C3 - GA

Spaţii vectoriale.
Sisteme liniar independente / dependente
Sisteme de generatori. Baze. Exemple.

Def (sp. vectorial)

$(\mathbb{K},+,\cdot)$ corp com., V mult. nevidă.

V are structură de spaţiu vectorial peste corpul $\mathbb{K}$ dacă

$+ : V\times V \longrightarrow V$

$\cdot : \mathbb{K}\times V \longrightarrow V$

a.î. A1) $(V,+)$ grup abelian.

A2) $a\cdot(b\cdot x)=(a\cdot b)\cdot x$

A3) $(a+b)\cdot x = a\cdot x+b\cdot x$

A4) $a\cdot(x+y) = a\cdot x + a\cdot y$

A5) $1_{\mathbb{K}}\cdot x = x$, $\forall x,y\in V$ (vectori)

$\forall a,b\in\mathbb{K}$ (scalari)

Not $(V,+,\cdot)_{/\mathbb{K}}$.

OBS

a) $0_{\mathbb{K}}\cdot x = 0_V$

b) $a\cdot 0_V = 0_V$

c) $(a-b)\cdot x = a\cdot x - b\cdot x$

d) $a\cdot(x-y) = ax-ay$, $\forall x,y\in V$, $\forall a,b\in\mathbb{K}$.

Exemple 1) $(\mathcal{V},+,\cdot)_{/\mathbb{R}}$ spaţiul vect. al vectorilor liberi din plan.

2) $(V_1,\oplus,\odot)_{/\mathbb{K}}$, $(V_2,\boxplus,\boxdot)_{/\mathbb{K}}$ sp. vect peste $\mathbb{K}$.

$(V_1\times V_2,+,\cdot)_{/\mathbb{K}}$ sp. vect

$+ : (V_1\times V_2)\times(V_1\times V_2)\longrightarrow V_1\times V_2$.

$(x_1,x_2)+(y_1,y_2)=(x_1\oplus y_1, x_2\boxplus y_2)$

$\cdot : \mathbb{K}\times(V_1\times V_2)\longrightarrow V_1\times V_2$

$a\cdot(x_1,x_2) = (a\odot x_1, a\boxdot x_2)$, $\forall (x_1,x_2),(y_1,y_2)\in V_1\times V_2$

$\forall a\in\mathbb{K}$.

Caz particular

$(\mathbb{R},+,\cdot)/_{\mathbb{R}}$ sp. vect , $(\mathbb{R}^n,+,\cdot)/_{\mathbb{R}}$ sp vect

$(x_1,..,x_n)+(y_1,..,y_n)=(x_1+y_1,...,x_n+y_n)$

$a\cdot(x_1,..,x_n)=(a\cdot x_1,.., a\cdot x_n), \forall a\in\mathbb{R}, \forall (x_1,..,x_n), (y_1,..,y_n)\in\mathbb{R}^n$.

3) $(\mathcal{M}_{m,n}(\mathbb{K}),+,\cdot)/_{\mathbb{K}}$ sp. vect.

$(a_{ij})_{\substack{i=\overline{1,m}\\ j=\overline{1,n}}} \longrightarrow (a_{11},.., a_{1n},.., a_{m1},.., a_{mn})\in\mathbb{K}^{n\cdot m}$

4) $(\mathbb{K}[X],+,\cdot)/_{\mathbb{K}}$ sp. vect.

$P=a_0+a_1X+...+a_nX^n$, grad $P=n$.

$(\mathbb{K}_n[X],+,\cdot)/_{\mathbb{K}}$ sp vect

$\mathbb{K}_n[X]=\{P\in\mathbb{K}[X],$ grad $P\le n\}$.

$P=a_0+a_1X+..+a_nX^n \longrightarrow (a_0,a_1,..,a_n)\in\mathbb{K}^{n+1}$

5) $I=[a,b]$, $a<b$.

$(\mathcal{C}=\{f:I\to\mathbb{R} \mid f$ cont$\},+,\cdot)/_{\mathbb{R}}$ sp vect

$(\mathcal{D}=\{f:I\to\mathbb{R} \mid f$ derivabilă$\},+,\cdot)/_{\mathbb{R}}$ sp vect

$(\mathcal{P}=\{f:I\to\mathbb{R} \mid f$ primitivabilă$\},+,\cdot)/_{\mathbb{R}}$ sp. vect

$(\mathcal{I}=\{f:I\to\mathbb{R} \mid f$ integrabilă$\},+,\cdot)/_{\mathbb{R}}$ sp vect.

