Seminar 13 GA.

I Aducerea la o formă canonică a conicelor cu $\delta = 0$.

II Cuodrice studiate pe ec. reduse

I

a) Fie conica $\Gamma : f(x) = \underline{3x_1^2 - 6x_1x_2 + 3x_2^2} + 2x_1 + 2x_2 - 2 = 0$

Să se aducă la o formă canonică, utilizând izometrii.

Reprezentare grafică

$$A = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} \qquad \delta = \det A = 0$$

$$\tilde{A} = \begin{pmatrix} 3 & -3 & 1 \\ -3 & 3 & 1 \\ 1 & 1 & -2 \end{pmatrix} \qquad \Delta = \det \tilde{A} = \begin{vmatrix} 0 & 0 & 2 \\ -3 & 3 & 1 \\ 1 & 1 & -2 \end{vmatrix} = -12$$

$\delta = 0 \quad \Delta \neq 0$, parabolă

$Q : \mathbb{R}^2 \to \mathbb{R} \quad Q(x) = 3x_1^2 - 6x_1x_2 + 3x_2^2$

$\det(A - \lambda I_2) = 0$

$$\begin{vmatrix} 3-\lambda & -3 \\ -3 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 6\lambda = 0 \qquad \lambda(\lambda - 6) = 0$$

$$\begin{cases} \lambda_1 = 6 \\ \lambda_2 = 0 \end{cases} \qquad Q(x) = 6x_1'^2$$

$V_{\lambda_1} = \{ x \in \mathbb{R}^2 \mid Ax = 6x \}$

$(A - 6I_2)X = 0$

$$\begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -3x_1 - 3x_2 = 0 \Rightarrow x_1 = -x_2$$

$V_{\lambda_1} = \{ \cancel{(-x_2, x_2)} \mid x_2 \in \mathbb{R} \}$
$(-x_1, -x_1)$

$e_1' = \frac{1}{\sqrt{2}}(+1, -1)$

$V_{\lambda_2} = \{ x \in \mathbb{R}^2 \mid AX = 0 \}$

$$\begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$3x_1 = 3x_2 \Rightarrow x_1 = x_2 \Rightarrow U_{\lambda_2} = \{ (x_1, x_1) \mid x_1 \in \mathbb{R} \}$

$e_2' = \frac{1}{\sqrt{2}}(1, 1)$

Obs!

$R = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ $\det = -1$ X

$\cancel{\theta}$: $X = RX' \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$

$\Rightarrow x_1 = \frac{1}{\sqrt{2}}(x_1' + x_2')$

$x_2 = \frac{1}{\sqrt{2}}(x_2' - x_1')$

$\theta(\Gamma): f(x) = 6x_1'^2 + \frac{2}{\sqrt{2}}(x_1' + x_2') + \frac{2}{\sqrt{2}}(x_2' - x_1') - 2 = 0$

$f(x) = 6x_1'^2 + \cancel{\frac{2}{\sqrt{2}} x_2}\ 2\sqrt{2}x_2' - 2 = 0 \Rightarrow 3x_1'^2 + \sqrt{2}x_2' - 1 = 0$

$x_1'^2 = \frac{-\sqrt{2}}{3} x_2' + \frac{1}{3}$

$x_1'^2 = 2 \cdot \underbrace{-\frac{\sqrt{2}}{6}}\left(x_2' - \frac{1}{\sqrt{2}} \right) = 2p\, x_2''$

$x_1' = x_1'$

$x_2'' = x_2' - \frac{1}{\sqrt{2}}$

$\tau : x' = x + x_0$

$x_0 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\tau(\theta(\Gamma)): \ x_1''^2 = 2p\, x_2''$

