Seminar 6

Aplicații liniare

① $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (2x_1 + 2x_2,\ x_1 + x_3,\ x_1 + 3x_2 - 2x_3)$

a) f nu este izomorfism de sp. vectoriale

b) $f|_{V'} : V' \to V''$ izomorfism, unde:

$$V' = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\}$$

$$V'' = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 3x_1 - 4x_2 - 2x_3 = 0\}.$$

c) $f(V' \cap V'') = ?$

d) $\mathbb{R}^3 = V' \oplus W$. Dați un exemplu de W

Fie $p: \mathbb{R}^3 \to \mathbb{R}^3$ proiecția pe V'

$s: \mathbb{R}^3 \to \mathbb{R}^3$ proiecția față de V''

$p(1,3,6)$ $\quad s(1,3,6)$

Soluție:

a) $f(x) = y \iff Y = AX \iff f$ liniară

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix} = [f]_{R_0, R_0},\ R_0 = \{e_1, e_2, e_3\}$$

obs: $f(e_1) = f(1,0,0) = (2,1,1) =$

$= 2e_1 - 1e_2 + 1e_3$

$f(e_2) = f(0,1,0) = (2,0,3) =$

$= 2e_1 + 0e_2 + 3e_3$

$f(e_3) = f(0,0,1) = (0,1,-2) =$

$= 0 \cdot e_1 + 1 \cdot e_2 - 2 \cdot e_3$

$\det A = 2 - 6 + 4 = 0 \Rightarrow rg\, A < 3 \Rightarrow f$ nu e bijecție

Obs!:

$\ker f = \{ X \in \mathbb{R}^3 \mid AX = 0 \} = S(A)$

$\dim \ker f = \dim \mathbb{R}^3 - rg\, A = 3 - 2 = 1$

Th. dimensiunii

$\dim \mathbb{R}^3 = \dim \ker f + \dim Im f \Rightarrow \dim Im f = 2$

b)

$V' = \{ (x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R} \} = < \underbrace{\{ (1,0,1), (0,1,1) \}}_{R'} >$

R' SG în V'

$\dim V' = 3 - 1 = 2 = card\, R'$ $\Bigg| \Rightarrow R'$ reper în V'

Obs:

$f|_{V'}$ bij $\Leftrightarrow$ transformă reper din V' în reper din V''

$f(R') = \{ f|_{V'}(1,0,1), f|_{V'}(0,1,1) \} = \{ \underbrace{(2,2,-1)}_{e_1''}, \underbrace{(2,1,1)}_{e_2''} \}$

$e_1'' \in V_1'' \Leftrightarrow 3 \cdot 2 - 4 \cdot 2 - 2(-1) = 0$ (A)

$e_2'' \in V'' \Leftrightarrow 3 \cdot 2 - 4 \cdot 1 - 2 \cdot 1 = 0$ (A)

$rg \begin{pmatrix} 2 & 2 \\ 2 & 1 \\ -1 & 1 \end{pmatrix} = 2 = \text{maxim} \xRightarrow{CLi} R''$ este SLi

$\dim V'' = 3 - 1 = 2 = card\, R''$ $\Bigg| \Rightarrow$

$\Rightarrow R''$ reper în V''

Deci $f|_{V'}$ bijecție $\Rightarrow f|_{V'}$ izomorfism

c) $f(U' \cap U'') = ?$

$$U' \cap U'' = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{cases} x_1 + x_2 - x_3 = 0 \\ 3x_1 - 4x_2 - 2x_3 = 0 \end{cases} \right\} = \langle \{(6,1,7)\} \rangle$$

$$\left(\boxed{\begin{matrix} 1 & 1 \\ 3 & -4 \end{matrix}} \begin{matrix} -1 \\ -2 \end{matrix} \right) \bigg| \begin{matrix} 0 \\ 0 \end{matrix} \qquad \dim(U' \cap U'') = 3 - 2 = 1 = \langle \{ (\tfrac{6}{7}, \tfrac{1}{7}, 1) \} \rangle$$

