Seminar 13 - 143

Relațiile lui Viète:

Fie $f \in K[X]$, $f = a_n X^n + a_{n-1} X^{n-1} + \ldots + a_1 X + a_0$, $a_n \neq 0$.
$x_1, \ldots, x_n$ rădăcinile lui f. Atunci:

$$\begin{cases} x_1 + x_2 + \ldots + x_n = -\dfrac{a_{n-1}}{a_n} \\ x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n = \dfrac{a_{n-2}}{a_n} \\ \vdots \\ x_1x_2\ldots x_n = (-1)^n \dfrac{a_0}{a_n} \end{cases}$$

Ex. 1: Fie $f = X^3 + 5X^2 - 2X + 3 \in \mathbb{C}[X]$ și $x_1, x_2, x_3 \in \mathbb{C}$ rădăcinile sale. Calculați:

a. $x_1^2 + x_2^2 + x_3^2$

b. $\dfrac{1}{x_1} + \dfrac{1}{x_2} + \dfrac{1}{x_3}$.

c. $x_1^3 + x_2^3 + x_3^3$.

Rez:

a. $(x_1 + x_2 + x_3)^2 = x_1^2 + x_2^2 + x_3^2 + 2(x_1x_2 + x_1x_3 + x_2x_3)$

$x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_1x_3 + x_2x_3)$.

$$\begin{cases} x_1 + x_2 + x_3 = -5 \\ x_1x_2 + x_1x_3 + x_2x_3 = -2 \\ x_1x_2x_3 = -3. \end{cases}$$

$x_1^2 + x_2^2 + x_3^2 = (-5)^2 - 2\cdot(-2) = 25 + 4 = 29$

b. $\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}=\frac{x_2x_3+x_1x_3+x_1x_2}{x_1x_2x_3}=\frac{-2}{-3}=\frac{2}{3}$.

c. $x_1^3+x_2^3+x_3^3$.

$(x_1+x_2+x_3)^3=\ldots$

$f=X^3+5X^2-2X+3$

x_1, x_2, x_3 sunt rădăcini

$\Rightarrow x_i^3+5x_i^2-2x_i+3=0$

$x_i^3=-5x_i^2+2x_i-3$.

$x_1^3+x_2^3+x_3^3=-5(x_1^2+x_2^2+x_3^2)+2(x_1+x_2+x_3)-9=$

$=-5\cdot 29+2\cdot(-5)-9=-5\cdot 31-9=-155-9=-164$.

Ex. 2: Fie $f=X^3-9X^2-X+9$, $x_1, x_2, x_3\in\mathbb{C}$ rădăcini.

a. Calculați câtul și restul împărțirii lui f la X^2-1.

b. Calculați $x_1^2+x_2^2+x_3^2$, $x_1^3+x_2^3+x_3^3$ și $x_1^3+x_2^3+x_3^3-9(x_1^2+x_2^2+x_3^2)$.

Rez:

a.
$$\begin{array}{l|l} X^3-9X^2-X+9 & X^2-1 \\ \hline -X^3 \qquad\quad +X & X-9 \\ \hline -9X^2 \quad +9 & \\ 9X^2 \quad -9 & \\ \hline = \qquad\quad = & \end{array}$$

$f=(X^2-1)(X-9)=(X-1)(X+1)(X-9)$

$\Rightarrow x_1=1,\ x_2=-1,\ x_3=9$

b. $\begin{cases} x_1+x_2+x_3=9 \\ x_1x_2+x_1x_3+x_2x_3=-1 \\ x_1x_2x_3=-9 \end{cases}$

$x_1^2+x_2^2+x_3^2=(x_1+x_2+x_3)^2-2(x_1x_2+x_1x_3+x_2x_3)=$
$=81-2\cdot(-1)=83.$

$x_1^3+x_2^3+x_3^3=9(x_1^2+x_2^2+x_3^2)+(x_1+x_2+x_3)-27$

$x_i^3=9x_i^2+x_i-9 \quad (\Leftrightarrow f(x_i)=0).$

$\Rightarrow x_1^3+x_2^3+x_3^3=9\cdot 83+9-27=729.$

$x_1^3+x_2^3+x_3^3-9(x_1^2+x_2^2+x_3^2)=x_1+x_2+x_3-27=-18.$

Ex. 3 : Determinați c.m.m.d.c. al polinoamelor $f=X^5+2X^3+X^2+X+1$, $g=X^5+X^4+2X^3+2X^2+2X+1$, $f,g\in\mathbb{Q}[X]$.

