Chapter 6 Principles of Radiative Transfer in Planetary Atmospheres

6.1 Introduction

6.1.1 A Brief History of Radiative Transfer

The subject of radiative transfer covers a variety of fields, including astrophysics, applied physics and optics, planetary sciences, atmospheric sciences, and meteorology, as well as various engineering disciplines. The notation that we use generally follows that developed by astrophysicists who pioneered the field at the beginning of the 20th century. The work of Schuster (1905) on the investigation of the transfer of radiation through a foggy atmosphere appears to be the first paper discussing the importance of multiple scattering. Schuster included the contribution of both the upward and downward light beams in the formulation, as cited in Section 3.4.2. This is the origin of the two-stream approximation for radiative transfer (Section 6.5.2). The two-beam concept was also employed by Schwarzschild (1906) to explain the limb darkening of the sun by substituting the two-beam solution into an integral equation. In this manner, a continuous distribution of radiation fields could be determined (see Section 4.6.3). In his effort to understand the physical structure of the interior of a star, Eddington (1916) developed an expansion of intensity using two-term Legendre polynominals, leading to the so-called Eddington's approximation for radiative transfer (Section 6.5.2). Furthermore, Schwarzschild (1914) introduced the concept that a medium could experience emission as well as absorption in the context of thermodynamic equilibrium, a subject concerning the transfer of thermal infrared radiation in molecular atmospheres assuming scattering can be neglected (Section 1.4.4). These pioneering papers in radiative transfer appear in a book volume in which Schwarzschild's two papers were translated into English (Menzel, 1966).

Prior to 1950, the subject of radiative transfer was studied principally by astrophysicists, although it was also an important research area in applied physics and nuclear engineering associated with neutron transport. In his landmark book, Chandrasekhar (1950) presented the subject of radiative transfer in plane-parallel atmospheres as a branch of mathematical physics and developed numerous solution methods and techniques, including the consideration of polarization. The principle of radiative transfer has also been extensively employed by planetary scientists, particularly in association with the remote sensing of planetary atmospheres by means of spectroscopy and polarimetry (Chamberlain and Hunten, 1987). In the first edition of this text, we introduced the subject of radiative transfer with reference to approximation methods and their application to the remote sensing of atmospheric composition and structure.

The underlying principles of radiative transfer have been firmly established in the papers and books cited earlier. However, new avenues, such as the analytic solution for the four-stream approximation, the delta-function adjustment for the phase function, and the efficient incorporation of line absorption in multiple scattering atmospheres, have been explored in recent years. In addition, radiative transfer in clouds consisting of nonspherical and spatially oriented ice crystals that occur in the upper troposphere, and radiative transfer in clouds that are finite and inhomogeneous in nature are both contemporary research subjects.

In Sections 1.1.4, 1.4.4, and 1.4.5, we introduced the concept of multiple scattering and presented the basic equations for radiative transfer. We also derived the source function in terms of the phase function and single-scattering albedo in Section 3.4.1. Following these discussions, we should first discuss here the fundamentals and exact solutions for radiative transfer, including the discrete-ordinates method, the principles of invariance, and the adding method. We will prove that the last two are, in principle, equivalent, and point out that the discrete-ordinates and adding methods are similar in terms of numerical calculations. Subsequently, we present various approximations for radiative transfer and the subject of radiative transfer including polarization. Finally, we discuss a number of advanced topics not covered by the plane-parallel assumption. In what follows, we present a number of fundamental equations for the plane-parallel condition and define the associated physical terms.

6.1.2 Basic Equations for the Plane-Parallel Condition

On the basis of Eqs. (3.4.5) and (3.4.6), the basic scalar equation for the transfer of radiation in plane-parallel atmospheres can be written in the form

$$\mu \frac{dI(\tau, \mathbf{\Omega})}{d\tau} = I(\tau, \mathbf{\Omega}) - \frac{\tilde{\omega}}{4\pi} \int_{4\pi} I(\tau, \mathbf{\Omega}') P(\mathbf{\Omega}, \mathbf{\Omega}')\, d\Omega' - \frac{\tilde{\omega}}{4\pi} F_{\odot} P(\mathbf{\Omega}, -\mathbf{\Omega}_0) e^{-\tau/\mu_0} + (1 - \tilde{\omega}) B[T(\tau)], \qquad (6.1.1)$$

where $\mathbf{\Omega} = (\mu, \phi)$ and $d\Omega = d\mu\, d\phi$ (other notations are defined in Section 3.4.1).

As shown in Section 3.4.1, the phase function may be numerically expanded in Legendre polynomials with a finite number of terms, N. And, in view of the definition of the scattering angle in Eq. (3.4.7), the phase function may be expressed by

$$P(\mu, \phi; \mu', \phi') = \sum_{\ell=0}^{N} \tilde{\omega}_\ell P_\ell[\mu\mu' + (1 - \mu^2)^{1/2}(1 - \mu'^2)^{1/2} \cos(\phi' - \phi)]. \qquad (6.1.2)$$

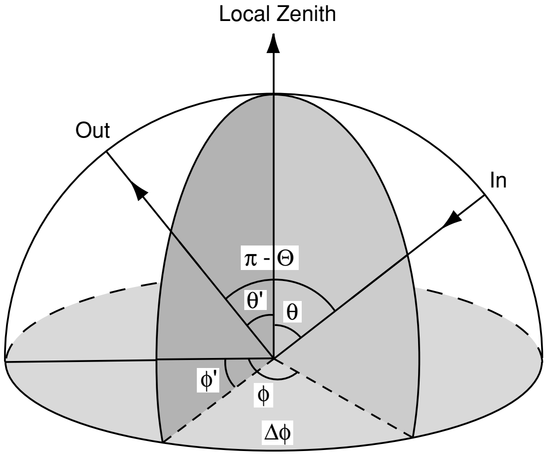

Figure 6.1 Relation of scattering (Θ), zenith (θ, θ'), and azimuthal angles (ϕ, ϕ') in a spherical atmosphere. In the discussion of multiple scattering of a light beam, the notation Θ is usually used to denote the scattering angle, while θ is used for the emergent angles. Note that in Chapter 5 dealing with single-scattering processes, θ is employed as the scattering angle.

The geometric relationship of the scattering, zenith, and azimuthal angles is shown in Fig. 6.1. The Legendre polynomials for the argument shown in Eq. (6.1.2) can be expanded by the addition theorem for spherical harmonics (see Appendix E) to give

$$P(\mu, \phi; \mu', \phi') = \sum_{m=0}^{N} \sum_{\ell=m}^{N} \tilde{\omega}_\ell^m P_\ell^m(\mu) P_\ell^m(\mu') \cos m(\phi' - \phi), \tag{6.1.3a}$$

where

$$\tilde{\omega}_\ell^m = (2 - \delta_{0,m})\tilde{\omega}_\ell \frac{(\ell - m)!}{(\ell + m)!} \qquad (\ell = m, \ldots, N, \quad 0 \le m \le N), \tag{6.1.3b}$$

$$\delta_{0,m} = \begin{cases} 1 & \text{if} \quad m = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{6.1.3c}$$

and P_ℓ^m denote the associated Legendre polynomials.

In view of the phase function expansion, we may also expand the intensity in the form

$$I(\tau; \mu, \phi) = \sum_{m=0}^{N} I^m(\tau, \mu) \cos m(\phi_0 - \phi). \tag{6.1.4}$$

On inserting Eqs. (6.1.3a) and (6.1.4) into Eq. (6.1.1) and noting the orthogonality of the associated Legendre polynomials, Eq. (6.1.1) splits up into $(N + 1)$ independent equations:

$$\mu\frac{dI^m(\tau,\mu)}{d\tau} = I^m(\tau,\mu) - (1+\delta_{0,m})\frac{\tilde{\omega}}{4}\sum_{\ell=m}^{N}\tilde{\omega}_\ell^m P_\ell^m(\mu)\int_{-1}^{1}P_\ell^m(\mu')I^m(\tau,\mu')\,d\mu'$$

$$-\frac{\tilde{\omega}}{4\pi}\sum_{\ell=m}^{N}\tilde{\omega}_\ell^m P_\ell^m(\mu)P_\ell^m(-\mu_0)F_\odot e^{-\tau/\mu_0}$$

$$+\,\delta_{0,m}(1-\tilde{\omega})B[T(\tau)] \qquad (m=0,1,\ldots,N). \tag{6.1.5}$$

Each equation may be solved independently for I^m. Consequently, from Eq. (6.1.4), I may be determined.

For $m = 0$, the intensity expressed in Eq. (6.1.4) corresponds to the azimuthal-independent case. We shall omit the superscript 0 for simplicity and rewrite Eq. (6.1.5) to yield

$$\mu\frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\tilde{\omega}}{2}\sum_{\ell=0}^{N}\tilde{\omega}_\ell P_\ell(\mu)\int_{-1}^{1}P_\ell(\mu')I(\tau,\mu')\,d\mu'$$

$$-\frac{\tilde{\omega}}{4\pi}\sum_{\ell=0}^{N}\tilde{\omega}_\ell P_\ell(\mu)P_\ell(-\mu_0)F_\odot e^{-\tau/\mu_0} + (1-\tilde{\omega})B[T(\tau)]. \tag{6.1.6}$$

For scattering atmospheres, the diffuse upward and downward solar flux densities are given, respectively, by

$$F_{\mathrm{dif}}^{\uparrow}(\tau) = \int_0^{2\pi}\int_0^1 I(\tau;\mu,\phi)\mu\,d\mu\,d\phi, \qquad \mu \ge 0, \tag{6.1.7a}$$

$$F_{\mathrm{dif}}^{\downarrow}(\tau) = \int_0^{2\pi}\int_0^{-1} I(\tau;\mu,\phi)\mu\,d\mu\,d\phi, \qquad \mu \le 0. \tag{6.1.7b}$$

Thus, by noting that

$$\int_0^{2\pi}\cos m(\phi_0-\phi)\,d\phi = 0, \qquad m \ne 0, \tag{6.1.8}$$

in Eq. (6.1.4), we obtain the upward and downward solar flux densities as follows:

$$F_{\mathrm{dif}}^{\uparrow\downarrow}(\tau) = 2\pi\int_0^{\pm 1} I(\tau,\ \mu)\mu\,d\mu. \tag{6.1.9}$$

Consequently, for calculations of solar fluxes in the atmosphere, the azimuthal dependence of the intensity expansion can be neglected, and Eq. (6.1.6) is sufficient for radiation studies.

Moreover, for azimuthal independent cases, we may define the phase function

$$P(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi}P(\mu,\phi;\mu',\phi')\,d\phi'. \tag{6.1.10}$$

In view of the phase function expansion represented by Eq. (6.1.3a), we have

$$P(\mu, \mu') = \sum_{\ell=0}^{N} \tilde{\omega}_\ell P_\ell(\mu) P_\ell(\mu'). \tag{6.1.11}$$

By virtue of this equation, the azimuthally independent transfer equation for diffuse radiation expressed in Eq. (6.1.6) can be rewritten as follows:

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\tilde{\omega}}{2} \int_{-1}^{1} I(\tau, \mu') P(\mu, \mu')\, d\mu' - \frac{\tilde{\omega}}{4\pi} F_\odot P(\mu, -\mu_0) e^{-\tau/\mu_0} + (1 - \tilde{\omega}) B[T(\tau)]. \tag{6.1.12}$$

We use the positive and negative μ's to denote the upward and downward light beams, as is evident in Eqs. (6.1.7a) and (6.1.7b). Hence the μ_0's that denote the direct solar radiation component are negative values. However, a positive μ_0 has been used for convenience, and $-\mu_0$ represents the fact that the direct solar radiation is downward. For the transfer of terrestrial infrared radiation in scattering atmospheres that are in local thermodynamic equilibrium, the direct solar term involving $F_\odot$ does not appear (Section 4.6).

For solar radiation, the equation of radiative transfer, however, only describes the diffuse component, i.e., light beams scattered more than once, and so we must include the direct component to account for the downward radiation. This is given by the simple Beer–Bouguer–Lambert law for extinction in the form

$$F_{\text{dir}}^{\downarrow}(\tau) = \mu_0 F_\odot e^{-\tau/\mu_0}. \tag{6.1.13}$$

The total upward and downward flux densities at a given τ are, respectively,

$$F^{\uparrow}(\tau) = F_{\text{dif}}^{\uparrow}(\tau) = 2\pi \int_0^1 I(\tau, \mu)\mu\, d\mu, \tag{6.1.14a}$$

$$F^{\downarrow}(\tau) = F_{\text{dif}}^{\downarrow}(\tau) + F_{\text{dir}}^{\downarrow}(\tau) = 2\pi \int_0^{-1} I(\tau, \mu)\mu\, d\mu + \mu_0 F_\odot e^{-\tau/\mu_0}. \tag{6.1.14b}$$

The net flux density for a given level is, therefore,

$$F(\tau) = F^{\downarrow}(\tau) - F^{\uparrow}(\tau). \tag{6.1.14c}$$

(See also Section 3.5 for a discussion of solar net flux and heating rate.) In the presentation of multiple scattering in planetary atmospheres, the thermal infrared emission term will be omitted.

6.2 Discrete-Ordinates Method for Radiative Transfer

The discrete-ordinates method for radiative transfer was elegantly developed by Chandrasekhar (1950) for application to the transfer of radiation in planetary atmospheres. Liou (1973a) demonstrated that the discrete-ordinates method is a useful and powerful method for the computation of radiation fields in aerosol and cloudy atmospheres. The method involves the discretization of the basic radiative transfer equation and the solution of a set of first-order differential equations. With the advance in numerical techniques for solving differential equations, the discrete-ordinates method has been found to be efficient and accurate for calculations of scattered intensities and fluxes. In presenting the fundamentals of this method, we first discuss the case involving isotropic scattering, including the law of diffuse reflection for semi-infinite atmospheres. That is followed by the general solution for anisotropic scattering. Finally, we present the application of the discrete-ordinates method for radiative transfer to nonhomogeneous atmospheres.

6.2.1 General Solution for Isotropic Scattering

For simplicity in introducing the discrete-ordinates method for radiative transfer, we shall first assume isotropic scattering, i.e., the scattering phase function $P(\mu, \phi; \mu', \phi') = 1$. In this case, the azimuthally independent intensity may be defined by

$$I(\tau, \mu) = \frac{1}{2\pi} \int_0^{2\pi} I(\tau; \mu, \phi)\, d\phi. \tag{6.2.1}$$

By neglecting the emission term, the equation of transfer given in Eq. (6.1.12) becomes

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\tilde{\omega}}{2} \int_{-1}^{1} I(\tau, \mu')\, d\mu' - \frac{\tilde{\omega} F_{\odot}}{4\pi} e^{-\tau/\mu_0}. \tag{6.2.2}$$

Now replacing the integral by a summation, according to Gauss's formula (see Section 6.5.2 for definition) and setting $I_i = I(\tau, \mu_i)$, we obtain

$$\mu_i \frac{dI_i}{d\tau} = I_i - \frac{\tilde{\omega}}{2} \sum_j I_j a_j - \frac{\tilde{\omega} F_{\odot}}{4\pi} e^{-\tau/\mu_0}, \qquad i = -n, \ldots, n, \tag{6.2.3}$$

where Σ_j denotes summation from $-n$ to n, i.e., $2n$ terms.

The solution of Eq. (6.2.3) may be derived by seeking first the general solution for the homogeneous part of the differential equation and then adding a particular solution. For the homogeneous part of the differential equation, we set

$$I_i = g_i e^{-k\tau}, \tag{6.2.4}$$

where g_i and k are constants. Substituting Eq. (6.2.4) into the homogeneous part of Eq. (6.2.3), we find

$$g_i(1 + \mu_i k) = \frac{\tilde{\omega}}{2} \sum_j a_j g_j. \tag{6.2.5}$$

This implies that g_i must be in the following form with a constant L:

$$g_i = L/(1 + \mu_i k). \tag{6.2.6}$$

With the expression of g_i given by Eq. (6.2.6), we obtain the characteristic equation for the determination of the eigenvalue k as follows:

$$1 = \frac{\tilde{\omega}}{2}\sum_j \frac{a_j}{1+\mu_j k} = \tilde{\omega}\sum_{j=1}^{n} \frac{a_j}{1-\mu_j^2 k^2}. \tag{6.2.7}$$

For $\tilde{\omega} < 1$, Eq. (6.2.7) admits $2n$ distinct nonzero eigenvalues, which occur in pairs as $\pm k_j (j = 1, \ldots, n)$. Thus, the general solution for the homogeneous part is

$$I_i = \sum_j \frac{L_j}{1+\mu_i k_j} e^{-k_j \tau}. \tag{6.2.8}$$

For a particular solution, we assume the following form:

$$I_i = \frac{\tilde{\omega} F_\odot}{4\pi} h_i e^{-\tau/\mu_0}, \tag{6.2.9}$$

where h_i are constants. Inserting Eq. (6.2.9) into Eq. (6.2.3) leads to

$$h_i(1+\mu_i/\mu_0) = \frac{\tilde{\omega}}{2}\sum_j a_j h_j + 1. \tag{6.2.10}$$

Hence, the constants h_i must be in the form

$$h_i = \gamma/(1+\mu_i/\mu_0), \tag{6.2.11}$$

where γ can be determined from Eq. (6.2.10) with the form

$$\gamma = 1 \Bigg/ \left[1 - \tilde{\omega}\sum_{j=1}^{n} a_j \Bigg/ \left(1 - \mu_j^2/\mu_0^2\right) \right]. \tag{6.2.12}$$

Adding the general and particular solutions, we obtain

$$I_i = \sum_j \frac{L_j}{1+\mu_i k_j} e^{-k_j \tau} + \frac{\tilde{\omega} F_\odot \gamma}{4\pi(1+\mu_i/\mu_0)} e^{-\tau/\mu_0}, \qquad i = -n, \ldots, n. \tag{6.2.13}$$

The unknown coefficients of proportionality L_j are determined from the boundary conditions imposed.

The next step is to introduce Chandrasekhar's H function to replace the constant γ. Consider the function

$$T(z) = 1 - \frac{\tilde{\omega} z}{2}\sum_j \frac{a_j}{z+\mu_j} = 1 - \tilde{\omega} z^2 \sum_{j=1}^{n} \frac{a_j}{z^2-\mu_j^2}, \tag{6.2.14}$$

which is a polynomial of degree $2n$ in z. We then compare this equation with the characteristic equation (6.2.7) and find that $z = \pm 1/k_j$ for $T(z) = 0$. Thus, we must have

$$\prod_{j=1}^{n} \left(z^2 - \mu_j^2\right) T(z) = \text{const} \prod_{j=1}^{n} \left(1 - k_j^2 z^2\right), \tag{6.2.15}$$

since the two polynomials of degree $2n$ have the same zeros. For $z = 0$, we find that

$$\text{const} = \prod_{j=1}^{n} \left(- \mu_j^2 \right).$$

Thus,

$$T(z) = (-1)^n \mu_1^2 \ldots \mu_n^2 \prod_{j=1}^{n} \left(1 - k_j^2 z^2\right) \Big/ \prod_{j=1}^{n} \left(z^2 - \mu_j^2\right). \tag{6.2.16}$$

The H function is defined by

$$H(\mu) = \frac{1}{\mu_1 \ldots \mu_n} \frac{\prod_{j=1}^{n} (\mu + \mu_j)}{\prod_{j=1}^{n} (1 + k_j \mu)}. \tag{6.2.17}$$

In terms of the H function, we have

$$\gamma = 1/T(\mu_0) = H(\mu_0) H(-\mu_0). \tag{6.2.18}$$

It follows that the complete solution to the isotropic, nonconservative radiative transfer equation in the nth approximation can now be expressed by

$$I_i = \sum_j \frac{L_j}{1 + \mu_i k_j} e^{-k_j \tau} + \frac{\tilde{\omega} F_\odot H(\mu_0) H(-\mu_0)}{4\pi (1 + \mu_i / \mu_0)} e^{-\tau/\mu_0}. \tag{6.2.19}$$

Figure 6.2 illustrates a distribution of eigenvalues for isotropic scattering with a single-scattering albedo $\tilde{\omega} = 0.95$ using four discrete streams. The characteristic equation denoted in Eq. (6.2.7) can be written in the form

$$f(k) = 1 - \tilde{\omega} \sum_{j=1}^{n} \frac{a_j}{1 - \mu_j^2 k^2}. \tag{6.2.20}$$

For $\tilde{\omega} \neq 0$, $f(k_j) \to \pm\infty$, as $k_j \to \mu_j^{-1}$. In this figure, the same intervals between each μ_j^{-1} were divided so that lines across the zeros can be clearly identified. The eigenvalues occur in pairs and there exists one, and only one, eigenvalue in each interval that can be mathematically proven. The eigenvalue in the discrete-ordinates method for radiative transfer may be physically interpreted as an effective extinction coefficient that, when multiplied by the normal optical depth, represents an effective optical path length in each discrete stream.

For conservative scattering, $\tilde{\omega} = 1$, we note that the characteristic equation (6.2.7) admits two zero eigenvalues, namely, $k^2 = 0$. Based on the relation

$$\sum_{j=-n}^{n} a_j \mu_j^\ell = \int_{-1}^{1} \mu^\ell \, d\mu = 2\delta_\ell / (2\ell + 1), \qquad \delta_\ell = \begin{cases} 1 & \text{even}, \\ 0 & \text{odd}, \end{cases}$$

we can show that

$$I_i = b(\tau + \mu_i + Q) \tag{6.2.21}$$

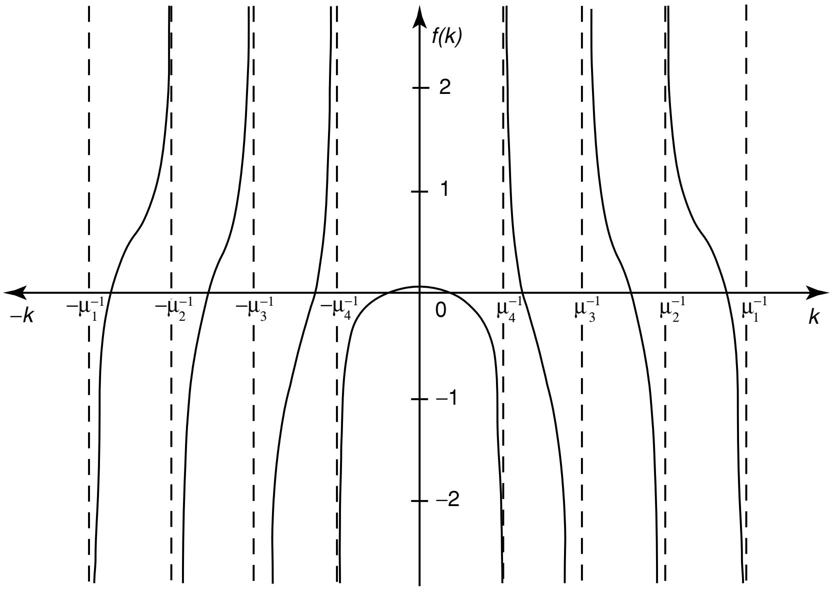

Figure 6.2 A typical distribution of eigenvalues (intercepts on the k-axis) for isotropic scattering with a single-scattering albedo of 0.95 for equal intervals between each $\mu_j^{-1}(j = \pm 1, \pm 2, \pm 3, \pm 4)$.

satisfies the homogeneous part of the differential equation, where b and Q are two arbitrary constants of integration. Thus, the complete solution to the isotropic radiative transfer equation in the nth approximation may be written as follows:

$$I_i = \sum_{j=-(n-1)}^{n-1} \frac{L_j}{1+\mu_i k_j} e^{-k_j \tau} + (\tau + \mu_i) L_{-n} + L_n + \frac{F_\odot H(\mu_0) H(-\mu_0)}{4\pi(1+\mu_i/\mu_0)} e^{-\tau/\mu_0}. \tag{6.2.22}$$

6.2.2 The Law of Diffuse Reflection for Semi-infinite Isotropic Scattering Atmospheres

Let us consider that there is no diffuse downward and upward radiation at the top ($\tau = 0$) and bottom ($\tau = \tau_1$) of a semi-infinite atmosphere so that

$$I(0, -\mu_i) = 0, \qquad I(\tau_1, +\mu_i) = 0. \tag{6.2.23}$$

Inserting the second boundary condition into the solution for the isotropic radiative transfer equation in the nth approximation denoted in Eq. (6.2.19), we obtain

$$I(\tau_1, +\mu_i) = 0 = \sum_{j=-n}^{-1} \frac{L_j}{1-\mu_i k_j} e^{k_j \tau_1}, \qquad i = 1, \ldots, n. \tag{6.2.24}$$

In order to satisfy the boundary condition for a semi-infinite atmosphere, we must have $L_j = 0$ $(j = -n, \ldots, -1)$. Thus,

$$I(\tau, \mu_i) = \sum_{j=1}^{n} \frac{L_j}{1 + \mu_i k_j} e^{-k_j \tau} + \frac{\tilde{\omega} F_\odot H(\mu_0) H(-\mu_0)}{4\pi(1 + \mu_i/\mu_0)} e^{-\tau/\mu_0}. \tag{6.2.25}$$

For convenience of analysis, we define

$$S(\mu) = \sum_{j=1}^{n} \frac{L_j}{1 - k_j \mu} + \frac{\tilde{\omega} F_\odot H(\mu_0) H(-\mu_0)}{4\pi(1 - \mu/\mu_0)}. \tag{6.2.26}$$

Thus,

$$S(\mu_i) = I(0, -\mu_i) = 0, \qquad i = 1, \ldots, n, \tag{6.2.27}$$

and the reflected intensity

$$I(0, \mu) = S(-\mu). \tag{6.2.28}$$

Moreover, we consider the function

$$(1 - \mu/\mu_0) \prod_{j=1}^{n} (1 - k_j \mu) S(\mu), \tag{6.2.29}$$

which is a polynomial of degree n in μ and vanishes for $\mu = \mu_i, i = 1, \ldots, n$. Hence, this function must be equal to $\prod_{j=1}^{n}(\mu - \mu_j)$ apart from a constant value. We may write

$$(1 - \mu/\mu_0) \prod_{j=1}^{n} (1 - k_j \mu) S(\mu) = \text{const} \cdot \frac{(-1)^n}{\mu_1 \ldots \mu_n} \prod_{j=1}^{n} (\mu - \mu_j). \tag{6.2.30}$$

Upon employing the definition of the H function defined in Eq. (6.2.17), we obtain

$$S(\mu) = \text{const} \cdot H(-\mu)/(1 - \mu/\mu_0). \tag{6.2.31}$$

To obtain the value of the constant, we note that

$$\lim_{\mu \to \mu_0} (1 - \mu/\mu_0) S(\mu) = \text{const} \cdot H(-\mu_0). \tag{6.2.32}$$

But from Eq. (6.2.26), we have

$$\lim_{\mu \to \mu_0} (1 - \mu/\mu_0) S(\mu) = \frac{1}{4\pi} \tilde{\omega} F_\odot H(\mu_0) H(-\mu_0). \tag{6.2.33}$$

Comparing Eq. (6.2.32) and Eq. (6.2.33), we find that

$$\text{const} = \frac{1}{4\pi} \tilde{\omega} F_\odot H(\mu_0), \tag{6.2.34}$$

and

$$S(\mu) = \frac{\tilde{\omega} F_\odot H(\mu_0) H(-\mu)}{4\pi(1 - \mu/\mu_0)}. \tag{6.2.35}$$

The reflected intensity for a semi-infinite, isotropic scattering atmosphere is then given by

$$I(0,\mu) = S(-\mu) = \frac{1}{4\pi}\tilde{\omega}F_{\odot}\frac{\mu_0}{\mu+\mu_0}H(\mu_0)H(\mu). \tag{6.2.36}$$

Thus, the diffuse reflection can be expressed in terms of the H function. This simple expression has been used to interpret the absorption line formation in cloudy atmospheres of other planets. Exercise 6.1 illustrates the applicability of the law of diffuse reflection.

6.2.3 General Solution for Anisotropic Scattering

To solve the general radiative transfer equation defined in Eq. (6.1.5), we first seek the solution for the homogeneous part of the differential equation and then add a particular solution for the inhomogeneous part. After some mathematical manipulation, the equation can be presented in the form

$$I^m(\tau,\mu_i) = \sum_j L_j^m \varphi_j^m(\mu_i)e^{-k_j^m \tau} + Z^m(\mu_i)e^{-\tau/\mu_0}, \tag{6.2.37}$$

where the eigenfunction derived from the associated homogeneous system is

$$\varphi_j^m(\mu_i) = \frac{1}{1+\mu_i k_j^m}\sum_{\ell=0}^{N}\tilde{\omega}_\ell^m \xi_\ell^m P_\ell^m(\mu_i), \tag{6.2.38}$$

and the Z function is

$$Z^m(\mu_i) = \frac{1}{4\pi}\tilde{\omega}F_{\odot}P_m^m(-\mu_0)\frac{H^m(\mu_0)H^m(-\mu_0)}{1+\mu_i/\mu_0}\sum_{\ell=0}^{N}\tilde{\omega}_\ell^m \xi_\ell^m\left(\frac{1}{\mu_0}\right)P_\ell^m(\mu_i). \tag{6.2.39}$$

The ξ function has the recursion form

$$\xi_{\ell+1}^m = -\frac{2\ell+1-\tilde{\omega}_\ell}{k(\ell-m+1)}\xi_\ell^m - \frac{\ell+m}{\ell-m+1}\xi_{\ell-1}^m. \tag{6.2.40}$$

Finally, the eigenvalues k_j^m can be determined from the characteristic equation described by

$$1 = \frac{\tilde{\omega}}{2}\sum_j \frac{a_j}{1+\mu_j k}\left[\sum_{\lambda=m}^{N}\tilde{\omega}_\lambda^m \xi_\lambda^m(k)P_\lambda^m(\mu_j)P_m^m(\mu_j)\right]. \tag{6.2.41}$$

Equation (6.2.41) is of order n in k^2 and admits, in general, $2n$ distinct nonvanishing eigenvalues that must occur in pairs. For strong anisotropic scattering having a sharp phase function, a number of eigenvalues are normally contained in the interval $(0, \mu_n^{-1})$ in which the eigenvalue pattern is highly unsymmetric and differs substantially from that displayed in Fig. 6.2.

The unknown coefficients L_j^m must be determined from the radiation boundary conditions. For simple boundary conditions given by Eq. (6.2.23), and in view of the intensity expansion in Eq. (6.1.4), we have

$$\left.\begin{aligned} I^m(0, -\mu_i) &= 0 \\ I^m(\tau_1, \mu_i) &= 0 \end{aligned}\right\} \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad m = 0, \ldots, N. \tag{6.2.42}$$

We may then determine L_j^m m times independently with the final result given by Eq. (6.1.4). At this point, the analytic solution for Eq. (6.1.5) is complete.

The solution expressed in Eq. (6.2.37) is valid only for nonconservative scattering because when $\tilde{\omega} = 1$, $k^2 = 0$ will satisfy the characteristic equation for $m = 0$, and $\xi_\ell^0(k)$ becomes indefinite. Thus, a different solution must be derived. Since there is no absorption for conservative scattering, the flux of radiation normal to the plane of stratification is constant. It can be shown that the transfer equation admits a solution of the form, for $m = 0$,

$$I^0(\tau, \mu_i) = \sum_{j=-(n-1)}^{n-1} L_j^0 \varphi_j^0(\mu_i) e^{-k_j^0 \tau} + [(1 - \tilde{\omega}_1/3)\tau + \mu_i] L_{-n}^0 + L_n^0 + Z^0(\mu_i)^{-\tau/\mu_0}. \tag{6.2.43}$$

A mathematical procedure has been developed to compute the eigenvalues k_j^m from a recurrence characteristic equation (Chandrasekhar, 1950). The eigenvectors $\varphi_j^m(\mu_i)$ may be expressed in terms of known functions, which contain the eigenvalues, while the particular solution is related to the known H function. The characteristic equation for the eigenvalues derived by Chandrasekhar is, however, mathematically as well as numerically ambiguous. The method is unstable for highly peaked phase functions, as pointed out by Liou (1973a), who discovered that the solution of the characteristic equation may be formulated as an algebraic eigenvalue problem. Further, Asano (1975) has shown that the degree of the characteristic equation for the eigenvalues can be reduced by a factor of 2 because the solution for the eigenvalues may be obtained by solving a characteristic polynomial of degree n for k^2. Both of these authors have expanded the matrix in polynomial form to solve the characteristic equation for the eigenvalues corresponding to the associated homogeneous system of the differential equations. However, the expansion in polynomial form is not a stable numerical scheme for obtaining eigenvalues. To solve the algebraic eigenvalue problem, a well-developed numerical subroutine found in the *IMSL User's Manual* (1987) can be used to compute the eigenvalues and eigenvectors of a real general matrix in connection with the discrete-ordinates method. Stamnes and Dale (1981) have shown that azimuthally dependent scattered intensities can be computed accurately using numerical methods.

