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$\boxed{\text{CURS 5}}$

Subspații vectoriale
Morfisme de spații vectoriale

Preliminarii

$(V, +, \cdot)/\mathbb{K}$, $\dim_{\mathbb{K}} V = n$

- $R = \{e_1, .., e_n\}$ reper (bază ordonată).
  $\forall x \in V$, $\exists!\ (x_1, .., x_n) \in \mathbb{K}^n$ (coord. în raport cu R)
  a.î. $x = x_1 e_1 + \ldots + x_n e_n$.
- Criteriu de LI
  $S = \{v_1, .., v_m\} \subset V$, $m \leq n$
  S este SLI $\Leftrightarrow$ $rg\, C = m =$ maxim.
  (C = matricea compon. vect din S în rap. cu un reper $\forall$)
- $V', V'' \subset V$ subsp. vect
  $V' + V'' = \langle V' \cup V'' \rangle = \{v' + v'',\ v' \in V',\ v'' \in V''\}$
  Suma este directă $V' \oplus V'' \Leftrightarrow V' \cap V'' = \{0_V\}$
  $\Leftrightarrow \forall v \in V' \oplus V''$ se scrie unic $v = \underset{V'}{\underset{\cap}{v'}} + \underset{V''}{\underset{\cap}{v''}}$
- Dc. $V = V' \oplus V''$, V'' = subsp. complementar lui V'
  R' reper în V', R'' reper în V'' $\Rightarrow R = R' \cup R''$ reper în V.
- Dc R reper în V și $R = R' \cup R''$ (o partiție)
  $\Rightarrow V' = \langle R' \rangle$, $V'' = \langle R'' \rangle$ avem $V = V' \oplus V''$.

Th. Grassmann

$$\dim(V' + V'') = \dim V' + \dim V'' - \dim(V' \cap V'')$$

Prop $A \in \mathcal{M}_{m,n}(\mathbb{R})$

$S(A) = \{x \in \mathbb{R}^n \mid AX = 0\} \subset \mathbb{R}^n$ subspațiu vectorial

și $\dim_{\mathbb{R}} S(A) = n - rg(A)$

Dem

$\forall x, y \in S(A)$, $\forall a, b \in \mathbb{R} \overset{?}{\Longrightarrow} ax + by \in S(A)$

$\left.\begin{array}{l} AX = 0 \\ AY = 0 \end{array}\right\} \Rightarrow A(aX + bY) = 0 \Rightarrow S(A) \subset \mathbb{R}^n$ subsp. vect.

Fie $rg A = r$. Fără a restrânge generalitatea,

$x_1, .., x_r$ = variabile principale.

$x_{r+1} = \lambda_1, .., x_n = \lambda_p$ = variabile secundare, $p = n - r$.

$$\begin{cases} x_1 = \alpha_{11}\lambda_1 + .. + \alpha_{1p}\lambda_p \\ x_r = \alpha_{r1}\lambda_1 + \ldots + \alpha_{rp}\lambda_p \end{cases}$$

Sol sist:

$$\circledast \begin{cases} (x_1, .., x_r, \lambda_1, .., \lambda_p) = \\ = (\alpha_{11}\lambda_1 + \ldots + \alpha_{1p}\lambda_p, \ldots, \alpha_{r1}\lambda_1 + \ldots + \alpha_{rp}\lambda_p, \lambda_1, .., \lambda_p) = \\ = \lambda_1 \underbrace{(\alpha_{11}, .., \alpha_{r1}, 1, 0, .., 0)}_{y_1} + .. + \lambda_p \underbrace{(\alpha_{1p}, \ldots, \alpha_{rp}, 0, .., 0, 1)}_{y_p} \end{cases}$$

$S(A) = \langle \{y_1, .., y_p\} \rangle$

$R = \{y_1, .., y_p\}$ SG pentru $S(A)$

Dem că R este SLI

Fie $\lambda_1, .., \lambda_p \in \mathbb{R}$ ai $\lambda_1 y_1 + \ldots + \lambda_p y_p = 0_{\mathbb{R}^n}$

