C_9 - GA

Spații vectoriale euclidiene

Procedeul Gram-Schmidt

Def $(V,+,\cdot)$ sp. vect. real.

$g: V\times V \to \mathbb{R}$ s.n. produs scalar $\iff$ 1) $g \in L^s(V,V;\mathbb{R})$
2) g poz. definită

OBS A da un produs scalar $\iff$ a declara un reper ortonormat $R=\{e_1,\dots,e_n\}$
i.e. $g(e_i,e_j)=\delta_{ij}$

"$\Rightarrow$" g produs scalar $\Rightarrow R=\{e_1,\dots,e_n\}$ reper cu $g(e_i,e_j)=\delta_{ij}$
$\forall i,j=\overline{1,n}$

"$\Leftarrow$" $R=\{e_1,\dots,e_n\}$ reper ortonormat
Construim $g\in L^s(V,V;\mathbb{R})$, p.def a.î. $g(e_i,e_j)=\delta_{ij}$
Prelungim prin liniaritate în ambele argumente:

$$g(x,y)=g\Big(\sum_{i=1}^n x_ie_i,\ \sum_{j=1}^n y_je_j\Big)=\sum_{i,j=1}^n x_iy_j\underbrace{g(e_i,e_j)}_{\delta_{ij}}=$$
$$=\sum_{i=1}^n x_iy_i$$

Exemplu $(\mathbb{R}^n,+,\cdot)|\mathbb{R}$, $g_0:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, $g_0(x,y)=\sum_{i=1}^n x_iy_i$

Not $g_0=\langle\cdot,\cdot\rangle=(\cdot,\cdot)$ produs scalar canonic.

$R_0=\{e_1,\dots,e_n\}$ reperul canonic este ortonormat i.e. $g_0(e_i,e_j)=\delta_{ij}$

Def (produs vectorial)
$((\mathbb{R}^3,+,\cdot),g_0)$ sp. vect. euclidian canonic
Fie $S=\{x,y\}\subset\mathbb{R}^3$ sistem de vectori.
Definim $z=x\times y$ (produsul vectorial) a.î.

1. Dc S este SLD, at $z=0$
2. Dc S este SLI, at
   a) $\|z\|^2=g_0(z,z)=Q(z)=\begin{vmatrix} g_0(x,x) & g_0(x,y)\\ g_0(y,x) & g_0(y,y)\end{vmatrix}$
   b) $z\perp x$, $z\perp y$ i.e. $g_0(z,x)=g_0(z,y)=0$
   c) $R=\{x,y,z\}$ reper pozitiv orientat

i.e. $\underset{\substack{\| \\ \{e_1, e_2, e_3\}}}{R_0} \xrightarrow{A} R$, $\det A > 0 \Leftrightarrow R_0, R$ sunt la fel orientate

reperul canonic

OBS $z = x \times y$ este un determinant „formal"

$$z = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = e_1 \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - e_2 \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + e_3 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

$$x = \sum_{i=1}^{3} x_i e_i, \quad y = \sum_{j=1}^{3} y_j e_j$$

$$\Rightarrow z = x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$$

OBS

$$z = \begin{vmatrix} x_1 & y_1 & e_1 \\ x_2 & y_2 & e_2 \\ x_3 & y_3 & e_3 \end{vmatrix}$$

Prop

a) $z = x \times y = - y \times x$

b) $(x \times y) \times z = \langle x, z \rangle y - \langle y, z \rangle x$

c) $\sum_{x,y,z} (x \times y) \times z = (x \times y) \times z + (y \times z) \times x + (z \times x) \times y = 0$

(Jacobi)

Def (produs mixt)

$((\mathbb{R}^3, +, \cdot), g_0 = \langle \cdot, \cdot \rangle)$ $\{x, y, z\} \subset \mathbb{R}^3$

$$z \wedge x \wedge y = \langle z, x \times y \rangle = \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

Exemplu $(\mathbb{R}^3, g_0)$, $g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g_0(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3$

$u = (1, -1, 2)$, $v = (0, 1, 3)$, $w = (1, 1, 0)$

a) $u \times v$; b) $w \wedge u \wedge v$

a) $u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{vmatrix} = e_1 \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} - e_2 \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} + e_3 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} =$

$u \times v = (-5, -3, 1)$ $\quad \{u, v\}$ SLI $\Leftrightarrow rg\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 3 \end{pmatrix} = 2 = \max.$

b) $w \wedge u \wedge v = \langle \underset{(1,1,0)}{w}, \underset{(-5,-3,1)}{u \times v} \rangle = 1 \cdot (-5) + 1 \cdot (-3) + 0 \cdot 1 = -5 - 3 = -8$

sau

$$w \wedge u \wedge v = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix} = -8$$

