

Appendix A: Derivation of the Planck Function

In accordance with Boltzmann statistics, if N_0 denotes the number of oscillators in any given energy state, then the number N in a state having energy higher by an amount ε is given by

$$N = N_0 e^{-\varepsilon/KT}, \quad (\text{A.1})$$

where K is Boltzmann's constant and T is the absolute temperature. On the basis of Planck's first postulation, an oscillator cannot have just any energy, but rather only energies given by Eq. (1.2.1). Thus, the possible values of E must be 0 , $h\tilde{\nu}$, $2h\tilde{\nu}$, and so on. If the number of oscillators with zero energy is N_0 , then by virtue of Eq. (A.1), the number with energy $h\tilde{\nu}$ is $N_0 e^{-h\tilde{\nu}/KT}$, the number with energy $2h\tilde{\nu}$ is $N_0 e^{-2h\tilde{\nu}/KT}$, and so on. The total number of oscillators with frequency $\tilde{\nu}$ for all states is therefore

$$\begin{aligned} N &= N_0 + N_0 e^{-h\tilde{\nu}/KT} + N_0 e^{-2h\tilde{\nu}/KT} + \dots \\ &= N_0 [1 + e^{-h\tilde{\nu}/KT} + (e^{-h\tilde{\nu}/KT})^2 + \dots] \\ &\cong N_0 / (1 - e^{-h\tilde{\nu}/KT}). \end{aligned} \quad (\text{A.2})$$

The total energy of these oscillators may be obtained by multiplying each term in Eq. (A.2) by the appropriate energy level:

$$\begin{aligned} E &= 0 \cdot N_0 + h\tilde{\nu} \cdot N_0 e^{-h\tilde{\nu}/KT} + 2h\tilde{\nu} \cdot N_0 e^{-2h\tilde{\nu}/KT} + 3h\tilde{\nu} \cdot N_0 e^{-3h\tilde{\nu}/KT} + \dots \\ &= h\tilde{\nu} N_0 e^{-h\tilde{\nu}/KT} [1 + 2e^{-h\tilde{\nu}/KT} + 3(e^{-h\tilde{\nu}/KT})^2 + \dots] \\ &\cong h\tilde{\nu} N_0 e^{-h\tilde{\nu}/KT} / (1 - e^{-h\tilde{\nu}/KT}). \end{aligned} \quad (\text{A.3})$$

The average energy per oscillator is then given by

$$\frac{E}{N} = \frac{N_0 h\tilde{\nu} e^{-h\tilde{\nu}/KT} / (1 - e^{-h\tilde{\nu}/KT})^2}{N_0 / (1 - e^{-h\tilde{\nu}/KT})} = h\tilde{\nu} / (e^{h\tilde{\nu}/KT} - 1). \quad (\text{A.4})$$

According to Planck's second postulation, the quanta of energy are emitted only when an oscillator changes from one to another of its quantized energy states. The average emitted energy of a group of oscillators is, therefore, given by Eq. (A.4), which is the factor appearing in Planck's formula.

To obtain the Planck function, we let $u_{\tilde{\nu}}$ denote the monochromatic energy density, i.e., the energy per unit volume per unit frequency interval in a cavity with temperature T . With this definition, we write

$$u_{\tilde{\nu}} = Ah\tilde{\nu}/(e^{h\tilde{\nu}/KT} - 1), \quad (\text{A.5})$$

where A is a constant to be determined. In accordance with the principle of equipartition of energy, the energy density in a cavity is given by the classical Rayleigh–Jeans formula:

$$u_{\tilde{\nu}} = (8\pi\tilde{\nu}^2/c^3)KT. \quad (\text{A.6})$$

This formula is valid when the temperature T is high and the frequency $\tilde{\nu}$ is small. By letting $h\tilde{\nu}/KT \rightarrow 0$ in Eq. (A.5), we find $A = 8\pi\tilde{\nu}^2/c^3$. Thus, the monochromatic energy density is

$$u_{\tilde{\nu}} = \frac{8\pi h\tilde{\nu}^3}{c^3(e^{h\tilde{\nu}/KT} - 1)}. \quad (\text{A.7})$$

For blackbody radiation, the emitted photons travel in all directions (4π solid angle) at the speed of light c . Thus, the emitted intensity (or radiance) in a cavity with a temperature T in units of energy/area/time/sr/frequency may be expressed by

$$B_{\tilde{\nu}}(T) = u_{\tilde{\nu}}c/4\pi. \quad (\text{A.8})$$

On substituting Eq. (A.7) into (A.8), we obtain the Planck function in the form

$$B_{\tilde{\nu}}(T) = \frac{2h\tilde{\nu}^3}{c^2(e^{h\tilde{\nu}/KT} - 1)}, \quad (\text{A.9})$$

as stated in Eq. (1.2.3).