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QUASI-STATIONARY DISTRIBUTIONS OF BIRTH-AND-DEATH PROCESSES

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Abstract

Let $q_n(t)$ be the conditioned probability of finding a birth-and-death process in state n at time t, given that absorption into state 0 has not occurred by then. A family $\{q_1(t), q_2(t), \cdots\}$ that is constant in time is a quasi-stationary distribution. If any exist, the quasi-stationary distributions comprise a one-parameter family related to quasi-stationary distributions of finite state-space approximations to the process.

BIRTH-AND-DEATH PROCESS: QUASI-STATIONARY DISTRIBUTION

1. Introduction

Consider a birth-and-death process on the non-negative integers with birth rates λ_i and death rates μ_i all positive except $\lambda_0 = 0$ (so that state 0 is absorbing). Assume an arbitrary initial distribution $\{p_n(0)\}$ and let $p_n(t)$ denote the probability that the system is in state n at time t. Further assume that $p_0(0) < 1$, and define

$$q_n(t) = \frac{p_n(t)}{(1-p_0(t))} \quad (n = 1, 2, 3, \cdots; t \geqq 0). \tag{1}$$

Thus $q_n(t)$ is the conditional probability of the system being in state n at time t, given that absorption has not occurred up until then. If the birth-and-death process is being used as a model of a biological population, $q_n(t)$ is the probability that a population encountered at epoch t has size n, for extinct populations are not encountered. The reasonable biological question, 'What is the probability of observing a population of size n, supposing that chance fluctuations have already been going on for a long while?' leads to the mathematical question, 'What is the behaviour of $q_n(t)$ as $t \to \infty$?' When the state space is finite, the answer is known. A theorem of Mandl then applies, telling us there is a unique stationary distribution $\{q_n\}$ with $q_n(t) \to q_n$ for each n regardless of the initial distribution. Moreover, each q_n is positive (Mandl (1960)). This is a pleasing analogue of the familiar limit theorems for unconditioned processes. Mandl used a Frobenius theorem to prove it. (The

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stationary distribution $\{q_n\}$ has sometimes been called a 'quasi-stationary distribution', but that term has also been used for the solutions of (18). Except in the title, I call $\{q_n\}$ a stationary distribution.)

In the case at hand, with infinite state space, Mandl's conclusion is not true. If one assumes that the initial distribution has all probability concentrated in a single state and also makes the 'usually harmless assumption' (as Feller calls it) that k exists for which

$$\int_0^\infty e^{ky} L\{dy\} = 1$$

where L is the (defective) renewal function of the absorbing process, then an honest probability distribution $\{q_n\}$ exists with each $q_n(t) \to q_n$. (Feller (1971), pp. 374–380.) But this result does not extend to all initial distributions. Moreover, the usually harmless assumption is impossible to verify in actual cases and, indeed, is sometimes false (as Proposition 14 below implies).

Seneta, Vere-Jones and others have undertaken to extend Mandl's proof by developing infinite-dimensional Frobenius theorems (Seneta and Vere-Jones (1966), Vere-Jones (1969), Seneta (1966), (1973)). They have found a larger class of initial conditions under which $q_n(t) \to q_n$ (Seneta and Vere-Jones (1966), p. 417), but the usually harmless assumption reappears in the hypotheses of the Frobenius theorem (Seneta (1973), Basic Assumption 3, p. 162). For some particular processes, much more has been learned by using methods peculiar to those processes. (See Section 5 below.) And so it is in this article.

In this article attention is restricted entirely to birth-and-death processes although a few results generalize easily. No use is made of Frobenius theory except for an appeal to Mandl's finite state-space result in Section 3. In Section 2 it is shown that the set of all positive stationary distributions is empty or is a one-parameter family. The connection of this family with finite state-space approximations to the process is established in Section 3. Section 4 deals with some connections of the conditioned process with a related reflecting process. Section 5 presents two illustrative examples.

2. The set of stationary distributions

The forward equations for $\{p_n(t)\}$ are (Feller (1968), p. 454, or Reuter and Ledermann (1953), Equation (59))

$$\begin{aligned} &\frac{d}{dt} p_0(t) = \mu_1 p_1(t) \\ (2)\qquad &\frac{d}{dt} p_n(t) = \lambda_{n-1} p_{n-1}(t) - (\lambda_n + \mu_n) p_n(t) + \mu_{n+1} p_{n+1}(t) \quad (n = 1, 2, 3, \cdots). \end{aligned}$$

An initial distribution with

$$\sum_{n=0}^{\infty} p_n(0)=1;\; p_0(0)<1;\; p_n(0)\geqq 0 \tag{3}$$

will always be assumed. Proposition 9 contains the condition that every solution of (2) and (3) satisfies

$$\sum_{n=0}^{\infty} p_n(t)=1 \quad (t\geqq 0). \tag{4}$$

I will evade the subtle question of when this condition holds, only observing here that the sufficient conditions for (4) of Reuter and Ledermann ((1953), Theorem 6) are founded on a restricted initial distribution which allows constants of integration to be absent from their Equation (73). Indeed, if $\lambda_0=0$; $\lambda_n=1$, $(n>0)$; $\mu_1=4$; $\mu_n=2^n+4$, $(n>1)$; then the functions

$$p_0(t)=2-2e^{-t}$$
$$p_n(t)=2^{-n}e^{-t} \quad (n\geqq 1)$$

are solutions of the forward equations (2). But in this case, (3) is true while (4) fails, even though the Reuter–Ledermann–Feller–Lundberg condition $\sum(1/\lambda_n)=\infty$ is satisfied. (The proof of Proposition 9 with $r_n=2^{-n-1}$ generates this example.)

