18.03.2021

SEMINAR 5

Subspații vectoriale. Aplicații liniare

OBS

- $A \in \mathcal{M}_{m,n}(\mathbb{R})$
  $S(A) = \{x \in \mathbb{R}^n \mid AX = 0\} \subset \mathbb{R}^n$ subsp. vect
  $\dim S(A) = n - \operatorname{rg} A$ $\qquad \underset{(m,n)}{A}\underset{(n,1)}{X} = \underset{(m,1)}{0}$
- $\forall V' \subseteq V$ subsp. vect,
  $\Rightarrow$ coord. vect din V', în raport cu $\forall$ reper, sunt soluțiile unui SLO i.e. A ai $V' = S(A)$

EX1 $(\mathcal{M}_2(\mathbb{C}), +, \cdot)/_{\mathbb{C}}$

a) $R = \left\{ J_2, P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, P_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$
reper în $\mathcal{M}_2(\mathbb{C})$ (matrice Pauli)

b) $R_0 \xrightarrow{A} R$, $A = ?$, R_0 = reperul canonic.

c) Să se afle coord lui M în rap cu R,
$M = \begin{pmatrix} 1 & 2i \\ 3 & i \end{pmatrix}$

d) $P_k^2 = J_2$, $\forall k = \overline{1,3}$, $P_a P_b = i\varepsilon_\sigma P_c$, $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$
$\varepsilon(\sigma) = (-1)^{m(\sigma)}$

e) Dați exemple de subspații care verifică
$\mathcal{M}_2(\mathbb{C}) = V_1 \oplus V_2 = W_1 \oplus W_2 \oplus W_3 = U_1 \oplus U_2 \oplus U_3 \oplus U_4$.

SOL

a) $R_0 = \left\{ E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$ matricea compon. lui R în rap. cu R_0

$J_2 = E_{11} + E_{22}$; $P_1 = E_{12} + E_{21}$, $P_2 = -iE_{12} + iE_{21}$
$P_3 = E_{11} - E_{22}$

$$\det A = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -i & 0 \\ 1 & i & 0 \\ 0 & 0 & -2 \end{vmatrix} = -2\begin{vmatrix} 1 & -i \\ 1 & i \end{vmatrix} = -4i \neq 0$$

$c_4' = c_4 - c_1$

$\left.\begin{array}{l} rg A = 4 = max \Longrightarrow \mathcal{R} \text{ este SLI} \\ \dim_{\mathbb{C}} \mathcal{M}_2(\mathbb{C}) = 4 = |\mathcal{R}| \end{array}\right\} \Rightarrow \mathcal{R}$ reper în $\mathcal{M}_2(\mathbb{C})$

b) $\mathcal{R}_0 \xrightarrow{A} \mathcal{R}$, A matrice de trecere

c) $M = \begin{pmatrix} 1 & 2i \\ 3 & i \end{pmatrix} = aJ_2 + bP_1 + cP_2 + dP_3$

$= a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + d\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} a+d & b-ci \\ b+ci & a-d \end{pmatrix}$

$\begin{cases} a+d = 1 \\ a-d = i \end{cases}$ $\oplus \ominus$

$2a = 1+i$

$2d = 1-i$

$\begin{cases} a = \frac{1+i}{2} \\ d = \frac{1-i}{2} \end{cases}$

$\begin{cases} b+ci = 3 \\ b-ci = 2i \end{cases} \Rightarrow$

$2b = 3+2i \Rightarrow b = \frac{3+2i}{2}$

$2ci = 3-2i \Rightarrow c = \frac{3-2i}{2i} = \frac{-i(3-2i)}{2}$

$c = \frac{-2-3i}{2}$

$(a,b,c,d) = \left(\frac{1+i}{2}, \frac{3+2i}{2}, \frac{-2-3i}{2}, \frac{1-i}{2}\right)$

coord lui M în rap cu $\mathcal{R}$

d) $P_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = J_2$

$P_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = J_2$

$P_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = J_2.$

• $P_1P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\cdot 1\overbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}^{P_3}$; $P_aP_b = i\,\varepsilon_\sigma P_c$

