$\triangle\ L_i = \{a^i b^i\}, i \geq 1.$

For any $i \geq 1$, is L_i a DFA language?

$\triangle\ K_j = \{a^i b^i : 0 \leq i \leq j\}, j \geq 1.$

For any $j \geq 1$, is K_j a DFA language ?

<u>Nondeterministic Finite Automata</u> (NFA)

$M = (Q, \Sigma, \delta, s, F)$

same as a DFA except
$\delta \subseteq Q \times \Sigma \times Q$**.**

δ **is a finite <u>transition relation</u>.**

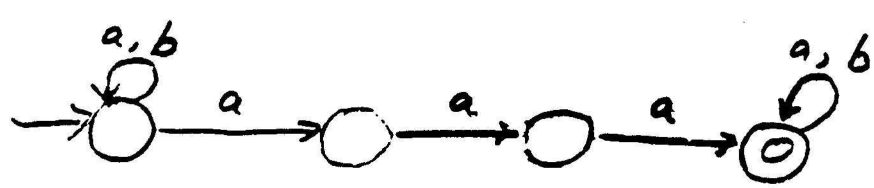
In a DFA
δ **is a transition function:**
$\delta : Q \times \Sigma \to Q$

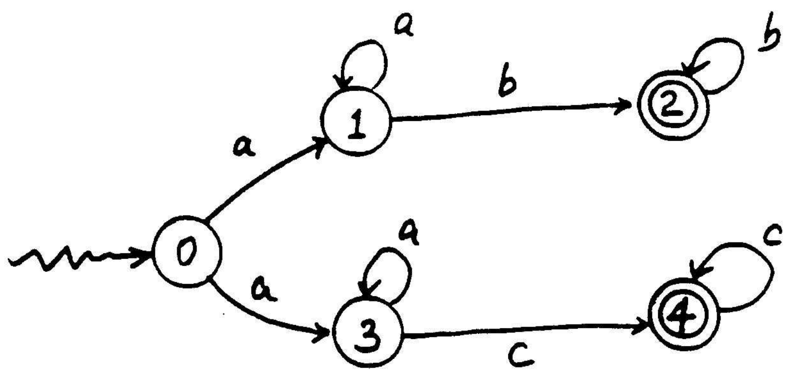
It can be viewed as a relation
$\delta : Q \times \Sigma \times Q$

In a NFA, δ **can be be viewed as a function:**
$\delta : Q \times \Sigma \to 2^Q$

<u>Examples</u>:

NFA for words in $\{a, b\}^*$ **that contain three consecutive a's.**





Both $(0, a, 1)$ **and** $(0, a, 3)$ **are in** δ**.**

We define acceptance by existence of a computation that leads to a final state.

Conversely, we define rejection by the nonexistence of any computation that leads to a final state.

The language of an NFA $M = (Q, \Sigma, \delta, s, F)$ **is defined by**

$L(M) = \{x \mid sx \vdash^* f$**, for some** f **in** F $\}$**.**

The <u>family of NFA languages</u> $\mathcal{L}_{NFA}$

is defined by:

$\mathcal{L}_{NFA} = \{L \mid L = L(M)$, for some NFA M $\}$.

Two NFAs M_1, and M_2 are <u>equivalent</u> if $L(M_1) = L(M_2)$.

<u>Why NFA</u>?

(i) easy to construct;
(ii) useful theoretically;
(iii) are of same power as DFA.

Note:

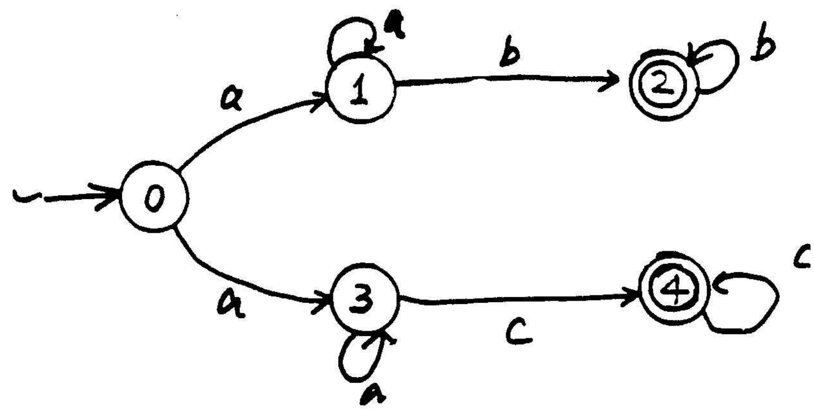
configurations are defined in the same way
Transition (move)

$$px \vdash qy$$

if $x = ay$, for some $a \in \Sigma$, and $(p, a, q) \in \delta$.

