2.03.2021.

CURS 3

Spații vectoriale.

Sisteme liniar independente, liniar dependente. Sisteme de generatori. Baze

Def Fie $(\mathbb{K},+,\cdot)$ corp comutativ și V mulțime $\neq \emptyset$

V are o structură de spațiu vectorial peste corpul $\mathbb{K}$

$\iff + : V\times V \longrightarrow V$ (lege internă)

$\cdot : \mathbb{K}\times V \longrightarrow V$ (lege externă)

aî 1) $(V,+)$ grup abelian

2) $a\cdot(b\cdot x) = (a\cdot b)x$

3) $(a+b)\cdot x = a\cdot x + b\cdot x$

4) $a\cdot(x+y) = ax + ay$

5) $1_{\mathbb{K}}\cdot x = x$, $\forall\, a,b \in \mathbb{K}$ (scalari)

$\forall\, x,y \in V$ (vectori)

Notăm $(V,+,\cdot)/\mathbb{K}$.

OBS a) $0_{\mathbb{K}}\cdot x = 0_V$

b) $a\cdot 0_V = 0_V$

c) $(a-b)\cdot x = a\cdot x - bx$

d) $a\cdot(x-y) = a\cdot x - ay$, $\forall\, a,b\in\mathbb{K}$, $x,y\in V$

Exemple

1) $(V,+,\cdot)/\mathbb{R}$ spațiul vectorilor liberi

2) $(\mathbb{K},+,\cdot)$ corp $\Rightarrow (\mathbb{K},+,\cdot)/\mathbb{K}$ sp. vect.

Cazuri particulare: $(\mathbb{R},+,\cdot)/\mathbb{R}$, $(\mathbb{C},+,\cdot)/\mathbb{C}$, $(\mathbb{Z}_p,+,\cdot)/\mathbb{Z}_p$
p = nr. prim

Scanned with CamScanner

3) $(\mathbb{K},+,\cdot)$ corp
$\mathbb{K}' \subset \mathbb{K}$ subcorp $\Big\} \Rightarrow (\mathbb{K},+,\cdot)/_{\mathbb{K}'}$ sp. vect.

<u>Cazuri part.</u>: $(\mathbb{R},+,\cdot)/_{\mathbb{Q}}$, $(\mathbb{C},+,\cdot)/_{\mathbb{R}}$, $(\mathbb{C},+,\cdot)/_{\mathbb{Q}}$.

4) $\underbrace{(V_1,\oplus,\odot)/_{\mathbb{K}},\ (V_2,\boxplus,\boxdot)/_{\mathbb{K}}}_{\text{sp. vect}} \Rightarrow (V_1\times V_2,+,\cdot)/_{\mathbb{K}}$ sp. vect

$+ : (V_1\times V_2)\times(V_1\times V_2) \longrightarrow V_1\times V_2$.

$(x_1,x_2)+(y_1,y_2) = (x_1\oplus y_1,\ x_2\boxplus y_2)$

$\cdot : \mathbb{K}\times(V_1\times V_2) \longrightarrow V_1\times V_2$

$a\cdot(x_1,x_2) = (a\odot x_1,\ a\boxdot x_2)$,

$\forall (x_1,x_2),(y_1,y_2)\in V_1\times V_2,\ \forall a\in\mathbb{K}$.

<u>Caz part</u> $(\mathbb{R},+,\cdot)/_{\mathbb{R}} \Rightarrow (\mathbb{R}^n,+,\cdot)/_{\mathbb{R}}$

$(x_1,\dots,x_n)+(y_1,\dots,y_n) = (x_1+y_1,\dots,x_n+y_n)$

$a\cdot(x_1,\dots,x_n) = (ax_1,\dots,ax_n)$,

$\forall (x_1,\dots,x_n),(y_1,\dots,y_n)\in\mathbb{R}^n,\ \forall a\in\mathbb{K}$.

5) $(\mathcal{M}_{m,n}(\mathbb{K}),+,\cdot)/_{\mathbb{K}}$.