$\overset{\circledast}{\Rightarrow} (x_1, .., x_r, \lambda_1, .., \lambda_p) = (0, .., 0) \Rightarrow \lambda_1 = .. = \lambda_p = 0 \Rightarrow$ SLI

$\mathcal{R} = \{y_1, ..., y_p\}$ este SG și SLI $\Rightarrow \mathcal{R}$ bază în $S(A)$

$\dim_{\mathbb{R}} S(A) = p = n - r = n - rg\,A$

Aplicație 2

$(\mathbb{R}^3, +, \cdot)$ $\quad V' = \left\{ x \in \mathbb{R}^3 \mid \begin{cases} x_1 - x_2 = 0 \\ x_1 + x_3 = 0 \end{cases} \right\} = S(A)$

a) $\dim_{\mathbb{R}} V' = 3 - 2 = 1.$ $\quad A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & +1 \end{pmatrix}$

V' = dreaptă care trece prin origine. $\quad \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \neq 0 \Rightarrow rg\,A = 2$

b) Precizați o bază în V'

c) Precizați un subspațiu complementar lui V'

$\mathbb{R}^3 = V' \oplus V''$

d) Să se descompună $x = (1,1,1)$ în raport cu $\mathbb{R}^3 = V' \oplus V''$

SOL

b) $\Delta_p = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \neq 0$ $\quad x_1, x_2$ = var. principale

x_3 = var. secundară.

$\begin{cases} x_1 - x_2 = 0 \\ x_1 = -x_3 \end{cases}$ $\quad x_1 = x_2 = -x_3$

$(x_1, x_2, x_3) = (-x_3, -x_3, x_3) = x_3(-1, -1, 1).$

$V' = \langle \{(-1, -1, 1)\} \rangle$

$\mathcal{R}' = \{(-1,-1,1)\}$ SG pt V' $\quad \Rightarrow \mathcal{R}'$ bază în V'

$(-1,-1,1) \neq 0_{\mathbb{R}^3} \Rightarrow \mathcal{R}'$ este SLI

c) $\det \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \neq 0 \overset{\text{CRIT}}{\underset{\text{LI}}{\Longrightarrow}} \mathcal{R} = \{(-1,-1,1), (1,0,0), (0,0,1)\}$ SLI

$\dim_{\mathbb{R}} \mathbb{R}^3 = 3 = |\mathcal{R}| \Rightarrow \mathcal{R}$ bază în $\mathbb{R}^3$

$V'' = \langle \mathcal{R}'' \rangle$, $\mathcal{R}'' = \{(1,0,0),(0,0,1)\}$

$\mathcal{R}'' \subset \mathcal{R}_0 \Rightarrow \mathcal{R}'' \text{ SLI}$

baza canonică (SLI)

$\mathcal{R}''$ bază în V'', $\dim_{\mathbb{R}} V'' = 2$ (plan)

d) $x = (1,1,1) = \underbrace{a(-1,-1,1)}_{\in V'} + \underbrace{b(1,0,0) + c(0,0,1)}_{\in V''}$

$(1,1,1) = (-a+b, -a, a+c)$

$$\begin{cases} -a+b = 1 \\ -a = 1 \Rightarrow a = -1 \\ a+c = 1 \end{cases} \quad b = 0, \quad c = 2$$

$x = (1,1,1) = \underbrace{(1,1,-1)}_{v' \in V'} + \underbrace{(0,0,2)}_{v'' \in V''}$

<u>OBS</u> $V' \subseteq V$ subspațiu vect

$\Rightarrow$ coordonatele vectorilor din V', în raport cu un reper, sunt soluțiile unui SLO, i.e $\exists A$ a.î.

$V' = S(A)$

<u>Aplicații</u>

① $(\mathbb{R}^4, +, \cdot)_{/\mathbb{R}}$, $V' = \langle \{(1,1,0,0),(1,0,1,-1)\} \rangle$

a) Să se descrie subsp. V' printr-un sistem de ecuații liniare.

b) $\mathbb{R}^4 = V' \oplus V''$, $V'' = ?$

<u>SOL</u>

a) $\forall x=(x_1,x_2,x_3,x_4)\in V'$, $\exists a,b\in\mathbb{R}$ a.î.