$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$

$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = e \Rightarrow \varepsilon(\sigma) = 1$

$$P_2P_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i \cdot 1 \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}^{P_1}$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \varepsilon(\sigma) = (-1)^2 = 1$$

Analog restul (sunt 6 relații în total)

e) $\mathcal{M}_2(\mathbb{C}) = V_1 \oplus V_2$, $V_1 = \langle\{I_2, P_1\}\rangle$, $V_2 = \langle\{P_2, P_3\}\rangle$

$\mathcal{M}_2(\mathbb{C}) = W_1 \oplus W_2 \oplus W_2$, $W_1 = \langle\{I_2\}\rangle$, $W_2 = \langle\{P_1\}\rangle$, $W_2 = \langle\{P_2, P_3\}\rangle$.

$\mathcal{M}_2(\mathbb{C}) = U_1 \oplus U_2 \oplus U_3 \oplus U_4$, $U_1 = \langle\{I_2\}\rangle$, $U_2 = \langle\{P_1\}\rangle$

$U_3 = \langle\{P_2\}\rangle$; $U_4 = \langle\{P_3\}\rangle$

OBS

$V = \langle S\rangle = \langle\{u, v\}\rangle = \{au + bv,\ a, b \in \mathbb{R}\}$

$\{u, v\}$ SG

EX2 $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$, $S = \{(1,2,3), (-1,1,5)\}$

$S' = \{(1,5,11), (2,1,-2), (3,6,9)\}$

a) $\dim\langle S\rangle$, $\dim\langle S'\rangle$

b) $\langle S\rangle = \langle S'\rangle = V'$

c) Să se descrie V' printr-un sistem de ec. liniare

d) Să se determine V'' un subsp. complementar lui V'

e) Să se descompună $x = (1,2,7)$ în rap cu $\mathbb{R}^3 = V' \oplus V''$.

SOL

a) $rg\begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 5 \end{pmatrix} = 2 \Rightarrow S$ este SLI

$\dim\langle S\rangle = 2$

$rg\begin{pmatrix} 1 & 2 & 3 \\ 5 & 1 & 6 \\ 11 & -2 & 9 \end{pmatrix} = 2 \Rightarrow S'$ este SLD

$c_3 = c_1 + c_2$

$S'' = \{(1,5,11), (2,1,-2)\}$ e SLI maximal în S'

$\langle S'\rangle = \langle S''\rangle$ $\dim S' = 2$

$\langle S' \rangle = \{ a(1,5,11) + b(2,1,-2) + c\underbrace{(3,6,9)}_{(1,5,11)+(2,1,-2)},\ a,b,c \in \mathbb{R}$

$= \{ \underbrace{(a+c)}_{a'}(1,5,11) + \underbrace{(b+c)}_{b'}(2,1,-2),\ a,b,c \in \mathbb{R} \}$

$= \{ a'(1,5,11) + b'(2,1,-2) \} = \langle S'' \rangle$

b) Dem că $\exists a,b \in \mathbb{R}$ a.î. $(1,5,11) = a(1,2,3) + b(-1,1,5)$

$\begin{cases} a-b=1 \\ 2a+b=5 \\ 3a+5b=11 \end{cases}$ $\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 1 & 5 \\ 3 & 5 & 11 \end{array}\right)$

SCD $\iff \Delta_c = 0 \iff \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 5 \\ 3 & 5 & 11 \end{vmatrix} = 0$

$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 3 \\ 3 & 8 & 8 \end{vmatrix} = 0$ $(c_2' = c_2 + c_1;\ c_3' = c_3 - c_1)$

Dem că $\exists a', b' \in \mathbb{R}$ a.î. $(2,1,-2) = a'(1,2,3) + b'(-1,1,5)$

SCD $\iff \Delta_c = 0 \iff \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 3 & 5 & -2 \end{vmatrix} = 0$

$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & -3 \\ 3 & 8 & -8 \end{vmatrix} = 0$

