Seminarul 10

Transformări ortogonale. End. sim

① $(\mathbb{R}^3, g_0)$ s.v.e.r. cu str. euclidiană canonică

$f \in End(\mathbb{R}^3)$, $A = [f]_{R_0,R_0} = \frac{1}{9}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix}$

R_0 = reperul canonic

a) Să se arate că $f \in O(\mathbb{R}^3)$, de speța 2 i.e

$f = s \circ R_\varphi$

b) Să se det $\sphericalangle$ de rotație φ și axa de rotație

c) Să se det un reper $R = \{e_1, e_2, e_3\}$ ortonormat

a.î. $[f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

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a) $A \cdot A^t = \frac{1}{81}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix}^2 =$

$= \frac{1}{81}\begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix} = I_3 \Rightarrow A \in O_3 \Rightarrow$

$\Rightarrow f \in O(\mathbb{R}^3)$

$$\det A = \frac{1}{9^3} \begin{vmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{vmatrix} = \frac{1}{9^3} \cdot (-9)^3 = -1$$ $\Rightarrow$

$\Rightarrow f$ este de speța 2.

b) $Tr(A) = -1 + 2\cos\varphi = \frac{9}{9} = 1 \Rightarrow \cos\varphi = 1$

$\varphi \in (-\pi, \pi] \Rightarrow \varphi = 0$.

Axa : $f(x) = -x$

$$f(x) = \frac{1}{9}(8x_1 + x_2 - 4x_3,\ x_1 + 8x_2 + 4x_3,\ -4x_1 + 4x_2 - 7x_3)$$

$$f(x) = -x \Rightarrow (8x_1 + x_2 - 4x_3,\ x_1 + 8x_2 + 4x_3,\ -4x_1 + 4x_2 - 7x_3)$$

$$= (-9x_1,\ -9x_2,\ -9x_3)$$

$$\begin{cases} 17x_1 + x_2 - 4x_3 = 0 \\ x_1 + 17x_2 + 4x_3 = 0 \\ -4x_1 + 4x_2 + 2x_3 = 0 \end{cases}$$

$$B = \begin{pmatrix} 17 & 1 & -4 \\ 1 & 17 & 4 \\ -4 & 4 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \det B = 0,\ rg\, B = 2$$

$$\begin{cases} 17x_1 + x_2 = 4x_3 \\ x_1 + 17x_2 = -4x_3 \end{cases} \quad (+)$$

$18x_1 + 18x_2 = 0 \Rightarrow x_1 = -x_2$

$x_1 = \frac{1}{4}x_3 \qquad x_2 = -\frac{1}{4}x_3$

$\left\{ \left(\frac{1}{4}x_3, -\frac{1}{4}x_3, x_3 \right) \mid x_3 \in \mathbb{R} \right\} =$

$= \left\{ \frac{x_3}{4} \underbrace{(1, -1, 4)}_{f_1} \mid x_3 \in \mathbb{R} \right\}$

$e_1 = \frac{f_1}{\|f_1\|} = \frac{(1, -1, 4)}{3\sqrt{2}}$ → versorul axei

e) $\langle e_1 \rangle^{\perp} = \{ x \in \mathbb{R}^3 \mid g_0(x, e_1) = 0 \} =$

$= \{ x \in \mathbb{R}^3 \mid x_1 - x_2 + 4x_3 = 0 \} = \{ (x_1, x_1 + 4x_3, x_3) \mid$

$x_1, x_3 \in \mathbb{R} \} = \langle \{ \underbrace{(1, 1, 0)}_{f_2}, \underbrace{(0, 4, 1)}_{f_3} \} \rangle$

$\{f_2, f_3\}$ reper în $\langle e_1 \rangle^{\perp}$ (SG + CLi)

Aplicăm Gram-Schmidt

$e_2' = f_2 = (1, 1, 0)$

$e_3' = f_3 - \frac{\langle f_3, e_2' \rangle}{\langle e_2', e_2' \rangle} e_2'$

$= (0, 4, 1) - \frac{4}{2} (1, 1, 0)$

$= (0, 4, 1) - (2, 2, 0)$

$= (-2, 2, 1)$

$\Rightarrow e_3 = \frac{1}{3}(-2,2,1) = \frac{e_3'}{\|e_3'\|}$

$R = \{ \frac{1}{3\sqrt{2}}(1,-1,4), \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{3}(-2,2,1) \}$

reper ortonormat în R^3 și $[f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$f \Rightarrow$ simetria ortogonală față de $\langle e_1 \rangle^{\perp}$

• problema 2:

(R^3, g_0) s.v.e.r., $u = (1,1,0)$

a) $\langle \{u\} \rangle^{\perp} = ?$. Precizați un reper ortonormat

b) Să se determine transf. ortogonală, de speța 1, care este rotație de $\measuredangle$ orientat $\varphi = \frac{\pi}{2}$ și axă $\langle \{u\} \rangle$

a) $\langle \{u\} \rangle^{\perp} = \{x \in R^3 \mid g_0(x,u) = 0\} = \{x \in R^3 \mid x_1 + x_2 = 0\} =$

$x_1 + x_2 = 0 \Downarrow x_2 = -x_1$

$= \{(x_1, -x_1, x_3) \mid x_1, x_3 \in R\} = \langle \{ \underbrace{(1,-1,0)}_{f_2}, \underbrace{(0,0,1)}_{f_3} \} \rangle$

$\{f_2, f_3\}$ este un reper arbitrar în $\langle \{u\} \rangle^{\perp}$ (dim $\langle \{u\} \rangle =$ card$\{f_2, f_3\} = 2$ și $\{f_2, f_3\}$ este S.G.)

