

5 Multiple Scattering: Elementary

This is the first of two chapters on multiple scattering. We could define a multiple-scattering medium as one for which the separate parts are excited to radiate not only by an external source of radiation but also by the radiation from each other. According to this unqualified definition, however, almost nothing, with the possible exception of a single (structureless) electron, is *not* a multiple-scattering medium. Multiple scattering is always a matter of degree, sometimes negligible, sometimes not. We pointed out in Section 3.4.9 that the light reflected by a glass of water is a consequence of scattering by a huge array of individual scatterers (molecules), separated by distances small compared with the wavelength and highly correlated in position. It follows that there is just as much multiple scattering in a water puddle as in a water cloud, both illuminated by sunlight. This is lost sight of – indeed, sometimes denied – because the usual theory of (specular) reflection by, say, a puddle, is based on continuum theory in which multiple scattering is not explicit. We call such a medium a *coherent* multiple-scattering medium. But from now on, unless stated otherwise, we confine ourselves to *incoherent* multiple-scattering media, ones for which, because of the arrangement of their separate parts, we are spared the drudgery of having to keep track of phases of scattered waves. For such media it is *as if* light were not a wave, which results in considerable simplification.

To understand what one sees in the natural world (e.g., the variation in color and brightness of the clear sky, the brightness and darkness of clouds, etc.) requires understanding multiple scattering, a topic conspicuous by its absence in optics textbooks. Because of this we sneak up on our subject rather than rush to embrace it, proceeding from a pile of plates to a two-stream theory, then, in the following chapter, N -stream theory, and finally the integro-differential equation of transfer. We first crawl, then walk, then run, and finally jump hurdles.

5.1 Multiple Scattering by a Pile of Parallel Plates

The simplest approach to a multiple-scattering medium, which demonstrates many of its characteristics, is by way of a pile of identical parallel plates, uniformly thick and infinite in lateral extent. We take the incident illumination to be perpendicular to the pile, although our analysis in no essential way depends on this. A single plate has a reflectivity R_1 and a transmissivity T_1 . What is the reflectivity R_2 and transmissivity T_2 of two such plates (Fig. 5.1)? We assume that we can ignore the consequences of interference, which is valid if the separation between plates is appreciably larger than the wavelengths of the illumination and the irradiances are not strictly monochromatic but integrated over a range of wavelengths. We could approach this problem as we did in Section 1.4.1 where we determined the radiation field between two identical emitting plates by summing an infinite series. But here we take a different approach

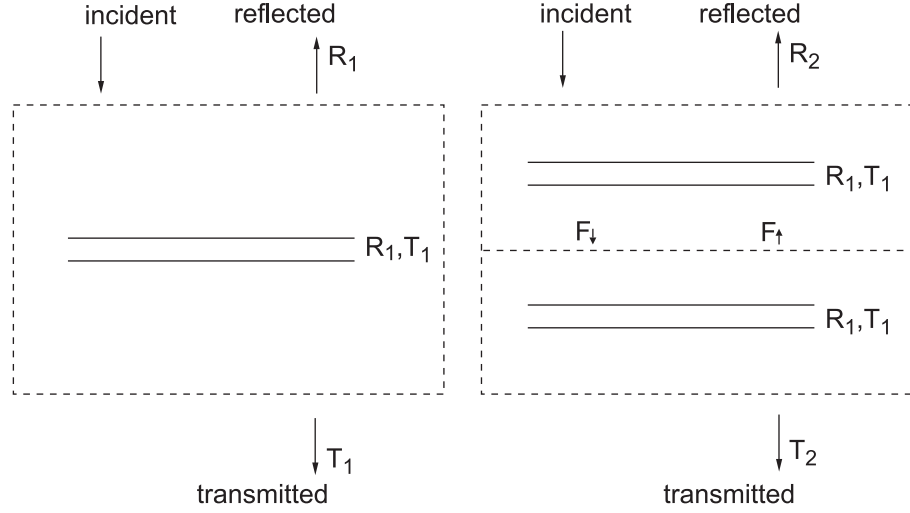


Figure 5.1: Given the reflectivity R_1 and transmissivity T_1 of an infinite (in lateral extent), uniform plate, the corresponding quantities for two such parallel plates, sufficiently far apart (relative to the wavelength of the incident illumination) that interference is negligible, follow by solving for the irradiances F_{\downarrow} and F_{\uparrow} between them.

to show that we can solve the same problem by different methods. The downward irradiance between the two plates is denoted by F_{\downarrow} , the upward irradiance by F_{\uparrow} ; the incident irradiance has magnitude 1 (the units are irrelevant). Downward and upward (the terms downwelling and upwelling are also used) are just convenient labels for two opposite directions and have nothing fundamental to do with the direction of a gravitational field. For our purposes, light is not affected by gravity. These two irradiances satisfy

$$F_{\downarrow} = T_1 + F_{\uparrow} R_1, \quad (5.1)$$

$$F_{\uparrow} = F_{\downarrow} R_1. \quad (5.2)$$

These two equations in the two unknown irradiances can be solved to obtain

$$F_{\downarrow} = \frac{T_1}{1 - R_1^2}, \quad (5.3)$$

$$F_{\uparrow} = \frac{R_1 T_1}{1 - R_1^2}. \quad (5.4)$$

The reflectivity and transmissivity of the two plates depend on these two irradiances (for unit incident irradiance):

$$R_2 = R_1 + F_{\uparrow} T_1, \quad (5.5)$$

$$T_2 = F_{\downarrow} T_1. \quad (5.6)$$

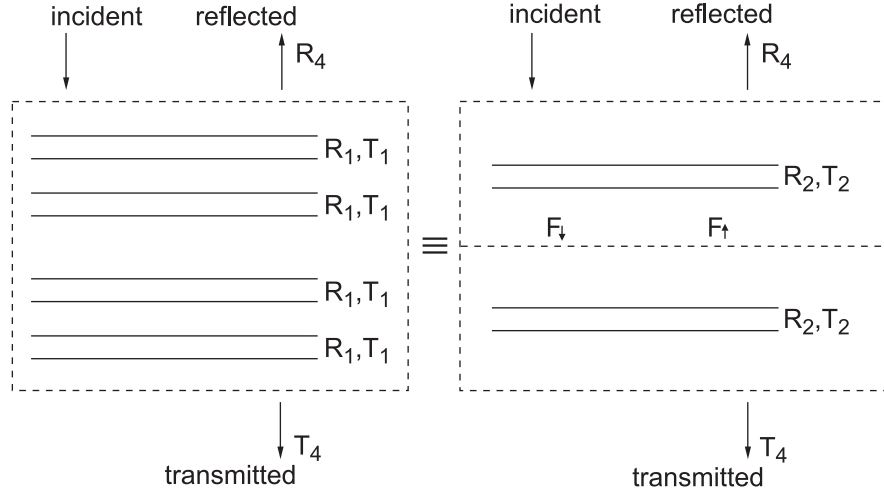


Figure 5.2: A pile of four identical parallel plates is equivalent to a pile of two composite plates, the reflectivity and transmissivity of which are determined by those of a single plate (Fig. 5.1). Thus the reflectivity and transmissivity of $2N$ identical plates readily follow from those of N plates.

Combining Eqs. (5.3)–(5.6) yields

$$R_2 = R_1 + \frac{T_1^2 R_1}{1 - R_1^2}, \quad (5.7)$$

$$T_2 = \frac{T_1^2}{1 - R_1^2}. \quad (5.8)$$

So far our results are general. Now we add the assumption that the plates are nonabsorbing ($R_1 + T_1 = 1$). With this assumption Eqs. (5.7) and (5.8) become

$$R_2 = \frac{2R_1}{1 + R_1}, \quad (5.9)$$

$$T_2 = \frac{1 - R_1}{1 + R_1}. \quad (5.10)$$

As required by conservation of radiant energy, $R_2 + T_2 = 1$.

We can look upon the two plates as a single composite plate with reflectivity given by Eq. (5.9), transmissivity by Eq. (5.10). Thus we have a simple way of obtaining the reflectivity R_4 and transmissivity T_4 of four identical plates (Fig. 5.2): in Eqs. (5.9) and (5.10) substitute R_2 for R_1 . The result for the reflectivity, after a bit of rearrangement, is

$$R_4 = \frac{4R_1}{1 + 3R_1}. \quad (5.11)$$

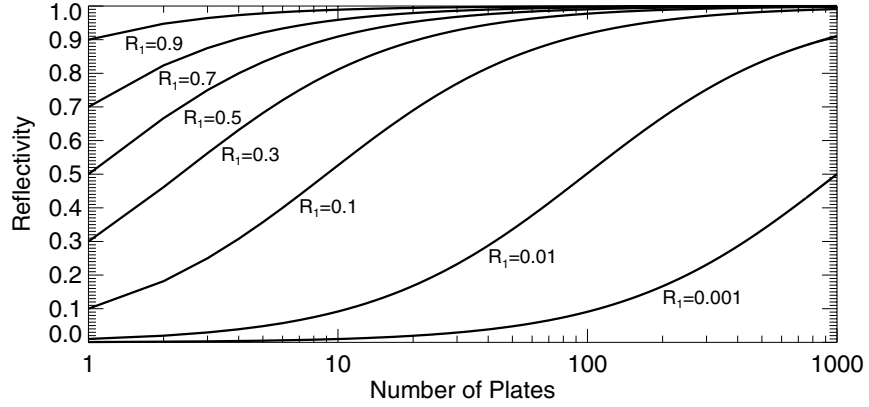


Figure 5.3: Reflectivity of a pile of identical, uniform, parallel plates, each with reflectivity R_1 , versus number of plates. Because they are taken to be nonabsorbing, the reflectivity of the pile asymptotically approaches 1.

Now we can consider the four identical plates to be a single composite plate, and substitute R_4 in Eq. (5.9) to obtain the reflectivity of eight plates, then 16, 32, 64 and so on *ad infinitum*. Although we assumed no absorption, this does not affect our general conclusion that if we can determine the reflectivity and transmissivity of only two identical plates, we can determine these quantities for any multiple of two.

The form of Eq. (5.11) suggests that

$$R_N = \frac{NR_1}{1 + (N-1)R_1}, N = 1, 2, 4, 8, \dots \quad (5.12)$$

Thus for any nonzero R_1 , R_N approaches 1 as N approaches infinity (Fig. 5.3). Even though the reflectivity of a single plate may be small, a sufficiently large number of plates is highly reflecting. You can demonstrate this with a pile of plastic bags or overhead transparencies or plastic food wrap. For example, put one transparency on a black background (black paper or cloth) and compare it with a pile of several transparencies. As you add more and more transparencies the pile will become brighter and brighter but eventually reach a point of diminishing returns in which the addition of another transparency yields no perceptible increase. Or just take a long sheet of plastic food wrap and fold it in half, then in half again, and again. What you'll see is the curve in Fig. 5.3 growing before your eyes.

If N is large compared with 1, Eq. (5.12) is approximately

$$R_N \approx \frac{NR_1}{1 + NR_1}. \quad (5.13)$$

From this equation it follows that by measuring R_N we cannot separately determine N and R_1 , only their product. The cumulative effect of many weakly reflecting plates is identical with that of a few highly reflecting plates. Measuring T_N would be of no help because it is

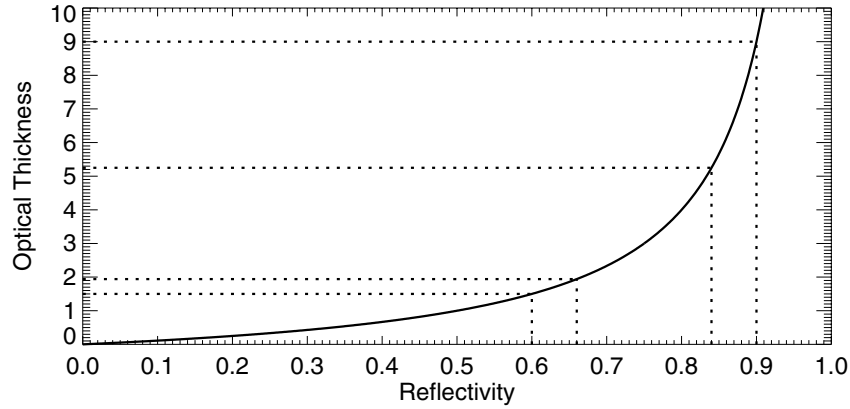


Figure 5.4: A fixed uncertainty (error) in reflectivity does not correspond to a fixed error in inferred optical thickness of a pile of nonabsorbing parallel plates. The greater the reflectivity, the greater the error in optical thickness. The error approaches infinity as the reflectivity approaches 1.

$1 - R_N$. Let's give the quantity NR_1 a symbol, τ , and a name, *optical thickness*, so that Eq. (5.13) can be written

$$R_N \approx \frac{\tau}{1 + \tau}. \quad (5.14)$$

Measuring one quantity (or quantities) to obtain another quantity (or quantities) is an example of an *inversion*. Here we invert the measured reflectivity to obtain the desired optical thickness. But we already have encountered one limitation (among several) of inversions: what we want may be unobtainable. If we want N and R_1 we are out of luck: we have to content ourselves with their product. Another limitation is that an inversion interposes a theory between measurement of one physical quantity (e.g., reflectivity) and inference of another (e.g., optical thickness). Thus for an inversion to have a hope of yielding accurate results the theory must be applicable to the system of interest, in this instance a pile of identical parallel plates for which interference can be neglected. To understand a third limitation of inversions, we invert Eq. (5.14) to obtain

$$\tau \approx \frac{R_N}{1 - R_N}. \quad (5.15)$$

If we were to measure R_N in a perfect world, one in which measurements are free of errors, we could determine optical thickness by substituting reflectivity in Eq. (5.15). But perfection is not of this world, and depending on where measurements lie on the τ - R_N curve, a small error in R_N may correspond to a huge error in τ . Note in Fig. 5.4 that for R_N near 1, even a small error in this quantity corresponds to a huge error (or uncertainty) in τ . The slope of this

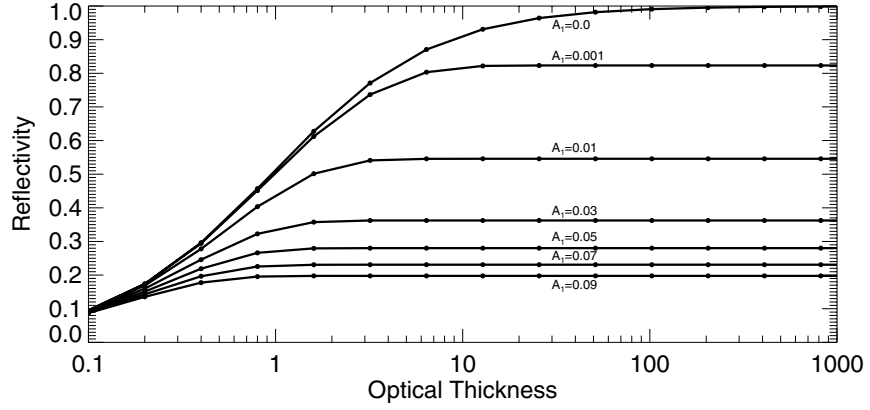


Figure 5.5: Reflectivity versus optical thickness for a pile of identical parallel plates, each with absorptivity A_1 .

curve

$$\frac{d\tau}{dR_N} \approx \frac{1}{(1 - R_N)^2} \quad (5.16)$$

is infinite at $R_N = 1$. The consequences of this can be made clearer by approximating the derivative in Eq. (5.16) by the ratio of differences and combining it with Eq. (5.15):

$$\frac{\Delta\tau}{\tau} \approx \frac{\Delta R_N}{R_N} \frac{1}{1 - R_N}. \quad (5.17)$$

If we interpret $\Delta\tau/\tau$ as the fractional error in optical thickness for a given fractional error in reflectivity, it follows from this equation that for reflectivities near 1 even a small error in reflectivity corresponds to a huge error in optical thickness. We have run head on into the limitations of an inversion scheme.

Now let us return to the general problem of N plates that may be absorbing (more physically realistic than no absorption). We already solved this problem by the *doubling method* embodied in Eqs. (5.7) and (5.8):

$$R_{2N} = R_N \left[1 + \frac{T_N^2}{1 - R_N^2} \right], \quad (5.18)$$

$$T_{2N} = \frac{T_N^2}{1 - R_N^2}, \quad N = 1, 2, 4, \dots \quad (5.19)$$

Figure 5.5 shows how the reflectivity increases with optical thickness for $R_1 = 0.05$ and different values of A_1 , the absorptivity of a single plate. On physical grounds it follows that as N approaches infinity, the transmissivity of the pile of plates approaches zero. If you need a proof of this, consider that for fixed R_1 , T_1 for an absorbing plate is always less than for

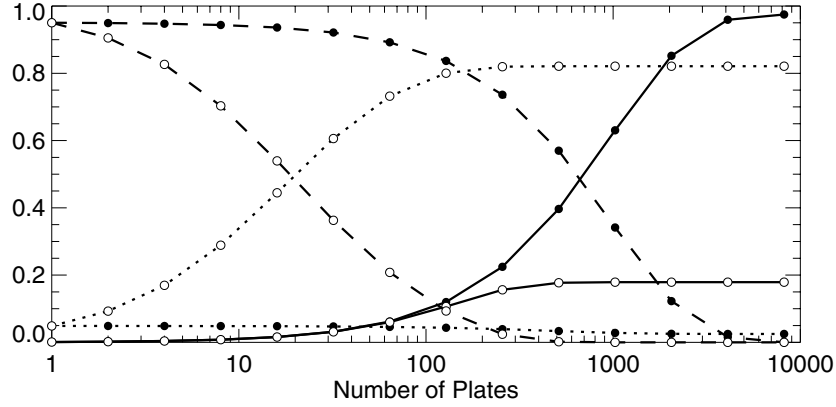


Figure 5.6: Absorptivity (solid curves), reflectivity (dotted curves), and transmissivity (dashed curves) of a pile of separated plates (open circles) and the same number of plates fused together (solid circles). The absorptivity of a single plate is 0.001 and the reflectivity because of a single interface of a plate is 0.025; normally incident radiation.

a nonabsorbing plate. This implies that R_2 and T_2 for an absorbing plate are less than the corresponding quantities when the plates are nonabsorbing. And similarly for any number of plates. We already showed that for nonabsorbing plates, the reflectivity approaches 1, and hence the transmissivity approaches 0 as the number of plates increases without limit. The transmissivity for a pile of absorbing plates, must approach the same limit. Define A_∞ as the limit of $1 - R_N - T_N$ as N approaches infinity, which is $1 - R_\infty$. Thus on a plot of reflectivity versus optical thickness (NR_1) for different values of A_1 , the difference between 1 and the asymptotic value for the reflectivity R_∞ is A_∞ (Fig. 5.5).

