Aufgabe 1 (Riemann-Integral über beschränkte Mengen, 4 + 5 + 5 + 6 = 20 Punkte)

(1) Skizzieren Sie die Menge $A := \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, x \le y \le 1\}$. Bestimmen Sie das Integral $\int_A \exp(y^2)\,\mathrm{d}(x,y)$.

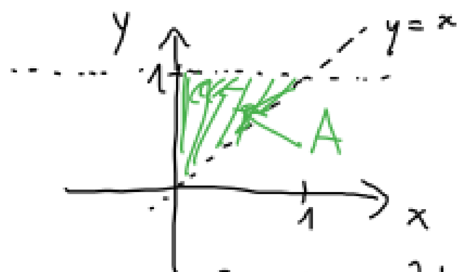
(2) Skizzieren Sie die Menge $B := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 25\}$. Bestimmen Sie das Integral $\int_B \exp(-(x^2+y^2))\,\mathrm{d}(x,y)$.

(3) Skizzieren Sie die Menge $C := \{(x,y) \in \mathbb{R}^2 : 0 \le y \le x^2, 0 \le x \le 1\}$. Bestimmen Sie das Integral $\int_C x^2 + y^2\,\mathrm{d}(x,y)$.

(4) Skizzieren Sie die Menge $D := \{(x,y) \in \mathbb{R}^2 : 1 \le x^2 + 4y^2, x^2 + y^2 \le 1\}$. Bestimmen Sie das Integral $\int_D |x| + |y|\,\mathrm{d}(x,y)$.

$\exp(y^2) = e^{y^2}$

1)



$$\int_A \exp(y^2)\,d(x,y) = \int_0^1\int_x^1 e^{y^2}\,dy\,dx = \int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1 \left[e^{y^2}\cdot x\right]_0^y dy = \int_0^1 e^{y^2}\cdot y\,dy = \left[\frac{1}{2}e^{y^2}\right]_0^1$$

$$= \frac{1}{2}e - \frac{1}{2}e^0 = \frac{1}{2}e - \frac{1}{2} = \frac{1}{2}(e-1)$$

$A = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1,\ 0 \le x \le y\}$

$$\int_0^y e^{y^2}dx = e^{y^2}(y-0) = e^{y^2}\cdot y$$

$$\left(\tfrac{1}{2}e^{y^2}\right)' = \cancel{\tfrac{1}{2}}e^{y^2}\cdot\cancel{2}y$$

$$(e^{y^2})' = e^{y^2}\cdot(y^2)' = e^{y^2}\cdot 2y$$

$$\int_a^b c\,dx = [c\cdot x]_a^b = c\cdot(b-a)$$

$$[F(x)]_a^b = F(b) - F(a)$$

$$\int x^2\,dx = \frac{1}{3}x^3 \qquad \int x^n\,dx = \frac{1}{n+1}x^{n+1}$$

$x^2+y^2 \le 25 \quad | -x^2$

$y^2 \le 25 - x^2 \quad | \sqrt{\ }$

$|y| \le \sqrt{25-x^2}$

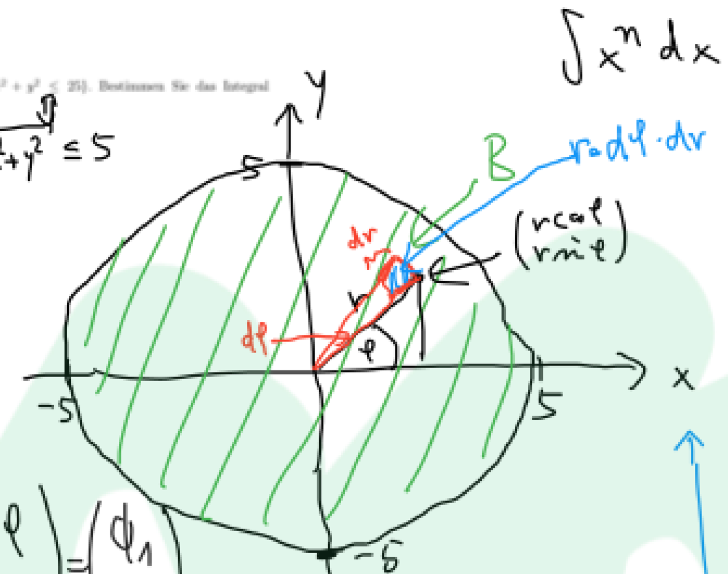
$\sqrt{25-x^2} = 5\cdot x$

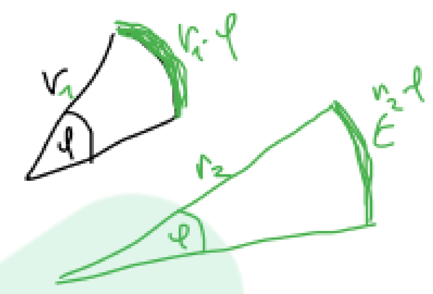
$\sqrt{a^2-b^2} \ne a-b \qquad \sqrt{y^2} = |y|$

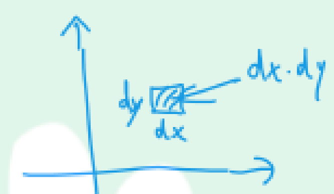
(2) Skizzieren Sie die Menge $B := \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 25\}$. Bestimmen Sie das Integral $\int_B \exp(-(x^2+y^2))\,\mathrm{d}(x,y)$.

