

Temă 2

$$1) (\mathbb{R}^4, +, \cdot) / \mathcal{R} : U = \{x \in \mathbb{R}^4 / x_2 - x_3 = 0; x_1 + x_4 = 0\}$$

a) REPER în U

b) $W \subseteq \mathbb{R}^4$ o.l. $\mathbb{R}^4 = U \oplus W$

c) $p: U \oplus W \rightarrow U \oplus W; p(0, 1, 2, -1) = ?$

Δ - SIMETRIC; $\Delta(0, 1, 2, -1) = ?$

$$a) U: \begin{cases} x_2 - x_3 = 0 \Rightarrow x_2 = x_3 \\ x_1 + x_4 = 0 \Rightarrow x_1 = -x_4 \end{cases} \Rightarrow U = \{(x_1, x_2, x_2, -x_1) / x_1, x_2 \in \mathbb{R}\} = \\ = \{x_1(1, 0, 0, -1) + x_2(0, 1, 1, 0) / x_1, x_2 \in \mathbb{R}\}$$

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \text{rang}(A) = 2 \Rightarrow \dim_{\mathbb{R}} U = 4 - 2 = 2$$

$$U = \mathcal{S}(A)$$

$$\mathcal{R} = \{(1, 0, 0, -1); (0, 1, 1, 0)\} \text{ SG. PT. } U \} \Rightarrow \mathcal{R} \text{ REPER în } U$$

$$|\mathcal{R}| = \dim_{\mathbb{R}} U = 2 \Rightarrow \mathcal{R} = \mathcal{S}U$$

b) EXTINDEM \mathcal{R} LA UN REPER în \mathbb{R}^4

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = (-1)^{4+1+1} \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0$$

$$W = \langle (0, 1, 1, 0); (0, 1, 0, 0) \rangle \Rightarrow \mathbb{R}^4 = U \oplus W$$

c) $X = (0, 1, 2, -1)$. DESCOMPUN X ÎN RAPORT CU $U \oplus W$.

$$\mathcal{R} = \{(1, 0, 0, -1); (0, 1, 1, 0); (1, 0, 0, 0); (0, 1, 0, 0)\}$$

$$X = (0, 1, 2, -1) = \underbrace{a(1, 0, 0, -1) + b(0, 1, 1, 0)}_{u \in U} + \underbrace{c(1, 0, 0, 0) + d(0, 1, 0, 0)}_{w \in W}$$

$$(0, 1, 2, -1) = (a+c, b+d, b, -a)$$

$$\begin{cases} a+c=0 \Rightarrow c=-1 \\ b+d=1 \Rightarrow d=-1 \\ b=2 \\ -a=-1 \Rightarrow a=1 \end{cases} \Rightarrow (1, 2, -1, -1) \text{ COORD. } X \text{ ÎN RAPORT CU } \mathcal{R}$$

$$u = 1(1, 0, 0, -1) + 2(0, 1, 1, 0) = (1, 2, 2, -1)$$

$$w = -1(1, 0, 0, 0) + (-1)(0, 1, 0, 0) = (-1, -1, 0, 0)$$

$$X = (0, 1, 2, -1) = (1, 2, 2, -1) + (-1, -1, 0, 0)$$

$$\rho(x) = \rho(u+w) = u = (1, 2, 2, -1)$$

$$\Delta = 2\rho - \text{id}_V = 2(1, 2, 2, -1) - (0, 1, 2, -1) = (2, 3, 2, -1)$$

2) $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3; f(x) = (x_1 + x_2 - x_3; 2x_1 + x_2 + x_3; x_1)$
 a) f LINIARĂ c) $\text{Ker } f = ?$ Im $f = ?$
 b) MATRICE f CU \mathbb{R}_0 d) REPER ÎN $\text{Ker } f$ Im f

a) $f(ax+by) = af(x) + bf(y) \rightarrow$ TREBUIE SĂ DEMONSTRĂM

$$ax+by = (ax_1+by_1; ax_2+by_2; ax_3+by_3)$$

$$f(ax+by) = (ax_1+by_1+ax_2+by_2-ax_3-by_3; 2ax_1+2by_1+ax_2+by_2+ax_3+by_3; ax_1+by_1) = a(x_1+x_2-x_3; 2x_1+x_2+x_3; x_1) + b(y_1+y_2-y_3; 2y_1+y_2+y_3; y_1) = af(x) + bf(y)$$

b) $y = AX$

$$\begin{pmatrix} x_1+x_2-x_3 \\ 2x_1+x_2+x_3 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \det(A) = 1+1=2 \neq 0 \Rightarrow$$

