15.03.2021

SEMINAR 9

Forme pătratice. Metoda Jacobi
Spații vectoriale euclidiene reale

- **Metoda Jacobi**

$Q: V \to \mathbb{R}$ formă pătratică reală și G matricea asociată în raport cu un reper R.

a) Dacă minorii diagonali

$$\Delta_1 = \det(g_{11}) \neq 0$$
$$\Delta_2 = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} \neq 0$$
$$\vdots$$
$$\Delta_n = \det G \neq 0$$

$\Longrightarrow$ $\exists$ un reper R' în V a.î. Q are o formă canonică

$$Q(x) = \frac{1}{\Delta_1} x_1'^2 + \frac{\Delta_1}{\Delta_2} x_2'^2 + \ldots + \frac{\Delta_{n-1}}{\Delta_n} x_n'^2.$$

b) Q este poz. def $\Leftrightarrow \Delta_i > 0, \forall i = \overline{1,n}$

- $(V,+,\cdot)/\mathbb{R}$, $g: V \times V \to \mathbb{R}$ produs scalar

$\Leftrightarrow$ 1) $g \in L^s(V,V;\mathbb{R})$

2) g poz. def.

(V,g) sp. vect. euclidian real

- $\|x\| = \sqrt{g(x,x)} = \sqrt{Q(x)}, \forall x \in V$

- $R = \{e_1, \ldots, e_n\}$ reper
  - ortogonal $\Leftrightarrow g(e_i, e_j) = 0, \forall i \neq j, i,j = \overline{1,n}$
  - ortonormat $\Leftrightarrow g(e_i, e_j) = \delta_{ij}, \forall i,j = \overline{1,n}$

$R \xrightarrow{A} R'$, $A \in O(n)$

repere ortonormate

- $(E, \langle \cdot, \cdot \rangle)$
  $U \subseteq E$ ssp vect $\Rightarrow U^{\perp} = \{ y \in E \mid \langle x, y \rangle = 0, \forall x \in U \}$

- $(\mathbb{R}^3, g_0)$ $\quad g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}, \ g_0(x, y) = x_1y_1 + x_2y_2 + x_3y_3$
  (produs scalar canonic)

$S = \{x, y\}$ SLI

a) $z = x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$, $\quad z \perp x$, $z \perp y$
(produs vectorial)

b) $u \wedge x \wedge y = \langle u, x \times y \rangle = \begin{vmatrix} u_1 & u_2 & u_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$
(produs mixt)

Teoremă (Gram-Schmidt)

$(E, \langle \cdot, \cdot \rangle)$, $R = \{f_1, \dots, f_n\}$ reper arb.

$\Rightarrow \exists R' = \{e_1, \dots, e_n\}$ reper ortogonal ai $Sp\{e_1, \dots, e_i\} = Sp\{f_1, \dots, f_i\}$, $i = \overline{1,n}$

$$\begin{cases} e_1 = f_1 \\ e_2 = f_2 - \dfrac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 \\ \vdots \\ e_n = f_n - \dfrac{\langle f_n, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 - \dots - \dfrac{\langle f_n, e_{n-1} \rangle}{\langle e_{n-1}, e_{n-1} \rangle} e_{n-1} \end{cases}$$

Ex1 Fie $Q : \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = x_1^2 + 3x_2^2 + 4x_2x_3$

Să se aducă la o formă canonică

SOL $R_0 = \{e_1, e_2, e_3\}$ reper canonic

$Q(x) = X^T G X = g_{11}x_1^2 + g_{22}x_2^2 + g_{33}x_3^2 + 2g_{12}x_1x_2 + 2g_{13}x_1x_3 + 2g_{23}x_2x_3$

Met Jacobi

$$G = \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

$\Delta_1 = 1 \neq 0$

$\Delta_2 = \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 3 \neq 0$

$\Delta_3 = \det G = 1 \begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix} = -4 \neq 0$

