which light exhibits under all its various modifications." These "general laws" had to wait a few more years for Fresnel and half a century for the glorious synthesis by Maxwell.

In a paper with the provocative title "Would Brewster recognize today's Brewster angle?", Akhlesh Lakhtakia critically examined the ways in which the term Brewster angle is used today. In modern textbooks it is defined most often by the condition $R_{\|} = 0$, but Brewster would not have known this and to him the angle now bearing his name is the angle of incidence (polarizing angle) for which reflected light is completely linearly polarized given incident unpolarized light. Moreover, Lakhtakia showed that there are isotropic media for which the polarizing angle and the angle of zero reflection for the parallel component are *not* the same.

You sometimes encounter the assertion that reflection by metals does not polarize incident unpolarized light. Although Eq. (7.96) does not have a real solution when the imaginary part of m is not negligible, metals do exhibit an angle of reflection, sometimes called the *pseudo-Brewster angle*, at which the degree of polarization is a maximum. And it can be surprisingly high, well over 50%, especially for metals (e.g., iron and chromium) with reflectivities lower than those of more conductive metals such as silver and aluminum. One reason for the misconception that light reflected by metals is unpolarized is that the pseudo-Brewster angle is typically within 10° or so from glancing incidence.

Yet another misconception about polarization is that light emitted by incandescent bodies is unpolarized, which may have its origins in the fact that radiation emitted by blackbodies (which don't exist) is unpolarized. Consider an optically smooth and homogeneous medium in air, sufficiently thick that transmission by it is negligible. With this assumption, emissivity is 1 minus reflectivity (see Sec. 1.4.1), and because reflectivity depends on polarization so does emissivity:

$$\varepsilon_{\|} = 1 - R_{\|}, \; \varepsilon_{\perp} = 1 - R_{\perp}. \tag{7.97}$$

These two emissivities are equal for normal and glancing directions, but for real bodies are unequal in all intermediate directions. Both are 1 in all directions for a blackbody ($R_{\|} = R_{\perp} = 0$).

Although the emphasis in this section is on polarization upon specular reflection, because it is relatively easy to observe, the state of polarization of incident radiation also can be changed upon transmission. The Fresnel coefficients for reflection, Eqs. (7.91) and (7.92), are accompanied by two for transmission:

$$\tilde{t}_{\|} = \frac{E_{\|\mathrm{t}}}{E_{\|\mathrm{i}}} = \frac{2\cos\vartheta_{\mathrm{i}}}{m\cos\vartheta_{\mathrm{i}} + \cos\vartheta_{\mathrm{t}}}, \tag{7.98}$$

$$\tilde{t}_{\perp} = \frac{E_{\perp\mathrm{t}}}{E_{\perp\mathrm{i}}} = \frac{2\cos\vartheta_{\mathrm{i}}}{\cos\vartheta_{\mathrm{i}} + m\cos\vartheta_{\mathrm{t}}}, \tag{7.99}$$

where the subscript t denotes transmission.

7.2.1 Scattering Interpretation of Specular Reflection and the Brewster Angle

According to Eq. (7.95) the degree of polarization of specularly reflected light is a maximum (100%) when the angle between the reflected and transmitted waves in a negligibly absorbing medium is $90°$. As we show in the following section, the degree of polarization of light scattered by a spherically symmetric dipole, again for unpolarized incident light, is also a maximum (100%) when the angle between the incident and scattered waves (scattering angle) is $90°$. This is not a coincidence but rather a consequence of the same underlying physics.

There are important differences between electromagnetic waves and, say, acoustic or water waves. The mathematics may be similar but the physics is not. People who study only the mathematics of wave motion see no fundamental difference between electromagnetic waves and other kinds, but the physical differences are profound. Acoustic and water waves require a material medium whereas electromagnetic waves do not: they propagate even in free space. And matter, appearances to the contrary, is almost entirely free space sparsely populated by charges, which can be acted upon by electromagnetic fields. What this means is that an incident electromagnetic wave exists everywhere in an illuminated medium. Contrast this with, say, acoustic waves incident on a wall in air. These waves do not literally penetrate the wall because they cannot carry their propagating medium (air) with them. But an incident electromagnetic wave penetrates an illuminated medium and excites *all* the molecules in it to radiate (scatter) waves. These scattered waves superpose in such a way that a total scattered wave is produced that interferes with the incident wave. We cannot observe these two waves separately, only their coherent superposition. When the illuminated medium is optically smooth and homogeneous the net result of this superposition is a reflected wave and a refracted wave with wavenumber different from that of the incident wave. This is only approximately true because a small amount of light also is scattered in directions other than the two special directions of reflection and refraction.

The Fresnel coefficients, Eqs. (7.91) and (7.92), usually are derived from continuum electromagnetic theory in which the graininess of matter is hidden from view. But that doesn't mean that this graininess isn't operating behind the scenes. Indeed, Bill Doyle showed that despite their macroscopic derivation the Fresnel coefficients can be made to betray their microscopic origins if properly interrogated. In particular, he showed that the Fresnel coefficients can be written as the product of a scattering function characteristic of an isolated dipole and a function characteristic of the coherently excited array of dipoles. Thus an illuminated medium is a *phased array* of dipolar antennas, which accounts for the highly directional radiation pattern of reflected and refracted waves. Radio antennas often are *designed* to produce highly directional beams; every specular reflector does this naturally.

Appearances to the contrary, specular reflection does not occur *at* an interface but rather *because* of it. An incident electromagnetic wave excites every molecule in a medium, not just those at its surface. As Doyle noted in his superb expository paper on the scattering interpretation of specular reflection, "In most discussions of Brewster's law the location of the dipoles responsible for the creation of the reflected beam, while usually not well defined, is put somewhere in the neighborhood of the interface. Actually, *all* of the dipoles together create the reflected beam." Thus notions about electromagnetic waves bouncing and bending are metaphorical, not to be taken literally. To do otherwise is to hobble one's thinking, not only getting the physics wrong but missing opportunities. Although it is difficult to derive the Fresnel coefficients from a microscopic point of view, knowing that scattering is ultimately the result of excitation of coherent arrays of dipoles can lead to solutions to problems of scattering by particles. Mie theory, discussed briefly in Section 3.5.3, is a continuum theory of scattering by a sphere, the solution to a boundary-value problem. But results similar to those given by Mie theory can be obtained from a discrete theory in which boundaries do not exist.

7.2.2 Transformations of Stokes Parameters: The Mueller Matrix

Up to this point we have determined the state of polarization of light transmitted by polarizers and retarders and their combinations by considering how these optical elements transform field components (amplitudes and phases). This was done purposely to give insight into what these optical elements do. But because we usually measure irradiances, we could have skipped fields and gone directly to irradiances. Thus if we denote by $\mathbf{I}_{\text{in}}$ the Stokes parameters of an input beam and by $\mathbf{I}_{\text{out}}$ those of an output beam that results from interaction of the input beam with any optical element, the two sets of Stokes parameters, written as column matrices, are related by

$$\mathbf{I}_{\text{out}} = \mathbf{M}\mathbf{I}_{\text{in}}, \qquad (7.100)$$

where $\mathbf{M}$ is a 4×4 matrix, sometimes called the *Mueller matrix*, characteristic of the optical element. The advantage of this approach is that for an input beam of *any* state of polarization we can determine that of the output beam by matrix multiplication without getting ensnarled in amplitudes and phases. Another use of this matrix algebraic approach is to determine how a series of optical elements (polarizing filters, retarders, reflectors, suspensions of particles, etc.) transforms the polarization state of a beam. Thus if $\mathbf{M}_1$, $\mathbf{M}_2$,..., $\mathbf{M}_n$ are the Mueller matrices for a series of optical elements encountered by an input beam in the order of the subscripts, by matrix multiplication we have

$$\mathbf{I}_{\mathrm{out}} = \mathbf{M}_n \mathbf{M}_{n-1} \dots \mathbf{M}_2 \mathbf{M}_1 \mathbf{I}_{\mathrm{in}}. \tag{7.101}$$

Underlying the strict validity of Eq. (7.101) is the assumption that the optical elements act independently of each other. Even if they are mutually incoherent, they still act in concert to some degree. For example, we show in Section 5.1 that the transmissivity of two identical (incoherent) plates is not exactly equal to the product of their transmissivities, although this may be a good approximation if the square of the reflectivity of a single plate is $\ll 1$. And if the optical elements are coherent, Eq. (7.101) may not even be approximately correct. An example is a multi-layer interference filter, a set of N negligibly absorbing identical double layers, each member with a suitably chosen (different) thickness (less than the wavelength) and refractive index. The transmissivity of the set decreases much more rapidly with N than that of a double layer raised to the power N. Indeed, this is how highly reflecting mirrors are made. Even the most reflective metallic mirror can't come nearly as close to 100% reflectivity as can a multi-layer interference filter the layers of which have optical properties quite different from those of metals. The key word here is "interference". If interference between optical elements is not negligible, Eq. (7.101) may not be valid. Nevertheless, if used with caution Mueller matrix algebra can save considerable effort and reduce the chances of errors in analysis. Its disadvantage is that one loses sight of what is happening physically, getting answers at the expense of understanding.

One of the simplest Mueller matrices is for specular reflection. From the Fresnel coefficients and the definition of the Stokes parameters we can obtain the reflected Stokes parameters as linear functions of the incident Stokes parameters. For example, it follows from Eqs. (7.15), (7.91) and (7.92) that

$$I_{\mathrm{r}} + Q_{\mathrm{r}} = \left|\tilde{r}_{\|}\right|^2 (I_{\mathrm{i}} + Q_{\mathrm{i}}),\ I_{\mathrm{r}} - Q_{\mathrm{r}} = \left|\tilde{r}_{\perp}\right|^2 (I_{\mathrm{i}} - Q_{\mathrm{i}}). \tag{7.102}$$

Similar algebraic manipulation yields the entire Mueller matrix for specular reflection:

$$\begin{pmatrix} I_{\mathrm{r}} \\ Q_{\mathrm{r}} \\ U_{\mathrm{r}} \\ V_{\mathrm{r}} \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & 0 & 0 \\ R_{12} & R_{11} & 0 & 0 \\ 0 & 0 & R_{33} & R_{34} \\ 0 & 0 & -R_{34} & R_{33} \end{pmatrix} \begin{pmatrix} I_{\mathrm{i}} \\ Q_{\mathrm{i}} \\ U_{\mathrm{i}} \\ V_{\mathrm{i}} \end{pmatrix}, \tag{7.103}$$

where

$$R_{11} = \frac{1}{2}(|\tilde{r}_{\|}|^2 + |\tilde{r}_{\perp}|^2) = \frac{1}{2}(R_{\|} + R_{\perp}), \tag{7.104}$$

$$R_{12} = \frac{1}{2}(|\tilde{r}_{\|}|^2 - |\tilde{r}_{\perp}|^2) = \frac{1}{2}(R_{\|} - R_{\perp}), \tag{7.105}$$

