Seminar 2 Transformări elementare. Forma eșalon. Sisteme liniare

① $\begin{cases} x + \alpha y + z = 1 \\ \alpha x - y + z = 1 \\ x + y - z = 2 \end{cases}$ Discuție după $\alpha \in \mathbb{R}$

$$A = \left(\begin{array}{ccc|c} 1 & \alpha & 1 & 1 \\ \alpha & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 \end{array}\right)$$

$$\det A = \begin{vmatrix} 1 & \alpha & 1 \\ \alpha & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & \alpha+1 & 0 \\ \alpha+1 & 0 & 0 \\ -1 & 1 & -1 \end{vmatrix} = -1\left(-(\alpha+1)^2\right) = (\alpha+1)^2$$

i) $\Delta \neq 0$ $(\alpha \in \mathbb{R} \setminus \{-1\})$

rang A = rang $\bar{A} = 3$ SCD

Aplicăm metoda Cramer

$$x = \frac{\Delta x}{\cancel{3}\Delta} \ ; \ \Delta x = \begin{vmatrix} 1 & \alpha & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & \alpha & 1 \\ 1 & -1 & 1 \\ 3 & 0 & 0 \end{vmatrix} = 3(\alpha+1)$$

$$x = \frac{3}{\alpha+1}$$

$$y = \frac{\Delta y}{\Delta} \ ; \ \Delta y = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} \boxed{1-\alpha} & 0 & 0 \\ \alpha & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} =$$

$$= (1-\alpha)(-3) = 3(\alpha-1) \Rightarrow y = \frac{3(\alpha-1)}{(\alpha+1)^2} = \cancel{\frac{3}{\alpha+1}}$$

$$\Delta z = \begin{vmatrix} 1 & \alpha & 1 \\ \alpha & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & \alpha-1 & -1 \\ \alpha & -1-\alpha & 1-2\alpha \\ \boxed{1} & 0 & 0 \end{vmatrix} =$$

$$= (\alpha-1)(1-2\alpha)-1-\alpha = \alpha-2\alpha^2-1+2\alpha-1-\alpha =$$

$$= -2\alpha^2+2\alpha-2 = 2(-\alpha^2+\alpha-1)$$

$$(x,y,z) = \left(\frac{3}{\alpha+1}, \frac{3(\alpha-1)}{(\alpha+1)^2}, \frac{2(-\alpha^2+\alpha-1)}{(\alpha+1)^2} \right)$$

ii) $\Delta = 0$ $\alpha = -1$ $A = \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 \end{array}\right)$

$$\Delta p = \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} = -2 \neq 0$$

$$\Delta c = \begin{vmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 0 & \boxed{1} \\ 3 & 3 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 2 & 0 \\ 3 & 3 \end{vmatrix} = 6 \neq 0$$

$\Rightarrow$ Si.

2) $\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ x + \alpha^2 z = 0 \end{cases}$, $\alpha \in \mathbb{R}$ $\Rightarrow A = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 1 & 0 & \alpha^2 & 0 \end{array}\right)$

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & \alpha^2 \end{vmatrix} = 5\alpha^2 + 12 - 15 - 8\alpha^2 = -3\alpha^2 - 3 = -3(\alpha^2+1) \neq 0$$

$\Rightarrow$ SCD $\Rightarrow \Delta x = \Delta y = \Delta z = 0 \Rightarrow (x,y,z) = \{(0,0,0)\}$

(4) fie ΔABC a,b,c lg. laturilor

a) $(\forall)$ ΔABC SCD

b) (x_0, y_0, z_0) verifică $x_0, y_0, z_0 \in (-1,1)$

$$\begin{cases} ay + bx = c \\ cx + az = b \\ bz + cy = a \end{cases}$$

$$A = \left(\begin{array}{ccc|c} b & a & 0 & c \\ c & 0 & a & b \\ 0 & c & b & a \end{array}\right)$$

$$\det A = \begin{vmatrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{vmatrix} = 0+0+0-0-abc-abc = -2abc \neq 0 \Rightarrow$$

$\Rightarrow$ SCD

$$x = \frac{\Delta x}{\Delta} \; ; \quad \Delta x = \begin{vmatrix} c & a & 0 \\ b & 0 & a \\ a & c & b \end{vmatrix} = 0+0+a^3-0-c^2a-b^2a = -a(-a^3+c^2+b^2)$$

$$x = \frac{-a(-a^2+b^2+c^2)}{-2abc} = \frac{b^2+c^2-a^2}{2bc} = \cos A \quad (\text{Th } \cos \; a^2 = b^2+c^2-2bc\cos A)$$

Analog $y = \cos B$, $z = \cos C$ $\Rightarrow$ $A, B, C \in (0, \pi)$ $\Rightarrow$