Def (subspațiu vectorial)

$(V,+,\cdot)/_{\mathbb{K}}$ sp. vect, $V'\subseteq V$ subm. nevidă

V' s.n. subspațiu vect $\Leftrightarrow$ subm. închisă la „+" vect (i.e. $\forall x,y\in V' \Rightarrow x+y\in V'$) și la „·" cu scalari (i.e. $\forall a\in\mathbb{K}, \forall x\in V' \Rightarrow a\cdot x\in V'$)

OBS $V'\subseteq V$ subsp vect $\Rightarrow (V',+,\cdot)/_{\mathbb{K}}$ sp vect (cu operațiile induse)

Prop $(V,+,\cdot)/\mathbb{K}$ sp vect, $V' \subseteq V$ subm. nevidă

V' subsp vect $\Leftrightarrow [\forall x,y \in V', \forall a,b \in \mathbb{K} \Rightarrow ax + b\cdot y \in V']$

$\Leftrightarrow [\forall x_1,\dots,x_n \in V', \forall a_1,\dots,a_n \in \mathbb{K} \Rightarrow a_1x_1+\dots+a_nx_n \in V']$

Dem

"$\Rightarrow$" $V' \subseteq V$ subsp. vect (ip)

$\forall a \in \mathbb{K}, \forall x \in V' \Rightarrow a\cdot x \in V'$

$b \in \mathbb{K}, \ y \in V' \Rightarrow b\cdot y \in V'$ $\Rightarrow a\cdot x + b\cdot y \in V'$

"$\Leftarrow$" $\forall x,y \in V', \forall a,b \in \mathbb{K} \Rightarrow a\cdot x + by \in V'$ (ip).

Fie $a = b = 1_{\mathbb{K}} \Rightarrow \underbrace{1_{\mathbb{K}}\cdot x + 1_{\mathbb{K}}\cdot y}_{\overset{A_5}{=}\, x+y} \in V', \forall x,y \in V'$

Fie $b = 0_{\mathbb{K}} \Rightarrow \underbrace{a\cdot x + \underbrace{0_{\mathbb{K}}\cdot y}_{0_V}}_{a\cdot x} \in V'$

<u>Exemple de subsp vect</u>

1) $(V,+,\cdot)/\mathbb{K}$. $\{0_V\}, V \subset V$ subsp. vect.

2) $n < m, \ m \geq 2$ $\quad \mathbb{R}^n \subset \mathbb{R}^m$ subsp vect.

3) $(\mathcal{M}_n(\mathbb{R}),+,\cdot)/\mathbb{R}$ sp vect.

a) $V' = \left\{ A = diag(a_1,\dots,a_n) = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \right\} \subset \mathcal{M}_n(\mathbb{R})$ ssp v

b) $V'' = \left\{ A = \begin{pmatrix} a_1 & \dots & a_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R}) \right\}$ ssp v

c) $V''' = \{A \in \mathcal{M}_n(\mathbb{R}) \,/\, A = A^T\}$ ssp v

d) $W = \{A \in \mathcal{M}_n(\mathbb{R}) \,/\, A = -A^T\}$ ssp v

e) $U = \{A \in \mathcal{M}_n(\mathbb{R}) \,/\, Tr(A) = 0\}$ ssp v.

OBS $GL(n,\mathbb{R}) = \{A \in \mathcal{M}_n(\mathbb{R}) \,/\, det(A) \neq 0\}$ nu e ssp vect

$O(n) = \{A \in \mathcal{M}_n(\mathbb{R}) \,/\, A\cdot A^T = I_n\}$

$SO(n) = \{A \in O(n) \,/\, det A = 1\}$. —//—

$SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \mid \det A = 1\}$.

$SO(n) = O(n) \cap SL(n,\mathbb{R})$

4) $W = \{(x,y) \in \mathbb{R}^2 \mid ax+by=0,\ a^2+b^2>0\} \subset \mathbb{R}^2$ sp v.

$W' = \{(x,y,z) \in \mathbb{R}^3 \mid ax+by+cz=0,\ a^2+b^2+c^2>0\} \subset \mathbb{R}^3$ —''—

$W'' = \{(x_1,..,x_n) \in \mathbb{R}^n \mid a_1x_1+\ldots+a_nx_n=0,\ \sum_{i=1}^{n} a_i^2>0\} \subset \mathbb{R}^n$ —''—

$U = S(A) = \{(x_1,..,x_n) \in \mathbb{R}^n \mid AX = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\} \subset \mathbb{R}^n$ sp vect. m. sol unui SLO

$\underset{(m,n)}{A}\ \underset{(n,1)}{X} = \underset{(m,1)}{0}$

($\cap$ a m hiperplane, care trec prin origine.

