CURS 9

22.04.2020

Spații vectoriale euclidiene reale

<u>Def</u> $(V,+,\cdot)_{/\mathbb{R}}$ sp. vect. real; $g: V\times V \to \mathbb{R}$ s.n. <u>produs scalar</u> $\Leftrightarrow$

1) g este formă biliniară, simetrică;

2) g este pozitiv definită

(i.e. $Q: V \to \mathbb{R}$ formă pătratică, $Q(x) = g(x,x), \forall x\in V$

poz def $\Leftrightarrow$ $Q(x) > 0, \forall x \in V \setminus \{0_V\}$

$Q(x) = 0 \Leftrightarrow x = 0_V$)

<u>Not</u> (V,g); $(E,<\cdot,\cdot>)$, $(E,(\cdot,\cdot))$ s.n. spațiu vectorial euclidian real

<u>Def</u> $\|x\| = \sqrt{g(x,x)} = \sqrt{Q(x)}$ <u>norma</u> lui x (s.v.e.r)

<u>Def</u> $(E,<\cdot,\cdot>)$ s.v.e.r, $\mathcal{R} = \{e_1,..,e_n\}$ reper în V

a) $\mathcal{R}$ s.n. <u>reper ortogonal</u> $\Leftrightarrow <e_i, e_j> = 0, \forall i,j=\overline{1,n}, i\neq j$

b) $\mathcal{R}$ s.n. <u>reper ortonormat</u> $\Leftrightarrow <e_i, e_j> = \delta_{ij}, \forall i,j = \overline{1,n}$

(vectorii sunt mutual $\perp$ și versori)

$\boxed{\text{OBS}}$

- $\mathcal{R} = \{e_1,..,e_n\} \xrightarrow{A} \mathcal{R}' = \{e_1',..,e_n'\}$ repere ortonormate $\Rightarrow A \in O(n)$ i.e $AA^T = A^TA = I_n$.

$$e_r' = \sum_{i=1}^{n} a_{ir} e_i, \forall r = \overline{1,n}$$

$$\underbrace{<e_r', e_s'>}_{\delta_{rs}} = <\sum_{i=1}^{n} a_{ir}e_i, \sum_{j=1}^{n} a_{js} e_j> = \sum_{i,j=1}^{n} a_{ir}a_{js}\underbrace{<e_i,e_j>}_{\delta_{ij}}$$

$$\Rightarrow \delta_{rs} = \sum_{i=1}^{n} a_{ir} a_{is} \Rightarrow I_n = A^TA \Rightarrow A \in O(n).$$

- Dacă $\mathcal{R}, \mathcal{R}'$ sunt la fel orientate ($\det A > 0$) $\Rightarrow$ $\Rightarrow \det A = 1$ și $A \in SO(n)$.

Prop $(E, \langle , \rangle)$ s.v.e.r. și $S = \{x_1, .., x_k\}$, $k \le n = \dim_{\mathbb{R}}$
Dacă $S \subseteq E$ este un sistem de vectori nenuli, mutual $\perp \Rightarrow S$ este SLI

Dem Fie $a_1, .., a_k \in \mathbb{R}$, a.î. $\sum_{i=1}^{k} a_i x_i = 0_E \mid \langle \cdot, x_1 \rangle$

$$a_1 \underbrace{\langle x_1, x_1 \rangle}_{\|x_1\|^2 \neq 0_{\mathbb{R}}} + a_2 \underbrace{\langle x_2, x_1 \rangle}_{0_{\mathbb{R}}} + \ldots + a_k \underbrace{\langle x_k, x_1 \rangle}_{0_{\mathbb{R}}} = 0_{\mathbb{R}}$$

$\Rightarrow a_1 \|x_1\|^2 = 0_{\mathbb{R}} \Rightarrow a_1 = 0_{\mathbb{R}}$.

Analog $\langle \cdot, x_j \rangle$, $\forall j = \overline{2,k} \Rightarrow a_j = 0$, $\forall j = \overline{2,k}$

$\Rightarrow S$ este SLI.

Exemplu $(\mathbb{R}^n, g_0)$, $g_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $g_0(x,y) = \sum_{i=1}^{n} x_i y_i$

g_0 = produs scalar canonic

- g_0 formă biliniară, simetrică: $g_0(x,y) = X^T I_n Y$, $I_n = I_n^T$
- g_0 poz def: $Q_0 : \mathbb{R}^n \to \mathbb{R}$, $Q_0(x) = g_0(x,x) = \sum_{i=1}^{n} x_i^2$

  signatura: $(n, 0)$ poz. def.