$A=(a_{ij})_{\substack{i=\overline{1,m}\\ j=\overline{1,n}}} \longrightarrow (a_{11},\dots,a_{1n},a_{21},\dots,a_{2n},\dots,a_{m1},\dots,a_{mn})\in\mathbb{R}^{mn}$

6) $(\mathbb{K}[X],+,\cdot)/_{\mathbb{K}}$.

$P = a_0+a_1X+\dots+a_nX^n \longrightarrow (a_0,a_1,\dots,a_n)\in\mathbb{K}^{n+1}$

7) $I=[a,b]$, $a<b$.

$(\mathcal{C}(I)=\{f:I\to\mathbb{R}\ /\ f \text{ cont}\},+,\cdot)/_{\mathbb{R}}$.

$(\mathcal{D}(I)=\{f:I\to\mathbb{R}\ /\ f \text{ derivabile}\},+,\cdot)/_{\mathbb{R}}$

$(\mathcal{P}(I)=\{f:I\to\mathbb{R}\ /\ f \text{ admite primitive}\},+,\cdot)/_{\mathbb{R}}$

$(\mathcal{I}(I)=\{f:I\to\mathbb{R}\ /\ f \text{ integrabile Riemann}\},+,\cdot)/_{\mathbb{R}}$

Scanned with CamScanner

Fie $(V,+,\cdot)/\mathbb{K}$, $V' \subseteq V$ subm. $\neq \emptyset$

V' s.n. <u>subspațiu vectorial</u> $\Leftrightarrow$ este închisă la „+" vectorilor și la „$\cdot$" cu scalari i.e.

- $\forall x, y \in V' \Rightarrow x+y \in V'$
- $\forall a \in \mathbb{K}, \forall x \in V' \Rightarrow ax \in V'$

<u>OBS</u>. $V' \subset V$ subsp. vect $\Leftrightarrow (V',+,\cdot)/\mathbb{K}$ sp. vect (cu operațiile induse)

<u>Prop</u> $(V,+,\cdot)/\mathbb{K}$ sp. vect, $V' \subseteq V$ subm. $\neq \emptyset$

V' subsp. vect $\Leftrightarrow \forall a,b \in \mathbb{K}, \forall x,y \in V' : ax+by \in V' \Leftrightarrow$

$\Leftrightarrow \forall a_1,..,a_n \in \mathbb{K}, \forall x_1,..,x_n \in V' : a_1x_1 + \ldots + a_nx_n = \sum_{i=1}^{n} a_i x_i \in V'$

<u>Dem</u>

„$\Rightarrow$" Ip: V' = sp. vect.

$\forall a \in \mathbb{K}, x \in V' \Rightarrow ax \in V'$
$\forall b \in \mathbb{K}, y \in V' \Rightarrow by \in V'$ $\Rightarrow ax+by \in V'$

„$\Leftarrow$" Ip: $ax+by \in V'$, $\forall a,b \in \mathbb{K}$, $\forall x,y \in V'$

Fie $a = b = 1_{\mathbb{K}} \Rightarrow 1_{\mathbb{K}} \cdot x + 1_{\mathbb{K}} y \in V' \overset{5)}{\Rightarrow} x+y \in V'$

Fie $a \in \mathbb{K}$, $b = 0_{\mathbb{K}} \Rightarrow a \cdot x + \underbrace{0_{\mathbb{K}} \cdot y}_{= 0_V} \in V' \Rightarrow ax \in V'$

<u>Exemple</u>

1) $(V,+,\cdot)/\mathbb{K}$ $\quad \{0_V\}, V \subset V$ ssp vect

2) $n < m, m \geq 2$ $\quad \mathbb{R}^n \underset{\text{ssp vect}}{\subset} \mathbb{R}^m$.

3) $(\mathcal{M}_n(\mathbb{R}),+,\cdot)/\mathbb{R}$.