In the discrete-ordinates method for radiative transfer, analytical solutions for diffuse intensity are explicitly given for any optical depth. Thus, the internal radiation field can be evaluated without additional computational effort. Moreover, analytic two- and four-stream approximations can be developed from this method for flux calculations. In the following, we present a matrix formulation of the discrete-ordinates method. We shall do so by considering the azimuth-independent component in the diffuse intensity component. On replacing the integral with a summation and omitting the emission term, Eq. (6.1.12) may be written in the form

$$\mu_i \frac{dI(\tau, \mu_i)}{d\tau} = I(\tau, \mu_i) - \frac{\tilde{\omega}}{2} \sum_{j=-n}^{n} I(\tau, \mu_j) P(\mu_i, \mu_j) a_j$$
$$- \frac{\tilde{\omega}}{4\pi} F_\odot P(\mu_i, -\mu_0) e^{-\tau/\mu_0}, \qquad i = -n, \ldots, n, \quad (6.2.44)$$

where we may select the quadrature weights and points that satisfy $a_{-j} = a_j (\Sigma_j a_j = 2)$ and $\mu_{-j} = -\mu_j$. To simplify this equation, we may define

$$c_{i,j} = \frac{\tilde{\omega}}{2} a_j P(\mu_i, \mu_j) = \frac{\tilde{\omega}}{2} a_j \sum_{\ell=0}^{N} \tilde{\omega}_\ell P_\ell(\mu_i) P_\ell(\mu_j), \qquad j = -n, \ldots -0, \ldots, n, \quad (6.2.45)$$

and

$$I(\tau, -\mu_0) = e^{-\tau/\mu_0} F_\odot / 2\pi, \quad (6.2.46)$$

where we set $a_{-0} = 1$ and the notation -0 is used to be consistent with the definition $\mu_{-0} = -\mu_0$. On the basis of the definition of Legendre polynomials, we have

$$c_{i,-j} = c_{-i,j}, \qquad c_{-i,-j} = c_{i,j}, \qquad i \neq -0. \quad (6.2.47)$$

Moreover, we define

$$b_{i,j} = \begin{cases} c_{i,j}/\mu_i, & i \neq j \\ (c_{i,j} - 1)/\mu_i, & i = j. \end{cases} \quad (6.2.48)$$

It follows that $b_{i,j} = -b_{-i,-j}$, and $b_{i,-j} = -b_{-i,j}$. Using the preceding definitions, Eq. (6.2.44) becomes

$$\frac{dI(\tau, \mu_i)}{d\tau} = \sum_j b_{i,j} I(\tau, \mu_j). \quad (6.2.49)$$

We may separate the upward and downward intensities in the forms

$$\frac{dI(\tau, \mu_i)}{d\tau} = \sum_{j=1}^{n} b_{i,j} I(\tau, \mu_i) + \sum_{j=0}^{n} b_{i,-j} I(\tau, -\mu_j), \quad (6.2.50a)$$

$$\frac{dI(\tau, -\mu_i)}{d\tau} = \sum_{j=1}^{n} b_{-i,j} I(\tau, \mu_j) + \sum_{j=0}^{n} b_{-i,-j} I(\tau, -\mu_j). \quad (6.2.50b)$$

In terms of a matrix representation for the homogeneous part, we write

$$\frac{d}{d\tau} \begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix} = \begin{bmatrix} \mathbf{b}^+ & \mathbf{b}^- \\ -\mathbf{b}^- & -\mathbf{b}^+ \end{bmatrix} \begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix}, \quad (6.2.51)$$

where

$$\mathbf{I}^{\pm} = \begin{bmatrix} I(\tau, \pm\mu_1) \\ I(\tau, \pm\mu_2) \\ \vdots \\ I(\tau, \pm\mu_n) \end{bmatrix}, \tag{6.2.52}$$

and $\mathbf{b}^{\pm}$ denotes the elements associated with $b_{i,j}$ and $b_{i,-j}$. Since Eq. (6.2.51) is a first-order differential equation, we may seek a solution in the form

$$\mathbf{I}^{\pm} = \varphi^{\pm} e^{-k\tau}. \tag{6.2.53}$$

Substituting Eq. (6.2.53) into Eq. (6.2.51) leads to

$$\begin{bmatrix} \mathbf{b}^{+} & \mathbf{b}^{-} \\ -\mathbf{b}^{-} & -\mathbf{b}^{+} \end{bmatrix} \begin{bmatrix} \varphi^{+} \\ \varphi^{-} \end{bmatrix} = -k \begin{bmatrix} \varphi^{+} \\ \varphi^{-} \end{bmatrix}. \tag{6.2.54}$$

Equation (6.2.54) may be solved as a standard eigenvalue problem. In the discrete-ordinates method for radiative transfer, the eigenvalues associated with the differential equations are all real and occur in pairs ($\pm k$), as pointed out previously. This property can also be understood from the symmetry of the **b** matrix. Thus, the rank of the matrix may be reduced by a factor of 2. To accomplish this reduction, we rewrite Eq. (6.2.54) in the forms

$$\mathbf{b}^{+}\varphi^{+} + \mathbf{b}^{-}\varphi^{-} = -k\varphi^{+}, \tag{6.2.55a}$$

$$\mathbf{b}^{-}\varphi^{+} + \mathbf{b}^{+}\varphi^{-} = k\varphi^{-}. \tag{6.2.55b}$$

Adding and subtracting these two equations yields

$$(\mathbf{b}^{+} - \mathbf{b}^{-})(\mathbf{b}^{+} + \mathbf{b}^{-})(\varphi^{+} + \varphi^{-}) = k^2(\varphi^{+} + \varphi^{-}). \tag{6.2.56}$$

It follows that the eigenvectors of the original system, $\varphi^{\pm}$, can now be obtained from the reduced system, $(\varphi^{+} + \varphi^{-})$, in terms of the eigenvalue k^2.

As discussed by Chandrasekhar (1950), the Gaussian quadrature formula for the complete angular range, $-1 < \mu < 1$, is efficient and accurate for the discretization of the basic radiative transfer equation. However, the Gaussian quadrature can also be applied separately to the half-ranges, $-1 < \mu < 0$ and $0 < \mu < 1$, which are referred to as *double-Gauss quadrature*. The division appears to offer numerical advantages when upward and downward radiation streams are treated separately.

6.2.4 Application to Nonhomogeneous Atmospheres

One of the fundamental difficulties in radiative transfer involves accounting for the nonhomogeneous nature of the atmosphere, which cannot be represented by a single single-scattering albedo $\tilde{\omega}$ and a phase function P. The radiative transfer equation for diffuse intensity must be modified to include variations in $\tilde{\omega}$ and P with optical depth.

The discrete-ordinates method for radiative transfer can be applied to nonhomogeneous atmospheres by numerical means (Liou, 1975). In the following analysis, consider the azimuthally independent transfer equation for diffuse radiation as follows:

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\tilde{\omega}(\tau)}{2} \int_{-1}^{1} I(\tau, \mu') P(\tau; \mu, \mu')\, d\mu' - \frac{\tilde{\omega}(\tau)}{4\pi} P(\tau; \mu, -\mu_0) F_{\odot} e^{-\tau/\mu_0}. \tag{6.2.57}$$

Since $\tilde{\omega}$ and P are functions of optical depth, analytic solutions for this equation are generally not possible. We may, however, devise a numerical procedure to compute the diffuse intensity in nonhomogeneous atmospheres.

As illustrated in Fig. 6.3, the atmosphere may be divided into N homogeneous layers, each of which is characterized by a single-scattering albedo, a phase function, and an extinction coefficient (or optical depth). The solution for the azimuthally independent diffuse intensity ($m = 0$), as given in Eq. (6.2.37), may be written for each individual layer ℓ in the form

$$I^{(\ell)}(\tau, \mu_i) = \sum_j L_j^{(\ell)} \varphi_j^{(\ell)}(\mu_i) e^{-k_j^{(\ell)}\tau} + Z^{(\ell)}(\mu_i) e^{-\tau/\mu_0}, \qquad \ell = 1, 2, \ldots, N. \tag{6.2.58}$$

At the top of the atmosphere (TOA) ($\tau = 0$), there is no downward diffuse flux, so that

$$I^{(1)}(0, -\mu_i) = 0. \tag{6.2.59}$$

Within the atmosphere, the upward and downward intensities must be continuous at the interface of each predivided layer. Thus, we have

$$I^{(\ell)}(\tau_\ell, \mu_i) = I^{(\ell+1)}(\tau_\ell, \mu_i), \tag{6.2.60}$$

where τ_ℓ denotes the optical depth from TOA to the bottom of the ℓ layer. At the bottom of the atmosphere, surface reflectance must be accounted for. The reflection of sunlight at the surface depends significantly on the latter's optical property with respect to the incident wavelength. For example, reflectance from vegetation and soil is highly wavelength-dependent. Also, surface reflectance patterns are generally anisotropic. For the computation of solar fluxes in the atmosphere, however, it is appropriate to use a Lambertian surface. Let the surface albedo be r_s. Then the upward diffuse intensities that are reflected from the surface may be expressed by

$$I^{(N)}(\tau_N, +\mu_i) = \frac{r_s}{\pi} \left[F^{\downarrow}(\tau_n) + \mu_0 F_{\odot} e^{-\tau_N/\mu_0} \right], \tag{6.2.61}$$

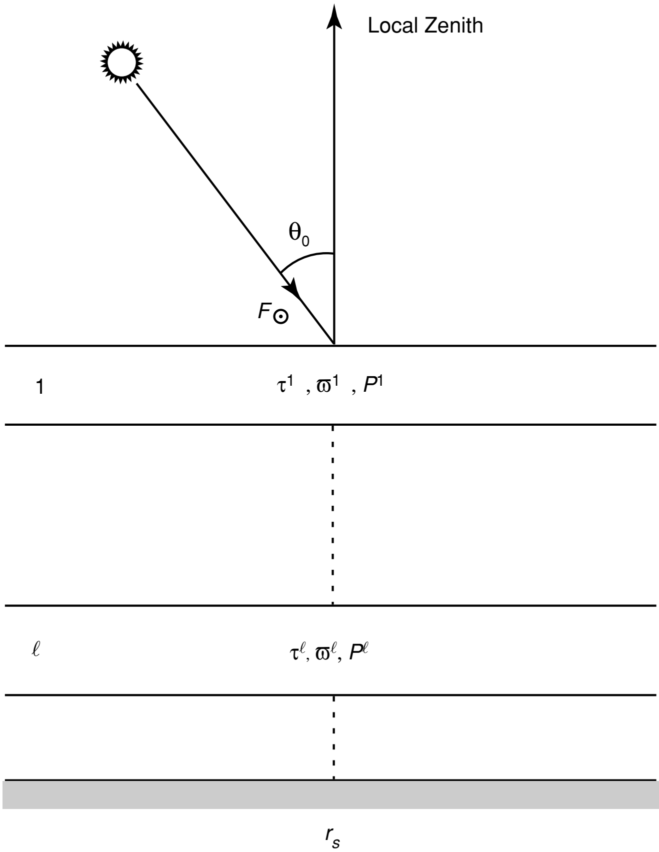

Figure 6.3 A nonhomogeneous atmosphere is divided into ℓ homogeneous layers with respect to the single-scattering albedo $\bar{\omega}$, phase function P, and optical depth τ. $F_\odot$ denotes the solar flux at the top of the atmosphere, θ_0 is the solar zenith angle, and r_s is the surface albedo.

where the downward diffuse flux reaching the surface is

$$F^{\downarrow}(\tau_N) = 2\pi \sum_{i=1}^{n} I^{(N)}(\tau_N, -\mu_i)\, a_i \mu_i. \tag{6.2.62}$$

Matching the required boundary and continuity conditions for the diffuse intensities, we obtain the following set of equations for the determination of the unknown coefficients:

$$\sum_j L_j^{(1)} \varphi_j^{(1)}(-\mu_i) = -Z^{(1)}(-\mu_i), \qquad i = 1, \ldots, n, \tag{6.2.63a}$$

$$\sum_j \left[L_j^{(\ell)} \gamma_j^{(\ell)}(\mu_i) + L_j^{(\ell+1)} \delta_j^{(\ell+1)}(\mu_i) \right] = -{}^{(\ell)}\eta^{(\ell+1)}(\mu_i),$$

$$i = -n, \ldots, -1, 1, \ldots, n, \qquad \ell = 1, 2, \ldots, N-1, \tag{6.2.63b}$$

$$\sum_j L_j^{(N)} \beta_j^{(N)}(+\mu_i) = -\varepsilon^{(N)}(+\mu_i), \qquad i = 1, \ldots, n, \tag{6.2.63c}$$

where

$$\gamma_j^{(\ell)}(\mu_i) = \varphi_j^{(\ell)}(\mu_i) e^{-k_j^{(\ell)} \tau_\ell}, \tag{6.2.64a}$$

$$\delta_j^{(\ell+1)}(\mu_i) = -\varphi_j^{(\ell+1)}(\mu_i) e^{-k_j^{(\ell+1)} \tau_\ell}, \tag{6.2.64b}$$

$${}^{(\ell)}\eta^{(\ell+1)}(\mu_i) = \left[Z^{(\ell)}(\mu_i) - Z^{(\ell+1)}(\mu_i) \right] e^{-\tau_\ell/\mu_0}, \tag{6.2.64c}$$

$$\beta_j^{(N)}(+\mu_i) = \left[\varphi_j^{(N)}(+\mu_i) - 2r_s \sum_{i=1}^{n} \varphi_j^{(N)}(-\mu_i) a_i \mu_i \right] e^{-k_j^{(N)} \tau_N}, \tag{6.2.64d}$$

$$\varepsilon^{(N)}(+\mu_i) = \left[Z^{(N)}(+\mu_i) - 2r_s \sum_{i=1}^{n} Z^{(N)}(-\mu_i) a_i \mu_i - \frac{r_s}{\pi} \mu_0 F_\odot \right] e^{-\tau_N/\mu_0}. \tag{6.2.64e}$$

Thus, we have $N \times 2n$ equations for the determination of $N \times 2n$ unknown coefficients, $L_j^{(\ell)}$. Equations (6.2.63a)–(6.2.63e) may be expressed in terms of a matrix representation in the form

$$\boldsymbol{\varphi} \mathbf{L} = \boldsymbol{\chi}. \tag{6.2.65a}$$

where

$$\mathbf{L} = \begin{bmatrix} L_{-n}^{(1)} \\ \vdots \\ L_n^{(1)} \\ L_{-n}^{(2)} \\ \vdots \\ L_n^{(2)} \\ \vdots \\ L_{-n}^{(N)} \\ \vdots \\ L_n^{(N)} \end{bmatrix}, \qquad -\boldsymbol{\chi} = \begin{bmatrix} Z^{(1)}(-\mu_n) \\ \vdots \\ Z^{(1)}(-\mu_1) \\ {}^{(1)}\eta^{(2)}(-\mu_n) \\ \vdots \\ {}^{(1)}\eta^{(2)}(+\mu_n) \\ \vdots \\ \varepsilon^{(N)}(+\mu_1) \\ \vdots \\ \varepsilon^{(N)}(+\mu_n) \end{bmatrix}, \tag{6.2.65b}$$

and

$$\varphi = \begin{bmatrix} \varphi_{-n}^{(1)}(-\mu_n) & \cdots & \varphi_n^{(1)}(-\mu_n) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \varphi_{-n}^{(1)}(-\mu_1) & \cdots & \varphi_n^{(1)}(-\mu_1) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \gamma_{-n}^{(1)}(-\mu_n) & \cdots & \gamma_n^{(1)}(-\mu_n) & \delta_{-n}^{(2)}(-\mu_n) & \cdots & \delta_n^{(2)}(-\mu_n) & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \gamma_{-n}^{(1)}(\mu_n) & \cdots & \gamma_n^{(1)}(\mu_n) & \delta_{-n}^{(2)}(\mu_n) & \cdots & \delta_n^{(2)}(\mu_n) & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \beta_{-n}^{(N)}(\mu_1) & \cdots & \beta_n^{(N)}(\mu_1) \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \beta_{-n}^{(N)}(\mu_n) & \cdots & \beta_n^{(N)}(\mu_n) \end{bmatrix}. \tag{6.2.65c}$$

Azimuthally dependent components can be formulated in a likely manner to obtain the angular intensity pattern. At this point, however, the discrete-ordinates method for radiative transfer has not been applied to cases involving polarization.

6.3 Principles of Invariance

6.3.1 Definitions of Scattering Parameters

The principles-of-invariance method for radiative transfer seeks certain physical statements and mathematical formulations regarding the fields of reflection and transmission of light beams. In this method, the radiation field is not derived directly from the transfer equation, as in the case of the discrete-ordinates method.

To introduce the principles of invariance and other multiple-scattering problems, it is necessary to define and clarify a number of parameters that have been used in the literature. We find it convenient to express the solutions to multiple-scattering problems in terms of the *reflection function R* and *transmission function T* in the forms

$$I_r(0;\ \mu, \phi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 R(\mu, \phi; \mu', \phi')I_\odot(-\mu', \phi')\mu' d\mu' d\phi', \tag{6.3.1a}$$

$$I_t(\tau_1;\ -\mu, \phi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 T(\mu, \phi; \mu', \phi')I_\odot(-\mu', \phi')\mu' d\mu' d\phi', \tag{6.3.1b}$$

where $I_\odot(-\mu, \phi)$ represents the intensity of sunlight incident on the top of the scattering layer. It suffices for most practical problems to approximate the intensity as monodirectional in the form

$$I_\odot(-\mu, \phi) = \delta(\mu - \mu_0)\delta(\phi - \phi_0)F_\odot, \tag{6.3.2}$$

where δ is the Dirac delta function, and $F_\odot$ denotes the solar flux density in the direction of the incident beam. Thus, from Eqs. (6.3.1a) and (6.3.1b), we have the definitions of the reflection and transmission functions in the forms

$$R(\mu, \phi; \mu_0, \phi_0) = \pi I_r(0; \mu, \phi)/(\mu_0 F_\odot), \tag{6.3.3a}$$

$$T(\mu, \phi; \mu_0, \phi_0) = \pi I_t(\tau_1; -\mu, \phi)/(\mu_0 F_\odot). \tag{6.3.3b}$$

Note here that $I_t(\tau_1; -\mu, \phi)$ represents the diffusely transmitted intensity, which does not include the directly transmitted solar intensity $F_\odot e^{-\tau_1/\mu_0}$. The direct component represents the attenuation of the incident solar beam that penetrates to the level τ_1. In the case where polarization is considered, in which the four Stokes parameters are required, R and T are composed of four rows and four columns and are referred to as *reflection* and *transmission matrices.* The reflection and transmission functions also have been referred to as reflection and transmission coefficients by Ambartzumian (1958) and Sobolev (1975). In satellite remote sensing, a parameter called *bidirectional reflectance,* which is analogous to the reflection function, is frequently used (Section 7.3.1).

On the basis of Eqs. (6.3.3a) and (6.3.3b), we may define the *reflection* r (also called the *local* or *planetary albedo*) and *transmission* (diffuse) t associated with reflected (upward) and transmitted (downward) flux densities in the forms

$$r(\mu_0) = \frac{F_{\mathrm{dif}}^{\uparrow}(0)}{\mu_0 F_\odot} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 R(\mu, \phi; \mu_0, \phi_0)\, \mu \, d\mu \, d\phi, \tag{6.3.4a}$$

$$t(\mu_0) = \frac{F_{\mathrm{dif}}^{\downarrow}(\tau_1)}{\mu_0 F_\odot} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 T(\mu, \phi; \mu_0, \phi_0)\, \mu \, d\mu \, d\phi. \tag{6.3.4b}$$

Note that the direct transmission is simply $e^{-\tau_1/\mu_0}$. In a similar manner, *absorption* of the atmosphere, bounded by the optical depths of 0 and τ_1, can be obtained from the net flux density divergence that includes the direct transmission component at levels of 0 and τ_1 and normalizes by $\mu_0 F_\odot$. To find the flux of energy reflected by the planet, we consider on the sphere a ring with radius a' and width da', where a' is the projected distance from the center of the disk, as shown in Fig. 6.4. Hence the flux of energy reflected by this ring is given by $r(\mu_0) F_\odot 2\pi a'\, da'$. But $a' = a \sin\theta_0$, and $da' = a \cos\theta_0 \, d\theta_0$. Thus, we may write this flux of energy as $2\pi a^2 F_\odot r(\mu_0)\mu_0 \, d\mu_0$. The flux of energy reflected by the entire planet is then given by

$$f^{\uparrow}(0) = 2\pi a^2 F_\odot \int_0^1 r(\mu_0)\, \mu_0 \, d\mu_0. \tag{6.3.5a}$$

The *spherical* (or *global*) *albedo,* which represents the ratio of the flux of energy reflected by the entire planet to the incident energy, is then given by

$$\bar{r} = \frac{f^{\uparrow}(0)}{\pi a^2 F_\odot} = 2 \int_0^1 r(\mu_0)\, \mu_0 \, d\mu_0. \tag{6.3.5b}$$

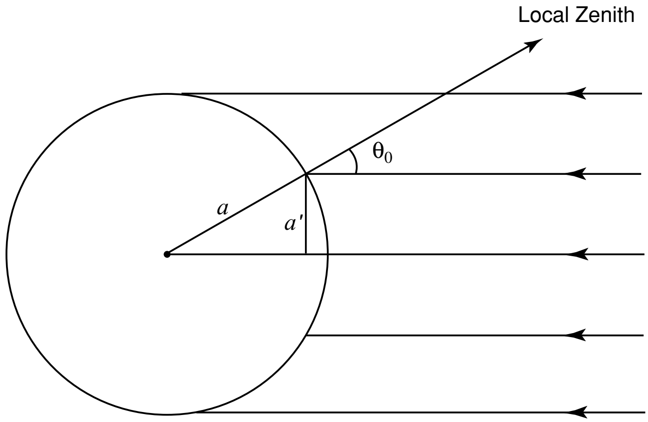

Figure 6.4 Geometry for the definition of the spherical albedo, where a is the radius of the earth and θ_0 is the solar zenith angle.

Likewise, the *global diffuse transmission* is

$$\bar{t} = \frac{f^{\downarrow}(\tau_1)}{\pi a^2 F_{\odot}} = 2\int_0^1 t(\mu_0)\,\mu_0\,d\mu_0 \tag{6.3.5c}$$

and the global direct transmission is simply $2\int_0^1 e^{-\tau_1/\mu_0}\mu_0\,d\mu_0$.

Chandrasekhar (1950) expressed the resulting laws of diffuse reflection and transmission for a finite atmosphere with an optical depth τ_1 in terms of the *scattering function S* and the *transmission function T_c* (diffuse), which differ from the parameters defined in Eqs. (6.3.1a) and (6.3.1b), in the forms

$$I_r(0;\mu,\phi) = \frac{1}{4\pi\mu}\int_0^{2\pi}\int_0^1 S(\mu,\phi;\mu',\phi')I_{\odot}(-\mu',\phi')\,d\mu' d\phi', \tag{6.3.6a}$$

$$I_t(\tau_1;-\mu,\phi) = \frac{1}{4\pi\mu}\int_0^{2\pi}\int_0^1 T_c(\mu,\phi;\mu',\phi')I_{\odot}(-\mu',\phi')\,d\mu' d\phi', \tag{6.3.6b}$$

where T_c is used to differentiate from T defined previously. On substituting Eq. (6.3.2) into Eqs. (6.3.6a) and (6.3.6b), we obtain the definitions of Chandrasekhar's scattering and transmission functions as follows:

$$S(\mu,\phi;\mu_0,\phi_0) = 4\mu\mu_0\pi I_r(0;\mu,\phi)/(\mu_0 F_{\odot}), \tag{6.3.7a}$$

$$T_c(\mu,\phi;\mu_0,\phi_0) = 4\mu\mu_0\pi I_t(\tau_1;-\mu,\phi)/(\mu_0 F_{\odot}). \tag{6.3.7b}$$

The introduction of the factor μ into the intensity parameters gives the required symmetry of S and T_c in the pair of variables (μ,ϕ) and (μ_0,ϕ_0) such that

$$S(\mu,\phi;\mu_0,\phi_0) = S(\mu_0,\phi_0;\mu,\phi), \tag{6.3.8a}$$

$$T_c(\mu,\phi;\mu_0,\phi_0) = T_c(\mu_0,\phi_0;\mu,\phi). \tag{6.3.8b}$$

6.3.2 Principles of Invariance for Semi-infinite Atmospheres

Consider a flux of parallel solar radiation $F_\odot$ in a direction defined by $(-\mu_0, \phi_0)$ ($-\mu_0$ denotes that the light beam is downward), incident on the outer boundary of a semiinfinite, plane-parallel atmosphere. The principles of invariance originally introduced by Ambartzumian (1942, 1958) stated that the diffuse reflected intensity from such an atmosphere cannot be changed if a plane layer of finite optical depth having the same optical properties as those of the original atmosphere is added. Let the optical depth of the added layer be $\Delta\tau$, which is so small that $(\Delta\tau)^2$ can be neglected when it is compared with $\Delta\tau$ itself. For simplicity in presenting the principles of invariance, we shall neglect the azimuthal dependence of the diffuse reflected intensity and define the *reflection function* in terms of the diffuse reflected intensity at the top of a semi-infinite atmosphere $I(0, \mu)$ in the form [see Eq. 6.3.3a)]

$$R(\mu, \mu_0) = \pi I(0, \mu)/(\mu_0 F_\odot). \tag{6.3.9}$$

In reference to Fig. 6.5, the reduction or increase of the reflection function, after the addition of an infinitesimal layer, can be defined by the following principles:

1. The differential attenuation of the reflection function in passing through $\Delta\tau$ downward, based on the first term on the right-hand side of Eq. (3.4.2) ($\Delta\tau = \beta_e \Delta z$), is given by

$$\Delta R_1' = -R(\mu, \mu_0)\Delta\tau/\mu_0. \tag{6.3.10}$$

The reflection function at $\tau = 0$ is now $(R + \Delta R_1')$, which is again attenuated in passing through $\Delta\tau$ upward. Thus,

$$\Delta R_1'' = -\left[R(\mu, \mu_0) + \Delta R_1'\right]\Delta\tau/\mu. \tag{6.3.11}$$

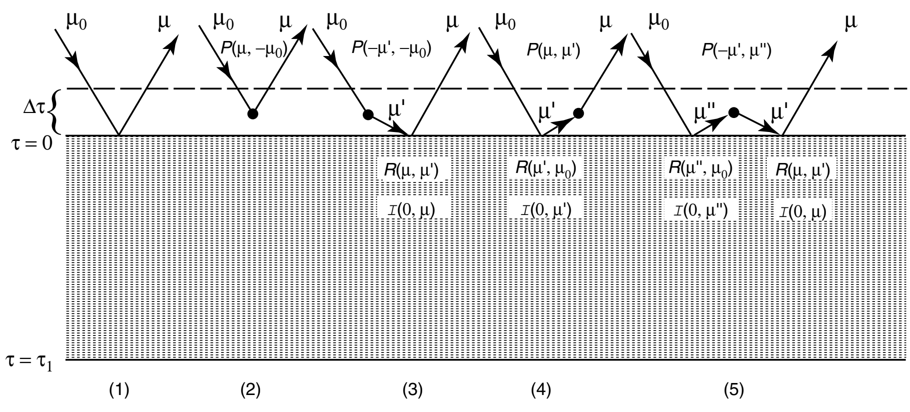

Figure 6.5 The principles of invariance for a semi-infinite, plane-parallel atmosphere. The μ and $-\mu$ denote the upward and downward directions, respectively. The black dots show that scattering events take place in which the phase function is required. The directional representation in the argument is such that the emergent angle is stated first and is then followed by the incident angle. A similar rule governs the argument of the reflection function.

The total attenuation is therefore

$$\Delta R_1 = \Delta R_1' + \Delta R_1'' = -R(\mu, \mu_0)\left[\Delta\tau\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right) - \frac{\Delta\tau^2}{\mu\mu_0}\right]$$
$$\approx -R(\mu, \mu_0)\Delta\tau\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right). \quad (6.3.12)$$

2. However, $\Delta\tau$ may scatter directly in the direction μ a part of the solar flux $F_\odot$ that is incident on it. Based on the second term on the right-hand side of Eq. (3.4.2), we find the additional reflection as follows:

$$\Delta R_2 = \frac{\pi}{\mu_0 F_\odot}\frac{\tilde{\omega}}{4\pi}F_\odot P(\mu, -\mu_0)\Delta\tau/\mu = \frac{\tilde{\omega}}{4}P(\mu, -\mu_0)\Delta\tau/(\mu\mu_0). \quad (6.3.13)$$

3. In addition, $\Delta\tau$ may scatter a part of the solar flux in the direction μ' onto the boundary $\tau = 0$. The diffuse light beam then undergoes reflection from this surface, and this additional reflection, analogous to the third term on the right-hand side of Eq. (3.4.2), is given by

$$\Delta R_3 = \frac{\pi}{\mu_0 F_\odot}\frac{\tilde{\omega}}{4\pi}\int_0^{2\pi} d\phi' \int_0^1 I(0, \mu)P(-\mu', -\mu_0)\, d\mu' \frac{\Delta\tau}{\mu'}$$
$$= \frac{\tilde{\omega}}{2}\frac{\Delta\tau}{\mu_0}\int_0^1 R(\mu, \mu')P(-\mu', -\mu_0)\, d\mu'. \quad (6.3.14)$$

4. Moreover, $\Delta\tau$, after attenuating a fraction of the light beam diffusely reflected from the boundary $\tau = 0$ in the direction μ', may scatter a part of it in the direction μ. This incremental reflection is given by

$$\Delta R_4 = \frac{\pi}{\mu_0 F_\odot}\frac{\tilde{\omega}}{4\pi}\int_0^{2\pi} d\phi' \int_0^1 P(\mu, \mu')I(0, \mu')\, d\mu' \frac{\Delta\tau}{\mu}$$
$$= \frac{\tilde{\omega}}{2}\frac{\Delta\tau}{\mu}\int_0^1 P(\mu, \mu')R(\mu', \mu_0)\, d\mu'. \quad (6.3.15)$$

5. Finally, the unscattered component of the solar flux $F_\odot$, which is reflected from the boundary $\tau = 0$ in the direction μ'', is scattered by $\Delta\tau$ back to $\tau = 0$ in the different direction μ', and again is reflected from the surface $\tau = 0$ in the direction μ. This additional contribution may be expressed by

$$\Delta R_5 = \frac{\pi}{\mu_0 F_\odot}\frac{\tilde{\omega}}{4\pi}\int_0^{2\pi} d\phi' \int_0^1 \frac{I(0, \mu)}{F_\odot}\, d\mu'$$
$$\times\left[\int_0^{2\pi} d\phi'' \int_0^1 P(-\mu', \mu'')\, I(0, \mu'')\, d\mu''\right]\frac{\Delta\tau}{\mu'}$$
$$= \tilde{\omega}\Delta\tau\int_0^1 R(\mu, \mu')\, d\mu'\left[\int_0^1 P(-\mu', \mu'')R(\mu'', \mu_0)\, d\mu''\right]. \quad (6.3.16)$$

On the basis of the principles of invariance stated above, we must have

$$\Delta R_1 + \Delta R_2 + \Delta R_3 + \Delta R_4 + \Delta R_5 = 0. \tag{6.3.17}$$

It follows that

$$\begin{aligned} R(\mu, \mu_0)\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right) = \frac{\tilde{\omega}}{4\mu\mu_0} \Bigg\{ & P(\mu, -\mu_0) + 2\mu \int_0^1 R(\mu, \mu')P(-\mu', -\mu_0)\, d\mu' \\ & + 2\mu_0 \int_0^1 P(\mu, \mu')R(\mu', \mu_0)\, d\mu' + 4\mu\mu_0 \int_0^1 R(\mu, \mu')\, d\mu' \\ & \times \left[\int_0^1 P(-\mu', \mu'')R(\mu'', \mu_0)\, d\mu''\right]\Bigg\}. \end{aligned} \tag{6.3.18}$$

For a simple case of isotropic scattering, Eq. (6.3.18) becomes

$$\begin{aligned} R(\mu, \mu_0)(\mu + \mu_0) &= \frac{\tilde{\omega}}{4}\left[1 + 2\mu \int_0^1 R(\mu, \mu')\, d\mu' + 2\mu_0 \int_0^1 R(\mu', \mu_0)\, d\mu' \right. \\ &\qquad \left. + 4\mu\mu_0 \int_0^1 R(\mu', \mu_0)\, d\mu' \int_0^1 R(\mu, \mu'')\, d\mu''\right] \\ &= \frac{\tilde{\omega}}{4}\left[1 + 2\mu \int_0^1 R(\mu, \mu')\, d\mu'\right]\left[1 + 2\mu_0 \int_0^1 R(\mu', \mu_0)\, d\mu'\right]. \end{aligned} \tag{6.3.19}$$

Inspection of Eq. (6.3.19) reveals that if it is satisfied by the function $R(\mu, \mu_0)$, it must also be satisfied by the function $R(\mu_0, \mu)$. And since this equation can have only one solution, we must have

$$R(\mu, \mu_0) \equiv R(\mu_0, \mu). \tag{6.3.20}$$

With this relationship, which is stated here without a rigorous mathematical proof, we may define

$$H(\mu) = 1 + 2\mu \int_0^1 R(\mu, \mu')\, d\mu', \tag{6.3.21}$$

such that

$$R(\mu, \mu_0) = \frac{\tilde{\omega}}{4}\frac{H(\mu)H(\mu_0)}{\mu + \mu_0}. \tag{6.3.22}$$

In reference to Section 6.2.2, we find that this expression is exactly the same as that in Eq. (6.2.36). It is indeed the exact solution for the semi-infinite isotropic scattering atmosphere. To examine the H function, we insert Eq. (6.3.22) into Eq. (6.3.21) to obtain

$$H(\mu) = 1 + \frac{\tilde{\omega}}{2}\mu H(\mu) \int_0^1 \frac{H(\mu')\, d\mu'}{\mu + \mu'}. \tag{6.3.23}$$

It is now clear that the solution of Eq. (6.3.19) is reduced to solving the H function. To do so, we may select an approximate value and then carry out appropriate iterations. We first seek the mean value of H in the form

$$H_0 = \int_0^1 H(\mu)\, d\mu. \tag{6.3.24}$$

From Eq. (6.3.23), we have

$$\int_0^1 H(\mu)\, d\mu = 1 + \frac{\tilde{\omega}}{2} \int_0^1 \int_0^1 \frac{H(\mu)H(\mu')\mu}{\mu + \mu'}\, d\mu\, d\mu'. \tag{6.3.25}$$

On interchanging μ with μ', Eq. (6.3.25) remains the same. Thus, we may write

$$\begin{aligned}\int_0^1 H(\mu)\, d\mu &= 1 + \frac{\tilde{\omega}}{4} \int_0^1 \int_0^1 \frac{H(\mu)H(\mu')\mu}{\mu + \mu'}\, d\mu\, d\mu' \\ &\quad + \frac{\tilde{\omega}}{4} \int_0^1 \int_0^1 \frac{H(\mu)H(\mu')\mu'}{\mu + \mu'}\, d\mu\, d\mu' \\ &= 1 + \frac{\tilde{\omega}}{4} \int_0^1 H(\mu)\, d\mu \int_0^1 H(\mu')\, d\mu'. \end{aligned} \tag{6.3.26}$$

Consequently, we obtain

$$H_0 = 1 + \frac{\tilde{\omega}}{4} H_0^2. \tag{6.3.27a}$$

This gives the solution of H_0 in the form

$$H_0 \equiv \int_0^1 H(\mu)\, d\mu = \frac{2}{\tilde{\omega}} \left(1 - \sqrt{1 - \tilde{\omega}}\right), \tag{6.3.27b}$$

where the positive root is found to be unrealistic because the albedo value becomes greater than unity (Exercise 6.2). To find $H(\mu)$ in Eq. (6.3.23), we may insert this zero-order approximation into the right-hand side of this equation to obtain a first approximation. The iterative procedure can be continued until a desirable degree of accuracy is achieved.

6.3.3 Principles of Invariance for Finite Atmospheres

In the last section, we described the principles of invariance for a semi-infinite atmosphere in which only the reflection function is involved. We now introduce the general principles of invariance for a finite atmosphere developed by Chandrasekhar (1950). Consistent with our previous discussions, we shall neglect the azimuthal dependence of the scattering parameters and use the reflection and transmission functions defined in Eqs. (6.3.3a) and (6.3.3b), instead of Chandrasekahr's scattering and transmission functions defined in Eqs. (6.3.7a) and (6.3.7b). We note, however, the relationships

$$S(\mu, \mu_0) = 4\mu\mu_0 R(\mu, \mu_0), \tag{6.3.28a}$$

$$T_c(\mu, \mu_0) = 4\mu\mu_0 T(\mu, \mu_0). \tag{6.3.28b}$$

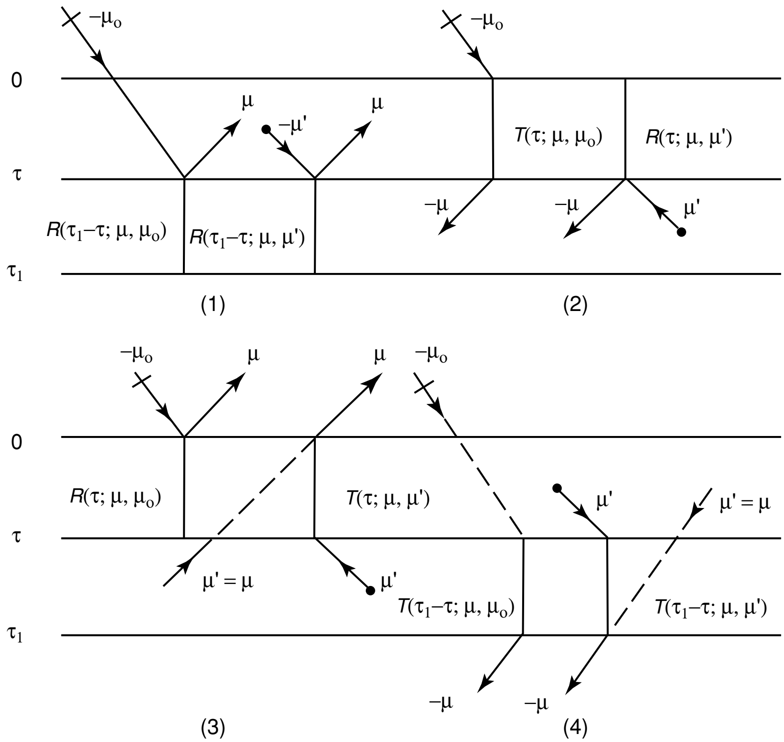

Figure 6.6 The four principles of invariance for a finite atmosphere (see text for explanation).