OBS $R = \{u, v, u \times v\}$ reper poz. orientat în $\mathbb{R}^3$

$R_0 = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\} \xrightarrow{A} R = \{u = (1,-1,2), v = (0,1,3), u \times v = (-5,-3,1)\}$

$$\det A = \begin{vmatrix} 1 & 0 & -5 \\ -1 & 1 & -3 \\ 2 & 3 & 1 \end{vmatrix} > 0$$

Problemă

$$R \longrightarrow R' \longrightarrow R''$$

reper arbitrar $\quad$ reper ortogonal $\quad$ reper ortonormat

Teoremă (procedeul Gram - Schmidt)

Fie $(E, \langle \cdot, \cdot \rangle)$ s.v.e.r.

$R = \{f_1, \dots, f_n\}$ reper arbitrar în E.

$\Rightarrow \exists R' = \{e_1, \dots, e_n\}$ reper ortogonal în E a.î. $Sp\{e_1, \dots, e_i\} = Sp\{f_1, \dots, f_i\}$ $\forall i = \overline{1,n}$

Dem. Dem este inductivă

$n = 1$ $\quad R = \{f_1\}$ reper arbitrar

$e_1 = f_1 \neq 0$

Construim $e_2 = f_2 + \alpha_{21} e_1$

$\langle e_2, e_1 \rangle = 0 \Rightarrow \langle f_2 + \alpha_{21} e_1, e_1 \rangle = 0 \Rightarrow$

$\langle f_2, e_1 \rangle + \alpha_{21} \langle e_1, e_1 \rangle = 0 \Rightarrow \alpha_{21} = -\dfrac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle}$

$e_2 = f_2 - \dfrac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 \Rightarrow \begin{cases} f_1 = e_1 \\ f_2 = \dfrac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + e_2 \end{cases} \Rightarrow$

$Sp\{f_1, f_2\} = Sp\{e_1, e_2\}$.

Ip. adev P_{k-1}. Dem P_k.

P_{k-1}: $\{e_1, .., e_{k-1}\}$ vect. mutual ortogonali și $Sp\{e_1, .., e_i\} = Sp\{f_1, .., f_i\}$, $i = \overline{1, k-1}$

Construim $e_k = f_k + \sum_{j=1}^{k-1} \alpha_{kj} e_j$

$0 = \langle e_k, e_i \rangle$, $\forall i = \overline{1, k-1}$ $= \langle f_k + \sum_{j=1}^{k-1} \alpha_{kj} e_j, e_i \rangle =$

$0 = \langle f_k, e_i \rangle + \sum_{j=1}^{k-1} \alpha_{kj} \langle e_j, e_i \rangle \Rightarrow$

$0 = \langle f_k, e_i \rangle + \alpha_{ki} \langle e_i, e_i \rangle \Rightarrow \alpha_{ki} = -\frac{\langle f_k, e_i \rangle}{\langle e_i, e_i \rangle}$, $i = \overline{1, k-1}$

$e_k = f_k - \sum_{j=1}^{k-1} \alpha_{kj} \frac{\langle f_k, e_j \rangle}{\langle e_j, e_j \rangle} \cdot e_j$

Am obț.

$$\begin{cases} f_1 = e_1 \\ f_2 = \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + e_2 \\ \vdots \\ f_k = \frac{\langle f_k, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \frac{\langle f_k, e_2 \rangle}{\langle e_2, e_2 \rangle} e_2 + \dots + \frac{\langle f_k, e_{k-1} \rangle}{\langle e_{k-1}, e_{k-1} \rangle} e_{k-1} + e_k \end{cases}$$

$Sp\{f_1, .., f_i\} = Sp\{e_1, .., e_i\}$, $\forall i = \overline{1, k}$.

Am construit.

