20.05.2021

SEMINAR 13

Drepte și plane în spațiu
Conice studiate pe ecuații reduse

Ex 1 $(\mathcal{E}_3, (E_3, \langle \cdot,\cdot \rangle), \varphi)$ sp. afin euclidian canonic

Fie dreptele:

$$\mathcal{D}_1 : \begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 - 1 = 0 \end{cases} \qquad \mathcal{D}_2 : \begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases}$$

a) Să se afle ec perpendicularei comune a dreptelor $\mathcal{D}_1, \mathcal{D}_2$

b) Să se determine dist$(\mathcal{D}_1, \mathcal{D}_2)$

Sol

a) $$\mathcal{D}_1 \begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 - 1 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -t \\ x_2 = t+1 \\ x_3 = t, \ t \in \mathbb{R} \end{cases}$$

$$\mathcal{D}_1 : \frac{x_1}{-1} = \frac{x_2 - 1}{1} = \frac{x_3}{1} = t, \quad u_1 = (-1, 1, 1)$$
$A_1(0, 1, 0)$

$$\mathcal{D}_2 : \begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = s \\ x_2 = 0 \\ x_3 = 0, \ s \in \mathbb{R} \end{cases}$$

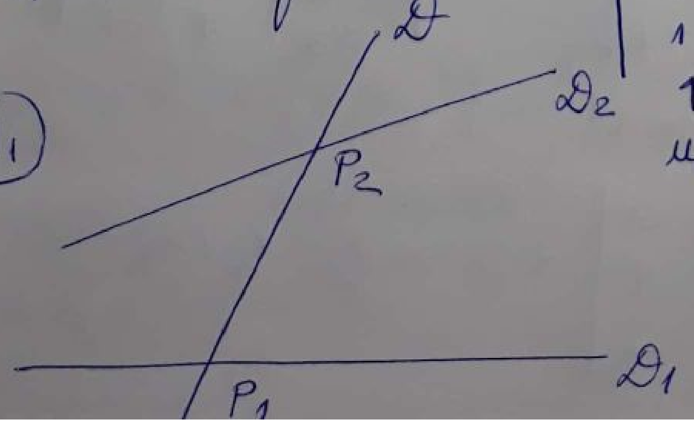
$$\mathcal{D}_2 : \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0} = s, \quad u_2 = (1, 0, 0)$$
$A_2(0, 0, 0)$

$\overrightarrow{A_1A_2} = (0, -1, 0)$

$\mathcal{D}_1, \mathcal{D}_2$ necoplanare $\iff \begin{vmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{vmatrix} \neq 0$

$\uparrow u_1 \quad \uparrow u_2 \quad \uparrow \overrightarrow{A_1A_2}$

M1



$P_1(-t,\ t+1,\ t) \in \mathcal{D}_1 \cap \mathcal{D}$

$P_2(s, 0, 0) \in \mathcal{D}_2 \cap \mathcal{D}$

$\overrightarrow{P_1P_2} = (s+t,\ -t-1,\ -t)$, $u_1 = (-1, 1, 1)$, $u_2 = (1, 0, 0)$

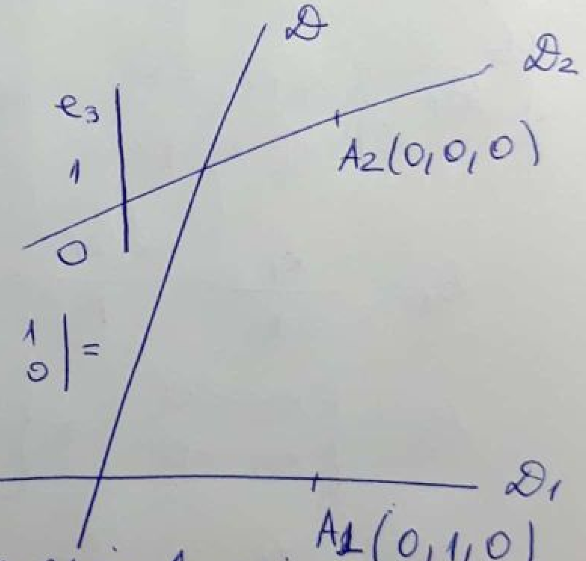
$$\begin{cases} \langle \overrightarrow{P_1P_2}, u_1 \rangle = 0 \\ \langle \overrightarrow{P_1P_2}, u_2 \rangle = 0 \end{cases} \Rightarrow \begin{cases} -s-t-t-1-t = 0 \\ s+t = 0 \end{cases} \Rightarrow \begin{cases} t = -\frac{1}{2} \\ s = \frac{1}{2} \end{cases}$$