 $\Rightarrow \text{rang}(A) = 3$

d+c) $\text{Ker } f = \{x \in \mathbb{R}^3 \mid f(x) = \overset{A}{0}_{\mathbb{R}^3}\}$

$$\begin{cases} x_1+x_2-x_3=0 \\ 2x_1+x_2+x_3=0 \\ x_1=0 \end{cases} \Rightarrow x_2=x_3 (=0)$$

$\text{Ker } f = S(A) \Rightarrow \dim_{\mathbb{R}} \text{Ker } f = 3-3=0$

$\text{Ker } f = \{(0, 0, 0)\} \Rightarrow \mathcal{R}_1 = \{(0, 0, 0)\}$

• Im $f = \{y \in \mathbb{R}^3 \mid (\exists) x \in \mathbb{R}^3 \text{ a. l. } f(x) = y\}$

⑦ $\begin{cases} x_1+x_2-x_3=y_1 \\ 2x_1+x_2+x_3=y_2 \\ x_1=y_3 \end{cases}$ ESTE COMPATIBIL; $\Delta_c = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0 \Rightarrow$
 $(\Delta_A = 2 \neq 0)$

$\Rightarrow y_3+y_2-y_1-2y_3=0 \Rightarrow y_2-y_1-y_3=0 \Rightarrow y_2=y_1+y_3$

Im $f = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_2=y_1+y_3\} = \{(y_1, y_1+y_3, y_3) \mid y_1, y_3 \in \mathbb{R}\}$
 $y_1(1, 1, 0) + y_3(0, 1, 1)$

$\mathcal{R}_2 = \{(1, 1, 0); (0, 1, 1)\}$ SG. PT. Im f
 $\text{rang}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 2(\text{MAX}) \Rightarrow \mathcal{R}_2 = \text{S.U.} \Rightarrow \mathcal{R}_2 \text{ REPER ÎN Im } f$

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$$1) (\mathbb{R}^4, +, \cdot) / \mathbb{R}; V_1 = \{(a, b, c, 0) \mid a, b, c \in \mathbb{R}\}; V_2 = \{(0, 0, d, e) \mid d, e \in \mathbb{R}\}$$

$$\overline{a) \mathbb{R}^4 = V_1 + V_2} \quad b) \text{ SUMA NU E DIRECTĂ}$$

$$a) V_1 = \{a(1, 0, 0, 0) + b(0, 1, 0, 0) + c(0, 0, 1, 0) \mid a, b, c \in \mathbb{R}\}$$

$$V_2 = \{d(0, 0, 1, 0) + e(0, 0, 0, 1) \mid d, e \in \mathbb{R}\}$$

$$\mathcal{R}_0 = \{e_1 = (1, 0, 0, 0); e_2 = (0, 1, 0, 0); e_3 = (0, 0, 1, 0); e_4 = (0, 0, 0, 1)\} \text{ (REPER CANONIC ÎN } \mathbb{R}^4)$$

$$\text{rang} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 3 \Rightarrow \left. \begin{array}{l} \{e_1, e_2, e_3\} = \text{S.U.} \\ \langle \{e_1, e_2, e_3\} \rangle = V_1 \end{array} \right\} \Rightarrow \mathcal{R}_1 = \{e_1, e_2, e_3\} \text{ REPER ÎN } V_1$$

$$\text{rang} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 \Rightarrow \left. \begin{array}{l} \{e_3, e_4\} = \text{S.U.} \\ \langle \{e_3, e_4\} \rangle = V_2 \end{array} \right\} \Rightarrow \mathcal{R}_2 = \{e_3, e_4\} \text{ REPER ÎN } V_2$$

$$\mathcal{R}_0 = \mathcal{R}_1 \cup \mathcal{R}_2$$

$$\text{dar } \mathcal{R}_1 \cap \mathcal{R}_2 = \{e_3\} \neq \emptyset \Rightarrow \mathbb{R}^4 = V_1 + V_2$$

$$\dim_{\mathbb{R}}(V_1 + V_2) = \dim_{\mathbb{R}} V_1 + \dim_{\mathbb{R}} V_2 - \dim_{\mathbb{R}}(V_1 \cap V_2) = 3 + 2 - 1 = 4 = \dim_{\mathbb{R}} \mathbb{R}^4$$