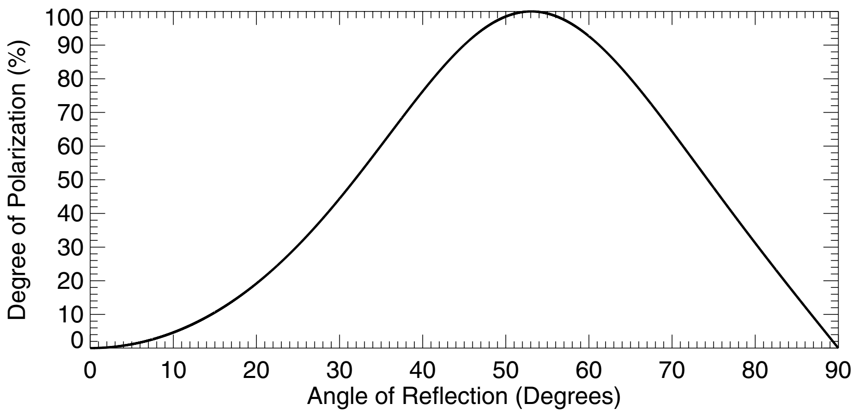

Figure 7.7: Degree of polarization of reflected light for unpolarized visible light incident at an air–water interface.

$$R_{33} = \frac{1}{2}(\tilde{r}_\parallel \tilde{r}_\perp^* + \tilde{r}_\parallel^* \tilde{r}_\perp) = \Re\{\tilde{r}_\parallel \tilde{r}_\perp^*\}, \tag{7.106}$$

$$R_{34} = \frac{i}{2}(\tilde{r}_\parallel^* \tilde{r}_\perp - \tilde{r}_\parallel \tilde{r}_\perp^*) = \Im\{\tilde{r}_\parallel \tilde{r}_\perp^*\}. \tag{7.107}$$

From these equations various results follow readily. For example, R_{11}, the arithmetic average of the two reflectivities for orthogonal polarization states, is the reflectivity for unpolarized incident light, which we could have guessed. The degree of linear polarization of reflected light, given incident unpolarized light, is

$$-\frac{R_{12}}{R_{11}} = \frac{R_\perp - R_\parallel}{R_\perp + R_\parallel}, \tag{7.108}$$

where the minus sign is introduced to make this quantity positive for reflection described by Eqs. (7.91) and (7.92).

Textbooks often are strangely silent about reflected light at angles other than the Brewster angle, the implication being that at such angles the reflected light is still unpolarized. But if reflected light were polarized only at the Brewster angle, polarizing sunglasses would not be very useful. Their function is to reduce reflection (glare) by non-metallic interfaces (e.g., water, painted hoods of automobiles) for directions in which the wearers of such sunglasses are likely to be looking. Brewsterists, members of an obscure religious sect, wear polarizing sunglasses and tilt their heads so that they see specular reflections only at the Brewster angle. But sunglasses are effective even for non-Brewsterists (called infidels) because the degree of polarization of specularly reflected light is high ($> 50\%$) over a wide range of angles around the Brewster angle. An example is shown in Fig. 7.7, the degree of polarization of visible light, for incident unpolarized light, reflected by water. You can verify the qualitative correctness of this figure from the waxing and waning of sun glint from water or glare from all kinds of smooth surfaces (polished floors, furniture, etc.) seen through a rotated polarizing filter.

Regardless of the illuminated medium and direction of incident light, it follows from Eq. (7.103) that reflection of unpolarized light cannot yield light with a degree of circular polarization ($V_\mathrm{r} \neq 0$). To obtain such light requires incident light (partially) polarized *obliquely* to the plane of incidence ($U_\mathrm{i} \neq 0$):

$$V_\mathrm{r} = -R_{34} U_\mathrm{i}. \tag{7.109}$$

But $R_{34} = 0$ for a negligibly absorbing medium (m real). Moreover, even for an arbitrary absorbing medium, $R_{34} = 0$ for normal (and glancing) incidence. Thus to obtain elliptically polarized reflected light requires obliquely polarized light incident on a metal, or material with appreciable absorption at the wavelength of interest, at non-normal incidence (but see Prob. 7.33 for an exception).

7.3 Polarization by Dipolar Scattering: Skylight

In the previous section, in which we consider specular reflection by an infinite interface, the plane of incidence is defined by the normal to the interface and the direction (wave vector) of the incident wave. But we could define this plane by the directions of the incident and reflected (or refracted) waves. Indeed, this is what is done in problems of scattering by finite objects: the *scattering plane* is defined by the direction of the incident wave and the Poynting vector of the scattered wave. In Sections 3.4 and 3.5 we discuss the frequency and angular but not polarization dependence of scattering. Here we tie up this loose end.

Consider a spherically symmetric electric dipole illuminated by a plane harmonic wave. We take the surrounding medium to be negligibly absorbing. We may consider the dipole to be a sphere of vanishingly small dimensions relative to the wavelength. The incident wave excites the dipole to radiate (i.e., scatter) in all directions. The direction of the incident wave and that of any Poynting vector of the scattered wave not parallel to the incident wave define a unique scattering plane, and we confine ourselves to all scattering directions lying in that plane. The electric fields of the incident and scattered waves lie in different planes perpendicular to the scattering plane (Fig. 7.8). Denote by $E_{\perp\mathrm{i}}$ and $E_{\|\mathrm{i}}$ the two components of the incident wave, perpendicular and parallel, respectively, to the scattering plane; $E_{\perp\mathrm{s}}$ and $E_{\|\mathrm{s}}$ are the components of the scattered field. Because of the assumed spherical symmetry of the dipole, incident light polarized perpendicular to the scattering plane can give rise to only a scattered field perpendicular to this plane. The same is true for incident light polarized parallel to the scattering plane. And because of the linearity of the equations of the electromagnetic field these scattered field components, like specularly reflected components, are proportional to the incident field components that excite them:

$$E_{\perp\mathrm{s}} \propto E_{\perp\mathrm{i}},\ E_{\|\mathrm{s}} \propto E_{\|\mathrm{i}}. \tag{7.110}$$

When the incident field is perpendicular to the scattering plane the field scattered by a dipole cannot depend on the scattering angle within that plane. You can grasp this by looking at a needle perpendicular to a sheet of paper. As you change your viewing direction within this sheet, the needle always looks the same. But two or more parallel needles separated by a distance that is not small compared with the wavelength would look different in different

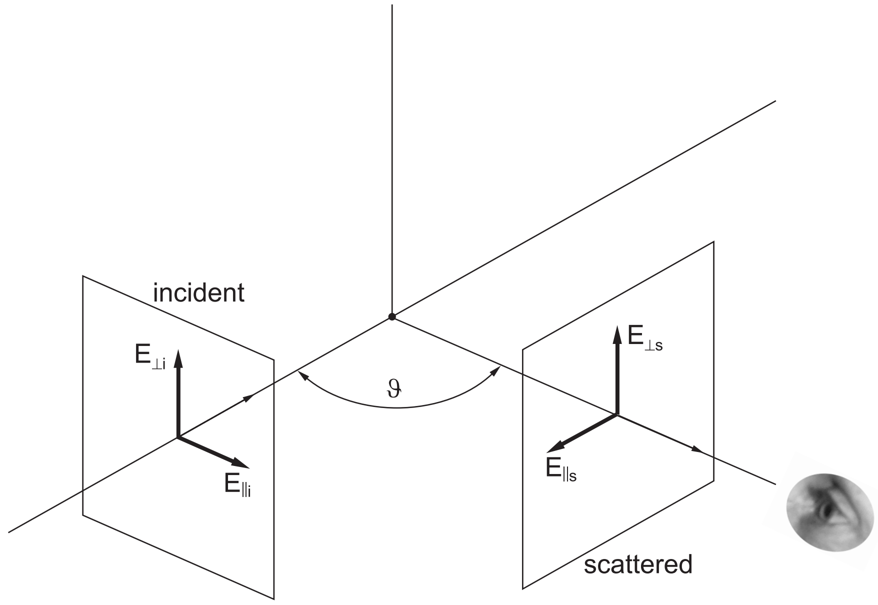

Figure 7.8: Two directions, those of incident and scattered waves, determine a plane, the scattering plane. Incident and scattered fields can be resolved into components perpendicular ($\perp$) and parallel ($\|$) to this plane. The basis vectors for the incident electric field are fixed whereas those for the scattered field change with scattering angle ϑ.

directions (see also Sec. 3.4). This independence of the field scattered by a dipole on direction when the incident field is perpendicular to the scattering plane is completely general, but when the incident field is parallel to the scattering plane, we have to impose an additional condition. Imagine a spherical polar coordinate system (r, ϑ, φ) to be centered on the dipole. At any point, the scattered field has a radial component (r-component) as well as tangential components. But the radial component decreases more rapidly with increasing r than do the tangential components. At sufficiently large distances relative to the wavelength, in what is called the *far field*, the radial component becomes negligible. We can make a simple, but not rigorous, argument for the angular dependence of the scattered far field when the incident radiation is parallel to the scattering plane. Figure 7.9 depicts the field $\mathbf{E}_\mathrm{d}$ at a dipole, excited by an incident parallel field, which gives rise to scattered field components both parallel (radial) E_dr and perpendicular (tangential) E_dt to the scattering direction. But in the far field the radial component is negligible, leaving only the tangential component, which is proportional to $E_\mathrm{d} \cos\vartheta$, where ϑ is the scattering angle. By this crude argument we conclude that in the far field

$$E_{\|\mathrm{s}} \propto E_{\|\mathrm{i}} \cos\vartheta. \tag{7.111}$$

We must mention an additional dependence of the field components even though it plays no essential role in our discussion of polarization. Imagine a sphere of arbitrary radius r

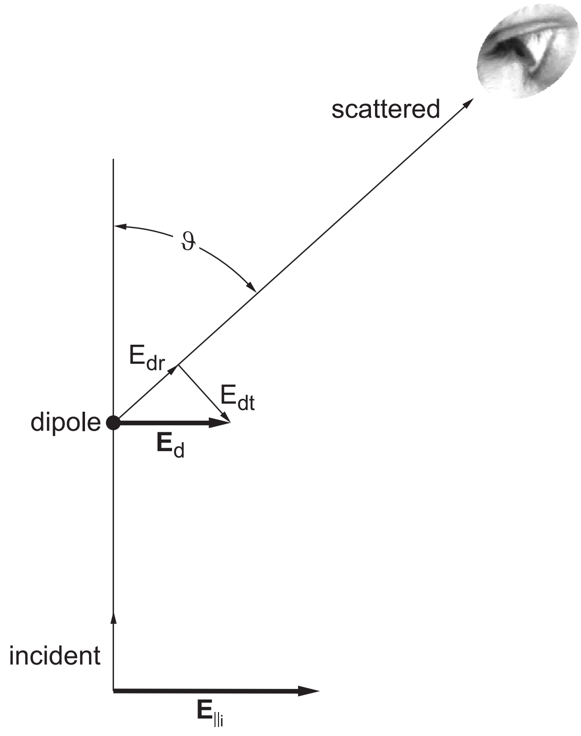

Figure 7.9: An incident electric field (in the plane of the diagram) excites a parallel field $\mathbf{E}_{\mathrm{d}}$ scattered by a spherically symmetric dipole. The radial (r) and tangential (t) components of this scattered field decrease with increasing distance at different rates. At sufficiently large distances (compared with the wavelength), the radial component is negligible compared with the tangential component.

centered on the dipole. The integral

$$\int_0^{2\pi} \int_0^{\pi} S_{\mathrm{r}} r^2 \sin\vartheta \; d\vartheta \; d\varphi, \tag{7.112}$$

over this spherical surface, where S_{r} is the radial component of the scattered Poynting vector, is the radiant energy scattered in all directions. Because the surrounding medium is negligibly absorbing, this energy is conserved, and hence Eq. (7.112) must be independent of r. A sufficient condition that this requirement be satisfied is that S_{r} be inversely proportional to r^2, which it is in the far field. This also implies that the electric field components are inversely proportional to r. But for our purposes we can write the relation between incident and scattered field components as

$$E_{\perp\mathrm{s}} = C E_{\perp\mathrm{i}}, \; E_{\|\mathrm{s}} = C \cos\vartheta E_{\|\mathrm{i}}, \tag{7.113}$$