From (1) and (2),

$$q_n'(t)=\frac{d}{dt}q_n(t)=\lambda_{n-1}q_{n-1}(t)-(\lambda_n+\mu_n)q_n(t)+\mu_{n+1}q_{n+1}(t)+\mu_1q_1(t)q_n(t) \tag{5}$$

follows by an easy calculation. If the functions $q_n(t)$ form a solution of (5), one may convert them into solutions of (2) by using the usual trick for Riccati systems (Bellman (1970), pp. 123–124, Reid (1972)). Let $p_0(t)$ be the solution of

$$\frac{d}{dt}p_0(t)=\mu_1q_1(t)(1-p_0(t)) \tag{6}$$
$$p_0(0)=0$$

and let

$$p_n(t)=q_n(t)(1-p_0(t)) \quad (n>0). \tag{7}$$

The functions $p_n(t)$ constructed in this way satisfy (2). Explicitly, they are

$$p_0(t) = 1 - \exp\left[-\int_0^t \mu_1 q_1(\tau)\,d\tau\right]$$

$$p_n(t) = q_n(t)\exp\left[-\int_0^t \mu_1 q_1(\tau)\,d\tau\right] \quad (n > 0).$$

If $q_1(t) \to q_1$ and $q_n(t) \to q_n$ as $t \to \infty$, then

$$(8) \qquad \begin{aligned} t^{-1}\log(1 - p_0(t)) &\to -\mu_1 q_1 \\ t^{-1}\log p_n(t) &\to -\mu_1 q_1. \end{aligned}$$

These latter relations may be found in a more general setting in Kingman (1963) and Vere-Jones (1969).

A sequence $\{q_1, q_2, q_3, \cdots\}$ will be called a *stationary distribution* if $q_n \geqq 0$ and

$$(9) \qquad \lambda_{n-1}q_{n-1} - (\lambda_n + \mu_n)q_n + \mu_{n+1}q_{n+1} + \mu_1 q_1 q_n = 0$$

for each $n \geqq 1$ (with the convention $q_0 = 0$ used to give (9) meaning when $n = 1$). To see that stationary distributions besides the trivial one $\{0, 0, 0, \cdots\}$ exist, let $\lambda_n = 1$ for $n = 1, 2, 3, \cdots$, let $\mu_1 = 1$ and let $\mu_n = 3$ for $n = 2, 3, 4, \cdots$. In this case, $\{1/2, 1/4, 1/8, 1/16, \cdots\}$ is a stationary distribution. Such examples are easy to find by starting with an arbitrary distribution and choice of $\{\lambda_i\}$ and solving (9) for μ_i, although careless choices of q_i or λ_i will yield negative μ_i. It is also simple to start with two distributions, $\{q_i\}$ and $\{r_i\}$ and solve for $\{\lambda_i\}$ and $\{\mu_i\}$ that make them both stationary distributions, but here great care must be taken in the choices in order to keep everything positive.

Given an arbitrary q_1, one can find $q_2, q_3, \cdots$ so that (9) holds (though some of these q_i may be negative). The following lemma makes this procedure explicit.

Lemma 1. Define a sequence $\langle f_n(x)\rangle$ of polynomials in the indeterminate x by

$$(10) \qquad \begin{aligned} f_1(x) &= x \\ f_n(x) &= \frac{\lambda_{n-1}f_{n-1}(x) + \mu_1 x\left[1 - \sum_{k=1}^{n-1} f_k(x)\right]}{\mu_n} \quad (n = 2, 3, 4, \cdots) \end{aligned}$$

A sequence $\{q_1, q_2, q_3, \cdots\}$ satisfies (9) if and only if $q_n = f_n(q_1)$.

Proof. Suppose the sequence $\{q_1, q_2, q_3, \cdots\}$ has $q_n = f_n(q_1)$. That is,

$$q_n = \frac{\lambda_{n-1}q_{n-1} + \mu_1 q_1\left[1 - \sum_{k=1}^{n-1} q_k\right]}{\mu_n}$$

or,

$$\mu_n q_n - \lambda_{n-1} q_{n-1} = \mu_1 q_1 \left(1 - \sum_{k=1}^{n-1} q_k\right) \tag{11}$$

so that

$$\begin{aligned}\lambda_{n-1}q_{n-1} - (\lambda_n + \mu_n)q_n + \mu_{n+1}q_{n+1} \\ &= (\mu_{n+1}q_{n+1} - \lambda_n q_n) - (\mu_n q_n - \lambda_{n-1}q_{n-1}) \\ &= \mu_1 q_1\left(1 - \sum_{k=1}^{n} q_k\right) - \mu_1 q_1\left(1 - \sum_{k=1}^{n-1} q_k\right) \\ &= -\mu_1 q_1 q_n\end{aligned}$$

which is (9).

