Seminar 6 - 143

Ex. 1: Det. elementele inversabile ale monoidului $(\mathbb{Z}_{12}, \cdot)$.

Rez:

În gen., $U(\mathbb{Z}_m) = \{ \hat{a} \in \mathbb{Z}_m \mid \underset{\text{comode}}{(a, m) = 1} \}$

Dem:

Th: Fie $a, b \in \mathbb{Z}$, $d = (a, b)$. Atunci $(\exists)\ m, n \in \mathbb{Z}$ a.î. $am + bn = d$.

„$\supseteq$" $(a, m) = 1 \overset{Th}{\Rightarrow} \exists\ k, l \in \mathbb{Z}$ $ak + m \cdot l = 1$.

Trecem la clase: $\hat{a} \cdot \hat{k} + \underset{=\hat{0}}{\hat{m}} \hat{l} = \hat{1} \iff \hat{a} \cdot \hat{k} = \hat{1} \iff \hat{a} \in U(\mathbb{Z}_m)$.

„$\subseteq$" $\hat{a} \in U(\mathbb{Z}_m) \Rightarrow \exists\ \hat{b} \in \mathbb{Z}_m$ $\hat{a} \cdot \hat{b} = \hat{1}$.

Pp. $(a, m) = d$

$\Rightarrow$ $a = d \cdot a'$ $\quad (a', m') = 1$.

$m = d \cdot m'$, $m' < m$

$\hat{a} \cdot \hat{b} = \hat{1} \iff \hat{d} \cdot \hat{a}' \cdot \hat{b} = \hat{1} \mid \cdot \hat{m}'$

$\Rightarrow$ $\underbrace{\hat{d} \cdot \hat{m}'}_{\hat{m} = \hat{0}} \cdot \hat{a}' \cdot \hat{b} = \hat{m}'$ $\Rightarrow$ $\hat{m}' = \hat{0}$ Ab.

$U(\mathbb{Z}_{12}) = \{\hat{1}, \hat{5}, \hat{7}, \hat{11}\}$

$a = 3$

$b = 12$ $\quad (a, b) = 3$.

$3 = 3 \cdot 1 + 12 \cdot 0$.

$a = 56$ $\quad d = 8$

$b = 48$

$d = a - b$

$a = 5$

$b = 7$ $\quad 1 = 5 \cdot k + 7 \cdot l$

$1 = 5 \cdot 3 - 7 \cdot 2$.

Ex. 2: Scrieți subgrupurile lui $(\mathbb{Z}_{12},+)$

Rez:

Obs: $x \in U(\mathbb{Z}_m) \Rightarrow \{\hat{k}x \mid k \in \mathbb{Z}\} = \mathbb{Z}_m$.

Triviale: $\{\hat{0}\}$, $\mathbb{Z}_{12}$.

$\{\hat{2}, \hat{4}, \hat{6}, \hat{8}, \hat{10}, \hat{0}\}$, $\{\hat{0}, \hat{3}, \hat{6}, \hat{9}\}$, $\{\hat{0}, \hat{4}, \hat{8}\}$, $\{\hat{0}, \hat{6}\}$

$\{0, \hat{5}, \hat{7}\}$ nu este subgrup

$\hat{5}+\hat{5} = \hat{10}$

Ex. 3: Fie G un grup în care $x^2 = 1$, $\forall x \in G$. Arătați că G este grup comutativ.

Rez:

$x^2 = 1 \iff x^{-1} = x$

Vrem: $\forall a, b \in G$ $\quad ab = ba$

$(ab)^2 = 1 \iff ab = (ab)^{-1} \iff ab = b^{-1}a^{-1}$

$\iff ab = ba$

G este comutativ.

Ex. 4: Fie mulțimile:

$G_1 = \left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=I_2}, \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{=A_1}, \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}_{=A_2}, \underbrace{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}_{=A_3} \right\}$,

$G_2 = \{1, -1, i, -i\}$,

$G_3 = \left\{ \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}}_{=e}, \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}}_{=\sigma_{12}}, \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}}_{=\sigma_{34}}, \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}}_{=\Gamma} \right\}$

a. Arătați că $(G_1, \cdot)$, $(G_2, \cdot)$, $(G_3, \circ)$ sunt grupuri comutative

b. Stabiliți dacă cele 3 grupuri sunt izomorfe cu $\mathbb{Z}_4$ sau $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Rez :

91. G_1:

$$\begin{cases} A_1^2 = I_2 \\ A_2^2 = I_2 \\ A_3^2 = I_2 \\ I_2^2 = I_2 \end{cases} \quad A_1A_2 = A_3 \quad A_1A_3 = A_2 = A_3A_1$$
$$A_2A_1 = A_3 \quad A_2A_3 = A_1 = A_3A_2.$$

G_1 grup cu $A^2 = I_2 \ \forall A \in G_1. \overset{Ex.3}{\Rightarrow}$ com.

$(G_2, \cdot) = \langle i \rangle$ grup comutativ.