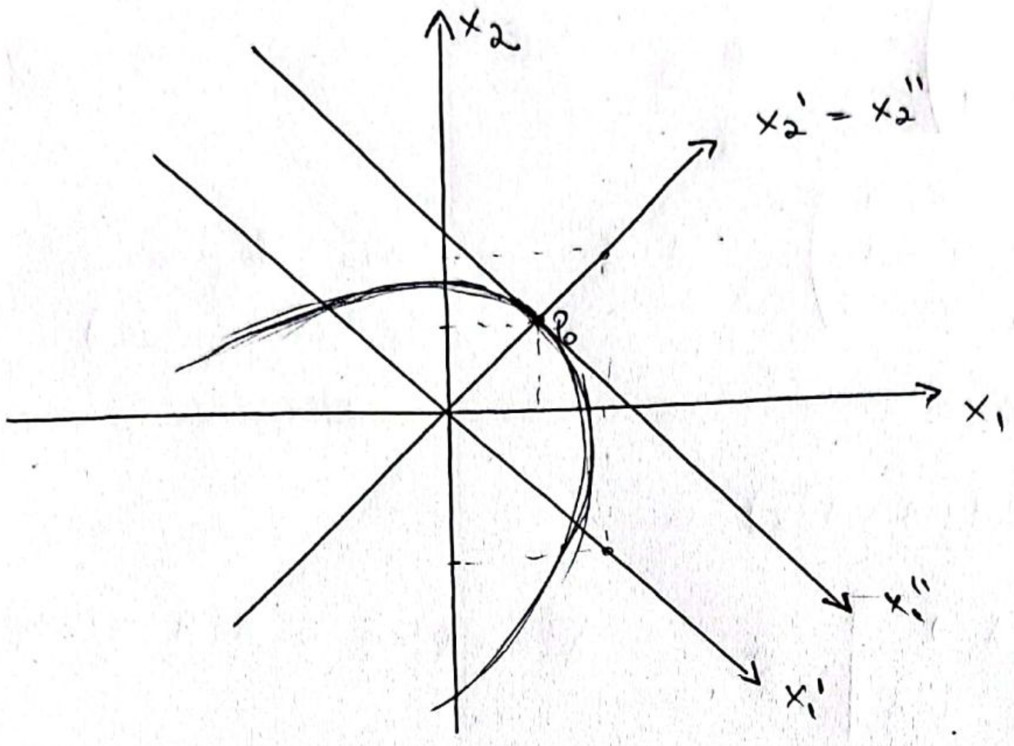
$p = \frac{-\sqrt{2}}{6} < 0$

$\left.\begin{array}{l} \theta : \ x = Rx' \\ \tau : \ x' = x'' + x_0 \end{array}\right| \Rightarrow \tau(\theta): \ x = Rx'' + Rx_0$

$Rx_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$P_0 \left(\frac{1}{2}, \frac{1}{2} \right)$ (în raport cu R)

$R = \{ O; e_1, e_2 \} \longrightarrow R' = \{ O; e_1', e_2' \} \rightarrow R'' = \{ P_0; e_1', e_2' \}$