Transforming NFA to DFA

Consider the NFA M_1 again



There are only limited number of choices. For example:

$0\underline{a}ab \vdash 1ab \vdash 1b \vdash 2$

$0aab \vdash 3ab \vdash 3b$

$\{0\}aab \vdash \{1,3\}ab \vdash \{1,3\}b \vdash \{2\}$

Why <u>limited</u> number of choices?

The state set is finite.

We summarize the choices at each step by combining all configuration sequences into one "<u>super-conf. sequence</u>".

$\{0\}aab \vdash \{1,3\}ab \vdash \{1,3\}b \vdash \{2\}$**.**

We now have a set of all possible states at each step. From this point of view the computation of the NFA on an input word is <u>deterministic</u>.

A <u>super-configuration</u> has the form

$$Kx$$

where $K \subseteq Q$ and $x \in \Sigma^*$.

Note that $\underline{\emptyset x}$ is a super-conf., it means that the NFA cannot be in any state at that point, i.e., an abort has occured.

We say that

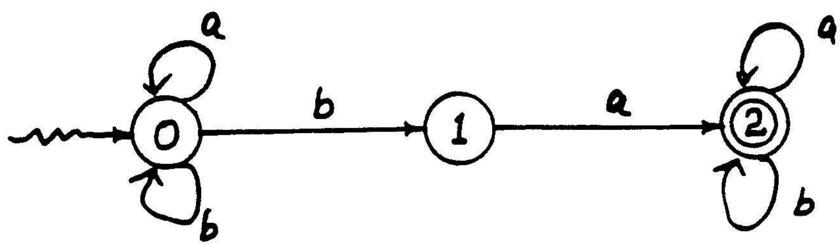
$$Kx \vdash Ny$$

if $x = ay$, for some $a \in \Sigma$, and

$N = \{q \mid (p, a, q) \in \delta$, **for some** $p \in K\}$

More examples on super-configurations

M: $L(M)$ is the set of all words that have "ba" as a subword.



The super-configuration sequence on input word "$abbaa$" is as follows:

$\{0\}abbaa \vdash \{0\}bbaa \vdash \{0,1\}baa \vdash \{0,1\}aa$
$\vdash \{0,2\}a \vdash \{0,2\}$

Notice that given a set $K \subseteq Q$ and an input symbol $a \in \Sigma$, the set $N \subseteq Q$ s.t. $Ka \vdash N$ is uniquely determined.

<u>Lemma</u> (2.3.1) (Determinism Lemma) Let $M = (Q, \Sigma, \delta, s, F)$ be an NFA. Then for all words $\underline{x}$ in Σ^* and for all $\underline{K} \subseteq Q$.

$\underline{Kx \vdash^* N \text{ and } Kx \vdash^* P}$

implies

$\underline{P = N}$.

<u>Lemma</u> (2.3.2) Let $M = (Q, \Sigma, \delta, s, F)$ be an NFA. Then for all words $\underline{x}$ in Σ^* and for all $\underline{q}$ in Q,

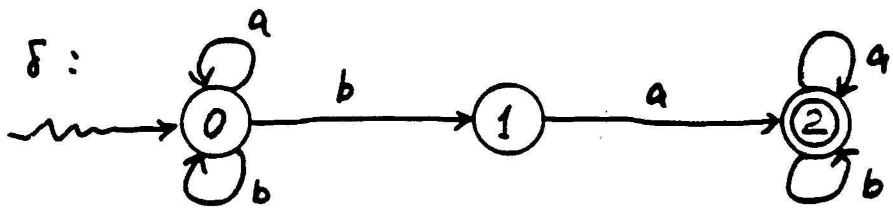
$\underline{qx \vdash^* p}$

iff $\underline{\{q\}x \vdash^* P}$, for some P with p in P.