$\exists$ un reper $R' = \{e_1', e_2', e_3'\}$ în $\mathbb{R}^3$ ai

$$Q(x) = \frac{1}{\Delta_1} x_1'^2 + \frac{\Delta_1}{\Delta_2} x_2'^2 + \frac{\Delta_2}{\Delta_3} x_3'^2 = 1 \cdot x_1'^2 + \frac{1}{3} x_2'^2 - \frac{3}{4} x_3'^2$$

$(2,1)$ signatura lui Q (nu e poz. def)

Met. Gauss

$$Q(x) = \underline{x_1^2} + \underline{\underline{3x_2^2 + 4x_2x_3}} = x_1^2 + 3\left(x_2^2 + \frac{4}{3}x_2x_3\right) =$$

$$= x_1^2 + 3\left(x_2 + \frac{2}{3}x_3\right)^2 - \frac{4}{3}x_3^2$$

Fie $\begin{cases} x_1' = x_1 \\ x_2' = x_2 + \frac{2}{3}x_3 \\ x_3' = x_3 \end{cases} \Rightarrow Q(x) = x_1'^2 + 3x_2'^2 - \frac{4}{3}x_3'^2$

$(2,1)$ signatura

Ex2 $(\mathbb{R}^2, +, \cdot)/\mathbb{R}$, $g: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = ax_1y_1 + bx_1y_2 + bx_2y_1 + cx_2y_2$

a) g formă biliniară simetrică

b) g produs scalar $\Leftrightarrow \begin{cases} a > 0 \\ ac - b^2 > 0. \end{cases}$

SOL

$R_0 = \{e_1, e_2\}$ reper canonic $\qquad G = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$\left.\begin{array}{l} g(x,y) = X^T G Y \Rightarrow g \text{ formă biliniară} \\ G = G^T \Rightarrow g \text{ simetrica} \end{array}\right\} \Rightarrow g \in L^s(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$

$Q: \mathbb{R}^2 \to \mathbb{R}$, $Q(x) = g(x,x) = ax_1^2 + cx_2^2 + 2bx_1x_2$

Met. Jacobi

$\Delta_1 = a$ $\qquad$ Q poz def $\Leftrightarrow \begin{cases} a > 0 \\ ac - b^2 > 0 \end{cases}$

$\Delta_2 = \det G = ac - b^2$

$\exists R' = \{e_1', e_2'\}$ reper în $\mathbb{R}^2$ a.î. $Q(x) = \frac{1}{a} x_1'^2 + \frac{a}{ac-b^2} x_2'^2$

Met Gauss

$Q(x) = \underline{a x_1^2 + 2b x_1 x_2} + c x_2^2 = \frac{1}{a}(a^2 x_1^2 + 2ab x_1 x_2) + c x_2^2 =$

$= \frac{1}{a}(a x_1 + b x_2)^2 - \frac{b^2}{a} x_2^2 + c x_2^2 = \frac{1}{a}(a x_1 + b x_2)^2 + \left(c - \frac{b^2}{a}\right) x_2^2$

$\begin{cases} x_1' = a x_1 + b x_2 \\ x_2' = x_2 \end{cases} \Rightarrow Q(x) = \frac{1}{a} x_1'^2 + \frac{ac-b^2}{a} x_2'^2$

Q poz def $\Leftrightarrow \begin{cases} \frac{1}{a} > 0 \\ \frac{ac-b^2}{a} > 0 \end{cases} \Leftrightarrow \begin{cases} a > 0 \\ ac - b^2 > 0 \end{cases}$

Ex3 $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$, $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ formă biliniară

$G = \begin{pmatrix} \boxed{3} & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ matricea asociată în rap. cu R_0

Este $(\mathbb{R}^3, g)$ spațiu vect. euclidian real?