Rez : Vom folosi Algoritmul lui Euclid pt. polinoame.
$\deg(f)=5$, $\deg(g)=5$

$$\begin{array}{l|l} X^5+X^4+2X^3+2X^2+2X+1 & X^5+2X^3+X^2+X+1 \\ \hline -X^5 \quad\quad -2X^3-X^2-X-1 & 1 \\ \hline X^4 \quad\quad +X^2+X & \end{array}$$

$g=f\cdot 1+\overbrace{(X^4+X^2+X)}^{r_0}$

$$\begin{array}{l|l} X^5+2X^3+X^2+X+1 & X^4+X^2+X \\ \hline -X^5-X^3-X^2 & X \\ \hline X^3 \quad\quad +X+1 & \end{array}$$

$f=r_0\cdot X+\underbrace{(X^3+X+1)}_{r_1}$

$$\begin{array}{l|l} x^4+x^2+x & x^3+x+1 \\ \hline -x^4-x^2-x & x \\ \hline \diagdown \quad \diagdown \quad \diagdown & \end{array}$$

$r_0 = r_1 \cdot x.$

$g = f \cdot 1 + r_0$

$f = r_0 \cdot x + \boxed{r_1}$ $\Rightarrow$ cmmdc$(f, g) = r_1 = x^3+x+1.$

$r_0 = \underline{\underline{r_1}} \cdot x$

$$\begin{array}{l|l} x^5+2x^3+x^2+x+1 & x^3+x+1 \\ \hline -x^5-x^3-x^2 & x^2+1 \\ \hline x^3 \qquad +x+1 & \\ -x^3 \qquad -x-1 & \\ \hline = \qquad = \quad = & \end{array}$$

$f = (x^3+x+1)(x^2+1).$

$$\begin{array}{l|l} x^5+x^4+2x^3+2x^2+2x+1 & x^3+x+1 \\ \hline -x^5 \qquad -x^3-x^2 & x^2+x+1 \\ \hline x^4+x^3+x^2+2x+1 & \\ -x^4 \qquad -x^2-x & \\ \hline x^3 \qquad +x+1 & \end{array}$$

$g = (x^3+x+1)(x^2+x+1).$

Ex. 4: Fie $f = (x-1)(x^2-1)(x^3-1)(x^4-1)$, $g = (x+1)(x^2+1)(x^3+1)(x^4+1)$, $f, g \in \mathbb{Q}[X]$.
Calculați cmmdc și cmmmc al lui f și g.

Rez:

$f = (x-1)[(x-1)(x+1)][(x-1)(x^2+x+1)][(x-1)(x+1)(x^2+1)]$

$g = (x+1)(x^2+1)[(x+1)(x^2-x+1)](x^4+1).$

$f = (x-1)^4 \cdot (x+1)^2 \cdot (x^2+1) \cdot (x^2+x+1) = (x+1)^2 (x^2+1) \cdot f_1$

$g = (x+1)^2 (x^2+1)(x^2-x+1)(x^4+1) = (x+1)^2 (x^2+1) \cdot g_1$

$\text{cmmdc}(f, g) = (x+1)^2 (x^2+1)\, \text{cmmdc}(f_1, g_1)$.

$f_1 = (x-1)^4 (x^2+x+1)$

$g_1 = (x^2-x+1)(x^4+1)$.

Obs: $x-a \mid f \Leftrightarrow f(a) = 0$.

$x-a$ prim (vezi curs 13).

$x-1 \nmid g_1 \Rightarrow (x-1, g_1) = 1 \Rightarrow ((x-1)^4, g_1) = 1$.

$\text{cmmdc}(f_1, g_1) = \text{cmmdc}(x^2+x+1, g_1) \overset{\text{ex}}{=} 1$.

$\text{cmmdc}(f, g) = (x+1)^2 (x^2+1)$.

$\text{cmmmc}[f, g] = (x+1)^2 (x^2+1) \cdot f_1 \cdot g_1$

$f \cdot g = (f, g) \cdot [f, g]$.