b) $\Gamma: f(x) = x_1^2 + 2x_1x_2 + x_2^2 + 2x_1 + 2x_2 - 3 = 0$

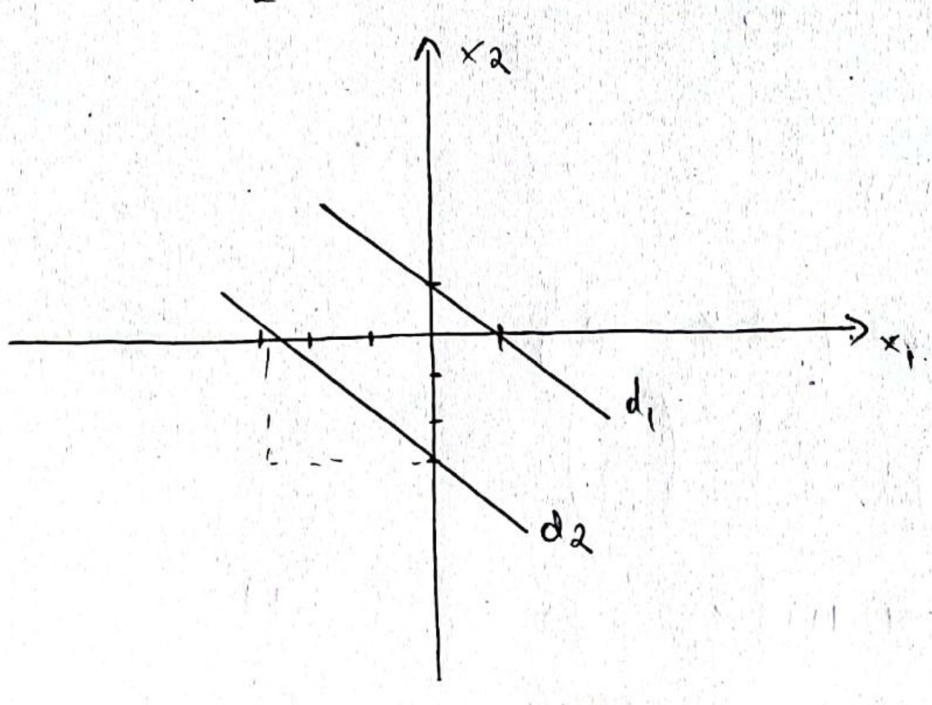
Obs: $f(x) = (x_1 + x_2)^2 + 2(x_1 + x_2) - 3 = 0$

$f(x) = (x_1 + x_2)^2 + 2(x_1 + x_2) + 1 - 4 = 0$

$f(x) = (x_1 + x_2 - 1)(x_1 + x_2 + x_3) = 0$

$d_1: x_1 + x_2 = 1$

$d_2: x_1 + x_2 = -3 \iff \frac{x_1}{-3} + \frac{x_2}{-3} = 1$



$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \delta = 0$$

$$\tilde{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -3 \end{pmatrix} \quad \Delta = 0 \Rightarrow \text{conică degenerată}$$

(dr ~~pct~~ paralele)

$P(\lambda) : \det(A - \lambda I_2) = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \qquad \lambda^2 - 2\lambda = 0 \Rightarrow \lambda(\lambda - 2) = 0$$

$$\begin{cases} \lambda_1 = 2 \\ \lambda_2 = 0 \end{cases}$$

$V_{\lambda_1} = \{ X \in \mathbb{R}^2 \mid AX = 2X \}$

$(A - 2I_2)X = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = x_2$$

$V_{\lambda_1} = \langle \{(1,1)\} \rangle$

$e_1' = \frac{1}{\sqrt{2}}(1,1)$

$V_{\lambda_2} = \{ X \in \mathbb{R}^2 \mid AX = 0 \}$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = -x_2$$

$V_{\lambda_2} = \langle \{(-1, +1)\} \rangle \qquad e_2' = \frac{1}{\sqrt{2}}(-1,1)$

$\theta: \; X = RX' \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$

$$\begin{cases} x_1 = \frac{1}{\sqrt{2}} (x_1' - x_2') \\ x_2 = \frac{1}{\sqrt{2}} (x_1' + x_2') \end{cases}$$

$R = \{O, e_1, e_2\} \longrightarrow R' = \{O, e_1', e_2'\} \Rightarrow \theta(\Gamma): 2x_1'^2 +$

$+ \frac{2}{\sqrt{2}} (x_1' - \cancel{x_2'}) + \frac{2}{\sqrt{2}} (x_1' + \cancel{x_2'}) - 3 = 0$