$P_1\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$, $P_2\left(\frac{1}{2}, 0, 0\right)$, $\overrightarrow{P_1P_2} = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$

$= \frac{1}{2}\underbrace{(0, -1, 1)}$

$$\mathcal{D}: \frac{x_1 - \frac{1}{2}}{0} = \frac{x_2 - 0}{-1} = \frac{x_3 - 0}{1}$$

$\operatorname{dist}(\mathcal{D}_1, \mathcal{D}_2) = \operatorname{dist}(P_1, P_2) = \|\overrightarrow{P_1P_2}\| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{2}}{2}$.



(M2) $N = u_1 \times u_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$

$= e_1 \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} - e_2 \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} + e_3 \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} =$

$= (0, 1, -1)$

Fie π_k planul care trece prin A_k și are vectorii directori u_k, N, $k = \overline{1,2}$.

$$\pi_1: \begin{vmatrix} x_1 - 0 & -1 & 0 \\ x_2 - 1 & 1 & 1 \\ x_3 - 0 & 1 & -1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x_1 & -1 & 0 \\ x_2 + x_3 - 1 & 2 & 0 \\ x_3 & 1 & -1 \end{vmatrix} = 0$$

$\pi_1: 2x_1 + x_2 + x_3 - 1 = 0$

$$\pi_2: \begin{vmatrix} x_1 - 0 & 1 & 0 \\ x_2 - 0 & 0 & 1 \\ x_3 - 0 & 0 & -1 \end{vmatrix} = 0 \Rightarrow \pi_2: x_2 + x_3 = 0.$$

$\mathcal{D} = \pi_1 \cap \pi_2 : \begin{cases} 2x_1 + x_2 + x_3 - 1 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2} \\ x_2 = -t \\ x_3 = t, \ t \in \mathbb{R}. \end{cases}$

$\mathcal{D}: \dfrac{x_1 - \frac{1}{2}}{0} = \dfrac{x_2}{-1} = \dfrac{x_3}{1} = t$

$Dist(\mathcal{D}_1, \mathcal{D}_2) = \dfrac{|\langle N, \overrightarrow{A_1A_2} \rangle|}{\|N\|} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$N = (0, 1, -1)$

$\overrightarrow{A_1A_2} = (0, -1, 0)$

Ex2 Fie dreptele $\mathcal{D}_1: \dfrac{x_1 - 1}{1} = \dfrac{x_2 - 2}{-1} = \dfrac{x_3 + 2}{2}$

$\mathcal{D}_2: \begin{cases} 2x_1 - x_3 - 1 = 0 \\ 2x_2 + x_3 + 3 = 0 \end{cases}$

a) Să se arate că $\mathcal{D}_1, \mathcal{D}_2$ necoplanare.

~~Să se scrie ec. planului det. de $\mathcal{D}_1, \mathcal{D}_2$~~

b) $Dist(\mathcal{D}_1, \mathcal{D}_2)$

Sol

a) $\mathcal{D}_1: \begin{cases} x_1 = 1 + t \\ x_2 = 2 - t \\ x_3 = -2 + 2t, \ t \in \mathbb{R} \end{cases}$ $\quad u_1 = (1, -1, 2)$, $\quad A_1(1, 2, -2)$

$\mathcal{D}_2: \begin{cases} 2x_1 - x_3 = 1 \\ 2x_2 + x_3 = -3 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2} + \frac{1}{2}t \\ x_2 = -\frac{3}{2} - \frac{1}{2}t \\ x_3 = t, \ t \in \mathbb{R} \end{cases}$

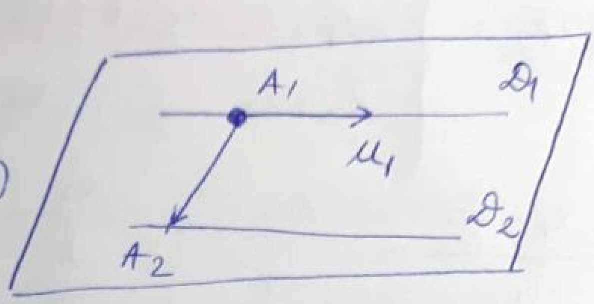
$\left(\frac{1}{2}, -\frac{1}{2}, 1\right) = \frac{1}{2}(\underbrace{1, -1, 2}_{u_2})$; $A_2(1, -2, 1)$

$V_{\mathcal{D}_1} = \langle \{u_1\} \rangle = \langle \{u_2\} \rangle = V_{\mathcal{D}_2} \Rightarrow \mathcal{D}_1 \parallel \mathcal{D}_2$

$\Rightarrow$ drepte coplanare.