Def (produs vectorial)
Fie $(\mathbb{R}^3, g_0)$ s.v.e.r. cu str. canonică
Fie $S = \{x, y\} \subset \mathbb{R}^3$. Fie $z = x \times y$ numit produs vectorial def. astfel:

1) Dacă S este SLD, atunci $z = 0_{\mathbb{R}^3}$
2) Dacă S este SLI, atunci:
   a) $\|z\|^2 = \begin{vmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle y, x \rangle & \langle y, y \rangle \end{vmatrix}$,
   b) $z \perp x$, $z \perp y$ i.e $\langle z, x \rangle = \langle z, y \rangle = 0$
   c) $\{x, y, z\}$ reper pozitiv orientat.
   (la fel orientat ca și reperul canonic).

„Determinant formal":

$$Z = x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & e_1 \\ x_2 & y_2 & e_2 \\ x_3 & y_3 & e_3 \end{vmatrix}$$

$$= e_1 \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - e_2 \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + e_3 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} =$$

$$= (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1)$$

Prop (R^3, g_0) s.v.e.r.can, $S = \{x, y\}$ un SLI.

a) $x \times y = -y \times x$

b) (nu este asociativ) $(x \times y) \times Z = \langle x, Z\rangle \cdot y - \langle y, Z\rangle x$

c) $\sum\limits_{x,y,Z}^{c} (x \times y) \times Z = 0$ (identitatea Jacobi)

SOL

a) $x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = - \begin{vmatrix} e_1 & e_2 & e_3 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = -y \times x$

b) $(x \times y) \times Z = \begin{vmatrix} e_1 & e_2 & e_3 \\ \underbrace{x_2y_3 - x_3y_2}_{Z_1} & \underbrace{x_3y_1 - x_1y_3}_{Z_2} & \underbrace{x_1y_2 - x_2y_1}_{Z_3} \\ & & \end{vmatrix}$

$= e_1 \cdot \alpha - e_2 \beta + e_3 \gamma = (\alpha, -\beta, \gamma)$.

$\langle x, Z\rangle y - \langle y, Z\rangle x =$

$= (x_1Z_1 + x_2Z_2 + x_3Z_3)(y_1e_1 + e_2y_2 + e_3y_3) -$

$-(y_1Z_1 + y_2Z_2 + y_3Z_3)(x_1e_1 + x_2e_2 + x_3e_3)$

$= e_1 \cdot \alpha - e_2 \cdot \beta + e_3\gamma = (\alpha, -\beta, \gamma)$.

c) $(x \times y) \times Z = \langle x, Z\rangle y - \cancel{\langle y, Z\rangle x}$

$(y \times Z) \times x = \cancel{\langle y, x\rangle Z} - \cancel{\langle Z, x\rangle y}$

$(Z \times x) \times y = \cancel{\langle Z, y\rangle x} - \cancel{\langle x, y\rangle Z}$ $\quad \oplus$

$\sum\limits_{x,y,Z}^{c} (x \times y) \times Z = 0$

Def (produs mixt)

$(\mathbb{R}^3, g_0)$ s.v.e.r.can, $S=\{x,y\}$ e SLI. Fie $z \in \mathbb{R}^3$

$z \wedge x \wedge y = \langle z, x \times y \rangle$
(produs mixt)

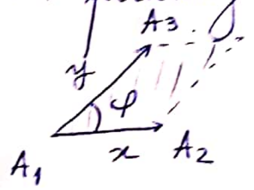
OBS $z \wedge x \wedge y = x \wedge y \wedge z \iff$

$$\begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

Aplicații $(\mathbb{R}^3, g_0)$

① $\|z\|^2 = \|x \times y\|^2 = \begin{vmatrix} \langle x,x \rangle & \langle x,y \rangle \\ \langle y,x \rangle & \langle y,y \rangle \end{vmatrix} = \begin{vmatrix} \|x\|^2 & \|x\|\cdot\|y\|\cos\varphi \\ \|x\|\|y\|\cos\varphi & \|y\|^2 \end{vmatrix} =$