SOL

$G = G^T \Rightarrow g$ simetrică

$Q: \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = 3x_1^2 + 4x_1x_2 + 2x_2^2 + 4x_2x_3 + x_3^2$

Met Jacobi $\Delta_1 = 3 > 0$

$\Delta_2 = \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 6 - 4 = 2 > 0$

$\Delta_3 = \det G = 3(2-4) - 2(2-0) = -6 - 4 = -10 < 0$

$(2,1)$ signatura lui $Q \Rightarrow g$ nu e poz. def

$(\mathbb{R}^3, g)$ nu este sp. vect euclidian

Ex4 $(\mathbb{R}^3, g_0)$, $g_0: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g_0(x,y) = x_1y_1 + x_2y_2 + x_3y_3$

$U = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\} = S(A)$, $A = (1 \ 1 \ -1)$

a) $U^\perp = ?$

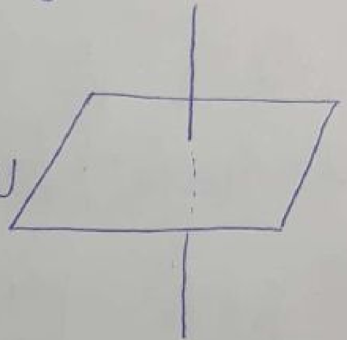
b) Să se det. un reper ortonormat în $\mathbb{R}^3$

$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, unde $\mathcal{R}_1$ reper ortonormat în U

$\mathcal{R}_2$ — " — $U^\perp$

SOL

a) $\mathbb{R}^3 = U \oplus U^\perp$ complement ortogonal



$U = \{x \in \mathbb{R}^3 \mid g_0((x_1, x_2, x_3), (1, 1, -1)) = 0\}$

$U^\perp = \langle\{(1,1,-1)\}\rangle$

$\dim U = 2$, $\dim U^\perp = 1$.

b) $U = \{(x_1, \underbrace{x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R}\}}_{\substack{\| \\ x_1(1,0,1) + x_2(0,1,1)}} = \{(x_1, 0, x_1) + (0, x_2, x_2) \mid x_1, x_2 \in \mathbb{R}\}$

$x_1 \underset{f_1}{(1,0,1)} + x_2 \underset{f_2}{(0,1,1)}$

$\left.\begin{array}{l}\{f_1, f_2\} \text{ e SG pt } U \\ \dim U = 2\end{array}\right\} \Rightarrow \{f_1, f_2\}$ reper arb.

Aplicăm procedeul Gram-Schmidt

$$\begin{cases} e_1 = f_1 = (1,0,1) \\ e_2 = f_2 - \dfrac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} \cdot e_1 = (0,1,1) - \dfrac{1}{2}(1,0,1) = \left(-\dfrac{1}{2}, 1, \dfrac{1}{2}\right) = \dfrac{1}{2}(-1,2,1) \end{cases}$$

$\langle f_2, e_1\rangle = g_0(f_2, e_1) = g_0((0,1,1), (1,0,1)) = 0\cdot 1 + 1 \cdot 0 + 1 \cdot 1 = 1$

$\langle e_1, e_1\rangle = g_0(f_1, f_1) = g_0((1,0,1), (1,0,1)) = 1 + 0 + 1 = 2$

$\underset{\text{reper } \forall}{\{f_1, f_2\}} \to \underset{\text{reper ortog.}}{\{e_1, e_2\}} \to \left\{\dfrac{e_1}{\|e_1\|}, \dfrac{e_2}{\|e_2\|}\right\} = \mathcal{R}_1$ reper ortonormat

$\|e_1\| = \sqrt{g_0(e_1, e_1)} = \sqrt{2}$

$\mathcal{R}_1 = \left\{\dfrac{1}{\sqrt{2}}(1,0,1), \dfrac{1}{\sqrt{6}}(-1,2,1)\right\}$ reper ortonormat în U.