$V_1 = \{ A = diag(a_1,..,a_n) \in \mathcal{M}_n(\mathbb{R}) \}$
$V_2 = \{ A \in \mathcal{M}_n(\mathbb{R}) \mid Tr(A) = 0 \}$

Scanned with CamScanner

$V_3 = \{ A \in \mathcal{M}_n(\mathbb{R}) \mid A = A^T \} = \mathcal{M}_n^s(\mathbb{R})$

$V_4 = \{ A \in \mathcal{M}_n(\mathbb{R}) \mid A = -A^T \} = \mathcal{M}_n^a(\mathbb{R})$

OBS $GL(n,\mathbb{R})$, $O(n)$, $SO(n)$ $\subset \mathcal{M}_n(\mathbb{R})$ nu sunt ssp. vect (nu sunt închise la „+")

4) $V' = \{ (x,y) \in \mathbb{R}^2 \mid ax+by=0,\ a^2+b^2>0 \} \subset \mathbb{R}^2$
(dreaptă care trece prin origine)

$V'' = \{ (x,y,z) \in \mathbb{R}^3 \mid ax+by+cz=0,\ a^2+b^2+c^2>0 \} \subset \mathbb{R}^3$
(plan $\ni O$)

$V''' = \{ (x_1,\dots,x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n a_i x_i = 0,\ \sum_{i=1}^n a_i^2 > 0 \} \subset \mathbb{R}^n$
(hiperplan $\ni O$)

$W = S(A) = \{ (x_1,\dots,x_n) \in \mathbb{R}^n \mid \underset{(m,n)(n,1)}{AX} = \underset{(m,1)}{O} \} \subset \mathbb{R}^n$.
($\cap$ a m hiperplane $\ni O$)

Subspațiul vectorial generat de o mulțime

Def $(V,+,\cdot)/\mathbb{K}$, $S \subset V$ (subm. $\neq \emptyset$)

$$\langle S \rangle = \left\{ x \in V \mid x = \sum_{i=1}^n a_i x_i,\ \begin{matrix} a_1,\dots,a_n \in \mathbb{K} \\ x_1,\dots,x_n \in S \end{matrix} \right\}$$

(comb. liniare finite de vectori din S cu scalari din $\mathbb{K}$)

- Dacă $V = \langle S \rangle$, atunci S s.n. sistem de generatori (SG)
- V s.n. spațiu vectorial finit generat $\Leftrightarrow \exists\ S$ un (SG) finit

OBS
a) $S \subset \langle S \rangle$
b) $\langle S \rangle$ = cel mai mic subsp. vect. care conține S
c) $\langle \emptyset \rangle = \{ 0_V \}$ (CONVENȚIE)

Scanned with CamScanner

$(V,+,\cdot)/\mathbb{K}$, $S \subset V$ subm $\neq \phi$

1) S. s.n. sistem liniar independent (SLI) $\Leftrightarrow$

$$\left[\begin{matrix}\forall a_1,..,a_n \in \mathbb{K} \\ \forall x_1,..,x_n \in S\end{matrix} : \sum_{i=1}^{m} a_i x_i = 0_V \Rightarrow a_1 = .. = a_n = 0\right]$$

($\forall$ comb. liniară nulă este trivială)

2) S s.n. sistem liniar dependent (SLD) $\Leftrightarrow$

$\exists x_1,..,x_n \in S$

$\exists a_1,..,a_n \in \mathbb{K}$, nu toți nuli aî $\sum_{i=1}^{n} a_i x_i = 0_V$

Prop. $(V,+,\cdot)/\mathbb{K}$, $x \in V \Rightarrow \{x\}$ este SLI, $x \neq 0_V$

Dem

Fie $a \in \mathbb{K}$ aî $a \cdot x = 0_V$

Pp. abs. $a \neq 0_{\mathbb{K}} \Rightarrow \exists a^{-1}$

$\underbrace{a^{-1} \cdot a}_{1_{\mathbb{K}}} \cdot x = \underbrace{a^{-1} \cdot 0_V}_{0_V} \Rightarrow x = 0_V$ $\not\!\!\! \not$ $(x \neq 0_V)$

Pp. este falsă $\Rightarrow \{x\}$ SLI

Def. $(V,+,\cdot)/\mathbb{K}$, $S \subset V$ subm $\neq \phi$

S s.n. bază $\Leftrightarrow \begin{cases} 1) \text{ S este SLI} \\ 2) \text{ S este SG.} \end{cases}$

Exemple

1) $(\mathbb{R}^n,+,\cdot)/\mathbb{R}$, $B_0 = \{e_1 = (1,0,..,0), e_2 = (0,1,0,..,0), ... e_n = (0,..,0,1)\}$ bază canonică

• B_0 este SLI

Fie $a_1,..,a_n \in \mathbb{R}$ aî $\sum_{i=1}^{n} a_i e_i = 0_{\mathbb{R}^n} \Rightarrow$

$a_1(1,0,..,0) + a_2(0,1,0,..,0) + ... + a_n(0,..,0,1) = 0_{\mathbb{R}^n}$.