In reference to Fig. 6.6, we find the following four principles governing the reflection and transmission of a light beam:

1. The reflected (upward) intensity at level τ is caused by the reflection of the attenuated incident solar flux density $F_\odot e^{-\tau/\mu_0}$, and the downward diffuse intensity incident on the surface τ from the finite optical depth $(\tau_1 - \tau)$ below [see Eqs. (6.3.1a) and (6.3.1b)]. Thus, we have

$$
\begin{aligned}
I(\tau, \mu) = {} & (\mu_0 F_\odot/\pi) e^{-\tau/\mu_0} R(\tau_1 - \tau; \mu, \mu_0) \\
& + 2 \int_0^1 R(\tau_1 - \tau; \mu, \mu') I(\tau, -\mu')\, \mu' d\mu'. \qquad (6.3.29)
\end{aligned}
$$

2. The diffusely transmitted (downward) intensity at level τ is due to the transmission of the incident solar flux density by the optical depth τ above, and the reflection of the upward diffuse intensity incident on the surface τ from below. Thus, we find

$$
I(\tau, -\mu) = (\mu_0 F_\odot/\pi) T(\tau; \mu, \mu_0) + 2 \int_0^1 R(\tau; \mu, \mu') I(\tau, \mu')\, \mu' d\mu'. \qquad (6.3.30)
$$

3. The reflected (upward) intensity at the top of the finite atmosphere ($\tau = 0$) is a result of the reflection by the optical depth τ of the atmosphere plus the transmission of the upward diffuse and direct intensities incident on the surface τ from below. Thus, we have

$$I(0, \mu) = (\mu_0 F_\odot/\pi) R(\tau; \mu, \mu_0) + 2 \int_0^1 T(\tau; \mu, \mu') I(\tau, \mu')\, \mu' d\mu' + e^{-\tau/\mu} I(\tau, \mu). \tag{6.3.31}$$

4. The diffusely transmitted (downward) intensity at the bottom of the finite atmosphere ($\tau = \tau_1$) is equivalent to the transmission of the attenuated incident solar flux density plus the transmission of the downward diffuse and direct intensities incident on the surface τ from above. Thus, we find

$$\begin{aligned} I(\tau_1, -\mu) &= (\mu_0 F_\odot/\pi) e^{-\tau/\mu_0} T(\tau_1 - \tau; \mu, \mu_0) \\ &\quad + 2 \int_0^1 T(\tau_1 - \tau; \mu, \mu') I(\tau_1, -\mu')\, \mu' d\mu' + e^{-(\tau_1 - \tau)/\mu} I(\tau, -\mu). \end{aligned} \tag{6.3.32}$$

In order to obtain the reflection and transmission functions of a finite atmosphere with an optical depth of τ_1, we first differentiate Eqs. (6.3.29)–(6.3.32) with respect to τ and evaluate the values at $\tau = 0$ and τ_1, where the boundary conditions stated in Eq. (6.2.23) can be applied. After differentiation with respect to τ, we set $\tau = 0$ and $\tau = \tau_1$ for principles 1 and 4 and for principles 2 and 3, respectively, to obtain the equations

$$\begin{aligned} \left.\frac{dI(\tau, \mu)}{d\tau}\right|_{\tau=0} &= (\mu_0 F_\odot/\pi) \left[-\frac{\partial R(\tau_1; \mu, \mu_0)}{\partial \tau_1} - \frac{1}{\mu_0} R(\tau_1; \mu, \mu_0) \right] \\ &\quad + 2 \int_0^1 \mu' d\mu' R(\tau_1; \mu, \mu') \left.\frac{dI(\tau, -\mu')}{d\tau}\right|_{\tau=0}, \end{aligned} \tag{6.3.33}$$

$$\begin{aligned} \left.\frac{dI(\tau, -\mu)}{d\tau}\right|_{\tau=\tau_1} &= (\mu_0 F_\odot/\pi) \frac{\partial T(\tau_1; \mu, \mu_0)}{\partial \tau_1} \\ &\quad + 2 \int_0^1 \mu' d\mu' R(\tau_1; \mu, \mu') \left.\frac{dI(\tau, \mu')}{d\tau}\right|_{\tau=\tau_1}, \end{aligned} \tag{6.3.34}$$

$$\begin{aligned} \left.\frac{dI(0, \mu)}{d\tau}\right|_{\tau=\tau_1} &= 0 = (\mu_0 F_\odot/\pi) \frac{\partial R(\tau_1; \mu, \mu_0)}{\partial \tau_1} \\ &\quad + 2 \int_0^1 \mu' d\mu'\, T(\tau_1; \mu, \mu') \left.\frac{dI(\tau, \mu')}{d\tau}\right|_{\tau=\tau_1} + e^{-\tau_1/\mu} \left.\frac{dI(\tau, \mu)}{d\tau}\right|_{\tau=\tau_1}, \end{aligned} \tag{6.3.35}$$

$$\left.\frac{dI(\tau_1,-\mu)}{d\tau}\right|_{\tau=0} = 0 = (\mu_0 F_\odot/\pi)\left[-\frac{\partial T(\tau_1;\mu,\mu_0)}{\partial \tau_1} - \frac{1}{\mu_0}T(\tau_1;\mu,\mu_0)\right]$$
$$+ 2\int_0^1 \mu' d\mu' T(\tau_1;\mu,\mu')\left.\frac{dI(\tau,-\mu')}{d\tau}\right|_{\tau=0}$$
$$+ e^{-\tau_1/\mu}\left.\frac{dI(\tau,-\mu)}{d\tau}\right|_{\tau=0}. \tag{6.3.36}$$

To eliminate the derivatives of the intensity, we utilize the azimuthally independent transfer equation [Eq. (6.1.12)] to obtain

$$\mu\left.\frac{dI(\tau,\mu)}{d\tau}\right|_{\tau=0} = (\mu_0 F_\odot/\pi)R(\tau_1;\mu,\mu_0) - \frac{\tilde{\omega}}{2}\int_0^1 P(\mu,\mu'')I(0,\mu'')\,d\mu''$$
$$- \frac{\tilde{\omega}}{4\pi}F_\odot P(\mu,-\mu_0), \tag{6.3.37}$$

$$-\mu\left.\frac{dI(\tau,-\mu)}{d\tau}\right|_{\tau=0} = 0 - \frac{\tilde{\omega}}{2}\int_0^1 P(-\mu,\mu'')I(0,\mu'')\,d\mu'' - \frac{\tilde{\omega}}{4\pi}F_\odot P(-\mu,-\mu_0), \tag{6.3.38}$$

$$\mu\left.\frac{dI(\tau,\mu)}{d\tau}\right|_{\tau=\tau_1} = 0 - \frac{\tilde{\omega}}{2}\int_0^1 P(\mu,-\mu'')I(\tau_1,-\mu'')\,d\mu''$$
$$- \frac{\tilde{\omega}}{4\pi}F_\odot P(\mu,-\mu_0)e^{-\tau_1/\mu_0}, \tag{6.3.39}$$

$$-\mu\left.\frac{dI(\tau,-\mu)}{d\tau}\right|_{\tau=\tau_1} = (\mu_0 F_\odot/\pi)T(\tau_1;\mu,\mu_0) - \frac{\tilde{\omega}}{2}\int_0^1 P(-\mu,-\mu'')I(\tau_1,-\mu'')\,d\mu''$$
$$- \frac{\tilde{\omega}}{4\pi}F_\odot P(-\mu,-\mu_0)e^{-\tau_1/\mu_0}. \tag{6.3.40}$$

In these four equations, we note that $\mu \geq 0$. We also note that $\pi I(0,\mu) = \mu_0 F_\odot \cdot R(\tau_1;\mu,\mu_0)$, and $\pi I(\tau_1,-\mu) = \mu_0 F_\odot T(\tau_1;\mu,\mu_0)$. Upon substituting Eqs. (6.3.37) and (6.3.38), (6.3.39) and (6.3.40), (6.3.39) and (6.3.38), and (6.3.38) into Eqs. (6.3.33)–(6.3.36), respectively, and rearranging the terms, we obtain

$$\frac{\partial R(\tau_1;\mu,\mu_0)}{\partial \tau_1} = -\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)R(\tau_1;\mu,\mu_0) + \frac{\tilde{\omega}}{4\mu\mu_0}P(\mu,-\mu_0)$$
$$+ \frac{\tilde{\omega}}{2\mu}\int_0^1 P(\mu,\mu'')R(\tau_1;\mu'',\mu_0)\,d\mu''$$
$$+ \frac{\tilde{\omega}}{2\mu_0}\int_0^1 R(\tau_1;\mu,\mu')P(-\mu',-\mu_0)\,d\mu'$$
$$+ \tilde{\omega}\int_0^1 R(\tau_1;\mu,\mu')\,d\mu'\left[\int_0^1 P(-\mu',\mu'')R(\tau_1;\mu'',\mu_0)\,d\mu''\right], \tag{6.3.41}$$

$$\begin{aligned}\frac{\partial T(\tau_1;\mu,\mu_0)}{\partial\tau_1} &= -\frac{1}{\mu}T(\tau_1;\mu,\mu_0) + \frac{\tilde{\omega}}{4\mu\mu_0}e^{-\tau_1/\mu_0}P(\mu,-\mu_0)\\ &+ \frac{\tilde{\omega}}{2\mu}\int_0^1 P(-\mu,-\mu'')T(\tau_1;\mu'',\mu_0)\,d\mu''\\ &+ \frac{\tilde{\omega}}{2\mu_0}e^{-\tau_1/\mu_0}\int_0^1 R(\tau_1;\mu,\mu')P(\mu',-\mu_0)\,d\mu'\\ &+ \tilde{\omega}\int_0^1 R(\tau_1;\mu,\mu')\,d\mu'\left[\int_0^1 P(\mu',-\mu'')T(\tau_1;\mu'',\mu_0)\,d\mu''\right],\end{aligned} \tag{6.3.42}$$

$$\begin{aligned}\frac{\partial R(\tau_1;\mu,\mu_0)}{\partial\tau_1} &= \frac{\tilde{\omega}}{4\mu\mu_0}\exp\left[-\tau\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right]P(\mu,-\mu_0)\\ &+ \frac{\tilde{\omega}}{2\mu}e^{-\tau_1/\mu}\int_0^1 P(\mu,-\mu'')T(\tau_1;\mu'',\mu_0)\,d\mu''\\ &+ \frac{\tilde{\omega}}{2\mu_0}e^{-\tau_1/\mu_0}\int_0^1 T(\tau_1;\mu,\mu')P(\mu',-\mu_0)\,d\mu'\\ &+ \tilde{\omega}\int_0^1 T(\tau_1;\mu,\mu')\,d\mu'\left[\int_0^1 P(\mu',-\mu'')T(\tau_1;\mu'',\mu_0)\,d\mu''\right],\end{aligned} \tag{6.3.43}$$

$$\begin{aligned}\frac{\partial T(\tau_1;\mu,\mu_0)}{\partial\tau_1} &= -\frac{1}{\mu_0}T(\tau_1;\mu,\mu_0) + \frac{\tilde{\omega}}{4\mu\mu_0}e^{-\tau_1/\mu}P(-\mu,-\mu_0)\\ &+ \frac{\tilde{\omega}}{2\mu}e^{-\tau_1/\mu}\int_0^1 P(-\mu,\mu'')R(\tau_1;\mu'',\mu_0)\,d\mu''\\ &+ \frac{\tilde{\omega}}{2\mu_0}\int_0^1 T(\tau_1;\mu,\mu')P(-\mu',-\mu_0)\,d\mu'\\ &+ \tilde{\omega}\int_0^1 T(\tau_1;\mu,\mu')\,d\mu'\left[\int_0^1 P(-\mu',\mu'')R(\tau_1;\mu'',\mu_0)\,d\mu''\right].\end{aligned} \tag{6.3.44}$$

Equations (6.3.41)–(6.3.44) represent four nonlinear integral equations that govern the complete radiation field at $\tau = 0$ and $\tau = \tau_1$ in plane-parallel atmospheres. In the preceding analysis, we neglected the azimuthal dependence on the reflection, transmission, and phase-function terms in the derivation of the four integral equations. However, it is simple to include ϕ and ϕ' terms in these four integral equations. Further, we note that as $\tau_1 \to \infty$, $\partial R/\partial\tau_1 \to 0$, and Eq. (6.3.41) reduces to Eq. (6.3.18) that was derived for a semi-infinite atmosphere. Equations (6.3.41) and (6.3.44) may be obtained by adding a thin layer ($\Delta\tau \ll 1$) to the top of a finite atmosphere following the principles outlined in Section 6.3.2. In addition, by adding a thin layer to the bottom of a finite atmosphere, Eqs. (6.3.42) and (6.3.43) may be derived. The addition of thin layers to a finite atmosphere is referred to as *invariant imbedding* (Bellman *et al.,* 1963), which is equivalent to the principles of invariance (Exercise 6.3).

6.3.4 The X and Y Functions

In Section 6.3.2, we showed that the reflection function of a semi-infinite atmosphere for isotropic scattering is given by the H function. In the following, we wish to demonstrate that the reflection and transmission functions of a finite atmosphere for isotropic scattering are governed by the X and Y functions. In the case of isotropic scattering, Eqs. (6.3.41)–(6.3.44) become

$$\begin{aligned}
&\frac{\partial R(\tau_1;\mu,\mu_0)}{\partial \tau_1} + \left(\frac{1}{\mu}+\frac{1}{\mu_0}\right) R(\tau_1;\mu,\mu_0) \\
&\quad = \frac{\tilde{\omega}}{4\mu\mu_0}\left[1 + 2\mu_0\int_0^1 R(\tau_1;\mu'',\mu_0)\,d\mu'' + 2\mu\int_0^1 R(\tau_1;\mu,\mu')\,d\mu'\right. \\
&\qquad \left. + 4\mu\mu_0\int_0^1 R(\tau_1;\mu,\mu')\int_0^1 R(\tau_1;\mu'',\mu_0)\,d\mu''\right],
\end{aligned} \tag{6.3.45}$$

$$\begin{aligned}
\frac{\partial R(\tau_1;\mu,\mu_0)}{\partial \tau_1} = \frac{\tilde{\omega}}{4\mu\mu_0}&\left[\exp\left\{-\tau_1\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right\} + 2\mu_0 e^{-\tau_1/\mu}\int_0^1 T(\tau_1;\mu'',\mu_0)d\mu''\right. \\
&+ 2\mu e^{-\tau_1/\mu_0}\int_0^1 T(\tau_1;\mu,\mu')\,d\mu' \\
&\left. + 4\mu\mu_0\int_0^1 T(\tau_1;\mu,\mu')\,d\mu'\int_0^1 T(\tau_1;\mu'',\mu_0)\,d\mu''\right],
\end{aligned} \tag{6.3.46}$$

$$\begin{aligned}
\frac{\partial T(\tau_1;\mu,\mu_0)}{\partial \tau_1} + \frac{1}{\mu}T(\tau_1;\mu,\mu_0) = \frac{\tilde{\omega}}{4\mu\mu_0}&\left[e^{-\tau_1/\mu_0} + 2\mu_0\int_0^1 T(\tau_1;\mu'',\mu_0)\,d\mu''\right. \\
&+ 2\mu e^{-\tau_1/\mu_0}\int_0^1 R(\tau_1;\mu,\mu')\,d\mu' \\
&\left. + 4\mu\mu_0\int_0^1 R(\tau_1;\mu,\mu')\,d\mu'\int_0^1 T(\tau_1;\mu'',\mu_0)\,d\mu''\right],
\end{aligned} \tag{6.3.47}$$

$$\begin{aligned}
\frac{\partial T(\tau_1;\mu,\mu_0)}{\partial \tau_1} + \frac{1}{\mu_0}T(\tau_1;\mu,\mu_0) = \frac{\tilde{\omega}}{4\mu\mu_0}&\left[e^{-\tau_1/\mu_0} + 2\mu\int_0^1 T(\tau_1;\mu,\mu')\,d\mu'\right. \\
&+ 2\mu_0 e^{-\tau_1/\mu_0}\int_0^1 R(\tau_1;\mu'',\mu_0)\,d\mu'' \\
&\left. + 4\mu\mu_0\int_0^1 T(\tau_1;\mu,\mu')\,d\mu'\int_0^1 R(\tau_1;\mu'',\mu_0)\,d\mu''\right].
\end{aligned} \tag{6.3.48}$$

From Eqs. (6.3.45)–(6.3.48), the integral terms may be expressed in terms of Chandrasekhar's X and Y functions in the forms

$$X(\mu) = 1 + 2\mu \int_0^1 R(\tau_1; \mu, \mu')\, d\mu', \tag{6.3.49}$$

$$Y(\mu) = e^{-\tau_1/\mu} + 2\mu \int_0^1 T(\tau_1; \mu, \mu')\, d\mu'. \tag{6.3.50}$$

It follows that Eqs. (6.3.45)–(6.3.48) may be rewritten as follows:

$$\frac{\partial R(\tau_1; \mu, \mu_0)}{\partial \tau_1} + \left(\frac{1}{\mu} + \frac{1}{\mu_0}\right) R(\tau_1; \mu, \mu_0) = \frac{\tilde{\omega}}{4\mu\mu_0} X(\mu) X(\mu_0), \tag{6.3.51}$$

$$\frac{\partial R(\tau_1; \mu, \mu_0)}{\partial \tau_1} = \frac{\tilde{\omega}}{4\mu\mu_0} Y(\mu) Y(\mu_0), \tag{6.3.52}$$

$$\frac{\partial T(\tau_1; \mu, \mu_0)}{\partial \tau_1} + \frac{1}{\mu} T(\tau_1; \mu, \mu_0) = \frac{\tilde{\omega}}{4\mu\mu_0} X(\mu) Y(\mu_0), \tag{6.3.53}$$

$$\frac{\partial T(\tau_1; \mu, \mu_0)}{\partial \tau_1} + \frac{1}{\mu_0} T(\tau_1; \mu, \mu_0) = \frac{\tilde{\omega}}{4\mu\mu_0} X(\mu_0) Y(\mu). \tag{6.3.54}$$

By eliminating $\partial R/\partial \tau_1$ from Eqs. (6.3.51) and (6.3.52), we obtain

$$\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right) R(\tau_1; \mu, \mu_0) = \frac{\tilde{\omega}}{4\mu\mu_0} [X(\mu)X(\mu_0) - Y(\mu)Y(\mu_0)], \tag{6.3.55}$$

and by eliminating $\partial T/\partial \tau_1$ from Eqs. (6.3.53) and (6.3.54), we have

$$\left(\frac{1}{\mu} - \frac{1}{\mu_0}\right) T(\tau_1; \mu, \mu_0) = \frac{\tilde{\omega}}{4\mu\mu_0} [X(\mu)Y(\mu_0) - X(\mu_0)Y(\mu)]. \tag{6.3.56}$$

Inserting Eqs. (6.3.55) and (6.3.56) into Eqs. (6.3.49) and (6.3.50), we find

$$X(\mu) = 1 + \mu \int_0^1 \frac{\Psi(\mu')}{\mu + \mu'} [X(\mu)X(\mu') - Y(\mu)Y(\mu')]\, d\mu', \tag{6.3.57}$$

$$Y(\mu) = e^{-\tau_1/\mu} + \mu \int_0^1 \frac{\Psi(\mu')}{\mu' - \mu} [X(\mu)Y(\mu') - X(\mu')Y(\mu)]\, d\mu', \tag{6.3.58}$$

where the characteristic function $\Psi(\mu') = \tilde{\omega}/2$. Thus, the exact solutions of the reflection and transmission functions are now governed by the X and Y functions, solutions of the two nonlinear integral equations. For a semi-infinite atmosphere $Y(\mu) = 0$ and the X function defined in Eqs. (6.3.49) and (6.3.57) is equivalent to the H function introduced in Eqs. (6.3.21) and (6.3.23). The characteristic function $\Psi(\mu')$ differs from problem to problem and has a simple algebraic form for the Rayleigh scattering phase function. For a more general case involving Lorenz–Mie scattering phase functions, however, the analytic characteristic functions $\Psi(\mu')$ appear to be extremely complicated and have not been derived for practical applications. The iteration procedure may be used to solve the preceding nonlinear integral equations for the X and Y functions. Extensive tables of these two functions for conservative and nonconservative isotropic scattering, as well as anisotropic phase functions, with as many as three terms, have been constructed.

6.3.5 Inclusion of Surface Reflection

For planetary applications, surface reflection plays an important role in reflected and transmitted sunlight. In this section, we introduce the inclusion of surface reflection in the scattered intensity and flux density equations. The ground is considered to reflect according to Lambert's law, with a reflectivity (or surface albedo) of r_s. Under this condition, the diffuse upward intensity is

$$I(\tau_1; \mu, \phi) = I_s = \text{const.} \tag{6.3.59}$$

Let $I^*(0; \mu, \phi)$ represent the reflected intensity including the contribution of surface reflection, and in reference to Fig. 6.7a, we find

$$I^*(0; \mu, \phi) = I(0; \mu, \phi) + \frac{1}{\pi} \int_0^{2\pi} \int_0^1 T(\mu, \phi; \mu', \phi') I_s \mu' d\mu' d\phi' + I_s e^{-\tau_1/\mu}. \tag{6.3.60}$$

The last two terms represent, respectively, the diffuse and direct transmission of the upward isotropic intensity I_s.

Equation (6.3.60) can be rewritten in terms of the reflection function and the direct and diffuse transmission defined in Section 6.3.1 in the form

$$I^*(0; \mu, \phi) = \mu_0 F_\odot R(\mu, \phi; \mu_0, \phi_0) + I_s \gamma(\mu), \tag{6.3.61}$$

where

$$\gamma(\mu) = e^{-\tau_1/\mu} + t(\mu), \tag{6.3.62}$$

and the diffuse transmission $t(\mu)$ is defined in Eq. (6.3.4b). The principle of reciprocity involving $T(\mu, \phi; \mu', \phi') = T(\mu', \phi'; \mu, \phi)$ is used to obtain $t(\mu)$.

The upward isotropic intensity from the surface also will be reflected by the atmosphere and will contribute to the downward intensity by an additional amount (see Fig. 6.7b)

$$I_s^R(-\mu) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 R(\mu, \phi; \mu', \phi') I_s \mu' \, d\mu' \, d\phi' = I_s r(\mu), \tag{6.3.63}$$

where again the principle of reciprocity, $R(\mu, \phi; \mu', \phi') = R(\mu', \phi'; \mu, \phi)$, is used. Thus, the total transmitted intensity, including the ground contribution, is given by

$$\begin{aligned} I^*(\tau_1; -\mu, \phi) &= I(\tau_1; -\mu, \phi) + I_s^R(-\mu) \\ &= \mu_0 F_\odot T(\mu, \phi; \mu_0, \phi_0) + I_s r(\mu). \end{aligned} \tag{6.3.64}$$

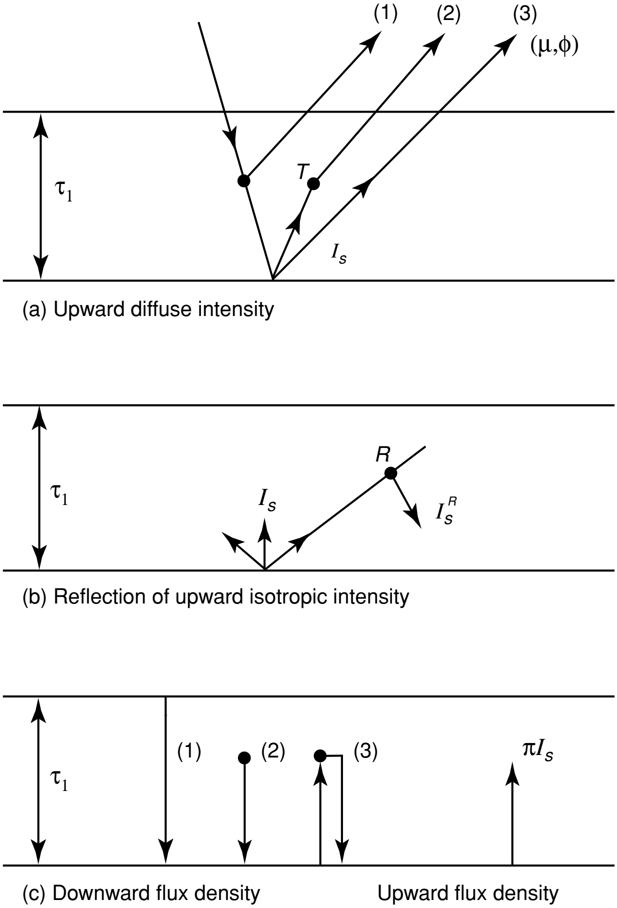

Figure 6.7 Scattering configuration for the inclusion of surface reflection: (a) upward diffuse intensity; (b) reflection of upward isotropic intensity; and (c) downward flux density and upward flux density.

We now require an equation to determine I_s. Since the upward flux density must be equal to the downward flux density multiplied by the surface albedo r_s, we have

$$\pi I_s = r_s \times \text{downward flux density.} \tag{6.3.65}$$

The downward flux density includes three components, as shown in Fig. 6.7c:

1. Direct transmission component:

$$\mu_0 F_\odot e^{-\tau_1/\mu_0}.$$

2. Diffuse transmission component:

$$\int_0^{2\pi}\int_0^1 I(\tau_1; -\mu, \phi)\mu \, d\mu \, d\phi = \int_0^{2\pi}\int_0^1 (\mu_0 F_\odot/\pi) T(\mu, \phi; \mu_0, \phi_0)\mu \, d\mu \, d\phi$$
$$= \mu_0 F_\odot t(\mu_0).$$

3. The component of I_s reflected by the atmosphere:

$$\int_0^{2\pi}\int_0^1 I_s^R(-\mu)\mu\, d\mu\, d\phi = \pi I_s \bar{r}.$$

From Eq. (6.3.65), we have the following equality at $\tau = \tau_1$:

$$\pi I_s = r_s\left[\mu_0 F_\odot e^{-\tau_1/\mu_0} + \mu_0 F_\odot t(\mu_0) + \pi I_s \bar{r}\right]. \tag{6.3.66}$$

We then rearrange these terms to yield

$$I_s = \frac{r_s}{1 - r_s \bar{r}} \frac{\mu_0 F_\odot}{\pi} \gamma(\mu_0). \tag{6.3.67}$$

It follows from Eqs. (6.3.61) and (6.3.64) that the reflected and transmitted intensities, including the ground reflection, are, respectively, given by

$$I^*(0; \mu, \phi) = I(0; \mu, \phi) + \frac{r_s}{1 - r_s \bar{r}} \frac{\mu_0 F_\odot}{\pi} \gamma(\mu)\gamma(\mu_0), \tag{6.3.68a}$$

$$I^*(\tau_1; \mu, \phi) = I(\tau_1; -\mu, \phi) + \frac{r_s}{1 - r_s \bar{r}} \frac{\mu_0 F_\odot}{\pi} \gamma(\mu_0) r(\mu). \tag{6.3.68b}$$

To obtain the reflected and transmitted flux densities, we may perform an integration of the intensity over the solid angle according to Eqs. (6.1.7a) and (6.1.7b) to yield

$$F^*(0) = F(0) + \frac{r_s}{1 - r_s \bar{r}} \mu_0 F_\odot \gamma(\mu_0) \bar{\gamma}, \tag{6.3.69a}$$

$$F^*(\tau_1) = F(\tau_1) + \frac{r_s}{1 - r_s \bar{r}} \mu_0 F_\odot \gamma(\mu_0) \bar{r}, \tag{6.3.69b}$$

where

$$\bar{\gamma} = \bar{t} + 2\int_0^1 e^{-\tau_1/\mu_0} \mu_0\, d\mu_0, \tag{6.3.70}$$

and $\bar{t}$ and $\bar{r}$ are defined in Eqs. (6.3.5b) and (6.3.5c). Further, by dividing $\mu_0 F_\odot$ and adding $e^{-\tau_1/\mu_0}$ to both sides in Eq. (6.3.69b), the preceding two equations become

$$r^*(\mu_0) = r(\mu_0) + f(\mu_0)\bar{\gamma}, \tag{6.3.71a}$$

$$\gamma^*(\mu_0) = \gamma(\mu_0) + f(\mu_0)\bar{r}, \tag{6.3.71b}$$

where

$$f(\mu_0) = \frac{r_s}{1 - r_s \bar{r}} \gamma(\mu_0). \tag{6.3.72}$$

Exercise 6.4 requires the derivation of Eqs. (6.3.71a) and (6.3.71b) by means of the ray-tracing technique.

6.4 Adding Method for Radiative Transfer

The principle of the adding method for radiative transfer was stated by Stokes (1862) in a problem dealing with reflection and transmission by glass plates. Peebles and Plesset (1951) developed the adding theory for application to gamma-ray transfer. van de Hulst (1980) presented a set of adding equations for multiple scattering that is now commonly used. Hansen (1971) applied the adding/doubling method to the interpretation of intensity and polarization of sunlight reflected from clouds. Takano and Liou (1989b) modified the adding method for radiative transfer by including polarization for application to randomly and horizontally oriented ice crystals. The exact adding/doubling method and its approximations appear to be a powerful tool for multiple scattering calculations, particularly with reference to remote-sensing applications from the ground, the air, and space.

6.4.1 Definitions of Physical Parameters

To introduce the adding principle for radiative transfer, we shall first define the reflection function R and the transmission function T. Consider a light beam incident from above, as shown in Fig. 6.8. The reflected and transmitted intensities of this beam may be expressed in terms of the incident intensity in the forms

$$I_{\text{out,top}}(\mu, \phi) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 R(\mu, \phi; \mu', \phi') I_{\text{in,top}}(\mu', \phi') \mu' \, d\mu' \, d\phi', \quad (6.4.1a)$$

$$I_{\text{out,bottom}}(\mu, \phi) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 T(\mu, \phi; \mu', \phi') I_{\text{in,top}}(\mu' \phi') \, \mu' \, d\mu' \, d\phi'. \quad (6.4.1b)$$

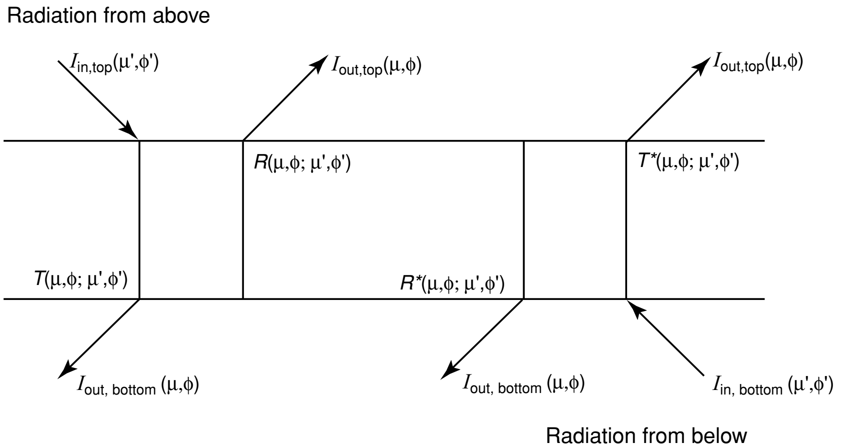

Figure 6.8 Configurations for radiation incident from above and below, and the definitions of the reflection and transmission functions for the adding method.

Likewise, if the light beam is incident from below (Fig. 6.8) we may write

$$I_{\text{out,bottom}}(\mu,\phi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 R^*(\mu,\phi;\,\mu',\phi')I_{\text{in,bottom}}(\mu',\phi')\,\mu'\,d\mu'\,d\phi', \quad (6.4.2\text{a})$$

$$I_{\text{out,top}}(\mu,\phi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 T^*(\mu,\phi;\,\mu',\phi')I_{\text{in,bottom}}(\mu',\phi')\,\mu'\,d\mu'\,d\phi'. \quad (6.4.2\text{b})$$

Equations (6.4.2a) and (6.4.2b) define R^* and T^*, respectively, where the superscript* denotes that the light beam comes from below.

Consider now the transfer of monochromatic solar radiation. The incident solar intensity, in the present notation, may be written in the form

$$I_{\text{in,top}}(-\mu_0,\phi_0) = \delta(\mu'-\mu_0)\delta(\phi'-\phi_0)F_\odot, \quad (6.4.3)$$

where δ is the Dirac delta function. Using Eq. (6.4.3), the reflection and transmission functions defined in Eqs. (6.4.1a) and (6.4.1b) are given by

$$R(\mu,\phi;\,\mu_0,\phi_0) = \pi I_{\text{out,top}}(\mu,\phi)/\mu_0 F_\odot, \quad (6.4.4\text{a})$$

$$T(\mu,\phi;\,\mu_0,\phi_0) = \pi I_{\text{out,bottom}}(\mu,\phi)/\mu_0 F_\odot. \quad (6.4.4\text{b})$$

Under the single-scattering approximation and neglecting the emission contribution, the source function defined in Eq. (6.1.1) may be written in the form [see also Eq. (3.4.10)]

$$J(\tau;\mu,\phi) = \frac{\tilde{\omega}}{4\pi}F_\odot P(\mu,\phi;-\mu_0,\phi_0)e^{-\tau/\mu_0}. \quad (6.4.5)$$

Assuming that there are no diffuse intensities from the top and bottom of the layer with an optical depth $\Delta\tau$, then the radiation boundary conditions are as follows:

$$\begin{aligned} I_{\text{in,top}}(\mu,\phi) &= 0,\\ I_{\text{in,bottom}}(\mu,\phi) &= 0. \end{aligned} \quad (6.4.6)$$

Subject to these boundary conditions, the reflected and transmitted diffuse intensities due to single scattering can be derived directly from the basic radiative transfer equation. Thus, the solutions for the reflection and transmission functions for an optical depth $\Delta\tau$ are given by

$$R(\mu,\phi;\,\mu_0,\phi_0) = \frac{\tilde{\omega}}{4(\mu+\mu_0)}P(\mu,\phi;-\mu_0,\phi_0)\left\{1-\exp\left[-\Delta\tau\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right]\right\}, \quad (6.4.7\text{a})$$

$$T(\mu,\phi;\,\mu_0,\phi_0) = \begin{cases} \dfrac{\tilde{\omega}}{4(\mu-\mu_0)}P(-\mu,\phi;-\mu_0,\phi_0)(e^{-\Delta\tau/\mu}-e^{-\Delta\tau/\mu_0}), & \mu\neq\mu_0\\[2ex] \dfrac{\tilde{\omega}\Delta\tau}{4\mu_0^2}P(-\mu,\phi;-\mu_0,\phi_0)e^{-\Delta\tau/\mu_0}, & \mu=\mu_0. \end{cases} \quad (6.4.7\text{b})$$

Consider a layer in which $\Delta\tau$ is very small (e.g., $\Delta\tau \approx 10^{-8}$). Equations (6.4.7a) and (6.4.7b) may then be further simplified to give

$$R(\mu, \phi; \mu_0, \phi_0) = \frac{\tilde{\omega}\Delta\tau}{4\mu\mu_0} P(\mu, \phi; -\mu_0, \phi_0), \tag{6.4.8a}$$

$$T(\mu, \phi; \mu_0, \phi_0) = \frac{\tilde{\omega}\Delta\tau}{4\mu\mu_0} P(-\mu, \phi; -\mu_0, \phi_0). \tag{6.4.8b}$$

Equations (6.4.7a) and (6.4.8a) were presented in the first subsection of Section 3.4.2. For a thin homogeneous layer, the reflection and transmission functions are the same regardless of whether the light beam comes from above or below. Thus, $R^* = R$ and $T^* = T$. However, when we proceed with the adding of layers, the reflection and transmission functions for combined layers will depend on the direction of the incoming light beam.

