Trajectoires spatiales - CH1

Thursday, March 20, 2025 10:46 PM

1.1.1

$\vec{C} = \vec{r} \wedge \dot{\vec{r}}$

We have : $\frac{d\vec{C}}{dt} = (\vec{r})' \wedge \dot{\vec{r}} + \vec{r} \wedge (\dot{\vec{r}})'$

$= \underbrace{\dot{\vec{r}} \wedge \dot{\vec{r}}}_{0} + \underbrace{\vec{r} \wedge \ddot{\vec{r}}}_{0}$

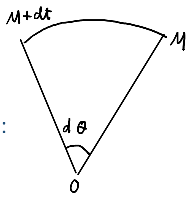
then $\boxed{\frac{d\vec{C}}{dt} = \vec{0}}$

By definition of central acceleration movement

1.1.3

The surface scanned by the vector $\vec{r}$ in one turn is : $A = \frac{\theta}{2} r^2$

We deduce : $\frac{dA}{dt} = \frac{r^2}{2}\dot{\theta}$



The infinitesimal surface scanned by unit of time is the area of the triangle :

This area can be obtained with : $\frac{1}{2} \det(\vec{OM}; \vec{OM+dt})$

$= \frac{1}{2} \begin{vmatrix} x & x+dx \\ y & y+dy \end{vmatrix} = \frac{1}{2}\left(xy + xdy - (yx + ydx)\right)$

$\boxed{dA = \frac{xdy - ydx}{2}}$

On the other hand : $\|\vec{C}\| = \|\vec{r} \wedge \dot{\vec{r}}\| = \begin{matrix} x \\ y \end{matrix} \wedge \begin{matrix} \dot{x} \\ \dot{y} \end{matrix} = x\dot{y} - y\dot{x}$

We find : $\frac{dA}{dt} = \frac{\|\vec{C}\|}{2}$

1.1.4 -- First part equation of motion

The acceleration in polar coordinates is given by : $\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\vec{u_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{u_\theta}$

By definition for central acceleration movement the acceleration is only along : $\vec{u_r}$

So we have : $\boxed{\begin{matrix} \ddot{r} - r\dot{\theta}^2 = \gamma \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \end{matrix}}$ as $\gamma = \|\ddot{\vec{r}}\|$

Detail on how $\ddot{\vec{r}}$ is obtained in polar coordinates :

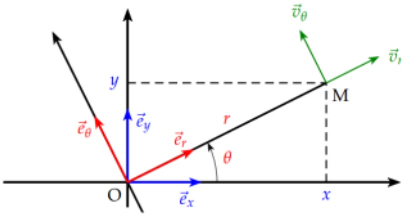
- In cartesian coordinates we can simply write $\vec{a} = \ddot{x}\,\vec{u_x} + \ddot{y}\,\vec{u_y}$
  but in polar coordinates the unit vector are not constant in time : $\frac{d\vec{u_r}}{dt} \neq 0$ and $\frac{d\vec{u_\theta}}{dt} \neq 0$
- Hence when we express : $\frac{d\vec{r}}{dt}$ in polar coordinates we have : $\frac{d\vec{r}}{dt} = \frac{d(r\,\vec{u_r})}{dt} = \dot{r}\,\vec{u_r} + r\,\frac{d\vec{u_r}}{dt}$
- Relations linking polar and cartesian coordinates can be used to obtain $\frac{d\vec{u_r}}{dt}$
- We have : $\vec{u_r} = \cos\theta\,\vec{u_x} + \sin\theta\,\vec{u_y}$
  $\vec{u_\theta} = -\sin\theta\,\vec{u_x} + \cos\theta\,\vec{u_y}$
- The following figure helps obtaining those relations :

1.3 Vecteur vitesse en coordonnées cartésiennes et coordonnées polaires

$u_{\theta} = -\sin\theta \, \vec{u_x} + \cos\theta \, \vec{u_y}$

- The following figure helps obtaining those relations :

1.3 Vecteur vitesse en coordonnées cartésiennes et coordonnées polaires



- $\dfrac{d\vec{u_r}}{dt} = \dfrac{d\vec{u_r}}{d\theta}\dfrac{d\theta}{dt} = \left(-\sin\theta\,\vec{u_x} + \cos\theta\,\vec{u_y}\right)\dot{\theta} = \vec{u_\theta}\,\dot{\theta}$
- $\dot{\vec{r}}$ is then $\boxed{\dot{\vec{r}} = \dot{r}\,\vec{u_r} + r\dot{\theta}\,\vec{u_\theta}}$
- The same process is then used to obtain : $\ddot{\vec{r}}$

$\ddot{\vec{r}} = \ddot{r}\,\vec{u_r} + \dot{r}\,\dot{\vec{u_r}} + (\dot{r}\dot{\theta} + r\ddot{\theta})\vec{u_\theta} + r\dot{\theta}\dfrac{d\vec{u_\theta}}{dt}$ and $\dfrac{d\vec{u_\theta}}{dt} = -\dot{\theta}\,\vec{u_r}$

$\boxed{\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\vec{u_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{u_\theta}}$

1.1.4 -- Second part Binet Equations

From the equation of $\dot{\vec{r}}$ in polar coordinates and by using the change of variable $u = \dfrac{1}{r}$

$\dot{\vec{r}} = \left(\dfrac{\dot{1}}{u}\right)\vec{u_r} + \dfrac{1}{u}\dot{\theta}\,\vec{u_\theta}$ Also, $\dfrac{du}{dt} = \dfrac{du}{d\theta}\dfrac{d\theta}{dt}$

$\dot{\vec{r}} = -\dfrac{du}{d\theta}\dfrac{\dot{\theta}}{u^2}\vec{u_r} + \dfrac{1}{u}\dot{\theta}\,\vec{u_\theta}$ Also, $\|\vec{C}\| = r^2\dot{\theta} = \dfrac{\dot{\theta}}{u^2}$

$\dot{\vec{r}} = -\dfrac{du}{d\theta}C\,\vec{u_r} + Cu\,\vec{u_\theta}$

$V^2 = \sqrt{\left(-\dfrac{du}{d\theta}C\right)^2 + (Cu)^2}^{\,2}$

$\boxed{V^2 = \left[\left(\dfrac{d}{d\theta}\cdot\dfrac{1}{r}\right)^2 + \dfrac{1}{r^2}\right]C^2}$

The time is obtained with $C = r^2\dot{\theta} = r^2\dfrac{d\theta}{dt} \Longleftrightarrow \boxed{dt = \dfrac{r^2\,d\theta}{C}}$

1.2.1

- We have nonhomogeneous linear second-order differential equation
- To find the general solution we solve the auxiliary polynomial : $r^2 + 1 = 0$, $r = i$ or $-i$
- The general solution form is : $e^{\alpha x}\left(C1\cos(\beta\theta) + C2\sin(\beta\theta)\right)$
  Where α and β are respectively the real and imaginary part of the conjugated solution from the auxiliary equation