$\overrightarrow{A_1A_2} = (0, -4, 3)$

$$\pi : \begin{vmatrix} x_1 - 1 & 0 & 1 \\ x_2 - 2 & -4 & -1 \\ x_3 + 2 & 3 & 2 \end{vmatrix} = 0$$



$\pi : (x_1 - 1)(-5) - (x_2 - 2)(-3) + (x_3 + 2) \cdot 4 = 0$

$\pi : 5x_1 - 3x_2 - 4x_3 - 7 = 0$

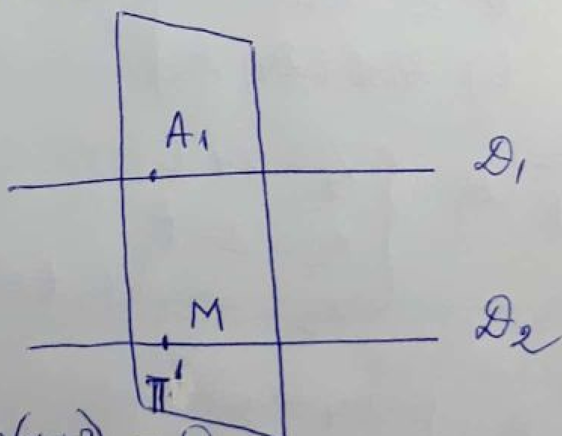
b) $\mathcal{D}ist(\mathcal{D}_1, \mathcal{D}_2) = \mathcal{D}ist(A_1, \mathcal{D}_2) = \dfrac{\| u_2 \times \overrightarrow{A_1A_2} \|}{\| u_2 \|} =$

$$u_2 \times \overrightarrow{A_1A_2} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & -1 & 2 \\ 0 & -4 & 3 \end{vmatrix} = (5, -3, -4)$$

$$\mathcal{D}ist(\mathcal{D}_1, \mathcal{D}_2) = \frac{\sqrt{25 + 9 + 16}}{\sqrt{1 + 1 + 4}} = \sqrt{\frac{50}{6}} = \frac{5\sqrt{3}}{3}$$

OBS

$\pi' \ni A_1, \ \pi' \perp \mathcal{D}_2$



$N_{\pi'} = u_2 = (1, -1, 2)$

$A_1(1, 2, -2)$

- $\pi' : 1(x_1 - 1) - (x_2 - 2) + 2(x_3 + 2) = 0$

$\pi' : x_1 - x_2 + 2x_3 + 5 = 0.$

$\mathcal{D}_2 \cap \pi' = \{M\} \Rightarrow \frac{1}{2} + \frac{1}{2}t + \frac{3}{2} + \frac{1}{2}t + 2t + 5 = 0$

$$\mathcal{D}_2 : \begin{cases} x_1 = \frac{1}{2} + \frac{1}{2}t \\ x_2 = -\frac{3}{2} - \frac{1}{2}t \\ x_3 = t, \ t \in \mathbb{R} \end{cases}$$

$6t + 14 = 0$

$t = -\frac{7}{3}$

$\Rightarrow M \Rightarrow dist(A_1, \mathcal{D}_2) = dist(A_1, M)$

Ex3 Fie $\mathcal{D}_1 : \frac{x_1-1}{2} = \frac{x_2-1}{-1} = \frac{x_3}{3}$

$\pi_1 : x_1 + x_2 + x_3 - 1 = 0$

$\pi_2 : x_1 - x_2 + x_3 = 0$, $M(1,2,-1)$.

a) Să se determine ec. dreptei $\mathcal{D}_2 = \pi_1 \cap \pi_2$

b) $\sphericalangle(\mathcal{D}_1, \mathcal{D}_2)$ ($\mathcal{D}_1, \mathcal{D}_2$ drepte orientate)

c) $\sphericalangle(\pi_1, \pi_2)$ (π_1, π_2 plane orientate)

d) Să se afle coord simetricului lui M față de π_1

SOL

a) $\pi_1 : x_1 + x_2 + x_3 - 1 = 0$ $\quad N_1 = (1,1,1)$

$\pi_2 : x_1 - x_2 + x_3 = 0$ $\quad N_2 = (1,-1,1)$

$\pi_1 \not\parallel \pi_2$ ($\nexists\, \alpha \in \mathbb{R}$ aî $N_2 = \alpha N_1$)

$$\mathcal{D}_2 : \begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 - x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 = 1 - t \\ x_1 - x_2 = -t \\ x_3 = t \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2} - t \\ x_2 = \frac{1}{2} \\ x_3 = t,\ t \in \mathbb{R} \end{cases}$$