$x = a(1,1,0,0)+b(1,0,1,-1)$

$(x_1,x_2,x_3,x_4)=(a+b,\ a,\ b,\ -b)$

$$\begin{cases} a+b = x_1 \\ a = x_2 \\ b = x_3 \\ -b = x_4 \end{cases} \qquad B=\left(\begin{array}{cc|c} \boxed{\begin{matrix}1 & 1\\ 1 & 0\end{matrix}} & & \begin{matrix}x_1\\ x_2\end{matrix} \\ \begin{matrix}0 & 1\\ 0 & -1\end{matrix} & & \begin{matrix}x_3\\ x_4\end{matrix}\end{array}\right)$$

$\underline{SCD}$ (a,b necunoscutele)

$\Delta_p=\begin{vmatrix}1 & 1\\ 1 & 0\end{vmatrix}\neq 0 \Rightarrow rg\, A=2$

$$\begin{cases}\Delta_{c_1}=0\\ \Delta_{c_2}=0\end{cases} \Rightarrow \begin{cases}\begin{vmatrix}1 & 1 & x_1\\ 1 & 0 & x_2\\ 0 & 1 & x_3\end{vmatrix}=0 \\ \begin{vmatrix}1 & 1 & x_1\\ 1 & 0 & x_2\\ 0 & -1 & x_4\end{vmatrix}=0\end{cases}$$

$$\begin{cases} x_1-x_2-x_3=0\\ x_1(-1)+x_2-x_4=0\end{cases}$$

$$V'=\left\{x\in\mathbb{R}^4 \;\middle|\; \begin{cases}x_1-x_2-x_3=0\\ -x_1+x_2-x_4=0\end{cases}\right\}=S(A)$$

$$A=\begin{pmatrix}1 & -1 & -1 & 0\\ -1 & 1 & 0 & -1\end{pmatrix}$$

b) $\mathbb{R}^4=V'\oplus V''$

$$\det\begin{pmatrix}1 & 1 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\end{pmatrix}\neq 0,\quad V''=\langle\{(0,0,0,1),(1,0,0,0)\}$$

② $(\mathbb{R}^4,+,\cdot)_{/\mathbb{R}}$ $\quad V'=\{(x,y,z,u)\in\mathbb{R}^4 \mid x+y-z-u=0\}$

$V''=\{(x,y,z,u)\in\mathbb{R}^4 / x-y-z+u=0\}$

a) $\mathbb{R}^4=V'+V''$

b) Este sumă directă?

Sol $V' = \{(x,y,z,u) \in \mathbb{R}^4 \mid x+y-z-u=0\}$ $A_{V'} = (1\ 1\ -1\ -1)$

$\dim V' = 4-1 = 3$ (hiperplan)

$V'' = \{(x,y,z,u) \in \mathbb{R}^4 \mid x-y-z+u=0\}$ $A_{V''} = (1\ -1\ -1\ 1)$

$\dim V'' = 4-1 = 3$ (hiperplan)

$$V' \cap V'' = \{(x,y,z,u) \in \mathbb{R}^4 \mid \begin{cases} x+y-z-u=0 \\ x-y-z+u=0 \end{cases}\}$$

$$A_{V'\cap V''} = \begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}} & \begin{matrix} -1 & -1 \\ -1 & 1 \end{matrix} \end{pmatrix}$$

$\dim(V' \cap V'') = 4-2 = 2$

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$$\left.\begin{array}{l} \dim(V'+V'') = 3+3-2 = 4 \\ V'+V'' \subset \mathbb{R}^4 \\ \dim \mathbb{R}^4 = 4 \end{array}\right\} \Rightarrow \mathbb{R}^4 = V'+V''$$

nu e ⊕ deoarece $V' \cap V'' \neq \{0_{\mathbb{R}^4}\}$.

$\boxed{\text{OBS}}$ $V' \subset V$, $\dim_{\mathbb{K}} V = n$

Dacă $\dim_{\mathbb{K}} V' = n$, atunci $V' = V$.