$\left.\begin{array}{l} \langle S' \rangle \subseteq \langle S \rangle \\ \dim \langle S' \rangle = \dim \langle S \rangle \end{array}\right\} \Rightarrow \langle S' \rangle = \langle S \rangle \overset{not}{=} V'$

c) $V' = \langle \{ (1,2,3), (-1,1,5) \} \rangle = \{ x \in \mathbb{R}^3 \mid 7x_1 - 8x_2 + 3x_3 = 0 \}$

$= S(A)$

$A = (7\ \ -8\ \ 3)$

$\forall x = (x_1, x_2, x_3) \in V'$, $\exists a'', b'' \in \mathbb{R}$ a.î.

$(x_1, x_2, x_3) = a''(1,2,3) + b''(-1,1,5)$

$\begin{cases} a'' - b'' = x_1 \\ 2a'' + b'' = x_2 \\ 3a'' + 5b'' = x_3 \end{cases}$ $\left(\begin{array}{cc|c} 1 & -1 & x_1 \\ 2 & 1 & x_2 \\ 3 & 5 & x_3 \end{array}\right)$

SCD $\iff \Delta_c = 0 \iff \begin{vmatrix} 1 & -1 & x_1 \\ 2 & 1 & x_2 \\ 3 & 5 & x_3 \end{vmatrix} = 0 \Rightarrow$

$x_1(10-3) - x_2(5+3) + x_3(1+2) = 0$

d) $\mathbb{R}^3 = V' \oplus V''$

$rg\begin{pmatrix} 1 & -1 & \boxed{0} \\ 2 & 1 & \boxed{0} \\ 3 & 5 & \boxed{1} \end{pmatrix} = 3$, $V'' = <\{(0,0,1)\}>$

e) $x = (1,2,7) = \underset{\in V'}{u'} + \underset{\in V''}{u''} = \overbrace{a(1,2,3) + b(-1,1,5)}^{\in V'} + \overbrace{c(0,0,1)}^{V''}$

$\begin{cases} a-b=1 \\ 2a+b=2 \\ 3a+5b+c=7 \end{cases}$ $\Rightarrow 3a=3 \Rightarrow a=1, b=0.$ $\Rightarrow c = 7-3 = 4$

$u' = (1,2,3)$, $u'' = (0,0,4)$

Ex $(\mathbb{R}^3,+,\cdot)_{/\mathbb{R}}$, $V' = \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, \begin{cases} x-y+2z=0 \\ 2x+y+z=0 \end{cases}\right\}$

a) Precizati un reper în V'

b) Det V'' un subsp. complementar lui V'

c) Să se descompună $x = (1,2,3)$ în rap cu $\mathbb{R}^3 = V' \oplus V''$.

SOL

a) $A = \begin{pmatrix} \boxed{\begin{matrix} 1 & -1 \\ 2 & 1 \end{matrix}} & \begin{matrix} 2 \\ 1 \end{matrix} \end{pmatrix}$, $V' = S(A)$

$\dim V' = 3 - rg A = 3 - 2 = 1$

$\begin{cases} x-y = -2z \\ 2x+y = -z \end{cases}$

$3x \quad / = -3z$

$x = -z$

$y = -z + 2z = z$

$V' = \{\underbrace{(-z,z,z)}_{= z(-1,1,1)} \mid z \in \mathbb{R}\} = <\{(-1,1,1)\}>$

$\left.\begin{array}{l} R' = \{(-1,1,1)\} \text{ un SG} \\ (-1,1,1) \neq 0_{\mathbb{R}^3} \Rightarrow R' \text{ e SLI} \end{array}\right\} \Rightarrow R' \text{ reper în } V'$

b) $\mathbb{R}^3 = V' \oplus V''$

$rg\begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 3 = max$, $V'' = <\{(0,1,0),(0,0,1)\}>$