$\Rightarrow (x_0, y_0, z_0) \in (-1,1) \times (-1,1) \times (-1,1)$

(5) $a,b,c \in \mathbb{R}$, distincte

$$\begin{cases} x + y + z = 0 \\ (b+c)x + (a+c)y + (a+b)z = 0 \\ bcx + acy + abz = 0 \end{cases}$$

$$A = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ b+c & a+c & a+b & 0 \\ bc & ac & ab & 0 \end{array}\right)$$

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ b+c & a+c & a+b \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ b+c & a-b & a-c \\ bc & c(a-b) & b(a-c) \end{vmatrix} =$$

$$= (a-b)(a-c) \begin{vmatrix} 1 & 1 \\ c & b \end{vmatrix} = (a-b)(a-c)(b-c) \neq 0 \Rightarrow$$

$\Rightarrow$ SCD $\Rightarrow \exists!\ (x,y,z) = (0,0,0)$

6. $\begin{cases} x + 2y = m+1 \\ 2x + 3y = m-1 \\ mx + y = 3 \end{cases}$ $m = ?$ a.î sistemul este incompatibil

$$A = \left(\begin{array}{cc|c} 1 & 2 & m+1 \\ 2 & 3 & m-1 \\ m & 1 & 3 \end{array}\right)$$ $rg\,A = 2$ şi $rg\,\bar{A} = 3$

$$\Delta p = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \qquad \Delta c = \begin{vmatrix} 1 & 2 & m+1 \\ 2 & 3 & m-1 \\ m & 1 & 3 \end{vmatrix} \neq 0$$

$$= \begin{vmatrix} m+4 & 2 & m+1 \\ m+4 & 3 & m-1 \\ m+4 & 1 & 3 \end{vmatrix} = (m+4) \begin{vmatrix} 1 & 2 & m+1 \\ 1 & 3 & m-1 \\ 1 & 1 & 3 \end{vmatrix} =$$

$$= (m+4) \begin{vmatrix} 1 & 2 & m+1 \\ 0 & 1 & -2 \\ 0 & -1 & 2-m \end{vmatrix} = (m+4) \begin{vmatrix} 1 & -2 \\ -1 & 2-m \end{vmatrix} =$$

$$= -(m+4)m \neq 0 \Rightarrow m \in \mathbb{R} \setminus \{-4, 0\}$$

(10)

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 4 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

a) Să se scrie A în forma eșalon respectiv forma eșalon redusă

b) rg A = ?

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 4 & 1 \\ 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & 1 \\ 3 & 1 & 2 \end{pmatrix} \underset{L_3 - 3L_1}{\sim} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & -2 & 2 \end{pmatrix} \underset{L_3 + \frac{1}{2}L_2}{\sim}$$

$$\sim \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{4} & 1 \\ 0 & 0 & \boxed{\frac{5}{2}} \end{pmatrix}$$ forma eșalon $$\sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ rg A = 3 forma eșalon redusă

(11) b) $$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$ $A^{-1} = ?$ utilizând algoritmul Gauss - Jordan

$$\Delta = \det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 2 \neq 0 \Rightarrow \exists A^{-1}$$

$$\left(\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \underset{\substack{L_2 - L_1 \\ L_3 - L_1}}{\sim} \left(\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & \boxed{-2} & 0 & -1 & 1 & 0 \\ 0 & 0 & \boxed{-1} & -1 & 0 & 1 \end{array}\right) \underset{\substack{L_2 \cdot (-\frac{1}{2}) \\ L_3 \cdot (-1)}}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \underset{L_1 - L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \underset{L_1 - L_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \qquad A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

(12) Să se rezolve, utilizând metoda eliminării Gauss-Jordan

$$\begin{cases} 3x_1 + x_2 - 3x_3 = 4 \\ x_1 + 3x_2 - 2x_3 = -5 \\ 2x_1 + 2x_2 + 5x_3 = 7 \end{cases}$$

$$\bar{A} = (A \mid B) = \left(\begin{array}{ccc|c} 3 & 1 & -3 & 4 \\ 1 & 3 & -2 & -5 \\ 2 & 2 & 5 & 7 \end{array}\right) \sim \left(\begin{array}{ccc|c} \boxed{1} & 3 & -2 & -5 \\ 3 & 1 & -3 & 4 \\ 2 & 2 & 5 & 7 \end{array}\right)$$