OBS $v = \alpha \cdot u,\ \alpha > 0$

$\|v\| = |\alpha| \cdot \|u\| = \alpha \|u\|$

$$\frac{v}{\|v\|} = \frac{\alpha\ u}{\alpha \|u\|} = \frac{u}{\|u\|}$$

$$e_2 = \underbrace{\frac{1}{2}}_{\alpha}(-1,2,1)$$

$\mathcal{R}_2 = \left\{ \frac{1}{\sqrt{3}}(1,1,-1) \right\}$ reper în $U^\perp$

$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 = \left\{ \frac{1}{\sqrt{2}}(1,0,1), \frac{1}{\sqrt{6}}(-1,2,1), \frac{1}{\sqrt{3}}(1,1,-1) \right\}$ reper în $\mathbb{R}^3$

OBS

$$f_1 \times f_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = e_1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - e_2 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + e_3 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$= (-1,-1,1) = -(1,1,-1)$

Ex5 $(\mathbb{C},+,\cdot)/\mathbb{R}$, $g: \mathbb{C}\times\mathbb{C} \to \mathbb{R}$ formă biliniară și $G = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ matricea asociată lui g în raport cu $\mathcal{R}_0 = \{1, i\}$

a) $(\mathbb{C}, g)$ este sp. vect. euclidian real

b) $u = 2 - i$ este versor în raport cu g

c) $\langle \{u\} \rangle^\perp$

d) Să se ortonormeze $\mathcal{R}_0$ în rap cu g

e) Să se afle intersecția dintre cercul unitate în $(\mathbb{C}, g_0)$ și în $(\mathbb{C}, g)$

SOL

a) $g: \mathbb{C}\times\mathbb{C} \to \mathbb{R}$, $\quad G = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$

$g(z, z') = x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + 5 x_2 y_2 = X^T G Y \Rightarrow$ g formă biliniară simetrică

$z = x_1 + i x_2 \equiv (x_1, x_2)$ $\quad \mathbb{C} \simeq \mathbb{R}^2$

$z' = y_1 + i y_2 \equiv (y_1, y_2)$

$G = G^T$

$Q: \mathbb{C} \to \mathbb{R}$, $Q(z) = g(z,z) = x_1^2 + 4x_1x_2 + 5x_2^2$

Met. Jacobi

$\Delta_1 = 1 > 0$

$\Delta_2 = 5-4 = 1 > 0$ $\Rightarrow \exists$ un reper R' în $\mathbb{C}$ aî $Q(z) = x_1'^2 + x_2'^2$

Met Gauss

$Q(z) = (x_1 + 2x_2)^2 + x_2^2 = x_1'^2 + x_2'^2$

$\begin{cases} x_1' = x_1 + 2x_2 \\ x_2' = x_2 \end{cases}$

$(2,0)$ signatura $\Rightarrow Q$ poz def $\Rightarrow g$ poz def.

g este produs scalar $\Rightarrow (\mathbb{C}, g)$ sp. vect. euclidian

b) $u = 2-i$ versor în rap cu g

$\|u\|_g = \sqrt{g(u,u)} = \sqrt{Q(u)} = \sqrt{4 + 4\cdot 2(-1) + 5(-1)^2} = \sqrt{4-8+5} = 1$

$u = 2-i$, $x_1 = 2, x_2 = -1$

c) $<\{u\}>^{\perp} = \{z \in \mathbb{C} \mid g(z,u) = 0\} = \{z = x_1 + x_2 i \mid -x_2 = 0\}$
$= \mathbb{R}$

$z = x_1 + ix_2 \equiv (x_1, x_2)$

$u = 2 - i = (2,-1) = (y_1, y_2)$

$g(z,z') = g((x_1,x_2),(y_1,y_2)) = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 5x_2y_2$

$g(z,u) = 2x_1 - 2x_1 + 4x_2 - 5x_2 = -x_2$

d) $R_0 = \{f_1 = 1, f_2 = i\}$

Aplicăm procedeul Gram-Schmidt

$\begin{cases} e_1 = f_1 = 1 \\ e_2 = f_2 - \dfrac{g(f_2,e_1)}{g(e_1,e_1)} e_1 = i - \dfrac{2}{1}\cdot 1 = -2+i \end{cases}$

$g(f_2,e_1) = g(\underset{0+1i}{i}, \underset{1+0i}{1}) = 0\cdot 1 + 2\cdot 0\cdot 0 + 2\cdot 1\cdot 1 + 5\cdot 1\cdot 0 = 2$

