Probleme

(1). $V_1=(1,2,1,1)$, $V_2=(1,0,1,2)$, $V_3=(0,1,1,-1)$

I

a) V_1, V_2, V_3 SLi

b) $V=(3,2,1,6)$, $V\in\langle\{V_1,V_2,V_3\}\rangle$

Dacă aparține, să găsim coord.

c) $V_0=?$ a.î. $\{V_0,V_1,V_2,V_3\}$ bază în $\mathbb{R}^4$

II

a) $$Rg\begin{pmatrix}1&1&0\\2&0&1\\1&1&1\\1&2&-1\end{pmatrix}=3 \text{ maxim} \xRightarrow{CLi} \{V_1,V_2,V_3\} \text{ SL}$$

$$\Delta_3=\begin{vmatrix}1&1&0\\2&0&1\\1&1&1\end{vmatrix}=1-3=-2\neq 0$$

!Regula Dacă $A \in M_{m,n}(\mathbb{R})$

$$rg\, A \le \min(m,n)$$

b) $v \in \langle \{v_1, v_2, v_3\} \rangle \iff \exists\, a,b,c \in \mathbb{R}$ a.î. $v = av_1 + bv_2 + cv_3$

$$(3,2,1,6) = a(1,2,1,1) + b(1,0,1,2) + c(0,1,1,-1)$$

$$(3,2,1,6) = (a+b,\ 2a+c,\ a+b+c,\ a+2b-c)$$

$$\begin{cases} a+b=3 \\ 2a+c=2 \\ a+b+c=1 \\ a+2b-c=6 \end{cases} \qquad \bar{A} = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 6 \end{array}\right)$$

SL

Vrem $rg\,\bar{A} = rg\,A$ pt a avea sol. (Kronecker Capelli)

Probleme

$$\det \bar{A} = \begin{vmatrix} 1 & 1 & 0 & 3 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 6 \end{vmatrix} \xrightarrow[c_4 - c_1]{\substack{c_2 - c_1 \\ c_3 - c_1}} \begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & -2 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & -2 & 5 \end{vmatrix}$$

$$= 1 \cdot (-1)^{3+1} \cdot \begin{vmatrix} 0 & -1 & 2 \\ -2 & -1 & 0 \\ 1 & -2 & 5 \end{vmatrix} = 8 - (-2 + 10) = 8 - 8 = 0$$

$$\Rightarrow rg\,\bar{A} \neq 4$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} \neq 0 \Rightarrow rg\,\bar{A} = rg\,A = 3 \Rightarrow \exists \text{ soluție} \Rightarrow v \in \langle \{v_1, v_2, v_3\} \rangle$$

$$\begin{cases} ① a+b=3 \\ ② 2a+c=2 \\ ③ a+b+c=1 \\ ④ a+2b-c=6 \end{cases}$$

①, ③ $\Rightarrow c = 1-3 = -2$

② $\Rightarrow a = \frac{2-c}{2} = \frac{4}{2} = 2$ $\Rightarrow$ v are coord $(2, 1, -2)$.

① $\Rightarrow b = 3 - a = 1$

verif ④ $\Rightarrow 2+2-(-2) = 6$

c) aleg $v_0 = e_1 = (1,0,0,0)$

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{vmatrix} = 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -2 + 2 - (1+4) = -5 \neq 0 \Rightarrow rg \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix} = 4 \text{ maxim}$$

CLi

$\Rightarrow B_0 = \{v_0, v_1, v_2, v_3\}$ SLI $\Big|$ $\Rightarrow B_0$ bază în $\mathbb{R}^4$

$\dim_{\mathbb{R}} \mathbb{R}^4 = 4 = \operatorname{card} B_0$

Probleme

② $f:\mathbb{R}^3\to\mathbb{R}^3$, $f(x,y,z)=(3x+2y,\ 2x+4y-2z,\ -2y+5z)$ $\forall x,y,z\in\mathbb{R}$

a) f apl. liniară

$A_f=[f]_{R_0R_0}$

b) f endomorfism diagonalizabil

c) bază în care f e diagonalizabilă

d) A_f^n $\forall n\in\mathbb{N}$

f aplicație liniară $\iff f(ax+bx',\ ay+by',\ az+bz')=$

$= af(x,y,z)+bf(x',y',z')$

$$E = \big(3(ax+bx') + 2(ay+by'),\ 2(ax+bx') + 4(ay+by') - 2(az+bz'),$$
$$-2(ay+by') + 5(az+bz')\big)$$

$$= a f(x,y,z) + b f(x',y',z') \Rightarrow f \text{ apl. liniară}$$

$$A_f = [f]_{R_0R_0} = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{pmatrix}$$

b) $P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 3-\lambda & 2 & 0 \\ 2 & 4-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda)(5-\lambda) - 4(3-\lambda) - 4(5-\lambda)$!