If the absorptivity of a single plate is zero, the absorptivity of any number of plates is also zero, and the asymptotic reflectivity is 1. But if the absorptivity of a single plate is as small as 0.001, the asymptotic reflectivity drops by almost 20%. Further increases in the absorptivity of a single plate cause the asymptotic reflectivity to drop even more, but not proportionately. That is, the magnitude of dR_∞/dA_1 (which is negative) is greatest at $A_1 = 0$.

How the reflecting, absorbing, and transmitting properties of a given amount of material are affected by how the material is dispersed is shown in Fig. 5.6, where these properties for piles of separated plates are compared with those for the same number of plates fused together. The plate separation is sufficiently large that interference can be neglected. The reflectivity of the separated plates greatly outstrips that for the fused plates, but at the cost of a lower transmissivity and absorptivity. Both transmissivities approach zero as the number of plates increases, but transmissivity of the separated plates is always lower, often much lower.

When absorption is not zero, measurement of reflection and transmission yields two distinct quantities. But, alas, we now have three quantities to be determined: number of plates and any two of the set $\{R_1, A_1, T_1\}$. So again inverting measurements cannot give us all that we might want.

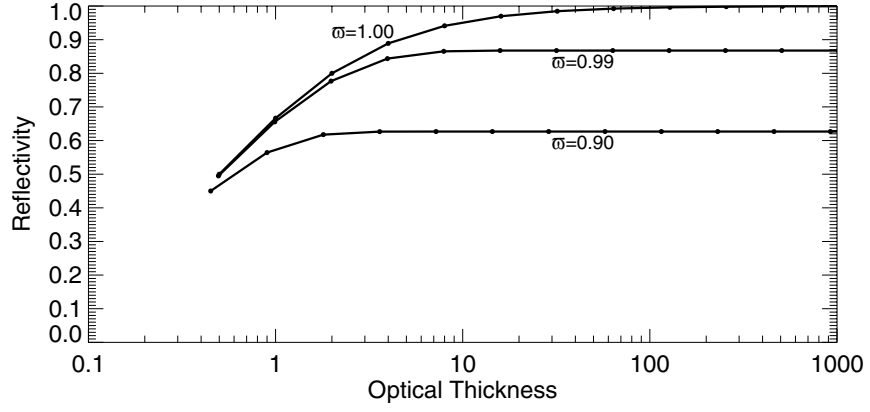


Figure 5.7: Reflectivity versus optical thickness for a pile of identical parallel plates for various values of the single-scattering albedo ϖ (the probability that an incident photon is not absorbed by a single plate) and asymmetry parameter $g = 0$.

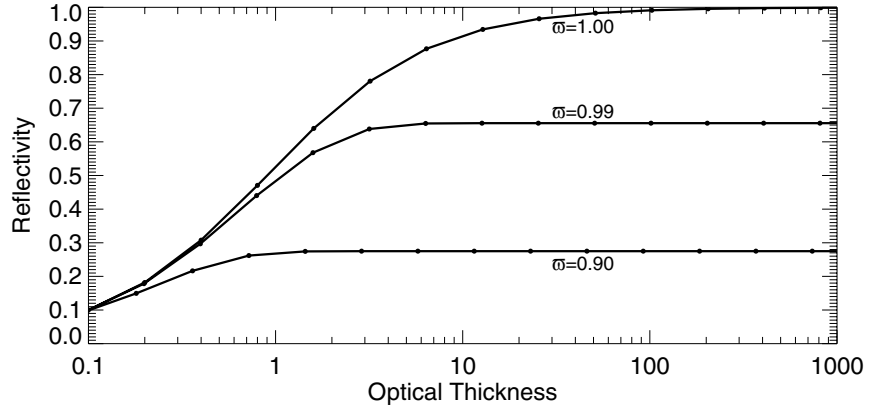


Figure 5.8: Same as Fig. 5.7 but with $g = 0.8$.

We can recast this in a different form. No new wine, just old wine in a new bottle. The quantity

$$\varpi = R_1 + T_1 = 1 - A_1 \quad (5.20)$$

is the probability that incident light is *not* absorbed by a single plate. We call this quantity the *single-scattering albedo* (the reason for this strange term will become apparent in following sections); its values lie between 0 and 1. Incident light not absorbed by a plate is either reflected, its direction changed by 180° , or transmitted, its direction not changed at all. This leads us to define the *asymmetry parameter* g as the average of the cosine of the angle by

which the incident light is changed:

$$g = \frac{T_1}{T_1 + R_1} \cos(0) + \frac{R_1}{T_1 + R_1} \cos(\pi) = \frac{T_1 - R_1}{T_1 + R_1}. \quad (5.21)$$

The asymmetry parameter lies between 1 (no reflection) and -1 (no transmission). We can solve Eqs. (5.20) and (5.21) for R_1 and T_1 in terms of ϖ and g :

$$R_1 = \varpi \frac{1 - g}{2}, \quad T_1 = \varpi \frac{1 + g}{2}. \quad (5.22)$$

The three independent parameters τ , ϖ , g are equivalent to the three independent parameters N and any two of the set $\{R_1, T_1, A_1\}$. Figures 5.7 and 5.8 show reflectivity versus optical thickness for various values of g and ϖ . Note that the consequence of increasing g (for fixed ϖ) is to reduce the reflectivity. The greater the value of g , the greater the transmissivity of a single plate and of a pile of plates.

5.1.1 Why We Sometimes Can Ignore Interference and Sometimes Not

We began the previous section by asserting that we can ignore the consequences of interference (phase differences) in determining reflection and transmission by a pile of plates if they are separated by distances large compared with the wavelength and if the reflectivities and transmissivities are averages over bands of wavelengths. Because this is such an important yet often misunderstood point we digress briefly to support our assertion.

Consider a single transparent (negligible absorption) plate of uniform thickness h and refractive index n illuminated at normal incidence by light of wavelength λ . If we ignore interference, the reflectivity of this plate is

$$R = \frac{2R_\infty}{1 + R_\infty}, \quad (5.23)$$

where R_∞ is the reflectivity of a single interface of the plate, also the reflectivity of an infinitely thick slab of the same material. The similarity of this equation to Eq. (5.9) is not an accident. A single plate has two identical interfaces, so is formally equivalent to two identical plates. Both equations were derived under the same assumption, namely that we can add irradiances of different beams of radiation without having to concern ourselves with phase differences. Note that the thickness of the plate is neither explicit nor implicit in Eq. (5.23), and wavelength is only implicit through the possible dependence of R_∞ on λ .

Now consider the reflectivity of this same plate but taking into account interference (the wave nature of light):

$$\tilde{R} = \frac{2R_\infty(1 - \cos \phi)}{1 - 2R_\infty \cos \phi + R_\infty^2}, \quad (5.24)$$

where $\phi = 4\pi nh/\lambda$. This equation can be derived by adding waves, taking account of phase differences, multiply reflected because of the two interfaces of the plate (see Prob. 7.55). We find the total reflected electromagnetic wave, which has amplitude and phase, then square it to obtain reflected irradiance. We call Eq. (5.24) the *coherent reflectivity* to emphasize that it was

derived taking interference into account and to distinguish it from the *incoherent reflectivity* [Eq. (5.23)]. These two equations are quite different. For example, the minimum value of \tilde{R} is zero, its maximum value is $4R_\infty/(1 + R_\infty)^2$. Both equations apply to the same plate. Which is correct? Well, they *both* are correct, although in a strict sense Eq. (5.24) is more correct. To make sense out of this seemingly paradoxical statement consider the average of Eq. (5.24):

$$\langle \tilde{R} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{2R_\infty(1 - \cos \phi)}{1 - 2R_\infty \cos \phi + R_\infty^2} d\phi. \quad (5.25)$$

This integral can be looked upon as either an average over one period (2π), any integral number of periods, or any non-integral number appreciably greater than 1. With a fair amount of labor, using tables of integrals or attacking Eq. (5.25) with hammer and tongs, we obtain

$$\langle \tilde{R} \rangle = \frac{2R_\infty}{1 + R_\infty} = R. \quad (5.26)$$

Note that because $\phi = 4\pi nh\nu/c$, where c is the free-space speed of light and ν is the frequency, an average over ϕ is essentially an average over a band of frequencies (if the band is sufficiently narrow that n does not vary appreciably). For a narrow range of frequencies $\Delta\nu$ the corresponding range of phases ϕ is

$$\Delta\phi = 4\pi \frac{nh}{\lambda} \frac{\Delta\nu}{\nu}, \quad (5.27)$$

which is many multiples of 2π only if $nh/\lambda \gg 1$. If this condition is satisfied, the average of \tilde{R} over this frequency interval is R to good approximation. But if the plate is thinner than or comparable with the wavelength, the average of \tilde{R} over the frequency interval is not likely to be R except by accident (see Fig. 5.9).

We could have obtained Eq. (5.23) by first deriving Eq. (5.24). This would have required us to obtain \tilde{R} from electromagnetic theory (a theory of waves with amplitudes and phases), which is not especially difficult but entails considerably more labor than deriving R . Then we would have had to evaluate the integral Eq. (5.25), which is difficult (but not impossible). This laborious procedure would have been like shooting an ant with a machine gun if we could have gone directly to Eq. (5.23) from the outset because the plate of interest is many wavelengths thick. We always are allowed to do more work than necessary, but life is short so we look for shortcuts. And keep in mind this was the easiest problem we could come up with to illustrate our point. A more complicated problem might not have been soluble exactly (i.e., within the framework of electromagnetic theory) or its solution might have been extremely tedious.

We hope that now you have a better understanding of why we could ignore interference when deriving the reflectivity of a pile of plates subject to the requirement that the separation between them is large compared with the wavelength. Note, however, that the plates themselves need not be thick compared with the wavelength. We could use Eq. (5.24) for the individual plates, then find the reflectivity of a pile of them using Eq. (5.12) if the separation between plates is large compared with the wavelength and if by reflectivity we mean a simple average over a range of frequencies.

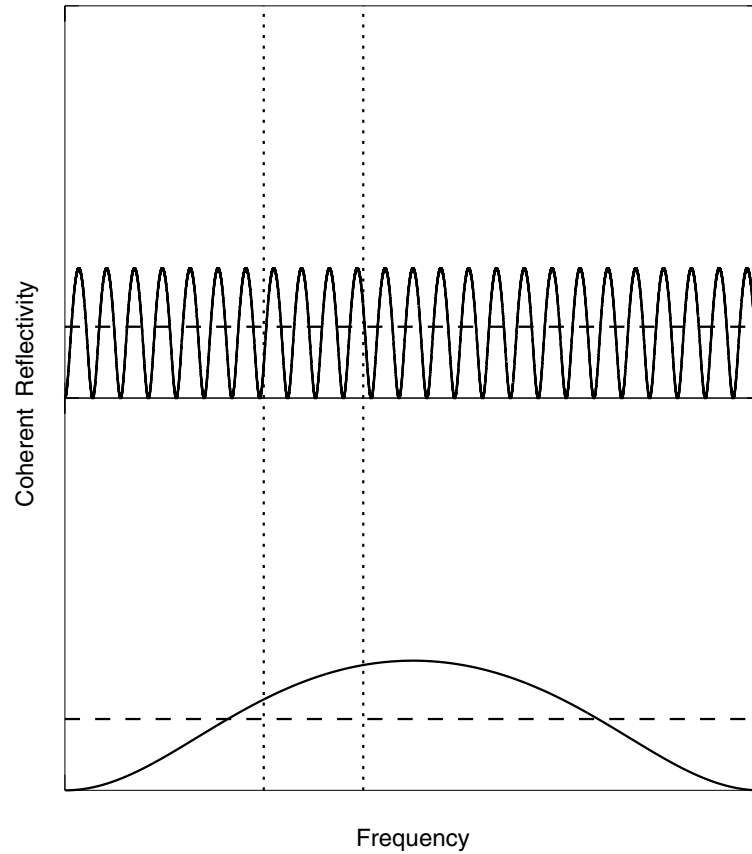


Figure 5.9: Coherent reflectivity of a single, uniformly thick, nonabsorbing plate versus frequency (solid line). The top curve is for a plate with thickness times refractive index about 25 times the wavelength of the incident illumination; the bottom curve is for a plate with thickness times refractive index of about one wavelength. The horizontal dashed lines are the incoherent reflectivity obtained by ignoring interference. Vertical dotted lines show the range of frequencies over which the coherent reflectivity is averaged. For the plate much thicker than the wavelength, this average is essentially the incoherent reflectivity, whereas this is not true for the plate with thickness comparable with the wavelength.

5.1.2 Radiative Transfer in Plane-Parallel Media

A pile of parallel plates is a *discrete* medium consisting of alternating regions with abrupt, regular, and fixed boundaries between them. Of course all media are discrete at some scale: air is composed of discrete molecules moving about in empty space; clouds are composed of discrete droplets suspended in air. But the fleeting and amorphous discreteness of air and

clouds is on a much smaller scale than that of a pile of plates. Moreover, atmospheres have no well-defined upper boundaries; like old soldiers, they just fade away.

In our analysis of a pile of plates we did not determine the radiation field everywhere, only the irradiances reflected and transmitted by the entire pile. All mathematical models are idealizations of reality, but if we hope to understand radiation fields at each point in an atmosphere or within clouds as well as above and below them, we need a model closer than a pile of plates to the reality of air and clouds. So we construct (on paper or in our minds) hypothetical *continuous* media the properties of which vary in only one direction. These media extend to infinity in directions in a plane, and hence are designated as *plane-parallel*. If there are boundaries within them we assume that they are abrupt (i.e., scattering and absorption properties change discontinuously at boundaries). No such media exist. An atmosphere, as its name implies, is spherical. Real clouds are finite, their properties vary in all directions, and they have jagged boundaries. Nevertheless, as long as we are careful to recognize the limitations of theories of radiative transfer in plane-parallel media (i.e., unreal media) they will help us understand radiative transfer in real media.

We assume that temperatures and wavelengths of interest are such that emission within media is negligible. All emission (i.e., external sources of radiation such as sunlight) is assumed to occur outside media of interest.

We also ignore the finite time it takes radiation to propagate, which is why time is absent from the equations of transfer in this book. Where time may enter is in the time-dependence of external sources (i.e., boundary conditions). If the boundary illumination changes with time, the consequences are assumed to be propagated instantaneously throughout the medium because of the huge value of the speed of light relative to distances and times of interest. For example, illumination of a cloud by sunlight may vary over times of order 10^2 s, whereas sunlight propagates thousands of meters in a cloud or air in times of order 10^{-5} s. What this means is that as sunlight changes, the radiation field everywhere in a cloud or in air adjusts almost instantaneously to these changes.

An example of when time lags may not be negligible is a cloud illuminated by a short pulse of radiation. A source (e.g., laser) is turned on suddenly, then turned off suddenly after a short time interval t . But the time interval during which scattered photons are received by a detector located at the source is greater than t because each photon has traveled a different path length. Moreover, the shape (in time) of the return pulse relative to that of the transmitted pulse depends on the characteristics in the clouds (see Sec. 6.4.2).

We assume further, as we did with the pile of plates, that the consequences of interference are negligible, which means that we can add radiant power from different sources. This is an inherent contradiction in radiative transfer theory. In deriving the equations of transfer in this book we take limits as distances shrink to zero, which implies that all quantities are mathematically continuous. But if media were truly continuous we could not apply to them a theory based on adding radiant power. We would have to add waves taking account of phase differences, then determine the radiant power. An example is a glass of water, which comes as close to a continuous medium as anything in nature. The water molecules are separated by distances small compared with the wavelengths of sunlight and their positions are strongly correlated (i.e., one molecule can't move without other molecules getting out of its way). Such a medium composed of a huge number of closely packed, correlated molecules is a multiple-scattering medium, just as a cloud of water droplets is. The difference between them lies only

in the theories applicable to them. The radiative transfer theory in this book applied to a glass of water does not yield the laws of specular reflection and refraction, not even approximately. To determine scattering by this coherent array of molecules we would have to find the waves scattered by all of them, add all these waves taking account of phase differences, then square the resultant to obtain the power. We therefore assume continuity for mathematical purposes when we derive the equation of transfer but apply it to discrete media because only for such media can we ignore phase differences. We can get away with this for air because even though its molecules are separated by distances small compared with the wavelengths of sunlight, they are in constant motion and their positions are essentially uncorrelated (see Sec. 3.4.9). And we can get away with this for clouds because the constantly changing separation between droplets is large compared with the wavelengths of solar and terrestrial radiation and also with their lateral coherence lengths (see Sec. 3.4.2).

Because we assumed incoherent scattering, the photon language is the natural one for discussing the radiative transfer theory considered here. We look upon photons as discrete blobs of energy without phases, so that the energy transported by two photons traveling in the same direction is the sum of the energy transported by each one separately. Our radiative transfer theory is a theory of multiple scattering of photons rather than of waves.

