$\boxed{C_1}$ G-A.

Matrice. Determinanți. Rang. Teorema Hamilton-Cayley. Teorema Laplace

$(\mathbb{K},+,\cdot)$ corp com.

$\det : \mathcal{M}_n(\mathbb{K}) \longrightarrow \mathbb{K}$

$$\det(A) = \sum_{\sigma \in S_n} \varepsilon_\sigma a_{1\sigma(1)} \cdot \ldots \cdot a_{n\sigma(n)}$$

$(S_n,\circ)$ grupul perm, $\sigma \in S_n$, $\sigma = \begin{pmatrix} 1 & \ldots & n \\ \sigma(1) & \ldots & \sigma(n) \end{pmatrix}$

$\varepsilon(\sigma) = (-1)^{m(\sigma)}$, $m(\sigma) =$ nr inversiuni

(i,j) inversiune $\Leftrightarrow \begin{cases} i < j \\ \sigma(i) > \sigma(j) \end{cases}$.

Def $A \in \mathcal{M}_n(\mathbb{K})$

1) A s.n. nesingulară $\Leftrightarrow \det(A) \neq 0$.

2) A s.n. inversabilă $\Leftrightarrow \exists A^{-1} \in \mathcal{M}_n(\mathbb{K})$ a.î. $A \cdot A^{-1} = A^{-1} A = I_n$.

PROP A nesingulară $\Leftrightarrow$ A inversabilă

OBS $A \longrightarrow A^T \longrightarrow A^*$, $A^*_{ij} = (-1)^{i+j} \Delta_{ij}$

(complementul algebric pt a_{ij})

$$A^{-1} = \frac{1}{\det A} \cdot A^* \quad (\det A \neq 0)$$

Prop a) $\det(A^{-1}) = \frac{1}{\det A}$

b) $\det(\alpha A) = \alpha^n \det A$, $A \in \mathcal{M}_n(\mathbb{K})$

c) $\det(A^*) = \det(A)^{n-1}$, $n \geq 2$.

Def (polinom caracteristic).

$A \in \mathcal{M}_n(\mathbb{K})$.

$P_A(X) = \det(A - X I_n) = (-1)^n \left[X^n - \sigma_1 X^{n-1} + \ldots + (-1)^n \sigma_n\right]$ polinom caracteristic asociat lui A

σ_k = suma minorilor diagonali de ordinul k, $k=\overline{1,n}$

$\sigma_1 = Tr(A)$, $\sigma_2 = \sum\limits_{i<j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$, $\sigma_3 = \sum\limits_{i<j<k} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}$, ...

$\sigma_n = \det(A)$.

Cazuri particulare

① $n=2$

$$\begin{vmatrix} a_{11}-X & a_{12} \\ a_{21} & a_{22}-X \end{vmatrix} = X^2 - \underbrace{(a_{11}+a_{22})}_{\sigma_1 = Tr(A)} X + \underbrace{a_{11}a_{22}-a_{12}a_{21}}_{\sigma_2 = \det(A)}.$$

$P_A(X) = X^2 - Tr(A)X + \det(A)$

② $n=3$ $P_A(X) = -(X^3 - \sigma_1 X^2 + \sigma_2 X - \sigma_3)$

$\sigma_1 = Tr(A)$, $\sigma_2 = Tr(A^*)$, $\sigma_3 = \det(A)$.

Teorema Hamilton - Cayley

$\forall A \in \mathcal{M}_n(\mathbb{K})$

$P_A(A) = O_n \Rightarrow A^n - \sigma_1 A^{n-1} + \dots + (-1)^n \sigma_n I_n = O_n$.

Aplicații

1) Calculul pt A^{-1}

Ex Fie $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow A^{-1} = ?$

TH-C: $A^3 - \sigma_1 A^2 + \sigma_2 A - \sigma_3 I_3 = O_3$.

$\sigma_1 = Tr(A) = 3$, $\sigma_2 = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1$

$\sigma_3 = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \underset{C_3-C_1}{=} \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0$

$A^3 - 3A^2 + A - I_3 = O_3 \quad | \cdot A^{-1}$

$A^2 - 3A + I_3 = A^{-1}$

2) Calcul prin recurență pt puterile unei matrice:

EX. $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$, $A^n = x_n A + y_n I_2$, $n \geq 1$.

