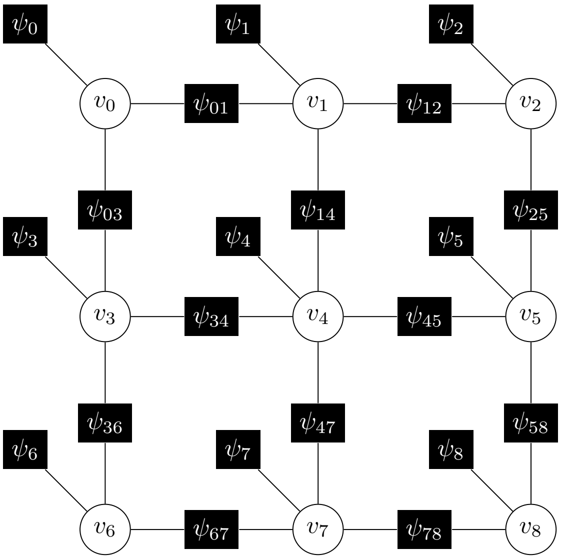
# Exercise 4

**Deadline: 19.05.2017, 2:15 pm**
**Regulations:** You should hand in the exercises in groups of two or three persons. Please send a *compressed* (!) directory or file containing your solutions including all graphics, descriptions and source code to *thorsten.beier@iwr.uni-heidelberg.de*. The subject line of this email should start with [MLCV17][EX04] followed by the full names of all group members. Please cross-reference your code files in your writeup, such that it is clear which file has to be run for each exercise.

## 1 Number of Configurations (5 points)




Consider a graphical model on a grid graph as in the picture above with the size $N \times N$ where each variable $x_i$ can take k states: $x_i \in \{0, 1, \ldots, k-1\}$.

- How many configurations can $X = \{x_0, x_1, \ldots, x_{N^2-1}\}$ take. Compare this number with the number of particles in the universe for some $N$ and $k$.
- We can formulate this graphical model as an integer linear program. How many indicator variables $\mu$ are needed for the graphical model above. What is the total number of configurations these binary indicator variables can take (ignoring the consistency constraints). How many of these configurations are allowed (Reminder: the consistency constraints rule out many configurations).

## 2 Construct $\psi$ (5 Points)

Consider a graphical model on a grid graph as in the picture above with the size $N \times N$ where each variable can take 2 states: $X \in \{0, 1\}^{N^2}$.

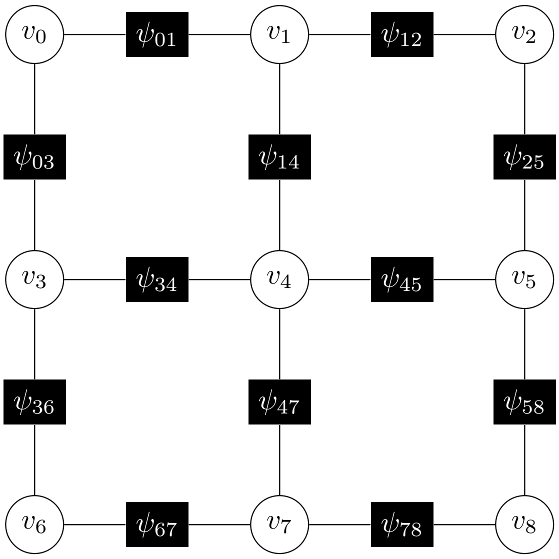
- Can we choose the potentials $\psi_i$ and $\psi_{ij}$ such that an arbitrary $X^*$ is the optimal labeling:

  $$X^* = \operatorname{argmin}_{X \in \{0,1\}^{N^2}} E(X). \tag{1}$$

  (In words: describe a strategy how to construct $\psi_i$ and $\psi_{ij}$ such that the optimal labeling takes an desired value)

# 3 Attractive vs. Repulsive Potts Model : (10 Points)

## 3.1 Grid Graphical Model




Consider a potts model on a grid graphical model without unary terms. The potts regularizer is given by:

$$\psi_{ij}(x_i, x_j) = \begin{cases} 0 & \text{if } x_i = x_j \\ \beta & \text{if } x_i \neq x_j \end{cases} \tag{2}$$

- Find a labeling $X$ such that $E(X) = 0$ for $\beta = 1$.
- Find a labeling $X$ such that $E(X) = 0$ for $\beta = -1$.
- Is it possible for to find an $X$ such that $E(X) = 0$ for $\beta = 1$ for all potts models with binary variables regardless of the structure / topology of the graphical model? Describe how.
- Is it possible for to find an $X$ such that $E(X) = 0$ for $\beta = -1$ for all potts models with binary variables regardless of the structure / topology of the graphical model? If not, which properties must be fulfilled to find an X such that $E(X) = 0$ for $\beta = -1$ .

## 3.2 Graphical Model with Irregular Structure

We again consider a potts model without unary terms. This time, the structure of the graphical model is arbitrary (not necessarily a grid). The potts regularizer is again given by:

$$\psi_{ij}(x_i, x_j) = \begin{cases} 0 & \text{if } x_i = x_j \\ \beta & \text{if } x_i \neq x_j \end{cases} \tag{3}$$

- Is it possible for to find an $X$ such that $E(X) = 0$ for $\beta = 1$ for all potts models with binary variables regardless of the structure / topology of the graphical model? Describe how.
- Is it possible for to find an $X$ such that $E(X) = 0$ for $\beta = -1$ for all potts models with binary variables regardless of the structure / topology of the graphical model? If not, which properties must be fulfilled to find an X such that $E(X) = 0$ for $\beta = -1$ .

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# 4 Bonus (15 Points)

All corners of the marginal polytope are integer solutions. Proof that the marginal polytope has no feasible integer solutions which are not corners of the polytope. **Hint:** The ingredients for the proof are given in exercise 2.