# Spații și subspații vectoriale

Spațiu vectorial: Corp comutativ $(\mathbb{K}, +, \cdot)$, mulțimea nevidă $V$

Avem legile $+ : V \times V \longrightarrow V$

$\cdot : \mathbb{K} \times V \longrightarrow V$

not. $(V, +, \cdot)_{/\mathbb{K}}$

care verifică

1. $(V, +)$ grup abelian

$+$ : Comutativă
Asociativă
El. neutru
Toate elem. sunt simetrizabile

2. $a(b \cdot x) = (a \cdot b) \cdot x$
3. $(a+b)x = a \cdot x + b \cdot x$
4. $a(x+y) = a \cdot x + a \cdot y$
5. $1_{\mathbb{K}} \cdot x = x$

$\forall\, a, b \in \mathbb{K}$ (scalari)

$\forall\, x, y \in V$ (vectori)

ex 1 $(\mathbb{R}, +, \cdot)$ corp comutativ, $V = \mathbb{R}^2$

$+ : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(x, y) + (x', y') = (x + x', y + y')$

$\cdot : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$: $a(x, y) = (0, ay)$

$(\mathbb{R}^2, +, \cdot)/\mathbb{R}$ sp. vectorial

$1 \cdot (x, y) = (0, y) \neq (x, y)$ (regula 5)

$\Rightarrow$ NU este sp. vect

ex2) $(\mathbb{C},+,\cdot)$ corp comutativ, $V=\mathbb{R}^2$

$+ : \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^2$, $(x,y)+(x',y') = (x+x', y+y')$

$\cdot : \mathbb{C}\times\mathbb{R}^2\to\mathbb{R}^2$, $(a+ib)\cdot(x,y) = (ax-by, ay+bx)$

$(\mathbb{R}^2,+,\cdot)/\mathbb{C}$ sp. vectorial

1. $(\mathbb{R}^2,+)$ grup abelian evident

2. $(a+ib)\left((c+id)\cdot(x,y)\right) = (a+ib)\left(\overbrace{cx-dy}^{x'}, \overbrace{cy+dx}^{y'}\right)$

$= (acx - ady - bcy - bdx, acy + adx + bcx - bdy)$

analog $\left((a+ib)(c+id)\right)(x,y)$

3. $((a+ib)+(c+di))(x,y) = (\underbrace{a+c}_{a'} + i\underbrace{(b+d)}_{b'})(x,y) = (ax+cx-by-dy,\ ay+cy+bx+dx)$

$= (ax-by,\ ay+bx) + (cx-dy,\ cy+dx)$

$= (a+ib)(x,y) + (c+id)(x,y)$

4. $(a+ib)((x,y)+(x',y')) = (a+ib)(x+x',\ y+y') = (ax+ax'-by-by',\ ay+ay'+bx+bx')$

$= (ax-by,\ ay+bx) + (ax'-by',\ ay'+bx')$

$= (a+ib)(x,y) + (a+ib)(x',y')$

5. $1\cdot(x,y) = (1+i0)(x,y) = (x-0,\ y+0) = (x,y)$

$\Longrightarrow (\mathbb{R}^2, +, \cdot)/\mathbb{C}$ sp. vect

Subspațiu vectorial: $V' \subset V$ submulțime nevidă:

$\forall a, b \in \mathbb{K}, \forall x, y \in V' \Rightarrow ax + by \in V'$

(ex 3) $V' = \{(x, y) \in \mathbb{R}^2 \mid tg\, y = 0\} \subset \mathbb{R}^2$, $(\mathbb{R}^2, +, \cdot)_{/\mathbb{R}}$

subspațiu vectorial ?

Fie $(x_1, y_1), (x_2, y_2) \in V'$, $a, b \in \mathbb{R}$ $\overset{?}{\Rightarrow}$ $a(x_1, y_1) + b(x_2, y_2) \in V'$

$a(x_1, y_1) + b(x_2, y_2) = (ax_1 + bx_2, ay_1 + by_2)$

$tg\, y_1 = 0$, $tg\, y_2 = 0 \Rightarrow \begin{cases} y_1 = k_1 \pi \\ y_2 = k_2 \pi \end{cases}$, $k_1, k_2 \in \mathbb{Z}$

$ay_1 + by_2 = ak_1\pi + bk_2\pi$

pentru $a=0$, $b=\frac{1}{2}$ obțin $\frac{k_2\pi}{2}$, $k_2 \in \mathbb{Z}$

pentru $k$ impar $\Rightarrow a(x_1, y_1) + b(x_2, y_2) \notin V'$ $\Rightarrow V'$ nu e subspațiu