Subspațiul vectorial generat de o mulțime

Def $(V,+,\cdot)/_{\mathbb{K}}$ sp vect, $S \subseteq V$ subm. nevidă

$\langle S \rangle = \{x \in V \mid x = a_1x_1 + \ldots + a_nx_n$, unde $a_1..a_n \in \mathbb{K}$, $x_1,..,x_n \in S\}$

subsp. vectorial generat de S.

S s.n sistem de generatori (SG) $\Leftrightarrow V = \langle S \rangle$

V s.n. spațiu vectorial finit generat dacă $\exists$ S un SG finit

OBS a) $S \subset \langle S \rangle$

b) $\langle S \rangle$ cel mai mic subsp. vect, care conține S.

c) $\langle \phi \rangle = \{0_V\}$ Convenție.

Def (SLI, SLD)

$(V,+,\cdot)/_{\mathbb{K}}$ sp vect, $S \subset V$ subm. nevidă

1) S s.n. sistem liniar independent (SLI) $\Leftrightarrow$

$\forall x_1,..,x_n \in S$ aî $\sum_{i=1}^{n} a_ix_i = 0_V \Rightarrow a_1 = .. = a_n = 0_{\mathbb{K}}$

$\forall a_1,..,a_n \in \mathbb{K}$

2) S s.n. sistem liniar dependent (SLD) $\Leftrightarrow$

$\exists x_1,..,x_n \in S$

$\exists a_1,..,a_n \in \mathbb{K}$, nu toți nuli aî $\sum_{i=1}^{n} a_ix_i = 0$

Prop Fie $x \neq 0_V \Rightarrow \{x\}$ este SLI

Dem Fie $a \in \mathbb{K}$ a.î. $a \cdot x = 0_V$

Sp. prin absurd că $a \neq 0_{\mathbb{K}} \Rightarrow \exists a^{-1}$ (în corpul $(\mathbb{K}, +, \cdot)$)

$a^{-1} \cdot a \cdot x = a^{-1} \cdot 0_V \Rightarrow 1_{\mathbb{K}} \cdot x = 0_V \Rightarrow x = 0_V$ Contrad.

Sp este falsă $\Rightarrow a = 0_{\mathbb{K}}$ și $\{x\}$ este SLI.

Def (bază) Fie $(V, +, \cdot)_{/\mathbb{K}}$ sp vect, $S \subseteq V$ subm. nevidă

S sn. bază $\Leftrightarrow \begin{cases} 1) \ S \text{ este SLI} \\ 2) \ S \text{ este SG.} \end{cases}$

Exemple

1) $(\mathbb{R}, +, \cdot)_{/\mathbb{R}}$. $B_0 = \{1\}$ este baza canonică.

$\{1\}$ este SLI $(1 \neq 0_{\mathbb{R}})$.

Dacă $x \in \mathbb{R}$, at $x = 1_{\mathbb{R}} \cdot x \in \langle 1 \rangle = \mathbb{R} \Rightarrow \{1\}$SG

B_0 bază

OBS $B = \{a\}$ bază, $\forall a \neq 0_{\mathbb{R}}$.

2) $(\mathbb{R}^2, +, \cdot)_{/\mathbb{R}}$, $B_0 = \{\underset{e_1}{(1,0)}, \underset{e_2}{(0,1)}\}$ baza canonică.

• SLI Fie $a, b \in \mathbb{R}$ a.î. $a(1,0) + b(0,1) = 0_{\mathbb{R}^2}$

$(a, 0) + (0, b) = (0,0) \Rightarrow (a, b) = (0,0) \Rightarrow \begin{cases} a = 0 \\ b = 0 \end{cases}$

• SG

$\forall x = (x_1, x_2) = (x_1, 0) + (0, x_2) = x_1(1,0) + x_2(0,1)$

$\in \langle \{e_1, e_2\} \rangle$

B_0 bază.

3) $(\mathbb{R}[X], +, \cdot)_{/\mathbb{R}}$. $B_0 = \{1, X, \ldots, X^n, \ldots\}$ bază

sp. vect care NU este finit generat

$(\mathbb{R}_n[X], +, \cdot)_{/\mathbb{R}}$, $B_0 = \{1, X, \ldots, X^n\}$ baza canonică

4) $(\mathcal{M}_{m,n}(\mathbb{R}), +, \cdot)/\mathbb{R}$. $E_{ij} = \begin{array}{c} \\ i \end{array}\begin{pmatrix} 0 & \overset{j}{\vdots} & 0 \\ \cdots & 1 & \\ 0 & \vdots & 0 \end{pmatrix}$ $i=\overline{1,m}$ $j=\overline{1,n}$

$B_0 = \{E_{ij},\ i=\overline{1,m},\ j=\overline{1,n}\}$ baza canonică.