5.1.3 Mean Free Path

What happens to photons as we imagine them launched from a particular point (call it the origin of the x -axis) in a medium? Photons obey statistical laws. We cannot determine what happens to a particular photon (photons are indistinguishable so the concept of a “particular photon” is meaningless) but we can determine what happens in a statistical sense to an ensemble of many photons. For example, what is the probability that a photon propagating along the x -axis beginning at $x = 0$ is absorbed between x and $x + \Delta x$? This is given by the integral of the probability distribution function $p(x)$

$$\int_x^{x+\Delta x} p(x) dx. \quad (5.28)$$

From Eq. (2.7) for exponential attenuation by absorption it follows that the probability a photon is *not* absorbed over a distance x from the origin is $\exp\{-\kappa x\}$, the probability it is *not* absorbed over a distance $x + \Delta x$ is $\exp\{-\kappa(x + \Delta x)\}$, and hence the probability it *is* absorbed in the interval Δx is the difference

$$\int_x^{x+\Delta x} p(x) dx = \exp\{-\kappa x\} - \exp\{-\kappa(x + \Delta x)\}. \quad (5.29)$$

If $p(x)$ is continuous and bounded then according to the mean value theorem of integral calculus the integral in Eq. (5.29) is

$$p(\bar{x})\Delta x = \exp\{-\kappa x\} - \exp\{-\kappa(x + \Delta x)\}, \quad (5.30)$$

where \bar{x} lies between x and $x + \Delta x$. Divide both sides of this equation by Δx and take the limit as $\Delta x \rightarrow 0$:

$$p(x) = -\frac{d}{dx} \exp(-\kappa x) = \kappa \exp(-\kappa x). \quad (5.31)$$

With this probability distribution, the integral of which over all x is 1, we can find the mean distance a photon propagates before being absorbed:

$$\langle x \rangle = \int_0^\infty x \kappa \exp(-\kappa x) dx = \frac{1}{\kappa} = \ell_a. \quad (5.32)$$

Thus the physical interpretation of κ is that its inverse is the mean free path for absorption ℓ_a . Attenuation by scattering also is exponential provided that a photon is removed from a beam of radiation if it is scattered once in any direction whatsoever. If we denote by β the scattering coefficient in the expression $\exp(-\beta x)$, the mean free path for scattering ℓ_s is $1/\beta$, and the total mean free path is

$$\ell_t = \frac{1}{\kappa + \beta}, \quad (5.33)$$

where

$$\frac{1}{\ell_t} = \frac{1}{\ell_a} + \frac{1}{\ell_s}. \quad (5.34)$$

We need one more result for the analysis that follows. What is the probability that given that a photon reaches x without being absorbed, it is absorbed in Δx ? What we are after here is a *conditional probability*, the probability of absorption in Δx given that a photon has reached x , which is related to, but not the same as Eq. (5.30). The probability that a photon propagates a distance x from the origin without absorption is $\exp(-\kappa x)$. The probability that a photon is absorbed in Δx is given by Eq. (5.30). So this probability divided by the previous one is the conditional probability we are after:

$$1 - \exp(-\kappa \Delta x). \quad (5.35)$$

This quantity is the probability that if a photon finds itself at x it is absorbed as it propagates a further distance Δx . If $\kappa \Delta x \ll 1$, this probability is approximately $\kappa \Delta x$. And similarly, the probability of scattering is $\beta \Delta x$ provided that $\beta \Delta x \ll 1$.

Although the absorption coefficient κ and scattering coefficient β are fundamental quantities in the theory of radiative transfer, determining their values lies outside this theory. We either have to obtain them from measurements or appeal to a theory that comes to grips with the discreteness of molecules and particles.

5.2 Two-Stream Theory of Radiative Transfer

The preceding sections were fairly general, applicable to all theories of radiative transfer. To proceed further we have to make some specific assumptions, and the simplest one is that the radiation field consists of irradiances F in two and only two directions (streams), denoted as upward and downward. This is an idealization given that strictly monodirectional irradiances do not exist; even a laser beam has a finite angular spread. Scattering can therefore occur in only these two directions: a photon directed downward can be scattered only downward or upward, and similarly for a photon directed upward. We also ignore the polarization state of the

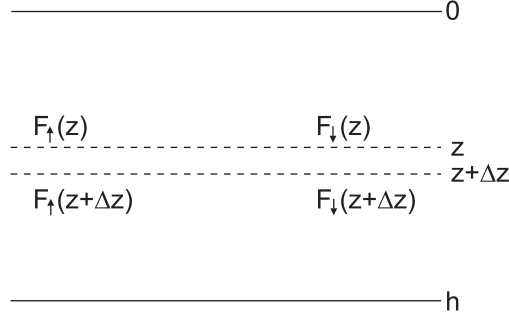


Figure 5.10: Downward (\downarrow) and upward (\uparrow) irradiances F are different at z and $z + \Delta z$ because of absorption and scattering within Δz . Note that the positive z -axis is downward.

radiation, which here makes no difference because the polarization state of light scattered by a spherically symmetric medium is not changed upon scattering in the forward and backward directions (see Sec. 7.4).

Conservation of downward radiant energy in a thin (relative to the mean free path) layer between z and $z + \Delta z$ (Fig. 5.10) yields

$$F_{\downarrow}(z + \Delta z) = F_{\downarrow}(z) - \kappa \Delta z F_{\downarrow}(z) - \beta \Delta z p_{\downarrow\uparrow} F_{\downarrow}(z) + \beta \Delta z p_{\uparrow\downarrow} F_{\uparrow}(z + \Delta z). \quad (5.36)$$

This is the mathematical form of the following statement. At the bottom of the layer the downward radiation is that incident at the top *decreased* by absorption ($\kappa \Delta z$ is the probability of absorption) and by scattering upward in Δz , but *increased* because upward radiation incident at the bottom of the layer is scattered downward in Δz . We can ignore the probability that a photon is scattered more than once in Δz (which depends on higher powers of $\beta \Delta z$) if we take $\beta \Delta z \ll 1$. The quantity $p_{\downarrow\uparrow}$ is the (conditional) probability that given that a downward photon is scattered, it is scattered in the upward direction, and similarly for $p_{\uparrow\downarrow}$. Divide both sides of Eq. (5.36) by Δz and take the limit as $\Delta z \rightarrow 0$:

$$\frac{dF_{\downarrow}}{dz} = -\kappa F_{\downarrow} - \beta p_{\downarrow\uparrow} F_{\downarrow} + \beta p_{\uparrow\downarrow} F_{\uparrow}. \quad (5.37)$$

A similar radiant energy conservation argument applied to the upward irradiance yields

$$\frac{dF_{\uparrow}}{dz} = \kappa F_{\uparrow} + \beta p_{\uparrow\downarrow} F_{\uparrow} - \beta p_{\downarrow\uparrow} F_{\downarrow}. \quad (5.38)$$

The sign reversal is a consequence of attenuation of upward radiation in the direction of decreasing z .

It is the presence of the third term on the right sides of Eqs. (5.37) and (5.38) that makes radiative transfer non-trivial. Were it not for this term, the solutions to both equations would be simple exponentials. What complicates matters is that downward radiation is a source for upward radiation and vice versa: the upward and downward irradiances are coupled.

Because photons can be scattered only upward or downward the sum of probabilities must satisfy

$$p_{\downarrow\uparrow} + p_{\downarrow\downarrow} = p_{\uparrow\downarrow} + p_{\uparrow\uparrow} = 1. \quad (5.39)$$

By using Eq. (5.39) we can rewrite Eqs. (5.37) and (5.38) as

$$\frac{dF_{\downarrow}}{dz} = -(\kappa + \beta)F_{\downarrow} + \beta(p_{\downarrow\downarrow}F_{\downarrow} + p_{\uparrow\downarrow}F_{\uparrow}), \quad (5.40)$$

$$\frac{dF_{\uparrow}}{dz} = (\kappa + \beta)F_{\uparrow} - \beta(p_{\uparrow\uparrow}F_{\uparrow} + p_{\downarrow\uparrow}F_{\downarrow}). \quad (5.41)$$

These equations are easier to interpret. The first term on the right side expresses all the ways radiation can be removed from a particular direction (expenses), the second term all the ways radiation can be added to that direction (income). If you can balance a checkbook you can understand these equations.

Now we further assume that the medium is *isotropic*: $p_{\downarrow\uparrow} = p_{\uparrow\downarrow}$, $p_{\downarrow\downarrow} = p_{\uparrow\uparrow}$. Such a medium is rotationally symmetric: rotate it by any amount and you can't tell that it has been rotated. A suspension of spherically symmetric scatterers is an isotropic medium, as is a suspension of randomly oriented asymmetric scatterers. An example of an anisotropic medium is a suspension of oriented, asymmetric scatterers.

We have to take some care with the term isotropic because it is used in different ways, sometimes in the same breath. An *isotropic radiation field* here is defined by $F_{\uparrow} = F_{\downarrow}$; *isotropic scattering* by $p_{\uparrow\downarrow} = p_{\downarrow\uparrow}$. Isotropic scattering does not necessarily imply an isotropic radiation field and conversely. To make matters more confusing, there are no isotropic scatterers of electromagnetic waves (acoustic waves are another story) in nature in the sense that they scatter equally in *all* directions (see Sec. 7.3). You can't beg, borrow, or steal such scatterers, but it is possible to find ones that scatter equally in two opposite hemispheres of directions (e.g., scattering of sunlight by air molecules).

As with the pile of plates, we define the asymmetry parameter g as the mean cosine of the scattering angle, which has only two values, 1 and -1 :

$$g = p_{\downarrow\downarrow}(1) + p_{\downarrow\uparrow}(-1). \quad (5.42)$$

With this definition and the assumption of an isotropic medium the various probabilities in Eqs. (5.40) and (5.41) can be expressed solely in terms of g :

$$p_{\downarrow\uparrow} = p_{\uparrow\downarrow} = \frac{1-g}{2}, \quad p_{\downarrow\downarrow} = p_{\uparrow\uparrow} = \frac{1+g}{2}. \quad (5.43)$$

It often is convenient to transform from physical depth z to *optical depth* τ :

$$\tau = \int_0^z (\kappa + \beta) dz = \tau_a + \tau_s. \quad (5.44)$$

We have to take care when we encounter this term because total optical depth τ is the sum of absorption optical depth τ_a and scattering optical depth τ_s . Authors often write optical depth as τ and leave it to readers to guess which one of the three possibilities is meant. If κ and β are independent of z it follows from Eq. (5.33) that τ is physical depth measured in units of total mean free path.

By using Eqs. (5.43) and (5.44) we can write Eqs. (5.40) and (5.41) as

$$\frac{dF_{\downarrow}}{d\tau} = -F_{\downarrow} + \varpi \left\{ \frac{1+g}{2} F_{\downarrow} + \frac{1-g}{2} F_{\uparrow} \right\}, \quad (5.45)$$

$$\frac{dF_{\uparrow}}{d\tau} = F_{\uparrow} - \varpi \left\{ \frac{1+g}{2} F_{\uparrow} + \frac{1-g}{2} F_{\downarrow} \right\}, \quad (5.46)$$

where the *single-scattering albedo* ϖ is $\beta/(\beta + \kappa)$. More compact versions of these equations can be obtained by adding and subtracting them:

$$\frac{d}{d\tau}(F_{\downarrow} - F_{\uparrow}) = -(1 - \varpi)(F_{\downarrow} + F_{\uparrow}), \quad (5.47)$$

$$\frac{d}{d\tau}(F_{\downarrow} + F_{\uparrow}) = -(1 - \varpi g)(F_{\downarrow} - F_{\uparrow}). \quad (5.48)$$

In general, optical depth τ , single-scattering albedo ϖ , and asymmetry parameter g depend on the frequency of the radiation. The latter two quantities also may vary with physical depth z or, equivalently, optical depth; ϖ lies between 0 and 1, g between -1 and 1, although the end points of these two intervals never occur in reality.

5.2.1 Conservative Scattering

A single-scattering albedo of 1 is sometimes referred to as *conservative scattering*, which is not a comment on the political affiliation of photons but rather signals that radiant energy is conserved; nonzero absorption implies that radiant energy is converted into other forms although, as always, total energy is conserved. As noted at the end of the previous paragraph $\varpi = 1$ does not exist in nature, but the assumption that it does sometimes leads to no serious errors and simplifies solutions to the equations of transfer. Indeed, with this assumption and the assumption that g is independent of τ , the difference of irradiances in Eq. (5.47) is constant, and their sum in Eq. (5.48) is a linear function of τ , which yields

$$F_{\downarrow} = B + C(1 - \tau^*), \quad F_{\uparrow} = B - C(1 + \tau^*), \quad (5.49)$$

where B and C are constants determined by conditions at the upper ($\tau = 0$) and lower ($\tau = \bar{\tau}$) boundaries of the medium. Although written without a subscript, here optical depth is scattering optical depth. The quantity $\tau^* = \tau(1 - g)$ is the *scaled optical depth*. Scaled here means scaled to isotropic scattering. Two media, one composed of isotropic scatterers ($g = 0$), the other of anisotropic scatterers ($g \neq 0$), are equivalent if the optical depth of the latter is the optical depth of the former divided by $1 - g$. The *optical thickness* of the entire medium (from $z = 0$ to $z = h$) is

$$\bar{\tau} = \int_0^h \beta \, dz. \quad (5.50)$$

Optical thickness and optical depth often are used interchangeably, but we try to be careful to reserve optical thickness for the optical depth of an entire medium.

5.2.2 Conservative Scattering: Equilibrium Solution

If the medium is infinite ($h \rightarrow \infty$), for the two irradiances in Eq. (5.49) to be finite requires $C = 0$, which in turn implies that the upward and downward irradiances are equal and independent of τ . Note that here the radiation field is isotropic even though scattering need

not be (g is arbitrary). Infinite media don't exist, so let's consider a medium illuminated by downward irradiance F_0 at its upper boundary, and with a mirror (reflectivity 1) at its lower boundary. The upper boundary condition is $F_{\downarrow} = F_0$, the lower boundary condition is $F_{\uparrow} = F_{\downarrow}$. Again the solution is $C = 0$, and the two irradiances are uniform and equal to the downward irradiance at the upper boundary. A uniform, isotropic radiation field is called the *equilibrium solution*. To understand why, consider the medium to be sandwiched between two mirrors (with reflectivity 1). Again, the solution to Eq. (5.49) is $C = 0$, but B is arbitrary. The interpretation of this result is that if photons are injected into the medium they can neither escape from nor be absorbed by it, and hence the radiation field eventually becomes uniform and isotropic. Strict equilibrium occurs when all gradients are ironed out.

5.2.3 Conservative Scattering: Reflection and Transmission

Although the solutions in the previous section are illuminating (forgive the pun), they are not especially realistic. So let's consider a more realistic problem in which photons that leak out the bottom boundary never return (i.e., $F_{\uparrow} = 0$ at $\bar{\tau}$), either because there is nothing to return them or they are absorbed before doing so; the downward irradiance at the top boundary is F_0 . With these boundary conditions the solution to Eq. (5.49) yields for the reflectivity R and transmissivity T

$$R = \frac{F_{\uparrow}(0)}{F_0} = \frac{\bar{\tau}(1-g)/2}{1 + \bar{\tau}(1-g)/2} = \frac{\bar{\tau}^*/2}{1 + \bar{\tau}^*/2}, \quad (5.51)$$

$$T = \frac{F_{\downarrow}(\bar{\tau})}{F_0} = \frac{1}{1 + \bar{\tau}(1-g)/2} = \frac{1}{1 + \bar{\tau}^*/2}. \quad (5.52)$$

As required by radiant energy conservation, $R + T = 1$. But wait! We've seen Eq. (5.51) before. It has the same form as Eq. (5.14), the reflectivity of a pile of plates, the only difference being that we called $\tau = NR_1 = N(1-g)/2$ [see Eq. (5.21)] the optical thickness. We did so because calling this quantity one-half the scaled optical thickness might have been perplexing. But now that you are inoculated against perplexity we can rewrite history and say that in Eq. (5.14) we really meant to write $\bar{\tau}$ for N . Does it make sense that N , the number of plates, is analogous to optical thickness? We think it does. Although N can take on only discrete values (1, 2, 3...) whereas $\bar{\tau}$ is continuous, $\bar{\tau} = 1$ corresponds (roughly) to a probability of 1 for a photon to be scattered at least once, $\bar{\tau} = 2$ corresponds to a probability of 1 for a photon to be scattered at least twice, and so on. But this is analogous to what happens when we add plates to a pile. Every plate added (in discrete steps) increases the probability of a photon being scattered. With 2 plates a photon is likely to be scattered (reflected) at least twice, with 3 plates, at least 3 times, and so on. With the wisdom of hindsight, and a cup of physical intuition, we could have solved the equation of transfer (without even being aware of its existence) to obtain Eqs. (5.51) and (5.52) knowing the solution to the pile-of-plates problem and understanding the physical meaning of optical thickness. Moreover, now we don't have to plot Eq. (5.51): we already did so (Figs. 5.7 and 5.8).

From Eq. (5.51) it follows that reflectivity asymptotically approaches 1 as optical thickness approaches infinity. But it is not necessary for the medium to be infinite in order for it to be *optically thick*, by which is meant indistinguishable from an infinite medium. For example, the

average contrast threshold of the human eye is 2%, which means that humans cannot detect the difference between two light sources with luminances different by less than 2%. If we adopt this criterion, a scaled optical thickness $\bar{\tau}(1 - g)$ of around 100 corresponds to a reflectivity of 0.98, which is within 2% of the reflectivity of an infinite medium.

With the help of Eq. (5.43) we can write Eq. (5.52) in a form that is easier to interpret:

$$T = \frac{1}{1 + \bar{\tau}p_{\downarrow\uparrow}}. \quad (5.53)$$

This equation tells us that attenuation of incident light is a consequence only of downward photons scattered upward. Downward photons scattered downward continue to contribute to the downward irradiance.

With the assumption that $\bar{\tau}^*/2 \ll 1$, Eq. (5.52) can be expanded in a power series and truncated after the second term:

$$T = 1 - \bar{\tau}^*/2. \quad (5.54)$$

But this is also the first two terms in the expansion of the exponential function:

$$T \approx \exp(-\bar{\tau}^*/2). \quad (5.55)$$

Exponential attenuation by scattering would prevail only if multiple scattering were negligible, that is, if photons scattered out of the forward direction never returned to that direction. Contrary to what the term “multiple” might lead you to think, attenuation for which multiple scattering is taken into account is always *less* than attenuation for which single scattering is assumed (Fig. 5.11). No equations or figures are needed to prove this. On the basis of single-scattering arguments, a photon scattered out of its original direction never returns to that direction, whereas multiple scattering gives the photon additional chances to be scattered back into that direction.