Def. $(V,+,\cdot)_{\mathbb{R}}$.

- $[v,w] = \{x \in V \mid x = (1-t)v + tw,\ t \in [0,1]\}$
- $C \subseteq V$ submulțime convexă $\Leftrightarrow [\forall v,w \in C \Rightarrow [v,w] \subseteq C]$

Prop a) $\forall V' \subset V$ subsp vect $\rightarrow V' =$ mult. convexă

b) $\{v_0, \dots, v_k\}$ sist. finit de vect din V

$\Rightarrow C = \{v = \sum_{i=0}^{k} \lambda_i v_i,\ \sum_{i=0}^{k} \lambda_i = 1\}$ convexă

im
a) $V' \subseteq V$ subsp. vect.

Dem că $\forall v, w \in V'$, $a = 1-t$, $b = t \in [0,1]$ $\Big\} \Rightarrow (1-t)v + tw \in V'$

$\Rightarrow [v,w] \subset V'$

b) $C = \left\{ v = \sum_{i=0}^{k} \lambda_i v_i \ , \ \sum_{i=0}^{k} \lambda_i = 1 \right\}$

$v = \sum_{i=0}^{k} \lambda_i v_i \ , \ \sum_{i=0}^{k} \lambda_i = 1.$

$w = \sum_{i=0}^{k} \alpha_i v_i \ , \ \sum_{i=0}^{k} \alpha_i = 1$

$$(1-t)v + tw = \sum_{i=0}^{k} \underbrace{\left[(1-t)\lambda_i + t\alpha_i\right]}_{\beta_i} v_i$$

$$\sum_{i=0}^{k} \beta_i = (1-t)\underbrace{\sum_{i=0}^{k}\lambda_i}_{=1} + t\underbrace{\sum_{i=0}^{k}\alpha_i}_{=1} = 1 - t + t = 1$$

$\Rightarrow (1-t)v + tw \in C \Rightarrow [v,w] \subseteq C \Rightarrow$

$C \subset V$ m. convexă.

Morfisme de spații vectoriale

Def $(V_i, +, \cdot)/\mathbb{K}_i$, $i = 1,2$ spații vect.

$f : V_1 \longrightarrow V_2$ s.n. aplicație semi-liniară

$\Leftrightarrow$ 1) $f(x+y) = f(x) + f(y)$, $\forall x, y \in V_1$

2) $\exists\, \theta : \mathbb{K}_1 \longrightarrow \mathbb{K}_2$ izom. de corpuri a.î.

$f(\alpha x) = \theta(\alpha) f(x)$, $\forall x \in V_1$, $\forall \alpha \in \mathbb{K}_1$

Dacă $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{K}$ și $\theta = id_{\mathbb{K}}$, atunci f s.n. <u>aplicație liniară</u> sau morfism de spații vect.

<u>Exemple</u>

① $(V_i, +, \cdot)/_{\mathbb{R}}$, $i = \overline{1,2}$ sp V.

Fie $\theta: \mathbb{R} \to \mathbb{R}$ autom. de corpuri $\Rightarrow \theta = id_{\mathbb{R}}$

$f: V_1 \to V_2$ semi-liniară $\Rightarrow$ liniară.

② $(\mathbb{C}^n, +, \cdot)/\mathbb{C}$

$f: \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $f(z) = \overline{z}$, $\forall z \in \mathbb{C}^n$

$\theta: \mathbb{C} \longrightarrow \mathbb{C}$, $\theta(\alpha) = \overline{\alpha}$, $\forall \alpha \in \mathbb{C}$

autom. de corpuri

$f(z+u) = \overline{z+u} = \overline{z} + \overline{u} = f(z) + f(u)$

$f(\alpha z) = \overline{\alpha z} = \overline{\alpha}\,\overline{z} = \theta(\alpha) f(z)$

$\Rightarrow f$ este semi-liniară (și nu e liniară).