$2x_1 = 1 - 2t \Rightarrow x_1 = \frac{1}{2} - t$

$x_2 = 1 - t - \frac{1}{2} + t = \frac{1}{2}$ $\quad u_2 = (-1, 0, 1)$

$$\mathcal{D}_2 : \frac{x_1 - \frac{1}{2}}{-1} = \frac{x_2 - \frac{1}{2}}{0} = \frac{x_3}{1} = t$$

b) $\sphericalangle(\mathcal{D}_1, \mathcal{D}_2) = \sphericalangle(u_1, u_2) = 4\varphi \in [0, \pi]$

$\mathcal{D}_1 : \frac{x_1-1}{2} = \frac{x_2-1}{-1} = \frac{x_3}{3} \Rightarrow u_1 = (2, -1, 3)$

$$\cos\varphi = \frac{\langle u_1, u_2 \rangle}{\|u_1\| \cdot \|u_2\|} = \frac{-2+0+3}{\sqrt{2} \cdot \sqrt{14}} = \frac{1}{2\sqrt{7}} = \frac{\sqrt{7}}{14}.$$

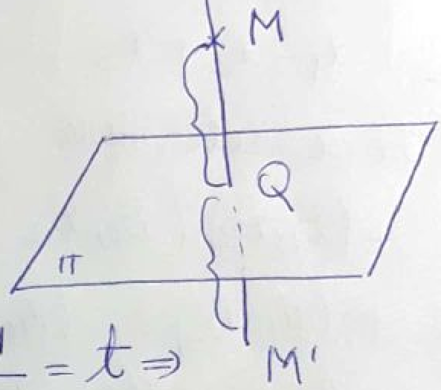
$\varphi = \arccos \frac{\sqrt{7}}{14}$.

c) $\angle(\pi_1, \pi_2) = \angle(N_1, N_2) = \angle \alpha$ $\qquad N_1 = (1,1,1)$, $N_2 = (1,-1,1)$

$$\cos\alpha = \frac{\langle N_1, N_2\rangle}{\|N_1\|\cdot\|N_2\|} = \frac{\cancel{1}-\cancel{1}+1}{\sqrt{3}\cdot\sqrt{3}} = \frac{1}{3}$$

$$\alpha = \arccos\left(\frac{1}{3}\right).$$

d) $MM' \perp \pi$



$u_{MM'} = N_1 = (1,1,1)$

$M(1,2,-1)$

$$MM': \frac{x_1-1}{1} = \frac{x_2-2}{1} = \frac{x_3+1}{1} = t \Rightarrow$$

$$\begin{cases} x_1 = 1+t \\ x_2 = 2+t \\ x_3 = -1+t, \ t\in\mathbb{R}. \end{cases}$$

$MM' \cap \pi = \{Q\}$ $\Rightarrow \cancel{1}+t+2+t-1+t = \cancel{1} \Rightarrow$

$\pi: x_1+x_2+x_3 = 1$

$\Rightarrow t = -\frac{1}{3} \Rightarrow Q\left(1-\frac{1}{3}, 2-\frac{1}{3}, -1-\frac{1}{3}\right)$

$Q\left(\frac{2}{3}, \frac{5}{3}, -\frac{4}{3}\right)$

$$Q \text{ mijl } [MM'] \Rightarrow \begin{cases} \frac{2}{3} = \frac{1}{2}(1+x_1') \\ \frac{5}{3} = \frac{1}{2}(2+x_2') \\ -\frac{4}{3} = \frac{1}{2}(-1+x_3') \end{cases}$$

$$\begin{cases} x_1' = \frac{4}{3} - 1 = \frac{1}{3} \\ x_2' = \frac{10}{3} - 2 = \frac{4}{3} \\ x_3' = -\frac{8}{3} + 1 = -\frac{5}{3} \end{cases} \Rightarrow M'\left(\frac{1}{3}, \frac{4}{3}, -\frac{5}{3}\right)$$

$$\overrightarrow{MQ} = \overrightarrow{QM'} \Rightarrow \left(\frac{2}{3}-1, \frac{5}{3}-2, -\frac{4}{3}+1\right) =$$
$$= \left(x_1' - \frac{2}{3}, x_2' - \frac{5}{3}, x_3' + \frac{4}{3}\right)$$