TH-C : $A^2 - 6A + 5I_2 = O_2 \Rightarrow \boxed{A^2 = 6A - 5I_2}$

$A^1 = 1 \cdot A + 0 I_2$. $\begin{cases} x_1 = 1 \\ y_1 = 0 \end{cases}$; $\begin{cases} x_2 = 6 \\ y_2 = -5 \end{cases}$

$A^{n+1} = A^n \cdot A$

$x_{n+1} A + y_{n+1} I_2 = x_n A^2 + y_n A$

$= x_n(6A - 5I_2) + y_n A = (6x_n + y_n) A - 5x_n I_2$

$\begin{cases} x_{n+1} = 6x_n + y_n \\ y_{n+1} = -5x_n \Rightarrow \boxed{y_n = -5x_{n-1}} \end{cases}$

$x_{n+1} - 6x_n + 5x_{n-1} = 0$, $\forall n \geq 2$

Ec. caracteristică: $t^2 - 6t + 5 = 0$ $\begin{matrix} t_1 = 1 \\ t_2 = 5 \end{matrix}$

- $x_n = C_1 t_1^n + C_2 t_2^n = C_1 + C_2 \cdot 5^n$, $\forall n \geq 1$

$n = 1 \Rightarrow \begin{cases} 1 = C_1 + 5C_2 \\ 6 = C_1 + 25C_2 \end{cases}$ $\ominus$ $\qquad C_2 = \frac{1}{4}$

$n = 2 \Rightarrow$

$5 = / + 20C_2$ $\qquad C_1 = 1 - \frac{5}{4} = -\frac{1}{4}$.

$x_n = -\frac{1}{4} + \frac{1}{4} \cdot 5^n = \frac{1}{4}(5^n - 1)$; $y_n = -\frac{5}{4}(5^{n-1} - 1)$

$\boxed{\text{OBS}}$ $\forall P \in \mathbb{K}[X]$ de grad $\forall$ de mare

$\exists R \in \mathbb{K}[X]$, grad $R \leq n-1$ aî

$P = P_A \cdot C + R \Rightarrow P(A) = R(A)$

EX $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$, $B = A^3 - A + I_2$

Să se afle $a, b \in \mathbb{R}$ aî $B = aA + bI_2$

SOL. $P = X^3 - X + 1$, $P_A = X^2 - 6X + 5$

$P = P_A \cdot C + R \Rightarrow \underset{\substack{\parallel \\ B}}{P(A)} = R(A)$

$$\begin{array}{l|l} X^3 \quad\quad -X+1 & X^2-6X+5 \\ \hline -X^3+6X^2-5X & X+6 \\ \hline /\quad 6X^2-6X+1 & \\ \quad -6X^2+36X-30 & \\ \hline \quad /\quad 30X-29 & \end{array}$$

$C = X+6$

$R = 30X-29$

$R(A) = 30A - 29I_2$

$$B = A^3 - A + I_2 = 30A - 29I_2 = aA + bI_2 \Rightarrow \begin{cases} a = 30 \\ b = -29 \end{cases}$$

$$\underset{P(A)}{\overset{\|}{}} = \underbrace{P_A(A)}_{O_2} C + R(A) = R(A)$$

3) Rezolvarea de ec. matriceale binome în $\mathcal{M}_2(\mathbb{C})$

Ex Rez ec $X^4 = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} = A$, $X \in \mathcal{M}_2(\mathbb{C})$

$\det(A) = 0 \Rightarrow \det(X^4) = (\det X)^4 = 0$

T.H-C: $X^2 = Tr(X)X \Rightarrow X^4 = (Tr X)^3 X$

$A = (Tr X)^3 X \mid Tr \Rightarrow Tr(A) = Tr^4(X) = 1 \Rightarrow Tr(X) \in \{\pm 1; \pm i\}$.

OBS $Tr(\alpha X) = \alpha Tr(X)$

a) $A = \pm X \Rightarrow X = \pm A$

b) $A = i^3 X = -iX \Rightarrow X = iA$

$A = -i^3 X = iX \Rightarrow X = -iA$

Def $A \in \mathcal{M}_{m,n}(\mathbb{K})$, $A \neq O_{m,n}$. $rg(A) = k$ $(k \leq \min\{m,n\})$

$\Leftrightarrow$ $\exists$ un minor de ord k nenul și toți minorii de ordin mai mare sunt nuli.

Conv $rg(O_{m,n}) = 0$.

OBS $\exists\ C_m^k \cdot C_n^k$ minori de ordin $k+1$ pt A

Teoremă $rg(A) = k \Leftrightarrow \exists$ un minor Δ_k de ordinul k nenul și toți minorii de ordin $k+1$ (dacă $\exists$) care îl conțin pe Δ_k sunt nuli.

OBS $\exists\ (m-k)(n-k)$ minori de ordin $k+1$ care îl conțin pe Δ_k (am optimizat).

Exemple

① $A = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$ $rg A = ?$ Discuție.