$$f : U' \cap U'' \longrightarrow \mathbb{R}^3$$

$$\dim U' \cap U'' = \dim Ker(f|_{U' \cap U''}) + \dim Im(f|_{U' \cap U''})$$

$$\begin{aligned} x_1 + x_2 &= x_3 \quad (-3 \\ 3x_1 - 4x_2 &= 2x_3 \quad \oplus \\ \hline -7x_2 &= -x_3 \Rightarrow x_2 = \tfrac{1}{7} x_3 \Rightarrow x_1 = \tfrac{6}{7} x_3 \end{aligned}$$

$$f(6,1,7) = (14, 13, -5) \neq 0_{\mathbb{R}^3}$$

$$f(U' \cap U'') = \langle \{ f(6,1,7) \} \rangle = \langle \{ (14, 13, -5) \} \rangle$$

Obs:

$$Ker(f|_{U' \cap U''}) = \{ 0_{\mathbb{R}^3} \}$$

d)

$rg\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = 3$, $w = \langle \{e_1\} \rangle$

p.s: $V' \oplus W \to V' \oplus W$ proiectia pe V' de-a lungul lui W

$p(v' \oplus w) = v'$

$s(v' \oplus w) = (2p - id_{\mathbb{R}^3})(v' + w) = 2v' - (v' + w) = v' - w$

simetria față de V'

$R = \{(1,0,1), (0,1,1), (1,0,0)\}$

$(1,3,6) = \underbrace{a(1,0,1) + b(0,1,1)}_{v'} + \underbrace{c \cdot (1,0,0)}_{w} =$

$= (a+c, b, a+b) = 3(1,0,1) + 3(0,1,1) + (-2)(1,0,0) =$

$\begin{cases} a + c = 1 \\ b = 3 \\ a + b = 6 \end{cases} \Rightarrow \begin{matrix} b = 3 \\ a = 3 \\ c = -2 \end{matrix}$

$= \underbrace{(3,3,6)}_{v'} + \underbrace{(-2,0,0)}_{w}$

$p(1,3,6) = (3,3,6)$

$s(1,3,6) = (5,3,6)$

kerf w

③ $f: \mathbb{R}_2[x] \to \mathbb{R}_1[x] \qquad f(P) = P'$

a) $[f]_{R,R'} = ?$ $\qquad R = \{x^2, 1+x, 2-x\}$ reper în $\mathbb{R}_2[x]$

$R' = \{x, 1+3x\}$ reper în $\mathbb{R}_1[x]$

b) $\mathbb{R}_2[x] = \ker f \oplus W$

$p_1: \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ proiecția pe $\ker f$

$p_2: \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ — " — pe W

$p_1(1-x+3x^2)$, $p_2(2x+x^2) = ?$

Soluție:

a) $f(x^2) = 2x = 2x + 0\cdot(1+3x)$

$f(1+x) = 1 = -3x + 1\cdot(1+3x) \qquad A = \begin{pmatrix} 2 & -3 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

$f(2-x) = -1 = 3x + (-1)\cdot(1+3x)$

$A = [f]_{R,R'}$

b) $\ker f = \{p \in \mathbb{R}_2[x] \mid f(p) = 0\} = \langle\{1\}\rangle$

$W = \langle\{x, x^2\}\rangle$

$1 - x + 3x^2 = \underset{\in \ker f}{1} + \underset{\in W}{(-x+3x^2)}$

$p_1(1-x+3x^2) = 1$

$2x + x^2 = \underset{\in \ker f}{0} + \underset{\in W}{2x+x^2} \qquad p_2(2x+x^2) = 2x+x^2$

(5) $(\mathbb{R}^3,+,\cdot)/_{\mathbb{R}}$ $\quad R=\{e_1,e_2,e_3\}$ reper canonic

$R'=\{e_1'=e_1+e_2+e_3,\ e_2'=e_1+e_2,\ e_3'=e_1\}$

$((\mathbb{R}^3)^*=\{f:\mathbb{R}^3\to\mathbb{R} \mid f \text{ liniar}\},+,\cdot)/_{\mathbb{R}}$ sp. ~~vect~~ dual