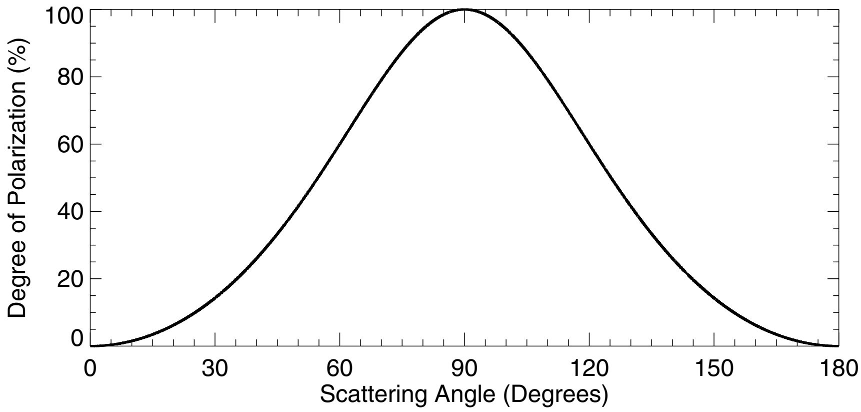

Figure 7.10: Degree of polarization of light scattered by a spherically symmetric dipole illuminated by unpolarized light.

where C incorporates all factors irrelevant to the polarization state (but not magnitude) of the scattered field. C is analogous to $\tilde{r}_\perp$, $C\cos\vartheta$ to $\tilde{r}_\|$, and so the Mueller matrix has the same form as Eq. (7.103):

$$\begin{pmatrix} \frac{1}{2}(\cos^2\vartheta+1) & \frac{1}{2}(\cos^2\vartheta-1) & 0 & 0 \\ \frac{1}{2}(\cos^2\vartheta-1) & \frac{1}{2}(\cos^2\vartheta+1) & 0 & 0 \\ 0 & 0 & \cos\vartheta & 0 \\ 0 & 0 & 0 & \cos\vartheta \end{pmatrix}, \tag{7.114}$$

where we omit any factor common to all matrix elements. The Mueller matrix for scattering by a particle or molecule is often called the *scattering matrix*.

From Eq. (7.114) it follows that the degree of linear polarization of scattered light, for incident unpolarized light, is

$$\frac{1-\cos^2\vartheta}{1+\cos^2\vartheta}, \tag{7.115}$$

which is shown in Fig. 7.10. At a scattering angle of 90° the scattered light is 100% polarized perpendicular to the scattering plane and is more than 50% polarized over a swath about 70° wide. For incident unpolarized light, the scattered irradiance varies with scattering angle according to

$$\frac{1}{2}(1+\cos^2\vartheta). \tag{7.116}$$

Scattering described by Eq. (7.116) is not isotropic, but is equal in the forward and backward hemispheres. If the incident light is polarized perpendicular to a particular scattering plane, scattering in that plane is indeed isotropic – but only in directions in that plane, not in all directions. As noted previously (Sec. 5.2), isotropic light scatterers do not exist, and now we can better understand why: light is a vector wave.

In Section 3.4 we make simple interference arguments that scattering in the forward direction should increase more rapidly with particle size than in any other direction. In Section 3.5 we support these arguments with calculations for spheres. Because Eq. (7.116) for a spherically symmetric electric dipole shows equal forward and backward scattering, we might be tempted to conclude that forward scattering is *never* less than backward scattering. But here again there is an exception, and it is related to the term "electric" qualifying dipole (absent in Ch. 3). Incident light also can excite *magnetic dipole radiation* by a molecule or particle small compared with the wavelength. An electric dipole (Sec. 2.6) is two charges equal in magnitude, opposite in sign, its (electric) dipole moment the magnitude of the charge times the distance between them. A magnetic dipole is a small current loop, its magnetic dipole moment the current times the area enclosed by the loop. Usually, magnetic dipole radiation from molecules and small particles is much smaller than electric dipole radiation. One exception is tiny metallic particles at far infrared frequencies. The conductivity of metals depends on frequency, and as a rule the lower the frequency the higher the conductivity. At low frequencies the high conductivity of, and hence high current in, a small metallic sphere may result in a magnetic moment sufficiently large to give rise to magnetic dipole radiation comparable with electric dipole radiation. Moreover, these two dipole fields interfere in such a way that forward scattering can be considerably *less* than backward scattering, almost a factor of 10. As far as we know, this has no consequences for the atmosphere, but it is well to be aware of at least one exception to the general rule that forward scattering is greater than or at most equal to backward scattering. There is always an exception to every rule, even the rule that there is always an exception.

7.3.1 Polarization of Skylight

You sometimes encounter assertions that skylight is 100% polarized at 90° from the sun, presumably because of Eq. (7.115). Skylight is *never* 100% polarized in any direction, and there is more to the story of skylight polarization than this simple equation tells. All the reasons for the departure from 100% are variations on the same theme, and hence we can explain them with the same analysis.

Suppose that two beams with Stokes parameters $\{I_\mathrm{k}, Q_\mathrm{k}, U_\mathrm{k}, V_\mathrm{k}\}$ and degree of polarization P_k, where $k = 1, 2$, are superposed incoherently. The degree of polarization of the resultant beam is

$$P_{12} = \frac{\sqrt{(Q_1 + Q_2)^2 + (U_1 + U_2)^2 + (V_1 + V_2)^2}}{I_1 + I_2}. \tag{7.117}$$

The Stokes parameters can be looked upon as specifying the coordinates of a point in a 4-dimensional space (Stokes space). If we denote a position vector in this space by $\mathbf{A}$ we can write Eq. (7.117) as

$$P_{12}^2 = \frac{P_1^2 I_1^2 + P_2^2 I_2^2 + 2\mathbf{A}_1 \cdot \mathbf{A}_2 - 2I_1 I_2}{(I_1 + I_2)^2}, \tag{7.118}$$

where $\mathbf{A}_1 \cdot \mathbf{A}_2 = I_1 I_2 + Q_1 Q_2 + U_1 U_2 + V_1 V_2$ is the scalar (dot) product of the two vectors. From the identity

$$2I_1 I_2 = (I_1 + I_2)^2 - I_1^2 - I_2^2 \tag{7.119}$$

and the inequality

$$\mathbf{A}_1 \cdot \mathbf{A}_2 \leq |\mathbf{A}_1||\mathbf{A}_2| = I_1 I_2 \sqrt{P_1^2 + 1}\sqrt{P_2^2 + 1} \tag{7.120}$$

it follows that

$$\sqrt{P_{12}^2 + 1} \leq R_1 \sqrt{P_1^2 + 1} + R_2 \sqrt{P_2^2 + 1}, \tag{7.121}$$

where $R_\mathrm{k} = I_\mathrm{k}/(I_1 + I_2)$. Equality holds only if both beams are polarized in the same way, that is, have identical vibration ellipses and degrees of polarization. If the two beams are not identical in this sense and we take

$$P_2 \geq P_1, \tag{7.122}$$

it follows from Eq. (7.121) that

$$\sqrt{P_{12}^2 + 1} < \sqrt{P_2^2 + 1}, \tag{7.123}$$

and because the degree of polarization is positive,

$$P_{12} < P_2. \tag{7.124}$$

Thus if two beams (other than those with the same polarization state) are combined incoherently, the degree of polarization of the resultant is less than that of the beam with the highest degree of polarization. And what is true of two beams is true for any number of beams. We now have the tools necessary to explain all the reasons why skylight is not 100% polarized at $90°$ from the sun or in any other direction.

First consider the least important reason. An observer looking in a direction at $90°$ to light in a particular direction from the sun receives sunlight scattered over a small range of angles (the angular width of the sun) near $90°$. Even if this scattered light were 100% polarized at $90°$, it is not at neighboring angles. Scattering at all these angles contributes to the light received by the observer, and hence from Eq. (7.124) the degree of polarization must be less than 100%.

Equation (7.113) is valid only for a spherically symmetric dipolar scatterer. But the dominant molecules in Earth's atmosphere are the *linear* molecules O_2 and N_2, which are not spherically symmetric. A linear molecule oriented perpendicular to the scattering plane does indeed result in 100% polarized scattered light at $90°$. But if the orientation of the molecule is changed, the degree of polarization of the scattered light in that direction is less than 100%. An observer looking in a particular direction receives the incoherent sum of light scattered by an ensemble of randomly oriented asymmetric molecules, because of which the degree of polarization must be less than 100%. Molecular asymmetry alone reduces the maximum degree of polarization of sunlight scattered by air to around 94%.

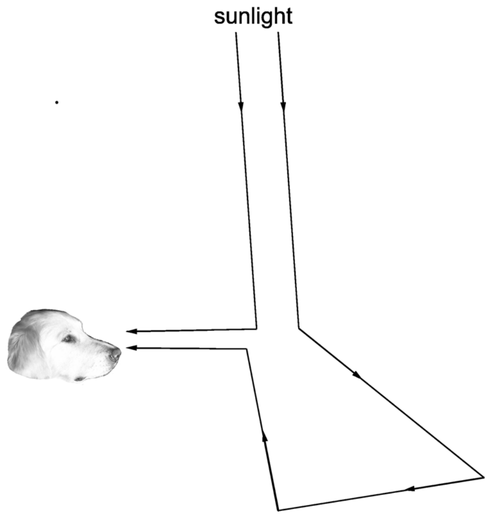

Figure 7.11: Light from the sky is the sum of successively decreasing contributions from single scattering, double scattering, triple scattering, and so on, of sunlight. Depicted here are the contributions to the total light at 90° from the direction of the sun as a result of single and quadruple scattering.

Multiple scattering of sunlight by the atmosphere is never completely absent. An observer looking in a direction 90° to light from the sun receives singly scattered light, doubly scattered light, triply scattered light, and so on (see Fig. 7.11). The degree of polarization of the singly scattered light is a maximum, that of the multiply scattered components being, in general, less. So again, by Eq. (7.124), multiple scattering in the atmosphere can only reduce the degree of polarization below the theoretical maximum value. Reflection by the ground is just another form of multiple scattering.

The atmosphere is rarely, if ever, entirely free of particles. Even scattering of unpolarized light by spheres (Sec. 7.4), if they are comparable with or larger than the wavelength, does not result in 100% polarization at 90° or any scattering angle. If the particles are small compared with the wavelength but nonspherical and randomly oriented, the degree of polarization of the light scattered by them is less than 100% at 90°. Moreover, *all* particles increase the scattering optical thickness of the atmosphere, and hence increase multiple scattering.

The finite angular width of the sun, molecular asymmetry, multiple scattering, reflection by the ground, and scattering by particles are all variations on the same underlying cause of why sunlight scattered at 90° is not 100% polarized: the incoherent superposition of beams with different degrees of polarization can only reduce the degree of polarization.

