Seminar 7 Vectori proprii. Valori proprii. Diagonalizare

5 $f: \mathbb{R}^4 \to \mathbb{R}^4$ $f(x) = (x_2 - x_3 + x_4, x_2 - x_3 + x_4, x_4, x_4)$

a) Să se afle valorile proprii

b) Precizați care sunt subspațiile proprii

c) $\exists$ un reper R în $\mathbb{R}^4$ a.î. $[f]_{R,R}$ este diagonală?

Soluție:

Algoritm:

1) $A = ?$

2) $p(\lambda) = ?$

3) $\lambda_1, \ldots, \lambda_r = ?$, $m_{\lambda_1}, \ldots, m_{\lambda_r} = ?$

4) $V_{\lambda_1} \ldots V_{\lambda_r} = ?$ $V = V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_r}$

$\downarrow$

$R_1 \ldots R_r$ repere

5) $R = R_1 \cup R_2 \cup \ldots \cup R_r$

a) $f(x) = y \Rightarrow Y = AX \Leftrightarrow \begin{pmatrix} x_2 - x_3 + x_4 \\ x_2 - x_3 + x_4 \\ \cancel{-x_3 + x_4} \quad x_4 \\ x_4 \end{pmatrix} =$

$$= \begin{pmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$p(\lambda) = \det(A - \lambda I_4) = 0 \iff \begin{vmatrix} -\lambda & 1 & -1 & 1 \\ 0 & 1-\lambda & -1 & 1 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\iff \lambda^2 (1-\lambda)^2 = 0 \iff \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 0 \end{cases} \text{ şi } \begin{cases} m\lambda_1 = 2 \\ m\lambda_2 = 2 \end{cases}$$

b) $V_{\lambda_1} = \{ x \in \mathbb{R}^4 \mid f(x) = \lambda_1 x \}$

$$A X = X \iff (A - I_4) X = O_{4,1}$$

$$\iff \begin{pmatrix} -1 & \boxed{1} & \boxed{-1} & 1 \\ 0 & \boxed{0} & \boxed{-1} & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\dim V_{\lambda_1} = 4 - rg(A - I_4) = 4 - 2 = 2 = m\lambda_1$$

$$\begin{cases} x_2 - x_3 = x_1 - x_4 \\ -x_3 = -x_4 \end{cases} \Rightarrow \begin{cases} x_2 = x_1 \\ x_3 = x_4 \end{cases} \Rightarrow V_{\lambda_1} = \{ (x_1, x_1, x_3, x_3) / x_1, x_3 \in \mathbb{R} \} =$$

$$= \langle \underbrace{\{(1,1,0,0), (0,0,1,1)\}}_{R_1 \text{ reper în } V_{\lambda_1}} \rangle$$

$$V_{\lambda_2} = \{ x \in \mathbb{R}^4 \mid f(x) = \lambda_2 x \} = \ker f$$

$$A X = O_{4,1}$$

$$\begin{pmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\dim V_{\lambda_2} = 4 - \operatorname{rg} A = 4 - 2 = 2 = m_{\lambda_2}$$

$$\begin{cases} -x_3 + x_4 = -x_2 \\ x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2 = x_3 \\ x_4 = 0 \end{cases} \Rightarrow V_{\lambda_2} = \{ (x_1, x_2, x_2, 0) \mid x_1, x_2 \in \mathbb{R} \}$$

$$= \langle \underbrace{\{ (1,0,0,0), (0,1,1,0) \}}_{R_2 \text{ reper in } V_{\lambda_2}} \rangle$$

Teorema

1) $\lambda_1 = 1$, $\lambda_2 = 0$

$m_{\lambda_1} = 2$, $m_{\lambda_2} = 2$

2) $\dim V_{\lambda_i} = m_{\lambda_i}$, $i = \overline{1,2}$

$\Rightarrow R = R_1 \cup R_2$ reper in $\mathbb{R}^4$ a.î. $[f]_{R,R}$ = diagonală

$$[f]_{R,R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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Fie $f \in End(\mathbb{R}^3)$, $R_0 = \{e_1, e_2, e_3\}$ reperul canonic în $\mathbb{R}^3$

a) $\begin{cases} f(e_1) = e_2 \\ f(e_2) = e_1 + e_2 + e_3 \\ f(e_3) = e_2 \end{cases}$

a) Precizați dacă există câte un reper R în $\mathbb{R}^3$ a.î. $[f]_{R,R}$ este matrice diagonală

b) $A^n = ?$ unde $A = [f]_{R_0, R_0}$

Soluție:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad f(x) = y \Leftrightarrow Y = AX$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 + x_2 + x_3 \\ x_2 \end{pmatrix}$$

