Drinfeld modular polynomials of level T

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Abstract

We investigate Drinfeld modular polynomials parametrizing T-isogenies between Drinfeld $\mathbb{F}_q[T]$-modules of rank $r \geq 2$. By providing an explicit classification of such isogenies, we derive explicit bounds on the T-degrees of the coefficients of the a associated modular polynomials. Numerical computations show that these bounds are often sharp.

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1 Introduction

Drinfeld modular polynomials classify isogenies between pairs of Drinfeld modules. The case of Drinfeld modules of rank $r = 2$ was first studied in [**?**] and closely parallels the case of classical modular polynomials parametrizing pairs of elliptic curves linked by a cyclic isogeny of given degree. When $r \geq 3$, the situation is complicated by the fact that now several isomorphism invariants are needed and the resulting modular polynomials have more than two variables, see [**?**, **?**]. In all cases, the size of the coefficients (as elements of $\mathbb{F}_q[T]$) grow very quickly, see e.g. [**?**, **?**, **?**].

In this article, we study Drinfeld modular polynomials of rank $r \geq 2$ parametrizing cyclic T-isogenies. Such isogenies can be parametrized by a single parameter, allowing a completely explicit description of such polynomials. We exploit this to deduce sharp bounds on the coefficients of such Drinfeld modular polynomials. One may think of this work as a higher rank continuation of [**?**]. At the end, we report on some computations showing that, at least in "small" cases, these bounds are mostly sharp.

1.1 Isomorphism invariants for rank r Drinfeld modules

Let $\mathbb{F}_q$ be the finite field of q elements. Let $A = \mathbb{F}_q[T]$ and let $r \geq 2$ be a positive integer. Let $g_1, g_2, \ldots, g_{r-1}, \Delta$ be algebraically independent over $k = \mathbb{F}_q(T)$.

We define the subring

$$C \subset A[g_1, \ldots, g_{r-1}, \Delta^{-1}]$$

generated by monomials of the form $a(T) g_1^{e_1} g_2^{e_2} \cdots g_{r-1}^{e_{r-1}} \Delta^{-e_r}$, where

$$\sum_{i=1}^{r-1} e_i (q^i - 1) = e_r (q^r - 1).$$

The morphism $C \to A[g_1, \ldots, g_{r-1}]; \Delta \mapsto 1$ is injective, and it is convenient to identify C with its image

$$C' \subset A[g_1, \ldots, g_{r-1}],$$

which is generated by monomials $a(T) g_1^{e_1} \cdots g_{r-1}^{e_{r-1}}$ satsifying

$$\sum_{i=1}^{r-1} e_i (q^i - 1) \equiv 0 \bmod (q^r - 1).$$

Then C (and thus also C') is the ring of isomorphism invariants of rank r Drinfeld A-modules, as follows.

Let

$$J = \sum_{e_1, \ldots, e_{r-1}} a_{e_1, \ldots, e_{r-1}}(T) g_1^{e_1} \cdots g_{r-1}^{e_{r-1}} \Delta^{-e_r} \in C$$

and let ϕ be a rank r Drinfeld module defined over an A-field $\gamma : A \to L$ by

$$\phi_T(X) = \gamma(T) X + \ell_1 X^q + \cdots + \ell_r X^{q^r}, \qquad \ell_1, \ldots, \ell_r \in L, \ell_r \neq 0.$$

We evaluate J at ϕ by

$$J(\phi) = \sum_{e_1, \ldots, e_{r-1}} \gamma(a_{e_1, \ldots, e_{r-1}}(T)) \ell_1^{e_1} \cdots \ell_{r-1}^{e_{r-1}} \ell_r^{-e_r} \in L.$$

Then by [**?**] (see also [**?**, Cor. 1.2]), two Drinfeld modules ϕ and ϕ' are isomorphic over an algebraically closed field if and only if $J(\phi) = J(\phi')$ for all $J \in C$.

1.2 The monic generic Drinfeld module

We define the rank r Drinfeld A-module φ over $K = \mathbb{F}_q(T, g_1, \ldots, g_{r-1})$ by

$$\varphi_T(X) = TX + g_1 X^q + \cdots + g_{r-1} X^{q^{r-1}} + X^{q^r}.$$

We call it the "monic generic" rank r Drinfeld module, since every other rank r Drinfeld module ϕ defined over an A-field $\gamma : A \to L$ is isomorphic, over a finite extension of L, to a specialization of φ, as follows.