$R'' = \{(0,1,0),(0,0,1)\}$ SG, $\dim V'' = |R''| = 2$ $\Rightarrow$ reper

$\mathcal{R} = \mathcal{R}' \cup \mathcal{R}''$ reper în $\mathbb{R}^3$

c) $(1,2,3) = \underset{\overset{\cap}{V'}}{u'} + \underset{\overset{\cap}{V''}}{u''} = (1,-1,-1) + (0,3,4)$

$(1,2,3) = \overbrace{a(-1,1,1)}^{u'} + \overbrace{b(0,1,0) + c(0,0,1)}^{u''}$

$= (-a, a+b, a+c)$

$-a = 1 \Rightarrow a = -1$

$a+b = 2 \Rightarrow b = 3$

$a+c = 3 \Rightarrow c = 4.$

Ex $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$, $V' = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$

$V'' = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = 0\}$ $\begin{cases} x+0y+0z=0 \\ 0\cdot x+y+0z=0 \end{cases}$

$\mathbb{R}^3 = V' \oplus V''$

SOL

$V' = S(A')$, $A' = (1 \ \ 1 \ \ 1)$, $\dim V' = 3-1 = 2$

$V'' = S(A'')$, $A'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\dim V'' = 3-2 = 1$

$V' \cap V'' = \{0_{\mathbb{R}^3}\} \Rightarrow$ "$\oplus$"

$\dim(V' \oplus V'') \overset{T.G}{=} \dim V' + \dim V'' = 2+1 = 3$

$\left.\begin{array}{l} V' \oplus V'' \subseteq \mathbb{R}^3 \\ \dim(V' \oplus V'') = \dim \mathbb{R}^3 = 3 \end{array}\right\} \Rightarrow \mathbb{R}^3 = V' \oplus V''$

Aplicații liniare

OBS $(V_i, +, \cdot)/\mathbb{K}$, $i = \overline{1,2}$ sp. vect

$f: V_1 \to V_2$ s.n. aplicație liniară $\Leftrightarrow$

1) $f(x+y) = f(x) + f(y)$

2) $f(ax) = a f(x)$, $\forall x, y \in V_1$, $\forall a \in \mathbb{K}$.

- f liniară $\Leftrightarrow f(ax+by) = a f(x) + b f(y)$, $\forall a, b \in \mathbb{K}$, $x, y \in V_1$

$\ker f = \{x \in V_1 \mid f(x) = 0_{V_2}\}$ nucleul lui f

- $\operatorname{Im} f = \{y \in V_2 \mid \exists x \in V_1$ a.î. $f(x) = y\}$ imaginea lui f

f inj $\Leftrightarrow \ker f = \{0_{V_1}\}$

f surj $\Leftrightarrow \dim_{\mathbb{K}} \operatorname{Im} f = \dim_{\mathbb{K}} V_2$

(T) $f: V_1 \to V_2$ lin

$\dim V_1 = \dim \ker f + \dim \operatorname{Im} f$

Ex $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1 + x_2, -x_2)$

a) f liniară + bij ($f \in \operatorname{Aut}(\mathbb{R}^2)$)

b) $\ker f$, $\operatorname{Im} f$

SOL

a) $f(x+y) = f(x_1+y_1, x_2+y_2) = (x_1+y_1+x_2+y_2, -(x_2+y_2))$

$= (x_1+x_2, -x_2) + (y_1+y_2, -y_2) = f(x) + f(y)$

$f(ax) = f(ax_1, ax_2) = (ax_1 + ax_2, -ax_2) = a(x_1+x_2, -x_2) = af(x)$

$\forall x, y \in \mathbb{R}^2, \forall a \in \mathbb{R}$

$\Rightarrow f$ liniară.

b) $\ker f = \{x \in \mathbb{R}^2 \mid f(x) = 0_{\mathbb{R}^2}\} = \left\{x \in \mathbb{R}^2 \,\middle|\, \begin{cases} x_1 + x_2 = 0 \\ -x_2 = 0 \end{cases}\right\}$

