**Homework Assignment #2**
**(Due October 18, 2022)**

**QMLE**

## 1. Introduction

In this assignment, you want to verify some results of Engle and Gonzalez-Rivera (1991). They claim that in a GARCH model with non-normal innovations, the QML estimator still works (i.e., provides a consistent estimator). However, this estimator is not efficient, with a loss of efficiency increasing with the departure from normality of the innovation distribution.

We consider the following simple GARCH(1,1) model:

$$r_t = \sigma_t z_t \tag{1}$$

$$\sigma_t^2 = (1-\alpha-\beta) + \alpha\, r_{t-1}^2 + \beta\, \sigma_{t-1}^2 \tag{2}$$

Notice that we replace the parameter $\omega$ by $(1-\alpha-\beta)$ to reduce the number of parameters to estimate. The distribution *f* of $z_t$ is in principle unknown, but we will consider two cases: a Normal *N*(0,1) or a t distribution with $\nu$ degrees of freedom.

To study the relative efficiency of QMLE and MLE, Engle and Gonzalez-Rivera compute the (complicated) analytical expression. We will use Monte-Carlo simulations to verify their results.

## 2. Relative Efficiency of QMLE

The relative efficiency (RE) of the QMLE estimate of a parameter $\theta$ is defined as the ratio of its asymptotic variance when the true density *f* is known to its asymptotic variance when normality is assumed:

$$RE_\theta = \frac{V(\hat{\theta}_{MLE})}{V(\hat{\theta}_{QMLE})} \qquad \text{where} \qquad \theta = (\alpha, \beta)'$$

### 2.a. Relative Efficiency under Normality

We start by assuming that the distribution *f* of $z_t$ is in fact a Normal *N*(0,1). The first step consists in simulating a sample of return and then estimating the GARCH process. To do so, simulate a sample of *T* = 1000 observations of model (1)-(2). This is done recursively:

i) Assume some values for $\alpha$ and $\beta$: $\alpha = 0.1$ and $\beta = 0.8$. As we have defined $\omega = (1-\alpha-\beta)$, by construction we have $\sigma^2 = 1$. Assume that the initial value for the variance is $\sigma_0^2 = 1$. Also assume that the initial value for the return is $r_0 = 0$.

ii) Simulate a first observation of $z_1$ drawn from a Normal $N(0,1)$ and compute the first observation for $r_1$ using equation (1) and for $\sigma_1^2$ using equation (2). Then iterate until $t = 1000$.

iii) This gives you a first sample $S^{(1)} = (r_1^{(1)},...,r_T^{(1)})'$ of a GARCH(1,1) process. You can now estimate the GARCH model (1)-(2) with ML. It gives you estimates $(\hat{\alpha}^{(1)},\hat{\beta}^{(1)})$ and variances $(\hat{\sigma}^{(1)2}_{\hat{\alpha},ML},\hat{\sigma}^{(1)2}_{\hat{\beta},ML})$. Store these estimates in a matrix.

iv) You can now simulate several samples ($i$ = 1, …, $N$, for $N$ = 1000) of the GARCH process (by iterating on steps ii and iii). For each sample $i$, you estimate and store $(\hat{\alpha}^{(i)},\hat{\beta}^{(i)})$ and variances $(\hat{\sigma}^{(i)2}_{\hat{\alpha},ML},\hat{\sigma}^{(i)2}_{\hat{\beta},ML})$.

v) With these simulations, you have a sample of parameter variances $(\hat{\sigma}^{(1)2}_{\hat{\alpha}},...,\hat{\sigma}^{(N)2}_{\hat{\alpha}})'$ and $(\hat{\sigma}^{(1)2}_{\hat{\beta}},...,\hat{\sigma}^{(N)2}_{\hat{\beta}})'$. Compute their empirical distribution and compute quantiles.

Now, you can proceed in the same way for the QMLE. Steps i) and ii) are the same as before, as you take the same simulated samples. For Step iii), you estimate the GARCH model (1)-(2) with QML, i.e., you compute the robust standard errors $(\hat{\sigma}^{(i)2}_{\hat{\alpha},QML},\hat{\sigma}^{(i)2}_{\hat{\beta},QML})$. You can also draw their empirical distribution or compute quantiles.

Finally, you compute, for each simulated sample, the relative efficiency

$$RE_{\alpha}^{(i)} = \frac{\hat{\sigma}^{(i)2}_{\hat{\alpha},QML}}{\hat{\sigma}^{(i)2}_{\hat{\alpha},ML}} \qquad \text{and} \qquad RE_{\beta}^{(i)} = \frac{\hat{\sigma}^{(i)2}_{\hat{\beta},QML}}{\hat{\sigma}^{(i)2}_{\hat{\beta},ML}}$$

Compute their empirical distribution, mean, variance, and quantiles. What do you observe?

**2.b. Relative Efficiency under t distribution**

Now that your code is working well under normality, you can switch to a t distribution and proceed in the same way. As in Engle and Gonzalez-Rivera, use degrees of freedom $\nu = 5, 8, 12$.

For each of these three degrees of freedom, simulate innovations $z_t$ from a t with $\nu$ degrees of freedom. Estimate the GARCH model with MLE and QMLE, compute the standard errors under both approaches, and deduce the relative efficiency for $\alpha$ and $\beta$.

Compute the empirical distribution, mean, variance, and quantiles of the relative efficiency. What do you observe?