$R = \{f_1, .., f_n\} \underset{A^{-1}}{\overset{A}{\rightleftarrows}} R' = \{e_1, .., e_n\}$ sistem de n vectori, mutual ortogonali $\Rightarrow$ SLI $\Rightarrow$ reper.

$$A^{-1} = \begin{pmatrix} 1 & \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} & \dots & \frac{\langle f_n, e_1 \rangle}{\langle e_1, e_1 \rangle} \\ 0 & 1 & & \vdots \\ \vdots & \vdots & & \frac{\langle f_n, e_{n-1} \rangle}{\langle e_{n-1}, e_{n-1} \rangle} \\ 0 & 0 & & 1 \end{pmatrix}; \ \det(A^{-1}) = \frac{1}{\det A} = 1$$

$\det(A) > 0$.

$$\underset{\text{reper } \forall}{R = \{f_1, .., f_n\}} \xrightarrow{A} \underset{\text{reper ortogonal}}{R' = \{e_1, .., e_n\}} \xrightarrow{B} \overset{\text{reper ortonormat}}{R'' = \left\{ \frac{e_1}{\|e_1\|}, .., \frac{e_n}{\|e_n\|} \right\}}$$

$$B = \begin{pmatrix} \frac{1}{\|e_1\|} & \dots & 0 \\ 0 & & \vdots \\ 0 & & \frac{1}{\|e_n\|} \end{pmatrix}$$

R, R', R'' la fel orientate. $\det B > 0$

$\tilde{e}_K = \frac{e_K}{\|e_K\|}$, $K=\overline{1,n}$ versori

$\|\tilde{e}_K\|^2 = \langle \tilde{e}_K, \tilde{e}_K \rangle = \langle \frac{e_K}{\|e_K\|}, \frac{e_K}{\|e_K\|} \rangle = \frac{1}{\|e_K\|^2} \underbrace{\langle e_K, e_K \rangle}_{\|e_K\|^2} = 1$.

Def $(E, \langle \cdot, \cdot \rangle)$ s.v.e.r.

a) $x \in E$, $\langle \{x\} \rangle^{\perp} = x^{\perp} = \{ y \in E / g(x,y) = \langle x, y \rangle = 0 \} \underset{\text{ssp v}}{\subset} E$

b) $\underset{\text{subsp vect}}{U \subset E}$, $U^{\perp} = \{ y \in E \mid \langle x, y \rangle = 0, \forall x \in U \} \underset{\text{ssp vect.}}{\subset} E$

Exemplu $(\mathbb{R}^3, g_0)$, $u = (1, 2, -1)$

a) $\langle \{u\} \rangle^{\perp} = u^{\perp} = ?$; b) Det. un reper ortonormat în $u^{\perp}$

Sol.

a) $u^{\perp} = \{ x \in \mathbb{R} \mid \underbrace{g_0(\overset{(x_1,x_2,x_3)}{x}, \overset{(1,2,-1)}{u})}_{x_1 + 2x_2 - x_3} = 0 \} = \{ x \in \mathbb{R} \mid x_1 + 2x_2 - x_3 = 0 \}$

$= \{ \underbrace{(x_1, x_2, x_1 + 2x_2)}_{(x_1, 0, x_1) + (0, x_2, 2x_2)} \mid x_1, x_2 \in \mathbb{R} \} = \langle \{ \underbrace{(1,0,1)}_{f_1}, \underbrace{(0,1,2)}_{f_2} \} \rangle$

$\{f_1, f_2\}$ reper arbitrar în $u^{\perp}$

Aplicăm Gram-Schmidt.

$e_1 = f_1 = (1,0,1)$

$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 = (0,1,2) - \frac{\cancel{2}}{\cancel{2}}(1,0,1)$

$= (0,1,2) - (1,0,1) = (-1,1,1)$

$\{e_1, e_2\}$ reper ortogonal în $u^{\perp}$ $\quad \|x\| = \sqrt{\langle x, x \rangle}$

$\tilde{e}_1 = \frac{e_1}{\|e_1\|} = \frac{1}{\sqrt{2}}(1,0,1)$, $\tilde{e}_2 = \frac{1}{\sqrt{3}}(-1,1,1)$

$\{\tilde{e}_1, \tilde{e}_2\}$ reper ortonormat în $u^{\perp}$

Prop $(E, \langle\cdot,\cdot\rangle)$ s.v.e.r. , $U \subset E$ subsp. vectorial

$E = U \oplus U^{\perp}$ (scrierea este unică)

$\hookrightarrow$ complement ortogonal (unic)

Dem

Fie $x \in U \cap U^{\perp}$

$\|x\|^2 = \langle x, x\rangle = 0 \Leftrightarrow x = 0_E \Rightarrow U \oplus U^{\perp} \subseteq E$ (din constr)

Dem că $E \subseteq U \oplus U^{\perp}$

Fie $v \in E$, $\dim U = k$, $R = \{e_1, \dots, e_k\}$ reper ortonormat în U

Fie $v' = v - \underbrace{\sum_{j=1}^{k} \langle v, e_j\rangle e_j}_{\in U}$. Dem că $v' \in U^{\perp}$