Suppose $\phi_T(X) = \gamma(T) X + \ell_1 X^q + \cdots + \ell_r X^{q^r}$ with $\ell_r \neq 0$. Then ϕ is isomorphic to $\phi' = c \phi c^{-1}$ over $L(c)$, where $c^{q^r - 1} = \ell_r$, and ϕ' is the image of φ under the map $K \to L(c)$; $T \mapsto \gamma(T)$, $g_k \mapsto c^{1-q^k} \ell_k$.

1.3 Drinfeld modular polynomials

Let $N \in A$ be monic and choose a basis for $\varphi[N] \cong (A/NA)^r$. Let $H \subset (A/NA)^r$ be an A-submodule. An isogeny $f : \varphi \to \varphi^{(f)}$ is said to be of type H if $\ker f \subset \varphi[N]$ is an element of the $\mathrm{GL}_r(A/NA)$-orbit of H.

Now for each invariant $J \in C$, we define the Drinfeld modular polynomial of type H for J by

$$\Phi_{J,H}(X) := \prod_{f:\varphi\to\varphi^{(f)} \text{ of type } H} \big(X - J(\varphi^{(f)})\big).$$

It is shown in [**?**] that $\Phi_{J,H}(X)$ has coefficients in C, and if we evaluate these coefficients at a Drinfeld module ψ, then the roots of the resulting polynomial $\Phi_{J(\psi),H}(X)$ are exactly the $J(\psi^{(f)})$ for isogenies $f : \psi \to \psi^{(f)}$ of type H.

In this paper, we consider the two cases $H = (A/TA)$, corresponding to outgoing T-isogenies $f : \varphi \to \varphi^{(f)}$, and $H = (A/TA)^{r-1}$, corresponding to incoming T-isogenies $f : \varphi^{(f)} \to \varphi$, since each incoming isogeny of type A/TA is dual to an outgoing isogeny of type $(A/TA)^{r-1}$.

The Drinfeld modular polynomials $\Phi_{J,(A/TA)}(X)$ and $\Phi_{J,(A/TA)^{r-1}}(X)$ both have degree

$$\psi_r(T) = \frac{q^r - 1}{q - 1}$$

in X. Our main result is the following sharp estimate on the T-degrees of the coefficients of these Drinfeld modular polynomials.

Theorem 1.1. *Let $J = g_1^{e_1} g_2^{e_2} \cdots g_{r-1}^{e_{r-1}} \Delta^{-e_r} \in C$ be a monomial invariant.*

1. *Consider the Drinfeld modular polynomial classifying outgoing T-isogenies,*

$$\Phi_{J,(A/TA)}(X) = \prod_{\substack{f:\varphi\to\varphi^{(f)}\\ \ker f \cong A/TA}} \big(X - J(\varphi^{(f)})\big) = a_0 + a_1 X + \cdots + a_{\psi_r(T)} X^{\psi_r(T)}.$$

The T-degrees of its coefficients satisfy

$$\begin{aligned} \deg_T a_0 &= \psi_r(t) w_1(J), \\ \deg_T a_k &\le (\psi_r(t) - k) w_1(J), \end{aligned}$$

for $k = 1, 2, \ldots, \psi_r(T)$, where

$$w_1(J) := q\left(\sum_{i=1}^{r-1} e_i - e_r\right).$$

2. *Consider the Drinfeld modular polynomial classifying incoming T-isogenies,*

$$\Phi_{J,(A/TA)^{r-1}}(X) = \prod_{\substack{f:\varphi^{(f)}\to\varphi\\ \ker f \cong A/TA}} \big(X - J(\varphi^{(f)})\big) = a_0 + a_1 X + \cdots + a_{\psi_r(T)} X^{\psi_r(T)}.$$