Example (Transformation of an NFA to a DFA)

$M = (Q, \Sigma, \delta, s, F)$ **where**

$Q = 0, 1, 2, \qquad \Sigma = a, b$
$s = 0, \qquad\qquad F = \{2\}$



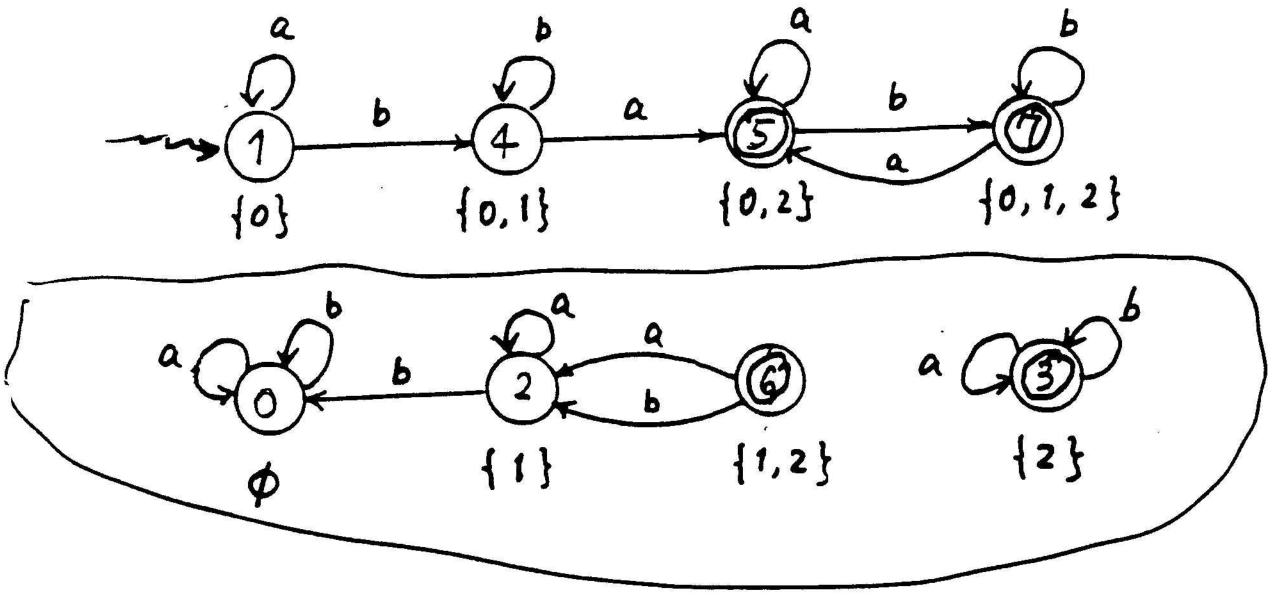
$M' = (Q', \Sigma, \delta', s', F')$ **where**

$$Q' = 2^Q = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$$

δ':

	current state \ input symbol	a	b
0	ϕ	ϕ	ϕ
1	$\{0\}$	$\{0\}$	$\{0,1\}$
2	$\{1\}$	$\{2\}$	ϕ
3	$\{2\}$	$\{2\}$	$\{2\}$
4	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
5	$\{0,2\}$	$\{0,2\}$	$\{0,1,2\}$
6	$\{1,2\}$	$\{2\}$	$\{2\}$
7	$\{0,1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\underline{\delta'(P, a) = \{q \mid (p, a, q) \in \delta \text{ and } p \in P\}}$



$s' = \{0\}$
$F' = \{$

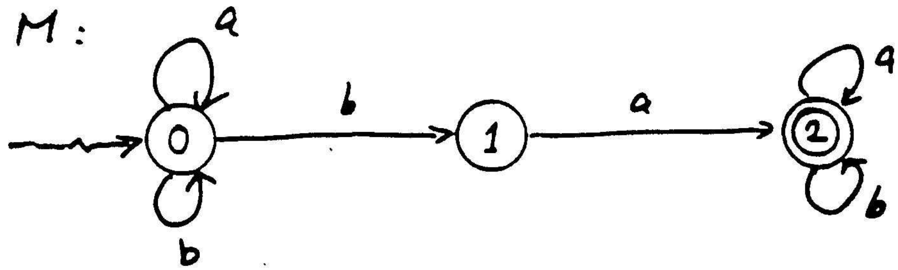
Algorithm NFA to DFA
—The Subset Construction

On entry: An NFA $M = (Q, \Sigma, \delta, s, F)$.