Clouds often are said to be white (strictly, the spectrum of light reflected by them is not appreciably different from that of the light illuminating them) because scattering of visible light by cloud droplets is independent of wavelength. This leads to the notion (which takes on various guises) that when a multiple-scattering medium is observed to be white, this infallibly signals the presence of “big” (compared with the wavelength) particles. Here we have a failure to distinguish between a necessary and a sufficient condition. It is indeed true that cloud particles (water droplets, ice particles) are sufficiently large compared with the wavelengths of visible light that they scatter more or less independently of wavelength (Fig. 3.12). This is a sufficient condition for the cloud to be white upon illumination by white light, but it is not necessary, which you can demonstrate for yourself. Paint the insides of two aluminum pie pans black. Fill them with water. To one add just a few drops of milk, to the other so much milk that the bottom of the pan is not visible. Observe these two pans side by side in bright sunshine. The dilute (optically thin) suspension has a bluish cast, whereas the more concentrated (optically thick) suspension is white. And yet the particles in both pans are the same. This is yet another example in which single-scattering arguments applied to a multiple-scattering medium lead to erroneous conclusions. The individual particles in milk are sufficiently small that they scatter more at the short-wavelength end of the visible spectrum

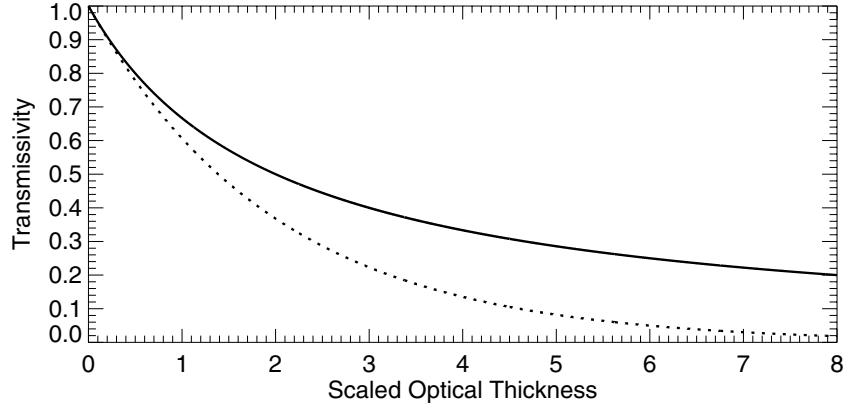


Figure 5.11: Because of multiple scattering, transmission by a plane-parallel medium decreases less rapidly (solid curve) with scaled optical thickness than exponentially (dashed curve).

than at the long-wavelength end. But in an optically thick medium of many such particles the cumulative effect of multiple scattering is to wash out this spectral dependence, which follows from differentiating Eq. (5.51) to obtain the wavelength dependence of reflectivity:

$$\frac{dR}{d\lambda} = \frac{dR}{d\bar{\tau}^*} \frac{d\bar{\tau}^*}{d\lambda} = \frac{1}{2(1 + \bar{\tau}^*/2)^2} \frac{d\bar{\tau}^*}{d\lambda}. \quad (5.56)$$

The spectral dependence of $\bar{\tau}^*$ is a consequence of that of the scattering coefficient β and of g (usually much weaker). Consider the two limits, optically thin and optically thick:

$$\frac{dR}{d\lambda} \approx \frac{d\bar{\tau}^*}{d\lambda} \quad (\bar{\tau}^*/2 \ll 1), \quad (5.57)$$

$$\frac{dR}{d\lambda} \approx 0 \quad (\bar{\tau}^*/2 \gg 1). \quad (5.58)$$

These are markedly different spectral dependences and yet the scatterers are the same. What is different is only their amount. For the optically thin medium [Eq. (5.57)], the spectral dependence of reflectivity is essentially that of the individual scatterers; for the optically thick medium [Eq. (5.58)], reflectivity is essentially independent of wavelength.

At least two caveats should accompany pronouncements about the whiteness of clouds. You may have noticed pastel colors, called *iridescence*, in thin clouds or at the edges of thick ones when looking toward the sun. Or you may even have seen colored rings a few degrees across, called the *corona*, around the sun or moon through thin, even imperceptible, clouds (see Sec. 8.4.1). The key word here is “thin”. Although *total* scattering (in all directions) of visible light by cloud droplets is to good approximation independent of wavelength, *angular* scattering is not: light is scattered more in some directions than in others depending on wavelength. We can express this within the context of our simple two-stream theory by saying

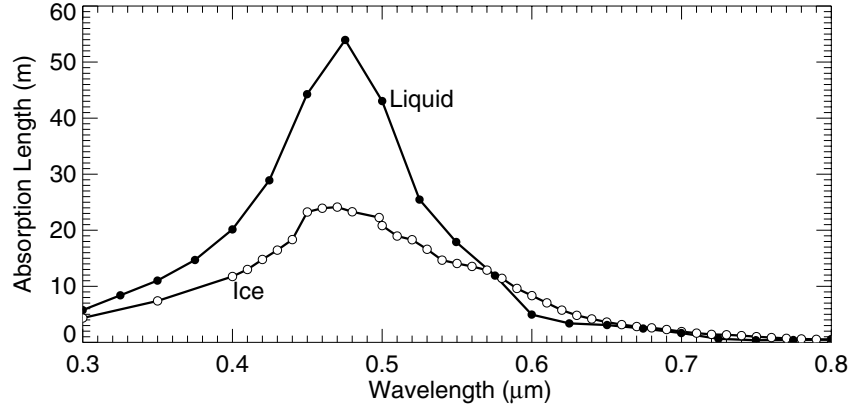


Figure 5.12: Absorption length (inverse absorption coefficient) of pure ice and liquid water over the visible spectrum. The data for liquid water were taken from Querry *et al.* (1991), those for ice from Warren (1984).

that g depends on wavelength. This dependence is weak but may be observable in thin clouds. Here is yet another example of how multiple scattering qualitatively changes what is observed.

Although clouds seen by *reflected* light are usually white (again, assuming incident white light), light *transmitted* by very thick clouds may have a bluish cast. This is not evident from Eq. (5.52) nor can it be because it was derived under the assumption that absorption is negligible. And it often is – but not always. Pure water, including ice, has an absorption minimum in the blue–green part of the visible spectrum and rises sharply toward the red (Fig. 5.12). But absorption by water at all visible wavelengths is weak in the sense that visible light has to be transmitted many meters through water before being perceptibly attenuated by absorption (which is why we don’t think of a glass of water as a glass of blue dye). Although the total liquid water paths of thick clouds are of order a centimeter or less, multiple scattering in very thick clouds can greatly magnify the effective transmission distance. We try to make this point clearer in Section 5.3 on multiple scattering in absorbing media.

5.2.4 Conservative Scattering: Diffuse Radiation

Now we turn our attention to the radiation field within and transmitted by a multiple-scattering medium such as a cloud illuminated at its upper boundary by irradiance F_0 . The irradiance of the light that has *not* been scattered, upward or downward, is an exponential function of optical depth into the medium:

$$F_{\downarrow}^u = F_0 \exp(-\tau), \quad (5.59)$$

where the superscript “u” indicates unscattered. By definition the *diffuse* downward irradiance D_{\downarrow} is light that *has* been scattered, and hence is the difference between the total downward

irradiance and the unscattered downward irradiance:

$$D_{\downarrow} = F_{\downarrow} - F_{\downarrow}^u. \quad (5.60)$$

The upward irradiance is necessarily diffuse because it is light that must have been scattered. For the sake of harmonious notation, we denote F_{\uparrow} as D_{\uparrow} . With the assumption that radiation transmitted through the lower boundary of the medium ($\tau = \bar{\tau}$) is not returned [$D_{\uparrow}(\bar{\tau}) = 0$], the solution to Eq. (5.49), combined with Eq. (5.60), yields the two diffuse irradiances at any optical depth τ within the medium:

$$\frac{D_{\downarrow}}{F_0} = \frac{1 + (\bar{\tau} - \tau)(1 - g)/2}{1 + \bar{\tau}(1 - g)/2} - \exp(-\tau), \quad (5.61)$$

$$\frac{D_{\uparrow}}{F_0} = \frac{(\bar{\tau} - \tau)(1 - g)/2}{1 + \bar{\tau}(1 - g)/2}. \quad (5.62)$$

If we assume that the medium is optically thick [$\bar{\tau}(1 - g)/2 \gg 1$], the downward irradiance, Eq. (5.61), at $\tau = 2/(1 - g)$ is approximately

$$\frac{D_{\downarrow}}{F_0} = 1 - \exp\{-2/(1 - g)\}. \quad (5.63)$$

The argument of the exponential ranges from -1 to $-\infty$, so for optical depths greater than $2/(1 - g)$ it is always a good approximation to neglect the exponential term in Eq. (5.61):

$$\frac{D_{\downarrow}}{F_0} = \frac{1 + (\bar{\tau} - \tau)(1 - g)/2}{1 + \bar{\tau}(1 - g)/2}. \quad (5.64)$$

At optical depths into the medium greater than about $2/(1 - g)$ the total irradiance is dominated by the diffuse irradiance. From Eqs. (5.62) and (5.64) the two diffuse irradiances are approximately equal if

$$(\bar{\tau} - \tau)(1 - g)/2 > 1 \text{ or } \tau < \bar{\tau} - 2/(1 - g). \quad (5.65)$$

At (optical) distances less than $2/(1 - g)$ from the lower boundary, the black underlying medium [i.e., $D_{\uparrow}(\bar{\tau}) = 0$] makes its presence felt.

Thus within the region lying at least $2/(1 - g)$ from the upper and lower boundaries, the diffuse radiation field is approximately isotropic. This makes sense if we recognize that over the optical distance $\tau p_{\uparrow\downarrow} = \tau p_{\downarrow\uparrow} = \tau(1 - g)/2 = 1$, a photon is nearly certain to have been turned around. And recall that the radiation field was isotropic for a medium sandwiched between two mirrors (equilibrium solution). The medium to which Eqs. (5.61) and (5.62) apply is also sandwiched between two mirrors, but they are *distributed* over an optical thickness (approximately) $2/(1 - g)$ rather than being *localized* at the upper and lower boundaries. So with a bit of physical intuition and an understanding of $\tau(1 - g)/2$ we could have bypassed Eqs. (5.61) and (5.62) and gone directly to the final result, namely an isotropic radiation field within the core of a multiple-scattering medium.

You can observe Eqs. (5.61) and (5.62) unfolding before your eyes as you descend in an airplane into a thick cloud. At first you can tell (optically) up from down. The light

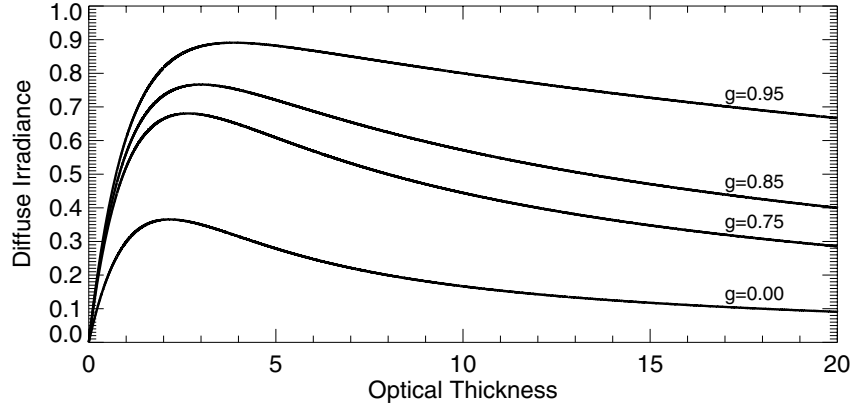


Figure 5.13: Diffuse irradiance beneath a plane-parallel, nonabsorbing medium as a function of its optical thickness. The higher values of the asymmetry parameter g are typical for clouds at visible and near-visible wavelengths; $g = 0$ is for molecular scattering.

above is brighter than that below. But as you descend farther you enter a world in which everything looks the same in all directions. Indeed, a former student of ours carried a small radiometer with him on an airplane flight and verified for himself that within a cloud the radiation field is more or less isotropic. Suspended in this featureless world, you may become a bit anxious, then breathe a sigh of relief when you finally notice darkness below, signaling that your airplane is about to leave the cloud.

The diffuse downward irradiance [Eq. (5.61)] *beneath* a cloud

$$\frac{D_{\downarrow}(\bar{\tau})}{F_0} = \frac{1}{1 + \bar{\tau}(1 - g)/2} - \exp(-\bar{\tau}) \quad (5.66)$$

rises steeply from 0 for an optical thickness $\bar{\tau} = 0$ to a maximum at approximately $\bar{\tau} = (1/g) \ln[2/(1 - g)]$ ($g > 0.5$), then gradually decreases for increasing optical thickness (Fig. 5.13). As with the isotropic radiation field within a cloud, you can readily observe Eq. (5.66) being played out in nature – and you don’t even have to board an airplane. The overhead sky (except, of course, directly toward the sun) is not nearly as bright as many clouds. This is evident when you observe patches of dark blue sky in the transitory openings in cloud decks (Fig. 1.9). Clouds increase the diffuse downward radiation – but only up to a point. With ever-increasing optical thickness clouds eventually reduce the downward radiation to less than what it would be from a clear sky. Clouds are both givers and takers of light. If not too thick, they can make the sky brighter than the clear sky, but very thick clouds result in a dark, gloomy day.

We sometimes encounter assertions that reflection by the ground can color the bottoms of clouds (e.g., the bottoms of clouds are green because of reflection by green grass). We can investigate these assertions within the framework of the two-stream theory by solving Eq. (5.49) subject to the lower boundary condition $F_{\uparrow}(\bar{\tau}) = R_s F_{\downarrow}(\bar{\tau})$, where R_s is the reflectivity of the

underlying surface:

$$D_{\downarrow}(\bar{\tau}) \approx F_{\downarrow}(\bar{\tau}) = \frac{F_0(1 - R)}{1 - RR_s}. \quad (5.67)$$

We used Eq. (5.51) for the reflectivity R over a non-reflecting surface to cast our solution in a simple form, easy to interpret: $1 - R$ is the transmissivity T of the cloud, $1/(1 - RR_s)$ is a consequence of multiple reflections between the cloud and the underlying ground. If we assume that $RR_s \ll 1$, we can expand the denominator in Eq. (5.67) and truncate the series after the second term:

$$D_{\downarrow}(\bar{\tau}) \approx F_0(1 - R)(1 + RR_s). \quad (5.68)$$

The spectrum of the diffuse downward radiation beneath the cloud is therefore not that of the ground [$F_0(1 - R)R_s$] but rather diluted, so to speak, as a consequence of multiplication by R and addition of 1. From this result it is difficult to fathom how reflection by colored ground could do more than tinge the light from the bottoms of more or less continuous clouds. Broken clouds are another matter (see Prob. 5.38).

Now consider an extreme case: $R_s = 1$ over some narrow band of wavelengths and 0 at other wavelengths. Over this band the downward irradiance [Eq. (5.67)] is F_0 , whereas outside this band is $F_0(1 - R)$. So indeed there could be a spike in the spectrum over the narrow band of wavelengths if $R \approx 1$. This result required invoking reflection by ground that does not generally exist in nature at visible wavelengths. But we must be careful here not to extrapolate results from the visible to other wavelengths (e.g., the near-infrared) where surface reflectivity may be much greater. Alexander Marshak and his colleagues have demonstrated that the increase in the reflectivity of vegetated surfaces from visible (0.65 μm) to near-infrared wavelengths (0.80 μm) is sufficient to give appreciable difference in the downward radiances from cloud base at these two wavelengths (see Prob. 5.35). Nevertheless, we conclude that if the bottoms of continuous clouds are markedly colored at visible wavelengths, we ought to look to causes other than reflection by the ground. One such cause, (selective) absorption, is briefly mentioned at the end of the previous section. Now we elaborate on this.

5.3 Multiple Scattering in an Absorbing Medium

Up to this point we have assumed that absorption is negligible and taken κ to be identically zero. But an absolutely nonabsorbing medium does not exist. What about free space? It, too, does not exist. Even the vast reaches of interstellar space are populated sparsely with molecules and particles. If you know how to make an absolute vacuum, run, do not walk, to the Patent Office. Moreover absorption is rarely if ever absolutely negligible: it may be negligible for some purposes but not others. So the term “nonabsorbing” should be read as shorthand for “negligibly absorbing for our particular purposes.” In this section we do not neglect absorption.

If we assume that ϖ and g are independent of τ and differentiate Eq. (5.47) with respect to τ , then substitute Eq. (5.48) in the resulting equation, and vice versa, we obtain two second-

order differential equations of the form

$$\frac{d^2 F}{d\tau^2} = K^2 F, \quad (5.69)$$

where

$$K = \sqrt{(1 - \varpi)(1 - g\varpi)} \quad (5.70)$$

and F is either the sum or difference of F_\uparrow and F_\downarrow . As a rule, differentiating differential equations to obtain ones of higher order is jumping out of the frying pan into the fire. But here the result was an equation [Eq. (5.69)] the solutions to which are simple exponential functions $A \exp(\pm K\tau)$, which can be verified by differentiating them twice; A is a constant of integration determined by boundary conditions. For simplicity, take the medium to be infinite, which means that for the irradiances to be finite we have to exclude solutions with positive exponent. Subject to the boundary condition that the downward irradiance at $\tau = 0$ is F_0 and the requirement that the sum and differences of the two irradiances be linked by Eq. (5.47) or, equivalently, by Eq. (5.48), the solution to Eq. (5.69) for an infinite medium is

$$F_\downarrow = F_0 \exp(-K\tau), \quad F_\uparrow = F_0 R_\infty \exp(-K\tau), \quad (5.71)$$

where the reflectivity $R_\infty = F_\uparrow(0)/F_0$ is

$$R_\infty = \frac{\sqrt{1 - \varpi g} - \sqrt{1 - \varpi}}{\sqrt{1 - \varpi g} + \sqrt{1 - \varpi}}. \quad (5.72)$$

As a check on the correctness of this solution, note that $R_\infty = 0$ when $g = 1$, which is what we expect on physical grounds: if scattering is entirely downward incident photons cannot contribute to the reflected (upward) irradiance.