$(a_1, a_2, .., a_n) = (0,..,0) \Rightarrow a_1 = ... = a_n = 0_{\mathbb{R}}$

• B_0 este SG.

$\forall x = (x_1,..,x_n) = \underbrace{(x_1,0,..,0)}_{x_1(1,0,..,0)} + .. + (0,..,0,x_n) = \underline{x_1} \cdot e_1 + ... + \underline{x_n} e_n$

Scanned with CamScanner

2) $(\mathbb{K}[X], +, \cdot)/\mathbb{K}$, $B_0 = \{1, X, X^2, \ldots\}$ nu e finit generat

$(\mathbb{K}_n[X] = \{P \in \mathbb{K}[X] \mid \text{grad } P \leq n\}, +, \cdot)/\mathbb{K}$

$B_0 = \{1, X, X^2, \ldots, X^n\}$ bază

- SLI
  $\forall a_0, \ldots, a_n \in \mathbb{K} : a_0 + a_1 X + \ldots + a_n X^n = 0 \Rightarrow a_0 = \ldots = a_n = 0$
- SG
  $\forall P = a_0 + a_1 X + \ldots + a_n X^n \in \langle B_0 \rangle$
  $\mathbb{K}_n[X] = \langle B_0 \rangle$

3) $(\mathcal{M}_{m,n}(\mathbb{K}), +, \cdot)/\mathbb{K}$.

$$B_0 = \left\{ E_{ij} = \begin{array}{cc} & j \\ i & \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & 1 & \\ 0 & \vdots & 0 \end{pmatrix} \end{array} \right\}_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}} \text{ bază.}$$

OBS a) $\forall$ subm. $\neq \emptyset$ a unui SLI este un SLI

b) $\forall$ supramulțime a unui SLD este un SLD

c) $\forall$ supramulțime a unui SG este SG.

Teorema schimbului

$(V, +, \cdot)/\mathbb{K}$ sp. vect. finit generat.

$\{x_1, \ldots, x_n\}$ un SG

$\{y_1, \ldots, y_n\}$ un SLI $\Rightarrow \{y_1, \ldots, y_n\}$ este SG

Dem

$V = \langle \{x_1, \ldots, x_n\} \rangle \Rightarrow \exists a_1, \ldots, a_n \in \mathbb{K}$ a.î. $y_1 = \sum_{i=1}^{n} a_i x_i$

$\cup$

y_1

Ip. abs $a_1 = \ldots = a_n = 0_{\mathbb{K}} \Rightarrow y_1 = 0_V$

$\left. \begin{array}{l} 1_{\mathbb{K}} \cdot 0_V + 0_{\mathbb{K}} \cdot y_2 + \ldots + 0_{\mathbb{K}} \cdot y_n = 0_V \\ \text{comb. lin. nulă, care nu e trivială} \end{array} \right\} \Rightarrow \{y_1, \ldots, y_n\}$ SLD

Ip. $a_1 \neq 0_{\mathbb{K}} \Rightarrow \exists a_1^{-1}$

$y_1 = a_1 \underline{x_1} + \ldots + a_n x_n \Rightarrow x_1 = a_1^{-1}(y_1 - a_2 x_2 - \ldots - a_n x_n)$

$\cap$

$<\{y_1, x_2, \ldots, x_n\}>$

$V = <\{x_1, x_2, \ldots, x_n\}> = \underline{<\{y_1, x_2, \ldots, x_n\}>}$

$\ni$

y_2

$y_2 = b_1 y_1 + a_2 x_2 + \ldots + a_n x_n$

$b_1, a_2, \ldots, a_n \in \mathbb{K}$.