6.4.2 Adding Equations

In reference to Fig. 6.9, consider two layers, one on top of the other. Let the reflection and total (direct and diffuse) transmission functions be denoted by R_1 and $\widetilde{T}_1$ for the first layer and R_2 and $\widetilde{T}_2$ for the second layer. We further define $\widetilde{D}$ and U for the combined total transmission and reflection functions between layers 1 and 2. In principle, the light beam may undergo an infinite number of scattering events. Accounting for multiple reflections of the light beam in the two layers, as shown in Fig. 6.9, the combined reflection and transmission functions are given by

$$\begin{aligned} R_{12} &= R_1 + \widetilde{T}_1^* R_2 \widetilde{T}_1 + \widetilde{T}_1^* R_2 R_1^* R_2 \widetilde{T}_1 + \widetilde{T}_1^* R_2 R_1^* R_2 R_1^* R_2 \widetilde{T}_1 + \ldots \\ &= R_1 + \widetilde{T}_1^* R_2 \left[1 + R_1^* R_2 + \left(R_1^* R_2\right)^2 + \ldots\right] \widetilde{T}_1 \\ &= R_1 + \widetilde{T}_1^* R_2 \left(1 - R_1^* R_2\right)^{-1} \widetilde{T}_1, \end{aligned} \tag{6.4.9a}$$

$$\begin{aligned} \widetilde{T}_{12} &= \widetilde{T}_2 \widetilde{T}_1 + \widetilde{T}_2 R_1^* R_2 \widetilde{T}_1 + \widetilde{T}_2 R_1^* R_2 R_1^* R_2 \widetilde{T}_1 + \ldots \\ &= \widetilde{T}_2 \left[1 + R_1^* R_2 + \left(R_1^* R_2\right)^2 + \ldots\right] \widetilde{T}_1 \\ &= \widetilde{T}_2 \left(1 - R_1^* R_2\right)^{-1} \widetilde{T}_1. \end{aligned} \tag{6.4.9b}$$

Likewise, the expressions U and $\widetilde{D}$ may be written as

$$\begin{aligned} U &= R_2 \widetilde{T}_1 + R_2 R_1^* R_2 \widetilde{T}_1 + R_2 R_1^* R_2 R_1^* R_2 \widetilde{T}_1 + \ldots \\ &= R_2 \left[1 + R_1^* R_2 + \left(R_1^* R_2\right)^2 + \ldots\right] \widetilde{T}_1 \\ &= R_2 \left(1 - R_1^* R_2\right)^{-1} \widetilde{T}_1, \end{aligned} \tag{6.4.10a}$$

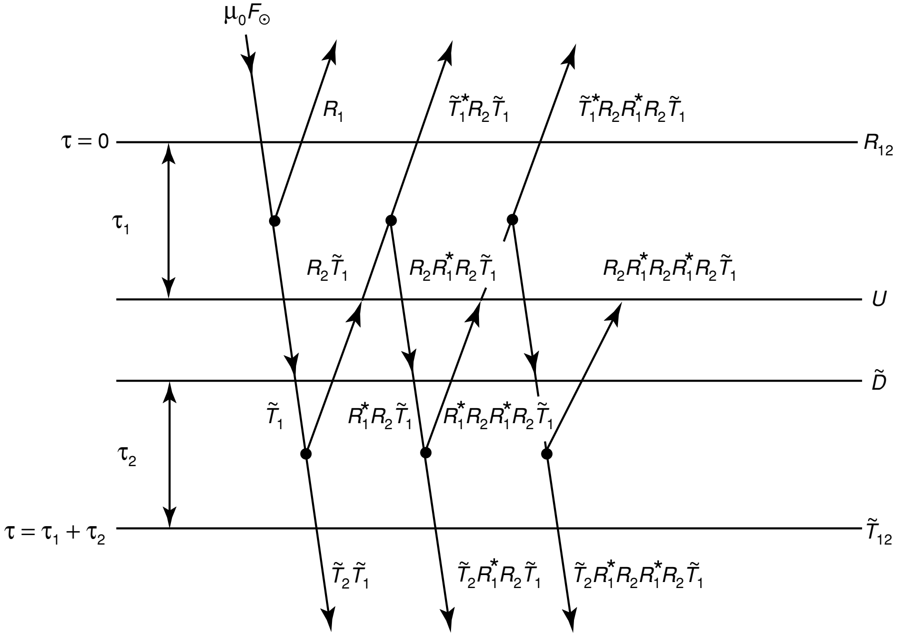

Figure 6.9 Configuration of the adding method. The two layers of optical depths τ_1 and τ_2 are rendered, for convenient illustration, as though they were physically separated (see text for the definitions of R, R^*, $\widetilde{T}$, and $\widetilde{T}^*$).

$$
\begin{aligned}
\widetilde{D} &= \widetilde{T}_1 + R_1^* R_2 \widetilde{T}_1 + R_1^* R_2 R_1^* R_2 \widetilde{T}_1 + \ldots \\
&= \left[1 + R_1^* R_2 + \left(R_1^* R_2\right)^2 + \ldots\right] \widetilde{T}_1 \\
&= \left(1 - R_1^* R_2\right)^{-1} \widetilde{T}_1. \qquad (6.4.10\text{b})
\end{aligned}
$$

In Eqs. (6.4.9a) and (6.4.9b), the infinite series can be replaced by a single inverse function. We may define an operator in the form

$$S = R_1^* R_2 \left(1 - R_1^* R_2\right)^{-1}, \qquad (6.4.11\text{a})$$

so that

$$\left(1 - R_1^* R_2\right)^{-1} = 1 + S. \qquad (6.4.11\text{b})$$

From the preceding adding equations, we have

$$R_{12} = R_1 + \widetilde{T}_1^* U, \qquad (6.4.12\text{a})$$

$$\widetilde{T}_{12} = \widetilde{T}_2 \widetilde{D}, \qquad (6.4.12\text{b})$$

$$U = R_2 \widetilde{D}. \qquad (6.4.12\text{c})$$

We shall now separate the diffuse and direct components of the total transmission function defined by

$$\widetilde{T} = T + e^{-\tau/\mu'}, \tag{6.4.13}$$

where $\mu' = \mu_0$ when transmission is associated with the incident solar beam, and $\mu' = \mu$ when it is associated with the emergent light beam in the direction μ. Using Eq. (6.4.13), we may separate the direct and diffuse components in Eqs. (6.4.10b) and (6.4.12b) to obtain

$$\begin{aligned}\widetilde{D} &= D + e^{-\tau_1/\mu_0} = (1+S)\big(T_1 + e^{-\tau_1/\mu_0}\big)\\ &= (1+S)T_1 + Se^{-\tau_1/\mu_0} + e^{-\tau_1/\mu_0},\end{aligned} \tag{6.4.14a}$$

$$\begin{aligned}\widetilde{T}_{12} &= \big(T_2 + e^{-\tau_2/\mu}\big)(D + e^{-\tau_1/\mu_0})\\ &= e^{-\tau_2/\mu} D + T_2 e^{-\tau_1/\mu_0} + T_2 D + \exp\left[-\left(\frac{\tau_1}{\mu_0} + \frac{\tau_2}{\mu}\right)\right]\delta(\mu - \mu_0),\end{aligned} \tag{6.4.14b}$$

where D, T_1, and T_2 denote the diffuse components only, and a delta function has been added to the exponential term to signify that the direct transmission function is a function of μ_0 only.

On the basis of the preceding analysis, a set of iterative equations for the computation of diffuse transmission and reflection involving the two layers may be written in the form

$$Q = R_1^* R_2, \tag{6.4.15a}$$

$$S = Q(1 - Q)^{-1}, \tag{6.4.15b}$$

$$D = T_1 + ST_1 + Se^{-\tau_1/\mu_0}, \tag{6.4.15c}$$

$$U = R_2 D + R_2 e^{-\tau_1/\mu_0}, \tag{6.4.15d}$$

$$R_{12} = R_1 + e^{-\tau_1/\mu} U + T_1^* U, \tag{6.4.15e}$$

$$T_{12} = e^{-\tau_2/\mu} D + T_2 e^{-\tau_1/\mu_0} + T_2 D. \tag{6.4.15f}$$

The direct transmission function for the combined layer is given by $\exp[-(\tau_1 + \tau_2)/\mu_0]$. In these equations, the product of the two functions implies an integration over the appropriate solid angle so that all possible multiple-scattering contributions can be accounted for, as in the following example:

$$R_1^* R_2 = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 R_1^*(\mu, \phi;\, \mu', \phi') R_2(\mu', \phi';\, \mu_0, \phi_0)\mu'\, d\mu'\, d\phi'. \tag{6.4.16}$$

In numerical computations, we may set $\tau_1 = \tau_2$. This is referred to as the *doubling method*. We may start with an optical depth $\Delta\tau$ on the order of 10^{-8} and use Eqs. (6.4.8a) and (6.4.8b) to compute the reflection and transmission functions. Equations (6.4.15a)–(6.4.15f) are subsequently employed to compute the reflection and transmission functions for an optical depth of $2\Delta\tau$. For the initial layers, $R_{1,2}^* = R_{1,2}$ and

$T^*_{1,2} = T_{1,2}$. Using the adding equations, the computations may be repeated until a desirable optical depth is achieved.

For a light beam incident from below, R^*_{12} and T^*_{12} may be computed from a scheme analogous to Eq. (6.4.15). Let the incident direction be μ'; then the adding equations may be written in the forms (Exercise 6.5)

$$Q = R_2 R_1^*, \tag{6.4.17a}$$

$$S = Q(1-Q)^{-1}, \tag{6.4.17b}$$

$$U = T_2^* + ST_2^* + Se^{-\tau_2/\mu'}, \tag{6.4.17c}$$

$$D = R_1^* U + R_1^* e^{-\tau_2/\mu'}, \tag{6.4.17d}$$

$$R^*_{12} = R_2^* + e^{-\tau_2/\mu} D + T_2 D, \tag{6.4.17e}$$

$$T^*_{12} = e^{-\tau_1/\mu} U + T_1^* e^{-\tau_2/\mu'} + T_1^* U. \tag{6.4.17f}$$

When polarization and azimuth dependence are neglected, the transmission function is the same regardless of whether the light beam is from above or below so that $T^*(\mu, \mu') = T(\mu', \mu)$. This relation can be derived based on the Helmholtz principle of reciprocity in which the light beam may reverse its direction (Hovenier, 1969).

For practical applications, we may begin with the computations of reflection and transmission functions given in Eqs. (6.4.8a) and (6.4.8b). The phase function must be expressed as a function of the incoming and outgoing directions via Eq. (6.1.3a) in the form

$$P(\mu, \phi; \mu', \phi') = P^0(\mu, \mu') + 2\sum_{m=1}^{N} P^m(\mu, \mu') \cos m(\phi' - \phi), \tag{6.4.18}$$

where $P^m(\mu, \mu')(m = 0, 1, \ldots N)$ denotes the Fourier expansion coefficients. The number of terms required in the expansion depends on the sharpness of the forward diffraction peak in phase function (see Fig. 3.13).

The preceding adding equations for radiative transfer have been written in scalar forms involving diffuse intensity. However, these equations can be applied to the case that takes polarization into account in which the light beam is characterized by the Stokes parameters and the phase function is replaced by the phase matrix. The phase matrix must be expressed with respect to the local meridian plane in a manner defined in Section 6.6. Finally, it should be noted that the numerical techniques referred to as *matrix formulation*, *matrix operator*, or *star product* are esssentially the same as the adding method, so far as the principle and actual computations are concerned.

6.4.3 Equivalence of the Adding Method and the Principles of Invariance

In reference to the principles of invariance for finite atmospheres defined in Eqs. (6.3.29)–(6.3.32) of Section 6.3.3, we replace τ by τ_1 and $\tau_1 - \tau$ by τ_2 and define the dimensionless upward and downward internal intensities as follows:

$$U(\mu, \mu_0) = \frac{\pi I(\tau_1, \mu)}{\mu_0 F_\odot}, \tag{6.4.19a}$$

$$D(\mu, \mu_0) = \frac{\pi I(\tau_1, -\mu)}{\mu_0 F_\odot}. \tag{6.4.19b}$$

The four principles of invariance may be expressed in terms of the reflection and transmission functions as follows:

$$\begin{aligned} U(\mu, \mu_0) = {} & R_2(\mu, \mu_0) e^{-\tau_1/\mu_0} \\ & + 2\int_0^1 R_2(\mu, \mu') D(\mu', \mu_0) \mu' d\mu', \end{aligned} \tag{6.4.20}$$

$$D(\mu, \mu_0) = T_1(\mu, \mu_0) + 2\int_0^1 R_1(\mu, \mu'') U(\mu'', \mu_0) \mu'' d\mu'', \tag{6.4.21a}$$

$$\begin{aligned} R_{12}(\mu, \mu_0) = {} & R_1(\mu, \mu_0) + e^{-\tau_1/\mu_0} U(\mu, \mu_0) \\ & + 2\int_0^1 T_1^*(\mu, \mu') U(\mu', \mu_0) \mu' d\mu', \end{aligned} \tag{6.4.22}$$

$$\begin{aligned} T_{12}(\mu, \mu_0') = {} & T_2(\mu, \mu_0) e^{-\tau_1/\mu_0} + e^{-\tau_2/\mu} D(\mu, \mu_0) \\ & + 2\int_0^1 T_2(\mu, \mu') D(\mu', \mu_0) \mu' d\mu'. \end{aligned} \tag{6.4.23}$$

The geometric configuration involving the basic variables is illustrated in Fig. 6.10. Although the preceding equations are written for azimuthally independent cases, these equations may be modified for general radiative transfer involving azimuthal terms

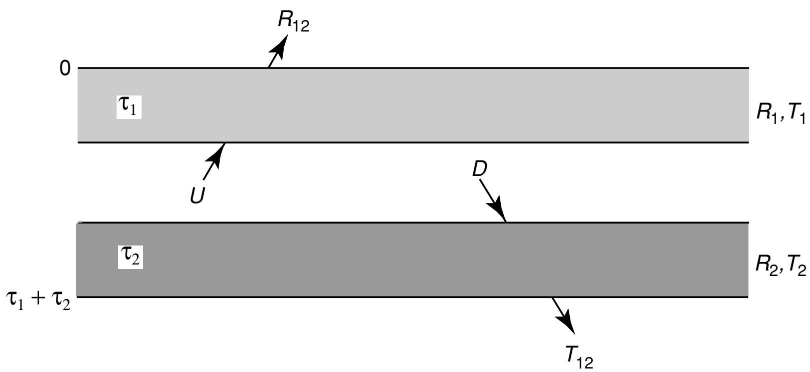

Figure 6.10 Geometric configuration for the reflection and transmission functions defined in Eqs. (6.4.19)–(6.4.23) based on the principles of invariance for a finite atmosphere. For illustration purposes, we have defined $\tau = \tau_1$ and $\tau_1 - \tau = \tau_2$ in Eqs. (6.3.29)–(6.3.32).

and polarization effects by replacing μ with (μ, ϕ) and the diffuse intensity with the Stokes parameters. Substituting Eq. (6.4.20) into Eq. (6.4.21a) leads to

$$D(\mu, \mu_0) = T_1(\mu, \mu_0) + S_{12}(\mu, \mu_0)e^{-\tau_1/\mu_0} + 2\int_0^1 S_{12}(\mu, \mu'')D(\mu'', \mu_0)\mu'' d\mu'', \tag{6.4.21b}$$

where

$$S_{12}(\mu, \mu'') = 2\int_0^1 R_1^*(\mu, \mu')R_2(\mu', \mu'')\mu' d\mu'. \tag{6.4.24}$$

In Eqs. (6.4.22) and (6.4.24), the superscript * denotes that radiation comes from below. Equation (6.4.21a) can be rewritten as follows:

$$2\int_0^1 \left[\delta(\mu - \mu'') - S_{12}(\mu, \mu'')\right]D(\mu'', \mu_0)\mu'' d\mu'' = T_1(\mu, \mu_0) + S_{12}(\mu, \mu_0)e^{-\tau_1/\mu_0}. \tag{6.4.21c}$$

In terms of the integral operator defined in Eq. (6.4.16), and noting again that $(1 - R_1^* R_2)^{-1} = 1 + S$ and $S_{12} = R_1^* R_2$, Eqs. (6.4.21c), (6.4.20), (6.4.22) and (6.4.23) can then be expressed by

$$D = T_1 + ST_1 + Se^{-\tau_1/\mu_0}, \tag{6.4.25a}$$

$$U = R_2 D + R_2 e^{-\tau_1/\mu_0}, \tag{6.4.25b}$$

$$R_{12} = R_1 + e^{-\tau_1/\mu} U + T_1^* U, \tag{6.4.25c}$$

$$T_{12} = e^{-\tau_2/\mu} D + T_2 e^{-\tau_1/\mu_0} + T_2 D. \tag{6.4.25d}$$

Equations (6.4.25a)–(6.4.25d) are exactly the same as Eqs. (6.4.15c)–(6.4.15f). We have, therefore, proved that the adding method for radiative transfer is "equivalent" to the principles of invariance for the case involving radiation from above. The principles of invariance can also be formulated for the case involving radiation from below, and the resulting equations would be "equivalent" to the adding equations presented in Eqs. (6.4.17c)–(6.4.17f).

6.4.4 Extension to Nonhomogeneous Atmospheres for Internal Fields

As demonstrated in the preceding sections, the adding principle is an efficient method for determining the radiation fields at the top and bottom of the atmosphere. To obtain the internal radiation fields, we may employ the following numerical procedures:

1. We may divide the atmosphere into N homogeneous layers, each of which is characterized by a single-scattering albedo, phase function, and optical depth. Let R_ℓ and $T_\ell (\ell = 1, 2, \ldots, N)$ denote the reflection and transmission functions for each homogeneous layer. Since homogeneous layers are considered, we have $R_\ell^* = R_\ell$ and $T_\ell^* = T_\ell$. R_ℓ and T_ℓ may be obtained from the doubling method described previously.

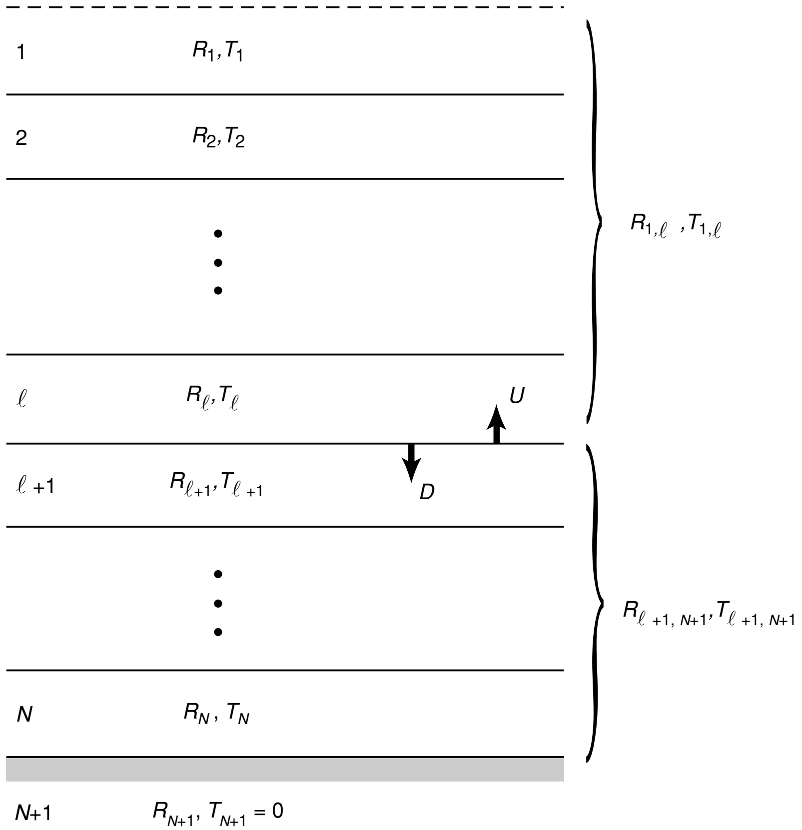

Figure 6.11 Illustrating diagram for the computation of internal intensities based on the adding principle for radiative transfer. The symbols U and D are defined in Eqs. (6.4.26a) and (6.4.26b).

The surface is considered to be a layer whose reflection function is R_{N+1} and whose transmission function is $T_{N+1} = 0$. If the surface is approximated as a Lambertian reflector, then R_{N+1} is the surface albedo r_s.

2. As shown in Fig. 6.11, the layers are added one at a time from TOA downward to obtain $R_{1,\ell}$ and $T_{1,\ell}$ for ℓ from 2 to $(N+1)$, and $R^*_{1,\ell}$, and $T^*_{1,\ell}$ for ℓ from 2 to N. For example, $R_{1,\ell}$ is the reflection function for composite layers 1 through ℓ, with the lower part of the atmosphere and surface absent.

3. The layers are then added one at a time from the surface upward to obtain $R_{\ell+1,N+1}$ and $T_{\ell+1,N+1}$ for ℓ from $(N-1)$ to 1.

4. We now consider the composite layers $(1, \ell)$ and $(\ell+1, N+1)$ and use the adding equations for the internal intensities noted in Eqs. (6.4.15c,d) to obtain

$$D = T_{1,\ell} + ST_{1,\ell} + S \exp(-\tau_{1,\ell}/\mu_0), \tag{6.4.26a}$$

$$U = R_{\ell+1,N+1}D + R_{\ell+1,N+1} \exp(-\tau_{1,\ell}/\mu_0), \tag{6.4.26b}$$

where

$$S = Q(1 - Q)^{-1}, \tag{6.4.27a}$$

$$Q = R^*_{1,\ell} R_{\ell+1,N+1}, \tag{6.4.27b}$$

and $\tau_{1,\ell}$ is the optical depth from TOA to the bottom of the layer.

To obtain the upward and downward fluxes at the interface between the ℓ and $\ell + 1$ layers, angular integrations can be performed. It is necessary to consider only the azimuth-independent condition, so that

$$F^{\uparrow} = \mu_0 F_{\odot} \left(2 \int_0^1 U(\mu, \mu_0)\mu \, d\mu \right), \tag{6.4.28a}$$

$$F^{\downarrow}_{\text{dif}} = \mu_0 F_{\odot} \left(2 \int_0^1 D(\mu, \mu_0)\mu \, d\mu \right). \tag{6.4.28b}$$

The downward direct solar flux is

$$F^{\downarrow}_{\text{dir}} = \mu_0 F_{\odot} \exp(-\tau_{1,\ell}/\mu_0). \tag{6.4.29}$$

Thus, the net flux is

$$F = \left(F^{\downarrow}_{\text{dif}} + F^{\downarrow}_{\text{dir}} \right) - F^{\uparrow}. \tag{6.4.30}$$

The heating rate can then be evaluated from the divergence of the net fluxes, as discussed in Section 3.5.

6.4.5 Similarity between the Adding and Discrete-Ordinates Methods

The adding equation can also be derived from an approach referred to as the *interaction principle,* which is based on a conservation relationship between the radiation emerging from a layer and the radiation incident to the boundary and emitted within the layer. For the present discussion, the emission contribution will be omitted.

Consider the geometric configuration illustrated in Fig. 6.9. Let the incoming and outgoing diffuse intensities at the top be $I^-(0)$ and $I^+(0)$, respectively. Similarly, the incoming and outgoing diffuse intensities at the level $\tau = \tau_1$ are denoted by $I^+(\tau_1)$ and $I^-(\tau_1)$, respectively. The $+$ or $-$ sign signifies that the intensity is upward or downward, respectively. The interaction principle states that the outgoing intensities $I^+(0)$ and $I^-(\tau_1)$ are related to the incoming intensities $I^-(0)$ and $I^+(\tau_1)$ via a linear conservation principle such that

$$I^+(0) = R_1 I^-(0) + \widetilde{T}^*_1 I^+(\tau_1), \tag{6.4.31a}$$

$$I^-(\tau_1) = \widetilde{T}_1 I^-(0) + R^*_1 I^+(\tau_1), \tag{6.4.31b}$$

where the notations R, R^*, $\widetilde{T}$, and $\widetilde{T}^*$ have been defined in Section 6.4.2. In matrix form, we write

$$\begin{bmatrix} I^+(0) \\ I^-(\tau_1) \end{bmatrix} = \begin{bmatrix} R_1 & \widetilde{T}_1^* \\ \widetilde{T}_1 & R_1^* \end{bmatrix} \begin{bmatrix} I^-(0) \\ I^+(\tau_1) \end{bmatrix}. \tag{6.4.32a}$$

Application of the interaction principle to the layer bounded by τ_1 and $\tau = \tau_1 + \tau_2$, as defined in Fig. 6.9, leads to

$$\begin{bmatrix} I^+(\tau_1) \\ I^-(\tau) \end{bmatrix} = \begin{bmatrix} R_2 & \widetilde{T}_2^* \\ \widetilde{T}_2 & R_2^* \end{bmatrix} \begin{bmatrix} I^-(\tau_1) \\ I^+(\tau) \end{bmatrix}. \tag{6.4.32b}$$

On eliminating the upward and downward diffuse intensities at the τ_1 level, $I^{\pm}(\tau_1)$, we can prove that

$$\begin{bmatrix} I^+(0) \\ I^-(\tau) \end{bmatrix} = \begin{bmatrix} R_1 + \widetilde{T}_1^* R_2 \left(1 - R_1^* R_2\right)^{-1} \widetilde{T}_1 & \widetilde{T}_1^* \left(1 - R_2 R_1^*\right)^{-1} \widetilde{T}_2^* \\ \widetilde{T}_2 \left(1 - R_1^* R_2\right)^{-1} \widetilde{T}_1 & R_2^* + \widetilde{T}_2 R_1^* \left(1 - R_2 R_1^*\right)^{-1} \widetilde{T}_2^* \end{bmatrix} \begin{bmatrix} I^-(0) \\ I^+(\tau) \end{bmatrix}. \tag{6.4.33a}$$

The interaction principle is general and can be applied to the combined layer, $\tau = \tau_1 + \tau_2$, so that

$$\begin{bmatrix} I^+(0) \\ I^-(\tau) \end{bmatrix} = \begin{bmatrix} R_{12} & \widetilde{T}_{12}^* \\ \widetilde{T}_{12} & R_{12}^* \end{bmatrix} \begin{bmatrix} I^-(0) \\ I^+(\tau) \end{bmatrix}. \tag{6.4.33b}$$

It follows that Eqs. (6.4.33a) and (6.4.33b) must be identical, leading to the following relationships:

$$R_{12} = R_1 + \widetilde{T}_1^* R_2 \left(1 - R_1^* R_2\right)^{-1} \widetilde{T}_1, \tag{6.4.34a}$$

$$\widetilde{T}_{12} = \widetilde{T}_2 \left(1 - R_1^* R_2\right)^{-1} \widetilde{T}_1, \tag{6.4.34b}$$

$$R_{12}^* = R_2^* + \widetilde{T}_2 R_1^* \left(1 - R_2 R_1^*\right)^{-1} \widetilde{T}_2^*, \tag{6.4.35a}$$

$$\widetilde{T}_{12}^* = \widetilde{T}_1^* \left(1 - R_2 R_1^*\right)^{-1} \widetilde{T}_2^*. \tag{6.4.35b}$$

The first two equations are exactly the same as those derived from the geometric ray-tracing technique for the light beam incident from above, denoted in Eqs. (6.4.9a) and (6.4.9b). We may also apply this technique to the light beam incident from below to obtain Eqs. (6.4.17e) and (6.4.17f) (Exercise 6.5). Equations (6.4.34a) and (6.4.34b) contain a term $(1 - R_1^* R_2)^{-1}$, which is associated with the inversion of a matrix. Moreover, the term $R_1^* R_2$, as defined in Eq. (6.4.16), can be expressed by summation according to Gaussian quadratures in the form

$$R_1^* R_2(\mu_i, \mu_0) = \frac{1}{2} \sum_{j=-n}^{n} R_1^*(\mu_i, \mu_j) R_2(\mu_j, \mu_0) \mu_j a_j, i = -n, n. \tag{6.4.36}$$

The discrete-ordinates method for radiative transfer presented in Section 6.2 directly solves the basic integrodifferential equation by means of discretization of the integral in terms of Gauss's formula. In line with the present discussion, we shall consider a homogeneous layer with an optical depth τ_1. We may begin with the upward and downward components of the diffuse intensity, as presented in Eq. (6.2.51), in matrix form as follows:

$$\frac{d}{d\tau}\begin{bmatrix} I^+(\tau) \\ I^-(\tau) \end{bmatrix} = \begin{bmatrix} b^+ & b^- \\ -b^- & -b^+ \end{bmatrix}\begin{bmatrix} I^+(\tau) \\ I^-(\tau) \end{bmatrix}. \tag{6.4.37}$$

The solutions of the reflected intensity at the top, $I^+(0)$, and the transmitted intensity at the bottom, $I^-(\tau_1)$, require the imposition of boundary conditions. For the sake of discussion, we may employ the vacuum boundary conditions such that there are no upward and downward diffuse intensities at the bottom and top layers, respectively. In this case, we have $I^+(\tau_1) = I^-(0) = 0$. The discrete-ordinates method, as presented in Eq. (6.4.37), consists of a differential operator and, hence, must be solved mathematically. The adding method that is derived from the interaction principle, as presented in Eqs. (6.4.33a) and (6.4.33b), however, entails calculations of the reflection and transmission functions in a direct way. Nevertheless, the two methods are similar to the extent that both are related to a 2×2 matrix, representing upward and downward radiation components.

Moreover, in the discrete-ordinates method, the solution of the diffuse intensity can be written as

$$I(\tau, \mu_i) = \sum_{j=-n}^{n} L_j \varphi_j(\mu_i) e^{-k_j \tau} + Z(\mu_i) e^{-\tau/\mu_0}, i = -n, n. \tag{6.4.38}$$

Equation (6.4.38) is similar to Eq. (6.2.58), except without the layer index. The unknown coefficients L_j must be determined from boundary conditions. We may follow the approach presented in Section 6.2.4 and write

$$\mathbf{L} = \boldsymbol{\varphi}^{-1}\boldsymbol{\chi}, \tag{6.4.39}$$

where $\mathbf{L}$ is a $2n$ column vector, $\boldsymbol{\varphi}$ is a $2n \times 2n$ matrix involving eigenvectors and eigenvalues, and $\boldsymbol{\chi}$ is a $2n$ column vector associated with the direct solar flux. When $n = 1$ and 2, the system of equations reduces to 2 and 4, referred to as the *two-stream* and *four-stream* approximations, respectively (Sections 6.5.2 and 6.5.4). Once L_j are determined, diffuse intensity can be computed at the top and bottom of a layer, as well as within the layer. For comparison with the adding method, we shall use the reflection function R_{12} defined by $\pi I^+(0, \mu_i)/\mu_0 F_\odot$. Thus, we have

$$R_{12} = (\pi/\mu_0 F_\odot)\left[\sum_{j=-n}^{n} L_j \varphi_j(\mu_i) + Z(\mu_i)\right], \tag{6.4.40a}$$

where L_j values are computed from the inversion of a matrix given by Eq. (6.4.39). On the other hand, using Eq. (6.4.36), the reflection function derived from the adding method can be expressed by

$$R_{12} = R_1 + \widetilde{T}_1^* R_2 \left[1 - \frac{1}{2} \sum_{j=-n}^{n} R_1^*(\mu_i, \mu_j) R_2(\mu_j, \mu_0) \mu_j a_j \right]^{-1} \widetilde{T}_1. \quad (6.4.40b)$$

It is not possible to match each term in Eqs. (6.4.40a) and (6.4.40b). However, to the extent that both involve summation over discrete streams and inversion of matrices, it appears appropriate to conclude that the adding and discrete-ordinates methods are similar in terms of numerical computation.

6.5 Approximations for Radiative Transfer

6.5.1 Successive-Orders-of-Scattering Approximation

In Section 6.4.1, we showed that under the single-scattering approximation and subject to the condition that there are no diffuse intensities from the top and bottom of the layer with an optical depth τ_1, the reflection and transmission functions are directly proportional to the phase function. Rewriting Eqs. (6.4.7a) and (6.4.7b) in terms of reflected and transmitted intensities, we have

$$I_1(0; \mu, \phi) = \frac{\tilde{\omega} \mu_0 F_\odot}{4\pi(\mu + \mu_0)} P(\mu, \phi; -\mu_0, \phi_0) \left\{ 1 - \exp\left[-\tau_1 \left(\frac{1}{\mu} + \frac{1}{\mu_0} \right) \right] \right\}, \quad (6.5.1a)$$

$$I_1(\tau_1; -\mu, \phi) = \begin{cases} \dfrac{\tilde{\omega} \mu_0 F_\odot}{4\pi(\mu - \mu_0)} P(-\mu, \phi; -\mu_0, \phi_0)(e^{-\tau_1/\mu} - e^{-\tau_1/\mu_0}), & \mu \neq \mu_0 \\ \dfrac{\tilde{\omega} \tau_1 F_\odot}{4\pi \mu_0} P(-\mu_0, \phi_0; -\mu_0, \phi_0) e^{-\tau_1/\mu_0}, & \mu = \mu_0. \end{cases} \quad (6.5.1b)$$

These two equations are fundamental in building the successive-orders-of-scattering (SOS) approximation, in which the intensity is computed individually for photons scattered once, twice, three times, and so forth. The total intensity is then obtained as the sum over all orders. Hence, for reflected and diffuse transmitted intensities, we may write, respectively,

$$I(\tau; \mu, \phi) = \sum_{n=1}^{\infty} I_n(\tau; \mu, \phi), \quad (6.5.2a)$$

$$I(\tau; -\mu, \phi) = \sum_{n=1}^{\infty} I_n(\tau; -\mu, \phi), \quad (6.5.2b)$$

where n denotes the order of scattering.