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad f(x) = (x_2, x_1 + x_2 + x_3, x_2)$$

$$P(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix}_{C_1 - C_3} = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ \lambda & 1 & -\lambda \end{vmatrix} =$$

$$= \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 2 & -\lambda \end{vmatrix} = -\lambda \cdot \begin{vmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - \lambda - 2)$$

$$= -\lambda(\lambda + 1)(\lambda - 2)$$

$$(-1)^3 [\lambda^3 - \sigma_1 \lambda^2 + \sigma_2 \lambda - \sigma_3] = 0$$

$\lambda_1 = 0$, $\lambda_2 = -1$ $\lambda_3 = 2$ $\Rightarrow m\lambda_1 = 1$ $m\lambda_2 = 1$ $m\lambda_3 = 1$

$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid f(x) = \lambda_1 x \} = \ker f$

$f(x) = 0 \iff \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\dim V_{\lambda_1} = 3 - rg\, A = 3 - 2 = 1 = m_{\lambda_1}$

$\begin{cases} x_2 = 0 \\ x_1 + x_2 = -x_3 \end{cases} \Rightarrow x_1 = -x_3$

$= \{ (-x_3), 0, x_3 \mid x_3 \in \mathbb{R} \} = \langle \{(-1, 0, 1)\} \rangle$

$V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid f(x) = \lambda_2 x \} \quad f(x) = -x \rightarrow$

$\Rightarrow AX = -X \Rightarrow (A + I_3) X = 0_{3,1}$

$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\dim V_{\lambda_2} = 3 - rg(A + I_3) = 3 - 2 = 1 = m_{\lambda_2}$

$\begin{cases} x_1 + x_2 = 0 \\ x_1 + 2x_2 = -x_3 \end{cases} \ominus$

$x_2 = -x_3 \Rightarrow x_1 = x_3$

$V_{\lambda_2} = \{ (x_3, -x_3, x_3) \mid x_3 \in \mathbb{R} \} = \langle \{(1, -1, 1)\} \rangle$

$V_{\lambda_3} = \{ x \in \mathbb{R}^3 \mid f(x) = \lambda_3 \cdot x \}$ $\quad f(x) = 2x$

$A x = 2X \Rightarrow (A - 2I_3) X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_2 = 2x_1 \\ -x_2 + x_3 = -x_1 \end{cases} \Rightarrow \begin{cases} x_2 = 2x_1 \\ x_3 = x_1 \end{cases}$$

$\dim V_{\lambda_3} = 3 - rg(A - 2I_3) = 3 - 2 = 1 = m_{\lambda_3}$

$V_{\lambda_3} = \{ (x_1, 2x_1, x_1) \} = \langle \{(1,2,1)\} \rangle$

Th.:

1) $-1, 0, 2 \in \mathbb{R}$

2) $\dim V_{\lambda_i} = m_{\lambda_i} \quad i = \overline{1,3}$

$R = R_1 \cup R_2 \cup R_3 = \{(-1,0,1), (1,-1,1), (1,2,1)\}$

reper in $\mathbb{R}^3$ a.î. $A' = [f]_{R,R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

b)

$R_0 \xrightarrow{C} R \qquad A' = C^{-1} A C \Rightarrow A = C A' C^{-1}$

$C = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \qquad A^n = (CA'C^{-1})(CA'C^{-1}) \ldots (CA'C^{-1}) =$