Unlike in a nonabsorbing (infinite) medium, the radiation field in an absorbing medium is not isotropic, the upward irradiance being less than the downward irradiance, although the radiation field is approximately isotropic if R_∞ is close to 1. Where the really striking differences occur, however, is in attenuation. According to Eq. (5.71) both irradiances are attenuated exponentially with (optical) depth. For ease of physical interpretation it will be more convenient to transform to physical depth z . If we take κ , β , and g to be independent of z we can write

$$K\tau = \sqrt{\kappa\{\kappa + \beta(1 - g)\}} z = \alpha z, \quad (5.73)$$

where we call α the *attenuation coefficient*. Note that K is dimensionless whereas α has the dimensions of inverse length. A necessary and sufficient condition for attenuation is that κ be nonzero, and hence attenuation is a consequence of absorption but with a twist: scattering amplifies this absorption. If $\beta(1 - g) = 0$, $\alpha = \kappa$, as we expect. But suppose that $\beta(1 - g)$ is not zero, indeed, suppose that it is much larger than κ . In this instance α can be considerably larger than κ :

$$\alpha = \kappa \sqrt{1 + \frac{\beta(1 - g)}{\kappa}} \approx \kappa \sqrt{\frac{\beta(1 - g)}{\kappa}} \approx \kappa \sqrt{\frac{1 - g}{1 - \varpi}}. \quad (5.74)$$

Another way of looking at this is that attenuation at physical depth \bar{z} in a medium for which $\beta(1 - g) \gg \kappa$ is the same at depth $z > \bar{z}$ in a medium for which $\alpha = \kappa$:

$$\bar{z} \approx z \sqrt{\frac{1 - \varpi}{1 - g}}. \quad (5.75)$$

Multiple scattering increases photon path lengths, and the longer the path, the greater the chance of absorption.

Perhaps the most striking difference between a nonabsorbing and an absorbing medium is in the evolution of the transmitted spectrum with depth. We showed previously that the radiation field in an infinite nonabsorbing medium is isotropic and constant with depth. From Eq. (5.52) the spectrum of the radiation transmitted by a finite nonabsorbing medium of physical thickness z is (for sufficiently large optical thicknesses)

$$F_{\downarrow}(z) \approx \frac{2F_0}{\beta(1 - g)z}, \quad (5.76)$$

if β is independent of z , whereas the spectrum of the downward irradiance at depth z in the infinite absorbing medium is

$$F_{\downarrow}(z) \approx F_0 \exp(-\alpha z). \quad (5.77)$$

Although the *magnitude* of the irradiance in Eq. (5.76) decreases with depth, the *shape* of the spectrum is invariant: F_0 modulated by the wavelength variation of $1/\{\beta(1 - g)\}$ at all depths. But in the absorbing medium the shape of the transmitted spectrum can change markedly with depth. For simplicity take F_0 to be constant. It follows from Eq. (5.77) that the ratio of the irradiance at the wavelength for which attenuation is a minimum (over the range of wavelengths of interest) to the irradiance at any other wavelength is

$$\exp\{(\alpha - \alpha_{\min})z\}, \quad (5.78)$$

where α_{\min} is the minimum value of the attenuation coefficient. Because $\alpha > \alpha_{\min}$ the limit of this ratio as z becomes indefinitely large is infinity. Thus with increasing depth the transmitted spectrum is dominated more and more by light at and around the wavelength of minimum attenuation (of course, at large depths the magnitude of the irradiance may be such that the light is imperceptible). This makes sense given that the light surviving to a depth z is that which has *not* been absorbed. The wavelength dependence of attenuation can be a consequence of the wavelength dependence of κ , of β , or of g , or of any two of these quantities, or of all three [Eq. (5.74)], although in the examples we consider in following sections the wavelength dependence of α is essentially that of κ .

Now let us contrast the spectra of reflected and transmitted light. In the spectrum of transmitted light one restriction on the medium is that κ is not zero. With the additional restriction that $\kappa \ll \beta(1 - g)$ (implying $\beta \gg \kappa$ and ϖ close to 1), the reflectivity R_{∞} [Eq. (5.72)] is approximately

$$R_{\infty} \approx 1 - 2\sqrt{\frac{\kappa}{\beta(1 - g)}}. \quad (5.79)$$

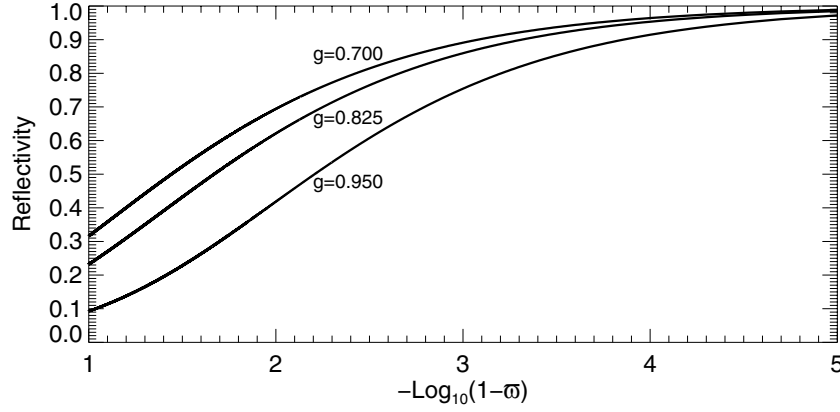


Figure 5.14: Reflectivity of an absorbing, plane-parallel medium of infinite depth for various values of the asymmetry parameter g . The horizontal axis is the negative logarithm of the ratio of absorption to absorption plus scattering.

It follows that R_∞ is close to 1, and hence the spectrum of the reflected light is essentially that of the incident light. This is much different from our result in the previous paragraph that the spectrum of light transmitted sufficiently deep in the medium can be markedly different from the incident spectrum. Why the difference?

To resolve this we turn to Eq. (5.51) for the reflectivity of a finite, nonabsorbing medium of physical thickness h , according to which the reflectivity is within 10% of its asymptotic value (1) when $\beta(1-g)h$ is about 20. This in turn implies that the reflectivity of the infinite absorbing medium [with $\kappa \ll \beta(1-g)$] is a consequence mostly of scattering within a distance h of the upper boundary, where

$$h = \frac{20}{\beta(1-g)}. \quad (5.80)$$

How much attenuation occurs in this layer? According to Eqs. (5.71) and (5.74) attenuation over a distance h is determined by the negative exponential of

$$\sqrt{\kappa\beta(1-g)} h = 20\sqrt{\frac{\kappa}{\beta(1-g)}}. \quad (5.81)$$

Thus if $\kappa/\beta(1-g)$ is sufficiently small, so is the argument [Eq. (5.81)] of the exponential function, and hence attenuation is negligible. As far as reflection is concerned most of the action occurs in a layer near the surface sufficiently thin that few incident photons traversing this layer are absorbed, whereas the deeper they penetrate into the medium the fewer escape absorption.

Before turning to particular observations you can make for yourselves that will breathe life and meaning into these mathematical results we consider some general conclusions that

follow from Eq. (5.72). The relative change of R_∞ with respect to ϖ ($g \neq 1$)

$$\frac{1}{R_\infty} \frac{\partial R_\infty}{\partial \varpi} = \frac{1}{\varpi \sqrt{1 - \varpi} \sqrt{1 - \varpi g}} \quad (5.82)$$

at $\varpi = 1$ ($R_\infty = 1$) is infinite. Thus the addition of a small amount of absorption (relative to scattering) to a nonabsorbing medium results in a relatively much greater decrease in its reflectivity: the purer you are, the more easily your corruption is evident for all to see. Equation (5.79) suggests that we should plot R_∞ versus the logarithm of $1 - \varpi$ (Fig. 5.14). We may interpret $1 - \varpi$ as the probability that a photon is absorbed in a single interaction with a molecule or particle, whereas $1 - R_\infty$ is the probability of absorption in many such interactions. From Fig. 5.14 it follows that even when the former is quite small, the latter can be many times greater. Once again we see that, as with the pile of absorbing plates (Sec. 5.1), a little bit of absorption goes a long way in a multiple-scattering medium.

5.3.1 Clouds, Snow, Paint, Frozen Waterfalls, Wet Sand, and Broken Beer Bottles

First, let's apply some of the results in the preceding section to clouds. For simplicity consider a uniform plane-parallel cloud of thickness h_c composed of N water droplets per unit volume, all with the same radius a , per unit volume of cloud. The total volume of water in this cloud per unit (horizontal) cross-sectional area is

$$Nvh_c = fh_c = h_w, \quad (5.83)$$

where v is the volume of a single cloud droplet, $f = Nv$ is the volume fraction of the cloud occupied by liquid water, and h_w is the liquid water path of the cloud expressed as the depth of water that would result if the cloud were compressed into a continuous slab of water. A large value for h_w is of order centimeters, and even this puny amount requires clouds thousands of meters thick. Thus the volume fraction f of water in clouds is quite small, of order 10^{-6} . Despite their apparent solidity, clouds are mostly air. It follows from Fig. 5.12 that a slab of liquid water (or ice) a few centimeters thick is insufficient to markedly attenuate visible light of any wavelength. But, as we noted previously, multiple scattering can greatly amplify photon paths. To find out how much we have to estimate the magnitude of the product of the attenuation coefficient α [Eq. (5.74)] and h_c . Because the single-scattering albedo for cloud droplets is very close to 1 over the visible spectrum (Fig. 5.15) we have

$$\alpha h_c \approx \kappa h_c \sqrt{\frac{\beta(1 - g)}{\kappa}}. \quad (5.84)$$

The absorption and scattering coefficients are

$$\kappa = NC_{\text{abs}}, \quad \beta = NC_{\text{sca}}, \quad (5.85)$$

where C_{abs} is the absorption cross section of a droplet and C_{sca} its scattering cross section (Secs. 2.8 and 3.5). Cloud droplets are large compared with the wavelengths of visible light

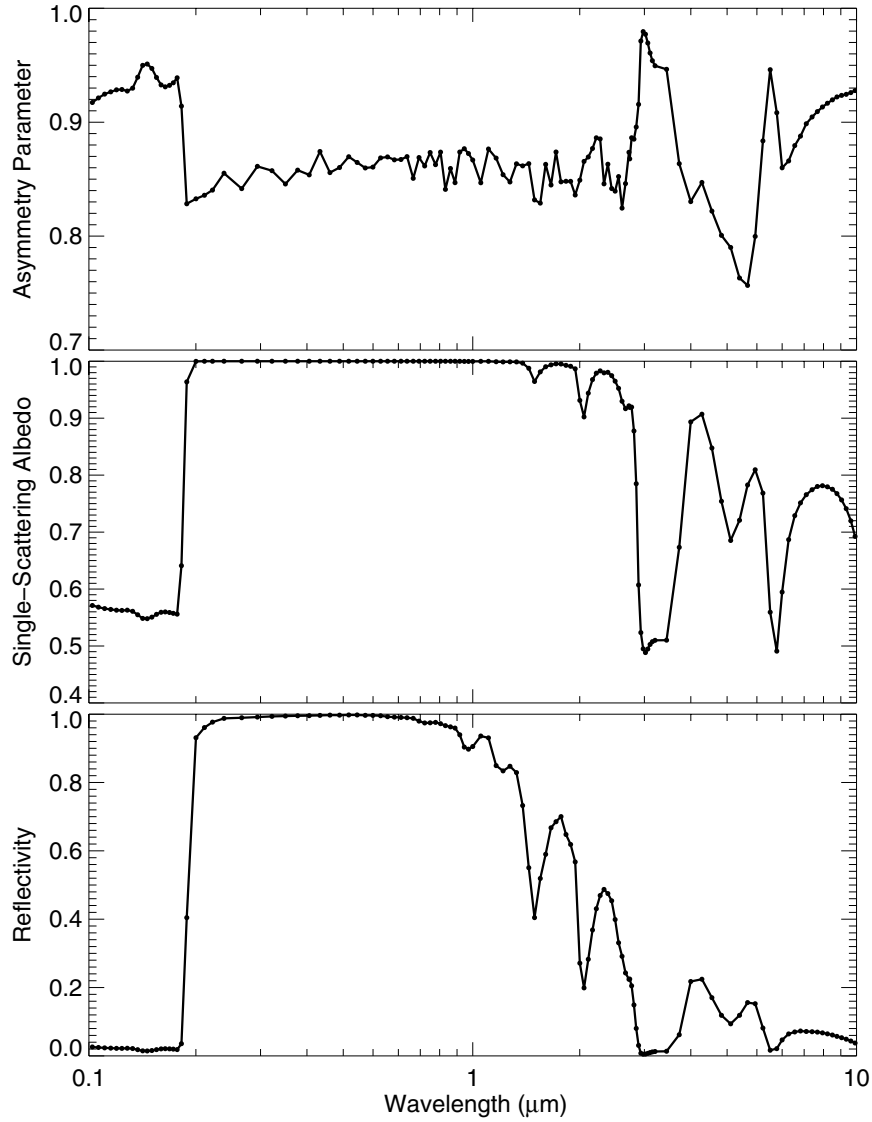


Figure 5.15: Asymmetry parameter and single-scattering albedo, from UV to IR, of a water droplet 10 μm in diameter. The reflectivity, from Eq. (5.72), is for an infinite medium composed of water droplets this size.

and weakly absorbing in that the product of droplet diameter and the absorption coefficient of water, κ_w , is much less than 1. Because of this the absorption cross section is approximately $\kappa_w v$, where v is the droplet volume (Sec. 2.8). This expression is not exact but is good enough for our purposes. The scattering cross section is approximately $2\pi a^2$. With these

approximations and a bit of algebra Eq. (5.84) can be written

$$\alpha h_c \approx f h_c \kappa_w \sqrt{\frac{3(1-g)}{\kappa_w d}} = h_w \kappa_w \sqrt{\frac{3(1-g)}{\kappa_w d}}, \quad (5.86)$$

where d is the droplet diameter. Although we took all droplets to be the same size (not true for most clouds, especially those with appreciable liquid water paths), the form of Eq. (5.86) is unchanged even if the droplets are distributed in size. All we need do is interpret d as the ratio $\langle d^3 \rangle / \langle d^2 \rangle$, where $\langle \rangle$ indicates an average over the distribution. Thus the cloud is equivalent (as far as attenuation is concerned) to a slab of water of thickness

$$h_w \sqrt{\frac{3(1-g)}{\kappa_w d}}. \quad (5.87)$$

Over the visible spectrum κ_w varies but is of order 0.1 m^{-1} (Fig. 5.12), whereas d is of order $10 \mu\text{m}$ (10^{-5} m) and $3(1-g)$ for cloud droplets is around 0.5. Thus the equivalent thickness of the cloud is amplified by nearly a factor of 1000, and a cloud with liquid water path 1 cm gives the same attenuation as a slab of water about 10 m thick. This much water *can* appreciably attenuate visible light. Because absorption by pure liquid water has a minimum in the blue-green part of the visible spectrum (Fig. 5.12), the bottoms of very thick clouds can (and do) have a bluish cast even though the spectrum of visible incident sunlight is nearly unchanged upon reflection. Although these conclusions are based on the assumption that clouds are composed of liquid water, the same is true for clouds composed of ice particles or of mixtures of both because the visible absorption coefficients for ice and liquid water are nearly the same (Fig. 5.12). But let's turn now to a medium that is composed of ice particles, snow, by which we mean a snowpack, snow on the ground rather than in the air.

Snow and Paint

Christmas cards notwithstanding, snowflakes do not long retain their intricate dendritic shapes, if they ever had them, after deposition but instead metamorphose into more or less rounded grains. Applying the radiative transfer theory in this book to snow has been criticized or, at the very least, has evoked astonishment that it agrees with measurements, because the grains in snow are touching. This is largely a red herring. For radiative transfer theory to be valid for an ensemble of particles requires that some characteristic length be *large compared with the wavelength*, as we saw in the discussion of Fig. 5.9. To expand on this point we first note that a particulate media, and here the particles could be molecules, is optically homogeneous if it contains many particles in a cubic wavelength. For N identical particles per unit volume this condition is

$$N \lambda^3 = N v \frac{\lambda^3}{v} = f \left(\frac{\lambda}{d} \right)^3 \gg 1, \quad (5.88)$$

where v is the volume of a particle and d its cube root. This condition is satisfied for air. Snow grains are of order 1 mm; the volume fraction f for snow varies but a high value would be

around 0.5. Thus the condition Eq. (5.88) is not satisfied for snow at visible wavelengths, and so it is not optically homogeneous.

But what is the condition under which scattered power is additive and hence the scattering coefficient increases linearly with N ? This is difficult to answer with certainty, although on the basis of physical reasoning we expect that we can add scattered powers when some characteristic linear dimension is large compared with the wavelength. According to the kinetic theory of gases, the collision mean free path ℓ of a molecule is $1/N\sigma_c$, where N is the number of molecules per unit volume and σ_c is the collision cross section of a molecule. Note that this mean free path is formally identical to the mean free photon paths discussed in Section 5.1.3. The molecular mean free path is the average distance a molecule travels before encountering another molecule. Although molecules are in motion, this is largely irrelevant. We can imagine all the molecules to be frozen in place. On average, ℓ is the distance one of these molecules can be translated before encountering another molecule. We can extend this to an ensemble of particles, which may be stationary, and take

$$\ell = \frac{1}{N\sigma_c} = \frac{v}{Nv\sigma_c} = \frac{d}{f} > \lambda \quad (5.89)$$

as the condition for the scattering coefficient to be proportional to N . The collision cross section is approximately the geometrical cross section d^2 . Equation (5.89) is approximately satisfied for air, the scattering coefficient for which is indeed proportional to N (see Sec. 3.4.9). And at visible wavelengths the condition Eq. (5.89) is certainly satisfied for snow.

But if snow grains were comparable with or smaller than the wavelengths of visible light, radiative transfer theory might not give such good results. Indeed, this is what happens in paint films. High-quality white paint is a suspension of titanium dioxide particles. The chapter entitled “Titanium” in Primo Levi’s *Periodic Table* begins with a “tall man” painting a closet white. A woman watching him asks, “Why is it so white?” After thinking for a while he replies, “Because it is titanium.” Titanium dioxide is inert, absorption by its particles is negligible at visible wavelengths, and its high refractive index yields considerable scattering per particle. This is what efficient opacifiers (i.e., whiteners) are intended to do. Titanium dioxide particles are used in all kinds of manufactured white objects: plastic cutlery, deck chairs – the list goes on and on. The titanium dioxide particles in paint are comparable with the wavelengths of visible light, about 0.2 μm , a size chosen to maximize the scattering cross section per unit particle volume.