Subject to the boundary conditions denoted in Eq. (6.4.6), the formal solution of the equations of transfer for upward and downward intensities is given by

$$I(\tau;\mu,\phi) = \int_{\tau}^{\tau_1} J(\tau';\mu,\phi)\exp[-(\tau'-\tau)/\mu]\frac{d\tau'}{\mu}, \tag{6.5.3a}$$

$$I(\tau;-\mu,\phi) = \int_{0}^{\tau} J(\tau';-\mu,\phi)\exp[-(\tau-\tau')/\mu]\frac{d\tau'}{\mu}, \tag{6.5.3b}$$

where τ_1 is the total optical depth. The source function for the first-order scattering is given by Eq. (6.4.5). Inserting this expression into the above formal solutions of the equation of transfer and integrating over the appropriate optical depths, the intensity due to photons scattered once may be obtained. It follows that the source functions and intensities may be derived successively by means of the following recursion relationships:

$$J_{n+1}(\tau;\mu,\phi) = \frac{\tilde{\omega}}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(\mu,\phi;\,\mu',\phi')I_n(\tau;\mu',\phi')\,d\mu'\,d\phi', \tag{6.5.4a}$$

$$I_n(\tau;\mu,\phi) = \int_{\tau}^{\tau_1} J_n(\tau';\mu,\phi)\exp[-(\tau'-\tau)/\mu]\frac{d\tau'}{\mu}, \qquad n \geq 1, \tag{6.5.4b}$$

$$I_n(\tau;-\mu,\phi) = \int_{0}^{\tau} J_n(\tau';-\mu,\phi)\exp[-(\tau-\tau')/\mu]\frac{d\tau'}{\mu}, \qquad n \geq 1, \tag{6.5.4c}$$

where the zero-order intensity is given by the Dirac δ function [see also Eq. (6.4.3)]

$$I_{\odot}(\tau;\mu',\phi') = F_{\odot}e^{-\tau/\mu_0}\delta(\mu'-\mu_0)\delta(\phi'-\phi_0). \tag{6.5.5}$$

Note that J_1 can be determined from Eq. (6.5.4a) and is given in Eq. (6.4.5). I_1 can be evaluated from Eqs. (6.5.4b) and (6.5.4c) and is given in Eqs. (6.5.1a) and (6.5.1b). Numerical techniques may be devised to carry out the integrations for a finite interval of τ in Eq. (6.5.4) to obtain the intensity distribution. Exercise 6.7 requires the derivation of the reflected intensity due to second-order scattering ($n = 2$).

The SOS method just outlined is an integral solution approach that can be directly applied to specific geometry without the requirement of solving the basic radiative transfer equation in differential form. In addition, the inhomogeneous structure of the medium can be incorporated in the calculation in a straightforward manner in terms of integration along the line path. However, it is well known, based on the experience of plane-parallel radiative-transfer calculations, that this method requires considerable computational effort to converge the intensity solution, especially for an optically thick medium.

6.5.2 Two-Stream and Eddington's Approximations

In Sections 3.4.2 and 4.6.3, we introduced the two-stream approximation for radiative transfer and pointed out the potential applicability of this approximation to the parameterization of radiative transfer for use in climate models. In this section, we present the details of this method.

In order to solve Eq. (6.1.6) analytically, the integral must be replaced by a summation over a finite number of quadrature points. In numerical integrations, Gauss's formula has been found to be superior to other formulas for quadratures in the interval $(-1, 1)$. For any function $f(\mu)$, Gauss's formula is expressed by

$$\int_{-1}^{1} f(\mu)d\mu \approx \sum_{j=-n}^{n} a_j f(\mu_j), \tag{6.5.6a}$$

where the weights are

$$a_j = \frac{1}{P'_{2n}(\mu_j)} \int_{-1}^{1} \frac{P_{2n}(\mu)}{\mu - \mu_j} d\mu, \tag{6.5.6b}$$

μ_j are the zeros of the even-order Legendre polynomials $P_{2n}(\mu)$, and the prime denotes the differentiation with respect to μ_j. Also,

$$a_{-j} = a_j, \qquad u_{-j} = -\mu_j, \qquad \sum_{j=-n}^{n} a_j = 2. \tag{6.5.6c}$$

Table 6.1 lists the Gaussian points and weights for the first four approximations.

Employing Gauss's formula, Eq. (6.1.6) can be written as

$$\mu_i \frac{dI(\tau, \mu_i)}{d\tau} = I(\tau, \mu_i) - \frac{\tilde{\omega}}{2} \sum_{\ell=0}^{N} \tilde{\omega}_\ell P_\ell(\mu_i) \sum_{j=-n}^{n} a_j P_\ell(\mu_j) I(\tau, \mu_j)$$

$$- \frac{\tilde{\omega}}{4\pi} F_\odot \left[\sum_{\ell=0}^{N} (-1)^\ell \tilde{\omega}_\ell P_\ell(\mu_i) P_\ell(\mu_0) \right] e^{-\tau/\mu_0}, \quad i = -n, n, \tag{6.5.7}$$

where $\mu_i(-n, n)$ represent the directions of the radiation streams. In the two-stream approximation, we take two radiation streams, i.e., $j = -1$ and 1, and $N = 1$. Note

Table 6.1

Gaussian Points and Weights

n	$2n$	$\pm\mu_n$	a_n
1	2	$\mu_1 = 0.5773503$	$a_1 = 1$
2	4	$\mu_1 = 0.3399810$	$a_1 = 0.6521452$
		$\mu_2 = 0.8611363$	$a_2 = 0.3478548$
3	6	$\mu_1 = 0.2386192$	$a_1 = 0.4679139$
		$\mu_2 = 0.6612094$	$a_2 = 0.3607616$
		$\mu_3 = 0.9324695$	$a_3 = 0.1713245$
4	8	$\mu_1 = 0.1834346$	$a_1 = 0.3626838$
		$\mu_2 = 0.5255324$	$a_2 = 0.3137066$
		$\mu_3 = 0.7966665$	$a_3 = 0.2223810$
		$\mu_4 = 0.9602899$	$a_4 = 0.1012285$

that $\mu_1 = 1/\sqrt{3}$ and $a_1 = a_{-1} = 1$. After rearranging terms and denoting $I^\uparrow = I(\tau, \mu_1)$ and $I^\downarrow = I(\tau, -\mu_1)$, we obtain two simultaneous equations as follows:

$$\mu_1 \frac{dI^\uparrow}{d\tau} = I^\uparrow - \tilde{\omega}(1-b)I^\uparrow - \tilde{\omega}bI^\downarrow - S^- e^{-\tau/\mu_0}, \tag{6.5.8a}$$

$$-\mu_1 \frac{dI^\downarrow}{d\tau} = I^\downarrow - \tilde{\omega}(1-b)I^\downarrow - \tilde{\omega}bI^\uparrow - S^+ e^{-\tau/\mu_0}, \tag{6.5.8b}$$

where

$$g = \frac{\tilde{\omega}_1}{3} = \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\cos\Theta\, d\cos\Theta = \langle\cos\Theta\rangle, \tag{6.5.9a}$$

$$b = \frac{1-g}{2} = \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\frac{1-\cos\Theta}{2} d\cos\Theta, \tag{6.5.9b}$$

$$S^\pm = \frac{F_\odot \tilde{\omega}}{4\pi}(1 \pm 3g\mu_1\mu_0). \tag{6.5.9c}$$

The asymmetry factor g was introduced in Section 3.4.1. It is the first moment of the phase function. Note that the zero moment of the phase function is simply equal to $\tilde{\omega}_0 (=1)$. For isotropic scattering, g is zero as it is for Rayleigh scattering. The asymmetry factor increases as the diffraction peak of the phase function becomes increasingly sharp. Conceivably, the asymmetry factor may be negative if the phase function is peaked in backward directions (90–180°). For Lorenz–Mie particles, whose phase function has a general sharp peak at a 0° scattering angle, the asymmetry factor denotes the relative strength of forward scattering. The parameters b and $(1-b)$ can be interpreted as the integrated fractions of energy backscattered and forward scattered, respectively. Thus, the multiple scattering contribution in the context of the two-stream approximation is represented by the upward and downward intensities weighed by the appropriate fraction of the forward or backward phase function, as illustrated in Eq. (6.5.8). The upward intensity is strengthened by its coupling with the forward fraction (0–90°) of the phase function plus the downward intensity that appears in backward fractions (90–180°) of the phase function. A similar argument applies to the downward intensity.

Equations (6.5.8a) and (6.5.8b) represent two first-order inhomogeneous differential equations. For the solutions, we first let $M = I^\uparrow + I^\downarrow$ and $N = I^\uparrow - I^\downarrow$ and note that $(1-2b) = g$. Subsequently, by subtracting and adding, these two equations become

$$\mu_1 \frac{dM}{d\tau} = (1 - \tilde{\omega}g)N - (S^- - S^+)e^{-\tau/\mu_0}, \tag{6.5.10a}$$

$$\mu_1 \frac{dN}{d\tau} = (1 - \tilde{\omega})M - (S^- + S^+)e^{-\tau/\mu_0}. \tag{6.5.10b}$$

Further, by differentiating both equations with respect to τ, we obtain

$$\mu_1 \frac{d^2 M}{d\tau^2} = (1 - \tilde{\omega} g)\frac{dN}{d\tau} + \frac{(S^- - S^+)}{\mu_0} e^{-\tau/\mu_0}, \tag{6.5.11a}$$

$$\mu_1 \frac{d^2 N}{d\tau^2} = (1 - \tilde{\omega})\frac{dM}{d\tau} + \frac{(S^- + S^+)}{\mu_0} e^{-\tau/\mu_0}. \tag{6.5.11b}$$

Upon inserting Eqs. (6.5.10b) and (6.5.10a) into Eqs. (6.5.11a) and (6.5.11b), respectively, we find

$$\frac{d^2 M}{d\tau^2} = k^2 M + Z_1 e^{-\tau/\mu_0}, \tag{6.5.12a}$$

$$\frac{d^2 N}{d\tau^2} = k^2 N + Z_2 e^{-\tau/\mu_0}, \tag{6.5.12b}$$

where the eigenvalue is given by

$$k^2 = (1 - \tilde{\omega})(1 - \tilde{\omega} g)/\mu_1^2, \tag{6.5.13}$$

and

$$Z_1 = -\frac{(1 - \tilde{\omega} g)(S^- + S^+)}{\mu_1^2} + \frac{S^- - S^+}{\mu_1 \mu_0}, \tag{6.5.14a}$$

$$Z_2 = -\frac{(1 - \tilde{\omega})(S^- - S^+)}{\mu_1^2} + \frac{S^- + S^+}{\mu_1 \mu_0}. \tag{6.5.14b}$$

Equations (6.5.12a,b) represent a set of second-order differential equations, which can be solved by first seeking the homogeneous part and then adding a particular solution. In seeking the homogeneous solution, the homogeneous parts of Eq. (6.5.10) need to be satisfied so that there are only two unknown constants involved. Straightforward analyses yield the solutions

$$I^{\uparrow} = I(\tau, \mu_1) = K v e^{k\tau} + H u e^{-k\tau} + \varepsilon e^{-\tau/\mu_0}, \tag{6.5.15a}$$

$$I^{\downarrow} = I(\tau, -\mu_1) = K u e^{k\tau} + H v e^{-k\tau} + \gamma e^{-\tau/\mu_0}, \tag{6.5.15b}$$

where

$$v = (1 + a)/2, \qquad u = (1 - a)/2, \tag{6.5.16a}$$

$$a^2 = (1 - \tilde{\omega})/(1 - \tilde{\omega} g), \tag{6.5.16b}$$

$$\varepsilon = (\alpha + \beta)/2, \qquad \gamma = (\alpha - \beta)/2, \tag{6.5.16c}$$

$$\alpha = Z_1 \mu_0^2 / \left(1 - \mu_0^2 k^2\right), \qquad \beta = Z_2 \mu_0^2 / \left(1 - \mu_0^2 k^2\right). \tag{6.5.16d}$$

The constants K and H are to be determined from the diffuse radiation boundary conditions at the top and bottom of the scattering layer. Assuming no diffuse components from the top and bottom, we have

$$K = -(\varepsilon v e^{-\tau_1/\mu_0} - \gamma u e^{-k\tau_1})/(v^2 e^{k\tau_1} - u^2 e^{-k\tau_1}), \tag{6.5.17a}$$

$$H = -(\varepsilon u e^{-\tau_1/\mu_0} - \gamma v e^{-k\tau_1})/(v^2 e^{k\tau_1} - u^2 e^{-k\tau_1}). \tag{6.5.17b}$$

Once the upward and downward intensities have been evaluated, the upward and downward diffuse flux densities are simply given by

$$F^{\uparrow}(\tau) = 2\pi\mu_1 I^{\uparrow}, \qquad F^{\downarrow}(\tau) = 2\pi\mu_1 I^{\downarrow}. \tag{6.5.18}$$

The preceding analyses constitute the two-stream approximation for radiative transfer.

The form of Eq. (6.5.8), without the direct solar source term, was first presented by Schuster (1905). The terms $\pm k$ in Eq. (6.5.13) are the eigenvalues for the solution of the differential equations, and u and v represent the eigenfunctions, which are defined by the similarity parameter a in Eq. (6.5.16b) (see Section 6.5.3 for discussion on the similarity principle in radiative transfer). For conservative scattering, $\tilde{\omega} = 1$. Simpler solutions can be derived from Eqs. (6.5.8a) and (6.5.8b) with one of the eigenvalues, $k = 0$ (Exercise 6.8). In practice, however, we may set $\tilde{\omega} = 0.999999$ in Eqs. (6.5.16a)–(6.5.16d) and obtain the results for conservative scattering.

To introduce *Eddington's approximation* for radiative transfer, we begin with the general approach of decomposing the equation of radiative transfer using the property of Legendre polynomials. In line with the Legendre polynomial expansion for the phase function denoted in Eq. (6.1.11), the scattered intensity may be expanded in terms of Legendre polynomials such that

$$I(\tau, \mu) = \sum_{\ell=0}^{N} I_\ell(\tau) P_\ell(\mu). \tag{6.5.19}$$

Using the orthogonal and recurrence properties of Legendre polynomials and omitting the emission term, Eq. (6.1.12) may be decomposed in N harmonics in the form

$$\begin{aligned} &\frac{\ell}{2\ell - 1}\frac{dI_{\ell-1}}{d\tau} + \frac{\ell + 1}{2\ell + 3}\frac{dI_{\ell+1}}{d\tau} \\ &\quad = I_\ell\left(1 - \frac{\tilde{\omega}\tilde{\omega}_\ell}{2\ell + 1}\right) - \frac{\tilde{\omega}}{4\pi}\tilde{\omega}_\ell P_\ell(-\mu_0) F_\odot e^{-\tau/\mu_0}, \\ &\quad \ell = 0, 1, 2, \ldots, N. \end{aligned} \tag{6.5.20}$$

The method of solving the basic radiative transfer equation using the aforementioned procedure is referred to as the *spherical harmonics method* (Kourganoff, 1952).

Eddington's approximation uses an approach similar to that of the two-stream approximation and was originally used for studies of radiative equilibrium in stellar atmospheres (Eddington, 1916). Letting $N = 1$, the phase function and intensity expressions may be written as follows:

$$\begin{aligned} P(\mu, \mu') &= 1 + 3g\mu\mu', \\ I(\tau, \mu) &= I_0(\tau) + I_1(\tau)\mu, \qquad -1 \le \mu \le 1. \end{aligned} \tag{6.5.21}$$

Consequently, Eq. (6.5.20) reduces to a set of two simultaneous equations in the form

$$\frac{dI_1}{d\tau} = 3(1-\tilde{\omega})I_0 - \frac{3\tilde{\omega}}{4\pi}F_\odot e^{-\tau/\mu_0}, \tag{6.5.22a}$$

$$\frac{dI_0}{d\tau} = (1-\tilde{\omega}g)I_1 + \frac{3\tilde{\omega}}{4\pi}g\mu_0 F_\odot e^{-\tau/\mu_0}. \tag{6.5.22b}$$

Differentiating Eq. (6.5.22b) with respect to τ and substituting the expression for $dI_1/d\tau$ from Eq. (6.5.22a) leads to

$$\frac{d^2 I_0}{d\tau^2} = k^2 I_0 - \chi e^{-\tau/\mu_0}, \tag{6.5.23}$$

where $\chi = 3\tilde{\omega}F_\odot(1+g-\tilde{\omega}g)/4\pi$, and the eigenvalues k^2 are exactly the same as they are for the two-stream approximation defined in Eq. (6.5.13). Equation (6.5.23) represents a well-known diffusion equation for radiative transfer. This diffusion approximation is particularly applicable to the radiation field in the deep domain of an optically thick layer (see Subsection 3.4.2.2).

Straightforward analysis yields the following solution for the diffusion equation:

$$I_0 = Ke^{k\tau} + He^{-k\tau} + \Psi e^{-\tau/\mu_0}, \tag{6.5.24a}$$

where

$$\Psi = \frac{3\tilde{\omega}}{4\pi}F_\odot\,[1+g(1-\tilde{\omega})]\big/\big(k^2 - 1/\mu_0^2\big).$$

Following a similar procedure, the solution for the second harmonic, I_1, is given by

$$I_1 = aKe^{k\tau} - aHe^{-k\tau} - \xi e^{-\tau/\mu_0}, \tag{6.5.24b}$$

where $a^2 = 3(1-\tilde{\omega})/(1-\tilde{\omega}g)$, defined in the two-stream approximation [Eq. (6.5.16b)], and

$$\xi = \frac{3\tilde{\omega}}{4\pi}\frac{F_\odot}{\mu_0}\left[1+3g(1-\tilde{\omega})\mu_0^2\right]\big/\big(k^2 - 1/\mu_0^2\big). \tag{6.5.24c}$$

The integration constants, K and H, are to be determined from proper boundary conditions. Finally, the upward and downward fluxes are given by

$$\left.\begin{matrix} F^\uparrow(\tau) \\ F^\downarrow(\tau) \end{matrix}\right\} = 2\pi\int_0^{\pm 1}(I_0 + \mu I_1)\mu\, d\mu = \begin{cases} \pi\left(I_0 + \frac{2}{3}I_1\right) \\ \pi\left(I_0 - \frac{2}{3}I_1\right). \end{cases} \tag{6.5.25}$$

In the following, we shall present the general two-stream equations. Using the radiative transfer equations denoted in Eq. (6.1.6) and the upward and downward diffuse fluxes defined in Eqs. (6.1.7a,b) and omitting the emission contribution, we may write the following equations:

$$\begin{aligned}\frac{1}{2\pi}\frac{dF^\uparrow(\tau)}{d\tau} &= \int_0^1 I(\tau,\mu)\,d\mu - \frac{\tilde{\omega}}{2}\int_0^1\int_{-1}^1 I(\tau,\mu)P(\mu,\mu')\,d\mu'\,d\mu \\ &\quad - \frac{\tilde{\omega}}{4\pi}F_\odot e^{-\tau/\mu_0}\int_0^1 P(\mu,-\mu_0)\,d\mu,\end{aligned} \tag{6.5.26a}$$

$$\frac{1}{2\pi}\frac{dF^{\downarrow}(\tau)}{d\tau} = -\int_0^1 I(\tau, -\mu)\,d\mu + \frac{\tilde{\omega}}{2}\int_0^1\int_{-1}^1 I(\tau, \mu')P(-\mu, \mu')\,d\mu'\,d\mu$$
$$+\frac{\tilde{\omega}}{4\pi}F_{\odot}e^{-\tau/\mu_0}\int_0^1 P(-\mu, -\mu_0)\,d\mu. \qquad (6.5.26b)$$

Thus, the generalized two-stream approximation may be expressed by

$$\frac{dF^{\uparrow}(\tau)}{d\tau} = \gamma_1 F^{\uparrow}(\tau) - \gamma_2 F^{\downarrow}(\tau) - \gamma_3\tilde{\omega}F_{\odot}e^{-\tau/\mu_0}, \qquad (6.5.27a)$$

$$\frac{dF^{\downarrow}(\tau)}{d\tau} = \gamma_2 F^{\uparrow}(\tau) - \gamma_1 F^{\downarrow}(\tau) + (1-\gamma_3)\tilde{\omega}F_{\odot}e^{-\tau/\mu_0}. \qquad (6.5.27b)$$

The differential changes in the upward and downward diffuse fluxes are directly related to the upward and downward diffuse fluxes, as well as the downward direct flux. The coefficients γ_i ($i = 1, 2, 3$) depend on the manner in which the intensity and phase function are approximated in Eq. (6.5.26). In the two-stream approximation, there are only upward and downward intensities in the directions μ_1 and $-\mu_1$ given by the Gauss quadrature formula, while the phase function is expanded in two terms in Legendre polynomials. In Eddington's approximation, both the intensity and phase functions are expanded in two polynomial terms. The coefficients γ_i can be directly derived from Eqs. (6.5.8a,b) and (6.5.22a,b) and are given in Table 6.2.

In Eqs. (6.5.26a,b), we let the last integral involving the phase function be

$$q = \frac{1}{2}\int_0^1 P(\mu, -\mu_0)\,d\mu. \qquad (6.5.28a)$$

Since the phase function is normalized to unity, we have

$$\frac{1}{2}\int_0^1 P(-\mu, -\mu_0)\,d\mu = 1 - q. \qquad (6.5.28b)$$

Equations (6.5.28a) and (6.5.28b) can be evaluated exactly by numerical means. We may take $\gamma_3 = q$ in the two-stream approximation. This constitutes the modified two-stream approximation proposed by Liou (1973b) and Meador and Weaver (1980). The two-stream approximation yields negative albedo values for a thin atmosphere when $\gamma_3 < 0$ (i.e., $g > \mu_1/\mu_0$). This also occurs in Eddington's approximation when $g > 0.67/\mu_0$. These negative albedo values can be avoided by using q, the full phase function integration for the direct solar beam, denoted in Eq. (6.5.28b). The accuracy of the two-stream approximation has been discussed in Liou (1973a). The overall accuracy of the two-stream and Eddington's approximations can be improved by

Table 6.2

Coefficients in Two-Stream Approximations

Method	γ_1	γ_2	γ_3
Two-stream	$[1 - \tilde{\omega}(1+g)/2]/\mu_1$	$\tilde{\omega}(1-g)/2\mu_1$	$(1 - 3g\mu_1\mu_0)/2$
Eddington's	$[7 - (4+3g)\tilde{\omega}]/4$	$-[1 - (4-3g)\tilde{\omega}]/4$	$(2 - 3g\mu_0)/4$

incorporating the δ-function adjustment for forward scattering and will be discussed in Section 6.5.3.

The solutions for the equations of the generalized two-stream approximation expressed in Eqs. (6.5.27a) and (6.5.27b) are as follows:

$$F^{\uparrow} = vKe^{k\tau} + uHe^{-k\tau} + \varepsilon e^{-\tau/\mu_0}, \tag{6.5.29a}$$

$$F^{\downarrow} = uKe^{k\tau} + vHe^{-k\tau} + \gamma e^{-\tau/\mu_0}, \tag{6.5.29b}$$

where K and H are unknown coefficients to be determined from the boundary conditions, and

$$k^2 = \gamma_1^2 - \gamma_2^2, \tag{6.5.30a}$$

$$v = \frac{1}{2}[1 + (\gamma_1 - \gamma_2)/k], \qquad u = \frac{1}{2}[1 - (\gamma_1 - \gamma_2)/k], \tag{6.5.30b}$$

$$\varepsilon = [\gamma_3(1/\mu_0 - \gamma_1) - \gamma_2(1 - \gamma_3)]\, \mu_0^2 \tilde{\omega} F_{\odot}, \tag{6.5.30c}$$

$$\gamma = -[(1 - \gamma_3)(1/\mu_0 + \gamma_1) + \gamma_2\gamma_3]\mu_0^2 \tilde{\omega} F_{\odot}. \tag{6.5.30d}$$

6.5.3 Delta-Function Adjustment and Similarity Principle

The two-stream and Eddington methods for radiative transfer are good approximations for optically thick layers, but they produce inaccurate results for thin layers and when significant absorption is involved. The basic problem is that scattering by atmospheric particulates is highly peaked in the forward direction. This is especially evident for cloud particles, for which the forward-scattered energy within $\sim 5^\circ$ scattering angles produced by diffraction is five to six orders of magnitude greater than it is in the side and backward directions (see, e.g., Figs 5.15 and 5.23). The highly peaked diffraction pattern is typical for atmospheric particulates. It is clear that a two-term expansion in the phase function is far from adequate.

To incorporate the forward peak contribution in multiple scattering, we may consider an adjusted absorption and scattering atmosphere, such that the fraction of scattered energy residing in the forward peak, f, is removed from the scattering parameters: optical depth, τ; single-scattering albedo, $\tilde{\omega}$; and asymmetry factor g. We use primes to represent the adjusted radiative parameters, as shown in Fig. 6.12. The optical (extinction) depth is the sum of the scattering (τ_s) and absorption (τ_a) optical depths. The forward peak is produced by diffraction without the contribution of absorption. Thus, the adjusted scattering and absorption optical depths must be

$$\tau_s' = (1 - f)\tau_s, \tag{6.5.31a}$$

$$\tau_a' = \tau_a. \tag{6.5.31b}$$

The total adjusted optical depth is

$$\tau' = \tau_s' + \tau_a' = (1 - f)\tau_s + \tau_a = \tau(1 - f\tilde{\omega}), \tag{6.5.32}$$

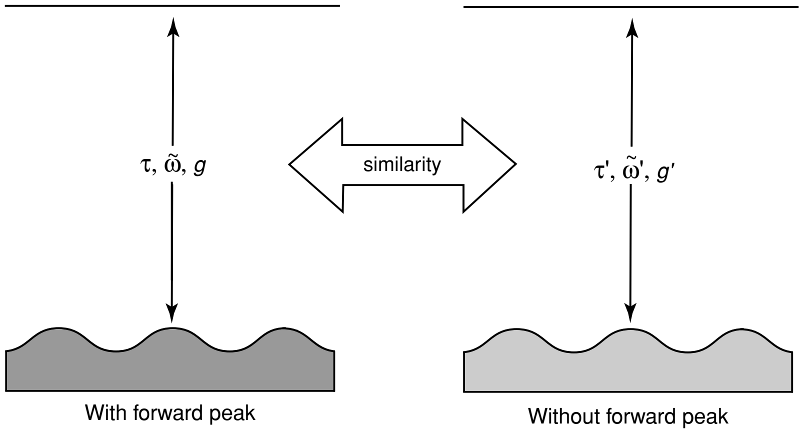

Figure 6.12 Similarity principle for radiative transfer. The prime system represents adjusted radiative parameters such that the forward diffraction peak in scattering processes is removed.

and the adjusted single-scattering albedo is then

$$\tilde{\omega}' = \frac{\tau_s'}{\tau'} = \frac{(1-f)\tau_s}{(1-f\tilde{\omega})\tau} = \frac{(1-f)\tilde{\omega}}{1-f\tilde{\omega}}. \tag{6.5.33}$$

Moreover, we multiply the asymmetry factor by the scattering optical depth to get the similarity equation

$$\tau_s' g' = \tau_s g - \tau_s f, \qquad \text{or} \qquad g' = \frac{g-f}{1-f}, \tag{6.5.34}$$

where we note that the asymmetry factor in the forward peak is equal to unity. In the diffusion domain, the solution for diffuse intensity is given by exponential functions with eigenvalues defined in Eq. (6.5.13). We may set the intensity solution in the adjusted atmosphere so that it is equivalent to that in the real atmosphere in the form

$$k\tau = k'\tau'. \tag{6.5.35}$$

From Eqs. (6.5.32)–(6.5.35), the similarity relations for radiative transfer can be expressed in the forms

$$\frac{\tau}{\tau'} = \frac{k'}{k} = \frac{1-\tilde{\omega}'}{1-\tilde{\omega}} = \frac{\tilde{\omega}'(1-g')}{\tilde{\omega}(1-g)}. \tag{6.5.36a}$$

Using the expression for the eigenvalue defined in Eq. (6.5.13), we also find the relation for the similarity parameter defined in Eq. (6.5.16b) as follows:

$$a = \left(\frac{1-\tilde{\omega}}{1-\tilde{\omega}g}\right)^{1/2} = \left(\frac{1-\tilde{\omega}'}{1-\tilde{\omega}'g'}\right)^{1/2}. \tag{6.5.36b}$$

The similarity principle can also be derived from the basic radiative transfer equation. We may begin with the following equation in the form

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\tilde{\omega}}{2} \int_{-1}^{1} I(\tau, \mu') P(\mu, \mu')\, d\mu'. \tag{6.5.37}$$

From Eq. (6.1.11), the phase function in the limit of the two-stream and Eddington's approximations is given by $P(\mu, \mu') = 1 + 3g\mu\mu'$. However, the phase functions involving cloud and aerosol particles are highly peaked in the forward direction, and two-term expansions do not adequately account for the strong forward scattering. Let the fraction of the energy scattered in the forward direction ($\Theta = 0°$) be denoted by f. The normalized phase function may be expressed in terms of this value as follows:

$$P(\mu, \mu') = 2f\delta(\mu - \mu') + (1 - f)(1 + 3g'\mu\mu'), \tag{6.5.38}$$

where $\mu = \mu'$ when $\Theta = 0$, δ is the delta function, and g' denotes the adjusted asymmetry factor. The phase function so defined is normalized to unity, and the asymmetry factor is given by

$$g = f + (1 - f)g'. \tag{6.5.39}$$

The second moment of the phase function is $\tilde{\omega}_2/5 = f$. Thus, the scaled asymmetry factor can be expressed by

$$g' = \frac{g - \tilde{\omega}_2/5}{1 - \tilde{\omega}_2/5}. \tag{6.5.40}$$

Substituting Eq. (6.5.38) into Eq. (6.5.37) leads to

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu)(1 - \tilde{\omega} f) - \frac{\tilde{\omega}(1 - f)}{2} \int_{-1}^{1} (1 + 3g'\mu\mu') I(\tau, \mu')\, d\mu'. \tag{6.5.41}$$

Consequently, if we redefine the optical depth, single-scattering albedo, and phase function such that

$$\tau' = (1 - \tilde{\omega} f)\tau, \tag{6.5.42a}$$

$$\tilde{\omega}' = \frac{(1 - f)\tilde{\omega}}{1 - \tilde{\omega} f}, \tag{6.5.42b}$$

$$P'(\mu, \mu') = 1 + 3g'\mu\mu', \tag{6.5.42c}$$

Eq. (6.5.41) then becomes

$$\mu \frac{dI(\tau', \mu)}{d\tau'} = I(\tau', \mu) - \frac{\tilde{\omega}'}{2} \int_{-1}^{1} I(\tau', \mu') P'(\mu, \mu')\, d\mu'. \tag{6.5.43}$$

Equation (6.5.43) is exactly the same as Eq. (6.5.37), except that g, τ, and $\tilde{\omega}$ have been replaced by g', τ', and $\tilde{\omega}'$. By redefining the asymmetry factor, optical depth, and single-scattering albedo, the forward-scattering nature of the phase function is approximately accounted for in the basic radiative transfer equation. In essence, we have incorporated the second moment of the phase function expansion in the formulation of the radiative transfer equation. The "equivalence" between Eqs. (6.5.43) and (6.5.37) represents the similarity principle stated previously.

The phase functions for aerosol and cloud particles require involved scattering calculations. For many applications to radiative transfer in planetary atmospheres, an analytic expression for the phase function in terms of the asymmetry factor has been proposed (Henyey and Greenstein, 1941):

$$\begin{aligned} P_{\mathrm{HG}}(\cos\Theta) &= (1-g^2)/\left(1+g^2-2g\cos\Theta\right)^{3/2} \\ &= \sum_{\ell=0}^{N}(2\ell+1)g^{\ell}P_{\ell}(\cos\Theta). \end{aligned} \tag{6.5.44}$$

This is referred to as the *Henyey–Greenstein phase function,* which is adequate for scattering patterns that are not strongly peaked in the forward direction. Using this expression, the second moment of the phase function is given by $\tilde{\omega}_2/5 = f = g^2$. Thus, in the limit of the Henyey–Greenstein approximation, the forward fraction of the scattered light is now expressed in terms of the asymmetry factor (Joseph *et al.*, 1976). Subsequently, the adjusted asymmetry factor, optical depth, and single-scattering albedo can now be expressed by

$$g' = \frac{g}{1+g}, \quad \tau' = (1-\tilde{\omega}g^2)\tau, \quad \tilde{\omega}' = \frac{(1-g^2)\tilde{\omega}}{1-\tilde{\omega}g^2}. \tag{6.5.45}$$

The similarity principle for radiative transfer was first stated by Sobolev (1975) for isotropic scattering. The general similarity relationships have been presented by van de Hulst (1980). The two-stream approximations are popular because they enable the derivation of the analytic solutions for upward and downward fluxes, leading to efficient numerical computations. The incorporation of the delta-function adjustment to account for the strong forward scattering of large size parameters in the context of two-stream approximations has led to a significant improvement in the accuracy of radiative flux calculations. As pointed out previously, the δ adjustment provides a third term closure through the second moment of the phase function expansion. The δ-two-stream and δ-Eddington approximations have the same accuracy. Relative errors of 15–20% could result for some values of optical depths, solar zenith angles, and single-scattering albedos.

6.5.4 Four-Stream Approximation

The four-stream approximation, as first derived in Liou (1974), is based on the general solution for the discrete-ordinates method for radiative transfer. In order to be able to understand the merit of the four-stream approximation, it is necessary to have some background in solving a set of differential equations based on Chandrasekhar's (1950) formulations. In particular, it is noted that the search for eigenvalues from the recurrence equation developed in the solution is both mathematically ambiguous and numerically troublesome (Section 6.2.3). More recently, a systematic development of the solution for this approximation has been presented by Liou *et al.* (1988). Specifically, this solution involves the computation of solar radiative fluxes using a relatively simple, convenient, and accurate method. Knowledge of the discrete-ordinates method for radiative transfer is desirable but not necessary. In addition, a wide range of accuracy checks for this approximation has been developed, including the δ-adjustment to account for the forward diffraction peak based on the generalized similarity principle for radiative transfer. In the following, we present the key equations for the four-stream approximation for readers who may wish to use this method for numerical calculations.

Consider two radiative streams in the upper and lower hemispheres (i.e., let $n = 2$). At the same time, expand the scattering phase function into four terms (i.e., $N = 3$) in line with the four radiative streams. On the basis of Eqs. (6.2.50a) and (6.2.50b), four first-order differential equations can then be written explicitly in matrix form:

$$\frac{d}{d\tau}\begin{bmatrix} I_2 \\ I_1 \\ I_{-1} \\ I_{-2} \end{bmatrix} = -\begin{bmatrix} b_{2,-2} & b_{2,-1} & b_{2,1} & b_{2,2} \\ b_{1,-2} & b_{1,-1} & b_{1,1} & b_{1,2} \\ -b_{1,2} & -b_{1,1} & -b_{1,-1} & -b_{1,-2} \\ -b_{2,2} & -b_{2,1} & -b_{2,-1} & -b_{2,-2} \end{bmatrix}\begin{bmatrix} I_2 \\ I_1 \\ I_{-1} \\ I_{-2} \end{bmatrix} - \begin{bmatrix} b_{2,-0} \\ b_{1,-0} \\ b_{-1,-0} \\ b_{-2,-0} \end{bmatrix} I_{\odot}, \tag{6.5.46}$$

where the terms $b_{i,j}(i = \pm 1, 2;\ j = -0, \pm 1, 2)$ are defined in Eq. (6.2.48). The four-by-four matrix represents the contribution of multiple scattering. Thus, the derivative of the diffuse intensity at a specific quadrature angle is the weighted sum of the multiple-scattered intensity from all four quadrature angles. The last term represents the contribution of the unscattered component of the direct solar flux at position τ.

