

C₁₃ - GA

- §1 Aducerea conicelor la o formă canonică ($\delta=0$)
 §2 Cuadrice studiate pe ecuații reduse.

$$\S 1. (\mathbb{R}^2, \mathbb{R}^2/\mathbb{R}, \varphi) / (\mathbb{R}^2, (\mathbb{R}^2, g_0), \varphi)$$

$$\Gamma: f(x) = X^T A X + 2BX + C = 0$$

$$f(x) = \underbrace{a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2}_{\delta} + \underbrace{2b_1x_1 + 2b_2x_2 + c}_{\delta=0} = 0$$

1. $\delta \neq 0$ (Γ are centru unic)

2. $\delta = 0$ (Γ nu are centru unic)

$$\mathcal{R} = \{0; e_1, e_2\} \rightarrow \mathcal{R}' = \{0; e'_1, e'_2\} \rightarrow \mathcal{R}'' = \{p; e'_1, e'_2\}$$

$$Q: \mathbb{R}^2 \rightarrow \mathbb{R}, Q(x) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

Aducem Q la o formă canonică

(în sp. afin: met Gauss, Jacobi)

În sp. punctual euclidian utilizăm met val pr.

$$P(\lambda) = \det(A - \lambda I_2) = 0$$

$$\lambda^2 - \text{Tr}(A)\lambda + \det A = 0$$

$$\lambda_1 \neq 0, \lambda_2 = 0.$$

$$\delta = \det A = 0, A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

e'_1, e'_2 versori proprii coresp. val. pr λ_1, λ_2 .

$$e'_k = (l_k, m_k), k = \overline{1, 2}$$

$$R = \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \in SO(2) \text{ (o alegem)}$$

Firotia: $\theta: X = RX'$

$$Q(x) = \lambda_1 x_1'^2 + 0 \cdot x_2'^2$$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\theta: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

$$\begin{cases} x_1 = l_1 x_1' + l_2 x_2' \\ x_2 = m_1 x_1' + m_2 x_2' \end{cases}$$

$$\theta(\Gamma): \lambda_1 x_1'^2 + 2b_1(l_1 x_1' + l_2 x_2') + 2b_2(m_1 x_1' + m_2 x_2') + c = 0$$

$$\lambda_1 x_1'^2 + 2b_1' x_1' + 2b_2' x_2' + c = 0$$

$$\Delta = \begin{vmatrix} \lambda_1 & 0 & b_1' \\ 0 & 0 & b_2' \\ b_1' & b_2' & c \end{vmatrix} = -b_2' \begin{vmatrix} \lambda_1 & 0 \\ b_1' & b_2' \end{vmatrix} = -\lambda_1 b_2'^2$$

① $\Delta \neq 0$ (Γ nedegenerat) $\Rightarrow b_2' \neq 0$.

$$\theta(\Gamma): \lambda_1 \left(x_1'^2 + 2 \frac{b_1'}{\lambda_1} x_1' + \frac{b_1'^2}{\lambda_1^2} \right) + 2b_2' x_2' + c' = 0$$

$$c' = c - \frac{b_1'^2}{\lambda_1}$$

$$\lambda_1 \left(x_1' + \frac{b_1'}{\lambda_1} \right)^2 + 2b_2' \left(x_2' + \frac{c'}{2b_2'} \right) = 0$$

$$\text{Firotia } \begin{cases} x_1'' = x_1' + \frac{b_1'}{\lambda_1} \\ x_2'' = x_2' + \frac{c'}{2b_2'} \end{cases}$$

Considerăm translatia $\zeta: X' = X'' + X_0, X_0 = \begin{pmatrix} -\frac{b_1'}{\lambda_1} \\ -\frac{c'}{2b_2'} \end{pmatrix}$

$$\zeta(\theta(\Gamma)): \lambda_1 x_1''^2 + 2b_2' x_2'' = 0 \text{ (parabolă)}$$

② $\Delta = 0$ (Γ degenerată) $\Rightarrow b_2' = 0$.

$\theta(\Gamma): \lambda_1 x_1'^2 + 2b_1' x_1' + c = 0$.

$\lambda_1 \left(x_1' + \frac{b_1'}{\lambda_1} \right)^2 + c' = 0$.

