Aplicații liniare

$f: V_1 \to V_2$ aplicație

liniară $\Leftrightarrow$ $f(ax+by) = af(x) + bf(y)$ $\forall\, x, y \in V_1$, $a, b \in K$

Fie $f: V_1 \to V_2$ ap. liniară

Nucleu / Defect: $\ker f = \{x \in V_1 \mid f(x) = 0_{V_2}\}$, $\dim \ker f = \text{defect}(f)$

$\ker f \subseteq V_1$ subsp. vectorial

Imaginea / Rangul: $\operatorname{Im} f = \{y \in V_2 \mid \exists x \in V_1 \text{ a.i. } f(x) = y\}$, $\dim \operatorname{Im} f = \text{rang}(f)$

$\operatorname{Im} f \subseteq V_2$ subsp. vect.

T. Dimensiunii / T. Rang - Defect

$(V_1, +, \cdot)_{/\mathbb{K}}$, $(V_2, +, \cdot)_{/\mathbb{K}}$ sp. vect.

$f: V_1 \to V_2$ ap. liniară

$$\boxed{\dim_{\mathbb{K}} V_1 = \dim_{\mathbb{K}} \mathrm{Im} f + \dim_{\mathbb{K}} \ker f}$$

$$\begin{cases} f \text{ injectivă} \iff \ker f = \{0_{V_1}\} \\ f \text{ surjectivă} \iff \dim \mathrm{Im} f = \dim V_2 \\ f \text{ bijectivă / izomorfism} \iff \begin{cases} \text{Var 1: } f \text{ inj} + f \text{ surj} \\ \text{Var 2: } \exists A^{-1} \end{cases} \end{cases}$$

I) Ora treate. Mat. de schimbare a reperului

$$R=\{e_1,\dots,e_n\} \xrightarrow[A]{} R'=\{e_1',\dots,e_n'\}$$

1. scriem vect. din R' în f de R

$$e_1' = a_1e_1 + b_1e_2 + \dots + z_1e_n$$

$$\vdots$$

$$e_n' = a_ne_1 + b_ne_2 + \dots + z_ne_n$$

2. scriem coef. pe coloane

$$A=\begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \\ \vdots & & \vdots \\ z_1 & \dots & z_n \end{pmatrix}$$

$$X = AX' \Rightarrow X' = A^{-1}X$$

coord din R

coord din R'

II) Ora asta: Mat. asociata aplicatiei liniare.

$$R=\{e_1,\dots,e_n\} \xrightarrow{f} R'=\{e_1',\dots,e_m'\}$$

1. scriem $f(e_i)$ in functie de R'

$$f(e_1)=a_1e_1'+b_1e_2'+\dots+z_1e_m'$$

$$\dots$$

$$f(e_n)=a_ne_1'+b_ne_2'+\dots+z_ne_m'$$

2. scriem coeficientii pe coloana

$$A=[f]_{RR'}=\begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \\ \vdots & & \vdots \\ z_1 & \dots & z_n \end{pmatrix}$$

$f: V_1 \to V_2$ ap. liniară

$$
\begin{tikzcd}
R_1 = \{e_1, \dots, e_n\} \arrow[r, "A"] \arrow[d, "C"] & R_2 = \{\bar{e}_1, \dots, \bar{e}_m\} \arrow[d, "D"] \\
R_1' = \{e_1', \dots, e_n'\} \arrow[r, "A'"] & R_2' = \{\bar{e}_1', \dots, \bar{e}_m'\}
\end{tikzcd}
$$

$A' : R_1' \to R_2'$

$A : R_1 \to R_2$

$C : R_1' \to R_1$

$D : R_2' \to R_2$

$\Rightarrow A' = D' \cdot A \cdot C$

! rangul mat. as. ap. liniare e invariant

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1) $f:\mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (\underbrace{x_1 + x_2 - x_3}_{y_1}, \underbrace{x_1 + x_2}_{y_2}, \underbrace{x_1 + x_2 + x_3}_{y_3})$, $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$

a) f ap. liniară

b) dim Ker f, dim Im f, câte un reper din fiecare.

a) $f(ax + by) = f(\overbrace{ax_1 + by_1}^{x_1'}, \overbrace{ax_2 + by_2}^{x_2'}, \overbrace{ax_3 + by_3}^{x_3'})$

$= (ax_1 + by_1 + ax_2 + by_2 - ax_3 - by_3, ax_1 + by_1 + ax_2 + by_2, ax_1 + by_1 + ax_2 + by_2 + ax_3 + by_3)$

$= (ax_1 + ax_2 - ax_3, ax_1 + ax_2, ax_1 + ax_2 + ax_3) + (by_1 + by_2 - by_3, by_1 + by_2, by_1 + by_2 + by_3)$