$= \{i, i^2, i^3, i^4\}$

$G_3 = \{e, \tau_{12}, \tau_{34}, \sigma\}.$

$\tau_{12} \circ \tau_{12} = e$

$\tau_{34} \circ \tau_{34} = e$ $\qquad \sigma = \tau_{12} \circ \tau_{34} = \tau_{34} \circ \tau_{12}$

$\sigma \circ \sigma = e$

$G_1 \simeq G_3 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$

$G_2 \not\simeq G_1, G_3$, $G_2 \simeq \mathbb{Z}_4$.

Obs : Pe $\mathbb{Z}$ considerăm rel. $x \sim y \Leftrightarrow i^x = i^y$

$\mathbb{Z}/\sim \ \simeq \mathbb{Z}_4$.

Ex. 5 : Găsiți toate morfismele de grupuri de la $(\mathbb{Z}, +)$ la $(\mathbb{Z}, +)$.

Rez : $f: \mathbb{Z} \to \mathbb{Z}$, $f(0) = 0$ și $f(m+n) = f(m) + f(n)$ $\forall m, n \in \mathbb{Z}$.

$f(m) = f(\overbrace{1+1+\ldots+1}^{m \text{ ori}}) = m \cdot f(1)$

$f(-1) = -f(1)$ deoarece $0 = f(0) = f(1-1) = f(1) + f(-1)$.

Obs. că $f(m) = m \cdot f(1)$, $\forall m \in \mathbb{Z}$.

$f(1) = a \in \mathbb{Z}$.

$f(m) = am$.

T: Găsiți toate morfismele $f : \mathbb{Z} \to \mathbb{Q}$, și $g : \mathbb{Q} \to \mathbb{Z}$.
Sunt cele 2 grupuri izomorfe? $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$.

Indicație: $1 = \underbrace{\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}}_{n \text{ ori}}$.

$g(1) = n \cdot g\left(\frac{1}{n}\right)$.

Ex. 6: Fie $m, n \in \mathbb{N}$, $m, n \geq 2$. Găsiți toate morfismele de grupuri $f : \mathbb{Z}_m \to \mathbb{Z}_n$, $(\mathbb{Z}_m, +), (\mathbb{Z}_n, +)$.

Rez: $f(\hat{0}) = \bar{0}$

Not.: $\hat{a} \in \mathbb{Z}_m$, $\bar{a} \in \mathbb{Z}_n$.

$f(\hat{0}) = \bar{0}$, $f(\hat{a} + \hat{b}) = f(\hat{a}) + f(\hat{b})$, $\forall \hat{a}, \hat{b} \in \mathbb{Z}_m$.

$f(\hat{a}) = f(\underbrace{\hat{1} + \hat{1} + \dots + \hat{1}}_{a}) = \bar{a} \cdot f(\hat{1})$, $\forall \hat{a} \in \mathbb{Z}_m$.

$f(\hat{a}) = \underline{\bar{k}} \cdot \bar{a}$, $\bar{k} \in \mathbb{Z}_n$.

! Buna def: $\hat{a} = \hat{b} \Rightarrow f(\hat{a}) = f(\hat{b})$ $\forall \hat{a} = \hat{b}$

$\hat{a} = \hat{b} \Leftrightarrow m \mid a - b \Leftrightarrow a - b = m \cdot t$, $t \in \mathbb{Z}$.

$f(\hat{a}) = f(\hat{b}) \Leftrightarrow \bar{k} \cdot \bar{a} = \bar{k} \cdot \bar{b} \Leftrightarrow \bar{k}(\bar{a} - \bar{b}) = \bar{0}$

$\Leftrightarrow n \mid k(a - b) \Leftrightarrow n \mid \underline{k \cdot m \cdot t}$ $\forall t \in \mathbb{Z}$.

$n \mid k \cdot m \cdot t$, $\forall t \in \mathbb{Z} \Rightarrow \boxed{n \mid km}$

cmmdc$(m, n) = d$

$n \mid km \Leftrightarrow \boxed{\frac{n}{d} \mid k}$

Exemple: $m = 3$ $\quad n = 5$

$f : \mathbb{Z}_3 \to \mathbb{Z}_5$, $f(\hat{x}) = \overline{k \cdot x}$, $5 \mid 3k \Rightarrow 5 \mid k$

$\bar{k} = \bar{0}$

$f(\hat{x}) = \bar{0}$, $\forall \hat{x}$

$m = 4 \qquad n = 6$

$g : \mathbb{Z}_4 \to \mathbb{Z}_6 \,, \; g(\hat{x}) = \overline{kx}$

cu propr. $6 \mid 4k \iff 3 \mid k$

$g_1(\hat{x}) = \overline{3x}$ sau $g_2(\hat{x}) = \bar{0}$.

g_1 morfism

$g_1(\hat{0}) = \bar{0}$

$g_1(\hat{1}) = \bar{3}$

$g_1(\hat{2}) = \bar{0}$

$g_1(\hat{3}) = \bar{3}$

$a - b = 4t$

$g_1(\hat{a}) = g_1(\hat{b})$

$\overline{3a} = \overline{3b} \iff 6 \mid 3(a-b)$

$2 \mid a-b$ ok.