The T-degrees of its coefficients satisfy

$$\begin{aligned}
\deg_t a_0 &= \psi_r(T) w_{r-1}(J), \\
\deg_t a_k &\leq (\psi_r(T) - k) w_{r-1}(J),
\end{aligned}$$

for $k = 1, 2, \ldots, \psi_r(T)$, where

$$w_{r-1}(J) := \sum_{i=1}^{r-1} e_i + e_r(q^r - q^{r-1} - 1).$$

2 T-Isogenies

In this section we classify all incoming and outgoing T-isogenies of a given Drinfeld module. Let L be an A-field of generic characteristic, i.e. one where $\gamma : A \hookrightarrow L$. Suppose two Drinfeld modules over L are given by

$$\phi_T(X) = TX + g_1 X^q + \cdots + g_{r-1} X^{q^{r-1}} + \Delta X^{q^r},$$

$$\tilde{\phi}_T(X) = TX + \tilde{g}_1 X^q + \cdots + \tilde{g}_{r-1} X^{q^{r-1}} + \tilde{\Delta} X^{q^r}.$$

Proposition 2.1. *Fix a Drinfeld module ϕ as above.*

1. *Consider the polynomial*

$$Q(x) = \sum_{i=0}^{r} (-1)^i g_{r-i} x^{\frac{q^r - q^{r-i}}{q-1}} \in L[x]$$

with the conventions $g_0 = T$ and $g_r = \Delta$. Then, up to isomorphism, all outgoing T-isogenies $f : \phi \to \tilde{\phi}$ are given by

$$f(X) = X + aX^q$$

where a ranges over the roots of $Q(x)$. Furthermore, for each such a, the coefficients of $\tilde{\phi}$ are given by

$$\begin{aligned}
&\tilde{g}_k = g_k + a g_{k-1}^q - a^{q^{k-1}} \tilde{g}_{k-1}, \quad k = 1, \ldots, r-1 \\
&\tilde{\Delta} = a^{1-q^r} \Delta^q,
\end{aligned}$$

with the conventions $g_0 = \tilde{g}_0 = T$.

2. *Consider the polynomial*

$$\tilde{Q}(x) := \sum_{i=0}^{r} (-1)^i g_{r-i}^{q^i} x^{\frac{q^i - 1}{q-1}} \in L[x]$$

with the conventions $g_0 = T$ and $g_r = \Delta$.

Then, up to isomorphism, all incoming T-isogenies $f : \tilde{\phi} \to \phi$ are given by

$$f(X) = X + aX^q$$

where a ranges over the roots of $\tilde{Q}(x)$. Furthermore, for each such a, the coefficients of $\tilde{\phi}$ are given by

$$\begin{aligned} \tilde{g}_k &= g_k + a^{q^{k-1}} g_{k-1} - a\tilde{g}_{k-1}^q, \quad k = 1, \ldots, r-1 \\ \tilde{\Delta}^q &= a^{q^r - 1}\Delta, \end{aligned}$$

with the conventions $g_0 = \tilde{g}_0 = T$.

Proof. To prove Part 1, we fix ϕ and consider an outgoing T-isogeny $f : \phi \to \tilde{\phi}$. Then $f(X)$ is $\mathbb{F}_q$-linear of degree q and up to isomorphism, we may assume the linear term is X, so $f(X) = X + aX^q$.

There exists a dual isogeny $\hat{f} : \tilde{\phi} \to \phi$,

$$\hat{f}(X) = b_0 X + b_1 X^q + \cdots + b_{r-1} X^{q^{r-1}},$$

for which

$$\hat{f} \circ f(X) = \phi_T(X). \tag{1}$$

We compare coefficients in (**??**) and obtain $b_0 = T$ along with

$$b_k = g_k - b_{k-1} a^{q^{k-1}}, \quad k = 1, 2, \ldots, r,$$

where we set $b_r = 0$ and $g_r = \Delta$. By induction on k, it follows that

$$b_k = \sum_{i=0}^{k} (-1)^i g_{k-i} a^{\frac{q^k - q^{k-i}}{q-1}}.$$

Now the $k = r$ case is just

$$Q(a) = 0,$$

as required.

Every T-isogeny $f : \phi \to \tilde{\phi}$ gives rise to a root a of $Q(x)$, and different roots a correspond to different factorisations of $\phi_T(X) = \hat{f} \circ f(X)$, hence correspond to non-isomorphic isogenies.