$= \|x\|^2 \cdot \|y\|^2 \underbrace{(1-\cos^2\varphi)}_{\sin^2\varphi} \Rightarrow \|z\| = A_{\text{paralelogram}}$



② $A_{\Delta A_1A_2A_3} = \frac{1}{2} \|\overrightarrow{A_1A_2} \times \overrightarrow{A_1A_3}\| =$

$$= \frac{1}{2} \left\| \begin{vmatrix} e_1 & e_2 & e_3 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} \right\|$$

$$= \frac{1}{2}\sqrt{\begin{vmatrix} y_2-y_1 & z_2-z_1 \\ y_3-y_1 & z_3-z_1 \end{vmatrix}^2 + \begin{vmatrix} x_2-x_1 & z_2-z_1 \\ x_3-x_1 & z_3-z_1 \end{vmatrix}^2 + \begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}^2}$$

③ Fie $\{u, v, w\}$ SLI în $\mathbb{R}^3$ i.e $R=\{u,v,w\}$ reper în $\mathbb{R}^3$

Fie $R' = \{u, v, u \times v\}$ reper (pozitiv orientat) în $\mathbb{R}^3$

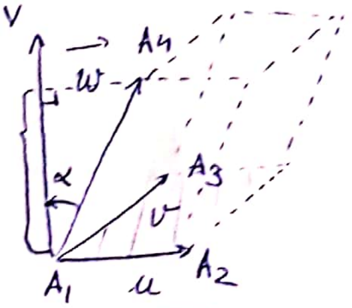
a) R, R' la fel orientate

$V_{\text{paralelipiped}} = \|u \times v\| \cdot \|w\| \cdot \cos\alpha =$

$\pm \langle w, u \times v \rangle = u \wedge v \wedge w$

b) R, R' nu sunt la fel orientate $\left(\alpha \in \left(\frac{\pi}{2}, \pi\right)\right)$

$V = |u \wedge v \wedge w|$



$$V_{A_1A_2A_3A_4} = \frac{1}{6}\left|\overrightarrow{A_1A_2}\wedge\overrightarrow{A_1A_3}\wedge\overrightarrow{A_1A_4}\right| =$$

$$= \frac{1}{6}\left|\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix}\right| = \frac{1}{6}\left|\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}\right|$$

$(L_2-L_1, L_3-L_1, L_4-L_1)$

<u>Exemplu</u> $(\mathbb{R}^3, g_0)$

$u=(1,2,-1)$; $v=(0,1,2)$, $w=(1,1,1)$

a) $u\times v$; b) $w\wedge u\wedge v$; c) $z=?$ a.î. $\{u,v,z\}$ reper ortogonal pozitiv orientat în $\mathbb{R}^3$

<u>SOL</u>

a) $u\times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \end{vmatrix} = (5,-2,1)$

b) $w\wedge u\wedge v = \langle w, u\times v\rangle = 1\cdot 5+1\cdot(-2)+1\cdot 1 = 4$

c) $\langle u,v\rangle = 1\cdot 0+2\cdot 1+(-1)2 = 0 \Rightarrow u\perp v.$

$z = u\times v = (5,-2,1)$ $\qquad$ $\{u,v,z\}$ reper ortog poz. orientat

<u>Teorema Cauchy-Buniakowski-Schwarz</u>

$(E,\langle\cdot,\cdot\rangle)$ sp. v. e, $x,y\in E \Rightarrow |\langle x,y\rangle| \le \|x\|\cdot\|y\|.$

„=" $\Leftrightarrow$ $\{x,y\}$ este SLD.

<u>Dem</u>

1) Dacă $x=0$ sau $y=0 \Rightarrow 0\le 0$ (A)

2) Dacă $x\neq 0$ și $y\neq 0$, fie $\lambda\in\mathbb{R}$ a.î.