Ip. prin absurd că $a_2 = \ldots = a_n = 0_{\mathbb{K}} \Rightarrow y_2 = b_1 y_1 \Rightarrow$

$b_1 y_1 - 1_{\mathbb{K}} \cdot y_2 + 0_{\mathbb{K}} \cdot y_3 + \ldots + 0_{\mathbb{K}} y_n = 0_V \Rightarrow$

$\{y_1, y_2, \ldots, y_n\}$ SLD ↯.

Considerăm $a_2 \neq 0_{\mathbb{K}} \Rightarrow \exists a_2^{-1}$

$y_2 = b_1 y_1 + \underline{a_2 x_2} + a_3 x_3 + \ldots + a_n x_n \mid a_2^{-1}$

$x_2 = a_2^{-1}(y_2 - b_1 y_1 - a_3 x_3 - \ldots - a_n x_n)$.

$V = <\{x_1, x_2, \ldots, x_n\}> = <\{y_1, x_2, x_3, \ldots, x_n\}> =$

$= <\{y_1, y_2, x_3, \ldots, x_n\}>$

Repetăm raționamentul și după n pași

$V = <\{y_1, \ldots, y_n\}> \Rightarrow \{y_1, \ldots, y_n\}$ SG

__Prop__

Card $\forall$ SG(finit) $\geq$ card $\forall$ SLI(finit)

__Dem__

Fie $\{x_1, \ldots, x_n\}$ SG.

Fie $\{y_1, \ldots, y_n, y_{n+1}\} \subseteq V$. Dem că este SLD

1) $\{y_1, \ldots, y_n\}$ SLI $\xrightarrow{Th\ Sch} \{y_1, \ldots, y_n\}$ SG

$y_{n+1} \in V = <\{y_1, \ldots, y_n\}> \Rightarrow y_{n+1} = a_1 y_1 + \ldots + a_n y_n$

Scanned with CamScanner

$a_1 y_1 + \ldots + a_n y_n - \underset{\substack{\neq \\ 0_{\mathbb{K}}}}{1_{\mathbb{K}}} \cdot y_{n+1} = 0 \Rightarrow \{y_1, \ldots y_n, y_{n+1}\}$ SLD

2. $\{y_1, \ldots, y_n\}$ SLD $\xRightarrow[\forall \text{ supram e SLD}]{} \{y_1, \ldots, y_n, y_{n+1}\}$ SLD

<u>Teoremă</u>

$(V, +, \cdot)/\mathbb{K}$ sp. vect. finit generat.

Dacă $B_1, B_2 \subset V$ sunt baze, atunci $|B_1| = |B_2| = \dim_{\mathbb{K}} V = n$

<u>Dem</u>

$$\left.\begin{array}{ll} 1) & B_1 \text{ SG} \\ & B_2 \text{ SLI} \end{array} \Rightarrow |B_1| \geq |B_2| \quad \begin{array}{ll} 2) & B_2 \text{ SG} \\ & B_1 \text{ SLI} \end{array} \Rightarrow |B_2| \geq |B_1| \right\} \Rightarrow |B_1| = |B_2| = n$$

<u>OBS</u>. $(V, +, \cdot)/\mathbb{K}$, $\dim_{\mathbb{K}} V = n$

$B = \{v_1, \ldots, v_n\} \subset V \quad (|B| = n)$

<u>UAE</u> 1) B bază

2) B este SLI

3) B este SG

<u>OBS</u>. $n = \dim_{\mathbb{K}} V$ = nr. max. de vectori care formează SLI

= nr. min. de vectori —"— SG

<u>OBS</u> a) $\forall$ SLI (finit) se poate completa la o bază

b) Din $\forall$ SG (finit, care conține cel puțin un vector $\neq 0_V$) se poate extrage o bază

Ex $(\mathbb{R}^2,+,\cdot)/\mathbb{R}$

a) $B=\{(1,2),(3,4)\}$ bază

b) $S=\{(1,2),(3,4),(4,2)\}$ este SLD, SG.

c) $S'=\{(1,4)\}$ este SLI, nu e SG.
Să se extindă la o bază

d) $S''=\{(1,-1),(2,3),(3,2),(1,4)\}$
Este SG; Să se extragă o bază.