Conversely, if $\{q_n\}$ satisfies (9), we can use induction to establish (11) from which (10) follows immediately. Trivially, $q_1 = f_1(q_1)$, and

$$-(\lambda_1 + \mu_1)q_1 + \mu_2 q_2 = -\mu_1 q_1^2$$

implies

$$\mu_2 q_2 - \lambda_1 q_1 = \mu_1 q_1 - \mu_1 q_1^2 = \mu_1 q_1(1 - q_1) = \mu_1 q_1\left(1 - \sum_{k=1}^{1} q_k\right).$$

For induction, assume

$$\mu_n q_n - \lambda_{n-1}q_{n-1} = \mu_1 q_1\left(1 - \sum_{k=1}^{n-1} q_k\right).$$

Then

$$\begin{aligned}\mu_{n+1}q_{n+1} - \lambda_n q_n &= \mu_{n+1}q_{n+1} - \lambda_n q_n - \mu_n q_n + \lambda_{n-1}q_{n-1} + (\mu_n q_n - \lambda_{n-1}q_{n-1}) \\ &= \mu_1 q_1 q_n + \mu_1 q_1\left(1 - \sum_{k=1}^{n-1} q_k\right) \\ &= \mu_1 q_1\left(1 - \sum_{k=1}^{n} q_k\right).\end{aligned}$$

This completes the proof.

Using the polynomials f_n we can describe the structure of the set of all stationary distributions.

***Theorem* 2.** There is a finite number $a \geqq 0$ such that

$$\{\{f_1(x), f_2(x), f_3(x), \cdots\} \mid 0 \leqq x \leqq a\}$$

is the set of all stationary distributions.

Proof. In view of Lemma 1, the only question we must settle involves the non-negativity of the components $f_i(x)$. Let a be the supremum of those numbers x which make $f_i(x) \geqq 0$ for every $i \geqq 1$. Since $f_i(0) = 0$ for each i, this a is defined and non-negative. If $x > a$ then some $f_i(x) < 0$, while $f_1(x) < 0$ if $x < 0$. The bound a cannot be infinite, since $f_2(x) < 0$ when $x > (\lambda_1 + \mu_1)/\mu_1$. Continuity of f_n ensures that each $f_n(a) \geqq 0$. From (9) it is easily seen that this actually means that each $f_n(a) > 0$, unless $a = 0$.

Suppose $0 < x < a$. Let $r_n = f_n(a)$ and $q_n = f_n(x)$. I must show that each $q_i \geqq 0$. Suppose there were a first negative $q_j = f_j(x)$. Then $j > 1$. Choose $n < j$ so that

$$\frac{q_{n-1}}{r_{n-1}} \leqq \frac{q_n}{r_n} \geqq \frac{q_{n+1}}{r_{n+1}}.$$

(If no such n exist, set $n = 1$ so that $q_n/r_n \geqq q_{n+1}/r_{n+1}$ and the following argument remains valid.) Since $\{r_i\}$ is a stationary distribution,

$$\lambda_{n-1} r_{n-1} - (\lambda_n + \mu_n) r_n + \mu_{n+1} r_{n+1} + \mu_1 r_1 r_n = 0$$

$$\lambda_{n-1} r_{n-1} \frac{q_n}{r_n} - (\lambda_n + \mu_n) r_n \frac{q_n}{r_n} + \mu_{n+1} r_{n+1} \frac{q_n}{r_n} + \mu_1 r_1 r_n \frac{q_n}{r_n} = 0$$

and since $r_{n-1},\ r_{n+1} > 0$,

$$\lambda_{n-1} r_{n-1} \frac{q_{n-1}}{r_{n-1}} - (\lambda_n + \mu_n) r_n \frac{q_n}{r_n} + \mu_{n+1} r_{n+1} \frac{q_{n+1}}{r_{n+1}} + \mu_1 r_1 r_n \frac{q_n}{r_n} \leqq 0$$

$$\lambda_{n-1} q_{n-1} - (\lambda_n + \mu_n) q_n + \mu_{n+1} q_{n+1} + \mu_1 r_1 q_n \leqq 0$$

$$\lambda_{n-1} q_{n-1} - (\lambda_n + \mu_n) q_n + \mu_{n+1} q_{n+1} + \mu_1 q_1 q_n < 0$$

contradicting Lemma 1. This shows that each $q_n \geqq 0$.

It is possible for a to be zero. See Proposition 14.

Seneta and Vere-Jones have found a one-parameter family of stationary distributions for branching processes, but it is not known whether that family contains all the stationary distributions. (For those branching processes that are birth-and-death processes the question is settled in Section 5.)

Corollary 3. If $\{q_1, q_2, \cdots\}$ and $\{r_1, r_2, \cdots\}$ are non-zero stationary distributions and $q_1 < r_1$, then

(i) $\dfrac{q_1}{r_1} < \dfrac{q_2}{r_2} \leqq \dfrac{q_3}{r_3} \leqq \dfrac{q_4}{r_4} \leqq \cdots;$

(ii) $\inf \{q_n/r_n\} > 0;$

(iii) $r \sum_{q_n} > q_1 \sum r_n;$

(iv) if $\sum r_n = \infty$, then $\sum q_n = \infty;$

(v) if $\sum q_n = \sum r_n < \infty$, then $\sum nq_n > \sum nr_n$ or both means are infinite.

Proof. Conclusion (i), which trivially implies (ii), follows from the proof of Theorem 2.