$R^*=\{e_1^*,e_2^*,e_3^*\}$, $\quad e_i^*(e_j)=\delta_{ij} \quad \forall\, i,j=\overline{1,3}$

$(R')^*=\{(e_1')^*,(e_2')^*,(e_3')^*\}$ $\quad e_i^*(e_j')=\delta_{ij}$ $(\forall)\, i,j = \overline{1,3}$

$R\xrightarrow{C}R'$

$R^*\xrightarrow{D}(R')^*$

Soluție:

$$C=\begin{pmatrix}1&1&1\\1&1&0\\1&0&0\end{pmatrix}$$

$e_1'^{*} = a\cdot e_1^* + b\cdot e_2^* + c\cdot e_3^*$ $\quad a,b,c\in\mathbb{R}$

~~$e_1'^{*} =$~~

$e_1'^{*}(e_1')=1$

$\overset{\shortparallel}{e_1'^{*}}(e_1+e_2+e_3) = a(\overset{1}{\overset{\shortparallel}{e_1^*(e_1)}} + \overset{0}{\overset{\shortparallel}{e_1^*(e_2)}} + \overset{0}{\overset{\shortparallel}{e_1^*(e_3)}}) +$

$+\, b(\overset{0}{\overset{\shortparallel}{e_2^*(e_1)}} + \overset{1}{\overset{\shortparallel}{e_2^*(e_2)}} + \overset{0}{\overset{\shortparallel}{e_2^*(e_3)}}) + c(\overset{0}{\overset{\shortparallel}{e_3^*(e_1)}} + \overset{0}{\overset{\shortparallel}{e_3^*(e_2)}} +$

$+\, \overset{1}{\overset{\shortparallel}{e_3^*(e_3)}})) = a+b+c$

$e_1'^{*}(e_2')=0$

$\overset{\shortparallel}{e_1'^{*}}(e_1+e_2) = a+b$

$e_1'(e_3) = 0 \qquad e_1'(e_1)=a$

$$\begin{cases} a+b+c=1 \\ a+b=0 \\ a=0 \end{cases} \Rightarrow \begin{cases} a=b=0 \\ c=1 \end{cases} \qquad D=\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$e_2'^{*} = a' \cdot e_1^{*} + b' \cdot e_2^{*} + c' \cdot e_3^{*}$$

$$e_2'(e_1') = 0$$

$$e_2'^{*}(e_1+e_2+e_3) = \cancel{a'(e_1^{*}(e_1)} \; a'+b'+c'$$

$$e_2'(e_2') = 1$$

$$\shortparallel$$

$$e_2'(e_1+e_2) = a'+b'$$

$$e_2'(e_3) = 0$$

$$\shortparallel$$

$$e_2'(e_1) = a'$$

$$\begin{cases} a'+b'+c'=0 \\ a'+b'=1 \\ a'=0 \end{cases} \Rightarrow \begin{cases} a'=0 \\ b'=1 \\ c'=-1 \end{cases}$$

$$e_3'^{*} = a'' e_1^{*} + b'' \cdot e_2^{*} + c'' \cdot e_3^{*}$$

$$e_3'^{*}(e_1') = 0$$

$$\shortparallel$$

$$e_3'^{*}(e_1+e_2+e_3) = a''+b''+c''$$

$$e_3'^{*}(e_2') = 0$$

$$\shortparallel$$

$$e_3'^{*}(e_1+e_2) = a''+b''$$

$$e_3'^{*}(e_3') = 1$$

$$\shortparallel$$

$$e_3'^{*}(e_1) = a''$$

$$\begin{cases} a''+b''+c''=0 \\ a''+b''=0 \\ a''=1 \end{cases} \Rightarrow \begin{cases} a''=1 \\ b''=-1 \\ c''=0 \end{cases}$$

⑧ $f \in End(\mathbb{R}^n)$ $\quad f^2 = f + id_{\mathbb{R}^n} \Rightarrow f \in Aut(\mathbb{R}^n)$

Soluție:

$$f \in End(\mathbb{R}^n) \Rightarrow \exists\, \underset{\substack{\| \\ [f]_{R,R}}}{A} \in \mathcal{M}_n(\mathbb{R}) \text{ a.î } \underset{\substack{\| \\ y}}{f(x)},\ Y = A \cdot X$$

$$\Rightarrow A^2 = A + I_n \Rightarrow A^2 - A = I_n \Rightarrow A(A - I_n) = I_n$$