According to Eq. (5.51) for the reflectivity of a nonabsorbing medium of thickness h , the reflectivity of a paint film daubed onto a black surface should steadily increase with particle volume fraction. To make this clear we rewrite Eq. (5.51) as

$$R = \frac{fh(1-g)(C_{\text{sca}}/v)}{2 + fh(1-g)(C_{\text{sca}}/v)}, \quad (5.90)$$

where the scaled scattering cross section per unit volume $(1-g)C_{\text{sca}}/v$ is fixed for a given particle. This equation shows why one wants to use particles that maximize $(1-g)C_{\text{sca}}/v$ for a fixed fh , the amount of material, per unit area, of the film, which is what you pay for. According to this equation R should increase steadily with increasing volume fraction (i.e., particle concentration). And this is what happens – at first. But as the concentration increases

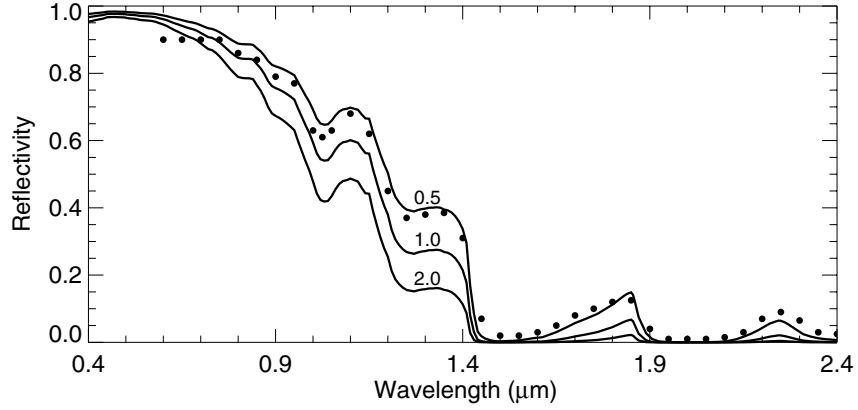


Figure 5.16: Reflectivity of snow calculated from the two-stream theory for three grain diameters (mm). The solid circles are measurements for snow of unspecified grain size made by O’Brien and Munis (1975).

a point is reached where the reflectivity begins to *decrease*. The paint folks refer to this as “crowding”, but there is just as much crowding in snowpacks as in paint films. The difference between them lies not in their volume fractions, which are similar, but in their particle sizes. For ice grains in snow this is many times the wavelengths of visible light, whereas for paint particles it is comparable with or even appreciably less.

Now that we understand why radiative transfer theory *should* work for snow, let’s show that it *does* by comparing theory with measurements and observations. From Eq. (5.79) and the same arguments that led to Eq. (5.86), the reflectivity of snow is approximately

$$R_{\infty} \approx 1 - 2\sqrt{\frac{\kappa_i d}{3(1-g)}}, \quad (5.91)$$

where κ_i is the absorption coefficient of ice and d is the grain diameter. Now we cheat (slightly) and “reverse approximate” Eq. (5.91):

$$R_{\infty} \approx \exp\left(-2\sqrt{\frac{\kappa_i d}{3(1-g)}}\right). \quad (5.92)$$

This equation is essentially the same as the previous one for small values of the argument but has the advantage of never giving negative reflectivities, whereas Eq. (5.91) pushed too far will. But when the reflectivity is sufficiently small we don’t care what its exact value is, and hence Eq. (5.92), although incorrect, is good enough for our purposes. Figure 5.16 shows R_{∞} calculated using Eq. (5.92) over the solar spectrum from 0.4 μm to 2.5 μm for various grain sizes. At visible wavelengths R_{∞} is close to 1 and nearly independent of wavelength. That is, snow is white (no surprise here), which we can see with our own eyes. What we can’t

see is the steep plunge of reflectivity beyond about $1.0 \mu\text{m}$ where absorption by ice greatly increases. As predicted by Eq. (5.92) the wiggles and bumps in the reflection spectrum of snow faithfully follow those in the absorption spectrum of ice (Fig. 2.2). Calculated reflection is consistent with measurements except that theory usually predicts higher values over the visible than those measured. This is likely a consequence of trace amounts of contaminants such as soot (a catchall name for the carbonaceous products of combustion). At visible wavelengths absorption by soot is around a million times greater than that by ice, so soot in snow in parts per million can lower its visible reflectivity. In the infrared, however, ice is sufficiently absorbing that the addition of a bit of soot to snow does not markedly change its reflectivity.

Figure 5.16 supports an assertion about snow that at first glance seems the product of a disturbed mind: snow is both the whitest natural substance on earth and the blackest. At visible wavelengths the reflectivity of clean, fine-grained snow is as high as that of anything you'll find in nature, whereas well into the infrared its reflectivity plunges to near zero. We are easily led astray by extrapolating outside the narrow range of visible wavelengths to which we are sensitive.

Frozen Waterfalls

Although snow is white, bubbly ice may not be. The scatterers in snow are ice grains suspended in air whereas the scatterers in bubbly ice are air grains suspended in (absorbing) ice. We can approximate the absorption coefficient of bubbly ice as $\kappa_i(1 - f)$, where f is the volume fraction of bubbles. By the same arguments that yielded Eqs. (5.91) and (5.92) the reflectivity of bubbly ice is

$$R_\infty = \exp\left(-2\sqrt{\frac{\kappa_i d(1 - f)}{3f(1 - g)}}\right), \quad (5.93)$$

where d is the bubble diameter. This equation, compared with Eq. (5.92), tells us that bubbly ice is equivalent to snow with an effective grain diameter

$$d_{\text{eff}} = \frac{(1 - f)d}{f}, \quad (5.94)$$

which makes physical sense. If f is low, say 0.01, the effective grains in bubbly ice are 100 times larger (for equal d) than the ice grains in snow. Because of the spectral dependence of κ_i (Fig. 5.12), light reflected by bubbly ice can have a bluish or blue-green cast. You can readily observe this if you live in a part of the world with real winters. Frozen waterfalls (icefalls) on road cuts are often perceptibly blue (Fig. 5.17). Moreover, this blueness is made more striking if the icefalls are partly covered by snow. You can see, side by side, the visual consequences of Eqs. (5.92) and (5.93): darker, bluish bubbly ice flanked by brighter, white snow.

Snow Holes and Crevasses

Although snow is usually white seen by reflection, the light transmitted into snow and other natural ice bodies is sometimes blue, the purity of which increases with increasing depth. The deep blueness of holes in snow, crevasses, and ice caves has been commented on by

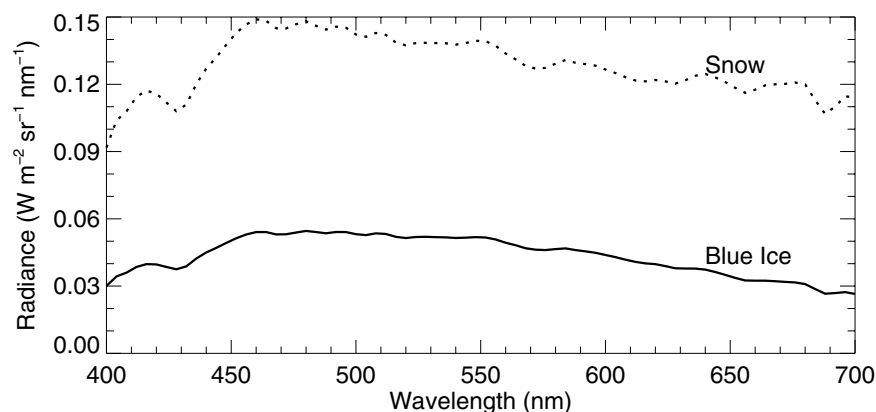


Figure 5.17: The measured radiance of blue ice is appreciably less than that of nearby snow. Although not immediately obvious from these measurements, the ice spectrum is slightly richer in shorter wavelength radiation. The ratio of blue (472 nm) to red (632 nm) radiances for the snow is about 1.2, whereas that for the ice is about 1.4; this difference is readily perceptible.

many writers. In 1923 Raman published a paper in which he attributed the blueness of ice to molecular scattering, that is, scattering described by Rayleigh's law, and such is Raman's prestige that this explanation, which is not difficult to demolish, is still widespread today.

Raman's argument, in essence, is that the scattering coefficient β of snow has the inverse fourth power of wavelength dependence of molecular scattering and that absorption plays no role. But the solution to Eq. (5.49) for an infinite (i.e., optically thick), nonabsorbing medium is a uniform, isotropic irradiance equal to the incident irradiance everywhere in the medium. Snow is often optically thick in the sense that not much need accumulate before its reflectivity does not change with a further increase in thickness. As a result, the reflectivity of the underlying surface is irrelevant. But suppose that the snow is not optically thick. Now Eq. (5.51) yields the reflected irradiance for an optically thin medium which is proportional to β [strictly, to $\beta(1-g)$, although for molecular scattering $g = 0$]. Also, Eq. (5.76) predicts that for a thick medium, the downward (transmitted) irradiance is *inversely* proportional to β and the shape of the transmitted spectrum does not change with increasing physical depth. Thus if the scattering coefficient of snow were the molecular scattering coefficient, the light reflected by a thin layer of snow should have the blueness of the sky and the light transmitted by a thick layer should be reddish but with no change in purity with depth. This is not observed. A word of caution is in order before we proceed. Snow in shadow sometimes is perceived to be blue when the dominant source of illumination is the blue sky. You can amuse yourselves by poring over art books or strolling through an art museum and carefully examining snowscapes. Artists usually paint shadows on snow blue, but of a purity not observed in nature, presumably to trumpet that they know shadows on snow can be blue.

The observation that the light in snow holes becomes an increasingly purer blue with increasing depth indicates that absorption must play a role, as evident from Eq. (5.71), which

shows an exponentially changing spectrum of both the upward and downward irradiances. For Raman's theory to have even a hope of being correct, the attenuation coefficient [Eq. (5.73)] must be dominated by β (with $g = 0$), and hence is given by

$$\alpha \approx \sqrt{\kappa\beta}. \quad (5.95)$$

But if β is the molecular scattering coefficient (and κ is independent of wavelength), attenuation is greatest at the short wavelength end of the visible spectrum and least at the long wavelength end. We therefore predict that the light transmitted into snow and other natural ice bodies should be a red the purity of which increases with increasing depth, which is contrary to what is observed.

Where did Raman go wrong in his explanation of blue ice resulting from molecular scattering? It is indeed true that the water molecules in ice – indeed, the molecules in anything – do the scattering, but the nature of this scattering depends on the distance over which they are correlated. Scattering by a single molecule is indeed proportional to inverse fourth power of wavelength, but this does not mean that an ice grain composed of many such molecules scatters this way. A grain is a coherent object: every molecule in it is correlated with the position of every other molecule over a characteristic distance, the size of the grain. The total light scattered by the grain is the superposition of all the waves scattered by its constituent molecules taking due account of the phase differences between them, which in turn depend on the size of the grain relative to the wavelength. This additional wavelength dependence of the scattering by a coherent object, if sufficiently large, is what effaces the wavelength dependence of the scattering by its individual molecules. Stated another way, scattering is a consequence of heterogeneity, be it on the molecular scale or larger. And in snow or bubbly ice scattering as a consequence of heterogeneity on the scale of grains or bubbles swamps scattering as a consequence of heterogeneity on the molecular scale.

This provides a beautiful example of how a medium can be both coherent and incoherent. A single ice grain (or cloud droplet) is a coherent scatterer in the sense that to determine the light scattered by such a scatterer one *must* take into account the phase differences between the waves scattered by the individual molecules. But a snowpack (or a cloud) is an incoherent scatterer in the sense that to determine the light scattered by many grains (or droplets) one *may* ignore the phase differences between the waves scattered by the individual grains.

Wet Sand

A common observation is that when sand and many other porous media are wet with water or other liquids they become darker. Why? To answer this let's first pretend that the sand is nonabsorbing and overlies a black substrate. From Eq. (5.51) the relative change in R with respect to g is

$$\frac{1}{R} \frac{\partial R}{\partial g} = \frac{-1}{(1-g)[1 + \bar{\tau}(1-g)/2]} \quad (5.96)$$

and hence R always decreases with increasing g . In the limit $g \rightarrow 1$, Eq. (5.96) approaches infinity. Thus the biggest change in reflectivity occurs when g is largest. Sand grains are much larger than the wavelengths of visible light. Scattering by such particles is highly peaked in the

forward direction. But although total (all directions) scattering is independent of the medium in which the sand grains are suspended the angular distribution of this scattering (i.e., g) is not. In particular, the more closely the surrounding medium optically matches the particles, the more scattering is peaked in the forward direction (g closer to 1). Thus we expect g for sand grains in water (wet sand) to be greater than for sand grains in air (dry sand), which from Eq. (5.96) implies a lower reflectivity. As g increases photons are more likely to penetrate to the black substrate and be absorbed there. But how do we know it isn't absorption by the water in a multiple-scattering medium that causes the reflectivity to decrease? Wet sand (composed of nonabsorbing grains in absorbing water) is similar to bubbly ice, and so we can apply Eq. (5.93) to such sand, where d is the size of the sand grains and κ_i is the absorption coefficient of liquid water (at visible wavelengths similar to that of ice). Sand grains are comparable in size to snow grains and the volume fraction f of grains in snow (as opposed to bubbles in ice) is of order 0.5, which yields only a small increase in effective size [Eq. (5.94)]. So if dry sand (optically thick) has a high reflectivity, that of the same sand when wet should not change much because of absorption by water.

Real sand is not strictly nonabsorbing. We once bought some clean, fine-grained aquarium sand at a pet shop. To our eyes it had a pastel reddish-orange cast. When we examined it under a microscope we discovered that it was composed mostly of large, transparent quartz grains seasoned with much smaller reddish-orange grains (presumably iron oxide). Scattering therefore was dominated by the quartz grains, absorption by the iron oxide grains. From Eq. (5.72) the relative change of R_∞ with g for an optically thick absorbing medium ($\varpi \neq 1$) is

$$\frac{1}{R_\infty} \frac{\partial R_\infty}{\partial g} = \frac{-\sqrt{1-\varpi}}{(1-g)\sqrt{1-\varpi g}}. \quad (5.97)$$

As with a nonabsorbing (finite) medium the greatest relative decrease in R_∞ occurs for g close to 1 (for fixed ϖ). The only difference between a finite nonabsorbing medium overlying a black surface and an optically thick (effectively infinite) absorbing medium is that in the former absorption occurs outside the medium whereas in the latter it occurs inside. For both media a decreased reflectivity upon wetting is a consequence of (single) scattering peaking more sharply in the forward direction. The experimental test of this is to wet sand with various transparent liquids of different refractive index ("transparent" means that you can see through a bottle of the liquid). The more closely the liquid matches the grains that dominate scattering, the darker the sand.

Broken Beer Bottles

And now we finally arrive at broken beer bottles. Many beer bottles are colored, some brown, some green. Indeed, it may very well be that only colored bottles contain beer worth drinking, but we leave that for readers to determine experimentally. Suppose that we recklessly, even violently, toss lots of green beer bottles into a dumpster, breaking most of them, with the result that the dumpster is piled with glass shards. This pile will still be seen as green. Now take a hammer and smash the shards more and more. We eventually end up with ground glass, but it is white, not green. What happened? Suffice it to say that pulverizing beer bottles doesn't change the chemical composition of the glass but it does decrease the size of the glass particles.

Suppose that $d (\gg \lambda)$ is a characteristic linear dimension of a glass particle (grain) and that κ_g , the absorption coefficient of bulk glass (not of glass particles), is such that $\kappa_g d \ll 1$. You can verify that this condition is satisfied by observing that a sliver of glass shows no color. With these restrictions the absorption cross section of a glass grain is proportional to its volume (d^3) and κ_g , and its scattering cross section is much larger and proportional to d^2 . It follows from Eqs. (5.72) and (5.79), also Fig. 5.14, that the reflectivity of an optically thick medium decreases with decreasing $1 - \varpi$, which for glass grains is

$$1 - \varpi = \frac{\kappa}{\kappa + \beta} \approx \frac{\kappa}{\beta} \approx C \frac{\kappa_g d^3}{d^2} = C \kappa_g d, \quad (5.98)$$

where C is all the constants of proportionality swept together. Absorption by a grain decreases more rapidly with decreasing size than does scattering, a consequence of which is that R_∞ increases with decreasing size.

5.4 Emission

Up to this point we have assumed that emission within a medium is negligible whereas scattering is not. Now we do the reverse: ignore scattering but allow for emission. The medium of interest could be the clear troposphere or a cloud. Around the wavelengths ($\sim 10 \mu\text{m}$) at which the Planck function has a maximum at typical terrestrial temperatures ($\sim 300 \text{ K}$), scattering by air is negligible compared with absorption. This follows from extrapolating the scattering optical thickness of the atmosphere as a function of wavelength (Fig. 8.2) into the infrared. Because scattering by air molecules is approximately inversely proportional to wavelength to the fourth power, the scattering optical thickness of the atmosphere at $10 \mu\text{m}$ is 10^4 times *smaller* than at $1 \mu\text{m}$. And the single-scattering albedo for a typical cloud droplet plunges from visible to infrared wavelengths (Fig. 5.15).