Conice studiate pe ecuații reduse

Ex1 $(\mathcal{E}_2, (E_2, \langle \cdot, \cdot \rangle), \varphi)$

Fie cercurile

$\mathcal{C}_1(O_1, R_1): x_1^2 + x_2^2 + 4x_1 + 6x_2 = 3$

$\mathcal{C}_2(O_2, R_2): x_1^2 + x_2^2 - 6x_1 + 6x_2 = -9$

a) Să se afle coord. centrelor O_1, O_2 și razele R_1, R_2

b) Să se determine $\mathcal{C}_1 \cap \mathcal{C}_2 = \{A, B\}$

Să se scrie ec. lui AB

SOL

a) $\mathcal{C}(A(a,b), R): (x_1 - a)^2 + (x_2 - b)^2 = R^2$

- $\mathcal{C}_1(O_1, R_1): x_1^2 + x_2^2 + 4x_1 + 6x_2 = 3$

$(x_1+2)^2 + (x_2+3)^2 = 3 + 4 + 9 = 16$

$O_1(-2, -3)$, $R_1 = 4$.

- $\mathcal{C}_2(O_2, R_2): x_1^2 + x_2^2 - 6x_1 + 6x_2 = -9$

$(x_1 - 3)^2 + (x_2 + 3)^2 = 9 + 9 - 9 = 9$

$O_2(3, -3)$, $R_2 = 3$.

b) $dist(O_1, O_2) = \sqrt{(3+2)^2 + 0} = 5 < R_1 + R_2 = 7$

$\Rightarrow$ cercurile sunt secante.

$$\mathcal{C}_1 \cap \mathcal{C}_2: \begin{cases} x_1^2 + x_2^2 + 4x_1 + 6x_2 = 3 \\ x_1^2 + x_2^2 - 6x_1 + 6x_2 = -9 \\ \hline 10x_1 \qquad = 12 \end{cases} \overset{\ominus}{\Rightarrow} x_1 = \frac{6}{5}$$

$x_2^2 + \frac{36}{25} + \frac{24}{5} + 6x_2 = 3 \Rightarrow x_2^2 + 6x_2 + \frac{36+120-75}{25} = 0$

$\Rightarrow x_{2_1} = -\frac{3}{5} \Rightarrow A\left(\frac{6}{5}, -\frac{3}{5}\right)$

$x_2 = -\frac{27}{5} \Rightarrow B\left(\frac{6}{5}, -\frac{27}{5}\right)$

$\frac{x_1 - \frac{6}{5}}{0} = \frac{x_2 + \frac{3}{5}}{-\frac{27}{5} + \frac{3}{5}}$

$$(x_2+3)^2 = 16 - \left(\frac{6}{5}+2\right)^2 = 16 - \left(\frac{16}{5}\right)^2 = 16\left(1-\frac{16}{25}\right) = \frac{1}{2}$$

$$= \frac{16}{25}$$

$x_2 + 3 = \frac{12}{5}$ sau $x_2 + 3 = -\frac{12}{5}$

$\Rightarrow x_2 = -\frac{3}{5}$ sau $x_2 = -\frac{27}{5}$

$AB: x_1 = \frac{6}{5}$

<u>Ex2</u> Fie $\mathcal{E}: \frac{x_1^2}{16} + \frac{x_2^2}{9} = 1$

a) Să se precizeze coord. vârfurilor, focarelor, excentricitatea, ec. directoarelor

b) Să se scrie ec tg în $A(4,0)$ la elipsă.