$g(e_1,e_1) = g(1,1) = Q(1) = 1^2 + 4\cdot 1\cdot 0 + 5\cdot 0^2 = 1$

Scanned with CamScanner

$\{1, i\} \rightarrow \{e_1 = 1,\ e_2 = -2 + i = u\}$

reper ∀ — reper ortonormat

$g(e_1, e_2) = 0$

$g(e_1, e_1) = 1, \quad g(e_2, e_2) = 1$

e) 1) $(\mathbb{C}, g_0)$, $g_0 : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{R}$, $g_0(z, z') = x_1 y_1 + x_2 y_2$.

$Q_0 : \mathbb{C} \longrightarrow \mathbb{R}$, $Q_0(z) = x_1^2 + x_2^2$

$$S^1_{g_0} = \{z \in \mathbb{C} \mid \|z\|_{g_0} = 1\} = \{\underset{x_1 + i x_2}{z} \in \mathbb{C} \mid x_1^2 + x_2^2 = 1\}$$

2) $(\mathbb{C}, g)$, $g : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{R}$, $g(z, z') = x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + 5x_2 y_2$

$Q : \mathbb{C} \longrightarrow \mathbb{R}$, $Q(z) = x_1^2 + 4x_1 x_2 + 5x_2^2$

$$S^1_g = \{z \in \mathbb{C} \mid \|z\|_g = 1\} = \{z \in \mathbb{C} \mid x_1^2 + 4x_1 x_2 + 5x_2^2 = 1\}$$

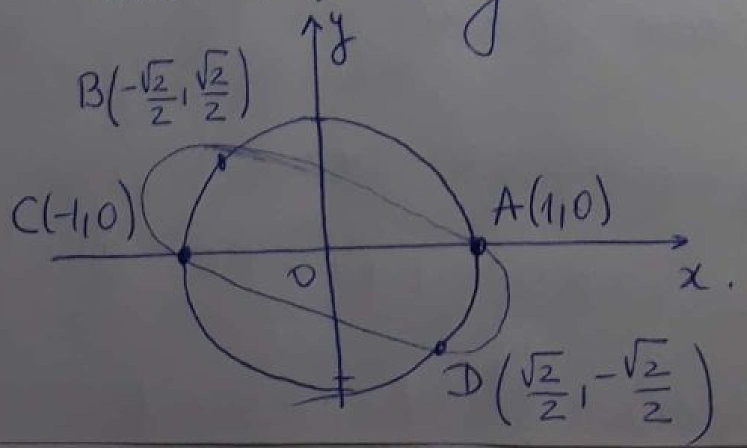
$$S^1_{g_0} \cap S^1_g : \begin{cases} x_1^2 + x_2^2 = 1 \\ x_1^2 + 4x_1 x_2 + 5x_2^2 = 1 \end{cases} \Rightarrow \begin{cases} x_1^2 + x_2^2 = 1 \\ 4x_1 x_2 + 4x_2^2 = 0 \end{cases}$$

$$4x_2(x_1 + x_2) = 0$$

$z = x_1 + i x_2 = \cos t + i \sin t$, $x_1^2 + x_2^2 = 1$.

$\sin t(\sin t + \cos t) = 0$.

- $\sin t = 0 \Rightarrow t = k\pi$, $k \in \mathbb{Z}$.
- $\sin t = -\cos t \Rightarrow \operatorname{tg} t = -1 \Rightarrow t = -\operatorname{arctg} 1 + k\pi = -\frac{\pi}{4} + k\pi$, $k \in \mathbb{Z}$.