Ex. 7: Se consideră mulțimea:

$$H = \left\{ \begin{pmatrix} m & n \\ \hat{0} & \hat{1} \end{pmatrix} \;\middle|\; m, n \in \mathbb{Z}_5 ,\; m \in \{\pm \hat{1}\} \right\}.$$

Arătați că $(H, \cdot)$ este grup.

Rez:

$$\begin{pmatrix} m & n \\ \hat{0} & \hat{1} \end{pmatrix} \begin{pmatrix} p & q \\ \hat{0} & \hat{1} \end{pmatrix} = \begin{pmatrix} \overset{\pm 1}{mp} & \overset{\in \mathbb{Z}_5}{mq + n} \\ \hat{0} & \hat{1} \end{pmatrix} \in H$$

Elem. neutru: I_2

Elem. simetrizabile:

$$\begin{vmatrix} m & n \\ \hat{0} & \hat{1} \end{vmatrix} = m \in \{\pm \hat{1}\}.$$

Obs: $A \in M_n(G)$, G grup comutativ

A este inv. $\iff$ $\det A$ este inv. în G.

În $M_2(\mathbb{Q})$, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ este inv.

$M_2(\mathbb{Z})$, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ nu este inv.

$A = \begin{pmatrix} m & n \\ \hat{0} & \hat{1} \end{pmatrix}$, $A^{-1} = \begin{pmatrix} m & \hat{0} \\ \hat{0} & \hat{1} \end{pmatrix}$

Este H comutativ?

$mq + n \neq np + q$

$p = m = 1$.

T: $G = \left\{ \begin{pmatrix} \hat{1} & m \\ \hat{0} & \hat{1} \end{pmatrix} \middle| m \in \mathbb{Z}_6 \right\}$

$(G, \cdot)$ este grup. Cu ce grup este izomorf G?

Ex. 8: Pe $\mathbb{C}$, $z \sim w \iff z - w \in \mathbb{R} \iff \operatorname{Im} z = \operatorname{Im} w$.

$z = a + bi$, $w = c + bi$

$f : \mathbb{C}/\sim \longrightarrow \mathbb{R}$.

$f(\hat{z}) = f(\hat{w})$

$f(\widehat{a+bi}) = f(\widehat{c+bi})$ $\forall a, b, c \in \mathbb{R}$.

$f(\widehat{a+bi}) = g(\underline{b})$, $g : \mathbb{R} \to \mathbb{R}$.

(b^2, $\sin b$, $\ln|b|$, $b^2 + 7b + 13$ etc.)

f bine def.

$f(\widehat{a+bi}) = \underline{b}^2 - 3\underline{b} + 2$

$f(\widehat{c+bi}) = b^2 - 3b + 2$

$\widehat{a+bi} = \{ x + bi \mid x \in \mathbb{R} \}$.

Ex. 9: Pe $\mathbb{N} \times \mathbb{N}/\sim$ $(a, b) \sim (c, d) \iff a + d = b + c$.

$\iff \underline{a-b} = \underline{c-d}$.

$g : \mathbb{N} \times \mathbb{N} \to G$, $g(a, b) = (a-b)^2 + 2(a-b) + a$

$g(c, d) = (c-d)^2 + 2(c-d) + c$

g bine def.

$g(a,b) = g(c,d) \Leftrightarrow a = c.$

Ex. 10: Pe $\mathbb{R}$, $x \sim y \Leftrightarrow x^2 - x = y^2 - y.$

$[x] = \{x, 1-x\}, \ [1] = [0]$

$f : \mathbb{R}/_\sim \to \mathbb{R}, \ f(x) = |x|.$

$x^3 - x = x(x^2-1) = x(x-1)(x+1).$

Dacă $x \sim y \Leftrightarrow x^3 - x = y^3 - y$ este rel. de echiv.

$x^3 - y^3 - (x-y) = 0.$

$(x-y)(x^2+xy+y^2-1) = 0.$

$x = y$ sau $x^2 + xy + y^2 - 1 = 0.$

$[x] = \{x, \{y^2 + xy + (x^2-1) = 0\}\}$

$[1] = \{1, 0, -1\}.$

$y^2 + y = 0$

$1 \sim y \Leftrightarrow 1 = y$ sau $1 + y + y^2 - 1 = 0$

$\Updownarrow$

$y^2 + y = 0 \Rightarrow y \in \{0, -1\}.$

$f(x) = |x|$

f bine def. $\Leftrightarrow |x| = |1-x|, \ \forall x.$

$x = 0$, $f(0) = 0$, $f(1-0) = 1. \Rightarrow f$ nu este bine def.

$g(x) = |x - \frac{1}{2}|$

$|x - \frac{1}{2}| = |1 - x - \frac{1}{2}| \Leftrightarrow |x - \frac{1}{2}| = |\frac{1}{2} - x|, \ \forall x$

$\Rightarrow g$ este bine def.