Fac $\begin{cases} x_1'' = x_1' + \frac{b_1'}{\lambda_1} \\ x_2'' = x_2' \end{cases}$

Fac translatia $\zeta: X' = X'' + X_0, X_0 = \begin{pmatrix} -\frac{b_1'}{\lambda_1} \\ 0 \end{pmatrix}$

$\zeta(\theta(\Gamma))?: \lambda_1 x_1''^2 + c' = 0$.

$\theta: X = RX'$

$\zeta: X' = X'' + X_0$

$X = RX'' + RX_0. \quad RX_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$P(\alpha, \beta)$ (în raport cu reperul canonic)

Δ (natura)	δ (genul)	Tipul conice
$\Delta \neq 0$	$\delta > 0$	Elipsă, \emptyset
	$\delta < 0$	Hiperbolă
	$\delta = 0$	Parabolă
$\Delta = 0$	$\delta > 0$	Punct dublu
	$\delta < 0$	Drepte concurente
	$\delta = 0$	Drepte confundate, \emptyset , Drepte //

Aplicatie Fie conica (în sp. punctual euclidian),
 $\Gamma: f(x) = x_1^2 - 4x_1x_2 + 4x_2^2 - 6x_1 + 2x_2 + 1 = 0$
 Să se aducă la o formă canonică, utilizând
 izometrie. Reprez. grafică.

SOL

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \quad \delta = 4 - 4 = 0 \quad (\text{centrul nu e unic})$$

$$\tilde{A} = \begin{pmatrix} 1 & -2 & -3 \\ -2 & 4 & 1 \\ -3 & 1 & 1 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 1 & -2 & -3 \\ -2 & 4 & 1 \\ -3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 0 & -5 \\ -3 & -5 & -5 \end{vmatrix} = -25 \neq 0$$

Γ nedegenerată (parabolă)

$$Q: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad Q(x) = x_1^2 - 4x_1x_2 + 4x_2^2$$

Aducem Q la o f. canonică (met. val. pr.)

$$\lambda^2 - 5\lambda + 0 = 0 \Rightarrow \lambda_1 = 5, \lambda_2 = 0.$$

$$V_{\lambda_1} = \{x \in \mathbb{R}^2 \mid AX = 5X\}$$

$$(A - 5I_2)X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - x_2 = 0 \Rightarrow x_2 = -2x_1$$

$$e_1' = \frac{1}{\sqrt{5}}(1, -2) \quad V_{\lambda_1} = \langle \{e_1'\} \rangle$$

$$V_{\lambda_2} = \{x \in \mathbb{R}^2 \mid AX = 0 \cdot X\}$$

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad x_1 - 2x_2 = 0.$$

$$e_2' = \frac{1}{\sqrt{5}}(2, 1), \quad V_{\lambda_2} = \langle \{e_2'\} \rangle$$