Conversely, the number of T-isogenies up to isomorphism is $\psi_r(T) = \deg_x(Q)$, so every root of $Q(x)$ corresponds to such an isogeny.

Finally, for a given isogeny $f(X) = X + aX^q$, we obtain the stated relations between the coefficients of ϕ and $\tilde{\phi}$ by comparing coefficients in $f \circ \phi_T(X) = \tilde{\phi}_T \circ f(X)$. This completes the proof of Part 1.

To prove Part 2, we fix ϕ and suppose $f : \tilde{\phi} \to \phi$ is an incoming T-isogeny. Then $f(X)$ is $\mathbb{F}_q$-linear of degree q and up to isomorphism, we may assume the linear term is X, so $f(X) = X + aX^q$.

We compare coefficients in

$$f \circ \hat{f}(X) = \phi_T(X), \tag{2}$$

where

$$\hat{f}(X) = b_0 X + b_1 X^q + \cdots + b_{r-1} X^{q^{r-1}}$$

is a dual isogeny, and obtain $b_0 = T$ and

$$b_k = g_k - a b_{k-1}^q, \quad k = 1, 2, \ldots, r,$$

with the conventions $b_r = 0$ and $g_r = \Delta$. By induction on k, it follows that

$$b_k = \sum_{i=0}^{k} (-1)^i g_{k-i}^{q^i} a^{\frac{q^i - 1}{q-1}}.$$

Again, the case $k = r$ is simply $0 = Q(a)$, as required.

The same argument as in Part 1 now concludes the proof. □

We record a variant of Proposition **??**, which is more useful when computing Drinfeld modular polynomials.

Proposition 2.2. *Fix a Drinfeld module ϕ as above.*

1. *Consider the polynomial*

$$Q(x) = \sum_{i=0}^{r} (-1)^i g_{r-i} x^{\frac{q^r - q^{r-i}}{q-1}} \in L[x]$$

with the conventions $g_0 = T$ and $g_r = \Delta$. Then, up to isomorphism, all outgoing T-isogenies $f : \phi \to \tilde{\phi}$ are given by

$$f(X) = a^{-1} X + X^q$$

where a ranges over the roots of $Q(x)$. Furthermore, for each such a, the coefficients of $\tilde{\phi}$ are given by

$$\begin{aligned} \tilde{g}_k &= a^{q^k} (a^{-1} g_k + g_{k-1}^q - \tilde{g}_{k-1}), \quad k = 1, \ldots, r-1 \\ \tilde{\Delta} &= \Delta^q, \end{aligned}$$

with the conventions $g_0 = \tilde{g}_0 = T$.

2. *Consider the polynomial*

$$\tilde{Q}(x) := \sum_{i=0}^{r} (-1)^i g_{r-i}^{q^{i-1}} x^{\frac{q^i - 1}{q-1}} \in L[x]$$

with the conventions $g_0 = T$ and $g_r = \Delta$.

Then, up to isomorphism, all incoming T-isogenies $f : \tilde{\phi} \to \phi$ are given by

$$f(X) = a^{-1} X + X^q$$

where a ranges over the roots of $\tilde{Q}(x)$. Furthermore, for each such a, the coefficients of $\tilde{\phi}$ are given by

$$\tilde{g}_k = a(g_{k-1} - \tilde{g}_{k-1}^q) + a^{1-q^k} g_k, \quad k = 1, \ldots, r-1$$
$$\tilde{\Delta}^q = \Delta,$$

with the conventions $g_0 = \tilde{g}_0 = T$.

Proof. The proof is similar to that of Proposition **??**. □

3 Bounding the coefficients

In this section, we apply our description of T-isogenies (Proposition **??**, which is easier to use here than Proposition **??**) to estimate the T-degrees of the coefficients of our Drinfeld modular polynomials.

Let $F = \mathbb{F}_q(g_1, g_2, \ldots, g_{r-1})$ and consider $K = \mathbb{F}_q(T, g_1, \ldots, g_{r-1})$ as a rational function field of transcendence degree one over F. We endow K with the valuation $v : K \to \mathbb{Z}$ with uniformizer $\frac{1}{T}$, i.e. $v(T) = -1$ and $v(x) = 0$ for all $x \in F$.