If we set $m = 1$ in Brewster's law [Eq. (7.96)] $\vartheta_{\mathrm{i}} = 45^\circ$. The angle between the incident and reflected waves can be considered a kind of scattering angle. Thus a scattering angle of 90° at which polarization is a maximum, expressed in the language of specular reflection, corresponds to a polarizing angle of 45°. Indeed, Brewster noted that "if we take the refractive power of air at 1.00031 the polarizing angle will be $45^\circ\,00'\,32''$, a result which agrees most strikingly with the observed angle." Well, yes and no. According to the Fresnel coefficients, which underlie Brewster's law, when $m = 1$ the interface disappears and reflection is zero. So although these equations correctly predict the polarizing angle for air, as we might expect given the scattering interpretation of reflection, they are powerless to say anything about the magnitude of scattering by air. Equation (7.115) applies to a *single*, spherically symmetric dipolar scatterer, whereas Eq. (7.108) applies to a coherent array of *many* dipolar scatterers. In the limit $m \rightarrow 1$ Eq. (7.115) approaches Eq. (7.108) if we call the scattering angle for specular reflection that between the reflected and transmitted waves. Brewster got the right answer by (slightly) wrong reasoning because of the similar underlying physical mechanisms for specular reflection (scattering by a coherent dipolar array) and for scattering of sunlight by air (scattering by an incoherent dipolar array).

An apparent conflict between polarization upon scattering by small particles and Brewster's law troubled John Tyndall, who did many of the early experimental investigations of light scattering by suspensions of particles. One of the opening paragraphs of Lord Rayleigh's famous 1871 paper "On the light from the sky, its polarization and colour" begins by addressing what Tyndall "felt as a difficulty": "Tyndall says, '…the polarization of the beam by the incipient cloud has thus far proved to be *absolutely independent of the polarizing-angle*. The law of Brewster does not apply to matter in this condition; and it rests with the undulatory theory to explain why. Whenever the precipitated particles are sufficiently fine, no matter what the substance forming the particles may be, the direction of maximum polarization is at right angles to the illuminating beam, the polarizing angle for matter in this condition being invariably $45°$. This I consider to be a point of capital importance…' Rayleigh responds: "As to the importance there will not be two opinions; but I venture to think that the difficulty is imaginary and is caused mainly by the misuse of the word reflection. Of course there is nothing in the etymology of reflection or refraction to forbid their application in this sense; but the words have acquired technical meanings, and become associated with certain well-known laws called after them. Now a moment's consideration of the principles according to which reflection and refraction are explained in the wave theory is sufficient to show that they have no application unless the surface of the disturbing body is larger than many square wave-lengths; whereas the particles to which the sky is supposed to owe its illumination must be *smaller* than the wave-length…The idea of polarization by reflection is therefore out of place;" and that 'the law of Brewster does not apply to matter in this condition' (of extreme fineness) is only what might have been inferred from the principles of the wave theory."

For this reason we are careful not to use reflection as a synonym for scattering by particles. Moreover, we are also careful to qualify reflection as specular when we have this kind in mind. The laws of specular reflection and refraction, the Fresnel coefficients, and Brewster's law, all have a limited range of validity.

Reduction of the maximum degree of polarization is not the only consequence of multiple scattering. According to Eq. (7.115) there should be two *neutral points* in the sky, directions in which skylight is unpolarized: directly toward the sun (*solar point*) and directly away from the sun (*antisolar point*). Because of multiple scattering, however, there are three such points. When the sun is higher than about 20° above the horizon there are neutral points within 20° of the sun, the *Babinet point* above it, the *Brewster point* below. They coincide when the sun is directly overhead and move apart as the sun descends. When the sun is lower than 20°, the *Arago point* is about 20° above the antisolar point.

7.4 Particles as Polarizers and Retarders

All the simple rules about polarization upon scattering are broken when we turn from molecules and small particles to particles comparable with or larger than the wavelength. The degree of polarization of light scattered by small particles is a simple function of scattering angle [Eq. (7.115)] but simplicity gives way to complexity as particles grow. The Mueller matrix for scattering (the scattering matrix) by an arbitrary homogeneous sphere must have the same symmetry as that for specular reflection [Eq. (7.103)] because incident light polarized perpendicular (parallel) to the scattering plane gives rise only to scattered light polarized perpendicular (parallel) to the scattering plane. So we immediately can write down the form of the scattering matrix for a sphere

$$\begin{pmatrix} S_{11} & S_{12} & 0 & 0 \\ S_{12} & S_{11} & 0 & 0 \\ 0 & 0 & S_{33} & S_{34} \\ 0 & 0 & -S_{34} & S_{33} \end{pmatrix}, \tag{7.125}$$

where the matrix elements S_{ij} depend on the size of the sphere relative to the wavelength of the illumination and its composition (i.e., complex refractive index). Unlike the Mueller matrix elements for specular reflection, these matrix elements are complicated functions not so readily calculated as sines and cosines. But many of the general statements we made about specular reflection also apply to scattering by a sphere. In particular, if the incident light is unpolarized, the scattered light is partially linearly polarized with degree of polarization given by an expression of the same form as Eq. (7.108):

$$-\frac{S_{12}}{S_{11}}. \tag{7.126}$$

Despite this similarity there is a difference. If unpolarized light is incident on a planar interface the reflected light is always partially polarized perpendicular to the plane of incidence ($R_\perp \geq R_\parallel$). But because a sphere has an additional degree of freedom, its size relative to the wavelength, light scattered by it can be partially linearly polarized either perpendicular or parallel to the scattering plane. Moreover, this can flip back and forth at different scattering angles, as shown in Fig. 7.12, the degree of polarization of light scattered by water droplets of different sizes.

We wrote Eq. (7.126) with a negative sign so that it reduces to Eq. (7.115) in the small particle limit. Although degree of polarization is strictly a positive quantity, allowing it to take on both positive and negative values is a simple way of showing whether the scattered light is partially polarized perpendicular to the scattering plane (positive) or parallel to this plane (negative).

The degree of polarization of light scattered by molecules or by small particles is nearly independent of wavelength. But this is not true for particles comparable with or larger than the wavelength. Scattering by such particles exhibits *dispersion of polarization*: the degree of polarization at, say, 90° may vary considerably over the visible spectrum. This is shown in Fig. 7.13, the degree of polarization of visible light scattered by a water droplet of diameter 0.5 μm. Dispersion of polarization is not necessarily a consequence of the dispersion of optical

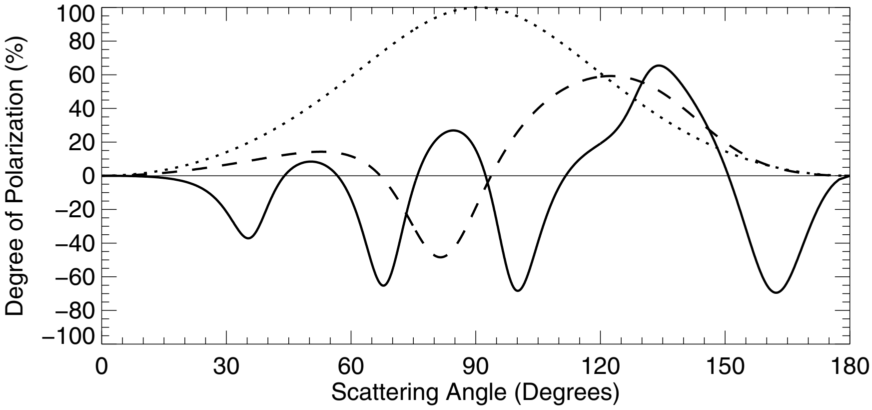

Figure 7.12: Degree of polarization of light scattered by water droplets of different size. The dotted curve is for a droplet of diameter 0.1 μm, the dashed curve for 0.5 μm, the solid curve for 1.0 μm; $\lambda = 0.55$ μm and $n = 1.33$. The incident light is unpolarized.

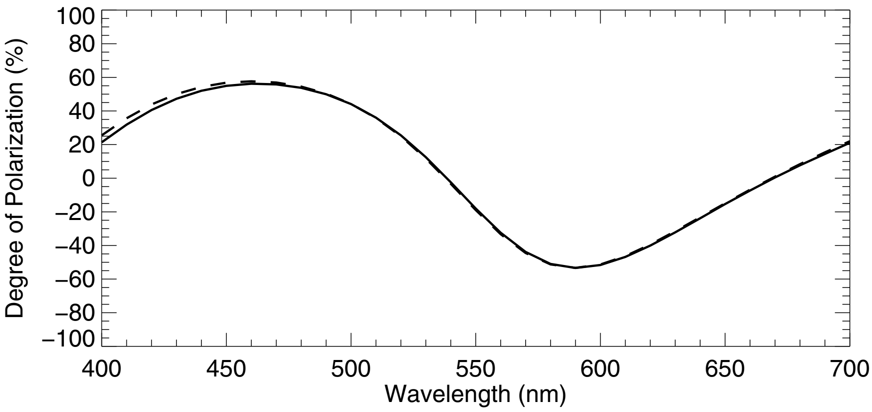

Figure 7.13: Degree of polarization of light scattered at $90°$ by a 0.5 μm diameter water droplet. The dashed line is for a fixed refractive index of 1.33, the solid line for a wavelength-dependent refractive index. The incident light is unpolarized.

constants. For the example shown in Fig. 7.13, the wavelength dependence of polarization is a consequence almost entirely of the dependence of phase differences on size relative to the wavelength.

Particles, unlike molecules, can act as polarizers and retarders. A linear polarizer transforms unpolarized light into partially polarized light by transforming the amplitudes of perpendicular field components differently, whereas a linear retarder transforms polarized light