$\langle x+\lambda y, x+\lambda y\rangle \geq 0$

$\|x\|^2+2\lambda\langle x,y\rangle+\lambda^2\|y\|^2$

$(*)$ $\lambda^2\|y\|^2+2\lambda\langle x,y\rangle+\|x\|^2 \geq 0, \forall\lambda\in\mathbb{R}$

$\Delta_\lambda \le 0 \Rightarrow 4\langle x,y\rangle^2 - 4\|x\|^2\|y\|^2 \le 0 \Rightarrow$

$|\langle x,y\rangle| \le \|x\|\cdot\|y\|.$

„$\Rightarrow$" „=" $\Leftrightarrow \Delta_\lambda = 0 \Leftrightarrow \exists\lambda_0\in\mathbb{R}$ a.î. $\underbrace{\langle x+\lambda_0 y, x+\lambda_0 y\rangle}_{Q(x+\lambda_0 y)} = 0$

$\xRightarrow{\text{poz def}} x+\lambda_0 y = 0 \Rightarrow \{x,y\}$ SLD.

"$\Leftarrow$" $\{x,y\} \in SLD \Rightarrow \exists a \in \mathbb{R}^*$ ai $y = ax$.

$|\langle x,y\rangle| = |\langle x, ax\rangle| = |a| \cdot \|x\|^2$.

$\|x\| \cdot \|y\| = \|x\| \cdot \|ax\| = |a| \cdot \|x\|^2$

$\|ax\| = \sqrt{\langle ax, ax\rangle} = \sqrt{a^2 \cdot \|x\|^2} = |a| \cdot \|x\|$

Teoremă (procedeul Gram-Schmidt)

Fie $(E, \langle\cdot,\cdot\rangle)$ s.v.e.r, $R = \{f_1, .., f_n\}$ reper în E

$\Rightarrow \exists R' = \{e_1, .., e_n\}$ reper ortogonal în E ai

$\langle\{e_1, .., e_i\}\rangle = \langle\{f_1, .., f_i\}\rangle, \forall i = \overline{1,n}$, $n = \dim E$

Dem Met. Inductivă.

$e_1 = f_1 \neq 0$

$\left.\begin{array}{l} e_2 = f_2 + \alpha f_1 \\ \langle e_2, e_1\rangle = 0 \end{array}\right\} \Rightarrow 0 = \langle f_2, e_1\rangle + \alpha \langle e_1, e_1\rangle = 0 \Rightarrow$

$\alpha = -\dfrac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle}$.

$e_2 = f_2 - \dfrac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} \cdot e_1 \Rightarrow$

$\begin{cases} f_1 = e_1 \\ f_2 = \dfrac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1 + e_2 \end{cases} \Rightarrow Sp\{e_1, e_2\} = Sp\{f_1, f_2\}$

Pp. adev P_k : $\{e_1, .., e_k\}$ sist vect. mutual ortog și

$\langle\{e_1, .., e_i\}\rangle = \langle\{f_1, .., f_i\}\rangle, \forall i = \overline{1,k}$

Dem P_{k+1} adev.

Fie $\cdot$ $e_{k+1} = f_{k+1} + \sum_{i=1}^{k} \alpha_{k+1\,i}\, e_i \;\Big|\; \langle\cdot, e_j\rangle, \forall j = \overline{1,k}$

$\underset{0}{\underset{\parallel}{\langle e_{k+1}, e_j\rangle}} = \langle f_{k+1}, e_j\rangle + \underbrace{\sum_{i=1}^{k} \alpha_{k+1\,i} \langle e_i, e_j\rangle}_{\alpha_{k+1\,j} \langle e_j, e_j\rangle}$.

$\alpha_{k+1\,j} = -\dfrac{\langle f_{k+1}, e_j\rangle}{\langle e_j, e_j\rangle}$.

$e_{k+1} = f_{k+1} - \sum_{i=1}^{k} \dfrac{\langle f_{k+1}, e_i\rangle}{\langle e_i, e_i\rangle} \cdot e_i$

$$\begin{cases} f_1 = e_1 \\ f_2 = \dfrac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1 + e_2 \\ \vdots \\ f_{k+1} = \sum_{i=1}^{k} \dfrac{\langle f_{k+1}, e_i\rangle}{\langle e_i, e_i\rangle} e_i + e_{k+1} \end{cases} \Rightarrow Sp\{e_1, .., e_i\} = Sp\{f_1, .., f_i\} \quad \forall i = \overline{1, k+1}$$

Se continuă raționamentul:

$$\begin{cases} f_1 = e_1 \\ f_2 = \dfrac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1 + e_2 \\ \vdots \\ f_n = \dfrac{\langle f_n, e_1\rangle}{\langle e_1, e_1\rangle} e_1 + \dfrac{\langle f_n, e_2\rangle}{\langle e_2, e_2\rangle} e_2 + \dots + \dfrac{\langle f_n, e_{n-1}\rangle}{\langle e_{n-1}, e_{n-1}\rangle} e_{n-1} + e_n. \end{cases}$$