The reason for (iii) and (iv) is

$$r_1 \sum q_n = r_1 \sum r_n q_n / r_n > r_1 \sum r_n q_1 / r_1 = q_1 \sum r_n.$$

To prove (v), use $q_1 < r_1$ and $\sum q_n = \sum r_n$ together with (i) to find an N such that $r_n - q_n > 0$ if $n \leqq N$ and $r_n - q_n \leqq 0$ if $n > N$. The difference of means

$$\begin{aligned}\sum nr_n - \sum nq_n &= \sum n(r_n - q_n)\\ &= \sum_{n=1}^{N} n(r_n - q_n) + \sum_{n=N+1}^{\infty} n(r_n - q_n)\\ &< \sum_{n=1}^{N} N(r_n - q_n) + \sum_{n=N+1}^{\infty} N(r_n - q_n)\\ &= N \sum (r_n - q_n)\\ &= 0.\end{aligned}$$

3. Finite-dimensional approximations

Here I digress to treat conditioned birth-and-death processes on finite state spaces, in particular the process on $\{1, 2, \cdots, k\}$ with the same $\lambda_0, \lambda_1, \cdots, \lambda_{k-1}$ and $\mu_1, \mu_2, \cdots, \mu_k$ as our infinite state-space process. Mandl's theorem states that there is a stationary distribution $\{q_1, q_2, \cdots, q_k\}$ for this process. The following analogue of Lemma 1 tells how to find it. (Make the conventions $\lambda_k = \mu_{k+1} = q_{k+1} = q_{k+1}(t) = 0$ so that (5) and (9) make sense.)

Lemma 4. Set

$$g_n(x) = \sum_{j=1}^{n} f_j(x).$$

A sequence $\{q_1, q_2, \cdots q_k\}$ in which not every q_i is zero satisfies (9) (under the conventions just made) if and only if both $g_k(q_1) = 1$ and $q_n = f_n(q_1)$ for $n \in \{1, 2, \cdots, k\}$.

Proof. The proof of Lemma 1 establishes all of this except that $g_k(q_1)$ must be 1 if $\{q_1, q_2, \cdots, q_k\}$ is a stationary distribution.

Assume then that $\{q_1, q_2, \cdots, q_k\}$ is a stationary distribution. Observe that $q_1 \neq 0$, for if $q_1 = 0$, then each $q_k = f_k(q_1) = f_k(0) = 0$. Write

$$\lambda_{n-1} q_{n-1} - (\lambda_n + \mu_n) q_n + \mu_{n+1} q_{n+1} + \mu_1 q_1 q_n = 0$$

and sum to get

$$\sum_{n=1}^{k} [\lambda_{n-1}q_{n-1} - (\lambda_n + \mu_n)q_n + \mu_{n+1}q_{n+1} + \mu_1 q_1 q_n] = 0$$

$$-\mu_1 q_1 + \mu_1 q_1 \sum_{n=1}^{k} q_n = 0$$

$$\sum_{n=1}^{k} q_n = 1.$$

But the left-hand side of this is $g_k(q_1)$, so the proof is finished.

Because (9) is an eigenvector problem with eigenvalue $\mu_1 q_1$, Lemma 4 amounts to no more than the statement of a characteristic polynomial.

The polynomial $g_k - 1$ is of degree k and generally has k roots. From Mandl's theorem we know that exactly one of these roots, x, makes $f_1(x)$ through $f_k(x)$ all positive. In fact, x is the smallest root of $g_k - 1$.

Proposition 5. If $\{q_1, q_2, \cdots, q_k\}$ is the non-zero stationary distribution (for the conditioned process on $\{1, 2, \cdots, k\}$ just described), then $q_n = f_n(x)$, where x is the smallest solution of $g_k(x) - 1 = 0$.

Proof. In view of Mandl's theorem and Lemma 4, all that remains to prove is that x is the *smallest* root. Suppose $\{q_1, q_2, \cdots, q_k\}$ and $\{r_1, r_2, \cdots, r_k\}$ both satisfy (9) and that $0 \neq q_1 < r_1$. I show that some r_i must be negative. Suppose, on the contrary, that each $r_i \geqq 0$ (which implies each $r_i > 0$). Choose n to maximize the ratio q_n/r_n. Thus $q_{n-1}/r_{n-1} \leqq q_n/r_n \geqq q_{n+1}/r_{n+1}$. Write

$$\lambda_{n-1}r_{n-1} - (\lambda_n + \mu_n)r_n + \mu_{n+1}r_{n+1} + \mu_1 r_1 r_n = 0$$

and use r_{n-1}, $r_{n+1} > 0$ to conclude, as in the proof of Theorem 2, that

$$\lambda_{n-1}q_{n-1} - (\lambda_n + \mu_n)q_n + \mu_{n+1}q_{n+1} + \mu_1 q_1 q_n < 0,$$

a contradiction that completes the proof.

Thus the very trick that in Theorem 2 proves there are infinitely many stationary distributions, proves here that there is only one.

Write the unique positive stationary distribution on k states as $\{q_{k,1}, q_{k,2}, \cdots, q_{k,k}\}$. We will pursue the limit

$$q_n = \lim_{k \to \infty} q_{k,n}.$$

(Of course, $q_{k,n}$ is defined only if $k \geqq n$.)