b) SUMA NU E DIRECTĂ PT. CĂ $V_1 \cap V_2 \neq \emptyset$, $\dim_{\mathbb{R}}(V_1 \cap V_2) > 1$ (de la punctul a)

$$2) (\mathbb{R}^4, +, \cdot) / \mathbb{R}; U_1 = \{x \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}$$

$$U_2 = \{x \in \mathbb{R}^4 \mid x_1 = x_2 = x_3 = 0\}$$

$$\overline{a) \mathbb{R}^4 = U_1 \oplus U_2}$$

b) SĂ SE DESCOMPUNĂ $x = (1, 2, 0, 1)$ ÎN RAPORT CU SUMA DIRECTĂ

$$a) U_1: x_1 + x_2 + x_3 + x_4 = 0$$

$$A_1 = (1 \ 1 \ 1 \ 1) \Rightarrow \text{rang}(A_1) = 1$$

$$U_1 = S(A_1) \Rightarrow \dim_{\mathbb{R}} U_1 = 4 - 1 = 3$$

$$U_2: \begin{cases} X_1 = 0 \\ X_2 = 0 \\ X_3 = 0 \end{cases} \quad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \text{rang}(A_2) = 3; U_2 = S(A_2)$$

$$\dim_{\mathbb{R}} U_2 = 4 - 3 = 1$$

$$U: \begin{cases} X_1 + X_2 + X_3 + X_4 = 0 \Rightarrow X_4 = 0 \\ X_1 = 0 \\ X_2 = 0 \\ X_3 = 0 \end{cases} \Rightarrow U = \{0_{\mathbb{R}^4}\} \Rightarrow$$

$$\Rightarrow \dim_{\mathbb{R}}(U_1 + U_2) = \dim_{\mathbb{R}} U_1 + \dim_{\mathbb{R}} U_2 + \dim_{\mathbb{R}} U = 3 + 1 = 4 = \dim_{\mathbb{R}} \mathbb{R}^4 \Rightarrow$$

$$\Rightarrow \mathbb{R}^4 = U_1 \oplus U_2$$

$$b) U_1 = \{(-X_2 - X_3 - X_4, X_2, X_3, X_4) \mid X_2, X_3, X_4 \in \mathbb{R}\} = \{X_2(-1, 1, 0, 0) + X_3(-1, 0, 1, 0) + X_4(-1, 0, 0, 1) \mid X_2, X_3, X_4 \in \mathbb{R}\}$$

$$\mathcal{R}_1 = \{(-1, 1, 0, 0); (-1, 0, 1, 0); (-1, 0, 0, 1)\} \text{ SG PT. } U_1 \Rightarrow \mathcal{R}_1 \text{ REPER}$$

$$|\mathcal{R}_1| = \dim_{\mathbb{R}} U_1 = 3 \Rightarrow \mathcal{R}_1 = SU^1$$

$$U_2 = \{(0, 0, 0, X_4) \mid X_4 \in \mathbb{R}\} = \{X_4(0, 0, 0, 1) \mid X_4 \in \mathbb{R}\}$$

$$\mathcal{R}_2 = \{(0, 0, 0, 1)\} \text{ SG PT } U_2$$

$$|\mathcal{R}_2| = \dim_{\mathbb{R}} U_2 = 1 \Rightarrow \mathcal{R}_2 = SU^1 \Rightarrow \mathcal{R}_2 \text{ REPER}$$

$$x = (1, 2, 0, 1) = a(-1, 1, 0, 0) + b(-1, 0, 1, 0) + c(-1, 0, 0, 1) + d(0, 0, 0, 1) = (-a - b - c, a, b, c + d)$$

$$\begin{cases} -a - b - c = 1 \Rightarrow c = -3 \\ a = 2 \\ b = 0 \\ c + d = 1 \Rightarrow d = 4 \end{cases}$$

$$\Rightarrow (2, 0, -3, 4) \text{ COORD. LUI } x \text{ ÎN RAPORT CU } \mathcal{R}_1, U_2$$

$$u = 2(-1, 1, 0, 0) + 0(-1, 0, 1, 0) = (-2, 2, 0, 0) \in U_1$$

$$v = -3(-1, 0, 0, 1) + 4(0, 0, 0, 1) = (3, 0, 0, 1) \in U_2$$

$$(1, 2, 0, 1) = (-2, 2, 0, 0) + (3, 0, 0, 1)$$