$= af(x) + bf(y) \to f$ ap. liniară

① $f:\mathbb{R}^3\to\mathbb{R}^3,\ f(x_1,x_2,x_3)=(\underbrace{x_1+x_2-x_3}_{f_1},\ \underbrace{x_1+x_2}_{f_2},\ \underbrace{x_1+x_2+x_3}_{f_3}),\quad (\mathbb{R}^3,+,\cdot)/\mathbb{R}$

a) $\ker f=\{x\in\mathbb{R}^3 \mid f(x)=0_{\mathbb{R}^3}\}$

$$\begin{cases} x_1+x_2-x_3=0 \\ x_1+x_2=0 \\ x_1+x_2+x_3=0 \end{cases} \qquad A=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

$\dim \ker f=\dim \mathbb{R}^3-\mathrm{rg}\,A=3-2=\underline{1}$

$\det A=0-0=0$

$$\left.\begin{array}{l} x_2=-x_1 \\ x_3=x_1+x_2=0 \end{array}\right| \Rightarrow \ker f=\{(x_1,-x_1,0)\mid x_1\in\mathbb{R}\}$$

$=\{x_1(1,-1,0)\mid x_1\in\mathbb{R}\}$

$=\langle\{(1,-1,0)\}\rangle$

$R_1=\{(1,-1,0)\}$ reper în $\ker f$

T Dimensiunii

$\dim \mathbb{R}^3 = \dim \ker f + \dim \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 3 - 1 = 2$

$\operatorname{Im} f = \{ y \in \mathbb{R}^3 \mid \exists x \in \mathbb{R}^3 \text{ a.î. } f(x) = y \}$

$$\begin{cases} x_1 + x_2 - x_3 = y_1 \\ x_1 + x_2 = y_2 \\ x_1 + x_2 + x_3 = y_3 \end{cases} \qquad A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & y_1 \\ 1 & 1 & 0 & y_2 \\ 1 & 1 & 1 & y_3 \end{array}\right)$$

(*) T. Kronecker - Capelli

Sistem $AX = B$ compatibil $\Leftrightarrow rg\, A = rg\, \overline{A}$

$rg\, A = 2 = rg\, \overline{A}$

$$\Delta_c = \begin{vmatrix} 1 & -1 & y_1 \\ 1 & 0 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = 0 \Rightarrow y_3 - 2y_2 + y_1 = 0$$

$\operatorname{Im} f = \{ y \in \mathbb{R}^3 \mid y_3 - 2y_2 + y_1 = 0 \}$

$(\mathbb{R}^3, +, \cdot)/\mathbb{R}$

① $f:\mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (\underbrace{x_1 + x_2 - x_3}_{y_1}, \underbrace{x_1 + x_2}_{y_2}, \underbrace{x_1 + x_2 + x_3}_{y_3})$;

$y_1 = 2y_2 - y_3$

$\operatorname{Im} f = \{(2y_2 - y_3, y_2, y_3) \mid y_2, y_3 \in \mathbb{R}\}$

$= \{(2y_2, y_2, 0) + (-y_3, 0, y_3) \mid y_2, y_3 \in \mathbb{R}\}$

$= \{y_2(2,1,0) + y_3(-1,0,1) \mid y_2, y_3 \in \mathbb{R}\}$

$= \langle \{(2,1,0), (-1,0,1)\} \rangle$

$R_2 = \{(2,1,0), (-1,0,1)\}$ reper în $\operatorname{Im} f$

=

(2) $f: \mathbb{R}_1[X] \to \mathbb{R}_2[X]$, $f(ax+b) = ax^2 + (a-2b)x + (a-b)$,

$R_0 = \{1, X\}$, $R' = \{X^2, X^2+X+1, X^2+X\}$ repere

a) $[f]_{R_0, R'}$

b) $\ker$, $\operatorname{Im} f$

a) $R_0 = \{1, X\} \xrightarrow{A} R' = \{X^2, X^2+X+1, X^2+X\}$

$f(1) = f(0 \cdot X + 1) = 2X - 1 = a \cdot X^2 + b(X^2+X+1) + c(X^2+X) \Rightarrow b = -1,\ c = 3,\ a = -2$

$f(X) = f(1 \cdot X + 0) = X^2 + X + 1 = aX^2 + b(X^2+X+1) + c(X^2+X) \Rightarrow a = 0,\ b = 1,\ c = 0$

$[f]_{R_0, R'} = \begin{pmatrix} -2 & 0 \\ -1 & 1 \\ 3 & 0 \end{pmatrix}$

(2) $f: \mathbb{R}_1[X] \to \mathbb{R}_2[X]$, $f(aX+b) = aX^2 + (a+2b)X + (a-b)$

$R_0 = \{1, X\}$, $R' = \{X^2, X^2+X+1, X^2+X\}$ repere

a)