After a lengthy and laborious derivation, the solution for Eq. (6.5.46) is given by

$$\begin{bmatrix} I_2 \\ I_1 \\ I_{-1} \\ I_{-2} \end{bmatrix} = \begin{bmatrix} \varphi_2^+ e2 & \varphi_1^+ e1 & \varphi_1^- e3 & \varphi_2^- e4 \\ \Phi_2^+ e2 & \Phi_1^+ e1 & \Phi_1^- e3 & \Phi_2^- e4 \\ \Phi_2^- e2 & \Phi_1^- e1 & \Phi_1^+ e3 & \Phi_2^+ e4 \\ \varphi_2^- e2 & \varphi_1^- e1 & \varphi_1^+ e3 & \varphi_2^+ e4 \end{bmatrix}\begin{bmatrix} G_2 \\ G_1 \\ G_{-1} \\ G_{-2} \end{bmatrix} + \begin{bmatrix} Z_2^+ \\ Z_1^+ \\ Z_1^- \\ Z_2^- \end{bmatrix} e^{-f_0\tau}, \tag{6.5.47}$$

where $f_0(\text{solar}) = 1/\mu_0$, $f_0(\text{thermal}) = -1/\tau_1 \ln(B_1/B_0)$, B_0 and B_1 are Planck functions evaluated at the top and bottom of the layer (Section 4.6.3), respectively, and the other terms except $G_{\pm 1,2}$ are defined as follows:

$$e1 = e^{-k_1\tau}, \qquad e2 = e^{-k_2\tau}, \tag{6.5.48a}$$

$$e3 = e^{-k_1(\tau_1-\tau)}, \qquad e4 = e^{-k_2(\tau_1-\tau)}, \tag{6.5.48b}$$

$$Z_{1,2}^{\pm} = \frac{1}{2}(\eta_{1,2} \pm \eta'_{1,2}), \tag{6.5.48c}$$

$$\varphi_{1,2}^{\pm} = \frac{1}{2}\left(1 \pm \frac{b_{11}^{-} - A_{1,2}b_{21}^{-}}{a^{-}}k_{1,2}\right), \tag{6.5.48d}$$

$$\Phi_{1,2}^{\pm} = \frac{1}{2}\left(A_{1,2} \pm \frac{A_{1,2}b_{22}^{-} - b_{12}^{-}}{a^{-}}k_{1,2}\right), \tag{6.5.48e}$$

where the parameters $k_{1,2}, A_{1,2}, \eta_{1,2}, \eta'_{1,2}, b_{11}^{-}, b_{21}^{-}, b_{22}^{-}, b_{12}^{-}, a^{-}$, and other related terms can be computed in successive order from the following equations:

$$\begin{aligned}
b_i(\text{solar}) &= \frac{\tilde{\omega}}{4\pi}F_{\odot}\sum_{\ell=0}^{3}\tilde{\omega}_\ell P_\ell(\mu_i)P_\ell(-\mu_0)/\mu_i,\\
b_i(\text{thermal}) &= (1-\tilde{\omega})B_0/\mu_i, \qquad i = -2,-1,1,2,\\
c_{i,j} &= \frac{\tilde{\omega}}{2}a_j\sum_{\ell=0}^{3}\tilde{\omega}_\ell P_\ell(\mu_i)P_\ell(\mu_j), \qquad i,j = -2,-1,1,2,\\
b_{i,j} &= \begin{cases} c_{i,j}/\mu_i, & i \neq j\\ (c_{i,j}-1)/\mu_i, & i = j,\end{cases}\\
b_{22}^{\pm} &= b_{2,2} \pm b_{2,-2}, \qquad b_{21}^{\pm} = b_{2,1} \pm b_{2,-1},\\
b_{12}^{\pm} &= b_{1,2} \pm b_{1,-2}, \qquad b_{11}^{\pm} = b_{1,1} \pm b_{1,-1},\\
b_{2}^{\pm} &= b_{2} \pm b_{-2}, \qquad b_{1}^{\pm} = b_{1} \pm b_{-1},\\
a_{22} &= b_{22}^{+}b_{22}^{-} + b_{12}^{+}b_{21}^{-}, \qquad a_{21} = b_{22}^{-}b_{21}^{+} + b_{21}^{-}b_{11}^{+},\\
a_{12} &= b_{12}^{-}b_{22}^{+} + b_{11}^{-}b_{12}^{+}, \qquad a_{11} = b_{12}^{-}b_{21}^{+} + b_{11}^{-}b_{11}^{+},\\
d_2 &= b_{22}^{-}b_{2}^{-} + b_{21}^{-}b_{1}^{-} + b_{2}^{+}f_0, \qquad d_1 = b_{12}^{-}b_{2}^{-} + b_{11}^{-}b_{1}^{-} + b_{1}^{+}f_0,\\
a'_{22} &= b_{22}^{-}b_{22}^{+} + b_{12}^{-}b_{21}^{+}, \qquad a'_{21} = b_{22}^{+}b_{21}^{-} + b_{21}^{+}b_{11}^{-},\\
a'_{12} &= b_{12}^{+}b_{22}^{-} + b_{11}^{+}b_{12}^{-}, \qquad a'_{11} = b_{12}^{+}b_{21}^{-} + b_{11}^{+}b_{11}^{-},\\
d'_2 &= b_{22}^{+}b_{2}^{+} + b_{21}^{+}b_{1}^{+} + b_{2}^{-}f_0, \qquad d'_1 = b_{12}^{+}b_{2}^{+} + b_{11}^{+}b_{1}^{+} + b_{1}^{-}f_0,\\
b &= a_{22} + a_{11}, \qquad c = a_{21}a_{12} - a_{11}a_{22},\\
a^{-} &= b_{22}^{-}b_{11}^{-} - b_{12}^{-}b_{21}^{-}, \qquad A_{1,2} = \left(k_{1,2}^2 - a_{22}\right)/a_{21},\\
k_1 &= \left[(b+\sqrt{b^2+4c})/2\right]^{1/2}, \qquad k_2 = \left[(b-\sqrt{b^2+4c})/2\right]^{1/2},\\
\eta_1 &= \left(d_1 f_0^2 + a_{12}d_2 - a_{22}d_1\right)/f', \qquad \eta_2 = \left(d_2 f_0^2 + a_{21}d_1 - a_{11}d_2\right)/f',\\
\eta'_1 &= \left(d'_1 f_0^2 + a'_{12}d'_2 - a'_{22}d'_1\right)/f', \qquad \eta'_2 = \left(d'_2 f_0^2 + a'_{21}d'_1 - a'_{11}d'_2\right)/f',\\
f' &= f_0^4 - bf_0^2 - c.
\end{aligned} \tag{6.5.49}$$

The coefficients $G_i (i = \pm 1, 2)$ are to be determined from radiation boundary conditions. If we consider a homogeneous cloud layer characterized by an optical depth τ_1 and assume that there is no diffuse radiation from the top and bottom of this layer, then the boundary conditions are

$$\left.\begin{aligned} I_{-1,-2}(\tau = 0) &= 0\\ I_{1,2}(\tau = \tau_1) &= 0\end{aligned}\right\}. \tag{6.5.50}$$

The boundary conditions can be modified to include nonzero diffuse radiation; G_i can be obtained by an inversion of a four-by-four matrix in Eq. (6.5.47). We note that the only difference in the four-stream formulation between infrared and solar wavelengths is the definition of f_0 and $b_i (i = \pm 1, 2)$. Finally, the upward and total (diffuse plus direct) downward fluxes at a given level τ are given by

$$F^{\uparrow}(\tau) = 2\pi(a_1\mu_1 I_1 + a_2\mu_2 I_2), \tag{6.5.51a}$$

$$F^{\downarrow}(\tau) = 2\pi(a_1\mu_1 I_{-1} + a_2\mu_2 I_{-2}) + \mu_0 F_{\odot} e^{-\tau/\mu_0}. \tag{6.5.51b}$$

We may also apply the four-stream solutions to nonhomogeneous atmospheres in the manner presented in Section 6.2.4.

The regular Gauss quadratures and weights in the four-stream approximation are $\mu_1 = 0.3399810$, $\mu_2 = 0.8611363$, $a_1 = 0.6521452$, and $a_2 = 0.3478548$, as defined in Table 6.1. When the isotropic surface reflection is included in this approximation or when it is applied to thermal infrared radiative transfer involving isothermal emission, double Gauss quadratures and weights ($\mu_1 = 0.2113248$, $\mu_2 = 0.7886752$, and $a_1 = a_2 = 0.5$) offer some advantage in flux calculations because $\sum_i a_i\mu_i = 0.5$ in this case. In the case of conservative scattering, $\tilde{\omega} = 1$, $\varphi_2^{\pm} = \Phi_2^{\pm} = 0.5$, the 4×4 matrix becomes 0 in Eq. (6.5.47). The solution for this equation does not exist. We may undertake direct formulation and solution from Eq. (6.4.46) by setting $\tilde{\omega} = 1$. However, we may also use $\tilde{\omega} = 0.999999$ in numerical calculations and obtain the results for conservative scattering. In the case $\tilde{\omega} = 0$, the multiple-scattering term vanishes.

We may incorporate a δ-function adjustment to account for the forward diffraction peak in the context of the four-stream approximation. In reference to Eq. (6.5.44), the normalized phase function expansion can be expressed by incorporating the δ-forward adjustment in the form

$$P_\delta(\cos\Theta) = 2f\delta(\cos\Theta - 1) + (1-f)\sum_{\ell=0}^{N} \tilde{\omega}'_\ell P_\ell(\cos\Theta), \tag{6.5.52a}$$

where $\tilde{\omega}'_\ell$ is the adjusted coefficient in the phase function expansion. The forward peak coefficient f in the four-stream approximation can be evaluated by demanding that the next highest order coefficient in the prime expansion, $\tilde{\omega}'_4$, vanish. Setting $P(\cos\Theta) = P_\delta(\cos\Theta)$ and utilizing the orthogonal property of Legendre polynomials, we find

$$\tilde{\omega}'_\ell = [\tilde{\omega}_\ell - f(2\ell + 1)]/(1-f). \tag{6.5.52b}$$

Letting $\tilde{\omega}'_4 = 0$, we obtain $f = \tilde{\omega}_4/9$. Based on Eq. (6.5.52b), $\tilde{\omega}'_\ell (\ell = 0, 1, 2, 3)$ can be evaluated from the expansion coefficients of the phase function, $\tilde{\omega}_\ell (\ell = 0, 1, 2, 3, 4)$.

The adjusted phase function from Eq. (6.5.52a) is then given by

$$P'(\cos\Theta) = \sum_{\ell=0}^{N} \tilde{\omega}'_\ell P_\ell(\cos\Theta). \tag{6.5.52c}$$

This equation, together with Eqs. (6.5.42a,b), constitutes the generalized similarity principle for radiative transfer. That is, the removal of the forward diffraction peak in scattering processes using adjusted single-scattering parameters is "equivalent" to actual scattering processes.

6.6 Radiative Transfer Including Polarization

6.6.1 Representation of a Light Beam

Electromagnetic waves are characterized by certain polarization configurations that are described by the vibration of the electric vector and by the phase difference between the two components of this vector. These components are commonly denoted by E_l and E_r, the electric fields parallel (l) and perpendicular (r) to a reference plane (defined in Section 5.2.3). This reference plane is commonly described as the plane containing the incident and scattered directions and is referred to as the scattering plane.

Electric fields are complex, oscillating functions and may be expressed by

$$E_l = a_l \exp[-i(\xi + \delta_l)], \tag{6.6.1a}$$

$$E_r = a_r \exp[-i(\xi + \delta_r)], \tag{6.6.1b}$$

where a_l and a_r are amplitudes, δ_l and δ_r are phases, $\xi = kz - \omega t$, $k = 2\pi/\lambda$, ω is the circular frequency, and $i = \sqrt{-1}$. From these two equations, we can show that the electric fields are defined by the equation of an ellipse (Exercise 6.12).

An electromagnetic wave can be represented by the amplitudes of its two electric components and their phase difference. Based on the Stokes parameters defined in Section 5.2.4, we have

$$I = E_l E_l^* + E_r E_r^* = a_l^2 + a_r^2, \tag{6.6.2a}$$

$$Q = E_l E_l^* - E_r E_r^* = a_l^2 - a_r^2, \tag{6.6.2b}$$

$$U = E_l E_r^* + E_r E_l^* = 2a_l a_r \cos\delta, \tag{6.6.2c}$$

$$V = i(E_r E_l^* - E_l E_r^*) = 2a_l a_r \sin\delta, \tag{6.6.2d}$$

where the superscript * denotes the complex conjugate and the phase difference $\delta = \delta_r - \delta_l$.

The Stokes parameters can be expressed in terms of the geometry defining an ellipse. Let β denote an angle whose tangent is the ratio of the axes of the ellipse traced by the endpoint of the electric vector, as displayed in Fig. 6.13. If the semimajor and -minor axes of the ellipse are given by a and b, respectively, then $\tan\beta = \pm b/a$. Also, let χ be the orientation angle between the major axis of the ellipse and the l direction. When the plane waves are time harmonics, we may express the electric field vectors along the l and r directions in terms of amplitude and phase using the

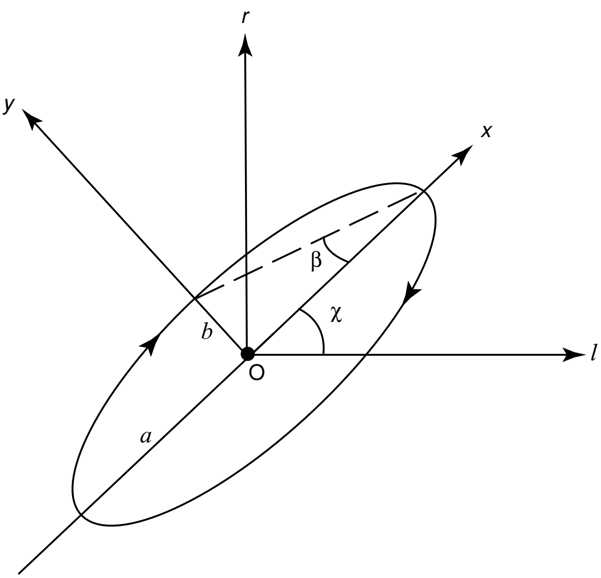

Figure 6.13 Geometric representation of elliptical polarization of a light beam in which the direction of propagation is into the paper, a and b are the lengths of the semimajor and –minor axes, respectively; χ is the orientation angle between the Ol and Ox axes; and β is the ellipticity angle whose tangent is the ratio of the ellipse traced by the endpoint of the electric vector, i.e., $\tan\beta = \pm\, b/a$, where + and − stand for the right- and left-handed polarization, respectively.

cosine representation in the forms

$$E_l = a_l \cos(\xi + \delta_l), \tag{6.6.3a}$$

$$E_r = a_r \cos(\xi + \delta_r). \tag{6.6.3b}$$

Let x and y denote the directions along the major and minor axes, respectively. Then the electric fields in the x–y plane may be written

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} E_l \\ E_r \end{pmatrix}, \tag{6.6.3c}$$

where E_x and E_y may also be expressed in terms of amplitudes (a, b) and an arbitrary phase δ_0 using cosine and sine representations such that they satisfy the elliptical equation in the forms

$$E_x = a \cos(\xi + \delta_0), \tag{6.6.3d}$$

$$E_y = \pm b \cos(\xi + \delta_0). \tag{6.6.3e}$$

After eliminating the propagation constant and all the phases using Eqs. (6.6.3a)–(6.6.3e), the Stokes parameters can be written in terms of the total intensity, and the

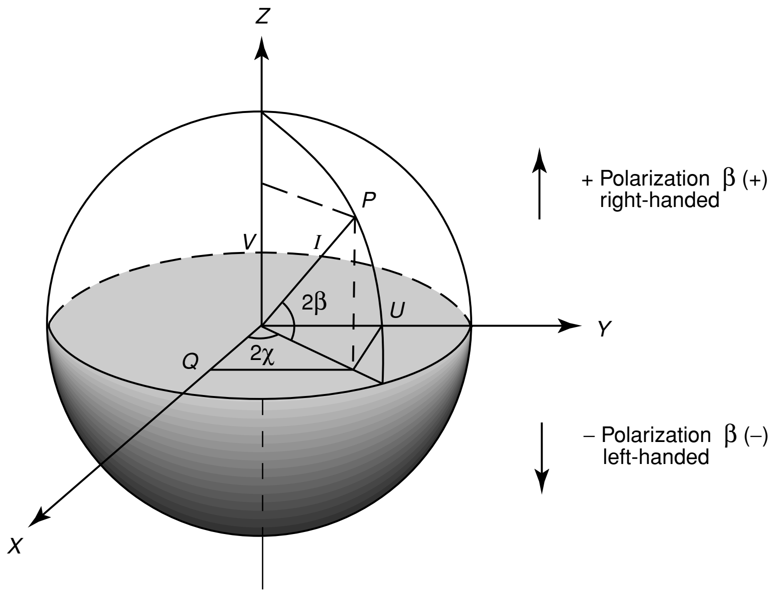

Figure 6.14 Polarization representation of the Stokes parameters (I, Q, U, V) on a Poincaré sphere. The angles β and χ are defined in Fig. 6.13. When β is positive (negative), the polarization configuration is referred to as right- (left-) handed polarization.

ellipticity and orientation angles in the forms (Exercise 6.13)

$$I = I_l + I_r, \tag{6.6.4a}$$

$$Q = I_l - I_r = I \cos 2\beta \cos 2\chi, \tag{6.6.4b}$$

$$U = I \cos 2\beta \sin 2\chi, \tag{6.6.4c}$$

$$V = I \sin 2\beta. \tag{6.6.4d}$$

It is noted that I and V are independent of the orientation angle χ. Equation (6.6.4) may be represented in Cartesian coordinates on a sphere called the *Poincaré sphere,* shown in Fig. 6.14. The radius of the sphere is given by I, and the zenithal and azimuthal angles are given by $\pi/2 - 2\beta$ and 2χ, respectively. Thus, Q, U, and V denote the lengths in the x, y, and z directions, respectively. On this sphere, the northern and southern hemispheres represent right-handed and left-handed elliptic polarizations, respectively. The north and south poles denote right-handed and left-handed circular polarizations, respectively, and points on the equatorial plane represent linear polarization. For a simple wave, we have $I^2 = Q^2 + U^2 + V^2$.

In representing the wave vibration using Eq. (6.6.2) we have assumed a constant amplitude and phase. However, the actual light beam consists of many simple waves in very rapid succession. Within a very short duration (on the order of, say, 1 second), millions of simple waves are collected by a detector. Consequently, measurable intensities are associated with the superimposition of many millions of simple waves with independent phases. Let the operator $\langle\ \rangle$ denote the time average for a time interval (t_1, t_2). Then the Stokes parameters of the entire beam of light for this time interval may be expressed by

$$\begin{aligned} I &= \langle a_l^2 \rangle + \langle a_r^2 \rangle = I_l + I_r, \\ Q &= \langle a_l^2 \rangle - \langle a_r^2 \rangle = I_l - I_r, \\ U &= \langle 2a_l a_r \cos\delta \rangle, \\ V &= \langle 2a_l a_r \sin\delta \rangle. \end{aligned} \tag{6.6.5}$$

In this case, we can prove that $I^2 \geq Q^2 + U^2 + V^2$. A light beam is generally characterized by partial elliptical polarization. If the phase differences between the two electric components are 0° or an integer order of 180° (i.e., $\beta = 0$), the light beam is linearly polarized ($V = 0$). If, on the other hand, the amplitudes of the two electric components are the same and their phase differences are an odd integer order of 90°, then the light beam is circularly polarized ($Q = U = 0$). When the ellipticity angle β is positive or negative, the circular polarization is said to be right- or left-handed. The degree of polarization of a light beam is defined by

$$PO = (Q^2 + U^2 + V^2)^{1/2}/I. \tag{6.6.6a}$$

If the ellipticity is not considered, we may define the degree of linear polarization in the form [see also Eq. (3.3.22)]

$$LP = -Q/I = -(I_l - I_r)/(I_l + I_r). \tag{6.6.6b}$$

From the measurement perspective, we may represent the Stokes parameters in terms of detectable variables. Referring to Fig. 6.15, we introduce a retardation ε in the r direction with respect to the l direction and consider the component of the electric field vector in the direction making an angle Ψ with the positive l direction.

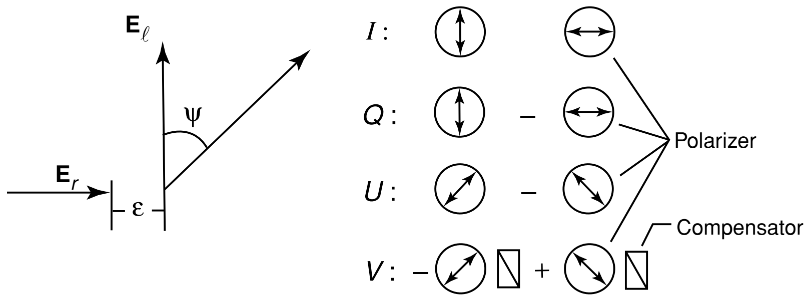

Figure 6.15 Representation of the electric field in terms of the retardation ε and the polarization angle Ψ.

Thus, for a simple wave at time t, the electric field may be represented in the form

$$\begin{aligned} E(t;\Psi,\varepsilon) &= E_l \cos\Psi + E_r e^{-i\varepsilon}\sin\Psi \\ &= a_l \cos\Psi e^{-i\xi} + a_r e^{-i(\delta+\varepsilon)-i\xi}\sin\Psi. \end{aligned} \tag{6.6.7}$$

The average intensity measured at a time interval (t_1, t_2) is then given by

$$\begin{aligned} I(\Psi,\varepsilon) &= \langle E(t;\Psi,\varepsilon)E^*(t;\Psi,\varepsilon)\rangle \\ &= \langle a_l^2\rangle \cos^2\Psi + \langle a_r^2\rangle \sin^2\Psi + \frac{1}{2}\langle 2a_l a_r \cos\delta\rangle \sin 2\Psi \cos\varepsilon \\ &\quad - \frac{1}{2}\langle 2a_l a_r \sin\delta\rangle \sin 2\Psi \sin\varepsilon. \end{aligned} \tag{6.6.8a}$$

By making use of Eq. (6.6.5) and noting that $I_l \cos^2\Psi + I_r \sin^2\Psi = (I + Q\cos 2\Psi)/2$, we obtain

$$I(\Psi,\varepsilon) = \frac{1}{2}[I + Q\cos 2\Psi + (U\cos\varepsilon - V\sin\varepsilon)\sin 2\Psi]. \tag{6.6.8b}$$

On the basis of Eq. (6.6.8b), the Stokes parameters may be expressed by the retardation and polarization angles as follows:

$$\begin{aligned} I &= I(0^\circ, 0) + I(90^\circ, 0), \\ Q &= I(0^\circ, 0) - I(90^\circ, 0), \\ U &= I(45^\circ, 0) - I(135^\circ, 0), \\ V &= -[I(45^\circ, \pi/2) - I(135^\circ, \pi/2)]. \end{aligned} \tag{6.6.9}$$

Thus, the Stokes parameters of a light beam can be measured by a combination of a number of polarizers and a compensator (e.g., a quarter-wave plate) as illustrated in Fig. 6.15.

We may now define natural light. It is the light whose intensity remains unchanged and is unaffected by the retardation of one of the orthogonal components relative to the other when resolved in any direction in the transverse plane. That is to say, for natural light it is required that $I(\Psi,\varepsilon) = I/2$. The intensity is then independent of Ψ and ε. Thus, the necessary and sufficient condition that light be natural is $Q = U = V = 0$. Under this condition, the percentage of the degree of polarization defined in Eq. (6.6.6a) for natural light is zero. As a consequence, natural light is also referred to as *unpolarized light*. Light emitted from the sun is unpolarized. However, after interacting with molecules and particles through scattering events, the unpolarized sunlight generally becomes partially polarized. Natural light characterized by $Q = U = V = 0$ can be shown to be equivalent to a mixture of any two independent oppositely polarized streams of half the intensity (Exercise 6.15).

In the atmosphere, light is generally partially polarized, and its Stokes parameters (I, Q, U, V) may be decomposed into two independent groups characterized by natural light and elliptically polarized light as follows:

$$\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} I - (Q^2 + U^2 + V^2)^{1/2} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} (Q^2 + U^2 + V^2)^{1/2} \\ Q \\ U \\ V \end{bmatrix}. \quad (6.6.10)$$

Moreover, from Eq. (6.6.4), the plane of polarization can be determined by $\tan 2\chi = U/Q$, and the ellipticity by $\sin 2\beta = V/(Q^2 + U^2 + V^2)^{1/2}$.

6.6.2 Formulation

On inserting Eq. (6.6.3c) into the Stokes parameters defined on the x–y coordinate and after some straightforward analysis, we have

$$\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix}_{x-y} = \mathbf{L}(\chi) \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix}, \quad (6.6.11)$$

where the transformation matrix for the Stokes parameters is given by

$$\mathbf{L}(\chi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\chi & \sin 2\chi & 0 \\ 0 & -\sin 2\chi & \cos 2\chi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (6.6.12)$$

From Eqs. (6.6.11) and (6.6.12), it is clear that I and V are invariant in the transformation process. We also note that $\mathbf{L}(\chi_1)\mathbf{L}(\chi_2) = \mathbf{L}(\chi_1 + \chi_2)$, and the inverse matrix $\mathbf{L}^{-1}(\chi) = \mathbf{L}(-\chi)$ (Exercise 6.16).

Having the transformation matrix defined, we can now formulate the transfer equation to include polarization. In Section 6.1, the transfer equation for plane-parallel atmospheres was presented for diffuse intensity, without taking into account the effect of polarization. In order to describe the radiation field completely at each point in space, we replace the scalar intensity I by the vector intensity $\mathbf{I} = (I, Q, U, V)$. The four Stokes parameters give, respectively, the intensity, the degree of polarization, the plane of polarization, and the ellipticity of the light waves, as functions of the incoming and outgoing directions.

The transfer equation given in Eq. (1.4.22) may now be written in vector form as follows:

$$\mu \frac{d\mathbf{I}(\tau; \mu, \phi)}{d\tau} = \mathbf{I}(\tau; \mu, \phi) - \mathbf{J}(\tau; \mu, \phi), \quad (6.6.13)$$

where the source function is a vector consisting of four elements. To obtain the expression for the vector source function, consider a differential increment $d\mathbf{J}(\tau; \mu, \phi; \mu', \phi')$, produced by multiple scattering involving a pencil of radiation of solid angle $d\Omega'$ in

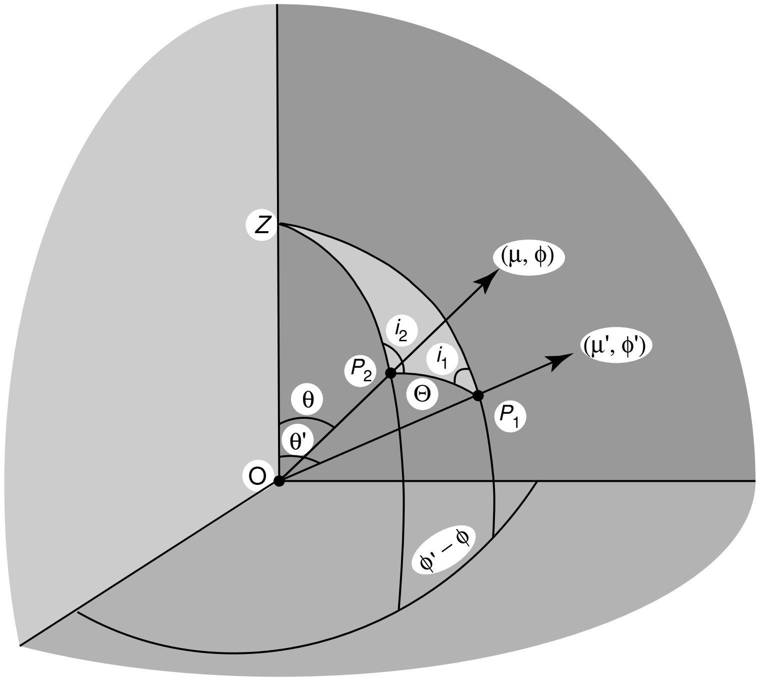

Figure 6.16 Scattering plane OP_1P_2 with respect to the meridian planes OP_1Z and OP_2Z (see text for the definitions of the angles).

the direction (μ', ϕ'). The diffuse intensity vector $\mathbf{I}(\tau;\ \mu', \phi')$, which generates the source term, is in reference to the meridian plane OP_1Z, as shown in Fig. 6.16. However, the phase matrix derived from the scattering theory [e.g., see Eq. (5.2.113)] is in reference to the plane of scattering OP_1P_2 that contains the incident and scattered beams. Thus, we must first transform $\mathbf{I}(\tau;\ \mu', \phi')$ to the plane of scattering in order to obtain the proper source function. We may first transform $\mathbf{I}(\tau; \mu', \phi')$ to the plane of scattering by applying the transformation matrix $\mathbf{L}(-i_1)$, where i_1 denotes the angle between the meridian plane OP_1Z and the plane of scattering OP_1P_2, and the minus sign signifies that the rotation of the plane is counterclockwise. Thus, the contribution to the source function with reference to the plane of scattering at P_2 is given by

$$\tilde{\omega}\mathbf{P}(\Theta)\mathbf{L}(-i_1)\mathbf{I}(\tau; \mu', \phi')\, d\Omega'/4\pi. \tag{6.6.14}$$

To transform this vector to the scattering direction (μ, ϕ), i.e., the meridian plane OP_2Z, we must again apply the transformation matrix $\mathbf{L}(\pi - i_2)$ through the angle $(\pi - i_2)$ clockwise, where i_2 denotes the angle between the meridian plane OP_2Z and the plane of scattering OP_1P_2. It follows that the desired differential source function due to the diffuse component is

$$d\mathbf{J}(\tau; \mu, \phi; \mu', \phi') = \tilde{\omega}\mathbf{L}(\pi - i_2)\mathbf{P}(\Theta)\mathbf{L}(-i_1)\mathbf{I}(\tau; \mu', \phi')\, d\Omega'/4\pi. \tag{6.6.15}$$

Thus, by performing the integration over all directions (μ', ϕ'), we obtain the source function vector for multiple scattering as follows:

$$\mathbf{J}(\tau;\,\mu,\phi;\mu',\phi') = \frac{\tilde{\omega}}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\mathbf{Z}(\mu,\phi;\,\mu',\phi')\mathbf{I}(\tau;\,\mu',\phi')\,d\mu' d\phi', \quad (6.6.16)$$

where the phase matrix is defined by

$$\mathbf{Z}(\mu,\phi;\mu',\phi') = \mathbf{L}(\pi - i_2)\mathbf{P}(\Theta)\mathbf{L}(-i_1). \quad (6.6.17)$$

Note that we differentiate **Z** and **P** by using the terms phase matrix and scattering phase matrix, respectively. From the spherical trigonometry, as illustrated in Appendix C, the angles i_1 and i_2 can be expressed by

$$\cos i_1 = \frac{-\mu + \mu'\cos\Theta}{\pm(1-\cos^2\Theta)^{1/2}(1-\mu'^2)^{1/2}}, \quad (6.6.18)$$

$$\cos i_2 = \frac{-\mu' + \mu\cos\Theta}{\pm(1-\cos^2\Theta)^{1/2}(1-\mu^2)^{1/2}}, \quad (6.6.19)$$

where the plus sign is to be used when $\pi < \phi - \phi' < 2\pi$ and the minus sign is to be used when $0 < \phi - \phi' < \pi$. Also note that $\cos\Theta$ has been defined in Eq. (3.4.7).

Following the same procedures, the direct component of the source function associated with the point source $\mathbf{I}_\odot(-\mu,\phi) = \delta(\mu-\mu_0)\delta(\phi-\phi_0)\mathbf{F}_\odot$ is given by

$$\mathbf{J}(\tau;\mu,\phi) = \frac{\tilde{\omega}}{4\pi}\mathbf{Z}(\mu,\phi;-\mu_0,\phi_0)\mathbf{F}_\odot e^{-\tau/\mu_0}. \quad (6.6.20)$$

Thus, the equation of transfer of sunlight including polarization can be written as follows:

$$\mu\frac{d\mathbf{I}(\tau;\mu,\phi)}{d\tau} = \mathbf{I}(\tau;\mu,\phi) - \frac{\tilde{\omega}}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\mathbf{Z}(\mu,\phi;\,\mu',\phi')\mathbf{I}(\tau;\,\mu',\phi')\,d\mu' d\phi'$$
$$- \frac{\tilde{\omega}}{4\pi}\mathbf{Z}(\mu,\phi;-\mu_0,\phi_0)\mathbf{F}_\odot e^{-\tau/\mu_0}. \quad (6.6.21)$$

Comparing Eq. (6.6.21) with Eq. (6.1.1) without the emission component, we see that the scalar intensity is now replaced by a vector intensity consisting of four elements. In the preceding formulation, we have assumed that an optical depth can be defined for the medium. This applies to spherical particles that are randomly located with the scattering phase matrix **P** given by Eq. (5.2.113), and to nonspherical particles that are randomly oriented in space with the scattering phase matrix **P** given by Eq. (5.4.31). In these cases, we may replace the phase function with the four-by-four scattering phase matrix to account for the full polarization effect and employ the adding method presented in Section 6.4 to proceed with numerical calculations.

The phase matrix **Z** obeys a number of unique properties associated with the symmetry principle of light beams. For the preceding cases, a light beam can reverse its direction with final results being the same such that

$$\mathbf{Z}(-\mu,-\phi;-\mu',-\phi') = \mathbf{Z}(\mu,\,\phi;\mu',\phi'). \quad (6.6.22)$$

Other relationships have been developed by Hovenier (1969). These relationships can be employed to optimize numerical computations involving multiple scattering processes in planetary atmospheres. Finally, it should be pointed out that a new formulation is required for nonspherical particles having a specific orientation (Section 6.7.1).