Let K_T denote the splitting field of $\varphi_T(X)$ over K and extend the valuation v to any valuation of K_T. By [**?**, §2] the roots of

$$\Phi_{J(A/TA)}(X), \Phi_{J,(A/TA)^{r-1}}(X) \in K[X]$$

lie in K_T. Since the T-degree of each coefficient equals the negative of its valuation, our approach is to prove Theorem **??** by computing the valuation of the roots of these Drinfeld modular polynomials.

Proof of Theorem **??**. Applying Proposition **??** to the T-isogenies $f : \varphi \to \varphi^{(f)}$, we see that each isogeny corresponds to a root a of

$$Q(x) = 1 + \left(\sum_{i=1}^{r-1} (-1)^i g_{r-i} x^{\frac{q^r - q^{r-i}}{q-1}} \right) + (-1)^r T x^{\frac{q^r-1}{q-1}} \in K[x].$$

We extend the valuation v to any valuation on the splitting field of $Q(x)$. The Newton polygon of $Q(x)$ consists of a single line segment from the origin to the point $\left(\frac{q^r-1}{r-1}, -1\right)$. It follows that each root a has valuation

$$v(a) = \frac{q-1}{q^r-1}.$$

By Proposition **??**, the coefficients of the corresponding target Drinfeld module $\tilde{\phi} = \varphi^{(f)}$ satisfy

$$\tilde{g}_1 = g_1 + a(T^q - T)$$
$$\tilde{g}_k = g_k + a g_{k-1}^q - a^{q^{k-1}} \tilde{g}_{k-1}, \quad k = 2, 3, \ldots, r.$$

In particular, we see by induction on k that

$$v(\tilde{g}_k) = \frac{q^k - 1}{q^r - 1} - q < 0 \quad \text{for} \quad k = 1, 2, \ldots, r.$$

Since $\sum_{i=1}^{r-1} e_i(q^i - 1) = e_r(q^r - 1)$ we compute:

$$\begin{aligned} v(J(\varphi^{(f)})) &= \sum_{i=1}^{r-1} e_i v(\tilde{g}_i) - e_r v(\tilde{\Delta}) \\ &= \sum_{i=1}^{r-1} e_i \left(\frac{q^i - 1}{q - 1} - q \right) + e_r(q - 1) \\ &= -q \left(\sum_{i=1}^{r-1} e_i - e_r \right) = -w_1(J). \end{aligned}$$

Since each coefficient a_i of X^i in $\Phi_{J,(A/TA)}(X)$ is an elementary symmetric polynomial of degree $\psi_r(T) - i$ in the roots $J(\varphi^{(f)})$, the estimate in Part 1 follows.

To prove Part 2, we perform a similar calculation, this time for the incoming T-isogenies $f : \varphi^{(f)} \to \varphi$. By Proposition **??**, these correspond to roots a of

$$\tilde{Q}(x) = 1 + \left(\sum_{i=1}^{r-1} (-1)^i g_{r-i}^{q^i} x^{\frac{q^i - 1}{q-1}} \right) + (-1)^r T^{q^r} x^{\frac{q^r - 1}{q-1}} \in K[x].$$

We again extend the valuation v to the splitting field of $\tilde{Q}(x)$. The Newton polygon of $\tilde{Q}(x)$ consists of a single line segment from the origin to the point $\left(\frac{q^r - 1}{q-1}, -q^r \right)$, hence each root a has valuation

$$v(a) = q^r \left(\frac{q - 1}{q^r - 1} \right).$$

Using Proposition **??**, we find again by induction on k that the coefficients of the source Drinfeld module $\tilde{\phi} = \varphi^{(f)}$ satisfy

$$\begin{aligned} v(\tilde{g}_k) &= \frac{q^k - 1}{q^r - 1} - 1 < 0, \quad k = 1, 2, \ldots, r - 1, \\ v(\tilde{\Delta}) &= q^{r-1}(q - 1). \end{aligned}$$