$S^1_{g_0} \cap S^1_g = \{z_A, z_B, z_C, z_D\}$

$z_A = 1$

$z_B = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$z_C = -1$

$z_D = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

Ex6 $(\mathbb{R}^3, g_0)$, $\mathcal{R} = \{f_1 = (1,2,3), f_2 = (0,1,1), f_3 = (1,2,5)\}$

a) $\mathcal{R}$ reper în $\mathbb{R}^3$. Să se ortonormeze

b) $f_1 \times f_2$; c) $f_1 \wedge f_2 \wedge f_3$

SOL

a) $\det\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 5 \end{pmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 2 \end{vmatrix} = 2 \neq 0 \Rightarrow$

$\left.\begin{array}{l} \Rightarrow \mathcal{R} \text{ este SLI în } \mathbb{R}^3 \\ \dim \mathbb{R}^3 = |\mathcal{R}| = 3 \end{array}\right\} \Rightarrow \mathcal{R}$ reper în $\mathbb{R}^3$

Aplicăm procedeul Gram-Schmidt

$$\begin{cases} e_1 = f_1 = (1,2,3) \\ e_2 = f_2 - \dfrac{g_0(f_2, e_1)}{g_0(e_1, e_1)} e_1 = (0,1,1) - \dfrac{5}{14}(1,2,3) = \left(\dfrac{-5}{14}, \dfrac{4}{14}, \dfrac{-1}{14}\right) \\ e_3 = f_3 - \dfrac{g_0(f_3, e_1)}{g_0(e_1, e_1)} e_1 - \dfrac{g_0(f_3, e_2)}{g_0(e_2, e_2)} e_2 \end{cases}$$

$g_0(f_2, e_1) = g_0(f_2, f_1) = g_0((0,1,1), (1,2,3)) = 0 + 2 + 3 = 5$

$g_0(e_1, e_1) = g_0((1,2,3), (1,2,3)) = 1 + 4 + 9 = 14$

$g_0(f_3, e_1) = g_0((1,2,5), (1,2,3)) = 1 + 4 + 15 = 20$

$g_0(f_3, e_2) = g_0\left((1,2,5), \left(\dfrac{-5}{14}, \dfrac{4}{14}, \dfrac{-1}{14}\right)\right) = \dfrac{-5}{14} + \dfrac{8}{14} - \dfrac{5}{14} = \dfrac{-2}{14} = \dfrac{-1}{7}$

$g_0(e_2, e_2) = \dfrac{1}{14^2} g_0((-5,4,-1), (-5,4,-1)) = \dfrac{1}{14^2}(25 + 16 + 1) =$

$= \dfrac{42}{14^2} = \dfrac{14 \cdot 3}{14^2} = \dfrac{3}{14}$.

$e_3 = (1,2,5) - \dfrac{20}{14}(1,2,3) - \dfrac{-\frac{1}{7}}{\frac{3}{14}} \cdot \dfrac{1}{14}(-5,4,-1) =$

$= (1,2,5) - \dfrac{10}{7}(1,2,3) + \dfrac{1}{21}(-5,4,-1) =$

$$e_3 = \left(1 - \frac{10}{7} - \frac{5}{21}, 2 - \frac{20}{7} + \frac{4}{21}, 5 - \frac{30}{7} - \frac{1}{21}\right) =$$

$$= \frac{1}{21}(21-30-5, 42-60+4, 105-90-1) =$$

$$= \frac{1}{21}(-14, -14, 14) = \frac{14}{21}(-1,-1,1) = \frac{2}{3}(-1,-1,1)$$

$\left\{e_1 = (1,2,3), e_2 = \frac{1}{14}(-5,4,1), e_3 = \frac{2}{3}(-1,-1,1)\right\}$ reper ortogonal

$\left\{\frac{1}{\sqrt{14}}(1,2,3), \frac{1}{\sqrt{42}}(-5,4,1), \frac{1}{\sqrt{3}}(-1,-1,1)\right\}$ reper ortonormat

b) $f_1 \times f_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = e_1 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} - e_2 \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} + e_3 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} =$

$= (-1, -1, 1)$

c) $f_1 \wedge f_2 \wedge f_3 = f_3 \wedge f_1 \wedge f_2 = g_0(f_3, f_1 \times f_2) = -1-2+5 = 2$

$f_3 = (1,2,5)$

SAU $\quad f_1 \wedge f_2 \wedge f_3 = \begin{vmatrix} 1 & 2 & 5 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = 2$