$\langle v', e_i\rangle = 0, \forall i = \overline{1,k}$

$\langle v - \sum_{j=1}^{k} \langle v, e_j\rangle e_j, e_i\rangle = \langle v, e_i\rangle - \underbrace{\sum_{j=1}^{k} \langle v, e_j\rangle \overbrace{\langle e_j, e_i\rangle}^{\delta_{ji}}}_{\langle v, e_i\rangle \langle e_i, e_i\rangle}$, $\langle e_i, e_i\rangle = 1$

$= \langle v, e_i\rangle - \langle v, e_i\rangle = 0, \forall i = \overline{1,k}$

Fie $x \in U$, $\langle v', x\rangle = \langle v', \sum_{i=1}^{k} x_i e_i\rangle =$

$= \sum_{i=1}^{k} x_i \underbrace{\langle v', e_i\rangle}_{=0} = 0 \Rightarrow v' \in U^{\perp}$

Deci $v = \underbrace{v'}_{\in U^{\perp}} + \underbrace{\sum_{j=1}^{k} \langle v, e_j\rangle e_j}_{\in U} \in U \oplus U^{\perp} \Rightarrow E = U \oplus U^{\perp}$

Exemplu $(\mathbb{R}^4, g_0)$, $U = \left\{x \in \mathbb{R}^4 \;\middle|\; \begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 + x_2 - x_4 = 0 \end{cases}\right\}$, $x_1 - x_2 + x_3 = \langle (x_1, x_2, x_3, x_4), (1, -1, 1, 0)\rangle$

a) $U^{\perp}$; b) $R = R_1 \cup R_2$ reper ortonormat în $\mathbb{R}^4$, unde

R_1 — // — U

R_2 — // — $U^{\perp}$

SOL

a) $U = \left\{x \in \mathbb{R}^4 \;\middle|\; \begin{cases} g_0(x, (1, -1, 1, 0)) = 0 \\ g_0(x, (1, 1, 0, -1)) = 0 \end{cases}\right\}$

$U^{\perp} = \langle\{f_1, f_2\}\rangle$, $f_1 = (1, -1, 1, 0)$, $f_2 = (1, 1, 0, -1)$

$\mathbb{R}^4 = U \oplus U^\perp$, $\dim U = 2$

$\dim U^\perp = 2$

$rg\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix} = 2 \max$ $\{f_1, f_2\}$ SLI $\Rightarrow \{f_1, f_2\}$ reper în $U^\perp$

$g_0(f_1, f_2) = 0 \Rightarrow \{f_1, f_2\}$ reper ortogonal în $U^\perp$

$R_2 = \left\{ \frac{1}{\sqrt{3}}(1,-1,1,0), \frac{1}{\sqrt{3}}(1,1,0,-1) \right\}$ reper ortonormat în $U^\perp$

$U = \left\{ x \in \mathbb{R}^4 \mid \begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 + x_2 - x_4 = 0 \end{cases} \right\}$ $\left(\left.\begin{matrix} \boxed{\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}} & \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \end{matrix} \right| \begin{matrix} 0 \\ 0 \end{matrix} \right)$

$\begin{cases} x_1 - x_2 = -x_3 \\ x_1 + x_2 = x_4 \end{cases}$ $x_1 = -\frac{1}{2}x_3 + \frac{1}{2}x_4$.

$2x_1 \;/ = -x_3 + x_4$

$x_2 = x_4 + \frac{1}{2}x_3 - \frac{1}{2}x_4 = \frac{1}{2}x_3 + \frac{1}{2}x_4$.

$U = \left\{ \left(-\frac{1}{2}x_3 + \frac{1}{2}x_4, \frac{1}{2}x_3 + \frac{1}{2}x_4, x_3, x_4 \right) \mid x_3, x_4 \in \mathbb{R} \right\}$

$\left(-\frac{1}{2}x_3, \frac{1}{2}x_3, x_3, 0\right) + \left(\frac{1}{2}x_4, \frac{1}{2}x_4, 0, x_4\right)$

$\frac{1}{2}x_3 \underbrace{(-1,1,2,0)}_{f_1'}$ $\quad \frac{1}{2}x_4 \underbrace{(1,1,0,2)}_{f_2'}$

$\{f_1', f_2'\}$ reper în $U \Rightarrow$ reper ortogonal în U

$g_0(f_1', f_2') = 0$

$R_1 = \left\{ \frac{1}{\sqrt{6}}(-1,1,2,0), \frac{1}{\sqrt{6}}(1,1,0,2) \right\}$ reper ortonormat în U

$R = R_1 \cup R_2$ reper ortonormat în $\mathbb{R}^4 = U \oplus U^\perp$