SOL

a) $(\mathbb{R}^2,+,\cdot)/_{\mathbb{R}}$ $B_0=\{e_1=(1,0), e_2=(0,1)\}$ bază canonica

$\dim_{\mathbb{R}}\mathbb{R}^2=2$

• $B=\{(1,2),(3,4)\}$ este SLI

Fie $a,b\in\mathbb{R}$ aî $a(1,2)+b(3,4)=0_{\mathbb{R}^2}$

$(a,2a)+(3b,4b)=(0,0) \Rightarrow (a+3b,2a+4b)=(0,0)$

$\begin{cases} a+3b=0 \\ 2a+4b=0 \end{cases}$ $A=\left(\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 4 & 0 \end{array}\right)$

$\det A\neq 0 \Rightarrow$ SLO are sol unică nula

$a=b=0 \Rightarrow B$ este SLI

OBS $\left.\begin{array}{l} \dim_{\mathbb{R}}\mathbb{R}^2=2 \\ |B|=2 \\ B \text{ este SLI} \end{array}\right\} \Rightarrow B$ bază

(sau dem că B este SG:

$\forall x=(x_1,x_2)=a(1,2)+b(3,4) \overset{(*)}{\Leftrightarrow} \begin{cases} a+3b=x_1 \\ 2a+4b=x_2 \end{cases}$

$A=\left(\begin{array}{cc|c} 1 & 3 & x_1 \\ 2 & 4 & x_2 \end{array}\right)$

$\det A\neq 0 \Rightarrow (*)$ este sol $\exists (a,b)\in\mathbb{R}\times\mathbb{R} \Rightarrow B$ SG

B SLI + SG $\Rightarrow$ bază)

b) $S=B\cup\{(4,2)\}$.
B este SLI, $2=$ nr. max de vect SLI $\Big|$ $\Rightarrow$ S este SLD
B SG $\Rightarrow$ S SG (supramulțime)

Scanned with CamScanner

c) $S' = \{(1,4)\}$

$(1,4) \neq (0,0) \Rightarrow S'$ SLI

$a(1,4) + b(1,0) = (0,0) \Rightarrow \begin{cases} a+b=0 \\ 4a=0 \end{cases}$

$\ \ \ \ \ \ \ \ \ \ \ \ \| $

$(a+b, 4a)$

$\det \begin{pmatrix} 1 & \boxed{1} \\ 4 & \boxed{0} \end{pmatrix} \neq 0$

$\left(\begin{matrix} 1 & 1 \\ 4 & 0 \end{matrix}\right) \Big| \begin{matrix} 0 \\ 0 \end{matrix}$

$\left.\begin{matrix} \{(1,4),(1,0)\} \text{ SLI} \\ 2 \text{ vectori} \end{matrix}\right| \Rightarrow$ o bază

d) $S'' = \{(1,-1), (2,3), (3,2), (1,4)\}$.

$A = \left(\boxed{\begin{matrix} 1 & 2 \\ -1 & 3 \end{matrix}} \begin{matrix} 3 & 1 \\ 2 & 4 \end{matrix}\right)$, $rg A = 2$

$\left.\begin{matrix} \{(1,-1),(2,3)\} \text{ SLI} \\ 2 \text{ vect} \end{matrix}\right\} \Rightarrow$ bază $\Rightarrow$ SG $\Rightarrow S''$ (supram) este SG.

SAU $\forall x = (x_1, x_2) = a(1,-1) + b(2,3) + c(3,2) + d(1,4)$

$\| $

$(a+2b+3c+d, -a+3b+2c+4d)$

$\begin{cases} a+2b+3c+d = x_1 \\ -a+3b+2c+4d = x_2 \end{cases}$

$\left(\boxed{\begin{matrix} 1 & 2 \\ -1 & 3 \end{matrix}} \begin{matrix} 3 & 1 \\ 2 & 4 \end{matrix}\right) \Big| \begin{matrix} x_1 \\ x_2 \end{matrix}$

SC dublu N