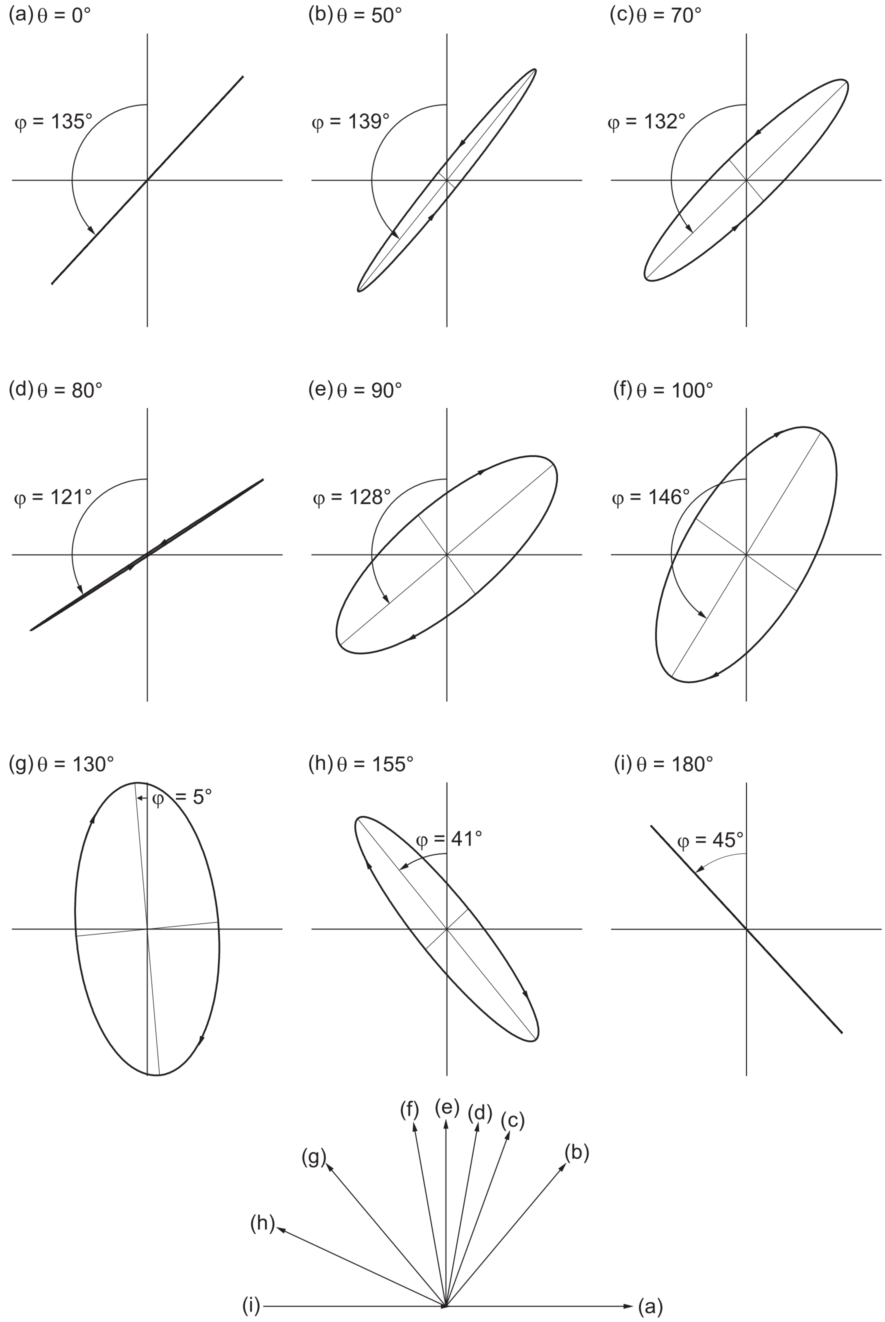

Figure 7.14: Vibration ellipses of visible light (0.55 μm) scattered in different directions by a water droplet of diameter 0.5 μm. The incident light is 100% linearly polarized at 45° to the scattering plane.

of one form into another by transforming the phases of perpendicular field components differently. But to do so a particle has to be sufficiently large relative to the wavelength to cause an appreciable phase shift. A point dipole cannot act as a retarder. If light linearly polarized obliquely to the scattering plane illuminates a spherically symmetric isotropic dipole, the scattered light in all directions is also linearly polarized, although the azimuth of the vibration ellipse (here, a line) varies with scattering angle. But for a sphere comparable with or larger than the wavelength, anything is possible. This can be shown graphically by drawing the vibration ellipses of light scattered in different directions by water droplets illuminated by light linearly polarized at $45°$ to the scattering plane (Fig. 7.14).

Up to this point we have considered only a single particle. Now consider N particles in a small volume illuminated by monodirectional light with irradiance I_{i}. By small here is meant that the linear dimensions of the volume are small compared with the distance to the observation point, which ensures that the scattering direction for each particle is approximately the same. If there is no fixed phase relation between the waves scattered by the particles (i.e., incoherent array), the Stokes parameters of the light scattered by each of them are additive, and hence the Stokes vector of the total scattered light at the observation point is

$$\mathbf{I}_{\mathrm{s}} = \mathbf{M}_1\mathbf{I}_{\mathrm{i}} + \mathbf{M}_2\mathbf{I}_{\mathrm{i}} + \ldots + \mathbf{M}_N\mathbf{I}_{\mathrm{i}} = \left(\sum_{j=1}^{N}\mathbf{M}_j\right)\mathbf{I}_{\mathrm{i}} = \mathbf{M}\mathbf{I}_{\mathrm{i}} = N\langle\mathbf{M}\rangle\mathbf{I}_{\mathrm{i}}, \tag{7.127}$$

where $\mathbf{M}_j$ is the scattering matrix for the j^{th} particle and $\langle\mathbf{M}\rangle$ is the average scattering matrix. If the incident light is 100% polarized, the light scattered in any direction by a single particle must be 100% polarized because such a particle is a coherent array. But the light scattered by an incoherent array of particles cannot be 100% polarized if the particles are not all identical. Two particles are different, and hence have different scattering matrices, if they are different in size or shape or composition or, if not spherically symmetric, orientation. Although the light scattered by each particle is 100% polarized, the ellipsometric parameters are, in general, different. Two or more beams with definite but different ellipsometric parameters cannot be superposed to obtain a beam with definite ellipsometric parameters. The elements of each scattering matrix $\mathbf{M}_j$ are such that if the incident light is 100% polarized, the Stokes parameters of the light scattered by each particle satisfy $I_{\mathrm{i}}^2 = Q_{\mathrm{i}}^2 + U_{\mathrm{i}}^2 + V_{\mathrm{i}}^2$ whereas the elements of the average scattering matrix are such that the Stokes parameters of the total scattered light do not. This is a variation on a theme in Section 7.3, where we show that adding beams incoherently can never increase the degree of polarization above that of the beam with the highest degree of polarization.

Although oriented nonspherical particles are not unknown in the atmosphere (e.g., falling ice crystals oriented by aerodynamic forces), we usually are most interested in randomly oriented particles or molecules. The average scattering matrix for a large (large enough for an average to mean something) number of randomly oriented particles that are superposable on their mirror images (no corkscrews, please) has the same block-diagonal symmetry as that for spheres [Eq. (7.125)], although, in general, $S_{22} \neq S_{11}$ and $S_{44} \neq S_{33}$. For such an array of particles, incident unpolarized light gives rise only to partially linearly polarized scattered light perpendicular or parallel to the scattering plane. And incident light partially linearly polarized either perpendicular or parallel to the scattering plane again gives rise only to scattered light similarly polarized. To obtain (partially) circularly polarized light ($V_\mathrm{s} \neq 0$) from incident linearly polarized light requires oblique polarization ($U_\mathrm{i} \neq 0$). Because of this it would seem that scattering of unpolarized sunlight by atmospheric particles could never yield light with a degree of circular polarization. This would be true if it were not for multiple scattering. Sunlight scattered by one particle or group of particles acquires a degree of linear polarization. This light then becomes incident light for a second particle, and if the scattering direction is such that the second scattering plane is not parallel to the first, this incident light can be obliquely polarized ($U_\mathrm{i} \neq 0$).

Let $(E_\|, E_\perp)$ be the field components of the light scattered by the first particle. These become the field components of the light illuminating the second particle. If the scattering direction is such that the scattering plane for the first particle does not coincide with that of the second, these field components must be transformed to those relative to the second scattering plane:

$$\begin{pmatrix} E'_\| \\ E'_\perp \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{pmatrix} \begin{pmatrix} E_\| \\ E_\perp \end{pmatrix}, \tag{7.128}$$

where ψ is the azimuthal angle of the second plane relative to the first (the two coincide if $\psi = \pi/2$) and the prime denotes transformed field components. From Eq. (7.128) it then follows that the Stokes parameters are transformed according to

$$\begin{pmatrix} I' \\ Q' \\ U' \\ V' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\cos 2\psi & \sin 2\psi & 0 \\ 0 & -\sin 2\psi & -\cos 2\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}. \tag{7.129}$$

This equation also reminds us that the Stokes parameters depend on the coordinate system. Suppose that the Stokes parameters of the light scattered by the first particle, illuminated by unpolarized light, are $(I_\mathrm{s}, Q_\mathrm{s}, 0, 0)$. These then become the Stokes parameters of the incident light for the second particle, which when transformed according to Eq. (7.129) become $(I_\mathrm{s}, -\cos 2\psi\, Q_\mathrm{s}, -\sin 2\psi\, Q_\mathrm{s}, 0)$. If $\sin 2\psi \neq 0$ this secondary incident light is partially polarized obliquely to the second scattering plane, and hence the degree of circular polarization of the secondary scattered light is

$$\frac{S_{34} \sin 2\psi\, Q_\mathrm{s}}{S_{11} I_\mathrm{s} - S_{12} \cos 2\psi\, Q_\mathrm{s}}. \tag{7.130}$$

This equation shows how geometry (ψ), linear polarization (Q_s) and the retarding properties of a scatterer (S_{34}) can conspire by way of multiple scattering to yield scattered light with a probably small, but measurable, degree of circular polarization, beginning with unpolarized light.

Now we are better equipped to discuss fine points skirted in previous chapters. We can ignore polarization in the two-stream equations of transfer (Sec. 5.2) because of the symmetry of the media of interest and of the radiation field. We take the media to be isotropic and constrain radiation to only two directions parallel to the normal to planar interfaces. With these restrictions, there is no distinction between light polarized parallel and perpendicular to a reference plane determined by two *different* non-collinear directions. This is evident for specular reflection: at normal incidence, the two reflectivities [Eq. (7.94)] are equal, and polarization, in effect, does not exist. These are examples of a general rule: changes of polarization state always require *asymmetry*: at the molecular level (e.g., birefringence, dichroism), the macroscopic level (e.g., non-spherical, oriented particles), of the illumination (e.g., oblique incidence), or of observation (e.g., scattering directions other than forward or backward).

We continue to assume isotropic media in Chapter 6 but no longer constrain radiation to only two directions. But again polarization is swept under the rug. Why? When polarization is taken into account, the phase *function* in Eq. (6.15) is replaced by a 4×4 phase *matrix* and the scalar radiance by a 4-component Stokes vector. For many applications this added complexity is not worth the effort. Radiative transfer theory in atmospheric science is most often applied to clouds, multiple-scattering media composed of droplets or ice particles distributed in size. In Section 7.3 we show that multiple scattering can only reduce the degree of polarization. And in this section we show that the degree of polarization of light scattered by water droplets varies greatly with their size, scattering angle, and wavelength (Figs. 7.12 and 7.13). Thus several factors conspire to make polarization irrelevant, usually, to radiative transfer in clouds. Cloud light is weakly polarized, which you can verify for yourself with a polarizing filter. But light from the clear sky may have a high degree of polarization (Sec. 7.3). Indeed, this difference between sky and clouds is one reason for polarizing filters on cameras. Such filters can greatly increase contrast (see Sec. 8.2) by reducing skylight more than cloud light. We have seen formless masses smothered in murk suddenly become transformed into clear and distinct clouds by observing them through a rotated polarizing filter. Try this yourself. To explore the consequences of multiple scattering for the polarization state of the clear sky, we cannot avoid going beyond Eq. (6.15) to a radiative transfer equation for the Stokes vector. But if our interest is only in irradiances under a clear sky, ignoring polarization does not lead to serious errors.

At least two charming myths in which the polarization state of cloudlight plays a role are widespread. One is that bees can, without qualification, navigate by polarized skylight. Remarkably clever though they may be, bees cannot do the impossible. The simple wavelength-independent relation between the position of the sun and the direction in which skylight is most highly polarized, which underlies navigation by means of polarized skylight, is obliterated when clouds cover the sky. Indeed this was the key to Karl von Fritsch's solving the puzzle of bee navigation: "Sometimes a cloud would pass across the area of sky… and the bees were unable to indicate the direction to the feeding place. Whatever phenomenon in the blue sky served to orient the dances, this experiment showed that it was seriously disturbed if the blue sky was covered by a cloud." But von Fritsch's words often are forgotten by those eager to spread the word about bee magic to those just as eager to believe what is charming even though untrue.