Lemma 6. For every n,

$$q_{n+1,1} < q_{n,1}.$$

Proof. By Lemma 4, $g_{n+1}(q_{n+1,1})=1$ and according to Proposition 5, $q_{n+1,1}$ is the least number for which this is so. Thus, since $g_{n+1}(0)=0$ and g_{n+1} is continuous, $g_{n+1}(x)\leqq 1$ whenever $0\leqq x\leqq q_{n+1,1}$. But

$$\begin{aligned} g_{n+1}(q_{n,1}) &= 1+f_{n+1}(q_{n,1}) \\ &= 1+\frac{\lambda_n f_n(q_{n,1})+\mu_1 q_{n,1}\left[1-\sum_{j=1}^{n} f_j(q_{n,1})\right]}{\mu_{n+1}} \\ &= 1+\frac{\lambda_n f_n(q_{n,1})+\mu_1 q_{n,1}(1-1)}{\mu_{n+1}} \\ &= 1+\frac{\lambda_n q_{n,n}}{\mu_{n+1}} \\ &> 1. \end{aligned}$$

Thus $q_{n,1}$ is not in the interval $[0, q_{n+1,1})$. Hence $q_{n+1,1}<q_{n,1}$.

Proposition 7. The limits $q_n=\lim_{k\to\infty} q_{k,n}$ exist and are non-negative. Moreover, $\{q_1, q_2, \cdots\}$ is a stationary distribution and $\sum_{n=1}^{\infty} q_n\leqq 1$.

Proof. The sequence $\langle q_{k,1}\rangle$ is a decreasing sequence from $[0, 1]$ and hence has a limit in $[0, 1)$. Since $q_{k,n}=f_n(q_{k,1})$ when $k\geqq n$, the continuity of f_n ensures that q_n exists, while $q_n\in[0, 1]$ because each $q_{k,n}\in[0, 1]$.

For $k>n+1$,

$$\lambda_{n-1}q_{k,n-1}=-(\lambda_n+\mu_n)q_{k,n}+\mu_{n+1}q_{k,n+1}+\mu_1 q_{k,1}q_{k,n}=0.$$

Going to the limit as $k\to\infty$, this gives

$$\lambda_{n-1}q_{n-1}-(\lambda_n+\mu_n)q_n+\mu_{n+1}q_{n+1}+\mu_1 q_1 q_n=0$$

so that this is indeed a stationary distribution.

By a Fatou inequality,

$$\sum_{n=1}^{\infty} \lim_{k\to\infty} q_{k,n}\leqq \lim_{k\to\infty} \sum_{n=1}^{\infty} q_{k,n}=\lim_{k\to\infty} 1=1.$$

Imitating the proof of Lemma 4 gives the following result when the state space is infinite.

Lemma 8. If $\{r_1, r_2, r_3, \cdots\}$ is a non-zero stationary distribution, then

$$\sum_{n=1}^{\infty} r_n=1-\frac{1}{\mu_1 r_1}\lim_{n\to\infty} (\mu_{n+1}r_{n+1}-\lambda_n r_n).$$

Proof. Since $r_1 \neq 0$, we may write

$$\sum_{k=1}^{n} \{\lambda_{k-1} r_{k-1} - (\lambda_k + \mu_k) r_k + \mu_{k+1} r_{k+1} + \mu_1 r_1 r_k\} = \sum_{k=1}^{n} 0$$

$$-\mu_1 r_1 - \lambda_n r_n + \mu_{n+1} r_{n+1} + \mu_1 r_1 \sum_{k=1}^{n} r_k = 0$$

$$\sum_{k=1}^{n} r_k = 1 - \frac{1}{\mu_1 r_1} (\mu_{n+1} r_{n+1} - \lambda_n r_n)$$

$$\sum_{n=1}^{\infty} r_n = \lim_{n\to\infty} \sum_{k=1}^{n} r_k = 1 - \frac{1}{\mu_1 r_1} \lim_{n\to\infty} (\mu_{n+1} r_{n+1} - \lambda_n r_n).$$

The limit on the left exists, though it may be infinite, so the limit on the right exists.

Lemma 8 implies, for example, that

$$0 \leqq \sum r_n \leqq 1 \quad \text{or} \quad \sum r_n = \infty \tag{12}$$

if the sequence $\langle \lambda_n \rangle$ is bounded. But we can do better.

Proposition 9. If every solution of (2) and (3) satisfies (4), then every stationary distribution $\{r_1, r_2, r_3, \cdots\}$ has

$$\sum_{n=1}^{\infty} r_n = 0 \quad \text{or} \quad 1 \quad \text{or} \quad \infty.$$

Proof. Let $s = \sum r_n$. Assume $0 < s < \infty$. The functions $q_n(t) \equiv r_n$ comprise a solution of (5). Hence, using (6) and (7),

$$\begin{aligned} p_0(t) &= s^{-1}[1 - e^{-\mu_1 r_1 t}] \\ p_n(t) &= s^{-1} r_n e^{-\mu_1 r_1 t} \end{aligned} \tag{13}$$

is a solution of (2) and (3). But as $t \to \infty$, the sum

$$\sum_{n=0}^{\infty} p_n(t) \to \frac{1}{s},$$

so s must be 1 if (4) is to hold.

Further significance of Proposition 7 now comes to light.

Theorem 10. If the sequence $\langle \lambda_n \rangle$ is bounded (or if any other sufficient condition for (12) holds) and if the limit q_1 of Proposition 7 is not zero, then q_1 is the largest number x such that $f_n(x)$ is positive for every n. That is, q_1 equals a of Theorem 2.