OBS $\underbrace{R = \{f_1, .., f_n\}}_{\text{reper } \forall} \xrightarrow{A} \underbrace{R' = \{e_1, .., e_n\}}_{\text{reper ortog}} \xrightarrow{B} \underbrace{R'' = \left\{\dfrac{e_1}{\|e_1\|}, .., \dfrac{e_n}{\|e_n\|}\right\}}_{\text{reper ortonormat}}$

$$A^{-1} = \begin{pmatrix} 1 & \dfrac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} & \cdots & \dfrac{\langle f_n, e_1\rangle}{\langle e_1, e_1\rangle} \\ 0 & 1 & & \vdots \\ \vdots & \vdots & & \dfrac{\langle f_n, e_{n-1}\rangle}{\langle e_{n-1}, e_{n-1}\rangle} \\ 0 & 0 & & 1 \end{pmatrix}; \quad B = \begin{pmatrix} \dfrac{1}{\|e_1\|} & & 0 \\ & \ddots & \\ 0 & & \dfrac{1}{\|e_n\|} \end{pmatrix}$$

$$\det(A^{-1}) = \frac{1}{\det A} = 1 \Rightarrow \det A = 1 > 0; \quad \det B = \frac{1}{\|e_1\|} \cdot \dots \cdot \frac{1}{\|e_n\|} > 0$$

R, R', R'' la fel orientate

Def $(E, \langle \cdot, \cdot \rangle)$ s.v.e.r.

a) $x \in E$, $\langle\{x\}\rangle^{\perp} = \{y \in E \mid \langle x, y\rangle = 0\}$ subspațiu ortogonal pe x.

b) $\underset{\text{ssp vect}}{U \subseteq E}$, $U^{\perp} = \{x \in E \mid \langle x, y\rangle = 0, \forall y \in U\}$.
subsp. ortogonal pe U.

OBS $U \subseteq \underset{\text{ssp. vect}}{W} \subseteq E \Rightarrow W^{\perp} \subseteq \underset{\text{subsp.}}{U^{\perp}} \subseteq E$.

Fie $x \in W^{\perp} \Rightarrow \langle x, y\rangle = 0, \forall y \in W$
$U \subseteq W \Rightarrow \langle x, u\rangle = 0, \forall u \in U \Rightarrow x \in U^{\perp}$
$\Rightarrow W^{\perp} \subseteq U^{\perp}$.

Teoremă $(E, \langle\cdot,\cdot\rangle)$

$U \subseteq E$ ssp vect $\Rightarrow E = U \oplus \underset{\substack{\downarrow \\ \text{complement ortogonal}}}{U^{\perp}}$ (scriere unică)

Dem

$\underset{\text{sspV}}{U, U^{\perp} \subseteq E} \Rightarrow \underset{\text{ssp V.}}{U + U^{\perp} \subseteq E}$

Fie $x \in U \cap U^{\perp} \Rightarrow x \in U$ și $x \in U^{\perp}$ $\Rightarrow \underset{\underset{Q(x)}{\parallel}}{\langle x, x\rangle} = 0 \xRightarrow{\text{poz def}} x = 0$

$\Rightarrow U \oplus U^{\perp} \subseteq E.$

Dem că $E \subseteq U \oplus U^{\perp}$

Fie $R = \{e_1, .., e_k\}$ reper ortonormat în U

Fie $v \in E$ și $v' = v - \sum_{i=1}^{k} \langle v, e_i\rangle e_i$

Arătăm că $v' \in U^{\perp}$

$$\langle v', e_1\rangle = \langle v, e_1\rangle - \underbrace{\sum_{i=1}^{k} \langle v, e_i\rangle \underbrace{\langle e_i, e_1\rangle}_{\delta_{i1}}}_{\langle v, e_1\rangle} = 0$$

$\vdots$

$\langle v', e_k\rangle = 0$

Fie $x \in U$, $\langle v', x\rangle = \langle v', \sum_{i=1}^{k} x_i e_i\rangle =$

$= \sum_{i=1}^{k} x_i \underset{\underset{0}{\parallel}}{\langle v', e_i\rangle} = 0 \Rightarrow v' \in U^{\perp}$