$$R = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \in SO(2)$$

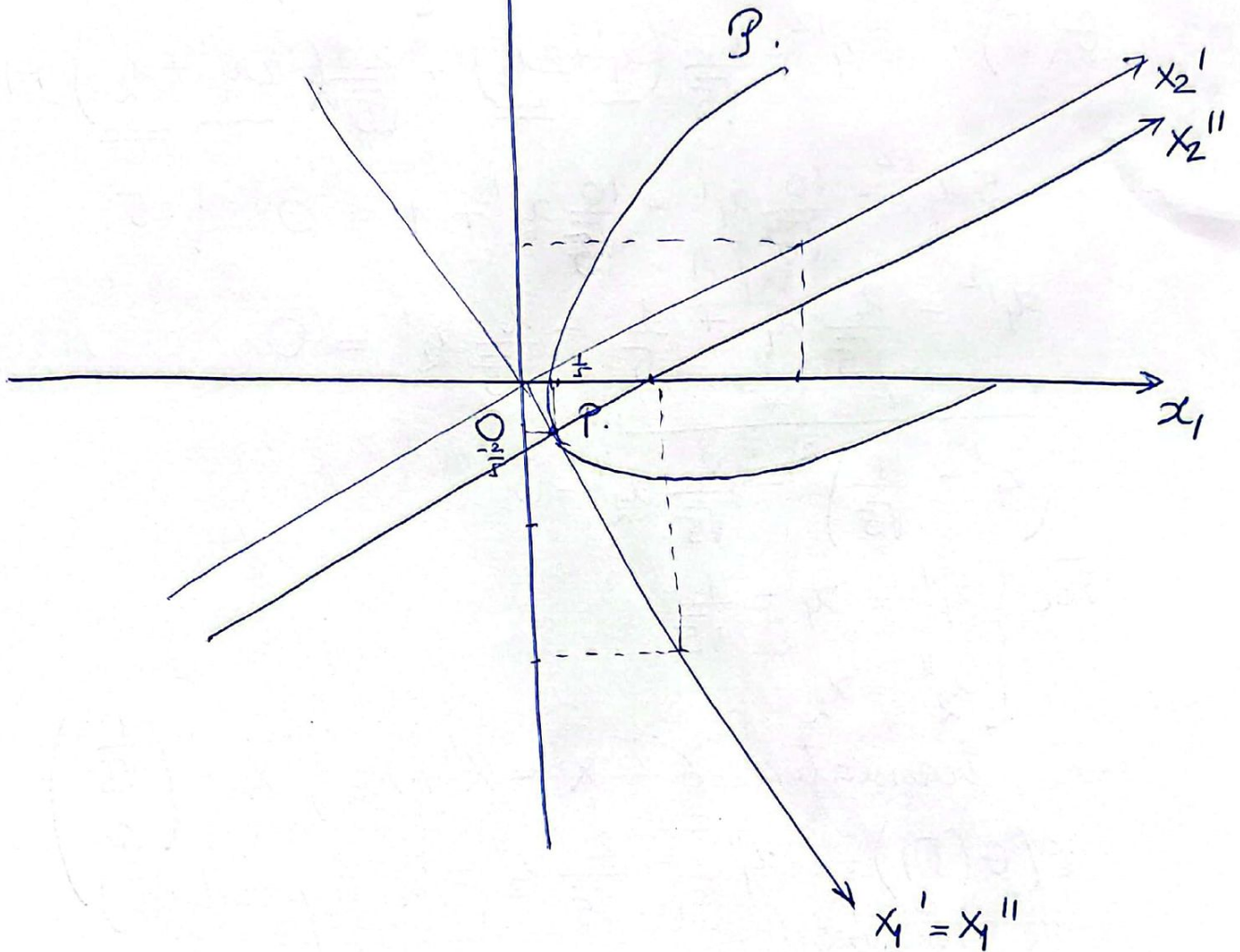
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$$\boxed{x_1''^2 = \frac{2}{\sqrt{5}} x_2''}$$

$$e_1' = \frac{1}{\sqrt{5}} (1, -2)$$

$$e_2' = \frac{1}{\sqrt{5}} (2, 1)$$

$$P\left(\frac{1}{5}, -\frac{2}{5}\right)$$



Five rotation $\theta: X = RX'$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \Rightarrow \begin{cases} x_1 = \frac{1}{\sqrt{5}} (x_1' + 2x_2') \\ x_2 = \frac{1}{\sqrt{5}} (-2x_1' + x_2') \end{cases}$$

$$\theta(\Gamma): 5x_1'^2 - \frac{6}{\sqrt{5}} (x_1' + 2x_2') + \frac{2}{\sqrt{5}} (-2x_1' + x_2') + 1 = 0.$$

$$5x_1'^2 - \frac{10}{\sqrt{5}} x_1' - \frac{10}{\sqrt{5}} x_2' + 1 = 0. \quad | :5$$

$$x_1'^2 - \frac{2}{\sqrt{5}} x_1' + \frac{1}{5} - \frac{2}{\sqrt{5}} x_2' = 0$$

$$\left(x_1' - \frac{1}{\sqrt{5}} \right)^2 - \frac{2}{\sqrt{5}} x_2' = 0$$

$$\text{Five } \begin{cases} x_1'' = x_1 - \frac{1}{\sqrt{5}} \\ x_2'' = x_2 \end{cases}$$

$$\text{Five translation } \gamma: X' = X'' + X_0, \quad X_0 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}$$

$$\gamma(\theta(\Gamma)): x_1''^2 - \frac{2}{\sqrt{5}} x_2'' = 0 \quad (\text{parabola}).$$