$\theta(\Gamma): \; 2x_1'^2 + \frac{4}{\sqrt{2}} x_1' - 3 = 0$

$\theta(\Gamma): \; x_1'^2 + \frac{2}{\sqrt{2}} x_1' + \frac{1}{2} - 2 = 0$

$\theta(\Gamma): \; \left(x_1' + \frac{1}{\sqrt{2}}\right)^2 - 2 = 0$

$x_1'' = x_1' + \frac{1}{\sqrt{2}}$

$x_2'' = x_2'$

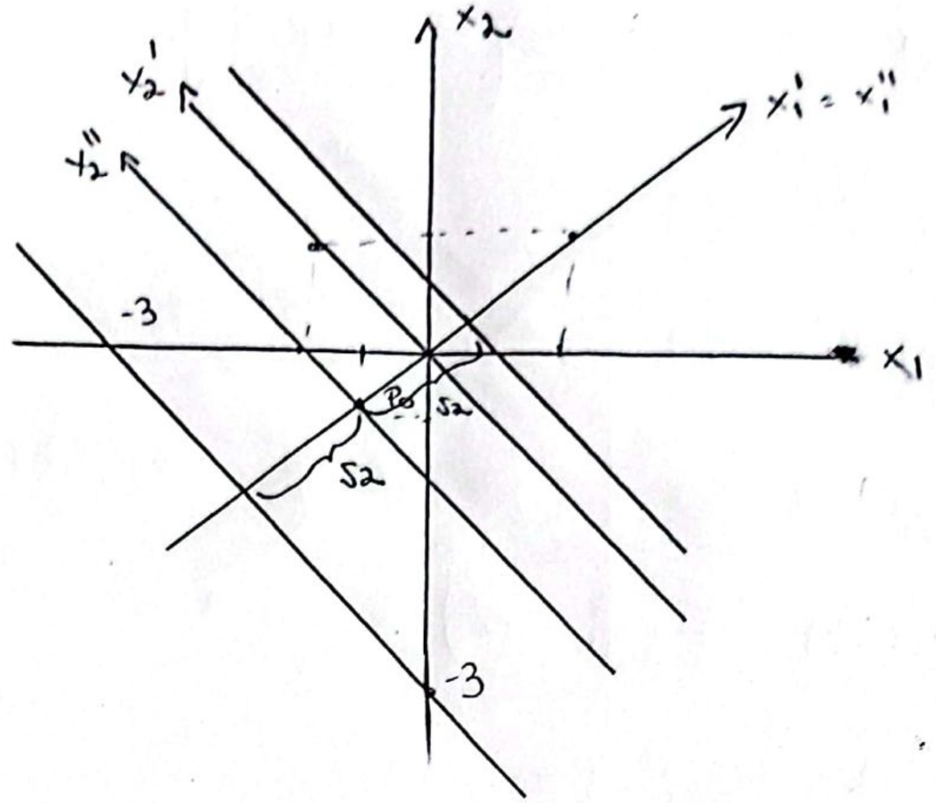
$\eta: \; X' = X'' + X_0 \qquad X_0 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$

$\eta(\theta): \; X = RX'' + RX_0$

$RX_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} =$

$= \frac{1}{2} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \Rightarrow P_0 = \left(-\frac{1}{2}, -\frac{1}{2}\right)$ în raport cu R