The other myth, namely that the Vikings could navigate on overcast days by somehow making use of polarized skylight, has been debunked by two modern-day Vikings, Curt Roslund and Claes Beckman. They begin their attack by noting that "Viking exploits in the North Atlantic have aroused people's imagination to a height where responsible judgement of extraordinary claims often seems to be suspended." Even if the Vikings had the means to detect the polarization state of the sky (and there is no evidence that they did), "On most overcast days the Vikings could most certainly not have used the polarization state of light to determine the location of the Sun. Although they might have been able to do so on partly cloudy days, there would have been no need to." The Vikings were indeed remarkable navigators, but not because of their precocity in polarimetry.

References and Suggestions for Further Reading

The treatise on polarization that has influenced us most is William A. Shurcliff, 1962: *Polarized Light: Production and Use*. Harvard University Press. Shurcliff approaches polarization from a fresh point of view. This is a well-written, critical book, mandatory reading for anyone interested in polarization. Another good book on polarization is David Clarke and John F. Grainger, 1971: *Polarized Light and Optical Measurements*. Pergamon. The authors carefully discuss conventions for the handedness of light. A more recent work, also more advanced, is Edward Collett, 1992: *Polarized Light: Fundamentals and Application*. Marcel Dekker.

An advanced treatise on polarization from a statistical point of view is Christian Brosseau, 1998: *Fundamentals of Polarized Light: A Statistical Optics Approach*. John Wiley & Sons. In previous chapters we skirted the entropic properties of radiation but did not meet them head on. For example, Problem 1.40 is a derivation of the form of the Stefan–Boltzmann law by way of the first and *second* laws of thermodynamics. In Section 4.1.6 we referred to specular reflection as reversible. And Eq. (7.124) has the faint odor of entropy about it, being a requirement that the degree of polarization of partially polarized beams can never increase upon superposition. Radiation does indeed possess entropy (see Prob. 7.49). In fact, Brosseau derives a more stringent form of Eq. (7.124) by entropy arguments (Appendix C), includes a section (3.4) on the entropy of radiation, and a brief appendix (E) on the history of the extension of the entropy concept to radiation. Entropy arguments have not been applied extensively to optics problems, possibly because they can be solved by other, more familiar, methods.

For a collection of 75 contributed chapters on various aspects of polarization see Tom Gehrels, Ed., 1974: *Planets, Stars and Nebulae Studied with Polarimetry*. University of Arizona Press.

For popular treatments of polarization, with many color plates, see Günther P. Können, 1980: *Polarized Light in Nature*. Cambridge University Press and David Pye, 2001: *Polarized Light in Science & Nature*, Institute of Physics.

For benchmark papers on polarization see William Swindell, Ed., 1975: *Polarized Light*. Dowden, Hutchinson & Ross. Two other collections edited by Bruce H. Billings are *Selected Papers on Polarization*, Vol. MS 23 (1990) and *Selected Papers on Applications of Polarized Light*, Vol. MS 27 (1992). SPIE Optical Engineering Press.

One of the Arago–Fresnel laws "deduced from direct experiment" long before (1819) the electromagnetic theory of light is "In the same condition in which two rays of ordinary light seem to destroy each other mutually, two rays polarized at right angles or in opposite senses exert on each other no appreciable action" (William Francis Magie, 1965: *A Source Book of Physics*. Harvard University Press, pp. 324–35).

For what Stokes wrote about the polarization parameters that now bear his name see George Gabriel Stokes, 1852: On the composition and resolution of streams of polarized light from different sources. *Transactions Cambridge Philosophical Society*, Vol. 9, pp. 399–416 (reprinted in Stokes's *Mathematical and Physical Papers*. Vol. 3, pp. 233–50 and in the collection edited by Swindell).

A fascinating paper on the development of sheet polarizing filters is Edwin H. Land, 1951: Some aspects of the development of sheet polarizers. *Journal of the Optical Society of America*, Vol. 41, pp. 957–63. This is a paper you must read. It is reprinted in the collections edited by Swindell and by Billings (Vol. MS 23). The October 1994 *Optics & Photonics News* is a special issue devoted to Edwin Land. He is held in such high esteem in the senior author's house that a yellow and brittle obituary of Land is still taped to the refrigerator.

For the discovery of "a strange and wonderful phenomenon. . . according to which objects seen through it appeared not with single images as with other transparent bodies, but with double images" by Erasmus Bartholinus see pp. 280–3 in Magie's *Source Book*. The original Latin version and a partial translation are in the collection edited by Swindell.

For the optics of naturally birefringent media see G. N. Ramachandran and S. Ramaseshan, 1961: Crystal optics, in *Handbuch der Physik*, Springer, Vol. 25/1, pp. 1–217.

For an analysis of the colors seen through airplane windows (with color plates) see Craig F. Bohren, 1991: On the gamut of colors seen through birefringent airplane windows. *Applied Optics*, Vol. 30, pp. 3474–8. These colors are also discussed without mathematics in Craig F. Bohren, 1991: *What Light Through Yonder Window Breaks?*, John Wiley & Sons, Chs. 3 & 4.

For what can be learned about hailstones because ice is naturally birefringent see Charles A. Knight and Nancy C. Knight, 1970: Hailstone embryos. *Journal of the Atmospheric Sciences*, Vol. 27, pp. 659–66; Lobe structure of hailstones, pp. 667–71; The falling behavior of hailstones, pp. 672–81.

For Malus's account of the discovery of polarization upon reflection see Magie's *Source Book*, pp. 315–18. The original French version and a partial translation are in the collection edited by Swindell.

If you dare to spell Snel's name correctly be prepared for a fight with copy editors and those who have misspelled it for a lifetime. Arm yourself with the biographical sketch of Willebrord Snel by his fellow countryman Dirk J. Struik in *Dictionary of Scientific Biography*. Snel's name has been misspelled for centuries because its latinized form is Snellius, which was delatinized (incorrectly) as Snell.

The Fresnel coefficients are derived in many books on optics and electromagnetic theory. See, for example, Julius Adams Stratton, 1941: *Electromagnetic Theory*. McGraw-Hill, pp. 490–511; Max Born and Emil Wolf, 1965: *Principles of Optics*, 3rd rev. ed. Pergamon, pp. 38–51;

John David Jackson, 1975: *Classical Electromagnetic Theory*, 2nd ed. John Wiley & Sons, pp. 278–82; John Lekner, 1987: *Theory of Reflection*. Martinus Nijhoff, pp. 1–10.

For what is now called Brewster's law see David Brewster, 1815: On the laws which regulate the polarisation of light by reflexion from transparent bodies. *Philosophical Transactions of the Royal Society*, Vol. 105, pp. 125–59. Excerpts from this paper are in the collection edited by Swindell. See also Akhlesh Laktakia, 1989: Would Brewster recognize today's Brewster angle? *Optics News*. Vol. 15, pp. 14–17.

For more on polarization of emitted radiation see Oscar Sandus, 1965: A review of emission polarization. *Applied Optics*, Vol. 4, pp. 1634–42. His derivation of the degree of polarization is different from ours but the result is the same. He presents experimental data compared with theory.

For a microscopic (scattering) interpretation of polarization upon reflection, but wrung out of macroscopic physics, see William T. Doyle, 1985: Scattering approach to Fresnel's equations and Brewster's law. *American Journal of Physics*, Vol. 53, pp. 463–68. Doyle's analysis should be included in every textbook derivation of the Fresnel coefficients.

For a proof that backscattering by metallic particles (strictly, ones with infinite refractive index) is 9 times forward scattering, and the physical reason for this, see Hendrik C. van de Hulst, 1957: *Light Scattering by Small Particles*, John Wiley & Sons, pp. 158–61.

Lord Rayleigh's 1871 paper, On the light from the sky, its polarization and colour, originally published in *Philosophical Magazine*, is reprinted in his *Scientific Papers*, Vol. I, Cambridge University Press (1899), pp. 88–103, also in Craig F. Bohren, Ed., 1989: *Selected Papers on Scattering in the Atmosphere*. SPIE Optical Engineering Press.

For an elementary discussion of the polarization of skylight see Craig F. Bohren, 1987: *Clouds in a Glass of Beer*, John Wiley & Sons, Ch. 19.

For a theoretical treatment of the polarization state of the clear sky and comparison with measurements see Subrahmanyan Chandrasekhar and Donna D. Elbert, 1954: The illumination and polarization of the sunlit sky on Rayleigh scattering. *Transactions of the American Philosophical Society*, Vol. 44, pp. 643–54. Reprinted in *Selected Papers on Scattering in the Atmosphere*.

A treatise devoted entirely to polarization of light from the sky is Kinsell L. Coulson, 1988: *Polarization and Intensity of Light in the Atmosphere*, A. Deepak Publishing. Especially recommend for many measurements of the degree of polarization of (clear) sky light (Chapter 5), which indicate that a degree of polarization greater than 85% is all but unattainable.

Section 7.4 is based, in part, on Craig F. Bohren, 1995: Optics, atmospheric, *Encyclopedia of Applied Physics*, Vol. 12, pp. 405–34.

We exclude in Section 7.4 particles that are not superposable on their mirror images. Including such particles (and molecules) opens up a vast field with many important applications, especially in chemistry and biology. For more on this see Akhlesh Lakhtakia, 1990: *Selected Papers on Natural Optical Activity*. SPIE Optical Engineering Press.

It should be evident on physical grounds that ignoring polarization in calculations for clear skies should result in larger radiance than irradiance errors. Irradiance is integrated radiance, the errors in which are about as likely to be positive as negative. This is indeed what is calculated and observed. Andrew A. Lacis, Jacek Chowdhary, Michael I. Mischenko, and Brian Cairns, 1998: Modeling errors in diffuse-sky radiation: Vector vs. scalar treatment. *Geophysical Research Letters*, Vol. 25, pp. 135–8 compare calculations both with and without accounting for polarization. Seiji Kato, Thomas P. Ackerman, Ellsworth G. Dutton, Nels Laulainen, and Nels Larson, 1999: A comparison of models and measured surface shortwave irradiance for a molecular atmosphere. *Journal of Quantitative Spectroscopy and Radiative Transfer*, Vol. 61, pp. 493–502 take a different approach. They compare irradiances calculated using a two-stream theory with measurements made in a very clean environment (Mauna Loa Observatory). They conclude that neglecting polarization introduces negligible errors in irradiance calculations.

For what bees can and, more important, cannot do see Karl von Fritsch, 1971: *Bees: Their Vision, Chemical Senses, and Language*, rev. ed., Cornell University Press, p. 116.

For a delightful debunking of fanciful notions about Viking photopolarimetry see Curt Roslund and Claes Beckman, 1994: Disputing Viking navigation by polarized light, *Applied Optics*, Vol. 33, pp. 4754–5.

Problems

7.1. Show that an unpolarized beam can be considered to be the incoherent superposition of two orthogonally elliptically polarized (100%) beams of equal irradiance.

HINT: This problem is a simple application of the Stokes parameters.

7.2. Show that the electric and magnetic fields of a plane harmonic wave are perpendicular to the direction of propagation only if the wave is homogeneous, and that the electric and magnetic fields are perpendicular to each other only if the medium is nonabsorbing.

7.3. Show by simple arguments that transmission of a beam of unpolarized light by an ideal linear retarder cannot change the state of polarization of the beam. No fancy mathematics is necessary, just simple arguments based on the nature of unpolarized light and the function of a retarder.

7.4. Show that the reflectivities for specular reflection, $R_{\|}$ and $R_{\perp}$, are both equal to 1 for an angle of incidence of 90° regardless of the refractive index of the medium.