(pătratică : mare $\rightarrow$ mic)

$$\Delta = \det A = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = \begin{vmatrix} a+2 & a+2 & a+2 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = (a+2) \begin{vmatrix} 1 & 0 & 0 \\ 1 & a-1 & 0 \\ 1 & 0 & a-1 \end{vmatrix}$$

$= (a+2)(a-1)^2$ $\quad l_1' = l_1 + l_2 + l_3$

a) $\Delta \neq 0 \Leftrightarrow a \in \mathbb{R} \setminus \{-2, 1\}$

$rg A = 3$

b) $\Delta = 0$

$b_1)$ $a = -2$ $\quad A = \begin{pmatrix} \boxed{\begin{matrix} -2 & 1 \\ 1 & -2 \end{matrix}} & \begin{matrix} 1 \\ 1 \end{matrix} \\ \begin{matrix} 1 & 1 \end{matrix} & -2 \end{pmatrix}$ $\quad rg A = 2$

$b_2)$ $a = 1$ $\quad A = \begin{pmatrix} \boxed{1} & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $\quad rg A = 1$.

② $A = \begin{pmatrix} \boxed{\begin{matrix} \boxed{1} & 1 \\ 0 & 1 \end{matrix}} & \begin{matrix} a & 1 \\ -1 & 2 \end{matrix} \\ \begin{matrix} 6 & 4 \end{matrix} & \begin{matrix} 8 & 3 \end{matrix} \end{pmatrix} \in \mathcal{M}_{3,4}(\mathbb{R})$ $\quad rg A = ?$

(A nu e pătratică : mic $\rightarrow$ mare)

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 6 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 6 & -2 & -3 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} 1 & 2 \\ -2 & -3 \end{vmatrix} = -3+4 = 1 \neq 0$$

$c_2' = c_2 - c_1$

$c_3' = c_3 - c_1$

$rg A = 3, \ \forall\ a \in \mathbb{R}$

③ $A \in \mathcal{M}_n(\mathbb{R})$ care verifică $A^3 - A - I_n = O_n$.

Să se det a) $rg A$; b) $rg(A + I_n)$

SOL

$A^3 - A = I_n \Rightarrow A(A^2 - I_n) = I_n \quad | \det$

$\det(A) \cdot \det(A^2 - I_n) = 1 \Rightarrow \det A \neq 0 \Rightarrow rg A = n$

$A^3 = A + I_n \quad | \det \Rightarrow (\det A)^3 = \det(A + I_n) \neq 0 \Rightarrow rg(A + I_n) = n$

$\boxed{\text{OBS}}$ $A \in \mathcal{M}_n(\mathbb{K})$

a) minor de ordin p.

$$\Delta_p = \det(A_{I,J}) = \begin{vmatrix} a_{i_1 j_1} & \dots & a_{i_1 j_p} \\ \vdots & & \\ a_{i_p j_1} & \dots & a_{i_p j_p} \end{vmatrix}$$

$I = \{i_1, \dots, i_p\}$ $\quad 1 \le i_1 < \dots < i_p \le n$

$J = \{j_1, \dots, j_p\}$ $\quad 1 \le j_1 < \dots < j_p \le n$.

b) minor complementar lui Δ_p (de ordin $n-p$)

$\det(A_{\overline{I},\overline{J}})$ $\quad \overline{I} = \{1, \dots, n\} \setminus I$

$\quad \overline{J} = \{1, \dots, n\} \setminus J$

(minorul lui A obținut prin suprimarea liniilor $l_{i_1}, \dots, l_{i_p}$ coloanelor $c_{j_1}, \dots, c_{j_p}$.

c) complement algebric pt Δ_p.

$$(-1)^{i_1 + \dots + i_p + j_1 + \dots + j_p} \det(A_{\overline{I},\overline{J}})$$

<u>Teorema Laplace</u>

$\det(A) =$ suma produselor minorilor de ordinul p cu complemenții algebrici corespunzători pentru p linii fixate, $l_{i_1} \dots l_{i_p}$ (respectiv p coloane fixate $c_{j_1} \dots c_{j_p}$)

$$\det(A) = \sum_{J} (-1)^{i_1 + \dots + i_p + j_1 + \dots + j_p} \det(A_{I,J}) \det(A_{\overline{I},\overline{J}})$$

$$= \sum_{I} (-1)^{i_1 + \dots + i_p + j_1 + \dots + j_p} \det(A_{I,J}) \det(A_{\overline{I},\overline{J}})$$

<u>OBS</u> $p = 1 \Rightarrow$ obț. dezv. unui determinant după o linie sau o coloană.

Exemplu

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{pmatrix} \in \mathcal{M}_4(\mathbb{R}).$$

Să se calculeze det(A) utilizând th. Laplace, pt $p=2$, ℓ_1, ℓ_2 fixate

SOL

$$\det(A) = (-1)^{1+2+1+2} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} + (-1)^{1+2+1+3} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \begin{vmatrix} 5 & -1 \\ -2 & 4 \end{vmatrix} +$$

$$+ (-1)^{1+2+1+4} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \cdot \begin{vmatrix} 5 & 1 \\ -2 & 2 \end{vmatrix} + (-1)^{1+2+2+3} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix}$$

$$+ (-1)^{1+2+2+4} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + (-1)^{1+2+3+4} \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix} = -5$$