$= S(A)$, $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

$\ker f = \{0_{\mathbb{R}^2}\} \Rightarrow f$ injectivă

- $\underline{T\dim}$ $\dim \mathbb{R}^2 = \underbrace{\dim \ker f}_{\overset{\shortparallel}{0}} + \dim \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = \underset{\dim \mathbb{R}^2}{\overset{\shortparallel}{2}}$

$\Rightarrow f$ surj, $\operatorname{Im} f = \mathbb{R}^2$

f lin $+ f$ bij $\Rightarrow f \in \operatorname{Aut}(\mathbb{R}^2)$

OBS $y \in \operatorname{Im} f \Rightarrow \exists x \in \mathbb{R}^2$ a.î. $f(x) = y \Rightarrow \begin{cases} x_1 + x_2 = y_1 \\ -x_2 = y_2 \end{cases}$ $\left(\begin{matrix} 1 & 1 \\ 0 & -1 \end{matrix} \,\middle|\, \begin{matrix} y_1 \\ y_2 \end{matrix}\right)$

SCD, $\forall y \in \mathbb{R}^2 \Rightarrow \operatorname{Im} f = \mathbb{R}^2$

Ex $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1,x_2,x_3) = (x_1+2x_2+x_3, 2x_1+5x_2+3x_3, -3x_1-7x_2$

a) f liniară
b) Ker f = ? Precizați un reper în Ker f
c) Im f = ? — " — Im f

SOL

a) $f(ax+by) = f(ax_1+by_1, ax_2+by_2, ax_3+by_3)$

$= (ax_1+by_1+2(ax_2+by_2)+ax_3+by_3, 2(ax_1+by_1)+5(ax_2+by_2)+3(ax_3+by_3),$
$-3(ax_1+by_1)-7(ax_2+by_2)-4(ax_3+by_3))$

$= a(x_1+2x_2+x_3, 2x_1+5x_2+3x_3, -3x_1-7x_2-4x_3) +$
$b(y_1+2y_2+y_3, 2y_1+5y_2+3y_3, -3y_1-7y_2-4y_3)$

$= af(x)+bf(y), \forall x,y \in \mathbb{R}^3, \forall a,b \in \mathbb{R}$

b) $\operatorname{Ker} f = \{x \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3}\} = \left\{x \in \mathbb{R}^3 \,\middle|\, \begin{cases} x_1+2x_2+x_3=0 \\ 2x_1+5x_2+3x_3=0 \\ -3x_1-7x_2-4x_3=0 \end{cases}\right\}$

$= S(A)$, $A = \begin{pmatrix} \boxed{\begin{matrix} 1 & 2 \\ 2 & 5 \end{matrix}} & \begin{matrix} 1 \\ 3 \end{matrix} \\ \begin{matrix} -3 & -7 \end{matrix} & -4 \end{pmatrix}$ $\quad l_3 = -(l_1+l_2)$

$\dim \operatorname{Ker} f = 3 - \operatorname{rg} A = 3-2 = 1;$ $\quad x_1 = x_3$

SCsN $\begin{cases} x_1+2x_2 = -x_3 & \mid \cdot(-2) \\ 2x_1+5x_2 = -3x_3 \end{cases} \oplus$

$\quad x_2 = -x_3$

$\operatorname{Ker} f = \{(x_3, -x_3, x_3),\ x_3 \in \mathbb{R}\} = \langle\{(1,-1,1)\}\rangle$

$R' = \{(1,-1,1)\}$ reper în Ker f (SG, $(1,-1,1) \neq 0_{\mathbb{R}^3} \Rightarrow$ SLi)
SAU
SG, $|R'| = \dim \operatorname{Ker} f = 1 \Rightarrow$ reper

c) $y \in \operatorname{Im} f \Rightarrow \exists x \in \mathbb{R}^3$ aî $f(x) = y$

$\begin{cases} x_1+2x_2+x_3 = y_1 \\ 2x_1+5x_2+3x_3 = y_2 \\ -3x_1-7x_2-4x_3 = y_3 \end{cases}$ $\quad \left(\begin{matrix} \boxed{\begin{matrix} 1 & 2 \\ 2 & 5 \end{matrix}} & \begin{matrix} 1 \\ 3 \end{matrix} \\ \begin{matrix} -3 & -7 \end{matrix} & -4 \end{matrix} \middle| \begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix}\right)$