<u>SOL</u>

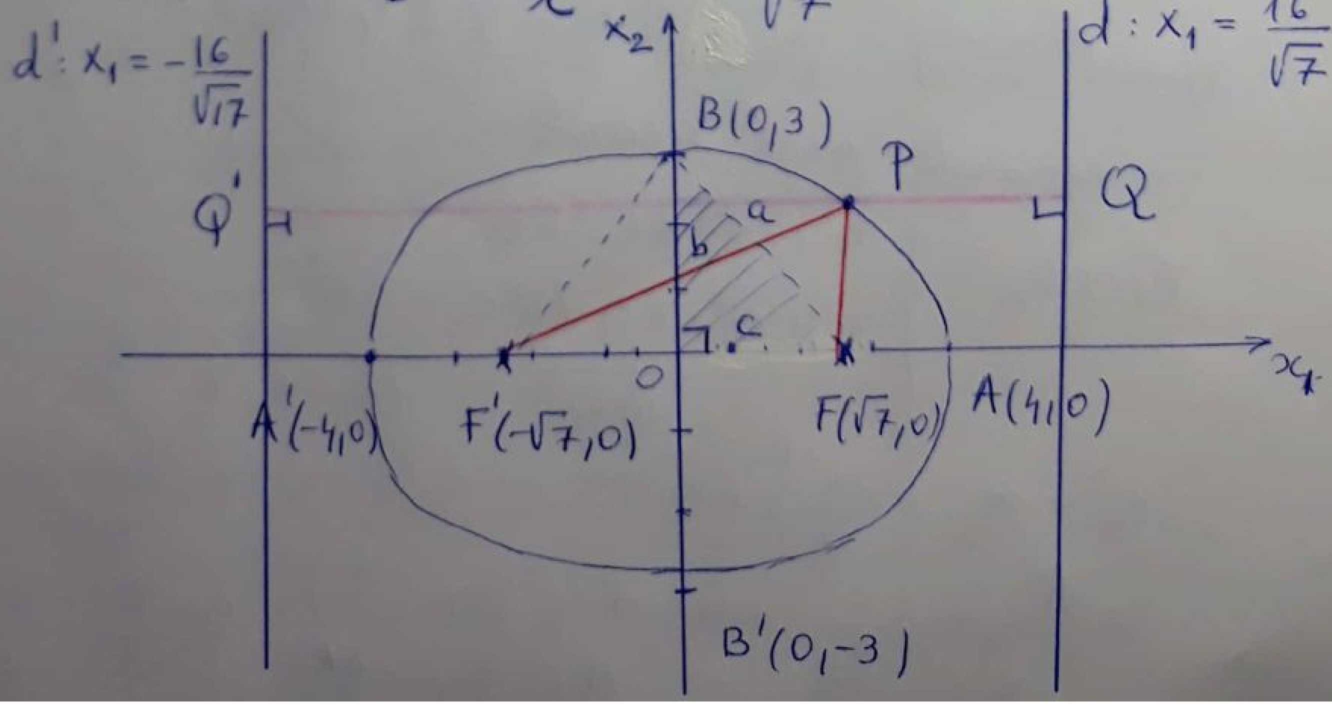
a) $\mathcal{E}: \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$, $a, b > 0$

$a^2 = b^2 + c^2 \Rightarrow c = \sqrt{a^2 - b^2}$

$a = 4$, $b = 3$, $c = \sqrt{16-9} = \sqrt{7}$

$e = \frac{c}{a} = \frac{\sqrt{7}}{4} < 1$

$d \cup d': x_1 = \pm \frac{a^2}{c} = \pm \frac{16}{\sqrt{7}}$



1) $PF + PF' = 2a = 8$ (suma razelor focale)
sau
2) $\frac{PF}{PQ} = \frac{PF'}{PQ'} = e = \frac{c}{a} = \frac{\sqrt{7}}{4}$

b) $\mathcal{E}: \frac{x_1 x_1}{16} + \frac{x_2 \cdot x_2}{9} = 1$

$A(4,0) \in \mathcal{E} \Rightarrow$ Ec. tg in A: $\frac{x_1 \cdot 4}{16} + \frac{x_2 \cdot 0}{9} = 1$

$\Rightarrow x_1 = 4$ (dedublare)

OBS

$M_0(x_1^0, x_2^0) \in \Gamma$ (conică)

tg in M_0 la Γ (procedeul de dedublare)

1) $\mathcal{E}: \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$

tg in M_0: $\frac{x_1 x_1^0}{a^2} + \frac{x_2 x_2^0}{b^2} = 1$

2) $\mathcal{H}: \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1$

tg in M_0: $\frac{x_1 x_1^0}{a^2} - \frac{x_2 x_2^0}{b^2} = 1$

3) $\mathcal{P}: x_2^2 = 2p x_1 = p(x_1 + x_1)$

tg in M_0: $x_2 x_2^0 = p(x_1 + x_1^0)$

Ex 3 a) Să se scrie ec. hiperbolei care trece prin $A(3,0)$ și are asimptotele $d_1 \cup d_2: x_2 = \pm 3x_1$

b) Precizați coord. vârfurilor, focarelor, excentricitatea și ec. directoarelor.