Ex. 5: Fie $a, b \in \mathbb{N}^*$, $d = (a, b)$. Arătați că $\text{cmmdc}(x^a - 1, x^b - 1) = x^d - 1$.

Rez: Folosim Alg. lui Euclid

$a = b \cdot q_0 + r_0$, $a > b$.

$$
\begin{array}{l|l}
x^a - 1 & x^b - 1 \\
\cline{2-2}
-x^a + x^{a-b} & x^{a-b} + x^{a-2b} + \ldots + x^{a-q_0 \cdot b} \\
\cline{1-1}
x^{a-b} - 1 & \\
-x^{a-b} + x^{a-2b} & \\
\cline{1-1}
x^{a-2b} - 1 &
\end{array}
$$

$a = b \cdot q_0 + r_0$, $a < r_0 < b$.

$\vdots$

$$\begin{array}{l} X^{a-(g_0-1)b} \quad -1 \\ -X^{a-(g_0-1)b} + X^{a-g_0\cdot b} \\ \hline X^{r_0} \quad -1 . \end{array}$$

$$X^a - 1 = (X^b - 1)\cdot g_0 + (X^{r_0} - 1)$$

$$X^b - 1 = (X^{r_0} - 1)\cdot g_1 + (X^{r_1} - 1), \quad b = r_0 \cdot g_1 + r_1$$

$a = b \cdot g_0 + r_0$

$b = r_0 \cdot g_1 + r_1$

$\vdots$

$r_{k-1} = r_k \cdot g_{k+1} + r_{k+1}$

$r_k = r_{k+1} \cdot g_{k+2}$ $\quad \neq 0. \quad \rightarrow X^{r_{k-1}} - 1 = (X^{r_k} - 1)\cdot g_{k+1} + (X^{r_{k+1}} - 1)$

$d = r_{k+1}$

$$(X^a - 1, X^b - 1) = X^d - 1.$$

Exemplu: $a = 16$, $b = 12$.

Alg. Euclid pt. numere:

$16 = 12 \cdot 1 + 4$

$12 = 4 \cdot 3$

$(16, 12) = 4.$

Alg. lui Euclid pt. polinoame:

$$\begin{array}{l|l} X^{16} \quad -1 & X^{12} - 1 \\ \cline{2-2} -X^{16} + X^4 & X^4 \\ \cline{1-1} X^4 - 1 & \end{array}$$

$$X^{16} - 1 = (X^{12} - 1)\cdot X^4 + (X^4 - 1)$$

$$
\begin{array}{l|l}
x^{12}-1 & x^4-1 \\
\hline
-x^{12}+x^8 & x^8+x^4+1 \\
\hline
x^8-1 & \\
-x^8+x^4 & \\
\hline
x^4-1 &
\end{array}
$$

$$x^{12}-1=(x^4-1)(x^8+x^4+1).$$

Exemplu: $a=40$, $b=7$.

$40=7\cdot 5+5$

$7=5\cdot 1+2$

$5=2\cdot 2+1$

$2=1\cdot 2$ . .

$$
\begin{array}{l|l}
x^{40}-1 & x^7-1 \\
\hline
-x^{40}+x^{33} & x^{33}+x^{26}+x^{19}+x^{12}+x^5 \\
\hline
x^{33}-1 & \\
-x^{33}+x^{26} & \\
\hline
x^{26}-1 & \\
-x^{26}+x^{19} & \\
\hline
x^{19}-1 & \\
-x^{19}+x^{12} & \\
\hline
x^{12}-1 & \\
-x^{12}+x^5 & \\
\hline
x^5-1 &
\end{array}
$$

$$
\begin{array}{l|l}
x^7-1 & x^5-1 \\
\hline
-x^7+x^2 & x^2 \\
\hline
x^2-1 &
\end{array}
\qquad
\begin{array}{l|l}
x^5-1 & x^2-1 \\
\hline
-x^5+x^3 & x^3+x \\
\hline
x^3-1 & \\
-x^3+x & \\
\hline
x-1 &
\end{array}
$$

$$
\begin{array}{l|l}
x^2-1 & x-1 \\
\hline
 & x+1.
\end{array}
$$

T: cmmdc$(x^4-4x^3+1,\ x^3-3x^2+1)$ în $\mathbb{R}[X]$.