6.7 Advanced Topics in Radiative Transfer

6.7.1 Horizontally Oriented Ice Particles

The scattering of a light beam by a nonspherical particle depends on the directions of the incoming and outgoing radiation, and on the orientation of the particle with respect to the incoming beam. To formulate the transfer of solar radiation in a medium composed of horizontally oriented nonspherical particles, we begin by assuming that such a medium is plane-parallel so that the intensity varies only in the z direction. In reference to Fig. 6.17, we select a fixed coordinate system xyz such that the z axis is in the zenith direction. Also, we let $x'y'z'$ represent a coordinate system in reference to the incoming light beam, which is placed on the z' axis. Angles ϕ', ϕ, γ', and γ are azimuthal angles corresponding to θ', θ, α', and α, denoted in the figure, and Θ is the scattering angle. The scattering parameters for a nonspherical particle, including the phase function, and the extinction and scattering cross sections may be expressed with respect to either of these two coordinate systems. Thus, we may write

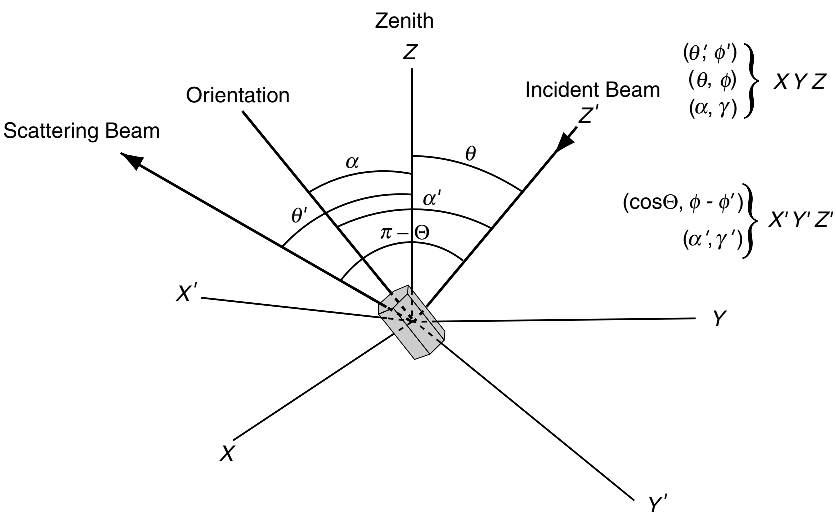

Figure 6.17 Geometry of single scattering involving a nonspherical particle. The coordinate system (x', y', z') is in reference to the incident light beam, while (x, y, z) is fixed in space. The angles (θ, θ') are zenith angles associated with the incident and scattered beams with respect to the (x, y, z) coordinates, while (ϕ, ϕ') are corresponding azimuthal angles. The angles (α, α') are zenith angles with respect to the orientation of the particle, while (γ, γ') are corresponding azimuthal angles. These angles can be transferred to the (x', y', z') coordinates in terms of a set of angles, $(\Theta, \phi - \phi')$ and (α', γ'), for the analysis of radiative transfer involving horizontally oriented ice crystals.

symbolically

$$P(\alpha, \gamma; \mu', \phi'; \mu, \phi) = P(\alpha', \gamma'; \cos\Theta, \phi - \phi'),$$
$$\sigma_{e,s}(\alpha, \gamma; \mu', \phi') = \sigma_{e,s}(\alpha', \gamma'). \tag{6.7.1}$$

Here we note that the phase function depends on the directions of the incident and scattered beams as well as on the orientation of the nonspherical particle. The extinction and scattering cross sections, however, depend only on the direction of the incident beam and the orientation of the particle.

For a sample of nonspherical particles randomly oriented in space, average single-scattering properties may be expressed in the forms

$$P(\cos\Theta, \phi - \phi') = \frac{1}{2\pi\sigma_s}\int_0^{2\pi}\int_0^{\pi/2} P(\alpha', \gamma'; \cos\Theta, \phi - \phi')\sigma_s(\alpha', \gamma')\sin\alpha'\, d\alpha' d\gamma',$$

$$\sigma_{e,s} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi/2} \sigma_{e,s}(\alpha', \gamma')\sin\alpha' d\alpha' d\gamma'. \tag{6.7.2}$$

It is clear that the extinction and scattering cross sections for randomly oriented nonspherical particles are independent of direction. Moreover, since $\cos\Theta$ can be expressed in terms of μ, ϕ, and μ', ϕ', as defined in Eq. (3.4.7), the source function in this case has the same form as that defined in Eq. (3.4.6). If all of the nonspherical particles have rotational symmetry (e.g., circular cylinders), then the phase function is independent of the azimuthal angle $\phi - \phi'$. Consequently, multiple scattering of the diffuse intensity in randomly oriented, symmetrical nonspherical particles can be formulated by following conventional procedures such as those presented for the adding method.

The spatial orientation of hexagonal and irregular ice crystals in cirrus clouds is a significant factor in the discussion of the transfer of radiation in the atmosphere. The fact that numerous halos and arcs have been observed demonstrates that specific orientation of ice particles must exist in some cirrus. Based on laboratory experiments, cylinders with a diameter-to-length ratio of less than 1 tend to fall with their long axes horizontally oriented. Observations of columnar and plate crystals in cirrus clouds have shown that these particles fall with their major axes parallel to the ground. The orientation of ice particles in cirrus clouds has been observed by numerous lidar measurements based on the depolarization technique in the backscattering direction (see Section 7.6.2). The depolarization ratio of the backscattered return from horizontally oriented plates is close to zero, but this ratio increases significantly as the lidar scans a few degrees off the vertical. Specific orientation occurs when the ice particles have relatively large sizes and defined shapes, such as columns and plates. However, if the ice crystals are irregular, such as aggregates, preferred orientation is unlikely to occur. Furthermore, smaller ice crystals in cirrus clouds where substantial turbulence occurs tend to orient in three-dimensional space. Finally, it has been noted that ice particle orientation and alignment are closely modulated by the electric field in clouds.

In the case of horizontally oriented ice crystals, their single-scattering parameters are dependent on the direction of the incident light beam. Thus, the conventional formulation for the multiple-scattering problem requires modification. The basic equation for the transfer of solar radiation in an optically anisotropic medium has been discussed and formulated in the first edition of this text. More recently, Takano and Liou (1989b) have used realistic scattering parameters and the Stokes vector for horizontally oriented ice crystals in association with the adding method for radiative transfer. Takano and Liou (1993) have further presented the theoretical formulation and numerical calculations involving the transfer of polarized thermal infrared radiation in optically anisotropic media with a specific application to horizontally oriented ice particles. In the following, we present a unified theoretical formulation that is applicable to both solar and thermal infrared radiative transfer, including polarization for horizontally oriented ice crystals.

In the case of ice particles randomly oriented in a horizontal plane, we have $\alpha = \pi/2$ from Eq. (6.7.1). Thus, the phase function and cross sections are dependent only on the incident angle and may be symbolically written in the forms

$$P(\mu', \phi'; \mu, \phi) = \frac{2}{\pi}\int_0^{\pi/2} P(\pi/2, \gamma; \mu', \phi'; \mu, \phi)\, d\gamma, \tag{6.7.3a}$$

$$\sigma_{e,s}(\mu) = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{\pi/2} \sigma_{e,s}(\pi/2, \gamma; \mu, \phi)\, d\gamma d\phi. \tag{6.7.3b}$$

With the preceding understanding of the incident direction of a light beam with respect to particle geometry, we may define a differential normal optical depth such that $d\tilde{\tau} = -\tilde{\beta}_e\, dz$, where the vertical extinction coefficient $\tilde{\beta}_e = \beta_e(\mu = 1)$ and z is the distance. The general equation governing the transfer of the Stokes vector may be expressed in the form

$$\mu \frac{d\mathbf{I}(\tilde{\tau}; \mu, \phi)}{d\tilde{\tau}} = \mathbf{k}(\mu)\mathbf{I}(\tilde{\tau}; \mu, \phi) - \mathbf{J}(\tilde{\tau}; \mu, \phi), \tag{6.7.4a}$$

where $\mathbf{I} = (I, Q, U, V)$ and the actual extinction coefficient normalized by the vertical extinction coefficient is defined by

$$\mathbf{k}(\mu) = \boldsymbol{\beta}_e(\mu)/\tilde{\beta}_e. \tag{6.7.4b}$$

For horizontally oriented particles the extinction coefficient is dependent on both the energy characteristics of the incident beam and its state of polarization, referred to as *dichroism* of the scattering medium. This generally occurs when the light beam passes through a cloud of aligned nonspherical particles associated with an electric and/or magnetic field, as noted previously. Because of dichroism, the extinction coefficients corresponding to the Stokes vector are represented by the 4×4 extinction matrix. For nonspherical particles randomly oriented in a plane, the extinction matrix may be written in the form (Martin, 1974; Mishchenko, 1991)

$$\boldsymbol{\beta}_e = \begin{bmatrix} \beta_e & \beta_{pol} & 0 & 0 \\ \beta_{pol} & \beta_e & 0 & 0 \\ 0 & 0 & \beta_e & \beta_{cpol} \\ 0 & 0 & -\beta_{cpol} & \beta_e \end{bmatrix}, \tag{6.7.5}$$

where β_{pol} and β_{cpol} are polarized and cross-polarized components, respectively, of the extinction coefficients with respect to the incident Stokes vector. For all practical purposes, we may use the scalar β_e for applications to ice-crystal cases.

The source function in the basic radiative transfer equation may be written as follows:

$$\begin{aligned}\mathbf{J}(\tilde{\tau};\mu,\phi) = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\mathbf{k}(\mu')\tilde{\omega}(\mu')\mathbf{Z}(\mu,\phi;\,\mu',\,\phi')\mathbf{I}(\tilde{\tau},\mu',\phi')\,d\mu' d\phi' \\ + \frac{1}{4\pi}\mathbf{k}(-\mu_0)\tilde{\omega}(-\mu_0)\mathbf{Z}(\mu,\phi;-\mu_0,\phi_0)\mathbf{F}_\odot \exp[-\mathbf{k}(-\mu_0)\tilde{\tau}/\mu_0] \\ + \mathbf{k}(\mu)[1-\tilde{\omega}(\mu)]B(T)\mathbf{I}_e, \qquad (6.7.6a)\end{aligned}$$

where the single-scattering albedo is defined by

$$\tilde{\boldsymbol{\omega}}(\mu) = \boldsymbol{\beta}_s(\mu)/\boldsymbol{\beta}_e(\mu), \qquad (6.7.6b)$$

where $\boldsymbol{\beta}_s$ is the scattering coefficient matrix, which has a form similar to the extinction coefficient matrix, $\mathbf{F}_\odot$ represents the Stokes vector for the incident solar irradiance, $B(T)$ is the Planck intensity at temperature T, and $\mathbf{I}_e = (I,\,Q_e,0,0)$, with $-Q_e$ the linear polarization component associated with emission. The phase matrix,

$$\mathbf{Z}(\mu,\phi;\mu',\phi') = \mathbf{L}(\pi - i_2)\mathbf{P}(\mu,\phi;\mu',\phi')\mathbf{L}(i_1), \qquad (6.7.7)$$

is duplicated here for the continuity of discussion [see also Eq. (6.6.17)], and the transformation matrix is given in Eq. (6.6.12). In general, the scattering phase matrix **P** consists of 16 elements as defined in Eq. (5.4.30). In Eq. (6.7.6a), the second and third terms on the right-hand side represent the contributions from direct solar radiation and thermal emission from a medium having a temperature T that is azimuthally independent. Also note that $\mathbf{k}\tilde{\omega} = \boldsymbol{\beta}_s/\tilde{\beta}_e$. For wavelengths shorter than about 3.7 μm, thermal emission within the earth–atmosphere system can be neglected in comparison to radiation from the sun. For wavelengths longer than 5 μm, the reverse is true. Between 3.7 and 5 μm, the relative importance of thermal emission and solar reflection for a cloud layer depends largely on the position of the sun and the cloud temperature. We have stated these important constraints in various parts of this text. If the particles are randomly oriented in space in such a manner that each one of them has a plane of symmetry and the law of reciprocity may be applied, the scattering phase matrix **P** consists of only six independent elements as shown in Eq. (5.4.32). In this case, $\mathbf{k}(\mu) = 1$ and β_s, β_e, $\tilde{\omega}$ are independent of μ.

We may approach the radiative transfer problem involving ice particles randomly oriented in a horizontal plane using the adding method introduced in Section 6.4.2. The phase function and single-scattering parameters are now dependent on the direction of the incident beam. From Eqs. (6.7.4a) and (6.7.6a), we may omit the multiple-scattering term and define the reflection and transmission functions based on single-scattering and optically thin approximations, as shown in Eqs. (6.4.8a,b). However, in the case of solar radiation, we need to use the normal optical depth $\Delta\tilde{\tau}$ and the single-scattering albedo, which is a function of the cosine of the solar zenith angle, i.e., $\tilde{\omega}(\mu_0)$. We must also distinguish between the reflection and transmission functions for radiation from above and below, since the phase functions for horizontally oriented ice particles differ in these two configurations. The adding equations will be the same, except that the optical depth is replaced by $k(\mu_0)\tilde{\tau}_{a,b}$, where k is the scalar ratio of the vertical extinction coefficient to the actual extinction coefficient [see Eq. (6.7.4b)].

Figure 6.18 shows an example of the bidirectional reflectances for horizontally oriented (2D) and randomly oriented (3D) columns using a representative cirrostratus ice crystal size distribution with an optical depth of 1 in the plane defined by the zenith (θ) and relative azimuthal ($\phi - \phi_0$) angles. The wavelength and the cosine of the solar zenith angle used are 0.63 μm and 0.5, respectively. Note that the domain for the reflectances is from 60° to 180° scattering angles, which does not cover the commonly observed halos located at 22° and 46°. In the 3D case, the maximum at $\theta \approx 80°$ and close to the principal plane $\phi - \phi_0 = 0°$, is related to the limb brightening. Otherwise, the reflectance variations are relatively small in the linear scale. In the case of 2D columns, we see numerous reflection maxima. The chief ones are: (1) the subsun located at $\theta \approx 60°$ in the principal plane produced by external reflections; (2) the lower tangent arc located at $\theta \approx 80°$ in the principal plane generated by two refractions; and (3) the antisolar peak located at $\theta \approx 60°$ and $\phi - \phi_0 = 180°$ caused by internal reflections. Much larger anisotropy occurs in this case as compared with the 3D case. In realistic cirrus clouds, we would anticipate that some of the large and defined ice particles are horizontally oriented. In Section 7.3.5, we demonstrate that the reflected polarization of sunlight contains information regarding ice crystal shape and orientation.

6.7.2 Three-Dimensional Nonhomogeneous Clouds

From satellite cloud pictures, as well as our day-to-day experience, we see that a portion of the clouds and cloud systems that cover the earth either are finite in extent or occur in the form of cloud bands. This is especially evident in the tropics and the midlatitudes in the summertime. One generally common feature is the presence of cumulus clouds whose horizontal dimensions are on the same order as their vertical dimensions. Satellite mapping of the optical depth in the midlatitude and tropical regions has illustrated that cirrus clouds are frequently finite in nature and display substantial horizontal variabilities. Vertical inhomogeneity of the ice crystal size distribution and ice water content is also demonstrated in the replicator sounding observations (see Fig. 5.3) and the time series of backscattering coefficients derived from lidar returns. Thus, the potential effects of cloud geometry and inhomogeneity on the transfer of radiation must be studied to understand their impact on the radiative properties of the atmosphere, as well as to accurately interpret radiometric measurements from the ground, the air, and space.

Following the discussion presented in Section 1.4.5, the basic steady-state radiative transfer diffuse intensity, I, can be expressed in the form

$$-\frac{1}{\beta_e(\mathbf{s})}(\mathbf{\Omega} \cdot \nabla) I(\mathbf{s}, \mathbf{\Omega}) = I(\mathbf{s}, \mathbf{\Omega}) - J(\mathbf{s}, \mathbf{\Omega}), \tag{6.7.8}$$

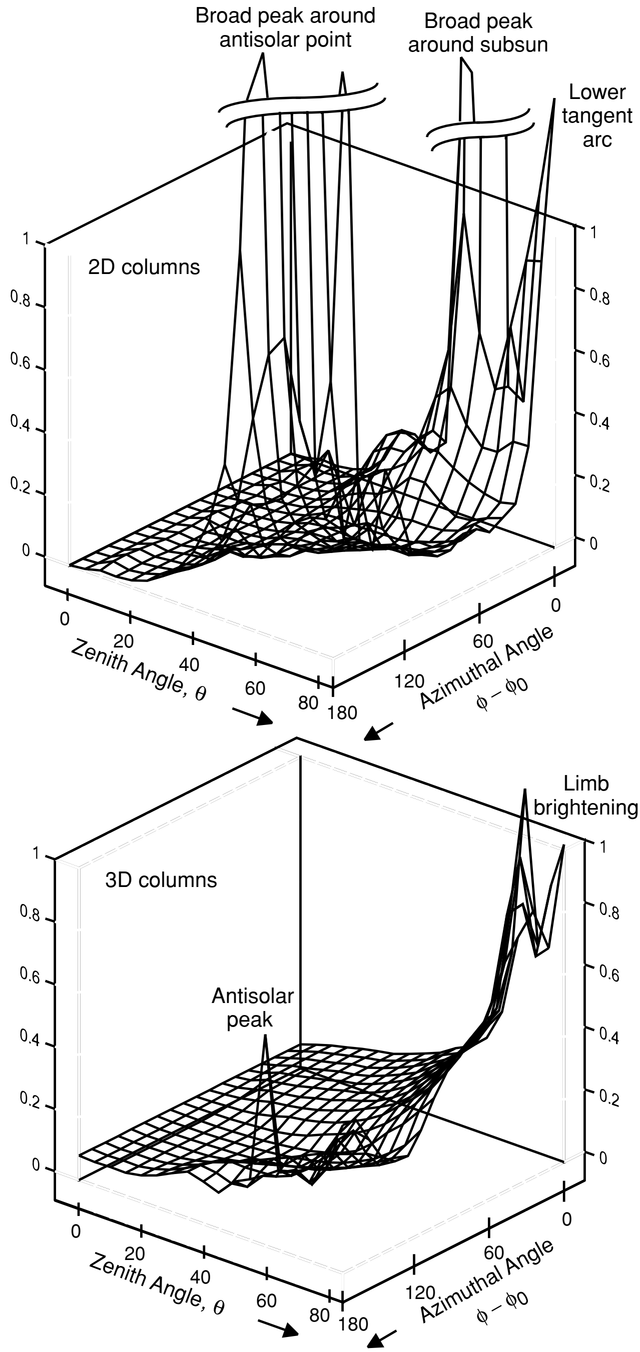

Figure 6.18 Bidirectional reflectances at a wavelength of 0.63 μm for ice columns randomly oriented in space (3D) and oriented parallel to the ground (2D columns) in the plane of zenith and azimuthal angles. The cosine of the solar zenith angle and the optical depth used are 0.5 and 1, respectively. For the 2D column case, three peak features are marked in the diagram.

where $\mathbf{s}$ is the position vector; $\mathbf{\Omega}$ is a unit vector representing the angular direction of scattering through the position vector; β_e is the extinction coefficient for cloud particles, which is a function of the position vector; and the source function, which is produced by the single scattering of the direct solar irradiance, multiple scattering of the diffuse intensity, and emission of the cloud, can be written as follows:

$$J(\mathbf{s}, \mathbf{\Omega}) = \frac{\tilde{\omega}(\mathbf{s})}{4\pi} \int_{4\pi} I(\mathbf{s}, \mathbf{\Omega}') P(\mathbf{s}; \mathbf{\Omega}, \mathbf{\Omega}')\, d\Omega' + \frac{\tilde{\omega}(\mathbf{s})}{4\pi} P(\mathbf{s}; \mathbf{\Omega}, \mathbf{\Omega}_0) F_\odot \exp\left[- \int_0^{s_0} \beta_e(\mathbf{s}')\, d\mathbf{s}' \right] + [1 - \tilde{\omega}(\mathbf{s})] B[T(\mathbf{s})], \tag{6.7.9}$$

where $\tilde{\omega} = \beta_s/\beta_e$ is the single-scatterng albedo with β_s the scattering coefficient; the phase function P is defined by the position of the light beam and the incoming and outgoing solid angles $\mathbf{\Omega}'(\mathbf{\Omega}_0)$ and $\mathbf{\Omega}$, respectively; $F_\odot$ is the incident solar irradiance; s_0 is defined as a spatial coordinate in the direction of the incident solar radiation; and $B(T)$ is the Planck function of temperature T. Solutions for I in Eqs. (6.7.8) and (6.7.9) in multidimensional space must be carried out numerically subject to the coordinate system imposed.

We shall consider the Cartesian coordinate system in which three-dimensional flux densities may be defined by

$$F_{\pm x_i}(x, y, z) = \int_{2\pi} I(x, y, z; \Omega)\Omega_{x_i}\, d\Omega, \tag{6.7.10}$$

where $x_1 = x$, $x_2 = y$, $x_3 = z$, $\Omega_x = (1 - \mu^2)^{1/2} \cos\phi$, $\Omega_y = (1 - \mu^2)^{1/2} \sin\phi$, and $\Omega_z = \mu$. There are six flux components corresponding to three coordinates. Angular integrations over the upward and downward hemispheres are the same as in the plane–parallel case, i.e., $(2\pi, 0)$ for ϕ and $(\pm 1, 0)$ for μ. The angular integrations in the x-direction are $(\pi/2, -\pi/2)$ and $(-1, 1)$, and $(3\pi/2, \pi/2)$ and $(-1, 1)$. In the y-direction, the angular integrations are $(\pi, 0)$ and $(-1, 1)$, and $(2\pi, \pi)$ and $(-1, 1)$. The local rate of change of temperature is produced by the 3D radiative flux divergence in the form

$$\frac{\partial T}{\partial t}(x, y, z) = -\frac{1}{\rho C_p} \mathbf{\nabla} \cdot \mathbf{F}, \tag{6.7.11a}$$

where

$$\mathbf{F} = \mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z, \tag{6.7.11b}$$

with $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ the unit vectors and F_x, F_y, and F_z the net flux densities in the x, y, and z directions, respectively.

A final note is in order. The conventional definition of the absorbed flux in a cloud layer from both the measurement and calculation perspectives is given by the divergence of net fluxes at the cloud top (z_t) and bottom (z_b) in the form

$$F_{\text{abs}} = \left[F^{\downarrow}(z_t) - F^{\uparrow}(z_t)\right] - \left[F^{\downarrow}(z_b) - F^{\uparrow}(z_b)\right]. \tag{6.7.12a}$$

Dividing by the downward solar flux at the cloud top, $F^{\downarrow}(z_t)$, we obtain the relative percentage as follows:

$$a = 1 - r - t, \tag{6.7.12b}$$

with the absorptance $a = F_{\text{abs}}/F^{\downarrow}(z_t)$, the reflectance $r = F^{\uparrow}(z_t)/F^{\downarrow}(z_t)$, and the net transmittance $t = [F^{\downarrow}(z_b) - F^{\uparrow}(z_b)]/F^{\downarrow}(z_t)$, where the upward flux at the cloud base is related to the contribution from the surface and the atmosphere below the cloud. The preceding definition of cloud absorption is correct if the horizontal extent of the cloud system is sufficiently large, such as those associated with large-scale frontal activities. However, for a cloud whose horizontal scale is comparable to or less than its vertical scale, such as tropical cumulus towers, the absorption definition based on the fluxes at the cloud top and bottom requires modification.

From a theoretical perspective, and if Cartesian coordinates are imposed, the side fluxes must be included in the discussion of cloud absorption. The problem is further complicated because the flux is a function of position (x, y, z) and appropriate domain averages are needed to obtain absorption in the cloud as a unit. Let the areas corresponding to the x, y, and z directions be A_x, A_y, and A_z, respectively. Based on the energy conservation principle, we must have

$$\begin{aligned}
&\int_{A_z} F_{\odot}\mu_0\, dxdy + \int_{A_x} F_{\odot}\left(1-\mu_0^2\right)^{1/2}\cos\phi_0\, dydz + \int_{A_y} F_{\odot}\left(1-\mu_0^2\right)^{1/2}\sin\phi_0\, dxdz \\
&\quad = \int_{A_z}\left(F_z^{\uparrow} + F_z^{\downarrow}\right) dxdy + \int_{A_x}\left(F_x^{\rightarrow} + F_x^{\leftarrow}\right) dydz \\
&\qquad + \int_{A_y}\left(F_y^{\rightarrow} + F_y^{\leftarrow}\right) dxdz + \int_V f_a\, dxdydz,
\end{aligned} \tag{6.7.13}$$

where V is the volume and f_a is the absorbed flux per volume. Averaging over the respective areas and volume, we obtain

$$\begin{aligned}
&F_{\odot}\left[\mu_0 A_z + \left(1-\mu_0^2\right)^{1/2}\cos\phi_0 A_x + \left(1-\mu_0^2\right)^{1/2}\sin\phi_0 A_y\right] \\
&\quad = \left(\bar{F}_z^{\uparrow} + \bar{F}_z^{\downarrow}\right)A_z + \left(\bar{F}_x^{\rightarrow} + \bar{F}_x^{\leftarrow}\right)A_x + \left(\bar{F}_y^{\rightarrow} + \bar{F}_y^{\leftarrow}\right)A_y + \bar{f}_a V,
\end{aligned} \tag{6.7.14}$$

where the flux notations are self-explanatory.

6.7.2.1 MONTE CARLO METHOD

The Monte Carlo method involves releasing photons from a source and tracing them through a medium that is divided into a suitable number of cubic cells. The absorption and scattering of photons can be considered stochastic processes in which the scattering phase function may be thought of as a transformation probability function that redistributes the photons in different directions. The single-scattering properties within the predivided cubic cells are prescribed. The Monte Carlo program then computes the free path length of a single photon from its initial entry point through these cells.

Consider a nonhomgeneous finite cloud and let the incident energy be $h\tilde{\nu}$. Let l denote the path length that a photon travels in the cloud before the first scattering, and let RN be a random number in the interval (0, 1) generated by computer. The mean path length l_0 between scattering must be inversely proportional to the scattering coefficient β_s (in units of per length) associated with the cloud particles. Using the Poisson distribution, we have

$$\mathrm{RN} = e^{-l/l_0}, \quad l = l_0 \ln(1/\mathrm{RN}). \tag{6.7.15a}$$

When the photon encounters scattering, its new direction in terms of the scattering and azimuthal angles may be determined by

$$\left.\begin{aligned} &\int_0^{\Theta} P(\cos\Theta)\sin\Theta\, d\Theta = \mathrm{RN}\cdot\int_0^{\pi} P(\cos\Theta)\sin\Theta\, d\Theta \\ &\phi = 2\pi\cdot\mathrm{RN} \end{aligned}\right\}, \tag{6.7.15b}$$

where P is the phase function and it converts the incident direction (μ', ϕ') to the scattered direction (μ, ϕ) via the definition of the scattering angle. With the new direction, a new path length for the next scattering is then determined from Eq. (6.7.15a) and this procedure continues until the photon is either absorbed in the cloud or escapes through a cloud boundary.

Analogous to the definition of the scattering path length, l_0, we may define the absorption and extinction path lengths as n_0 and m_0, respectively. Thus, we have $m_0/n_0 = 1 - \tilde{\omega}$, and the mean number of scattering events, which is an integer, is given by

$$N_0 \geq \frac{n_0}{m_0} = \frac{1}{1-\tilde{\omega}}. \tag{6.7.15c}$$

In the extreme case involving conservative scattering, $\tilde{\omega} = 1$, there will be infinite numbers of scattering events and the computation continues until the photon departs the cloud. In the other extreme case when $\tilde{\omega} = 0$, scattering does not occur and the photon is absorbed immediately. The number of scattering events before absorption is also determined from the Poisson distribution and is given by

$$N = N_0 \ln(1/\mathrm{RN}). \tag{6.7.15d}$$

The Monte Carlo program contains codes that sample the relevant behavior of each photon as a function of depth in the cloud. It will keep a record of the number and direction of photons that cross a predivided plane in a cloud. The intensity may be collected within preset solid angles from which the flux density can be evaluated.

In principle, the Monte Carlo method may be applied to the transfer of radiation in a medium with any geometric configuration (see, e.g., Marchuk *et al.,* 1980). The calculation involves technical input/output operations. The results computed from the Monte Carlo method are subject to statistical fluctuations, which decrease in magnitude as the square root of the number of photons used in the calculation increases. Hence, enormous amounts of computer time may be required in order to achieve reliable accuracy. The Monte Carlo method has long been employed to simulate the transfer of solar radiation in planetary atmospheres (Plass and Kattawar, 1968). Applications to three-dimensional (3D) cloud problems have also been made by many researchers (e.g., Cahalan *et al.,* 1994).

6.7.2.2 Successive-Orders-of-Scattering (SOS) Approach

In Section 6.5.1, we pointed out that the SOS approach can be directly applied to specific geometry without the requirement of solving the basic radiative transfer equation in differential form. In addition, the nonhomogeneous structure of a medium can be incorporated in the calculation in a straightforward manner in terms of integration along the line path. To begin this method, we perform the line integration along the spatial coordinates in Eq. (6.7.8) to obtain

$$I(s, \Omega) = I(0, \Omega)\exp[-\tau(s)_\Omega] + \int_0^s \beta_e(s')J(s', \Omega)\exp\{-[\tau(s) - \tau(s')]_\Omega\}\, ds', \tag{6.7.16a}$$

where $I(0, \Omega)$ is the incident diffuse intensity at position $s = 0$. The effective optical depth $\tau(s)$, or $\tau(s')$, or $\tau(s_0)$ is defined by

$$\tau(s)_\Omega = \int_0^s \beta_e(s')\, ds'. \tag{6.7.16b}$$

For simplicity of presentation, we shall assume that there is no diffuse downward, upward, or inward intensity at the top, base, and sides of a finite cloud layer so that $I(0, \Omega) = 0$. Moreover, using the index n to denote each order-of-scattering event, we can write

$$I_n(s, \Omega) = \int_0^s \beta_e(s')J_n(s', \Omega)\exp\{-[\tau(s) - \tau(s')]_\Omega\}\, ds', \quad n \geq 1. \tag{6.7.17}$$

Based on the SOS principle, the source function defined in Eq. (6.7.9) can be decomposed into the forms

$$J_1(s, \Omega) = \frac{\tilde{\omega}(s)}{4\pi} P(s; \Omega, -\Omega_0)F_\odot \exp[-\tau(s)_{\Omega_0}], \tag{6.7.18a}$$

$$J_n(s, \Omega) = \frac{\tilde{\omega}(s)}{4\pi} \int_{4\pi} I_{n-1}(s, \Omega')P(s; \Omega, \Omega')\, d\Omega', \qquad n \geq 2. \tag{6.7.18b}$$

Equations (6.7.17) and (6.7.18) are iterative equations in which the intensity and source function for each order of scattering can be computed successively, beginning with $n = 1$. The total diffuse intensity is then

$$I(s, \Omega) = \sum_{n=1}^{M} I_n(s, \Omega), \tag{6.7.19}$$

where the index M represents the order-of-scattering events such that $|I_M - I_{M-1}|/I_{M-1} < \varepsilon$, a prescribed small number depending on the accuracy requirement.

The SOS method has been employed by a number of researchers for applications to radiative transfer concerning specific geometry. Liou *et al.* (1990a) used the method to study the transmission of thermal infrared radiation for a target–detector system. Weinman (1976) applied it to compute the backscattering return in a collimated lidar system. Herman *et al.* (1994) utilized the general principle of the SOS approach to calculate the radiation field in spherical atmospheres. However, the method has not been widely applied to radiative transfer in plane-parallel atmospheres, primarily because of the substantial computer time requirement to achieve the solution convergence for optically thick media. Liou and Rao (1996) employed the SOS method to investigate the effects of cloud geometry and nonhomogeneity on the reflection and transmission of sunlight with verifications based on the plane-parallel adding method and the Monte Carlo method for 3D cloud fields.

For 3D nonhomogeneous radiative transfer problems, we may define normalized scattered intensities in terms of the conventional reflection function (bidirectional reflectance) and transmission function in the forms

$$R(x, y, 0; \mu, \mu_0, \Delta\phi) = \pi I(x, y, 0; \mu, \mu_0, \Delta\phi)/\mu_0 F_\odot, \tag{6.7.20a}$$

$$T(x, y, z_c; \mu, \mu_0, \Delta\phi) = \pi I(x, y, z_c; \mu, \mu_0, \Delta\phi)/\mu_0 F_\odot, \tag{6.7.20b}$$

where z_c is the cloud-base height and the cloud-top height is set at $z = 0$. Normalized side intensities can also be defined in a similar manner. For fluxes, we define the reflectance and transmittance in the forms

$$r(x, y, 0; \mu_0) = F_z^{\uparrow}(x, y, 0; \mu_0)/\mu_0 F_\odot, \tag{6.7.21a}$$

$$t(x, y, z_c; \mu_0) = F_z^{\downarrow}(x, y, z_c; \mu_0)/\mu_0 F_\odot. \tag{6.7.21b}$$

Figure 6.19 shows the difference patterns for reflection and transmission functions (upper two diagrams) and reflectance and transmittance (lower two diagrams) defined in Eqs. (6.7.20)–(6.7.21) as functions of latitude and longitude for a visible wavelength. The 3D extinction coeffcient field for a finite cirrus cloud field with a horizontal dimension of 20 km $\times$ 30 km and a vertical thickness of 3.5 km was constructed from the optical depth retrieved from AVHRR and the ice crystal size distribution determined from point vertical replicator sounding. The single-scattering albedo and phase function were assumed to be the same for the cloud domain. The differences are between results from 3D nonhomogeneous and pixel-by-pixel plane-parallel models. For the reflection and transmission functions, the emergent angles used were 30° and 150°, respectively, and the results are presented in the principal plane ($\Delta\phi = 0$). The relatively large positive differences are associated with the low sun angle ($\theta_0 = 71°$), at which a significant amount of solar flux is available to two cloud sides in addition to the cloud top, as well as with the specific emergent angles used in the calculation. For other incoming and outgoing directions, differences can be either positive or negative, revealing the complexity of the intensity field associated with 3D nonhomogeneous clouds. Because of the larger horizontal dimension (compared to the vertical) used in the calculation, the absolute differences of reflectance and transmittance associated with fluxes are relatively small.

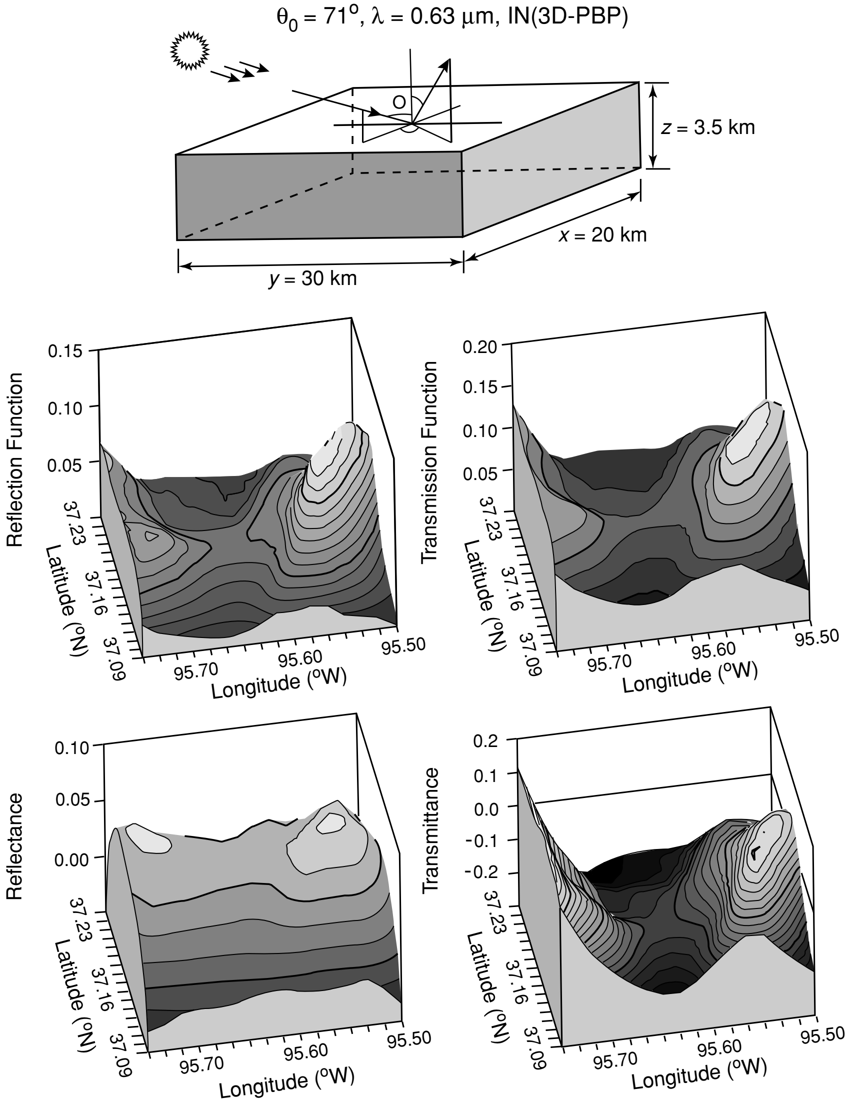

Figure 6.19 Differences of the reflection function, transmission function, reflectance, and transmittance distributions computed from a 3D inhomogeneous (3DIH) model with those computed from a pixel-by-pixel (PBP) plane-parallel model for the 0.63-μm wavelength. The geometrical parameters are given in the figure. The 3D extinction coefficient field was constructed on the basis of the optical depth retrieved from satellite radiometers and the point vertical ice crystal profile determined from a balloon sounding (after Liou and Rao, 1996).