Before proceeding we need Kirchhoff's law *within* a medium. All the assumptions in Section 1.4 underlying the derivation of this law apply here. Consider a thin slab of thickness Δz and area A (Δz much less than the dimensions of A) in an opaque cavity at temperature T . We assume that Δz is sufficiently small that the emissivity of the slab is proportional to Δz : $\varepsilon = \chi \Delta z$. That is, the emissivity is linear in the amount of matter as long as the matter has little chance of absorbing the radiation it emits. This is the first term in a Taylor series expansion of the emissivity in powers of Δz . At equilibrium, emission by the slab (neglecting the edges) at a given frequency is balanced by absorption of incident radiation:

$$2P_e \chi \Delta z A = 2P_e \kappa \Delta z A, \quad (5.99)$$

where P_e is the Planck function (Sec. 1.2). The factor 2 arises because the slab emits in and is illuminated from two directions. From this equation it follows that

$$\chi = \kappa, \quad (5.100)$$

which is closer to the original Kirchhoff's law than the form obtained in Section 1.4. His form was that the ratio of what he called the emissive power (our χP_e) to what he called the absorptive power (our κ) is a universal function of wavelength and temperature, which much later

was shown to be the Planck function. Kirchhoff's terminology was poorly chosen because the dimensions of these two quantities are not the same and κ is solely a material property whereas χP_e is not. Planck called χP_e the coefficient of emission and κ the coefficient of absorption, terminology as objectionable as Kirchhoff's and for the same reasons. Equation (5.100) is what is needed to determine upward and downward emission by a thin slab within a medium:

$$\kappa \Delta z P_e. \quad (5.101)$$

First consider the downward irradiance, the governing equation for which is Eq. (5.40) with the scattering terms on the right side removed and an emission term [from Eq. (5.101)] added:

$$\frac{dF_{\downarrow}}{dz} = -\kappa F_{\downarrow} + \kappa P_e \{T(z)\}, \quad (5.102)$$

where $T(z)$ is the absolute temperature profile in the medium. As before we transform from physical depth z to (absorption) optical depth τ [Eq. (5.44)], although we omit the subscript a :

$$\frac{dF_{\downarrow}}{d\tau} + F_{\downarrow} = P_e \{T(\tau)\}. \quad (5.103)$$

At the top of the medium ($\tau = 0$) the downward irradiance is zero (no external source of radiation). Multiply both sides of Eq. (5.103) by $\exp(\tau)$:

$$e^{\tau} \frac{dF_{\downarrow}}{d\tau} + e^{\tau} F_{\downarrow} = \frac{d}{d\tau} e^{\tau} F_{\downarrow} = e^{\tau} P_e \{T(\tau)\} \quad (5.104)$$

and integrate the result from the top of the medium to its bottom ($\bar{\tau}$):

$$F_{\downarrow}(\bar{\tau}) = \int_0^{\bar{\tau}} \exp\{-(\bar{\tau} - \tau)\} P_e \{T(\tau)\} d\tau. \quad (5.105)$$

The interpretation of this equation is so straightforward that we could have obtained it without solving any differential equation. At each optical depth τ an amount of radiation per unit optical depth $P_e \{T(\tau)\}$ is emitted downward. The fraction of this emitted radiation that reaches the bottom of the medium is $\exp\{-(\bar{\tau} - \tau)\}$, where $\bar{\tau} - \tau$ is the optical distance from the point of emission within the medium to its bottom. The sum (integral) over all points is the total radiation at the bottom.

As a check on this result consider a medium with a uniform temperature. This is not a bad approximation for the troposphere or clouds when we realize that variations in *absolute* temperature within them are only about 10–20%. With this assumption Eq. (5.105) is readily integrated:

$$F_{\downarrow}(\bar{\tau}) = \{1 - \exp(-\bar{\tau})\} P_e(T), \quad (5.106)$$

which implies that the emissivity of the medium is

$$\varepsilon = 1 - \exp(-\bar{\tau}). \quad (5.107)$$

The sum of the reflectivity, transmissivity, and absorptivity must be 1. Because scattering was assumed to be zero, the reflectivity is 0. The transmissivity is $\exp(-\bar{\tau})$, and hence the absorptivity is $1 - \exp(-\bar{\tau})$. But by Kirchhoff's law, absorptivity equals emissivity, which leads to Eq. (5.107) by the route that led to Eq. (2.24). If at a particular wavelength $\bar{\tau} \gg 1$, then $\varepsilon \approx 1$, and at that wavelength the medium can be said to be black.

We could set up another differential equation for the upward irradiance, but this is hardly necessary because we can go straight to its solution just as we could have gone straight to Eq. (5.105), bypassing Eqs. (5.102)–(5.104):

$$F_{\uparrow}(0) = F_{\uparrow}(\bar{\tau}) \exp(-\bar{\tau}) + \int_0^{\bar{\tau}} \exp(-\tau) P_e\{T(\tau)\} d\tau. \quad (5.108)$$

The upward irradiance at the top of the medium is the sum of transmitted upward irradiance emitted at the bottom boundary plus all the unattenuated radiation emitted upward at each point of the medium.

Equations (5.105) and (5.108) are at the heart of remote sensing of atmosphere temperature profiles, either from the ground or from a satellite. According to these equations the radiation at the top or bottom of the atmosphere depends on the temperature profile, which is buried within an integral. The challenge is to dig it out by making spectral radiation measurements.

References and Suggestions for Further Reading

The doubling equation for reflectivity [Eq. (5.12)] is getting on in years. It can be found in an 1862 paper by Sir George Gabriel Stokes, On the intensity of the light reflected from or transmitted through a pile of plates, in *Proceedings of the Royal Society*. A more readily obtainable source for this paper than the original journal is Stokes's *Mathematical and Physical Papers*, Vol. IV, pp. 143–56 (Cambridge University Press, 1904). In a footnote Stokes states that in an 1856 paper “I find that the formulae for the particular case of perfect transparency have already been given by M. Neumann. His demonstration does not appear to have been published.” To this is added, by the editor of Stokes's papers, Sir Joseph Larmor, that the “efficiency of a pile of plates was also considered by Fresnel himself; his solution, which neglects opacity, was published posthumously” (in 1868). Fresnel died in 1827.

Although the doubling method in Section 5.1 is applied to (implicitly) optically homogeneous plates, it is just as valid for optically inhomogeneous (plane-parallel) media (see Prob. 5.33). Indeed, this method has been applied to media described by the equation of radiative transfer in Section 6.1.2; for a good review see James E. Hansen and Larry D. Travis, 1974: Light scattering in planetary atmospheres. *Space Science Reviews*, Vol. 16, pp. 527–610. This method has been extended to media containing spatially non-uniform sources of radiation (e.g., thermal radiation); see Warren J. Wiscombe, 1976: Extension of the doubling method to inhomogeneous sources. *Journal of Quantitative Spectroscopy and Radiative Transfer*, Vol. 16, pp. 477–89.

Equation (5.24) is derived in, for example, O. S. Heavens, 1991: *Optical Properties of Thin Solid Films*, Dover, pp. 55–9, Max Born and Emil Wolf, 1965: *Principles of Optics*, 3rd

rev. ed., Pergamon, pp. 61–2. This equation can be derived without explicitly solving the equations of the electromagnetic field (see Prob. 7.41).

It should become evident in the following chapter that an infinite number of two-stream equations could result from approximating the integro-differential equation of radiative transfer for a plane-parallel medium. The two-stream theory of Section 5.2, derived by physical arguments rather than by approximating the equation of transfer, is discussed in an expository paper on multiple scattering by Craig F. Bohren, 1987: Multiple scattering of light and some of its observable consequences. *American Journal of Physics*, Vol. 55, pp. 524–33, reprinted in the collection edited by Craig F. Bohren, 1989: *Scattering in the Atmosphere*, SPIE Optical Engineering Press. The granddaddy of two-stream theories is that by Arthur Schuster, 1905: Radiation through a foggy atmosphere. *Astrophysical Journal*, Vol. 21, pp. 1–22, reprinted in the collection edited by Donald H. Menzel, 1966: *Selected Papers on Transfer of Radiation*, Dover.

The absorption length for liquid water in Fig. 5.12 is from Table 1 in Marvin R. Querry, David M. Wieliczka, and David J. Segelstein, 1991: Water (H_2O), pp. 1059–77 in Edward D. Palik, Ed., 1991: *Handbook of Optical Constants of Solids*, Vol. II, Academic Press. That for ice is from the compilation by Stephen G. Warren, 1984: Optical constants of ice from the ultraviolet to the microwave. *Applied Optics*, Vol. 23, pp. 1206–25.

The reflectivity measurements for snow in Fig. 5.16 are from Harold W. O’Brien and Richard H. Munis, 1975: Red and near-infrared spectral reflectance of snow. CRREL Research Report 332, Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

The method alluded to in Section 5.2.4 for determining cloud properties, using ground-based measurements, by exploiting the quite different reflection properties of vegetation at visible and near-infrared wavelengths is discussed in Alexander Marshak, Yuri Knyazikhin, Anthony B. Davis, Warren J. Wiscombe, and Peter Pilewskie, 2000: Cloud–vegetation interaction: use of Normalized Difference Cloud Index for estimation of cloud optical thickness. *Geophysical Research Letters*, Vol. 27, pp. 1695–98; Alexander Marshak, Yuri Knyazikhin, Keith D. Evans, Warren J. Wiscombe, 2004: The “RED versus NIR” plane to retrieve broken-cloud optical depth from ground-based measurements. *Journal of the Atmospheric Sciences*, Vol. 61, pp. 1911–25.

For a comprehensive review of theories of multiple scattering by snow (on the ground) and comparison with measurements see Stephen G. Warren, 1982: Optical properties of snow. *Reviews of Geophysics and Space Physics*, Vol. 20, pp. 67–89.

For a discussion of the colors of natural ice bodies, accompanied by color photographs of icefalls and holes in snow, see Craig F. Bohren, 1983: Colors of snow, frozen waterfalls, and icebergs. *Journal of the Optical Society of America*, Vol. 73, pp. 1646–52.

The blue of natural ice bodies discussed in this chapter should not be confused with the blueness of shadows on snow. For more on this see Michael E. Churma, 1994: Blue shadows: physical, physiological, and psychological causes. *Applied Optics*, Vol. 33, pp. 4719–22.

C. V. Raman's incorrect explanation of the colors of glaciers is in his 1923 paper Thermal opalescence in crystals and the colour of ice in glaciers, *Nature*, Vol. 111, pp. 13–14.

For detailed calculations of the optimum diameter of titanium dioxide particles ($0.22\ \mu\text{m}$), based on the two-stream theory of this chapter, see Bruce R. Palmer, Penelope Stamatakis, Craig F. Bohren, and Gary G. Salzman, 1989: A multiple-scattering model for opacifying particles in polymer films. *Journal of Coatings Technology*, Vol. 61, pp. 41–7.

Primo Levi's story about titanium is in *The Periodic Table* (1984), Schoken Books, p. 166.

For theory and measurements of the darkening of sand when wet by various liquids see Sean A. Twomey, Craig F. Bohren, and John L. Mergenthaler, 1986: Reflectance and albedo differences between wet and dry surfaces. *Applied Optics*, Vol. 25, pp. 431–37. For a non-mathematical account of the brightness differences between dry and wet sand, also sand of different grain sizes, see Craig F. Bohren, 1987: *Clouds in a Glass of Beer*, John Wiley & Sons, Ch. 15. For a different interpretation of why some things are darker when wet see John Lekner and Michael C. Dorf, 1988: Why some things are darker when wet, *Applied Optics*, Vol. 27, pp. 1278–80.

Related to the question raised in Problem 5.15 is one that never seems to go away: Can you get a sunburn through glass? For an experimental answer to this question see Richard Bartels and Fred Loxsom, 1995: Can you get a sunburn through glass? Theory and an experiment. *The Physics Teacher*, Vol. 33, pp. 466–70.

Problems

5.1. Prove Eq. (5.12).

HINT: You can do this proof by induction. Assume that the truth of Eq. (5.12) for arbitrary N implies the truth of it for $2N$ using Eq. (5.9), the general rule for finding the reflectivity of $2N$ plates given that for N plates. We know that Eq. (5.12) is true for $N = 2$ and $N = 4$.

5.2. In Section 5.1.3 we determine the mean free path for absorption (scattering). What is the root-mean-square free path? Determine the n^{th} -root-mean free path, that is, $\langle x^n \rangle^{1/n}$. What is the median free path? What is the most probable free path?

5.3. To convince yourself that the denominator in Eq. (5.67) is a consequence of multiple reflections between cloud and ground, derive this equation by summing the infinite series of such reflections. Treat the cloud as a slab with reflectivity R and transmissivity T overlying a surface with reflectivity R_s . You can check this solution by solving the equations of radiative transfer subject to suitable boundary conditions.

5.4. A magazine advertisement for Bermuda once caught our eye. The ad showed a couple in bathing suits sitting together romantically on a beach. A distinct line running parallel to the water separated bright white sand from darker pink sand (at water's edge), and the couple was sitting on the pink sand. The caption read "Does pink sand feel softer?" Don't bother answering this question but instead explain the abrupt change in color (and brightness) of the beach.

HINT: It might help to learn a bit about coral.

5.5. Estimate the wavelength of incident radiation such that a snowpack can be considered optically homogeneous. Take the snow volume fraction f to be 0.3, a typical value.

5.6. We once were asked the following question by a professor at another university: “The bottoms of thick clouds often are dark or gray. Yet scattering by cloud droplets is strongly peaked in the forward direction. Doesn’t this imply that the bottoms of clouds should be bright?” Answer this question.

5.7. We once received an anguished message from a scientist interested in transmission of visible light by suspensions of particles. He stated that “as the beam marches through the sample it is assumed to be extinguished (scattering + absorption) exponentially as $\exp(-\alpha_{\text{ext}}h)$,” where h is the sample thickness and $\alpha_{\text{ext}} = fC_{\text{ext}}/v$, f is the volume fraction of particles in the suspension, C_{ext} is the extinction cross section, and v is the volume of a single particle.

This scientist stated that using values of extinction for a (nonabsorbing) sphere of diameter $1.05 \mu\text{m}$, he computed a value for α_{ext} of 0.443 mm^{-1} ($f = 10^{-4}$). He further noted that “for our path length of 23 mm, the incident beam [according to calculations] is attenuated to about 3.8×10^{-5} of its original value (i.e., almost totally extinguished). This attenuation is far too great. We know [from laboratory observations] that a 10^{-4} suspension is very transparent [i.e., visible light is transmitted by it]. What are we doing wrong?”

Respond to this message. If necessary, draw upon examples in the atmosphere to help this person.

5.8. Within the simple two-stream model of multiple scattering, it is not possible to determine how the reflectivity of a cloud depends on the angle of incidence of the illumination (angle between the normal to the cloud and the incident beam). Nevertheless, on the basis of physical intuition acquired from the two-stream model you should be able to guess how the reflectivity of a cloud depends on this angle. Sketch this dependence and briefly explain your reasoning.

5.9. This problem is related to the previous one. Using a simple extension of the two-stream theory of reflection by clouds, and guided by physical intuition, you should be able to do more than just make a crude sketch of cloud reflectivity versus solar zenith angle. You should be able to determine the slope of this curve and its limiting value.

5.10. Find the most general expression for the rate (per unit volume) at which radiant energy (of a given frequency) is deposited (transformed) within a medium according to the two-stream theory.

HINT: It might help to contemplate Fig. 5.10 and review the section on flux divergence (2.3).

5.11. Given the result of Problem 5.10, find the rate of transformation of radiant energy at each point of an infinite, absorbing medium according to the two-stream theory. The medium is illuminated from above. As a way of checking your result integrate this local energy transformation rate over the entire medium. You should obtain an expression you could have written down immediately without doing any integration.

5.12. This problem is related to the previous two. Pure lake or seawater (water containing no solid particle or biological organisms) is heated by absorption of solar radiation through a depth of tens of meters (see Figs. 2.2 and 5.12). Suppose that particles are added to the water. What happens to the radiative heating rate just below the surface (assuming no change in the incident solar radiation)? You may assume that the particles are nonabsorbing at solar wavelengths and that the reflectivity of the water is zero. Your first reaction to this question

might be that because reflection by the water is increased by the particles, the heating rate decreases. Although this is true *globally* (for the medium as a whole) show either by simple mathematical or physical arguments (or some combination of both) that the *local* heating rate may not necessarily decrease. How might the particles change the temperature and its gradient in the water? Devise a simple experiment for testing your conclusions.

5.13. If the atmosphere were isothermal but not uniform in composition (i.e., the concentration of infrared active gases varies with height), would emission by the atmosphere to space be the same as, greater than or less than emission by the atmosphere to the ground? Here emission to space does not include radiation from the ground transmitted by the atmosphere. You may assume that the atmosphere emits only up or down.

5.14. On a sunny day we once noticed small clouds scattered over the horizon in all directions. The sun was perhaps $30 - 40^\circ$ above the horizon. Clouds toward the sun were bright, but those away from the sun were darker. Moreover, it was possible to find pairs of clouds of apparent similar size, although in different directions, one of the pair dark, the other bright. We also observed that among the dark clouds (those opposite the sun) the brighter ones were also the largest. Explain. A diagram here will be helpful.

5.15. You occasionally come across the assertion that it is easier to get a sunburn on a cloudy day than on a clear day. The explanation sometimes offered for this is that clouds “transmit more ultraviolet radiation” (by “more” here presumably is meant more than on a clear day). Discuss this explanation. If you don’t believe it, and can back this up with physical arguments, put forward an alternative explanation (under the assumption that the assertion is true).

5.16. Einstein’s formula (see references at the end of Ch. 3) for the scattering coefficient (which he called an absorption coefficient because it determines the *apparent* absorption of a beam of light) of a nonabsorbing, one-phase, homogeneous medium can be written

$$\beta_{\text{mol}} = \frac{k_B T}{6\pi} \frac{(n^2 - 1)^2 (n^2 + 2)^2}{9} \kappa_T \left(\frac{2\pi}{\lambda} \right)^4,$$

where n is the refractive index of the medium and κ_T its isothermal compressibility. This is the scattering coefficient of a medium as a consequence of its irreducible graininess at the molecular level. With the use of this formula, critically discuss the notion that the blueness of the deep sea is a consequence solely of molecular scattering. The isothermal compressibility of water is about $5 \times 10^{-10} \text{Pa}^{-1}$.

5.17. Suppose that you are faced with the task of measuring the imaginary part of the refractive index of pure water at, say, visible wavelengths by measuring (relative) transmission of a beam. What is the irreducible error in the imaginary index? Is it positive or negative? By irreducible here is meant that there is *no* error in the transmission measurement. Suppose that you needed to know the imaginary index to within 10%. What lower limit does this then set on the actual value of the imaginary index? That is, what is the value of the actual imaginary index below which it is impossible to attain the desired accuracy by measuring transmission? HINT: This problem is related to the previous one.