SCsN $\Leftrightarrow \Delta_c = 0 \Rightarrow \begin{vmatrix} 1 & 2 & y_1 \\ 2 & 5 & y_2 \\ -3 & -7 & y_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 2 & y_1 \\ 2 & 5 & y_2 \\ 0 & 0 & y_1+y_2+y_3 \end{vmatrix} = 0$

$y_1+y_2+y_3 = 0.$

Scanned with CamScanner

$\operatorname{Im} f = \{y \in \mathbb{R}^3 \mid y_1 + y_2 + y_3 = 0\} = S(A)$, $A = (1 \ 1 \ 1)$

$\dim \operatorname{Im} f = 3 - 1 = 2$ $\left[\dim \mathbb{R}^3 = \underset{\substack{\| \\ 1}}{\dim \operatorname{Ker} f} + \dim \operatorname{Im} f\right]$

$\operatorname{Im} f = \{(-y_2 - y_3, y_2, y_3) \mid y_2, y_3 \in \mathbb{R}\} = \langle\{(-1,1,0),(-1,0,1)\}\rangle$

$\left.\begin{array}{l} R'' = \{(-1,1,0),(-1,0,1)\} \text{ SG} \\ |R''| = \dim \operatorname{Im} f = 2 \end{array}\right\} \Rightarrow R''$ reper în $\operatorname{Im} f$

<u>Ex</u> $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x) = (3x_1 - 2x_2, 2x_1 - x_2, -x_1 + x_2)$

a) f lin; b) f inj; c) $\operatorname{Im} f$

<u>SOL</u>

a) $f(ax + by) = af(x) + bf(y)$, $\forall x, y \in \mathbb{R}^2$, $\forall a, b \in \mathbb{R}$

b) $\operatorname{Ker} f = \{x \in \mathbb{R}^2 \mid f(x) = 0_{\mathbb{R}^3}\} = \left\{x \in \mathbb{R}^2 \;\middle|\; \begin{cases} 3x_1 - 2x_2 = 0 \\ 2x_1 - x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases}\right\}$

$= S(A)$ $\quad A = \left(\begin{array}{cc|c} 3 & -2 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right)$

$\operatorname{Ker} f = \{0_{\mathbb{R}^2}\} \Rightarrow f$ inj

c) $y \in \operatorname{Im} f \Rightarrow \exists x \in \mathbb{R}^2$ a.î. $f(x) = y$

$\begin{cases} 3x_1 - 2x_2 = y_1 \\ 2x_1 - x_2 = y_2 \\ -x_1 + x_2 = y_3 \end{cases}$ $\quad A = \left(\begin{array}{cc|c} 3 & -2 & y_1 \\ 2 & -1 & y_2 \\ -1 & 1 & y_3 \end{array}\right)$

SCD $\Leftrightarrow \Delta_c = 0 \Leftrightarrow \begin{vmatrix} 3 & -2 & y_1 \\ 2 & -1 & y_2 \\ -1 & 1 & y_3 \end{vmatrix} = 0 \Rightarrow y_1(2-1) - y_2(3-2) + y_3(-3+4) = 0$

$\operatorname{Im} f = \{y \in \mathbb{R}^3 \mid y_1 - y_2 + y_3 = 0\}$

$= \{(y_1, y_1 + y_3, y_3),\ y_1, y_3 \in \mathbb{R}\} = \langle\underbrace{\{(1,1,0),(0,1,1)\}}_{R}\rangle$

$\left.\begin{array}{l} R \text{ este SG} \\ R \text{ SLI} \end{array}\right\} \Rightarrow$ reper $\quad \dim \operatorname{Im} f = 2$

<u>OBS</u> $\dim \mathbb{R}^2 = \underset{\substack{\| \\ 0}}{\dim \operatorname{Ker} f} + \dim \operatorname{Im} f$