Proof. Suppose $r > q_1$ has $f_n(r) > 0$ for all n. Choose $q_{k,1}$ so that $q_1 < q_{k,1} < r$.

By Corollary 3 (iii),

$$\sum_{n=1}^{\infty} f_n(q_{k,1}) < \frac{q_{k,1}}{q_1} \sum_{n=1}^{\infty} q_n,$$

and since the right-hand side is finite, $\sum_{n=1}^{\infty} f_n(q_{k,1}) \leqq 1$ because of (12). But

$$g_k(q_{k,1}) = \sum_{j=1}^{k} q_{k,j} = 1,$$

so that

$$\sum_{j=1}^{\infty} f_j(q_{k,1}) > 1,$$

a contradiction.

It must be observed here that if (12) holds and $q_1 = 0$, then either $a = 0$ or we are in a probabilistically uninteresting case where $\sum r_n = \infty$ for every non-zero stationary distribution $\{r_1, r_2, r_3, \cdots\}$. In fact, if $\sum f_n(a) \leqq 1$, then each $g_k(a) \leqq 1$ so that $q_{k,1} > a$ and $0 = q_1 \geqq a$. On the other hand, if $a > 0$ and $\sum f_n(a) > 1$ then $\sum f_n(a) = \infty$ and Corollary 3 (iv) shows that $\sum f_n(x) = \infty$ whenever $0 < x \leqq a$. For an example, let $q_n = n/2$, let $\lambda_n = 1$ and define μ_n recursively by $\mu_1 = 1$, $\mu_{n+1} = (2 + n\mu_n + n/2)/(n+1)$.

4. The conditioned probabilities and the reflecting process

Now make a new unconditioned, reflecting birth-and-death process the same as our original process except that state zero is deleted (and with it, λ_0 and μ_1). Give the letter p a new meaning for the moment. Let $p_n(t)$ be the probability of this new process being in state n at time t. The forward equations

$$\frac{d}{dt} p_n(t) = \lambda_{n-1} p_{n-1}(t) - (\lambda_n + \mu_n) p_n(t) + \mu_{n+1} p_{n+1}(t) \quad (n = 1, 2, 3, \cdots) \tag{14}$$

hold with the conventions $\lambda_0 = p_0(t) = \mu_1 = 0$ used to give (14) meaning when $n = 1$. Assume as always that $\sum p_n(0) = 1$. The following connection between $\{p_n(t)\}$ and $\{q_n(t)\}$ will give information about stationary distributions in Theorem 12. (The same old $q_n(t)$ is meant, with the *old* $p_n(t)$ and $p_0(t)$ in (1).)

Proposition 11. If for every n,

$$\sum_{k=1}^{n} q_k(0) \leqq \sum_{k=1}^{n} p_k(0)$$

then for every $t \geqq 0$ and every n,

$$\sum_{k=1}^{n} q_k(t) \leqq \sum_{k=1}^{n} p_k(t).$$

Proof. Let $\tilde{p}_k(t)=(1+\varepsilon)p_k(t)$ where $\varepsilon>0$. Let t_0 be the infimum of those t for which an n exists having

$$\sum_{k=1}^{n} q_k(t) \geqq \sum_{k=1}^{n} \tilde{p}_k(t).$$

Then even when $t=t_0$ there is an n that makes this inequality true, as we show next.

For each t, let $M(t)$ be the greatest integer for which $\sum_{k=1}^{M} q_k(t) \geqq \sum_{k=1}^{M} \tilde{p}_k(t)$ if such an integer exists and zero otherwise. Then a sequence $\langle t_n \rangle$ exists with $t_n \searrow t_0$ and $M(t_n)>0$. Let $\sigma(t)$ be the least integer σ having $\sum_{k=1}^{\sigma} \tilde{p}_k(t)>1$. Thus $M(t)<\sigma(t)$. Moreover,

$$\limsup_{t \searrow t_0} \sigma(t)<\infty$$

for if it were infinite there would be for each N a sequence $\langle t_j^N \rangle$ with $t_j^N \searrow t_0$ and $\sigma(t_j^N)>N$ so that

$$\sum_{k=1}^{N} \tilde{p}_k(t_0)=\lim_{j\to\infty} \sum_{k=1}^{N} \tilde{p}_k(t_j^N) \leqq 1$$

contradicting $\sum_{k=1}^{\infty} \tilde{p}_k(t_0)=1+\varepsilon$. Set

$$n=\limsup_{t \searrow t_0} M(t).$$

Then

$$0<n \leqq \limsup_{t \searrow t_0} \sigma(t)<\infty$$

and $\sum_{k=1}^{n} q_k(t_0) \geqq \sum_{k=1}^{n} \tilde{p}_k(t_0)$ because a sequence $\langle t_i \rangle$ exists with $\sum_{k=1}^{n} q_k(t_i) \geqq \sum_{k=1}^{n} \tilde{p}_k(t_i)$ and $t_i \searrow t_0$.