7.5. Show that as m becomes indefinitely large, the degree of polarization of specularly reflected light (given unpolarized incident light) approaches zero for all angles of incidence.

7.6. Many years ago a scheme for reducing glare from the headlights of oncoming automobiles was seriously considered. Linear polarizing filters were to be placed on the headlights and windshields of all automobiles. How should these filters be oriented so that drivers do not see the light from oncoming cars but do see light reflected by the headlights of their own cars? You may assume that the state of polarization of light from headlights is not changed upon reflection. Can you think of at least one reason why this scheme was never adopted?

7.7. Estimate the extent to which the maximum degree of polarization of skylight is reduced because of the finite width (about half a degree) of the sun. You may assume that air molecules are spherically symmetric.

7.8. By how much does an ideal linear polarizing filter reduce the irradiance of an incident unpolarized beam? By how much does the filter reduce the irradiance of an incident beam partially polarized with degree P perpendicular to the transmission axis?

HINT: The first question is easier than the second. You can guess the answer to the second by considering the two limiting cases. To check your guess note that the Mueller matrix for an ideal linear polarizing filter has the same form as the Mueller matrix for reflection in which one of the reflectivities is 1, the other 0.

7.9. It should be evident on physical grounds that incident 100% polarized light yields specularly reflected light that also is 100% polarized although with possibly different degrees of linear and circular polarization. Prove this. That is, show that if $I_{\mathrm{i}}^2 = Q_{\mathrm{i}}^2 + U_{\mathrm{i}}^2 + V_{\mathrm{i}}^2$ it follows that $I_{\mathrm{r}}^2 = Q_{\mathrm{r}}^2 + U_{\mathrm{r}}^2 + V_{\mathrm{r}}^2$. In constructing this proof you will also discover that the Mueller matrix elements for specular reflection are not independent (indeed, this result is an essential part of the proof).

7.10. The transmission axis of a linearly polarizing filter in, for example, polarizing sunglasses, is not perceptible to the human eye. How would you quickly determine the direction of this axis?

7.11. We discussed only the degree of polarization of specularly reflected light, for incident unpolarized light, as a function of angle of incidence for reflection by an infinitely thick medium. Such media are thin on the ground. How do you expect the degree of polarization to vary for light incident on a medium of *finite* thickness? Take it to be a nonabsorbing plate of uniform thickness sufficiently large relative to the wavelength that the consequences of interference need not be taken into account. For sake of visualization, consider the plate to be a microscope slide. Equations in Section 5.1 will help you derive an expression for the degree of polarization of the plate as a function of the reflectivities of the infinite medium. But you should first try to determine by physical reasoning if the degree of polarization increases, decreases, or remains the same (relative to the infinite medium) and how this depends on angle of incidence.

7.12. Find the degree of polarization of light specularly reflected by an incoherent pile of $N = 2, 4, \ldots$ identical plates like those in the previous problem. The plates are sufficiently far apart (relative to the wavelength) that coherence need not be taken into account. Find the degree of polarization of the reflected light and the reflectivity for unpolarized light as a function of the angle of incidence in the limit of indefinitely large N. You can do this by analysis or by physical reasoning. Sketch the reflectivity and degree of polarization as a function of angle in this limit.

7.13. Determine the degree of polarization of light *transmitted* by the pile of plates in the previous problem as a function of the angle of incidence in the limit of an indefinitely large number of plates. Sketch the degree of polarization of the transmitted light and reflectivity for unpolarized light as function of angle of incidence in this limit. First try to make these sketches by only physical reasoning. Compare these sketches with those for reflection. What do you conclude about using a pile of plates as a polarizing filter?

7.14. Show that the surfaces of constant amplitude for an inhomogeneous plane wave that results from illumination of a planar, optically smooth interface between a negligibly absorbing medium and an absorbing medium are planes parallel to the interface. Show that when the imaginary part of the refractive index of the absorbing medium is small compared with the real part, the spatial rate of attenuation of the transmitted field is *approximately* $\exp(-2\pi n_\mathrm{i} s/\lambda)$, where n_i is the imaginary part of the complex refractive index of the absorbing medium and s is the distance into this medium *along the direction of refraction*.

7.15. An article in *Science News* (July 3, 2003) about beetles navigating by moonlight begins with the assertion that "an international team of researchers has turned up evidence that the insect aligns its path by detecting the polarization of moonlight." The article goes on to say that the researchers "found that beetles active during the day depend on sunlight polarization patterns." These statements, taken literally (which is the only way we can take them) are incorrect. Why? How would you rewrite them to make them correct?

7.16. On what kind of imaginary but physically allowable planet with what kind of atmosphere illuminated by what kind of sun would skylight be 100% polarized at 90° from the sun?

7.17. Show that in the limit $m \to 1$ Eq. (7.108) for the degree of polarization of specularly reflected light (given incident unpolarized light) approaches Eq. (7.115), the degree of polarization of light scattered by a spherically symmetric electric dipole if by the scattering angle in Eq. (7.108) is meant that between the reflected and transmitted waves.

HINTS: You will need L'Hospital's rule, basic theorems about limits of products and quotients, and trigonometric identities for sums of angles and half angles.

7.18. The angular dependence of scattering (in a given plane) of unpolarized light by a spherically symmetric dipolar scatterer is given by Eq. (7.116). Because this scattering is azimuthally symmetric, you should be able to determine the corresponding (normalized) phase function (the probability of scattering, per unit solid angle, in any direction).

7.19. Show that any optical element that transforms both amplitudes and phases of perpendicular electric field components differently can transform unpolarized light only into partially linearly polarized light. This is a more complicated version of Problem 7.3.

HINT: A proof follows from the definition of the Stokes parameters for light of arbitrary polarization state together with simple trigonometric identities.

7.20. We state without proof in Section 2.2.1 that "for many materials over many wavelength intervals, reflectivity changes hardly at all even with huge increases in absorption coefficient. And if there is a change, it is likely to result in a decrease in absorptivity..." You now have all the ingredients to prove this statement, namely the Fresnel coefficients and the discussion of the complex refractive index (see Sec. 3.5.2).

HINT: Consider specular reflection at normal incidence.

7.21. A plane harmonic wave incident on the interface between two negligibly absorbing media and originating in the (first) medium of higher refractive index is *totally internally reflected* at an angle of incidence, called the *critical angle* ϑ_c, such that the angle of transmission is $\pi/2$ (see Sec. 4.2.1). Show that the reflectivity at the critical angle for incident light polarized parallel or perpendicular to the plane of incidence is 1. Take the second medium to be air with refractive index 1.

HINT: Snel's law [Eq. (7.90)] and the cosine form of the Fresnel coefficients [Eqs. (7.91) and (7.92)] should be helpful.

7.22. This problem is an extension of the previous one. There is no law prohibiting waves from being incident at angles *greater* than the critical angle. What are the reflectivities for incident waves perpendicular and parallel to the plane of incidence at these angles?

HINT: The key to this problem is to recognize that ϑ_t in $\sin\vartheta_t$ and $\cos\vartheta_t$ in Eq. (7.88) does not necessarily correspond to a real angle. That is, $k_t \sin\vartheta_t$ and $k_t \cos\vartheta_t$ are simply the components of what is called the transmitted wavevector, the only requirement being that $\sin^2\vartheta_t + \cos^2\vartheta_t = 1$, and Snel's law is a mathematical relation that is always satisfied but not always easy to interpret geometrically. The cosine form of the Fresnel coefficients again should be helpful.

7.23. This problem is an extension of the previous one. Show that for angles of incidence greater than the critical angle, the reflected light is elliptically polarized for incident light linearly polarized obliquely to the plane of incidence.

7.24. We stated in Section 7.1 that the concept of a surface of constant phase is, in general, meaningless. Show this.

HINT: By a surface of constant phase is meant a single such surface. Write down the expression for an arbitrary (complex) vector field and the proof should be obvious.

7.25. Blackbody radiation is unpolarized and isotropic but can be considered the incoherent superposition of two sources of equal magnitude but orthogonally linearly polarized (see Prob. 7.1). From this result, derive an expression for the emissivity of an opaque slab, optically smooth and homogeneous (in air, say), in terms of the reflectivities for incident radiation polarized parallel ($R_{\|}$) and perpendicular ($R_{\perp}$) to the plane of incidence.

7.26. Derive an expression for the degree of polarization as a function of direction for radiation emitted by the slab in the previous problem. It may help to write the Stokes parameters of the emitted radiation. What is the largest degree of polarization (and in what direction) of 10 μm radiation emitted by a layer of water in air?

HINTS: We chose our words carefully here: "largest degree" rather than maximum degree. The easiest way to do this problem is to use the Fresnel coefficients Eq. (7.91) and Eq. (7.92) to obtain an explicit expression for the degree of polarization as a function of the angles ϑ_i and ϑ_t. Use the form of these equations containing trigonometric functions of the sum and differences of angles. Even though water is absorbing at 10 μm, the imaginary part of its refractive index is sufficiently small compared with its real part (about 1.2) that you can ignore the imaginary part. With a bit of algebra and a simple trigonometric identity, you can obtain a fairly simple expression for the degree of polarization.

7.27. Our proof of the inequality Eq. (7.57) was based on the (unproven) assumption that $I_p \leq I$, whereas a rigorous proof must start from Eqs. (7.51)–(7.54). You will need the

Cauchy-Schwarz inequality in the form $\left(\int fg\,dt\right)^2 \leq \int f^2 \;\; dt \int g^2 \;\; dt$ and the identity $\cos^2\delta + \sin^2\delta = 1$. We confess that we needed the help of two mathematicians, George Greaves and V. I. Burenkov, with a proof, which need not be long.

7.28. Convince yourself that Eq. (7.8) does indeed correspond to two orthogonally polarized waves. All that is needed is a crude sketch.

7.29. We once observed through a polarizing (camera) filter daylight reflected by a polished floor near the Brewster angle. As we rotated the filter, the brightness of the reflection greatly diminished, as expected. But the reflection could not be made to completely disappear. We always observed a dark reflection of a strikingly pure blue. At first we thought this had something to do with illumination by the blue sky. But this hypothesis was quickly discarded when we noticed that the source of illumination was light from an overcast sky. Explain.

HINT: Although it is not necessary to write down the Mueller matrix for an *non-ideal* linear polarizing filter, doing so, or at least thinking about doing so, is likely to help.

7.30. The *Umov effect* or *Umov's law (rule)* is a reciprocal relationship between reflectivity and degree of polarization of light reflected by rough surfaces or granular media (e.g., soils, snow, powders): the higher the reflectivity the lower the degree of polarization and *vice versa*. Although this rule appears to be fairly well known to (planetary) astronomers, a simple, short explanation of it is hard to find. And yet it can be explained adequately in a sentence or two. Provide such an explanation, concise, correct, and easy to understand.

7.31. Obtaining reflectivities from the Fresnel coefficients (ratios of fields, not irradiances) for reflection is straightforward because the incident and reflected waves make the same angle with the normal to the interface and both are in the same medium. Indeed, we presented these reflectivities without fanfare. Transmissivities require a bit more care. Derive expressions for the two transmissivities (ratio of transmitted irradiance to incident irradiance) as a function of angle of incidence for an infinite negligibly absorbing medium in air.