Again, we compute

$$\begin{aligned} v(J(\tilde{\phi})) &= \sum_{i=1}^{r-1} e_i v(\tilde{g}_i) - e_r v(\tilde{\Delta}) \\ &= \sum_{i=1}^{r-1} e_i \left(\frac{q^i - 1}{q^r - 1} - 1 \right) - e_r q^{r-1}(q - 1) \\ &= -\sum_{i=1}^{r-1} e_i - e_r(q^r - q^{r-1} - 1) = -w_{r-1}(J). \end{aligned}$$

Part 2 now follows. ☐

Remark 3.1. *One may similarly obtain bounds on the g_i-degrees of the terms in $\Phi_{J,(A/TA)^s}(X)$ by considering a valuation v_i with uniformizer $1/g_i$. However, the resulting bounds are more complicated and less sharp. This may be a topic for future work. Similar bounds are obtained in [**?**] via analytic methods.*

4 Examples

In this section we compute some examples of Drinfeld modular polynomials and verify the bounds in Theorem **??**.

Let $J = g_1^{e_1} \cdots g_{r-1}^{e_{r-1}} \Delta^{-e_r} \in C$ be a monomial invariant. It will be more convenient to work with the isomorphic ring C' of invariants, where $\Delta = 1$ and $J = g_1^{e_1} \cdots g_{r-1}^{e_{r-1}}$.

To compute the Drinfeld modular polynomials $\Phi_{J,A/TA}(X)$, we follow the approach of [**?**]. By Proposition **??**, the isogenies $f : \varphi \to \varphi^{(f)}$ are given by $f(X) = a^{-1}X + X^q$, where a ranges over the roots of $Q(x)$. It will be convenient to instead consider the roots $v = a^{-1}$ of the reciprocal polynomial

$$P(x) = x^{\psi_r(T)} Q(x^{-1}).$$

Let M_v be an $r \times r$ matrix over K with characteristic polynomial $P(x)$ and compute its inverse $M_a = M_v^{-1}$. This plays the role of our roots a (more precisely, its eigenvalues are the a's).

Next, compute the matrices $M_{\tilde{g}_i}$ corresponding to the coefficients of the target Drinfeld module by substituting the matrix M_a into a in the expressions for $\tilde{g}_i$ given in Proposition **??**.

Finally, our polynomial $\Phi_{J,A/TA}(X)$ is the characteristic polynomial of the matrix $M_{\tilde{J}}$ corresponding to the invariant $\tilde{J} = J(\phi^{(f)})$:

$$\begin{aligned} M_{\tilde{J}} &= M_{\tilde{g}_1}^{e_1} \cdots M_{\tilde{g}_{r-1}}^{e_{r-1}}, \\ \Phi_{J,A/TA}(X) &= \det(IX - M_{\tilde{J}}) \\ &= a_0 + a_1 X + \cdots + a_{\psi_r(T)-1} X^{\psi_{r-1}(T)} + X^{\psi_r(T)} \in C'[X]. \end{aligned}$$

The polynomial $\Phi_{J,(A/TA)^{r-1}}(X)$ is computed similarly, this time using $\tilde{Q}(x)$ instead of $Q(x)$. Note that the constant coefficient of $\tilde{Q}(x)$ is $\Delta^{1/q} = 1$.

The easiest coefficient to compute is $a_{\psi_r(T)-1} = -\mathrm{Tr}\big(M_{\tilde{J}}\big)$; its degree is bounded by the weight $w_1(J)$ or $w_{r-1}(J)$ from Theorem **??**. Computing this first and checking that the result is indeed in $C'[X]$ is a valuable sanity check before attempting to compute the complete polynomial.

This approach via the characteristic polynomial has the advantage of being conceptually simple and easy to implement. It is, however, very slow, with both the time and space requirements growing rapidly with $\psi_r(T)$ and $w_s(J)$. The "small" cases reported below were computed on a mid-level (2024 era) gaming laptop using Pari/GP [**?**] and took at most a few hours each. The code can be found in [**?**].