$\operatorname{Ker} f = \{ x \in \mathbb{R}_1[X] \mid f(x) = 0_{\mathbb{R}_2[X]} \}$

$f(aX+b) = 0$

$\operatorname{Ker} f = \{ 0_{\mathbb{R}_1[X]} \}$

$\dim \operatorname{Ker} f = 0$

T.d. $\dim_{\mathbb{R}} \mathbb{R}_1[X] = \dim_{\mathbb{R}} \operatorname{Ker} f + \dim_{\mathbb{R}} \operatorname{Im} f \Rightarrow 2 = \dim_{\mathbb{R}} \operatorname{Im} f$

$\operatorname{Im} f = \{ y \in \mathbb{R}_2[X] \mid \exists x \in \mathbb{R}_1[X],\ f(x) = y \} \Rightarrow \begin{cases} a - b = y_1 \\ a + 2b = y_2 \\ a = y_3 \end{cases}$ $A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}$

$\operatorname{rg} A = \operatorname{rg} \bar{A} = 2 \Rightarrow \Delta_c = \begin{vmatrix} 1 & -1 & y_1 \\ 1 & 2 & y_2 \\ 1 & 0 & y_3 \end{vmatrix} = 0$

$\Rightarrow 3y_3 - y_2 - 2y_1 = 0$

$\operatorname{Im} f = \{ y \in \mathbb{R}_2[X] \mid 3y_3 - y_2 - 2y_$

$\mathbb{R}_2[X]$

T.d. $\dim_{\mathbb{R}} \mathbb{R}_1[X] = \dim_{\mathbb{R}} \ker f + \dim_{\mathbb{R}} \operatorname{Im} f \Rightarrow 2 = \dim_{\mathbb{R}} \operatorname{Im} f$

$\operatorname{Im} f = \{ y \in \mathbb{R}_2[X] \mid \exists x \in \mathbb{R}_1[X],\ f(x) = y \} \Rightarrow \begin{cases} a - b = y_1 \\ a + 2b = y_2 \\ a = y_3 \end{cases} \quad A = \begin{pmatrix} 1 & -1 \\ \boxed{\begin{matrix} 1 & 2 \\ 1 & 0 \end{matrix}} \end{pmatrix} \Bigg| \begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix}$

$\operatorname{rg} A = \operatorname{rg} \bar{A} = 2 \Rightarrow \Delta_c = \begin{vmatrix} 1 & -1 & y_1 \\ 1 & 2 & y_2 \\ 1 & 0 & y_3 \end{vmatrix} = 0$

$\Rightarrow 3y_3 - y_2 - 2y_1 = 0$

$\operatorname{Im} f = \{ y \in \mathbb{R}_2[X] \mid 3y_3 - y_2 - 2y_1 = 0 \}$

$:=2$

2) $f: \mathbb{R}_1[X] \to \mathbb{R}_2[X]$, $f(aX+b) = aX^2 + (a-2b)X + (a-b)$,
$R_0 = \{1, X\}$, $R' = \{X^2, X^2+X+1, X^2+X\}$ repere

$y_2 = -2y_1 + 3y_3$

$\operatorname{Im} f = \{(y_1, -2y_1+3y_3, y_3) \mid y_1, y_3 \in \mathbb{R}\} = \{y_1(1,-2,0) + y_3(0,3,1) \mid y_1, y_3 \in \mathbb{R}\} = \langle\{(1,-2,0),(0,3,1)\}\rangle$

$R_2 = \{(1,-2,0),(0,3,1)\}$ în $\operatorname{Im} f$

$\operatorname{Ker} f = \{0_{\mathbb{R}_1[X]}\} \Rightarrow f$ injectivă

$\dim \operatorname{Im} f = 2 \neq \dim \mathbb{R}_2[X] \Rightarrow f$ nu e surjectivă

⑤ $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (2x_1 + 2x_2, x_1 - x_3, x_1 + 3x_2 - 2x_3)$

f nu e izomorfism

alegem R_0 repr canonic afăm $[f]_{R_0, R_0} \overset{not}{=} A$

$$R_0 = \{e_1, e_2, e_3\} \xrightarrow{A} R_0 = \{e_1, e_2, e_3\}$$

$$f(e_1) = f(1,0,0) = (2,1,1) = a e_1 + b e_2 + c e_3 \Rightarrow a = 2, b = 1, c = 1$$

$$f(e_2) = f(0,1,0) = (2,0,3) \Rightarrow a = 2, b = 0, c = 3$$

$$f(e_3) = f(0,0,1) = (0,1,-2) \Rightarrow a = 0, b = 1, c = -2$$

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix}$$

$$\det A = 2 - 2 = 0 \Rightarrow \nexists A^{-1} \Rightarrow f \text{ nu e izomorfism}$$