$(c+id)\cdot$

## Sisteme liniare. Sisteme de generatori. Baze

Sistem de generatori (SG) = mul. $S$, $V = \langle S \rangle$

$$\langle S \rangle = \{ x \in V \mid x = \sum_{i=1}^{n} a_i x_i ,\ a_i \in K,\ x_i \in S \}$$

$\longrightarrow$ generează spațiul

Sistem liniar independent (SLI): $\forall\, a_1, \dots, a_n \in K$, $\forall\, x_1, \dots, x_n \in S$

$$\sum_{i=1}^{n} a_i x_i = 0_V \Leftrightarrow a_1 = \dots = a_n = 0$$

Sistem liniar dependent (SLD): are și soluții nenule

$\exists\, a_1, \dots, a_n \in K$ (nu toți nuli)

$\exists\, x_1, \dots, x_n \in S$ a.î. $\sum_{i=1}^{n} a_i x_i = 0_V$

Bază = SG și SLI

Dimensiunea spațiului: $\dim_{\mathbb{K}} V$ = nr. de vectori necesari pentru o bază.

Card $\forall$ SG $\geq$ Card $\forall$ SLI (finit)

$n = \dim_{\mathbb{K}} V$ $\begin{cases} \text{nr. max. de vectori pt SLI} \\ \text{nr. min de vect pt SG} \end{cases}$

egalitate $\Rightarrow \left.\begin{cases} S = SG \\ S = SLI \\ S = \text{Bază} \end{cases}\right\}$ echivalente

Sisteme liniare. Sisteme de generatori. Baze

Baze canonice

1. $(\mathbb{R}^3, +, \cdot)_{/\mathbb{R}}$ $\quad \mathcal{B}_0 = \{(1,0,0), (0,1,0), (0,0,1)\}$, $\dim_{\mathbb{R}} \mathbb{R}^3 = 3$

2. $(\mathbb{C}, +, \cdot)_{/\mathbb{R}}$ $\quad \mathcal{B}_0 = \{1, i\}$, $\dim_{\mathbb{R}} \mathbb{C} = 2$

3. $(M_{2,2}, +, \cdot)_{/\mathbb{R}}$ $\quad \mathcal{B}_0 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$\dim_{\mathbb{R}} M_{2,2} = 4$

4. $(\mathbb{R}_2[x], +, \cdot)_{/\mathbb{R}}$ $\quad \mathcal{B}_0 = \{x^2, x, 1\}$

$\dim_{\mathbb{R}} \mathbb{R}_2[x] = 3$

Sisteme liniare. Sisteme de generatori. Baze

(ex 4) $S=\{(1,m,1),(m,1,1),(1,0,m)\}$
$(\mathbb{R}^3,+,\cdot)|\mathbb{R}$ sp vect.

expr. m pt SLI, SLD

Fie $a_1, a_2, a_3 \in \mathbb{R}$ a.î. $a_1x_1 + a_2x_2 + a_3x_3 = (0,0,0)$

$a_1(1,m,1) + a_2(m,1,1) + a_3(1,0,m) = (0,0,0)$

$(a_1 + ma_2 + a_3, \; ma_1 + a_2, \; a_1 + a_2 + ma_3) = (0,0,0)$

$$\begin{cases} a_1 + ma_2 + a_3 = 0 \\ ma_1 + a_2 = 0 \\ a_1 + a_2 + ma_3 = 0 \end{cases}$$

$$\bar{A} = \left(\begin{array}{ccc|c} 1 & m & 1 & 0 \\ m & 1 & 0 & 0 \\ 1 & 1 & m & 0 \end{array}\right)$$

$$\det A = (m-1)(m^2+m-1) = 0$$

SLI $\Leftrightarrow \exists!$ sol $(0,0,0) \Leftrightarrow \det A \neq 0 \Leftrightarrow m \in \mathbb{R} \setminus \left\{1, \frac{-1\pm\sqrt{5}}{2}\right\}$

SLD $\Leftrightarrow \exists$ sol. nenule $\Leftrightarrow \det A = 0 \Leftrightarrow m \in \left\{1, \frac{-1\pm\sqrt{5}}{2}\right\}$

SLI $\longrightarrow$ Vect pe coloană

*Dacă vrem să aflăm SLI, scriem vectorii pe coloană