6.7.2.3 Delta Four-Term (Diffusion) Approximation

In Section 6.5.2, we presented the two-stream approximation that is particularly useful for applications to broadband flux calculations. In the following, we describe a similar approximation for 3D radiative transfer. Referring to Eqs. (6.7.8) and (6.7.9), the phase function and diffuse intensity may be expressed in terms of the spherical harmonics expansion as follows:

$$P(\mathbf{s}, \mathbf{\Omega}, \mathbf{\Omega}') = \sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} \tilde{\omega}_\ell Y_\ell^m(\mathbf{\Omega}) Y_\ell^{m*}(\mathbf{\Omega}'), \tag{6.7.22a}$$

$$I(\mathbf{s}, \mathbf{\Omega}) = \sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} I_\ell^m(\mathbf{s}) Y_\ell^m(\mathbf{\Omega}), \tag{6.7.22b}$$

where $\tilde{\omega}_\ell$ are certain coefficients, and N denotes the number of terms in the spherical harmonics expansion defined by

$$Y_\ell^m(\theta, \phi) = (-1)^{(m+|m|)/2} \left(\frac{(\ell - |m|)!}{(\ell + |m|)!} \right) P_\ell^{|m|}(\cos\theta) e^{im\phi}, \tag{6.7.23a}$$

where P_ℓ^m is the associated Legendre polynomial defined in Appendix E, $|m|$ is the absolute value of m, and $i = \sqrt{-1}$. The complex conjugates of the spherical harmonics are given by

$$Y_\ell^{m*}(\theta, \phi) = Y_\ell^{-m}(\theta, \phi)/(-1)^m. \tag{6.7.23b}$$

The spherical harmonics are normalized such that

$$\frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{1} Y_\ell^m(\mu, \phi) Y_\alpha^{\beta*}(\mu, \phi)\, d\mu\, d\phi = \delta_\ell^\alpha \delta_m^\beta \big/ (2\ell + 1), \tag{6.7.23c}$$

where δ_ℓ^α and δ_m^β are Kronecker delta functions.

To decompose Eq. (6.7.9) in accordance with spherical harmonics, we may insert Eqs. (6.7.22a,b) into Eqs. (6.7.8) and (6.7.9) to obtain

$$\begin{aligned} \frac{1}{\beta_e(\mathbf{s})} (\mathbf{\Omega} \cdot \nabla) \sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} I_\ell^m(\mathbf{s}) Y_\ell^m(\mathbf{\Omega}) = &- \sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} \gamma_\ell I_\ell^m(\mathbf{s}) Y_\ell^m(\mathbf{\Omega}) \\ &+ \frac{\tilde{\omega}(\mathbf{s})}{4\pi} \sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} \tilde{\omega}_\ell Y_\ell^m(\mathbf{\Omega}) Y_\ell^{m*}(\mathbf{\Omega}_0) F_\odot e^{-\tau_s} \\ &+ [1 - \tilde{\omega}(\mathbf{s})] B[T(\mathbf{s})], \end{aligned} \tag{6.7.24}$$

where $\gamma_\ell = 1 - \tilde{\omega}\tilde{\omega}_\ell/(2\ell + 1)$. Subsequently, we perform the following successive integrations:

$$\int_{4\pi} \text{Eq. (6.7.24)} \times Y_\alpha^{\beta*}(\mathbf{\Omega})\, d\Omega, \qquad \alpha = 0, 1, \ldots, N; \beta = -\alpha, \ldots, \alpha. \tag{6.7.25}$$

We find that

$$-\frac{1}{\beta_e(\mathbf{s})}\sum_{\ell=0}^{N}\sum_{m=-\ell}^{\ell}\int_{4\pi}(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})Y_\ell^m(\boldsymbol{\Omega})Y_\alpha^{\beta*}(\boldsymbol{\Omega})I_\ell^m(\mathbf{s})\,d\Omega$$

$$= -\gamma_\alpha I_\alpha^\beta(\mathbf{s})\frac{4\pi}{2\alpha+1} + \frac{\tilde{\omega}(\mathbf{s})\tilde{\omega}_\ell}{2\alpha+1}Y_\alpha^{\beta*}(\boldsymbol{\Omega}_0)F_\odot e^{-\tau_s} + 4[1-\tilde{\omega}(\mathbf{s})]\pi B[T(\mathbf{s})]. \quad (6.7.26)$$

The left-hand side of this equation may be decomposed by using the recursion relationships in each coordinate system (Ou and Liou, 1982; Evans, 1993).

For application to the finite homogeneous cloud problem, we may make a first-order approximation (i.e., $N = 1$). Using Cartesian coordinates and the definition of spherical harmonics and their recursion relationships, we obtain the following four partial differential equations:

$$\frac{\partial I_1^0}{\partial z} + \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)I_1^{-1} - \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)I_1^1$$

$$= -\beta_e I_0^0(1-\tilde{\omega}) + \frac{\tilde{\omega}}{4\pi}\beta_e F_\odot e^{-\tau_s}, \quad (6.7.27a)$$

$$\beta_e(1-\tilde{\omega}g)I_1^{-1} = -\frac{1}{3\sqrt{2}}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)I_0^0 + \frac{3}{\sqrt{2}}\frac{\tilde{\omega}g}{4\pi}\left(1-\mu_0^2\right)^{1/2}(\cos\phi_0 + i\sin\phi_0)F_\odot e^{-\tau_s}, \quad (6.7.27b)$$

$$\beta_e(1-\tilde{\omega}g)I_1^0 = -\frac{1}{3}\frac{\partial I_0^0}{\partial z} + \frac{3\tilde{\omega}g}{4\pi}\mu_0 F_\odot e^{-\tau_s}, \quad (6.7.27c)$$

$$\beta_e(1-\tilde{\omega}g)I_1^1 = \frac{1}{3\sqrt{2}}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)I_0^0 - \frac{1}{\sqrt{2}}\frac{3\tilde{\omega}g}{4\pi}\left(1-\mu_0^2\right)^{1/2}(\cos\phi_0 - i\sin\phi_0)F_\odot e^{-\tau_s}. \quad (6.7.27d)$$

On substituting Eqs. (6.7.27b–d) into Eq. (6.7.27a), the following 3D nonhomogeneous diffusion equation may be derived:

$$\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}I_0^0/\beta_t\right) - 3\alpha_t I_0^0 = -F_t + \boldsymbol{\Omega}_0\cdot\boldsymbol{\nabla}(F_t g/\beta_t), \quad (6.7.28a)$$

where

$$\beta_t = \beta_e(1-\tilde{\omega}g), \quad (6.7.28b)$$

$$\alpha_t = \beta_e(1-\tilde{\omega}), \quad (6.7.28c)$$

$$F_t = \begin{cases} 3\beta_e F_\odot e^{-\tau_s}/4\pi & \text{solar} \\ 3\beta_e(1-\tilde{\omega})B(T) & \text{IR.} \end{cases} \quad (6.7.28d)$$

In these equations, all the variables are functions of the coordinates (x, y, z). In the case when the single-scattering parameters β_e, $\tilde{\omega}$, and g are independent of (x, y, z), a general diffusion equation in radiative transfer can be obtained (Exercise 6.17) from which an analytic solution for the diffuse intensity can be derived (Liou, 1992).

Equation (6.7.28a) represents a second-order nonhomogeneous partial differential equation, which must be solved numerically. From the spherical harmonics expansion for diffuse intensity denoted in Eq. (6.7.22b), we have

$$I(x, y, z; \mathbf{\Omega}) = I_0^0 + I_1^{-1} Y_1^{-1}(\mathbf{\Omega}) + I_1^0 Y_1^0(\mathbf{\Omega}) + I_1^1 Y_1^1(\mathbf{\Omega}). \qquad (6.7.29a)$$

Substituting Eqs. (6.7.27b–d) into Eq. (6.7.29a), we obtain

$$I(x, y, z; \mathbf{\Omega}) = I_0^0 - \frac{3}{2h} \sum_{j=1}^{3} \frac{\partial I_0^0}{\partial x_j} \Omega_{xj} + \frac{9q}{2h} (\mathbf{\Omega} \cdot \mathbf{\Omega}_0) e^{-\tau_s}, \qquad (6.7.29b)$$

where $x_1 = x, x_2 = y, x_3 = z, h = 3(1 - \tilde{\omega} g)/2,\ q = \tilde{\omega} g F_\odot / 12\pi, \Omega_z = \mu,\ \Omega_x = (1 - \mu^2)^{1/2} \cos\phi$, and $\Omega_y = (1 - \mu^2)^{1/2} \sin\phi$. For thermal infrared, the last term vanishes. To increase computational accuracy, we may apply the similarity principle for radiative transfer introduced in Section 6.5.3 for each computational point such that $\beta_e' = \beta_e(1 - \tilde{\omega} f)$, $\tilde{\omega}' = \tilde{\omega}(1 - f)(1 - \tilde{\omega} f)$, and $g' = (g - f)/(1 - f)$. The fractional energy in the diffraction peak of the phase function f can be taken to be $\tilde{\omega}_2/5$, where $\tilde{\omega}_2$ is the second moment in the phase function expansion.

Contrail cirrus are a typical example of finite clouds. Observations from lidar backscattering and depolarization demonstrate that these clouds are also highly nonhomogeneous (see Section 8.3.3 for a discussion of contrail cirrus). At this point, a 3D extinction coefficient field for cirrus derived directly from observations is not available. Figure 6.20 illustrates a hypothetical extinction coefficient field constructed in the x–z plane. For radiative transfer calculations, the extinction coefficients are assumed to be the same in the y-direction. The solar and emergent angles used are 10° and 40°, respectively, with a relative azimuthal angle of 140°. The reflectance pattern is presented in the x–y plane at the cloud top. Maximum values close to the edge are shown because of the position of the sun and larger extinction coefficients. Similar patterns are displayed along the y direction. By using a mean extinction coefficient in the x–z plane, the reflectance pattern now corresponds to a homogeneous cloud. Except near the left edge, associated with the finite geometry, the pattern is uniform. This example demonstrates the significance and intricacy of the finite and nonhomogeneous cloud structure with respect to its radiative properties (see also Fig. 6.19).

6.7.3 Spherical Atmospheres

When calculating the transfer of solar radiation involving low sun, such as in twilight, or when the limb extinction technique is used to infer ozone, aerosols, and trace gases (Section 7.2.3), the effect of spherical geometry must be accounted for. In reference to Fig. 6.21, the spatial operator in conventional spherical coordinates (the SO system)

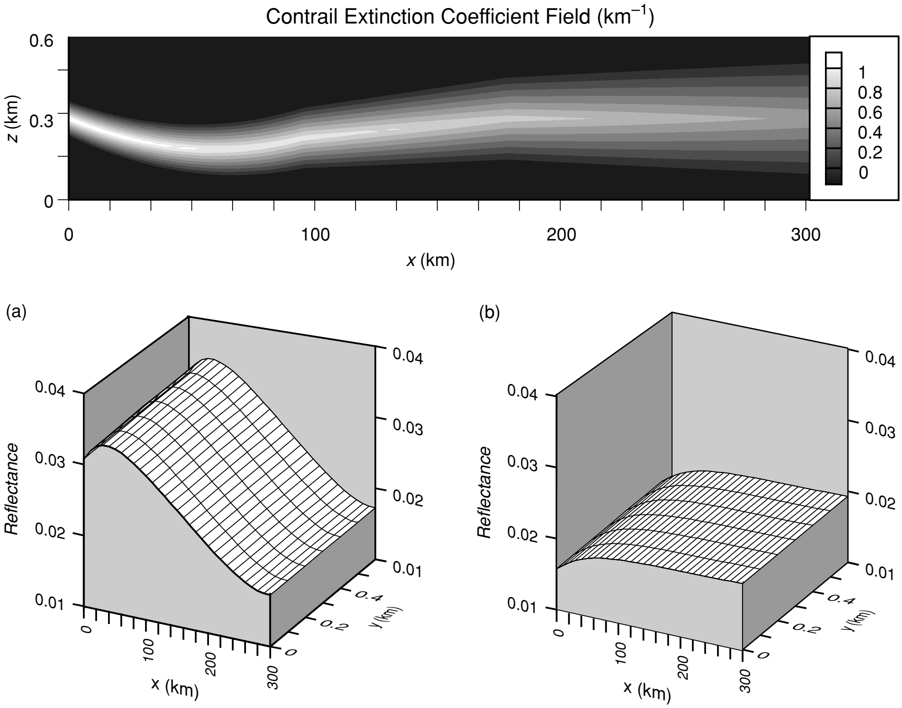

Figure 6.20 The top panel illustrates an extinction coefficient field mimicking a contrail in the x–z plane. The lower panel displays the bidirectional reflectances for (a) inhomogeneous and (b) homogeneous (mean extinction value) contrail fields in the x–y plane. The solar and emergent angles and the azimuthal difference used in the calculations are 10°, 40°, and 140°, respectively.

may be written in the form

$$\boldsymbol{\Omega} \cdot \boldsymbol{\nabla} = \Omega_r \frac{\partial}{\partial r} + \Omega_{Z_0} \frac{\partial}{r \partial Z_0} + \Omega_{A_0} \frac{\partial}{r \sin Z_0 \partial A_0}, \tag{6.7.30}$$

where the directional cosines may be derived by a transformation from those in polar coordinates and are given by

$$\begin{bmatrix} \Omega_r \\ \Omega_{Z_0} \\ \Omega_{A_0} \end{bmatrix} = \begin{bmatrix} \sin Z_0 \cos A_0 & \sin Z_0 \sin A_0 & \cos Z_0 \\ \cos Z_0 \cos A_0 & \cos Z_0 \sin A_0 & -\sin Z_0 \\ -\sin A_0 & \cos A_0 & 0 \end{bmatrix} \begin{bmatrix} \sin Z \cos A \\ \sin Z \sin A \\ \cos Z \end{bmatrix}, \tag{6.7.31}$$

where Z_0 and Z are the zenith angles and $A_0 - A$ denotes the azimuthal difference in the SO system.

However, the system of equations for radiative transfer in plane-parallel atmospheres that has been developed is in reference to the local zenith (see Fig. 6.1). Thus, we must transform the SO system of coordinates to those in the PO system with respect to the local zenith, as defined in Fig. 6.21. In this system, the scattered intensity

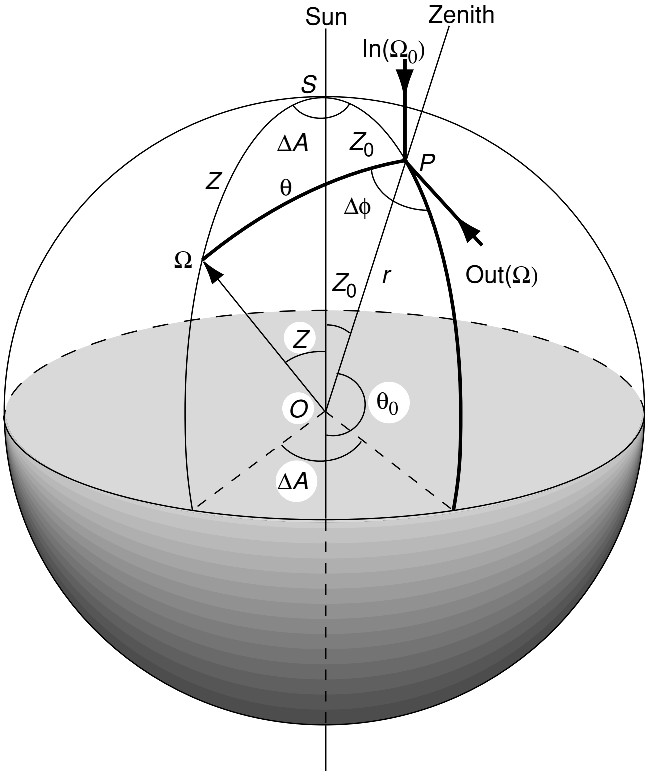

Figure 6.21 Spherical coordinate systems with respect to the sun and the center of the earth (SO), and with respect to the zenith and the center of the earth (PO), which can be compared with the plane-parallel system. The former involves the zenith angles Z_0 and Z and the azimuthal angle difference $\triangle A$, while the latter involves the zenith angles $\theta_0\,(= \pi - Z_0)$ and θ and the azimuthal angle difference $\triangle\phi$. The point P is the location where scattering takes place, O is the center of the sphere, r (OP) is the radius, θ_0 is the solar zenith angle associated with the solid angle Ω_0, and θ is the emergent angle associated with the solid angle Ω.

is a function of four variables: $\theta, \theta_0, \Delta\phi$, and r. The spatial operator can, therefore, be written as follows:

$$\mathbf{\Omega}\cdot\mathbf{\nabla} = \Omega_r\frac{\partial I}{\partial r} + \Omega_\theta\frac{\partial I}{\partial \theta} + \Omega_{\theta_0}\frac{\partial I}{\partial \theta_0} + \Omega_{\Delta\phi}\frac{\partial I}{\partial \Delta\phi}. \tag{6.7.32}$$

Determination of the directional consines in this coordinate system is quite involved. After considering a number of geometric relationships for the angles on the sphere as shown in Fig. 6.21, we find (Exercise 6.18)

$$\Omega_r = \frac{dr}{ds} = \cos\theta = \mu, \tag{6.7.33a}$$

$$\Omega_\theta = \frac{d\theta}{ds} = -\frac{\sin\theta}{r}, \tag{6.7.33b}$$

$$\Omega_{\theta_0} = \frac{d\theta_0}{ds} = -\frac{\sin\theta\cos\Delta\phi}{r}, \tag{6.7.33c}$$

$$\Omega_{\Delta\phi} = \frac{d\Delta\phi}{ds} = \frac{\sin\theta\cos\theta_0\sin\Delta\phi}{r\sin\theta_0}, \tag{6.7.33d}$$

where ds is an arbitrary differential distance in space with respect to the local zenith direction r. It follows that the radiative transfer equation for spherical atmospheres with reference to the local zenith may be written in the form

$$\begin{aligned}\mu\frac{\partial I}{\partial r} &+ \left(1-\mu^2\right)\frac{\partial I}{r\partial\mu} + \left(1-\mu^2\right)^{1/2}(1-\mu_0)^{1/2}\cos\Delta\phi\frac{\partial I}{r\partial\mu_0}\\ &+ \frac{\left(1-\mu^2\right)^{1/2}\mu_0\sin\Delta\phi}{\left(1-\mu_0^2\right)^{1/2}}\frac{\partial I}{r\partial\Delta\phi} = -\beta_e(r)[I(r;\mu,\mu_0,\Delta\phi) - J(r;\mu,\mu_0,\Delta\phi)],\end{aligned} \tag{6.7.34}$$

where the source function is defined in Eq. (6.7.9) and we note that the extinction coefficient β_e and the single-scattering albedo $\tilde{\omega}$ are functions of position only and are not dependent on the incoming and outgoing directions.

We may decompose Eq. (6.7.34) in a manner similar to that developed in Section 6.7.2. For this purpose, we may expand the diffuse intensity in the form

$$I(r;\ \mu,\phi;\ \mu_0,\phi_0) = \sum_{m=0}^{N} I_m(r;\mu,\mu_0)\cos m\Delta\phi. \tag{6.7.35a}$$

The expansion of the phase function was introduced in Section 6.1 and is given by

$$P(r;\mu,\phi;\mu',\phi') = \sum_{m=0}^{N}\sum_{\ell=m}^{N}\tilde{\omega}_\ell^m(r)P_\ell^m(\mu)P_\ell^m(\mu')\cos m\Delta\phi. \tag{6.7.35b}$$

Performing the integration $\int_0^{2\pi}$ [Eq. (6.7.34)] $d\Delta\phi$ leads to

$$\begin{aligned}\mu\frac{\partial I_0}{\partial r} &+ \frac{(1-\mu^2)}{r}\frac{\partial I_0}{\partial\mu} = -\beta_e(I_0 - J_0)\\ &- \frac{1}{2}\left[\frac{(1-\mu^2)^{1/2}\left(1-\mu_0^2\right)^{1/2}}{r}\frac{\partial I_1}{\partial\mu_0} - \frac{(1-\mu^2)^{1/2}\mu_0}{\left(1-\mu_0^2\right)^{1/2}r}I_1\right],\end{aligned} \tag{6.7.36}$$

where the azimuthally averaged source function is given by

$$\begin{aligned}J_0 = \int_0^{2\pi} J\ d\Delta\phi &= \frac{\tilde{\omega}}{2}\int_{-1}^{1} P_0(\mu,\mu')I_0(r;\mu,\mu')\,d\mu'\\ &+ \frac{\tilde{\omega}}{4\pi}P_0(\mu,-\mu_0)F_\odot\exp[-\tau\mathrm{Ch}(r,\mu_0)] + (1-\tilde{\omega})B(T),\end{aligned} \tag{6.7.37}$$

where Ch denotes the Chapman function (see Exercise 3.3), and the single-scattering properties (β_e, $\tilde{\omega}$, and P_0) as well as the temperature field can be functions of position.

Equation (6.7.36) contains two variables: I_0 and I_1. The solution requires one additional equation for I_1. Consequently, the intensity will be a function of $\Delta\phi$, the azimuthal difference of the incoming and outgoing beams, even in the case of isotropic scattering, i.e., $P(\mu, \phi; \mu', \phi') = 1$.

It appears that Lenoble and Sekera (1961) were the first to present the basic radiative transfer equation in spherical atmospheres with reference to the local zenith. Chandrasekhar (1950) briefly discussed radiative transfer problems with spherical symmetry, stating "And when, further, no radiation from the outside is incident, the intensity and the source function will be functions only of the distance r and the inclination θ to the radius vector." This is the situation when the source function is given by the emission in Eq. (6.7.37) without the direct solar radiation term. In this case the last two terms in Eq. (6.7.36) can be omitted, leading to Eq. (133) in Chandrasekhar's Section 14. Dahlback and Stamnes (1991) also presented correct equations for radiative transfer in spherical atmospheres with applications to the calculation of atmospheric photodissociation and heating rates in middle atmospheres in which the I_1 terms were neglected. The importance of azimuthal dependence in the case of spherical atmospheres, even with isotropic scattering, has been pointed out in Herman *et al.* (1995).

To complete the solution for Eq. (6.7.36), we may carry out an approach similar to that presented in Section 6.7.2 by truncating the intensity expansion to the I_1 term. By performing the integration $\int_0^{2\pi}$ [Eq. (6.7.34)] $\cos\Delta\phi\, d\Delta\phi$ and by setting $I_2 = 0$, we obtain

$$\mu\frac{\partial I_1}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I_1}{\partial \mu} = -\beta_e(I_1 - J_1) - \frac{1}{2}\frac{(1-\mu^2)^{1/2}\left(1-\mu_0^2\right)^{1/2}}{r}\frac{\partial I_0}{\partial \mu_0}, \qquad (6.7.38)$$

where

$$J_1 = \int_0^{2\pi} J\cos\Delta\phi\, d\Delta\phi = \frac{\tilde{\omega}}{2}\int_{-1}^{1} P_1(\mu,\mu')I_1(r;\mu,\mu')\,d\mu'$$
$$+ \frac{\tilde{\omega}}{4\pi}P_1(\mu,-\mu_0)F_\odot \exp[-\tau \mathrm{Ch}(r,\mu_0)], \qquad (6.7.39)$$

where P_1 is the first Fourier component of the phase function ($m = 1$). In principle, Eqs. (6.7.36) and (6.7.38) can be used to solve for I_0 and I_1 simultaneously. The theoretical subject of radiative transfer in spherical atmospheres appears not to have been fully explored at this point.

Exercises

6.1 A satellite radiometer measures the solar radiation reflected from a semi-infinite, isotropic-scattering atmosphere composed of particulates and gases near the vicinity of an absorption line whose line shape is given by the Lorentz profile and whose absorption coefficient can be written as

$$k_\nu = \frac{S}{\pi}\frac{\alpha}{(\nu-\nu_0)^2+\alpha^2}.$$

Assuming that the particulates are nonabsorbing and that the scattering optical depth is equal to the gaseous absorption optical depth at the line center, calculate the reflected intensity as a function of wavenumber ν using the two-stream approximation. Do the problem by formulating (a) the single-scattering albedo as a function of ν; and (b) the reflected intensity in terms of the two-stream approximation.

6.2 For a semi-infinite, isotropic-scattering atmosphere, show that the planetary albedo

$$r(\mu_0) = 1 - H(\mu_0)\sqrt{1 - \tilde{\omega}}$$

and the spherical albedo

$$\bar{r} = 1 - 2\sqrt{1 - \tilde{\omega}} \int_0^1 H(\mu_0)\mu_0 \, d\mu_0.$$

Using the first approximation for the H function and assuming single-scattering albedos of 0.4 and 0.8, compute the planetary albedo for μ_0 of 1 and 0.5 and the spherical albedo.

6.3 An optically thin layer $\Delta\tau$ is added to a finite atmosphere with an optical depth of τ_1, and all the possible transmissions of the incident beam due to the addition of the thin layer are displayed in Fig. 6.22. Formulate Eq. (6.3.44) using the principles of invariance discussed in Section 6.3.2. The method is also referred to as *invariant imbedding*. In this diagram, the dotted lines represent direct transmission.

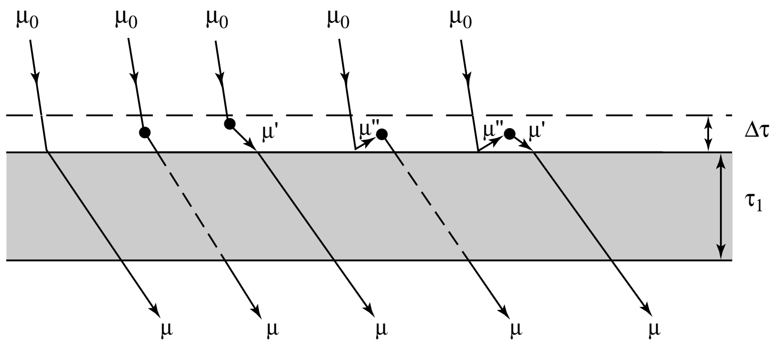

Figure 6.22

6.4 Consider a cloud layer having a total global transmission of $\bar{\gamma}$ and a global reflection (spherical albedo) of $\bar{r}$ above a Lambertian surface with an albedo of r_s. Assuming no atmosphere between the cloud and surface, derive Eqs. (6.3.69a) and (6.3.69b) by means of geometric ray-tracing for multiple reflections between the cloud and the surface.

6.5 Based on the geometric configuration shown in Fig. 6.9, derive the adding equations [Eqs. (6.4.17a)–(6.4.17f)] for a light beam incident from below the layer defined by μ'.

6.6 Neglecting the ground reflection effect, use the single-scattering approximation to compute and plot the reflected intensity (reflection) at the top of nonabsorbing molecular atmospheres whose optical depths are assumed to be 0.1 and 1 for $\mu_0 = 0.8$.

6.7 Derive an analytical expression for the diffuse reflection at the top of the atmosphere using the second-order scattering approximation (neglect the surface reflection). Carry out the analysis for $\mu \neq \mu'$ and $\mu = \mu'$.

6.8 Derive the two-stream solution for conservative scattering and calculate the reflection and transmission, assuming an asymmetry factor of 0.75 for optical depths of 0.25, 1, 4, and 16. Plot the results as functions of the cosine of the solar zenith angle μ_0.

6.9 Derive the one-dimensional diffuse equation for radiative transfer denoted in Eq. (6.5.23) from Eqs. (6.5.22a) and (6.5.22b), and solve for I_1 using the radiation boundary condition defined in Eq. (6.2.23).

6.10 By employing the orthogonality property of the Legendre polynomials and the recurrence formula

$$\mu P_\ell(\mu) = \frac{\ell + 1}{2\ell + 1} P_{\ell+1} + \frac{\ell}{2\ell + 1} P_{\ell-1},$$

show that Eq. (6.1.6) without the emission term can be reduced to the set of first-order differential equations given in Eq. (6.5.20), based on the intensity expansion given in Eq. (6.5.19). For simple cases of $\ell = 0,1$, Eq. (6.5.20) leads to Eqs. (6.5.22a,b).

6.11 Formulate the transfer of thermal infrared radiation in a scattering atmosphere having an isothermal temperature T in local thermodynamic equilibrium, assuming the intensity is azimuthally independent. By means of the discrete-ordinates method for radiative transfer and assuming isotropic scattering, show that the scattered intensity is given by

$$I(\tau, \mu_i) = \sum_{\alpha=-n}^{n} \frac{L_\alpha}{1 + \mu_i k_\alpha} e^{-k_\alpha \tau} + B_\nu(T),$$

where L_α are unknown constants of proportionality, μ_i denote the discrete streams, k_α are the eigenvalues, and B_ν represents the Planck function.

6.12 From Eqs. (6.6.1a) and (6.6.1b), show that an electromagnetic wave is defined by the equation of an ellipse. The wave is said to be *elliptically polarized*. If the phase difference between the two electric vectors $\delta = \delta_r - \delta_l$ is of the order of π, show that the vibration of the wave is governed by two lines, referred to as *linearly polarized*. If, on the other hand, it is of the order of $\pi/2$ and if the amplitudes $a_l = a_r$, show that the vibration of the wave is governed by a circle where the waves are referred to as *circularly polarized*.

6.13 From Eqs. (6.6.3a)–(6.6.3e) derive the Stokes parameters in terms of the ellipticity and orientation angles given by Eqs. (6.6.4a)–(6.6.4d).

6.14 Any time-average quantity may be represented by the summation of individual components, e.g., $\langle x \rangle = \sum_{n=1}^{N} t_n x_n$. Using this principle, show that $I^2 \geq Q^2 + U^2 + V^2$ based on the relationships given in Eq. (6.6.5). In doing this exercise, let $N = 2$ for simplicity.

6.15 (a) Show that elliptically polarized light can be decomposed into a circularly polarized part and a linearly polarized part. Then rotate the linearly polarized beam through the angle χ and show that χ, which makes the intensity maximum (or minimum) in the direction y (see Fig. 6.13), is given by $\tan 2\chi = U/Q$.

(b) Assuming a light beam with 50% linear polarization in the r direction and another independent light beam also with 50% right-handed circular polarization: (1) What would be the Stokes parameters for the mixture and the resulting total intensity and percentage polarization? (2) What would be the measured intensity if a polarizer having a plane of polarization along the r direction is used? And (3) sketch a diagram to denote the resultant polarization.

(c) With reference to (b), decompose the partially polarized light beam into natural light and 100% elliptically polarized light and compute the plane of polarization χ and ellipticity angle β for the polarized component.

(d) Natural light is equivalent to any two independent oppositely polarized beams of half the intensity. By virtue of this principle, evaluate the Stokes parameters for these two polarized beams based on the results obtained from (c).

(e) Upon combining the polarized beams derived from (c) and (d), what would be the Stokes parameters corresponding to two independent polarized beams?

6.16 Derive Eq. (6.6.11) from Eq. (6.6.3c) and prove that $\mathbf{L}(\chi_1)\mathbf{L}(\chi_2) = \mathbf{L}(\chi_1 + \chi_2)$ and $\mathbf{L}^{-1}(\chi) = \mathbf{L}(-\chi)$.

6.17 Assuming that the single-scattering parameters β_e, $\tilde{\omega}$, and g are independent of the coordinate system in Eqs. (6.7.27a)–(6.7.27d), derive the general diffusion equation for radiative transfer. Compare the final result with Eq. (6.5.23).

6.18 On the basis of the geometric configuration shown in Fig. 6.21 and the trigonometric relationships among various angles depicted in this figure, derive Eqs. (6.7.33a)–(6.7.33d).

6.19 Prove that

$$\frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{1} P(\mu, \phi; \mu', \phi')\mu \, d\mu d\phi = \mu' g,$$

where g is the asymmetry factor defined in Eq. (6.5.9a). In carrying out this exercise, expand the phase function in the form presented in Eq. (6.1.3a) and use the orthogonal property of the Legendre polynomial.

Suggested Reading

Ambartzumian, V. A. (1958). *Theoretical Astrophysics*. Pergamon Press, New York. Chapters 33 and 34 contain fundamental discussions of the principles of invariance for a semi-infinite, isotropic atmosphere.

Chandrasekhar, S. (1950). *Radiative Transfer*. Dover, New York. Chapters 1, 3, 6, and 7 contain basic materials for the discrete-ordinates method for radiative transfer and the principles of invariance.

Kourganoff, V. (1952). *Basic Methods in Transfer Problems*. Oxford University Press, London. Chapter 3 contains some elementary discussions of the discrete-ordinates and spherical harmonics methods for radiative transfer.

Lenoble, J. (1993). *Atmospheric Radiative Transfer*. A. Deepak Publishing, Hampton, Virginia. Chapters 13 and 14 consist of useful discussions of radiative transfer methodologies and polarization.

Liou, K. N. (1992). *Radiative and Cloud Processes in the Atmosphere. Theory, Observation, and Modeling*. Oxford University Press, New York. Chapter 3 presents various radiative transfer methods and approximations with a specific emphasis on flux calculations.

Sobolev, V. V. (1975). *Light Scattering in Planetary Atmospheres*. Pergamon Press, New York. Chapters 2 and 3 contain discussions of the principles of invariance.

Thomas, G. E., and Stamnes, K. (1999). *Radiative Transfer in the Atmosphere and Ocean*. Cambridge University Press, Cambridge, U.K. Chapters 7 and 8 discuss useful approximations and numerical solutions for radiative transfer.

van de Hulst, H. C. (1980). *Multiple Light Scattering. Tables, Formulas, and Applications*, Vol. 1 and 2. Academic Press, New York. Chapter 4 in Vol. 1 contains a general review of various methods for radiative transfer, including the adding/doubling method. Chapter 13 in Vol. 2 provides some useful numerical values that can be used to check various radiative transfer calculations.