On exit: A DFA $M' = (Q', \Sigma, \delta', s', F')$
satisfying $L(M) = L(M')$**.**

begin Let $Q' = 2^Q, s' = \{s\}$ **and**
$F' = \{K \mid K \in Q',$ **and** $K \cap F \neq \emptyset\}$
We define $\delta' : Q' \times \Sigma \to Q'$ **by**
For all $K \in Q'$ **and for all** $a \in \Sigma$**,**
$\delta'(K, a) = N$**, if** $Ka \vdash N$ **in** M**.**
end of Algorithm

if $N = \{q \mid (p, a, q) \in \delta$ **and** $p \in K\}$

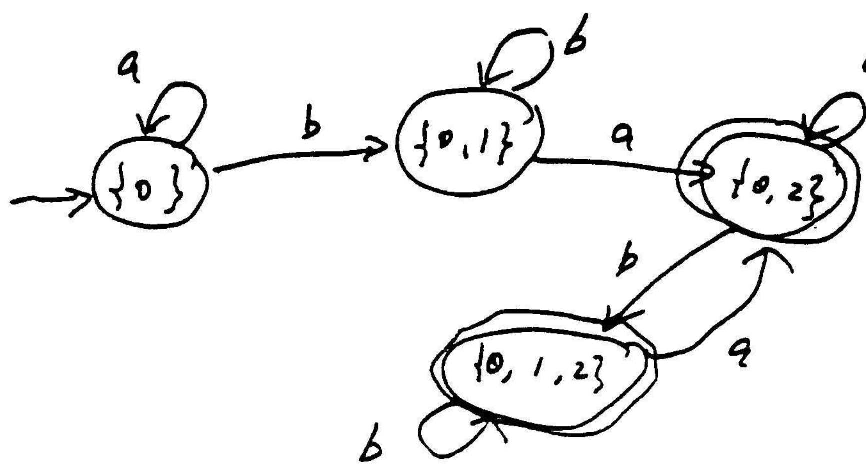


$s' = \{0\}$

current state \ input symbol	a	b
$\{0\}$	$\{0\}$	$\{0,1\}$
$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
$\{0,2\}$	$\{0,2\}$	$\{0,1,2\}$
$\{0,1,2\}$	$\{0,2\}$	$\{0,1,2\}$

	a	b
0	$\{0\}$	$\{0,1\}$
1	$\{2\}$	$\emptyset$
2	$\{2\}$	$\{2\}$

Algorithm NFA to DFA 2
—<u>The Iterative Subset Construction</u>



Theorem Given an NFA $M = (Q, \Sigma, \delta, s, F)$, then the DFA $M' = (Q', \Sigma', \delta', s', F')$ obtained by either subset construction satisfies $L(M') = L(M)$.

Proof:

By Lemma 2.3.2, for all $x \in \Sigma^*$ in M

$$sx \vdash^* p, \text{ iff } \{s\}x \vdash^* P \text{ for some } P \text{ with } p \in P$$

By the construction of M',
$\{s\}x \vdash^* P$ in M iff
$\{s\}x \vdash^* P$ in M'.

$$\begin{aligned} x \in L(M) &\Leftrightarrow sx \vdash^* f, \text{ for some } f \in F \\ &\Leftrightarrow \{s\}x \vdash^* P,\ f \in P, \text{ in } M \\ &\Leftrightarrow \{s\}x \vdash^* P, \text{ in } M' \text{ and } P \cap F \neq \emptyset \\ &\Leftrightarrow s'x \vdash^* P,\ P \in F \\ &\Leftrightarrow x \in L(M') \end{aligned}$$

Theorem

Every NFA Language is a DFA language and conversely.

$(\mathcal{L}_{NFA} = \mathcal{L}_{DFA})$

Example

Every finite language is accepted by a DFA.

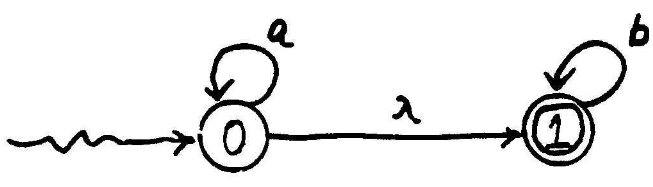
λ–NFA

It is useful to loosen the definition of NFA even more by allowing the read head to remain over the same symbol of the input and read nothing.