Let M be the greatest integer for which

$$\sum_{k=1}^{M} q_k(t_0) \geqq \sum_{k=1}^{M} \tilde{p}_k(t_0).$$

This M exists because

$$1=\sum_{k=1}^{\infty} q_k(t_0)<\sum_{k=1}^{\infty} \tilde{p}_k(t_0)=1+\varepsilon.$$

Because M is finite, the continuity of the functions $q_k(t)$ and $p_k(t)$ ensures that $t_0>0$. Let m be the least integer with

$$\sum_{k=1}^{m} q_k(t_0) \geqq \sum_{k=1}^{m} \tilde{p}_k(t_0).$$

Continuity ensures that

$$\sum_{k=1}^{m} q_k(t_0) = \sum_{k=1}^{m} \tilde{p}_k(t_0), \tag{15}$$

and the minimality of m thus gives $\tilde{p}_m(t_0) < q_m(t_0)$. As a consequence of the definitions of t_0 and m,

$$\frac{d}{dt}\sum_{k=1}^{m} q_k(t) \geqq \frac{d}{dt}\sum_{k=1}^{m} \tilde{p}_k(t) \quad \text{at} \quad t = t_0,$$

or

$$\mu_{m+1}q_{m+1}(t_0) - \lambda_m q_m(t_0) - \mu_1 q_1(t_0)\left[1 - \sum_{k=1}^{m} q_k(t_0)\right] \geqq \mu_{m+1}\tilde{p}_{m+1}(t_0) - \lambda_m \tilde{p}_m(t_0)$$

so

$$\mu_{m+1}q_{m+1}(t_0) - \mu_1 q_1(t_0)\left[1 - \sum_{k=1}^{m} q_k(t_0)\right]$$

$$> \mu_{m+1}\tilde{p}_{m+1}(t_0)$$

$$q_{m+1}(t_0) > \tilde{p}_{m+1}(t_0). \tag{16}$$

Adding (15) and (16) gives

$$\sum_{k=1}^{m+1} q_k(t_0) > \sum_{k=1}^{m+1} \tilde{p}_k(t_0),$$

and by continuity this holds in a neighborhood of t_0, contradicting the definition of t_0.

Thus

$$\sum_{k=1}^{n} q_k(t) < (1+\varepsilon) \sum_{k=1}^{n} \tilde{p}_k(t)$$

for all $t > 0$ and n. Since ε was arbitrary, this implies

$$\sum_{k=1}^{n} q_k(t) \leqq \sum_{k=1}^{n} p_k(t)$$

as desired.

To say that the unconditioned reflecting process is not positive recurrent is equivalent to saying that an $n > 0$ and an initial distribution exist such that $p_n(t) \to 0$ as $t \to \infty$ or to saying that $p_n(t) \to 0$ as $t \to \infty$ for every n no matter what the initial distribution may be. A necessary and sufficient condition for this is that

$$\sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{\lambda_k}{\mu_{k+1}} = \infty. \tag{17}$$

When this is the case, each $q_n(t) \to 0$ for each initial distribution.

Theorem 12. If the unconditioned, reflecting process is not positive recurrent, then

$$\lim_{t \to \infty} q_j(t) = 0$$

for any initial distribution and every j.

Proof. Set $\tilde{q}_k(t) = q_k(t+1)$ for each k. The functions $\tilde{q}_k(t)$ satisfy (3) and each $\tilde{q}_k(0) > 0$. Set

$$z_0 = 0$$

$$z_n = \left[\sum_{k=1}^{n} \tilde{q}_k(0) \right]^{1/2}.$$

Set $p_n(0) = z_n - z_{n-1}$. Then for each $n \geqq 1$,

$$0 < \sum_{k=1}^{n} \tilde{q}_k(0) < z_n = \sum_{k=1}^{n} p_k(0)$$

where the inequalities are strong because $\tilde{q}_k(0) > 0$. Thus an initial distribution $\{p_1(0)\}$ depending on $\{q_1(0)\}$ has been constructed for which Proposition 11 applies. Proposition 11 indicates that $\sum_{k=1}^{n} \tilde{q}_k(t)$ remains bounded by $\sum_{k=1}^{n} p_k(t)$ which in turn vanishes. Thus

$$\lim_{t \to \infty} q_k(t) = \lim_{t \to \infty} \tilde{q}_k(t) = 0$$

for every k.

Corollary 13. If the unconditioned reflecting process is not positive recurrent, then there is no stationary distribution $\{r_1, r_2, r_3, \cdots\}$ with

$$\sum_{n=1}^{\infty} r_n = 1.$$

Proposition 14. If the probabilities $p_n(t)$ of the reflecting process tend to non-zero limits, the conditioned probabilities $q_n(t)$ need not do the same. That is, the converse of Corollary 13 is false.

Proof. From (9), any stationary distribution must have $\mu_1 q_1 < \lambda_n + \mu_n$ for each n. Set $\lambda_n = 1/n$ and $\mu_{n+1} = 2/n$ for $n = 1, 2, 3, \cdots$ and $\mu_1 = 1$. Then the only stationary distribution is $\{0, 0, 0, \cdots\}$. On the other hand, it is easily verified that the reflecting stationary distribution is $p_n = 2^{-n} = \lim p_n(t)$.

The criterion (17) for the reflecting $p_n(t)$ tending to zero is also necessary and sufficient for the expected time to absorption being infinite in the absorbing process. Thus this is a case where the mean time to absorption is finite but $q_n(t)$ does not have a positive limit. (See Karlin (1969), Theorem 7.1, p. 205.)