$\Rightarrow A$ inversabilă $\Rightarrow f$ bijectivă $\Rightarrow f \in Aut(\mathbb{R}^2)$

② $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1 + x_2 - x_3, -x_1 - x_2 + x_3, x_1 + x_2 + x_3)$

a) $[f]_{R,R}$ $\quad R = \{e_1' = e_1 + e_2 + e_3,\ e_2' = e_1 + e_3,\ e_3' = e_1 + e_2\}$

b) $\mathbb{R}^3 = Im f + W$

$s: \mathbb{R}^3 \to \mathbb{R}^3$ simetria față de W

$s(0,1,1) = ?$

c) $\mathbb{R}^3 = f(V') \oplus U$

$$V' = \left\{ x \in \mathbb{R}^3 \mid \begin{cases} x_1 + 2x_2 + x_3 = 0 \\ -x_1 + x_2 + 2x_3 = 0 \end{cases} \right\}$$

$p: \mathbb{R}^3 \to \mathbb{R}^3$ proiecția pe $f(V')$

$p(2,-1,3) = ?$

Soluție:

a) $f(x)=y \Leftrightarrow Y=AX$ (f liniară)

$$A=\begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = [f]_{R,R}$$

$$R_0 \xrightarrow{C} R \qquad C=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

b) $$A^{-1} = [f]_{R,R} = C^{-1}AC = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 0 & 4 \\ -2 & -2 & 0 \end{pmatrix}$$

sau

$$f(e_1') = f(1,1,1) = (1,-1,3) = a(1,1,1)+b(1,0,1)+c(1,1,0) = (\underbrace{a+b+c}_{=1}, \underbrace{a+c}_{=-1}, \underbrace{a+b}_{=3}) \Rightarrow$$

$$\Rightarrow \begin{cases} b=2 \\ a=1 \\ c=-2 \end{cases}$$

$$f(e_2') = f(1,0,1) = (0,0,2) = (\underbrace{a'+b'+c'}_{=0}, \underbrace{a'+c'}_{=0}, \underbrace{a'+b'}_{=2})$$

$$\Rightarrow \begin{cases} b'=0 \\ a'=2 \\ c'=-2 \end{cases}$$

$$f(e_3') = f(1,1,0) = (2,-2,2) = (\underbrace{a''+b''+c''}_{=2}, \underbrace{a''+c''}_{=-2}, \underbrace{a''+b''}_{=2}) \Rightarrow \begin{cases} c''=0 \\ a''=-2 \\ b''=4 \end{cases}$$

$y \in \operatorname{Im} f \Rightarrow \exists\ x \in \mathbb{R}^3$ a.î $f(x) = y$

$$A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & y_1 \\ -1 & -1 & 1 & y_2 \\ 1 & 1 & 1 & y_3 \end{array} \right)$$

$$\Delta c = 0 = \begin{vmatrix} 1 & -1 & y_1 \\ -1 & 1 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & y_1 + y_2 \\ -1 & 1 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = (y_1 + y_2) \cdot (-2) = 0$$

$$\Rightarrow \operatorname{Im} f = \{ y \in \mathbb{R}^3 \mid y_1 + y_2 = 0 \} = \{ (y_1, -y_1, y_3) \} =$$

$$= \langle \{ (1,-1,0), (0,0,1) \} \rangle$$

$$\operatorname{rg} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 3 \Rightarrow W = \langle \{ e_1 \} \rangle$$

$\mathbb{R}^3 = \operatorname{Im} f + W$

$x = x' + x''$

$s(x) = -x' + x''$

$$(0,1,1) = a(1,-1,0) + b(0,0,1) + c(1,0,0)$$

$$= (a + c, -a, b) \Rightarrow a = -1,\ b = 1,\ c = 1$$

$$s(0,1,1) = (1,-1,0) - (0,0,1) + (1,0,0)$$

$$= (2,-1,-1)$$