SOL $\mathcal{H}: \dfrac{x_1^2}{a^2} - \dfrac{x_2^2}{b^2} = 1, \; a, b > 0.$

$$d_1 \cup d_2 : \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 0 \Rightarrow x_2 = \pm \frac{b}{a} x_1 .$$

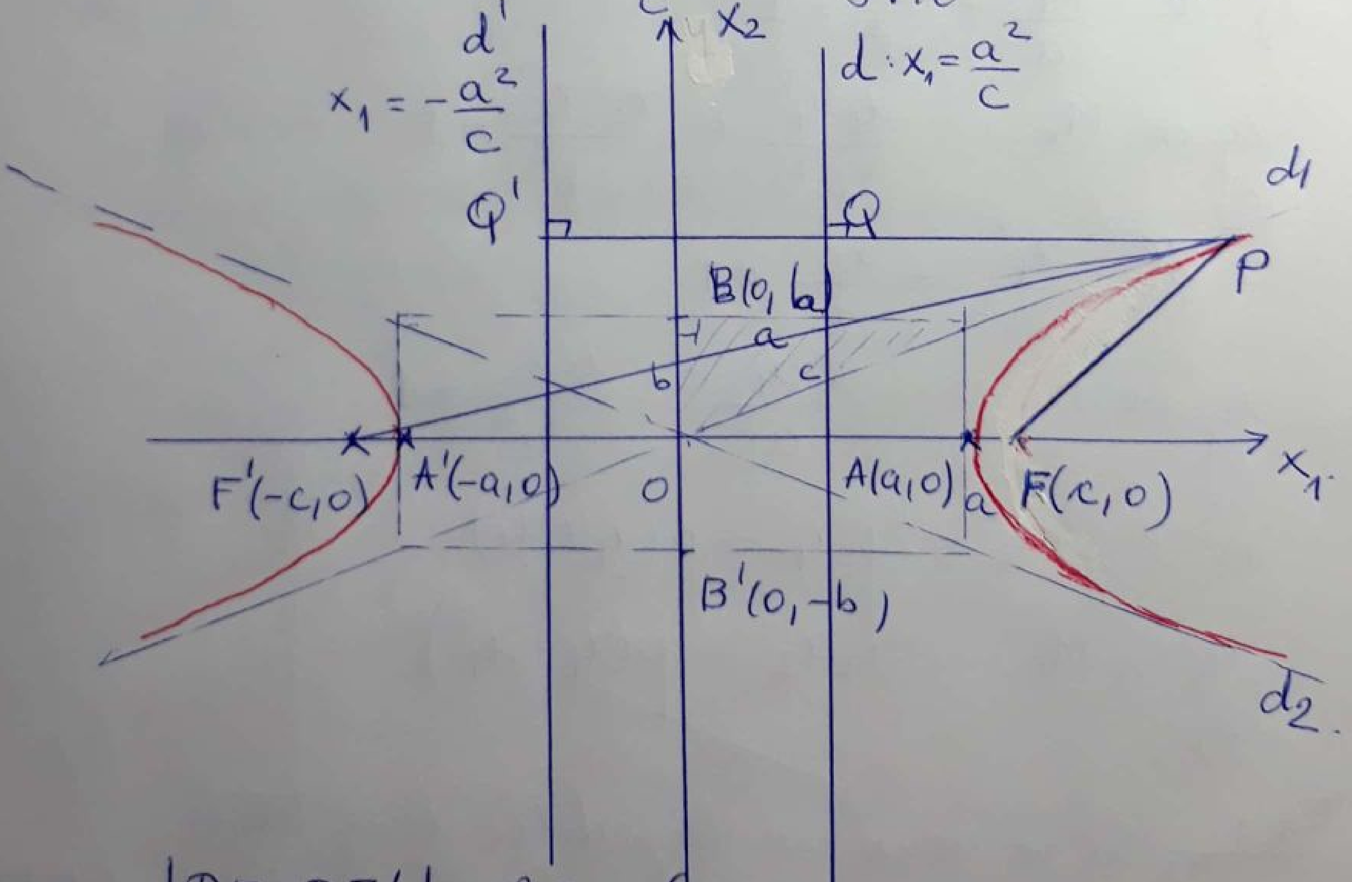
$$\left.\begin{aligned} &A(3,0) \in \mathcal{H} \Rightarrow \frac{9}{a^2} = 1 \Rightarrow a = 3 \\ &d_1 \cup d_2 : x_2 = \pm 3x_1 \Rightarrow \frac{b}{a} = 3 \end{aligned}\right\} \Rightarrow b = 9$$