Aplicăm Gram-Schmidt:

$e_2' = f_2 = (1,-1,0) \Rightarrow e_2 = \frac{1}{\sqrt{2}}(1,-1,0)$

$e_3' = f_3 - \frac{\langle f_3, e_2' \rangle}{\langle e_2', e_2' \rangle} \cdot e_2' = (0,0,1) - \frac{0}{\dots} e_2' = (0,0,1)$

$\Rightarrow e_3 = (0,0,1)$

$$e_1 = \frac{u}{\|u\|} = \frac{1}{\sqrt{2}}(1,1,0)$$

$$\mathcal{R} = \left\{ e_1 = \frac{1}{\sqrt{2}}(1,1,0),\ e_2 = \frac{1}{\sqrt{2}}(1,-1,0),\ e_3 = (0,0,1) \right\}$$

reper ortonormat.

$$A' = [f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ 0 & \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathcal{R}_0 = \{e_1^0, e_2^0, e_3^0\} \xrightarrow{C} \mathcal{R} = \{e_1, e_2, e_3\}$$

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$A = [f]_{\mathcal{R}_0,\mathcal{R}_0}; \quad A' = C^{-1} A C = C^t A C$$

$$\boxed{A = C A' C^t}$$

3) $(\mathbb{R}^3, g_0)$, $f \in End(\mathbb{R}^3)$

$$A = [f]_{\mathcal{R}_0,\mathcal{R}_0} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

a) Dem că $f \in Sim(\mathbb{R}^3)$. Determinați f

b) Să se afle $Q: \mathbb{R}^3 \to \mathbb{R}$ formă pătratică asociată

c) Să se aducă Q la o formă canonică, efectuând o transformare ortogonală h. (i.e. o schimbare de repere ortonormate)

a) $A = A^t \Rightarrow f \in \text{Sim}(\mathbb{R}^3)$

$f(x) = (x_1 + x_3, x_2, x_1 + x_3)$

$Q(x) = X^t A X = \sum_{i,j=1}^{3} a_{ij} x_i \cdot x_j$

$Q(x) = x_1^2 + 2x_1x_3 + x_2^2 + x_3^2$

Aplicăm metoda valorilor proprii

I Polinomul caracteristic

$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} =$

$= (1-\lambda) \cdot (-1)^4 \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)((1-\lambda)^2 - 1) =$

$= (1-\lambda)(1 - 2\lambda + \lambda^2 - 1) = (1-\lambda)(\lambda^2 - 2\lambda) =$

$= \lambda(1-\lambda)(\lambda - 2)$

II. $\lambda_1 = 1, \quad m_1 = 1$

$\lambda_2 = 0, \quad m_2 = 1$

$\lambda_3 = 2, \quad m_3 = 1$

$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid A \cdot X = \lambda_1 \cdot X = X \}$

$(A - I_3) \cdot X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\det = 0 \Rightarrow rg = 1$

$$\begin{cases} x_3 = 0 \\ x_1 = 0 \end{cases}$$

$V_{\lambda_1} = \{(0, x_2, 0) \mid x_2 \in \mathbb{R}\} = \langle\{(0,1,0)\}\rangle$

$e_1 = (0,1,0)$

$V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid A \cdot X = X \cdot \lambda_2\}$

$A \cdot X = 0_{3,1}$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -x_3 \\ x_2 = 0 \end{cases}$$

$V_{\lambda_2} = \langle\{(1,0,-1)\}\rangle$

$e_2 = \frac{1}{\sqrt{2}}(1,0,-1)$

$V_{\lambda_3} = \{x \in \mathbb{R}^3 \mid A \cdot X = X \cdot \lambda_3\}$

$$(A - 2I_3) \cdot X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x_1 + x_3 = 0 \Rightarrow x_1 = x_3 \\ -x_2 = 0 \\ x_1 - x_3 = 0 \end{cases}$$

$V_{\lambda_3} = \{(x_1, 0, x_1) \mid x_1 \in \mathbb{R}\} = \langle\{(1,0,1)\}\rangle$

$e_2 = \frac{1}{\sqrt{2}}(1,0,1)$

$R = \{(0,1,0), \frac{1}{\sqrt{2}}(1,0,-1), \frac{1}{\sqrt{2}}(1,0,1)\} \Rightarrow$ reper ortonormat

$$[f]_{R,R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = A'$$

$Q(x) = x_1'^2 + 2x_3'^2$

signatura: $(2,0) \Rightarrow$ nu e pozitiv definită
(trebuie să fie 3 și 0)

$R_0 = \{e_1^0, e_2^0, e_3^0\} \xrightarrow[\perp]{C \in O(3)} R = \{e_1, e_2, e_3\}$
ortogonale

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 \\ \sqrt{2} & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

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$R \in O(\mathbb{R}^3)$

$R(e_i^0) = e_i \quad (\forall)\ i = \overline{1,3}$