$$\theta: X = RX'$$

$$\gamma: X' = X'' + X_0 \Rightarrow X = RX'' + \underbrace{RX_0}$$

$$RX_0 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$P\left(\frac{1}{5}, -\frac{2}{5}\right) \text{ (în raport cu } \mathcal{R} \text{)}$$

$$\mathcal{R} = \{0; e_1, e_2\} \rightarrow \mathcal{R}' = \{0; e_1', e_2'\} \rightarrow \mathcal{R}'' = \{P; e_1', e_2'\}$$

$$e_1' = \frac{1}{\sqrt{5}} (1, -2)$$

$$e_2' = \frac{1}{\sqrt{5}} (2, 1)$$

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§2. Cuadrice studiate pe ecuații reduse.

1) Sfera

$$S(A(a, b, c), R): (x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 - R^2 = 0.$$

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 2ax_1 - 2bx_2 - 2cx_3 + d = 0$$

$$d = a^2 + b^2 + c^2 - R^2$$

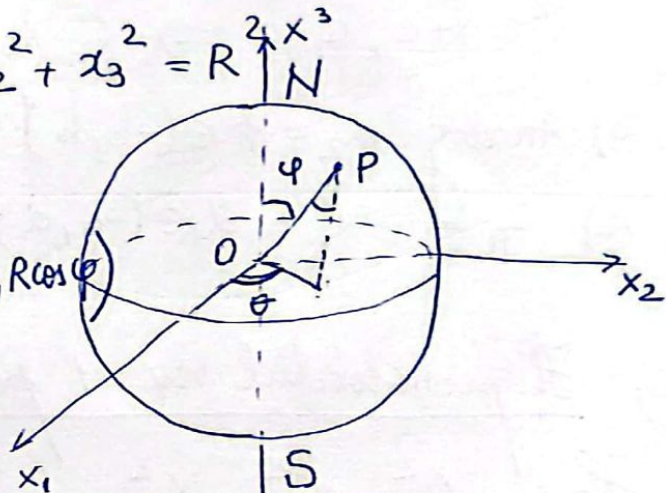
În particular, $A = 0$

$$S(0, R): x_1^2 + x_2^2 + x_3^2 = R^2$$

$$P(R \sin \varphi \cos \theta, R \sin \varphi \sin \theta, R \cos \varphi)$$

$$\theta \in [0, 2\pi)$$

$$\varphi \in [0, \pi]$$



\cap cu plane // cu planele de coord.

$$x_3 = \gamma \in (-R, R) \Rightarrow x_1^2 + x_2^2 = R^2 - \gamma^2 \text{ cerc.}$$

$$\text{Dacă } \gamma \in \{-R, R\} \Rightarrow N(0, 0, R), S(0, 0, -R)$$

$$\text{Analog pt } \begin{matrix} x_2 = \beta \\ x_1 = \alpha \end{matrix}$$

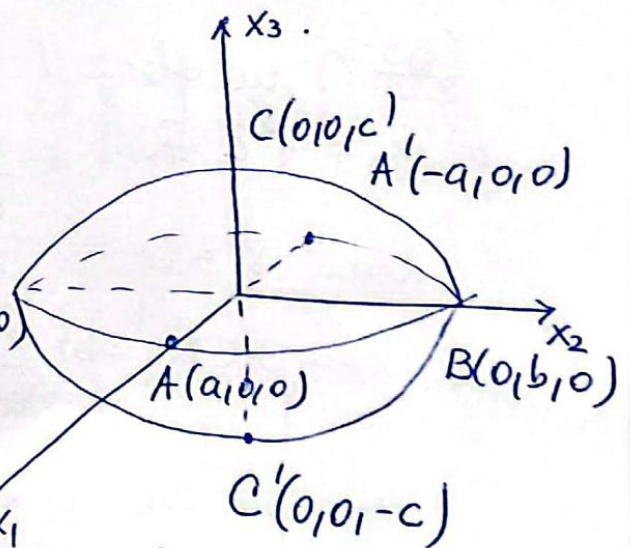
2) Elipsoidul

$$\mathcal{E}: \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$$

$$a, b, c > 0 \text{ (semiaxe)}$$

$$\begin{cases} x_1 = a \sin \varphi \cos \theta \\ x_2 = b \sin \varphi \sin \theta \end{cases}$$

$$x_3 = c \cos \varphi, \quad \varphi \in [0, \pi], \theta \in [0, 2\pi)$$



OBS

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- a) planele de simetrie = planele de coordonate.
 - b) axe de simetrie = axele de coordonate.
 - c) centrul de simetrie = orig. axelor.