HINTS: You need the Fresnel coefficients for transmission as well as the relation between the Poynting vector and irradiance. We previously ignored a constant of proportionality between the magnitude of the Poynting vector (irradiance) and the square of electric fields. But for this problem we cannot ignore this constant because it is different for different media. For a plane, homogeneous wave in a negligibly absorbing medium $|\mathbf{S}| = n|\mathbf{E}\cdot\mathbf{E}^*|/2Z_\mathrm{o}$, where n is the (real) refractive index of the medium and Z_o is a universal constant called the impedance of free space.

7.32. To check the correctness of the result obtained in the previous problem, show that $R + T = 1$, where R is the reflectivity and T is the transmissivity (either polarization state). A further check is the reciprocal relation $T(\vartheta_\mathrm{i}, n) = T(\vartheta_\mathrm{t}, 1/n)$, which follows from the reversibility of rays.

7.33. In Section 7.2.2 we assert that to obtain elliptically polarized light by specular reflection requires obliquely polarized light incident on a metal (or a material with appreciable absorption at the wavelength of interest) at non-normal incidence. This statement is strictly true only for an *infinite* medium because if it is nonabsorbing $R_{34} = 0$. Show by simple arguments that for oblique incidence R_{34} is not necessarily zero for a *finite* nonabsorbing single layer or many layers (the form of the Mueller matrix is the same as that for an infinite medium). This problem is a variation on the theme that high reflectivity can be obtained either by a metal (appreciably absorbing) or by a multi-layer interference filter (negligibly absorbing).

7.34. Derive the Mueller matrix for an ideal linear polarizing filter with its transmission axis at an arbitrary angle to the reference coordinate system in which the Stokes parameters are defined.

HINT: Begin with Eqs. (7.64) and (7.65).

7.35. Derive the Mueller matrix for an ideal linear retarder with arbitrary retardance and with its fast axis at an arbitrary angle to the reference coordinate system in which the Stokes parameters are defined. This is considerably more difficult than the previous problem. You have to resolve the field components in the reference system along the fast and slow axes of the retarder, introduce different phase shifts, then transform back to the reference system. Trigonometric identities for the cosine and sine of the sum of angles can be used to simplify results.

7.36. As noted at the end of Section 7.1.3, although the Stokes parameters were defined by way of a set of hypothetical measurements not all of which are feasible, once these parameters are defined we can devise ways to measure them. Suppose that you have an irradiance detector, a linear polarizing filter, and a linear retarder with variable retardance. How many measurements and which kind would you have to make in order to determine the Stokes parameters of an arbitrary beam? Do you need both a polarizing filter and a retarder? Try to devise the simplest (easiest to describe) set of measurements. The results of the previous two problems can help you specify in detail what kind of measurements to make.

7.37. Show that if an incoherent suspension of spheres, regardless of their distribution in size and composition but sufficiently thin (optically) that multiple scattering is negligible, is illuminated by light linearly polarized perpendicular (parallel) to the scattering plane, the scattered light in this plane is 100% polarized. You can show this by simple physical arguments, but a mathematical proof (by way of the scattering matrix) leads you into the second part of this problem. By measuring the polarization properties of light scattered by a suspension of particles, how can it be determined if they are nonspherical even if randomly oriented? What scattering matrix elements or combination of elements should be measured and how (we can think of at least two possibilities)?

HINT: Results in Section 7.4 are necessary, and although the solution to Problem 7.34 is not, it might be helpful. For what scattering angles is the quantity measured likely to be most sensitive to departures of the particles from sphericity? You might review Section 3.4.8 before addressing this last question.

7.38. Based on the discussion of fluorescence in the references at the end of Chapter 1 and the discussion in Section 7.4 of the essential role that asymmetry plays in yielding polarized light, what do you expect the state of polarization (at any angle) of fluorescent light excited in gases and liquids to be? What about solids (with other than cubic symmetry)? No mathematical analysis is necessary. This problem tests your physical understanding of polarization and emission by gases and solids.

7.39. Does it seem contradictory that the light scattered at, say, 90°, by randomly oriented nonspherical molecules (for incident unpolarized light) can be partially polarized?

HINT: Consider a spheroidal molecule that differs only slightly from a spherical molecule.

7.40. Show that for negligible absorption, the reflection coefficient Eq. (7.91) or (7.92) reverses sign when the rays are reversed. That is, if we denote $\tilde{r}_{12}$ as the reflection coefficient for light incident at angle ϑ_i from medium 1 onto medium 2, and $\tilde{r}_{21}$ as the reflection coefficient for light incident from medium 2 onto medium 1, where the angles of incidence and refraction are reversed, show that $\tilde{r}_{21} = -\tilde{r}_{12}$. Then show that $1 + \tilde{r}_{12}\tilde{r}_{21} = \tilde{t}_{12}\tilde{t}_{21}$, where $\tilde{t}_{12}$ is the transmission coefficient [Eq. (7.98) or (7.99)], corresponding to $\tilde{r}_{12}$. What is the physical interpretation of this equation?

7.41. We did not derive Eq. (5.24), but you now should be able to do so in a way similar to summing the infinite series in Section 1.4. Although this equation was for normal incidence, no more effort is necessary to derive it for arbitrary incidence.

HINTS: Equation (5.7) suggests the form of the solution. For this problem you have to add fields, taking due account of phase shifts, then take the product of the resultant with its complex conjugate to obtain the reflectivity. Problem 7.40 also is needed for this problem. And you will need the difference in phase between a transmitted wave at $z = 0$ and at $z = h$, which you can obtain from Eq. (7.88).

7.42. With the results of Problem 7.41 you now should be able to explain why different thin film interference colors sometimes are seen in different directions.

7.43. Suppose that the thin film of Problem 7.41 is illuminated by a laser beam of diameter d at oblique incidence. What criterion must be satisfied for the reflectivity obtained in that problem to be applicable to reflection of the laser beam? The purpose of this problem is to underscore the (possible) differences between reflection of an *infinite* plane wave and a *finite* laser beam.

7.44. A single particle is a coherent object, and hence if illuminated by 100% polarized light, the scattered light is also 100% polarized (although not necessarily with all the same ellipsometric parameters). This is not difficult to understand. But we also argue in Section 3.4.2 that a piece of paper is a coherent object, and yet if it is illuminated by 100% polarized light the reflected light is at best weakly polarized. Why the difference? Devise a simple experiment to show that light reflected by a piece of white paper illuminated by 100% polarized light is essentially unpolarized. Try the same experiment with an optically smooth object such as glassware.

7.45. This problem is related to the previous one. If 100% polarized light illuminates a piece of white paper, the reflected light is weakly polarized but not with a degree of polarization of exactly 0%. If 100% polarized light illuminates a piece of glass, the reflected light is not exactly 100% polarized. Explain. Why do we specify "white" paper? What happens when black paper is illuminated by 100% polarized light (see Prob. 7.30)? Do a simple experiment to find out.

7.46. Unlike scattering by a small particle (see Sec. 3.5), reflection because of an interface (i.e., the Fresnel coefficients in Sec. 7.2) does not depend explicitly on wavelength (although it does depend implicitly on wavelength by way of the refractive index). Give a simple explanation why.

HINT: This is not a problem in electromagnetic theory but rather requires invoking a fundamental characteristic of all equations in which the variables are dimensional.

7.47. We note at the end of the chapter that neglecting polarization results in errors in radiance calculations for clear air illuminated by sunlight. Based strictly on physical reasoning you should be able to estimate the (normal) optical thickness for which the error is a maximum.

7.48. In the references for Chapters 2 and 3 we cite (quite favorably) Tony Rothman's *Everything's Relative*. But that doesn't mean that we'll let him get away with the footnote on page 22 of this book: "every time you look through a pair of Polaroid sunglasses you are using Malus' discovery. Polaroids work because reflected light viewed through them loses half its intensity." What's wrong with this (other than the use of intensity for luminance)?

7.49. If you did Problem 1.40 then you already have done almost everything you need to find the specific entropy of blackbody radiation. Do so.

7.50. In what simple, intuitive sense can specular reflection be said to be reversible whereas diffuse reflection is irreversible.

HINT: Consider rays.

7.51. Based strictly on your intuition about entropy determine what happens to the entropy of radiation upon specular reflection and refraction, diffuse reflection, scattering of a beam by a particle, and the incoherent superposition of partially polarized beams.

7.52. Section 7.2 begins with a discussion of reflection and refraction because of illumination of a smooth interface between two different media, the first of which can be taken to be air. Equation (7.86) is the wave vector of the transmitted wave in an arbitrary illuminated medium. Show that if this medium is absorbing, the surfaces of constant amplitude are parallel to the interface and the surfaces of constant phase are not, except for normal incidence.

7.53. Show that for a plane harmonic wave the time-averaged Poynting vector is *not*, in general, parallel to the real part of the complex wave vector. Under what conditions is the Poynting vector parallel to the real part of the wave vector? You may take the constant C in Eq. (7.5) to be real.

7.54. Consider reflection and refraction because of a smooth interface between two dissimilar infinite media. A wave is incident at an arbitrary direction from medium 1, giving rise to a refracted wave in medium 2 and a reflected wave in medium 1. Denote by $\tilde{t}_{12}$ the amplitude of the wave transmitted from 1 to 2 and by $\tilde{r}_{12}$ the amplitude of the reflected wave (for unit incident amplitude). The polarization state is either parallel or perpendicular to the plane of incidence. Now consider the reverse: a wave is incident from medium 2 along the direction of the refracted wave. Denote by $\tilde{t}_{21}$ and $\tilde{r}_{21}$ the corresponding ratios of amplitudes. Show that

$$\tilde{t}_{12}\tilde{t}_{21} = 1 - \tilde{r}_{12}^2,$$

$$\tilde{r}_{12} = -\tilde{r}_{21}.$$

These relations seem to have been derived first by Stokes. You can obtain them by physical arguments about the reversibility of the waves or analytically from the Fresnel equations. A simple diagram is essential. You may find statements in textbooks that these relations are valid only for nonabsorbing media. Show that this is not true.

7.55. With the results of the previous problem derive Eq. (5.24), the normal-incidence reflectivity of a nonabsorbing slab of uniform thickness, by adding all the multiply reflected waves taking account of all phase shifts. The equation you obtain for the reflected amplitude will be valid even for an absorbing medium, illuminated at arbitrary angle of incidence, and for either polarization state (parallel or perpendicular).

7.56. Find the transmitted amplitude for the slab considered in the previous problem. As a check on your result, show that the corresponding transmittivity plus the reflectivity is equal to 1.

7.57. We implicitly assumed in the previous problems that the medium on both sides of the slab is the same. Find the reflectivity of a slab by the series summation method for medium 1 on one side, medium 3 on the other side (all media negligibly absorbing).

7.58. Consider a negligibly absorbing slab between two different infinite media, also negligibly absorbing. Denote the media by subscripts 1, 2, and 3, where 2 denotes the slab. Show that the reflectivity for radiation incident from medium 1 is the same as that for radiation from medium 3. Keep in mind that this *reciprocity principle* requires the angle of incidence for radiation from medium 3 to be the same as the angle of transmission for the slab when illuminated from medium 1.

7.59. Use the results of Problem 7.57 to design an anti-reflection coating. That is, for normal-incidence radiation of wavelength λ, determine the refractive index n_2 and thickness h of a thin layer to be deposited on a substrate with refractive index n_3 such that the reflectivity of the system is zero.

7.60. We state that infinite, monodirectional plane electromagnetic waves do not exist. Prove this.

HINT: About the only physical law that we can always depend upon is conservation of energy.