Example

$L_1 = \{a^i b^j \mid i \geq 0,\ j \geq 0\}$

M:



$$L(M) = L_1$$

$$\begin{array}{rl} \delta: & (0, a, 0) \\ & \underline{(0, \lambda, 1)} \quad \lambda - transition \\ & (1, b, 1) \end{array}$$

$$\begin{array}{rl} 0aab & \vdash\ 0ab\ \vdash\ 0b\ \vdash\ 1b\ \vdash\ 1 \\ 0bb & \vdash\ 1bb\ \vdash\ 1b\ \vdash\ 1 \\ 0a & \vdash\ 0\ \vdash\ 1 \end{array}$$

Formally, a λ-NFA $M = (Q, \Sigma, \delta, s, F)$ where $Q,\ \Sigma,\ s,\ F$ are as before, but δ is a finite transition relation for which

$$\delta \subseteq Q \times (\Sigma \cup \{\lambda\}) \times Q$$

Configurations are as before.
$\underline{\vdash}$ is defined by

$$px \vdash qy$$

if either $\underline{x = ay}$ for $a \in \Sigma$ and $(p, a, q) \in \delta$
or $\underline{x = y}$ and $(p, \lambda, q) \in \delta$

<u>Example</u>

Given FA M_1 and M_2, construct a FA M_3 such that
$L(M_3) = L(M_1) \cup L(M_2)$

Transforming λ-NFA to NFA

Two steps:
Step I: $\lambda-$ **completion**

Step II: $\lambda-$ **transition removal**

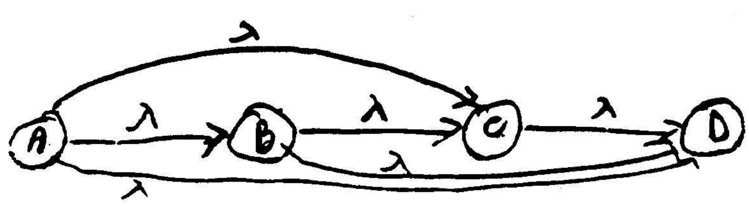
(I). **λ-Completion**

Given a λ-NFA $M = (Q, \Sigma, \delta, s, F)$
perform the following process:

For all $p, q, r \in Q$**:**
  whenever $(p, \lambda, q), (q, \lambda, r)$ **are in** δ
    add (p, λ, r) **to** δ
until no new transitions are added to δ
and let this be $\underline{\delta'}$**.**

Let the new $\lambda-$NFA be
$M' = (Q, \Sigma, \delta', s, F')$
where $F' = F \cup \{p \mid (p, \lambda, f) \in \delta$ **and** $f \in F\}$
and $\delta' = \delta \cup \{(p, \lambda, q) \mid p \vdash^{+} q\}$

Example:



Claim 1: **For any** $p,\ q \in Q$,

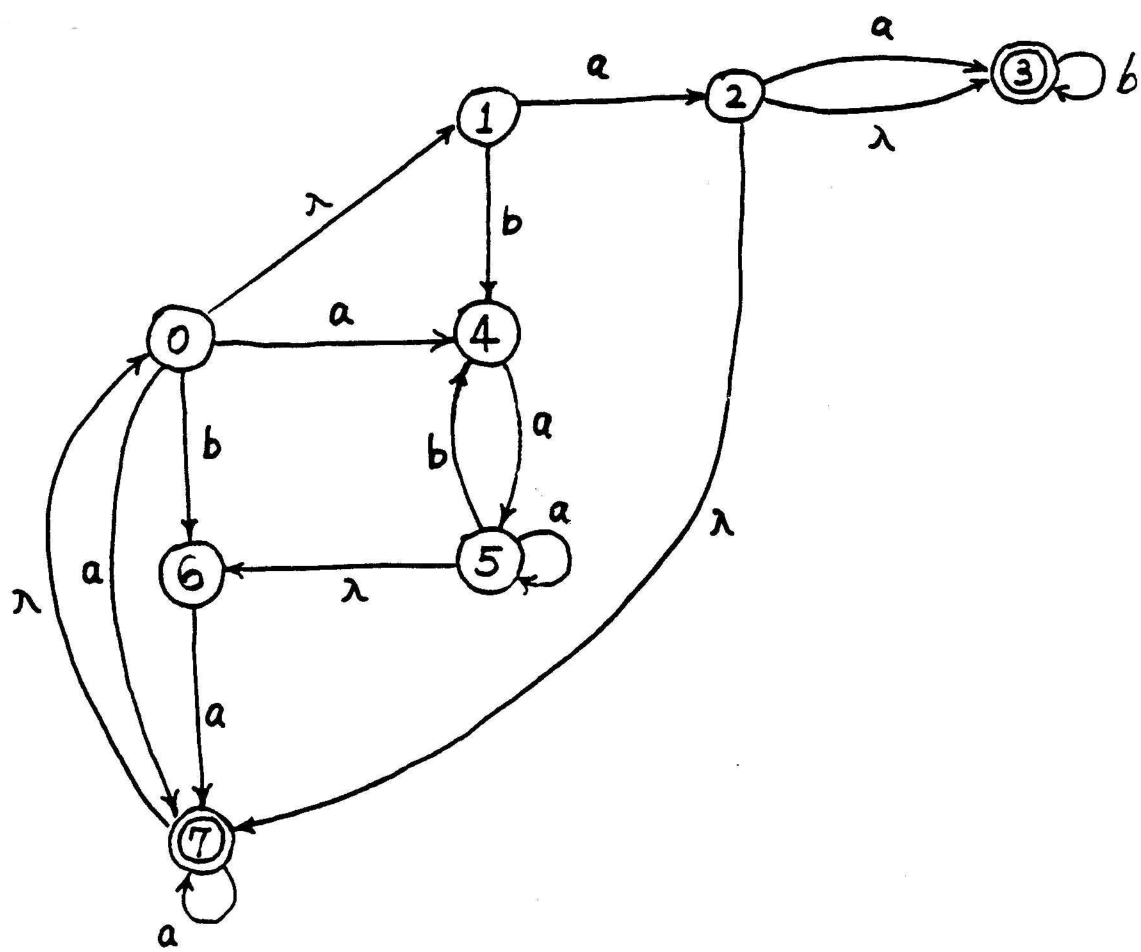
$$p \ \vdash_M^+ \ q \text{ if and only if } p \ \vdash_{M'} \ q$$

Claim 2: **For any** $p,\ q \in Q,\ x \in \Sigma^*$,

$$px \ \vdash_M^* \ q \text{ if and only if } px \ \vdash_{M'}^* \ q$$

Theorem: $L(M') = L(M)$

Example:



(II) λ–Transition Removal

Given a λ-completed λ-NFA

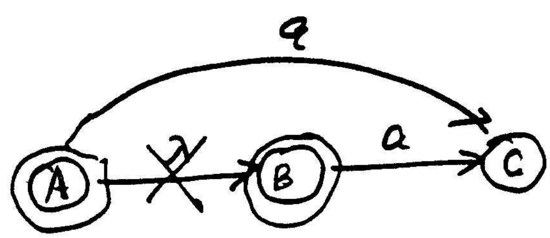
$$M = (Q,\ \Sigma,\ \delta,\ s,\ F),$$

perform the following process:

$$\begin{array}{rl} (0) & \delta' = \delta; \\ (i) & \textbf{For all } p, q, r \in Q, \\ & \qquad \textbf{if } \underline{(p, \lambda, q)} \textbf{ and } \underline{(q, a, r)} \textbf{ in } \delta \\ & \qquad \textbf{then } \underline{\textbf{add}}\ \underline{(p, a, r)} \textbf{ to } \delta'; \\ (ii) & \underline{\textbf{Delete}} \textbf{ all } \underline{\lambda\textbf{-transitions}} \textbf{ from } \delta'. \end{array}$$

Now we got $M' = (Q,\ \Sigma,\ \delta',\ s,\ F)$
where $\delta' = (\delta \cup \{(p, a, r) \mid (p, \lambda, q), (q, a, r) \in \delta\})$
$-\{(p, \lambda, q) \mid p, q \in Q\}$

Example



<u>Claim</u> Whenever

$$sx \ \vdash_M^* \ f$$

for some $f \in F$, we have

$$sx \ \vdash_{M'}^* \ f$$

and vice versa.

<u>Claim</u> $L(M') = L(M)$

<u>Theorem</u>

$$\mathcal{L}_{\lambda-NFA} = \mathcal{L}_{NFA}$$