Let $\{p_n\}$ be the stationary probabilities for the reflecting birth-and-death process. These numbers satisfy $\mu_{n+1}p_{n+1} = \lambda_n p_n$ or

$$\frac{p_n}{p_{n+1}} = \frac{\mu_{n+1}}{\lambda_n} \tag{18}$$

while

$$\lambda_n q_n = \mu_{n+1} q_{n+1} - \mu_1 q_1 \left(1 - \sum_{k=1}^{n-1} q_k\right)$$

for any stationary distribution $\{q_n\}$. This gives

$$\frac{q_n}{q_{n+1}} = \frac{\mu_{n+1}}{\lambda_n} - \frac{\mu_1 q_1\left(1 - \sum_{k=1}^{n-1} q_k\right)}{\lambda_n} = \frac{p_n}{p_{n+1}} - \frac{\mu_1 q_1\left(1 - \sum_{k=1}^{n-1} q_n\right)}{\lambda_n} < \frac{p_n}{p_{n+1}} \tag{19}$$

if $\sum q_k \leqq 1$ and every p_n and q_n is positive. This leads to an inequality that could also be proved using Proposition 11, as Corollary 13 was.

Proposition 15. If $\{q_n\}$ is any positive stationary distribution having $\sum q_k \leqq 1$, then

$$\sum_{j=1}^{n} q_j < \sum_{j=1}^{n} p_j$$

for every n.

Proof. If $\sum_1^n q_j \geqq \sum_1^n p_j$, then $q_i \geqq p_i$ for some $i \leqq n$. Since $q_{i+1}/q_i > p_{i+1}/p_i$, this gives $q_{i+1} > p_{i+1}$. Repeating this argument leads to $q_{i+2} > p_{i+2}$ and, indeed, $q_j > p_j$ for $j > i$. Thus $\sum_{n+1}^{\infty} q_j > \sum_{n+1}^{\infty} p_j$; and adding this inequality to $\sum_1^n q_j \geqq \sum_1^n p_j$ gives $\sum_1^{\infty} q_j > \sum_1^{\infty} p_j$, which is a contradiction, since the right side equals 1. (For, if each p_j were zero, then, by Theorem 12, so would be each q_j.)

5. The linear case and the random walk

The linear birth-and-death process (in which $\lambda_n = n\lambda$ and $\mu_n = n\mu$) happens also to be a branching process. In case the initial distribution is $\{1, 0, 0, \cdots\}$, it is well known that $\{q_n(t)\}$ has a limit, which, in our birth-and-death case, has generating function

$$G(s)=\sum_{n=1}^{\infty} q_n s^n = 1-\frac{\mu(1-s)}{(\mu-\lambda s)}.$$

(Apply Theorem 9.1, p. 317–318 of Karlin (1969).) Seneta and Vere-Jones have shown that the family of generating functions

$$1-[1-G(s)]^{\alpha} \quad (0\leqq\alpha\leqq 1) \tag{20}$$

generates a family of stationary distributions. (Seneta and Vere-Jones (1966), p. 423. Nothing is different in continuous time.) Moreover, if the initial distribution has finite mean, the conditioned probabilities $q_n(t)$ converge, as $t\to\infty$, to the distribution generated by $G(s)$.

This result illustrates Theorem 2 and Corollary 3. In fact (20) describes a one-parameter family of stationary distributions having $\{0, 0, \cdots\}$ as one limit and the distribution generated by $G(s)$ as the other limit. To see that (20) gives all the stationary distributions use Corollary 3(v). If there were a stationary distribution beyond the one given by $G(s)$, it would have a finite mean, and would therefore converge in time toward $G(s)$, a contradiction. The first component of the distribution generated by $G(s)$ is

$$q_1 = a = \frac{d}{ds}\, G(s)\,\bigg|_{s=0} = \frac{\mu-\lambda}{\mu}.$$

For the other stationary distributions, $q_1=\alpha(\mu-\lambda)/\mu$.

Another birth-and-death process for which the set of stationary distributions can be explicitly calculated is the 'random walk' in which $\lambda_n=\lambda$ and $\mu_n=\mu$ for each n. In particular, assume $\mu=1$. There is no real loss in this; it amounts to a mere change of scale of time. Further assume $\lambda<1$, for otherwise Theorem 12 says no positive stationary distributions exist. The method here is to regard (9) as a homogeneous linear difference equation with constant coefficients

$$q_{n+2}+(q_1-1-\lambda)q_{n+1}+\lambda q_n = 0.$$

Standard calculations (Richardson (1954), pp. 107–110) then give

$$q_n = \frac{q_1}{c}\left[\left(\frac{\lambda+1-q_1+c}{2}\right)^n - \left(\frac{\lambda+1-q_1-c}{2}\right)^n\right]$$

where $c=[(q_1-\lambda-1)^2-4\lambda]^{1/2}$. The validity of this solution is assured if and only if c is real, and this happens when $q_1<\lambda+1-2\lambda^{1/2}$. In the limiting case when $q_1=\lambda+1-2\lambda^{1/2}=a$ (so $c=0$), the stationary distribution is

$$q_n = nq_1\left(\frac{\lambda+1-q_1}{2}\right)^{n-1}.$$

In case c is imaginary, there will be some negative q_n. Seneta (1966) has shown that if the initial distribution has all probability concentrated in a single state, then convergence to the limiting distribution $\{f_1(a), f_2(a), \cdots\}$ can be expected for this process.

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