$\sqrt{x^2+y^2} \le 5$

$B = \{(x,y) \in \mathbb{R}^2 : -5 \le x \le 5,\ -\sqrt{25-x^2} \le y \le \sqrt{25-x^2}\}$







$$\phi : (0,5)\times(0,2\pi) \to B$$

$$(r,\varphi) \mapsto \begin{pmatrix} r\cdot\cos\varphi \\ r\cdot\sin\varphi\end{pmatrix} = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

$$\int_B f(x,y)\,d(x,y) = \int_{(0,5)\times(0,2\pi)} f(\phi(r,\varphi))\cdot\left|\det J_\phi\begin{pmatrix}r\\ \varphi\end{pmatrix}\right|\,d(r,\varphi)$$

Jacobi-Matrix

$$\int_B e^{-(x^2+y^2)}\,d(x,y) = \int_0^5\int_0^{2\pi} e^{-((r\cos\varphi)^2+(r\sin\varphi)^2)}\,r\cdot d\varphi\cdot dr = \int_0^5\int_0^{2\pi} e^{-(r^2\cos^2\varphi + r^2\sin^2\varphi)}\,r\,d\varphi\,dr$$

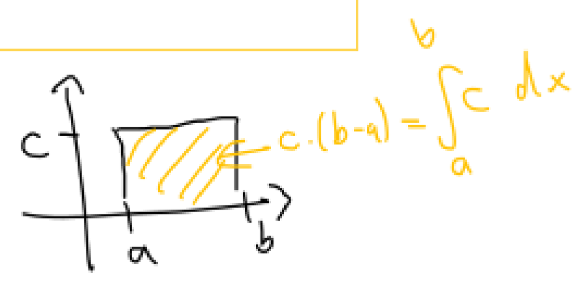
Transformationsformel mit Polarkoordinaten

$\left|\det J_\phi\begin{pmatrix}r\\ \varphi\end{pmatrix}\right|$

$r^2(\cos^2\varphi + \sin^2\varphi)$

$$= \int_0^5\int_0^{2\pi} e^{-r^2}\cdot r\,d\varphi\,dr = 2\pi\cdot\int_0^5 e^{-r^2}\cdot r\,dr = \cancel{2}\pi\cdot\left[-\cancel{\tfrac{1}{2}}e^{-r^2}\right]_0^5 = \pi\cdot\left(-e^{-25}+1\right) = \pi\cdot\left(1-e^{-25}\right)$$

$$\boxed{\int_a^b c\,dx = (b-a)\cdot c}$$



$$\left(-\tfrac{1}{2}e^{-r^2}\right)' = -\cancel{\tfrac{1}{2}}e^{-r^2}\cdot(-\cancel{2}r)$$

$$[F(x)]_a^b = F(b) - F(a)$$

$$(a + f(x))' = f'(x)$$

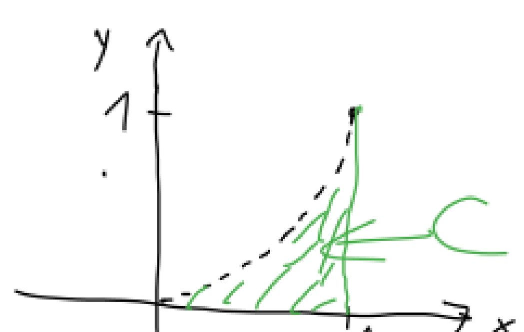
$$(a\cdot f(x))' = a\cdot f'(x)$$

$$J_\phi\begin{pmatrix}r\\ \varphi\end{pmatrix} = \begin{pmatrix}\frac{\partial\phi_1}{\partial r} & \frac{\partial\phi_1}{\partial\varphi}\\ \frac{\partial\phi_2}{\partial r} & \frac{\partial\phi_2}{\partial\varphi}\end{pmatrix} = \begin{pmatrix}\cos\varphi & r\cdot(-\sin\varphi)\\ \sin\varphi & r\cdot\cos\varphi\end{pmatrix}$$

$$\det J_\phi\begin{pmatrix}r\\ \varphi\end{pmatrix} = \cos(\varphi)\cdot r\cdot\cos\varphi - \sin(\varphi)\cdot r\cdot(-1)\cdot\sin\varphi = r\cdot\cos^2\varphi + r\sin^2\varphi = r\cdot\underbrace{(\cos^2\varphi + \sin^2\varphi)}_{=1} = r$$

$$\det\begin{pmatrix}a & b\\ c & d\end{pmatrix} = a\cdot d - c\cdot b$$

(3) Skizzieren Sie die Menge $C := \{(x,y) \in \mathbb{R}^2 : 0 \le y \le x^2, 0 \le x \le 1\}$. Bestimmen Sie das Integral $\int_C x^2+y^2\,\mathrm{d}(x,y)$.



$y = x^2$

$\int c\,dy =$

$$\int_C x^2+y^2\,d(x,y) = \int_0^1\int_0^{x^2} x^2+y^2\,dy\,dx = \int_0^1\left[\tfrac{1}{3}y^3 + x^2y\right]_0^{x^2}dx$$

$$= \int_0^1 \tfrac{1}{3}x^6 + x^4\,dx$$

$$= \left[\tfrac{1}{21}x^7 + \tfrac{1}{5}x^5\right]_0^1 = \frac{1}{21} + \frac{1}{5} = \frac{26}{105}$$