OBS

\cap cu plane \parallel cu planele de coord.

a) $x_3 = \gamma \in (-c, c) : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 - \frac{\gamma^2}{c^2}$ Elipsă.

$\gamma \in \{-c, c\} \Rightarrow C \text{ si } C'$

b) Analog $x_2 = \gamma \in (-b, b)$ Elipsă.

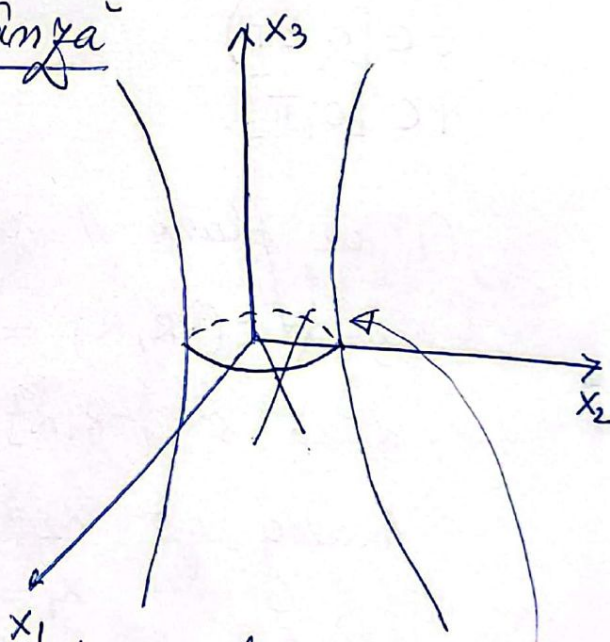
c) \perp $x_1 = \alpha \in (-a, a)$ Elipsă.

3). Hyperboloidul cu 1 pânză

$H_b: \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 1$

$$\begin{cases} x_1 = a \cos \theta \operatorname{ch} \varphi \\ x_2 = b \sin \theta \operatorname{ch} \varphi \\ x_3 = c \operatorname{sh} \varphi \end{cases}$$

$\varphi \in \mathbb{R}, \theta \in [0, 2\pi]$



OBS \cap cu plane \parallel cu planele de coord.

a) $x_3 = \gamma \in \mathbb{R} : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 + \frac{\gamma^2}{c^2}$ Elipsă

Dacă $\gamma = 0 \Rightarrow$ Elipsă colier: $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$

b) $x_2 = \beta \in \mathbb{R} \setminus \{\pm b\} : \frac{x_1^2}{a^2} - \frac{x_3^2}{c^2} = 1 - \frac{\beta^2}{b^2}$ Hyperbolă

$x_2 = \beta = \pm b : \frac{x_1^2}{a^2} - \frac{x_3^2}{c^2} = 0 \Rightarrow x_3 = \pm \frac{c}{a} x_1$
(drepte)

$$e) x_1 = \alpha \in \mathbb{R} \setminus \{\pm a\} \quad \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 1 - \frac{\alpha^2}{a^2} \quad \text{Hiperbolă.}$$

$$x_1 = \alpha = \pm a \quad \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 0 \Rightarrow x_3 = \pm \frac{c}{b} x_2 \quad (\text{Drepte})$$

Teorema \mathcal{H}_1 = cuadrică dublu reglată i.e. \exists 2 familii de generatoare G_1 și G_2 și prin fiecare punct al cuadricii trece câte o dreaptă din fiecare familie.