$$\underset{\substack{L_2 - 3L_1 \\ L_3 - 2L_1}}{\sim} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -5 \\ 0 & -8 & 3 & 19 \\ 0 & -4 & 9 & 17 \end{array}\right) \underset{L_3 - \frac{1}{2}L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -5 \\ 0 & -8 & 3 & 19 \\ 0 & 0 & \frac{15}{2} & \frac{15}{2} \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & -2 & -5 \\ 0 & 1 & -\frac{3}{8} & -\frac{19}{8} \\ 0 & 0 & 1 & 1 \end{array}\right) \underset{\substack{L_2 + \frac{3}{8}L_3 \\ L_1 + 2L_3}}{\sim} \left(\begin{array}{ccc|c} 1 & 3 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \underset{L_1 - 3L_2}{\sim}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \Leftrightarrow \begin{cases} x_1 = 3 \\ x_2 = -2 \\ x_3 = 1 \end{cases}$$

$$\begin{cases} 3x_1 + 2x_2 + 5x_3 + 4x_4 = -1 \\ 2x_1 + x_2 + 3x_3 + 3x_4 = 0 \\ x_1 + 2x_2 + 3x_3 = -3 \end{cases}$$

Să se rezolve folosind algoritmul Gauss - Jordan

$$A = \left(\begin{array}{cccc|c} 3 & 2 & 5 & 4 & -1 \\ 2 & 1 & 3 & 3 & 0 \\ 1 & 2 & 3 & 0 & -3 \end{array}\right) \sim \left(\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 0 & -3 \\ 2 & 1 & 3 & 3 & 0 \\ 3 & 2 & 5 & 4 & -1 \end{array}\right) \underset{\substack{L_2 = L_2 - 2L_1 \\ L_3 = L_3 - 3L_1}}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & -3 \\ 0 & -3 & -3 & 3 & 6 \\ 0 & -4 & -4 & 4 & 8 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & -3 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -1 & -2 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 0 & -3 \\ 0 & \boxed{1} & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

~~x_1, x_2 var principale~~

~~$x_2 = \alpha \quad x_3 = \beta$~~

$\Rightarrow \underset{L_1 - 2L_2}{\sim}$

$$\Rightarrow \left(\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 2 & 1 \\ 0 & \boxed{1} & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

x_1, x_2 var principale

$x_2 = \alpha \quad x_3 = \beta$

$\Rightarrow$

$$\Rightarrow \quad x_1 = 1 - \alpha - 2\beta$$

$$x_2 = -2 - \alpha + \beta$$

(7) $\sum_{i=1}^{k}(1+i)x_i + \sum_{i=1}^{4-k} i\,x_{i+k} = 0 \quad (\forall)\, k=\overline{1,3} \iff$

$i+k=j \Rightarrow i=j-k$

$i=1 \Rightarrow j=k+1$

$i=4-k \Rightarrow j=4$

$\iff \sum_{i=1}^{k}(1+i)x_i + \sum_{j=k+1}^{4}(j-k)x_j = 0 \quad (\forall)\, k=\overline{1,3} \iff$

$\iff$ Când $\cdot\, k=1,\ 2x_1+x_2+2x_3+3x_4=0$

$\cdot\, k=2,\ 2x_1+3x_2+x_3+2x_4=0 \quad \Rightarrow$

$\cdot\, k=3,\ 2x_1+3x_2+4x_3+x_4=0$

$$\Rightarrow \left(\begin{array}{cccc|c} 2 & 1 & 2 & 3 & 0 \\ 2 & 3 & 1 & 2 & 0 \\ 2 & 3 & 4 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 2 & 1 & 2 & 3 & 0 \\ 2 & 3 & 1 & 2 & 0 \\ 2 & 3 & 4 & 1 & 0 \end{array}\right)$$

$$\underset{\substack{L_2-L_1 \\ L_3-L_2}}{\sim} \left(\begin{array}{cccc|c} 2 & 1 & 2 & 3 & 0 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & 0 & 3 & -1 & 0 \end{array}\right) \underset{L_2+\frac{1}{3}L_3}{\sim} \left(\begin{array}{cccc|c} 2 & 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & 3 & -1 & 0 \end{array}\right) \underset{L_1-\frac{2}{3}L_3}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} 2 & 1 & 0 & \frac{11}{3} & 0 \\ 0 & 2 & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & 3 & -1 & 0 \end{array}\right) \underset{L_1-\frac{1}{2}L_2}{\sim} \left(\begin{array}{cccc|c} 2 & 0 & 0 & \frac{13}{3} & 0 \\ 0 & 2 & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & 3 & -1 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{13}{6} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \end{array}\right)$$

$x_{1,2,3}$ var principale

$x_4=\alpha$ var secundară

$$\begin{cases} x_1 = -\frac{13}{6}\alpha \\ x_2 = \frac{2}{3}\alpha \\ x_3 = \frac{1}{3}\alpha \end{cases}, \ \alpha \in \mathbb{R}$$