$\Rightarrow v = \underset{\underset{U^{\perp}}{\cap}}{v'} + \underset{\underset{U}{\cap}}{\sum_{i=1}^{k} \langle v, e_i\rangle e_i} \Rightarrow E \subseteq U \oplus U^{\perp}$

Deci $E = U \oplus U^{\perp}$

Exercițiu $(\mathbb{R}^3, g_0)$ s.v.e.r. can ; $u = (1, 2, -1)$

a) $\langle\{u\}\rangle^{\perp}$; b) precizați un reper ortonormat în $\langle\{u\}\rangle^{\perp}$, utilizând procedeul Gram-Schmidt

SOL

a) $\langle\{u\}\rangle^{\perp} = \{x \in \mathbb{R}^3 \mid g_0(\underset{(x_1,x_2,x_3)}{x}, \underset{(1,2,-1)}{u}) = 0\} = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 - x_3 = 0\}$

$\dim \langle\{u\}\rangle^{\perp} = 2$

$x_3 = x_1 + 2x_2 \Longrightarrow \langle\{u\}\rangle^{\perp} = \{(x_1, x_2, x_1 + 2x_2) \mid x_1, x_2 \in \mathbb{R}\}$

$$x_1\underbrace{(1,0,1)}_{f_1} + x_2\underbrace{(0,1,2)}_{f_2}$$

$\left.\begin{array}{l} R = \{f_1, f_2\} \text{ SG} \\ |R| = \dim\langle\{u\}\rangle^{\perp} = 2 \end{array}\right\} \Rightarrow R = \{f_1, f_2\}$ reper $\forall$

Aplicăm procedeul Gram-Schmidt.

$e_1 = f_1 = (1,0,1)$

$e_2 = f_2 - \dfrac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1 = (0,1,2) - \dfrac{2}{2}(1,0,1) =$

$= (0,1,2) - (1,0,1) = (-1,1,1)$.

$R' = \{e_1, e_2\}$ reper ortogonal

$R'' = \left\{ \dfrac{e_1}{\|e_1\|} = \dfrac{1}{\sqrt{2}}(1,0,1), \dfrac{e_2}{\|e_2\|} = \dfrac{1}{\sqrt{3}}(-1,1,1) \right\}$ reper ortonormat.

$\boxed{\text{OBS}}$ A da un produs scalar $\Longleftrightarrow$ a declara un reper ortonormat.

- $g: V \times V \to \mathbb{R}$ produs scalar, $R = \{e_1, .., e_n\}$ reper ortonormat $\Longleftrightarrow g(e_i, e_j) = \delta_{ij}, \forall i, j = \overline{1,n}$
- $R = \{e_1, .., e_n\}$ reper ortonormat, $g(e_i, e_j) = \delta_{ij}$, $g: V \times V \to \mathbb{R}$ p. scalar.

Prelungim g prin liniaritate:

$$g(x,y) = g\Big(\sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{n} y_j e_j\Big) = \sum_{i,j=1}^{n} x_i y_j \underbrace{g(e_i, e_j)}_{\delta_{ij}} =$$

$$= \sum_{i=1}^{n} x_i y_i$$

$\boxed{\text{OBS}}$ $(V, \langle\cdot,\cdot\rangle_V) \longrightarrow (V^*, \langle\cdot,\cdot\rangle_{V^*})$, $V^* = \{f: V \to \mathbb{R} \mid f \text{ lin}\}$ (sp. vect. dual)

$R = \{e_1, .., e_n\}$ reper ortonormat în V, $\langle e_i, e_j\rangle_V = \delta_{ij}$ | $R^* = \{e_1^*, .., e_n^*\}$, $e_i^*: V \to \mathbb{R}$ lin, $e_i^*(e_j) = \delta_{ij}, \forall i,j = \overline{1,n}$

$\langle e_i^*, e_j^*\rangle_{V^*} = \delta_{ij}$.

$\langle\cdot,\cdot\rangle_{V^*}: V^* \times V^* \to \mathbb{R}$. Extindem prin liniaritate

$$\langle f, h\rangle_{V^*} = \Big\langle \sum_{i=1}^{n} f_i e_i^*, \sum_{j=1}^{n} h_j e_j^* \Big\rangle = \sum_{i=1}^{n} f_i h_i$$