4.1 The case $r = 3$, $q = 2$.

When $r = 3$ and $A = \mathbb{F}_2[T]$, we have $\psi_3(t) = 7$ and invariants are of the form

$$J_{e_1\,e_2} = g_1^{e_1} g_2^{e_2} \Delta^{-e_3}, \quad \text{where} \quad e_1 + 3e_2 = 7e_3.$$

The relevant weights are

$$w_1(J_{e_1\,e_2}) = 2(e_1 + e_2 - e_3) \quad \text{and} \quad w_2(J_{e_1\,e_2}) = e_1 + e_2 + 3e_3.$$

It it shown in [?] that the four invariants in the following table generate the ring C of invariants:

J	e_1	e_2	e_3	$w_1(J)$	$w_2(J)$
J_{12}	1	2	1	4	6
J_{41}	4	1	1	8	8
J_{70}	7	0	1	12	10
J_{07}	0	7	3	8	16

The polynomials $\Phi_{J,(A/TA)^2}(X)$ for these invariants were computed in [?] and one finds that all of the degree bounds from Theorem **??** are in fact equalities.

We similarly computed the polynomials $\Phi_{J,(A/TA)}(X)$ and found again that all bounds in Theorem **??** are sharp for these four invariants.

4.2 The case $r = 4$, $q = 2$.

In this case our modular polynomials have degree $\psi_2(T) = 15$. Invariants are given by

$$J_{e_1\,e_2\,e_3} = g_1^{e_1} g_2^{e_2} g_3^{e_3} \Delta^{-e_4}, \quad \text{where} \quad e_1 + 3e_2 + 7e_3 = 15e_4.$$

The relevant weights are

$$w_1(J_{e_1\,e_2\,e_3}) = 2(e_1 + e_2 + e_3 - e_4) \quad \text{and} \quad w_3(J_{e_1\,e_2\,e_3}) = e_1 + e_2 + e_3 + 7e_4.$$

We list some small invariants in C:

J	e_1	e_2	e_3	e_4	$w_1(J)$	$w_3(J)$
J_{102}	1	0	2	1	4	10
J_{050}	0	5	0	1	8	12
J_{221}	2	2	1	1	8	12

Once again, we find that the bounds in Theorem **??** are sharp in the cases of $\Phi_{J_{102},A/TA}(X), \Phi_{J_{221},A/TA}(X), \Phi_{J_{050},(A/TA)}(X)$ and $\Phi_{J_{102},(A/TA)^3}(X)$.

4.3 The case $r = 3$, $q = 3$.

In this case our modular polynomials have degree $\psi_3(T) = 13$. We list some small invariants in C:

J	e_1	e_2	e_3	$w_1(J)$	$w_2(J)$
J_{13}	1	3	1	9	21
J_{52}	5	2	1	18	24
J_{91}	9	1	1	27	27
J_{26}	2	6	2	18	42

In this case, the degree bounds from Theorem **??** are not all sharp:

	$\Phi_{J_{13},(A/TA)}(X)$		$\Phi_{J_{52},(A/TA)}(X)$		$\Phi_{J_{26},(A/TA)}(X)$	
i	$\deg_T(a_i)$	$9(13-i)$	$\deg_T(a_i)$	$18(13-i)$	$\deg_T(a_i)$	$18(13-i)$
0	117	117	234	234	234	234
1	108	108	216	216	216	216
2	**94**	99	**193**	198	**193**	198
3	90	90	180	180	180	180
4	81	81	162	162	162	162
5	**58**	72	**133**	144	**130**	144
6	**49**	63	**115**	126	**112**	126
7	**40**	54	**97**	108	**94**	108
8	**31**	45	**79**	90	**76**	90
9	36	36	72	72	72	72
10	27	27	54	54	54	54
11	**13**	18	**31**	36	**31**	36
12	9	9	18	18	18	18

	$\Phi_{J_{13},(A/TA)^2}(X)$		$\Phi_{J_{52},(A/TA)^2}(X)$	
i	$\deg_T(a_i)$	$21(13-i)$	$\deg_T(a_i)$	$24(13-i)$
0	273	273	312	312
1	252	252	288	288
2	**228**	231	**255**	264
3	210	210	240	240
4	189	189	216	216
5	**159**	168	**180**	192
6	**138**	147	**159**	168
7	**117**	126	**132**	144
8	**96**	105	**108**	120
9	84	84	96	96
10	63	63	72	72
11	**39**	42	**39**	48
12	21	21	24	24

It is very curious that in all five cases above, the coefficients where the bound fails to be sharp are $a_2, a_5, a_6, a_{13-6}, a_{13-5}$ and a_{13-2}.

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