! $\boxed{\text{OBS}}$ a) $\forall$ subm $\neq \emptyset$ a unui SLI este un SLI

$S = \{x_1, .., x_n\}$ SLI $\Rightarrow S' = \{x_1, .., x_{n-1}\}$ SLI.

$a_1 x_1 + \ldots + a_{n-1} x_{n-1} = 0_V \Rightarrow a_1 x_1 + \ldots + a_{n-1} x_{n-1} + 0_{\mathbb{K}} \cdot x_n = 0$

$S \Downarrow$ SLI

$a_1 = .. = a_{n-1} = 0_{\mathbb{K}}$.

b) $\forall$ supramulțime a unui SLD este un SLD?

$S = \{x_1, .., x_n\}$ SLD $\Rightarrow S' = S \cup \{x_{n+1}\}$ SLD.

Fie. $a_1 x_1 + \ldots + a_n x_n + 0 \cdot x_{n+1} = 0_V$

$\quad \overset{\shortparallel}{a_1 x_1 + .. + a_n x_n}$. $a_1, .., a_n$ nu toți nuli.

c) $\forall$ supramulțime a unui SG este un SG.

$S = \{x_1, .., x_n\}$, $S' = S \cup \{x_{n+1}\}$

$V = \langle S \rangle$, $x = a_1 x_1 + .. + a_n x_n = a_1 x_1 + .. + a_n x_n + 0_{\mathbb{K}} x_{n+1}$

$V \subset \langle S' \rangle$
dar $\langle S' \rangle \subset V$ (din def) $\Big| \Rightarrow V = \langle S' \rangle$. $\hat{\langle S' \rangle}$.

d) $0_V \in S \Rightarrow S$ nu e SLI

<u>Teorema schimbului</u> $(V, +, \cdot)/\mathbb{K}$ sp vect finit generat.

Fie $\{x_1, .., x_n\}$ SG
$\{y_1, .., y_n\}$ SLI $\Big| \Rightarrow \{y_1, .., y_n\}$ SG.

<u>Dem</u> $V = \langle \{x_1, .., x_n\} \rangle \Rightarrow y_1 = a_1 x_1 + \ldots + a_n x_n$.

$\underset{y_1}{\overset{\Psi}{}}$ Pp abs $a_1 = .. = a_n = 0_{\mathbb{K}} \Rightarrow y_1 = 0_V \Rightarrow$

$\{0, y_2, .., y_n\}$ SLD ↯

$\exists\, a_1 \neq 0_{\mathbb{K}}$ (eventual renumerotăm) $\Rightarrow$

$a_1 x_1 = y_1 - a_2 x_2 - .. - a_n x_n \Rightarrow x_1 = a_1^{-1}(y_1 - a_2 x_2 - .. - a_n x_n)$

$V = \langle \{x_1, \dots, x_n\} \rangle = \langle \{y_1, x_2, \dots, x_n\} \rangle \Rightarrow$

$\ni y_2$

$y_2 = b_1 y_1 + a_2 x_2 + \dots + a_n x_n$.

Pp. abs $a_2 = \dots = a_n = 0_{\mathbb{K}} \Rightarrow y_2 = b_1 y_1 \Rightarrow$

$b_1 y_1 - 1 \cdot y_2 + 0 \cdot y_3 + \dots + 0 y_n = 0_V \Rightarrow \{y_1, \dots, y_n\}$ SLD abs

$\exists\, a_2 \neq 0_{\mathbb{K}}$ (ev. renum.)

$a_2 x_2 = y_2 - b_1 y_1 - a_3 x_3 - \dots - a_n x_n$.

- $x_2 = a_2^{-1}(y_2 - b_1 y_1 - a_3 x_3 - \dots - a_n x_n)$

$V = \langle \{x_1, \dots, x_n\} \rangle = \langle \{y_1, x_2, \dots, x_n\} \rangle = \langle \{y_1, y_2, x_3, \dots x_n\} \rangle$

Repetăm raționamentul și după un nr finit de pași $\Rightarrow V = \langle \{y_1, \dots, y_n\} \rangle \Rightarrow \{y_1, \dots, y_n\}$ SG.

T $(V, +, \cdot)/\mathbb{K}$ sp v. f g.

Dacă B_1, B_2 baze, at $\operatorname{card} B_1 = \operatorname{card} B_2 = n = \dim_{\mathbb{K}} V$ (invariant).