Mnat

$C_3 + C_1$
$L_1 + L_3$

$$\begin{vmatrix} 3-\lambda & 0 & 8-2\lambda \\ 2 & 4-\lambda & 0 \\ 0 & -2 & 5-\lambda \end{vmatrix} = (3-\lambda)\cdot(-1)^{1+1}\cdot\begin{vmatrix} 4-\lambda & 0 \\ -2 & 5-\lambda \end{vmatrix} + 2\cdot(-1)^{1+2}\cdot\begin{vmatrix} 0 & 8-2\lambda \\ -2 & 5-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda)(5-\lambda) - 8(4-\lambda)$$

$$= (4-\lambda)\big((3-\lambda)(5-\lambda) - 8\big) = (4-\lambda)(\lambda^2 - 8\lambda + 15 - 8) = (4-\lambda)(\lambda-7)(\lambda-1) = 0$$

Probleme

$$\Rightarrow \lambda_1 = 1,\ m_1 = 1.$$
$$\lambda_2 = 1,\ m_2 = 1$$
$$\lambda_3 = 1,\ m_3 = 1.$$

$$V_{\lambda_1} = \{X \in \mathbb{R}^3 \mid f(x) = \lambda_1 x\}$$

$$A \cdot X = \lambda_1 X \Rightarrow (A - \lambda_1 I_3) X = O_{3,1}$$

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -x+2y \\ 2x-2z \\ -2y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} x &= 2y \\ x &= z \\ z &= 2y \end{aligned}$$

$$V_{\lambda_1} = \{(2y, y, 2y) \mid y \in \mathbb{R}\} = \langle \{(2,1,2)\} \rangle,\ B_1 = \{(2,1,2)\}$$

bază în V_{λ_1}

$V_{\lambda_2} = \{X \in \mathbb{R}^3 \mid f(X) = \lambda_2 X\}$

$AX = 7X \Rightarrow (A - 7I_3)X = O_{3,1} \Rightarrow \begin{pmatrix} -4 & 2 & 0 \\ 2 & -3 & -2 \\ 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -4x+2y \\ 2x-3y-2z \\ -2y-2z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2x = y \\ y = z \end{cases}$

$2x-3y-2z = y-3y-2y = 0$

$V_{\lambda_2} = \{(x, 2x, 2x) \mid x \in \mathbb{R}\} = \langle \{(1,2,2)\} \rangle$, $B_2 = \{(1,2,2)\}$ bază în V_{λ_2}

$V_{\lambda_3} = \{X \in \mathbb{R}^3 \mid f(X) = \lambda_3 X\}$

$AX = 1 \cdot X \Rightarrow (A - I_3)X = O_{3,1} \Rightarrow \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & -2 \\ 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 2x+2y \\ 2x+3y-2z \\ -2y+4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x = -y \\ y = 2z \end{cases}$

$2x+3y-2z = -2y+3y-y = 0$

$V_{\lambda_3} = \{(-2z, 2z, z) \mid z \in \mathbb{R}\} = \langle \{(-2,2,1)\} \rangle$, $B_3 = \{(-2,2,1)\}$ bază în V_{λ_3}

Probleme

$$\left.\begin{array}{l} \dim V_{\lambda_1} = 1 = m_1 \\ \dim V_{\lambda_2} = 1 = m_2 \\ \dim V_{\lambda_3} = 1 = m_3 \end{array}\right| \Rightarrow f \text{ endomorfism diagonalizabil}$$

c) $B = B_1 \cup B_2 \cup B_3 = \{(2,1,2), (1,2,2), (-2,2,1)\}$ bază în care

$$[f]_{B,B} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

d) Reamintim!

$$\begin{tikzcd} R_0 \arrow[r, "A_f"] \arrow[d, "C"] & R_0 \arrow[d, "C"] \\ B \arrow[r, "A"] & B \end{tikzcd}$$

$$A = C^{-1} A_f C$$

$$A_f = C A C^{-1}$$

$$A_f^{\,n} = \underbrace{C A C^{-1} C A C^{-1} \ldots C A C^{-1}}_{\text{de } n \text{ ori}}$$

$$A_1^n = C A^n C^{-1} \quad \forall n \in \mathbb{N}$$

$$C = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$A^n = \begin{pmatrix} 5^n & 0 & 0 \\ 0 & 7^n & 0 \\ 0 & 0 & 1^n \end{pmatrix} \quad \forall n \in \mathbb{N}$$