Dem $\mathcal{H}_1: \frac{x_1^2}{a^2} - \frac{x_3^2}{c^2} = 1 - \frac{x_2^2}{b^2} \Rightarrow$

$$\underbrace{\left(\frac{x_1}{a} - \frac{x_3}{c}\right)}_{f_1} \underbrace{\left(\frac{x_1}{a} + \frac{x_3}{c}\right)}_{f_2} = \underbrace{\left(1 - \frac{x_2}{b}\right)}_{g_1} \underbrace{\left(1 + \frac{x_2}{b}\right)}_{g_2}$$

$$\mathcal{D}: \begin{cases} \alpha f_1 = \beta g_1 \\ \beta f_2 = \alpha g_2 \end{cases} \quad \alpha \beta f_1 f_2 = \alpha \beta g_1 g_2$$

$$\mathcal{D} \subset \mathcal{H}_1.$$

$$1) \alpha = 0 \quad \begin{cases} g_1 = 0 \\ f_2 = 0 \end{cases} \quad \text{"} \lambda$$

$$2) \alpha \neq 0 \quad \begin{cases} f_1 = \left(\frac{\beta}{\alpha}\right) g_1 \\ \frac{\beta}{\alpha} f_2 = g_2 \end{cases}$$

$$G_1: d_\lambda: \begin{cases} \frac{x_1}{a} - \frac{x_3}{c} = \lambda \left(1 - \frac{x_2}{b}\right) \\ \lambda \left(\frac{x_1}{a} + \frac{x_3}{c}\right) = 1 + \frac{x_2}{b} \end{cases}$$

$$d_\infty: \begin{cases} 1 - \frac{x_2}{b} = 0 \\ \frac{x_1}{a} + \frac{x_3}{c} = 0 \end{cases}$$

Analog obținem familia.

$$G_2: \bar{d}_\mu: \begin{cases} \frac{x_1}{a} - \frac{x_3}{c} = \mu \left(1 + \frac{x_2}{b}\right) \\ \mu \left(\frac{x_1}{a} + \frac{x_3}{c}\right) = 1 - \frac{x_2}{b} \end{cases}$$

$$\bar{d}_\infty: \begin{cases} 1 + \frac{x_2}{b} = 0 \\ \frac{x_1}{a} + \frac{x_3}{c} = 0 \end{cases}$$

$$\lambda \left(1 - \frac{x_2}{b}\right) = \mu \left(1 + \frac{x_2}{b}\right) \Rightarrow \frac{x_2}{b} (\mu + \lambda) = \lambda - \mu.$$

$$\bullet \lambda + \mu = 0 \Rightarrow \lambda - \mu = 0 \Rightarrow \lambda = \mu = 0.$$

$$d_0: \begin{cases} \frac{x_1}{a} - \frac{x_3}{c} = 0 \\ 1 + \frac{x_2}{b} = 0 \end{cases} ; \bar{d}_0: \begin{cases} \frac{x_1}{a} - \frac{x_3}{c} = 0 \\ 1 - \frac{x_2}{b} = 0 \end{cases}$$

$$d_0 \cap \bar{d}_0 = \emptyset$$

$$\bullet \lambda + \mu \neq 0 \Rightarrow \frac{x_2}{b} = \frac{\lambda - \mu}{\lambda + \mu}$$

$$1 + \frac{x_2}{b} = 1 + \frac{2-\mu}{2+\mu} = \frac{2}{1+\mu}$$

$$1 - \frac{x_2}{b} = 1 - \frac{\lambda - \mu}{\lambda + \mu} = \frac{2\mu}{\lambda + \mu}$$

$$\begin{cases} \frac{x_1}{a} - \frac{x_3}{c} = \frac{2\lambda\mu}{\lambda+\mu} \\ \frac{x_1}{a} + \frac{x_3}{c} = \frac{2}{\lambda+\mu} \end{cases}$$

$$\frac{x_1}{a} \quad / \quad = \quad \frac{2\mu+1}{2+\mu}$$

$$\frac{x_3}{c} = \frac{1 - \lambda \mu}{\lambda + \mu}$$

$$d_\lambda \cap \bar{d}_\mu: \mathbb{P}\left(a \cdot \frac{\lambda + \mu + 1}{\lambda + \mu}, b \cdot \frac{\lambda - \mu}{\lambda + \mu}, c \cdot \frac{1 - \lambda - \mu}{\lambda + \mu}\right)$$

$$d_2 \cap d_\infty \left[\left(a \frac{\frac{\lambda + \frac{1}{\mu}}{\lambda} + 1}{\frac{\lambda}{\mu} + 1}, b \frac{\frac{\lambda}{\mu} - 1}{\frac{\lambda}{\mu} + 1}, c \frac{\frac{1}{\mu} - \lambda}{\frac{\lambda}{\mu} + 1} \right) \right]$$