$$\mathcal{H} : \frac{x_1^2}{9} - \frac{x_2^2}{81} = 1 .$$

$$c^2 = a^2 + b^2 = 9 + 81 = 90 \Rightarrow c = 3\sqrt{10} > a$$

$$e = \frac{c}{a} = \frac{\cancel{3}\sqrt{10}}{\cancel{3}} = \sqrt{10} > 1$$

$$d \cup d' : x_1 = \pm \frac{a^2}{c} = \pm \frac{9}{3\sqrt{10}} = \pm \frac{3\sqrt{10}}{10} .$$



$$|PF - PF'| = 2a = 6$$

$$\frac{PF}{PQ} = \frac{PF'}{PQ'} = e = \sqrt{10} .$$

Ex $\mathcal{P}: x_2^2 = 16x_1$, $\mathcal{D}: x_1 + x_2 - 5 = 0$

a) Să se afle coord focarului, ec. directoarei

b) $\mathcal{P} \cap \mathcal{D} = \{A, B\}$

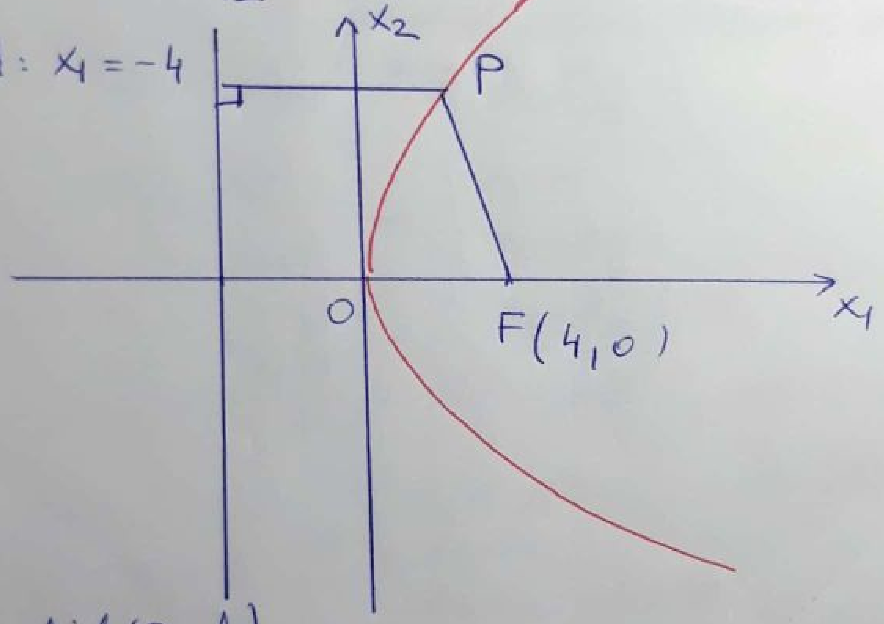
Ec tg în A la $\mathcal{P}$, unde $x_2^A > 0$

Sol

a) $\mathcal{P}: x_2^2 = 2px_1 \Rightarrow 2p = 16 = 8$

$F\left(\frac{p}{2}, 0\right) \Rightarrow F(4,0)$

$d: x_1 = -\frac{p}{2} \Rightarrow x_1 = -4$



$PF = dist(P, d)$.

b) $\begin{cases} x_2^2 = 16x_1 \Rightarrow x_2^2 = 16(5 - x_2) \Rightarrow x_2^2 + 16x_2 - 80 = 0 \\ x_1 + x_2 - 5 = 0 \Rightarrow x_1 = 5 - x_2 \end{cases}$

$(x_2 + 20)(x_2 - 4) = 0$

1) $x_2 = 4 \Rightarrow x_1 = 1 \Rightarrow A(1,4)$

2) $x_2 = -20 \Rightarrow x_1 = 25 \Rightarrow B(25, -20)$

Tg în $A(1,4)$ $\quad \mathcal{P}: x_2^2 = 16x_1$

$\underline{x_2} \cdot x_2 = 8(\underline{x_1} + x_1)$

$x_2 \cdot 4 = 8(x_1 + 1) \Rightarrow x_2 = 2x_1 + 2 \Rightarrow 2x_1 - x_2 + 2 = 0$