C^{-1} = de făcut acasă

Probleme

③ Fie $T: \mathbb{R}^n \to \mathbb{R}^n$ aplicație liniară cu matricea

$$[T]_{R_0 R_0} = A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

a) $\ker(T)$, $\mathrm{Im}(T)$, bază pt. $\ker(T)$

b) $u = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} \in \mathrm{Im}(T)$

TEMĂ c) val proprii ale lui T, vect proprii corespunzători val proprii

Obs: pt. verificarea val pr / vect pr. intrăm pe "Mathway"

introducem $[T]_{R_0 R_0}$: Eigenvalues

Eigenvectors

a) $\ker(T) = \{x \in \mathbb{R}^4 \mid f(x) = 0\}$

$$A = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{array}\right) \qquad \begin{cases} x_1 + x_4 = 0 \\ x_2 + x_3 = 0 \\ x_3 + x_4 = 0 \\ x_1 + x_2 = 0 \end{cases}$$

$$\det A = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{vmatrix} = 1 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} + 1 \cdot (-1)^{1+4} \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1 - 1 = 0 \Rightarrow \operatorname{rg} A \neq 4$$

$$\Delta_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow \operatorname{rg} A = 3$$

$$\dim \ker(T) = \dim \mathbb{R}^4 - \operatorname{rg} A = 4 - 3 = 1$$

$$\left.\begin{array}{l} x_1 = -x_4 \\ x_2 = -x_3 \\ x_3 = -x_4 \\ x_1 = -x_2 \end{array}\right| \Rightarrow \ker(T) = \{(x_1, -x_1, x_1, -x_1) \mid x_1 \in \mathbb{R}\} = \langle \{(1, -1, 1, -1)\} \rangle$$

$B_1 = \{(1, -1, 1, -1)\}$ bază în $\ker(T)$

Probleme

T. Dimensiunii: $\dim \mathbb{R}^4 = \dim \ker(T) + \dim \operatorname{Im}(T)$

$\Rightarrow \dim \operatorname{Im}(T) = 4 - 1 = 3$

$\operatorname{Im}(T) = \{ y \in \mathbb{R}^4 \mid \exists x \in \mathbb{R}^4,\ f(x) = y \}$

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \left| \begin{matrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{matrix} \right.$$

$rg\, A = 3 \Rightarrow$ pt. a avea soluții, $\Delta_c = 0$

$$\Delta_c = \begin{vmatrix} 1 & 0 & 0 & y_1 \\ 0 & 1 & 1 & y_2 \\ 0 & 0 & 1 & y_3 \\ 1 & 1 & 0 & y_4 \end{vmatrix} = 0 \Rightarrow 1 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 1 & y_2 \\ 0 & 1 & y_3 \\ 1 & 0 & y_4 \end{vmatrix} + y_1 \cdot (-1)^{1+4} \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} =$$

$$= y_4 + y_3 - y_2 - y_1 (\overbrace{1 - 0}^{1}) = 0 \Rightarrow y_2 = -y_1 + y_3 + y_4$$

b) $m \in$ ...

$(1, 2$ ...

$$\mathrm{Im}(T) = \{(y_1, -y_1+y_3+y_4, y_3, y_4) \mid y_1, y_3, y_4 \in \mathbb{R}\}$$

$$\parallel$$

$$(y_1, -y_1, 0, 0) + (0, y_3, y_3, 0) + (0, y_4, 0, y_4)$$

$$\mathrm{Im}(T) = \langle \{(1,-1,0,0), (0,1,1,0), (0,1,0,1)\} \rangle$$

b) $u \in \mathrm{Im}(T) \iff \exists\, a, b, c \in \mathbb{R}$ a.î.

$$(1,2,-1,3) = a(1,-1,0,0) + b(0,1,1,0) + c(0,1,0,1)$$

$$\begin{cases} a = 1 \\ -a+b+c = 2 \\ b = -1 \\ c = 3 \end{cases}$$

$$-a+b+c = -1+(-1)+3 = 1 \neq 2 \iff \nexists \text{ soluție} \Rightarrow u \notin \mathrm{Im}(T)$$

$y_1 + y_3 + y_4$