5.18. Show that total emission to the surface in the radial (vertical) direction from a molecular species with an absorption coefficient that decreases exponentially with scale height (e -folding distance) H is the same as that for a uniform, finite layer of thickness H and absorption coefficient equal to the surface value. Take the (slab) atmosphere to be isothermal.

5.19. We assert in Section 2.2 that the increasing infrared ($\sim 10\ \mu\text{m}$) brightness temperature of the clear atmosphere with increasing zenith angle is a consequence mostly of an increasing path length (emissivity). Yet the average temperature along a path in the atmosphere does increase with increasing zenith angle because temperature usually decreases with height. Show that this is not the major contribution to the variation of brightness temperature with zenith angle. That is, estimate the maximum correction to emission as a result of assuming that the atmosphere is isothermal with a temperature equal to the surface value. Before tackling this problem first decide on physical grounds the direction for which the error is greatest, then do your analysis for this direction. You may assume that temperature decreases linearly with height at a rate of $6.5\ ^\circ\text{C}/\text{km}$.

HINT: Do the previous problem and Problem 2.12 first.

5.20. Suppose that scatterers are added to an absorbing ($\varpi = 0$) plane-parallel medium. What happens to its emissivity? How does your answer depend on the asymmetry parameter? Detailed analysis and calculations are not necessary to answer these questions.

5.21. First we derived two-stream equations of transfer, Eqs. (5.45) and (5.46), under the assumption of no emission. Then we derived an equation for the downward irradiance, Eq. (5.103), under the assumption of no scattering. Without much effort you should be able to write down two-stream equations for a medium for which both emission and scattering must be taken into account.

5.22. Solve the equations derived in the previous problem to obtain the emissivity of a uniform, isothermal, optically thick plane-parallel medium. This problem requires a fair amount of labor, so before tackling it make some plausible estimates, based on physical reasoning, of the functional dependence of emissivity on single-scattering albedo and asymmetry parameter.

5.23. More than half a century ago, long before artificial satellites circled Earth, its albedo for solar radiation was measured using visual observations of a natural satellite, the moon (see Prob. 4.10). The albedo for solar radiation, which is mostly a consequence of clouds, especially thick ones, is the ratio of solar radiation incident on Earth to the solar radiation reflected by Earth. These measurements gave a value for the average albedo of around 0.38 (38%). Many years later, measurements from artificial satellites yielded the now accepted value of around 0.30 (30%). Why the discrepancy?

HINT: The answer is not friction or “experimental error” or “better instruments.” You may assume that the uncertainties in albedos measured by the two different methods are smaller than the difference between albedos obtained by these two methods.

5.24. On examinations students have hazarded the guess that the difference between the albedos in the previous problem is a consequence of “atmospheric attenuation.” Show that this guess is not correct.

5.25. Estimate the optical thickness of a cloud such that its transmitted (diffuse) radiance is equal to the attenuated (by the cloud) radiance of direct sunlight. This is an estimate of how optically thick a cloud must be such that the disc of the sun cannot be seen through it.

HINT: Assume that the cloud is sufficiently thick that the downward radiance emerging from the cloud is isotropic.

5.26. Estimate the least optical thickness of a cloud layer such that it is as bright (as seen from below cloud) as the clear (zenith) sky. Then estimate the physical thickness of such a cloud.

All that is wanted here is a rough estimate: meters, tens of meters, hundreds of meters...? Estimate the cloud optical thickness beyond which the light emerging from the bottom of a cloud is *less* bright than the clear sky. And again estimate the physical thickness of such a cloud. Ignore absorption and make any reasonable simplifying assumptions.

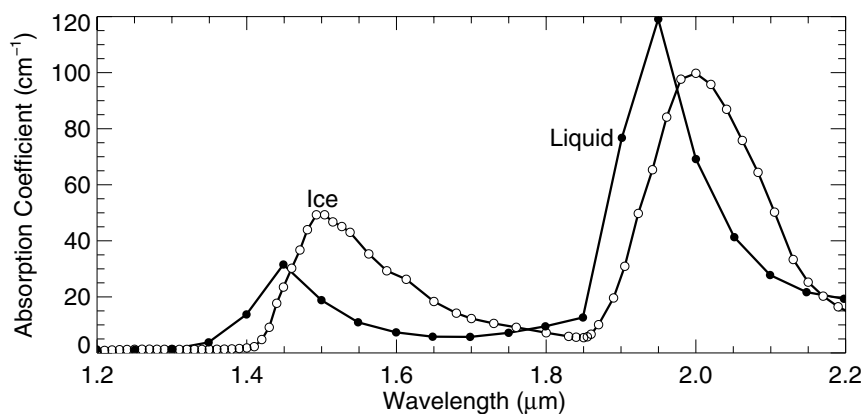
5.27. You are flying over a vast snow-covered expanse above clouds and notice that the clouds are not as bright as the snow. This may seem puzzling. Explain.

5.28. In the preceding problem, the snow was brighter than the clouds. But suppose that you observe the clouds to be brighter than the snow. Under what viewing conditions and cloud optical thicknesses might this be possible?

5.29. According to the *Guinness Book of World Records*, aerogel is the world's lightest (lowest density) solid. Composed of tiny pores in a silicon dioxide (glass) matrix, it is 99.8% air. We once used aerogel in an examination. We placed two sheets of paper, one black, one white, side by side on a desk, with a slab of aerogel (perhaps a few centimeters thick) straddling the sheets. The aerogel over the black sheet had a definite blue cast, whereas that over the white sheet was yellow. Explain.

5.30. In Section 1.3 we noted that the emissivity of a single *particle* can be greater than 1 at some frequencies. What about the emissivity of a suspension of many such particles (take the lateral dimensions of the suspension to be much larger than the wavelength)?

5.31. We cited in the references at the end of Chapter 2 a method for determining the thermodynamic phase (ice or water) of cloud particles by making use of differences between near-infrared absorption by ice and water (see accompanying figure). A criticism of this method when first presented was that a change in cloud reflectivity could be a consequence of a change in particle size. And this is a valid criticism given that the product of absorption coefficient and particle size [e.g., Eq. (5.91)] determines the amount to which reflectivity is reduced below unity. What experimental evidence, in addition to different magnitudes of reflectivity, would be necessary to rebut this criticism?



5.32. Suppose that all the droplets in a cloud coalesce to form rain. By how much does the scattering optical thickness of the cloud change at visible wavelengths? The typical size of a cloud droplet is of order 10 μm, that of a raindrop is of order 1 mm.

5.33. The reflectivity [Eq. (5.51)] for a plane-parallel medium of arbitrary optical thickness was obtained by solving the two-stream equations of transfer. But it could be obtained even if you had never seen these equations by using the doubling formula for a pile of plates [Eq. (5.12)] beginning with the reflectivity for an optically thin plane-parallel medium. Do so.

HINTS: Divide the medium of interest into N layers of equal optical thickness. By making N arbitrarily large the optical thickness of a layer can be made arbitrarily small. The reflectivity of a single thin layer is readily obtained from single-scattering theory.

5.34. Show that, in principle, one can determine the visible reflectivity of clouds by making measurements from the ground of downward radiation at two different wavelengths, one in the visible and one in the near-infrared, provided that the reflectivity of the ground is quite different at these two wavelengths whereas that of the cloud is not. For simplicity, assume that the reflectivity of the ground is 0 in the visible and 1 in the infrared.

5.35. Solve Eq. (5.69) for a finite medium with K given by Eq. (5.70) for downward irradiance F_0 at $\tau = 0$ and upward irradiance $F_{\uparrow} = R_g F_{\downarrow}$ at $\tau = \bar{\tau}$, where the reflectivity of the ground is R_g .

5.36. Interference colors often are seen in light reflected by *thin* films (e.g., oil slicks illuminated by sunlight). Why must they be “thin”? And what exactly is meant by “thin”? After all, Eq. (5.24) is for a film of *arbitrary* thickness.

HINT: Analysis is simplified if you take $R_{\infty} \ll 1$ and ignore the wavelength dependence of refractive index.

5.37. At the end of Section 5.2.4 we assert that “it is difficult to fathom how reflection by colored ground could do more than tinge the light from the bottoms of more or less continuous clouds. Broken clouds are another matter.” Can you imagine a cloud cover and ground radiance spectrum (physically realistic even if improbable) such that the bottom of the cloud (at least part of it) would have the color of the ground?

HINT: See Problem 4.58.

5.38. We state in Section 3.4.9 that scattering (of visible radiation) per molecule by air is much greater than that by liquid water. By how much (at standard pressure and temperature)?

HINT: Problem 5.16 is useful. You also need a bit of elementary thermodynamics and some refractive index data.

5.39. We state in Section 3.4.9 that scattering per molecule by air is much greater than that by liquid (and ask you to estimate by how much in Problem 5.38). We have encountered the following explanation for why scattering per molecule in condensed matter (such as liquid water) is *less* than that in a gas. For each molecule in liquid water one is almost certain to find another molecule such that the waves scattered by the two (in the same direction, of course) are exactly out of phase, and so the waves interfere destructively. Do this for every pair of molecules and you end up with essentially no (laterally) scattered light. Do you accept this explanation? Why or why not?

5.40. For what kind of medium does the Einstein equation in Problem 5.16 predict identically zero lateral scattering? Does such a medium exist?

5.41. We state in Section 3.4.9 that the Einstein theory of scattering (see Prob. 5.16) predicts infinite scattering at the critical point. Why?

HINT: To answer this question requires some familiarity with thermodynamic phase diagrams.

5.42. Consider a thin suspension of particles with scattering cross section nearly independent of wavelength but absorption cross section strongly dependent on wavelength and not negligible compared with the scattering cross section (this is neither unphysical nor unusual). What is the reflection spectrum of this suspension if multiple scattering is identically zero? If you have difficulty answering this question, retreat to the simpler problem of a pile of plates.

5.43. Problem 5.35 required you to obtain an expression, using the two-stream theory, for the reflectivity of a finite medium. Use that expression to test the correctness of your answer to Problem 5.42. That is, consider the reflectivity in the limit of small optical thickness for a medium with a scattering coefficient that does not depend on wavelength but an absorption coefficient that does.

5.44. In Section 5.3.2 we showed that attenuation in a multiple-scattering medium is not necessarily exponential. This is true for irradiance and for plane-parallel illumination, by which is meant that the illumination is uniform over an infinite plane. As a result, if we use a detector with line of sight perpendicular to the plane-parallel medium to measure the irradiance, its location in the plane perpendicular to its line of sight does not affect the irradiance it measures. But what about for a narrow, collimated beam illuminating the plane-parallel medium and a detector with a small aperture and a narrow field of view and pointed directly at the beam?

5.45. Show that the optical thickness of a medium composed of (incoherent) scatterers does not change if the medium is uniformly compressed (but not to the point where the scatterers fuse together) or expanded.

5.46. It is often said that Mie theory (Sec. 3.5.1) is “exact”. This cannot be literally true because this theory is based on the (false) assumption that matter is homogeneous at all scales (continuum electromagnetic theory). Estimate how inexact Mie theory is by considering the ratio of intrinsic scattering by a given volume of water to scattering predicted by continuum theory for that same volume of water in the form of a droplet in air. Take the droplet diameter to be typical of that for cloud droplets and the wavelength in the middle of the visible spectrum. You will need the Einstein equation in Problem 5.16. How does your result help to understand Raman’s incorrect explanation of the blueness of natural ice bodies (Sec. 5.3.1)?

5.47. This problem is related to the previous one, in which we considered a droplet much larger than the wavelength. Now consider the other extreme: a droplet much smaller than the wavelength. For what size droplet will scattering on the molecular scale be comparable with that on the macroscopic scale? Try to guess the answer (on physical grounds) before doing the calculations. You’ll need results from Section 3.5.1.

5.48. Derive the expression Eq. (5.23) for the incoherent reflectivity of a nonabsorbing plate by adding up multiple reflections (see Sec. 1.4.1). When is it obvious that this equation cannot be correct? Go one step further and derive the incoherent transmissivity T of an absorbing plate of thickness h and absorption coefficient κ . What condition would have to be satisfied in order to have a hope of inferring the absorption coefficient from the slope of $-\ln T$ versus h and R_∞ from the intercept of this curve? Even if this condition were satisfied, what are some of the pitfalls one would face in trying to use this curve to infer R_∞ and κ by measuring transmission by plates of the same material but different thicknesses?

5.49. Soot is a catchall name for the carbonaceous products of combustion. It has no definite chemical composition but is highly absorbing. Typically, soot particles are much smaller than the wavelengths of visible light. The (visible) reflectivity of clean, fine-grained snow is close to 1. How much soot added to a snowpack would reduce its reflectivity sufficiently that the human observer could notice a difference between clean and dirty thick snowpacks (observed side by side)? Express your answer as the ratio of soot volume to ice volume. The real part of the refractive index of substances called soot is around 2, its imaginary part around 1, and both don't vary much over the visible spectrum. The exact values are meaningless because soot is not an exact substance. Make as many simple approximations as you can in order to quickly estimate how much soot is needed (parts per thousand, million, billion?) to visibly change a snowpack.

5.50. To use the doubling method for a scattering and absorbing medium of arbitrary optical thickness requires the reflectivity and transmissivity of a very optically thin slice of that medium. You should be able to obtain them (in the two-stream approximation) by strictly physical arguments. You can check your result by considering the solution to Problem 5.35 (for $R_g = 0$) in the limit of small optical thickness. Be careful to take this limit consistently. You need a correct power series expansion of the reflectivity and transmissivity to first order in optical thickness. If you are ambitious you can write a doubling program to use the reflectivity and transmissivity you obtain to calculate that of an arbitrary medium and compare the computed (approximate) results with the exact (two-stream) theory. Note in particular how the error changes with increasing optical thickness. Also, try to reduce the error by reducing the thickness of the initial slab.

5.51. There are three ways of determining reflection and transmission by an arbitrary finite slab (within the two-stream theory) overlying a plane with an arbitrary reflectivity R_g . Solve the appropriate differential equations [Eqs. (5.47) and (5.48)] subject to suitable upper and lower boundary conditions. Or begin with the solution to reflection and transmission by the slab over a non-reflecting plane and then either follow the same approach that yielded Eqs. (5.7) and (5.8) or sum the same kind of infinite series (infinite multiple reflections) as in Section 1.4.1. Try all three methods. Which is easier? Which gives you the most physical insight? One of the tests of the correctness of equations for reflectivity and transmissivity of a slab (over a black surface) is to examine whether their sum is 1 for no absorption. Can this same test be applied if the slab overlies a surface with arbitrary reflectivity? Why or why not? If the usual test fails, come up with one that will not.

5.52. Carbon black is a suspension of very small carbon particles. Absorption by these particles is much greater than scattering, and you can take their absorption cross section to be inversely proportional to wavelength. What is the color of an infinitely thick carbon black film? Suppose the film is over a substrate with reflectivity 1 at visible wavelengths. How do the color and luminance of the film vary as its thickness varies from 0 to a sufficiently large value that it can be considered infinitely thick? You can answer these questions using the two-stream theory or by physical reasoning.

5.53. Derive the reflectivity and transmissivity of an absorbing slab of uniform thickness h and absorption coefficient κ , infinite in lateral extent and normally illuminated. The slab is optically smooth and optically homogeneous. R is the reflectivity and T the transmissivity at the slab *interface*. Ignore interference. Do this problem two ways: by summing infinite series

of rays reflected and transmitted by the *slab* (and not just by its interface with the medium in which it is embedded); and by solving the two-stream equation of radiative transfer with $\beta = 0$ and carefully choosing the boundary conditions. Keep in mind that the two-stream solution applies only to interior points of the medium ($0 < z < h$).

5.54. For the slab in Problem 5.53, show by physical arguments that $R + T = 1$. The absorptivity of the slab \bar{A} is $1 - (\bar{R} + \bar{T})$. Take κh to be fixed. If \bar{A}_0 is the absorptivity for a non-reflecting slab ($R = 0$), show that the absorptivity \bar{A} for any $R > 0$ is less than \bar{A}_0 .

5.55. In Chapter 2 we derived an approximate expression for the volumetric absorption cross section of a uniformly thick slab, normally illuminated, for “weak absorption” [Eq. (2.148)] and “strong absorption” [Eq. (2.149)]. We neglected reflection by the slab. Use the results of Problem 5.54 to remove this restriction. Does reflection increase or decrease the volumetric absorption cross section?

5.56. Clouds do not have optically smooth surfaces (at least not at visible and near-visible wavelengths). But there are some scattering media that do, for example, bubbly ice and water containing suspended particles. Derive the reflectivity of a scattering and absorbing medium the illuminated surface of which is optically smooth (i.e., specularly reflecting and transmitting). For simplicity take the medium to be infinitely thick. You can do this problem two ways. The hard way is to solve the two-stream equations subject to the boundary conditions in Problem 5.53 then interpret the result physically. The easy way is to use physical arguments to solve the problem, then write down the solution. Does the absorptivity of the infinite slab increase or decrease because of the addition of scatterers?

5.57. This problem is related to Problems 5.53–5.56, but is more of a small term project than a problem to be dashed off in a few hours. Mie theory is fundamentally incapable of treating absorption by spheres that are themselves cloudy. And yet such spheres exist: milk drops, for example. Perhaps we can obtain some insights by considering a simpler problem, absorption by a cloudy, uniform slab, much larger than the wavelength, and illuminated at normal incidence. Is there any combination of reflectivity R , absorption coefficient κ , scattering coefficient β , and thickness h such that the absorptivity of a cloudy slab is greater than that of a clear slab, all else being equal. For simplicity, take the asymmetry parameter of the scatterers in the slab to be zero. Also assume that the absorption coefficient with and without scatterers is the same. The problem of absorption by a cloudy sphere was considered by Sean A. Twomey, 1987: Influence of internal scattering on the optical properties of particles and drops in the near infrared. *Applied Optics*, Vol. 26, pp. 1342–47. Twomey concluded that “absorption by a weakly absorbing particle can be increased substantially beyond that in the absence of internal scattering.” What do you conclude on the basis of calculations for a slab?

5.58. Reflections between parallel plates vanish when the separation between them is zero and the two plates fuse into one. But this separation need not be identically zero before they fuse optically and reflections become negligible. Obtain a criterion for when the plates fuse optically.

HINT: Consider a planar air gap between two infinite, identical media and use Eq. (5.24).