$\eta(\theta(\Gamma)): \; x_1''^2 - 2 = 0 \Rightarrow x_1'' = \pm\sqrt{2}$



II

① $P_h : \dfrac{x_1^2}{6} - \dfrac{x_2^2}{4} = 3x_3$

$\pi : x_2 = 2$

$P_h \cap \pi = ?$

$\dfrac{x_1^2}{6} - 1 = 3\, x_3 \Rightarrow x_1^2 = 18x_3 + 6$

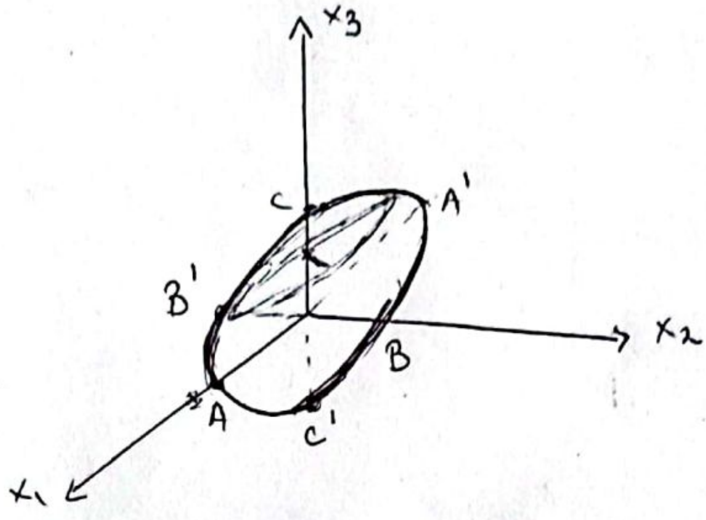
$x_1^2 = 18\left(x_3 + \dfrac{1}{3}\right)$ parabolă în planul π

② $\mathcal{E} : \dfrac{x_1^2}{64} + \dfrac{x_2^2}{49} + \dfrac{x_3^2}{25} - 1 = 0 \Rightarrow a = 8$

$b = 7$

$c = 5$

$\pi : x_3 = 4$



$$\frac{x_1^2}{64} + \frac{x_2^2}{49} = 1 - \frac{16}{25} = \frac{9}{25} \Rightarrow \frac{25x_1^2}{64\cdot 9} + \frac{25x_2^2}{49\cdot 9} = 1 \quad \text{elipsa}$$

$$\Rightarrow a' = \frac{24}{5} \ ; \ b' = \frac{21}{5}.$$

3

$$\mathcal{E}: \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$$

$$P_e: \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 2x_3$$

$\mathcal{E} \cap P_e = ?$

$$2x_3 + \frac{x_3^2}{c^2} = 1$$

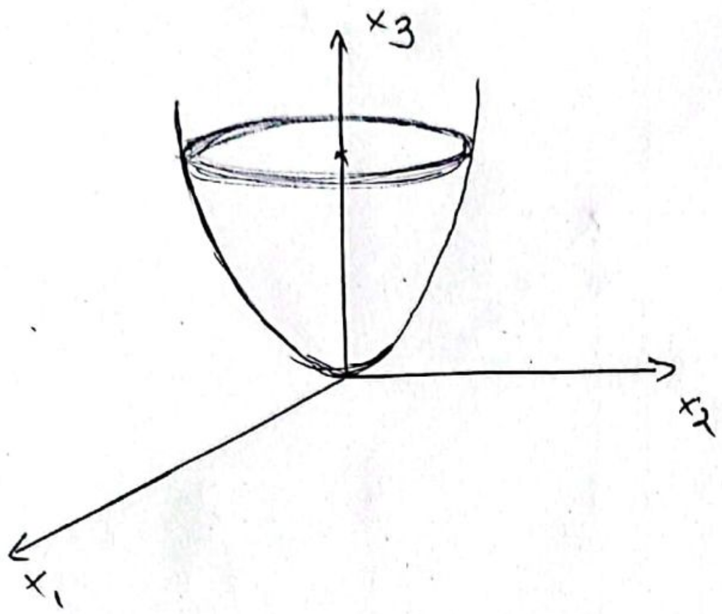
$$\frac{2c^2x_3 + x_3^2}{c^2} = 1$$

$$2c^2x_3 + x_3^2 = c^2$$

$$x_3^2 + 2c^2x_3 - c^2 = 0$$

$$(x_3 + c^2)^2 - c^4 - c^2 = 0 \Rightarrow x_3 + c^2 = \pm\sqrt{c^4 + c^2} = \pm\sqrt{c^2+1}$$

$x_3 = -c^2 \pm c\sqrt{c^2+1}$



$x_3 > 0$

$x_3 = \frac{-c^2}{+c\sqrt{c^2+1}} = \alpha$

$c\sqrt{c^2+1} - c^2 < c$

$\sqrt{c^2+1} < c+1$

$2c^2+1 < c^2+2c+1$ (A)

$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 - \frac{\alpha^2}{c^2}$ elipsă.