$= C (A')^n C^{-1}$

$$(A')^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} \qquad C^{t} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$\det C = \begin{vmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} \overset{C_3 + C_1}{=} \begin{vmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 2 & 2 \end{vmatrix}$$

$$= (-1)^{1+1} \cdot (-1) \cdot \begin{vmatrix} -1 & 2 \\ 2 & 2 \end{vmatrix} = (-2-4) = 6$$

$$C^{*} = \begin{pmatrix} -3 & 0 & 3 \\ 2 & -2 & -2 \\ 1 & 2 & 1 \end{pmatrix} \qquad C^{-1} = \frac{1}{6} \cdot \begin{pmatrix} -3 & 0 & 3 \\ 2 & -2 & +2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$A' = C^{-1} A C$$

$$A = C A' C^{-1}$$

$$A^n = \frac{1}{6} \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

! $\boxed{A^n = C A'^n C^{-1}}$

12 $f \in End(\mathbb{R}^3)$

Dacă $\lambda_1 = 3$, $\lambda_2 = -2$, $\lambda_3 = 1$ sunt valorile proprii

și $v_1 = (-3,2,1)$, $v_2 = (-2,1,0)$, $v_3 = (-6,3,1)$

sunt vectorii proprii coresp, atunci care este matricea

$A = [f]_{R_0,R_0} = ?$

$R = \{v_1, v_2, v_3\}$ reper în $\mathbb{R}^3$ a.î. $A' = [f]_{R,R} =$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A' = C^{-1} A C \Rightarrow A = C A' C^{-1}$$

$$C^{-1} = \frac{1}{\det C} \cdot C^*$$

$$R_0 \xrightarrow{C} R$$

$$C = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\det C = -3 \cancel{-6} \cancel{+6} + 4 = 1$$

$$C^t = \begin{pmatrix} -3 & 2 & 1 \\ -2 & 1 & 0 \\ -6 & 3 & 1 \end{pmatrix}$$

$$C^* = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & -3 \\ -1 & -2 & 1 \end{pmatrix}$$

$$C^{-1} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & -3 \\ -1 & -2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & -3 \\ -1 & -2 & 1 \end{pmatrix}$$

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$f: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ liniară

$$A = [f]_{R_0, R_0} = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 2 & -1 & 0 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix}$$

a) Să se afle valorile proprii și subsp. proprii coresp.

b) $U = \langle \{e_1 + 2e_2, e_2 + e_3 + 2e_4\} \rangle$. Să se arate că U este subsp. invariant al lui f i.e $f(U) \subset U$

Soluție:

a) $$p(\lambda) = \begin{pmatrix} 1-\lambda & 0 & 2 & -1 \\ 0 & 1-\lambda & 4 & -2 \\ 2 & -1 & -\lambda & 1 \\ 2 & -1 & -1 & 2-\lambda \end{pmatrix} \underset{L_4 - L_3}{=}$$

$$= \begin{pmatrix} 1-\lambda & 0 & 2 & -1 \\ 0 & 1-\lambda & 4 & -2 \\ 2 & -1 & -\lambda & 1 \\ 0 & 0 & \lambda-1 & 1-\lambda \end{pmatrix} \underset{C_3 + C_4}{=} \begin{pmatrix} 1-\lambda & 0 & 1 & -1 \\ 0 & 1-\lambda & 2 & -2 \\ 2 & -1 & 1-\lambda & 1 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} =$$

$$= (1-\lambda) \cdot \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 2 \\ 2 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda) \cdot (1-\lambda)^3 = (1-\lambda)^4$$

$\Rightarrow \lambda_1 = 1 \Rightarrow m\lambda_1 = 4$

b) $V_{\lambda_1} = \{ x \in \mathbb{R}^4 \mid f(x) = x \}$

$$(A - I_4)\, x = O_{4,1} \Leftrightarrow \begin{pmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & 4 & -2 \\ 2 & -1 & -1 & 1 \\ 2 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 4x_3 = 2x_4 \\ -x_2 - x_3 = -x_4 - 2x_1 \end{cases} \Leftrightarrow \begin{cases} x_3 = \frac{1}{2} x_4 \\ x_2 = 2x_1 + \frac{1}{2} x_4 \end{cases}$$

$$V_{\lambda_1} = \{ (x_1, 2x_1 + \frac{1}{2} x_4, \frac{1}{2} x_4, x_4) \mid x_1, x_2, x_3, x_4 \in \mathbb{R} \}$$

f nu se poate diagonaliza.