$$\mathbb{P}(a\lambda, -b, -c\lambda)$$

$$d_\infty \cap \bar{d}_\mu \left[p \left(a \frac{\mu + \frac{1}{\lambda}}{1 + \frac{\mu}{\lambda}}, b \frac{1 - \frac{\mu}{\lambda}}{1 + \frac{\mu}{\lambda}}, c \frac{\frac{1}{\lambda} - \mu}{1 + \frac{\mu}{\lambda}} \right) \right]$$

$$P(a\mu, b, -c\mu)$$

Aplicatie

$$\mathcal{H}_1: x_1^2 + 3x_2^2 - x_3^2 = 1.$$

Să se scrie ec. generatoarelor care trec prin $M(1,0,0)$.

SOL

$$a=1, b=\frac{1}{\sqrt{3}}, c=1$$

$$\left(\frac{\lambda\mu+1}{\lambda+\mu}, \frac{1}{\sqrt{3}} \frac{\lambda-\mu}{\lambda+\mu}, \frac{1-\lambda\mu}{\lambda+\mu} \right) = (1, 0, 0).$$

$$\begin{cases} \lambda - \mu = 0 \Rightarrow \lambda = \mu \\ 1 - \lambda\mu = 0 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1. \end{cases}$$

1) $\lambda = \mu = 1 \Rightarrow \frac{2}{2} = 1 \quad \checkmark$

2) $\lambda = \mu = -1 \Rightarrow \frac{-2}{-2} = -1$ nu conv.

Generatoarele sunt:

$$d_1: \begin{cases} x_1 - x_3 = 1 - \frac{x_2}{\frac{1}{\sqrt{3}}} \\ x_1 + x_3 = 1 + \frac{x_2}{\frac{1}{\sqrt{3}}} \end{cases}$$

$$\bar{d}_1: \begin{cases} x_1 - x_3 = 1 + \frac{x_2}{\frac{1}{\sqrt{3}}} \\ x_1 + x_3 = 1 - \frac{x_2}{\frac{1}{\sqrt{3}}} \end{cases}$$

OBS $M(1,0,0) \notin d_\infty, M(1,0,0) \notin \bar{d}_\infty.$

4) Hyperboloid cu 2 pânze.

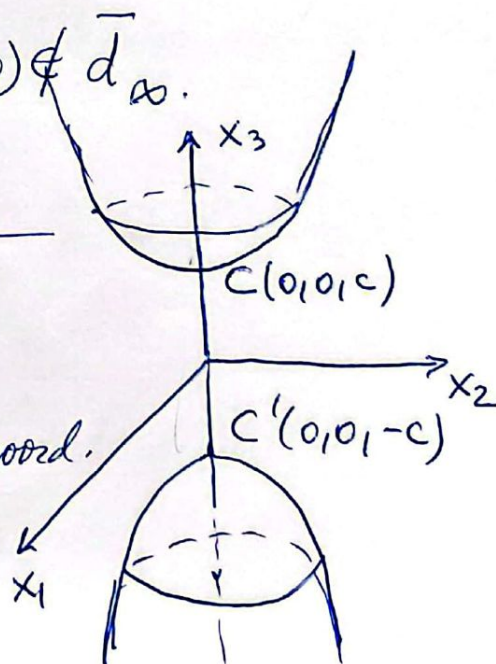
$$\mathcal{H}_2: \frac{-x_1^2}{a^2} - \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1.$$

\cap cu plane // cu planele de coord.

a) $x_3 = y \in (-\infty, -c) \cup (c, \infty)$

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = \frac{y^2}{c^2} - 1 \quad \text{Elipsa}$$

$$y \in \{-c, c\} \Rightarrow C, C'$$



$$b) x_2 = \beta$$

$$c) x_1 = \alpha$$

$$-\frac{x_1^2}{a^2} + \frac{x_3^2}{c^2} = 1 + \frac{\beta^2}{b^2}$$

Hiperbola

$$-\frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 + \frac{\alpha^2}{a^2}$$

Hiperbola

M