<u>Aplicații liniare</u>

- $f: V_1 \longrightarrow V_2$ apl. lin.

  f s.n. izomorfism dacă e bijectivă

- $(V, +, \cdot)/\mathbb{K}$ sp. vect

  $f \in End(V) \Leftrightarrow \begin{cases} f: V \to V \\ f \text{ liniară} \end{cases}$ (endomorfism de sp. vect)

  $f \in Aut(V) \Leftrightarrow f: V \to V$ liniară + bij (automorfism de sp. vect)

<u>OBS</u>. a) $f: V_1 \to V_2$ apl. lin $\Rightarrow$

$f: (V_1, +) \longrightarrow (V_2, +)$ morf. de grupuri $\Rightarrow f(0_{V_1}) = 0_{V_2}$

•) $$\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[rr, bend right, "h = g\circ f"'] & V_2 \arrow[r, "g"] & V_3 \end{tikzcd}$$ f, g liniare $\Rightarrow h$ liniară

Exemple

a) $f: V \longrightarrow V$, $f(x) = 0_V$ sau $f(x) = x$

b) $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $f(x) = y$

$Y = AX$, $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$, $A = (a_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$

c) $f: \mathcal{M}_n(\mathbb{R}) \longrightarrow \mathbb{R}$, $f(X) = Tr(X)$

Este $f(X) = \det(X)$ liniară? NU

Prop (caracterizare)

$f: V_1 \longrightarrow V_2$ aplicație

f liniară $\Leftrightarrow f(ax+by) = af(x) + bf(y)$
$\forall a, b \in \mathbb{K}, \forall x, y \in V_1$

Dem

"$\Rightarrow$" Ip: f liniară

$a \in \mathbb{K}, x \in V_1 \Rightarrow ax \in V_1$
$b \in \mathbb{K}, y \in V_1 \Rightarrow by \in V_1$ $\Rightarrow f(ax+by) = f(ax) + f(by)$
$= af(x) + bf(y)$

"$\Leftarrow$" Ip $f(ax+by) = af(x) + bf(y)$, $\forall a, b \in \mathbb{K}$
$\forall x, y \in V_1$

• $a = b = 1_{\mathbb{K}}$

$f(1_{\mathbb{K}} x + 1_{\mathbb{K}} y) = f(x+y) = 1_{\mathbb{K}} f(x) + 1_{\mathbb{K}} f(y) = f(x) + f(y)$

• $b = 0_{\mathbb{K}} \Rightarrow f(ax) = af(x)$

$\boxed{\text{OBS}}$ $f: V_1 \to V_2$ liniară

$V' \subset V_1$ subsp vect $\Rightarrow f(V') \subset V_2$ subsp. vect.

$\forall y_1, y_2 \in f(V')$, $\exists x_1, x_2 \in V'$ a.î. $y_1 = f(x_1)$, $y_2 = f(x_2)$

$\forall a, b \in \mathbb{K} \overset{?}{\Longrightarrow} a y_1 + b y_2 \in f(V')$

$$\underset{a f(x_1) + b f(x_2)}{\underset{\|}{a y_1 + b y_2}} \overset{lin}{=} f(a x_1 + b x_2) = f(x)$$

$x \in V'$ (V' subsp. vect).

Def $f: V_1 \to V_2$ apl. liniară

$\text{Ker} f = \{x \in V_1 \mid f(x) = 0_{V_2}\}$ nullspace, Kernel (nucleul lui f)

$\text{Im} f = \{y \in V_2 \mid \exists x \in V_1 : f(x) = y\}$ (imag. lui f)

Prop $f: V_1 \to V_2$ lin

a) $\text{Ker} f$, $\text{Im} f$ subsp vect în V_1, resp V_2

b) f inj $\Leftrightarrow \text{Ker} f = \{0_{V_1}\}$

c) f surj $\Leftrightarrow \dim \text{Im} f = \dim V_2$

Dem

a) $\forall x_1, x_2 \in \text{Ker} f \Rightarrow a x_1 + b x_2 \in \text{Ker} f$
$\forall a, b \in \mathbb{K}$

$$f(a x_1 + b x_2) \overset{f\, lin}{=} a \underset{0_{V_2}}{\underset{\|}{f(x_1)}} + b \underset{0_{V_2}}{\underset{\|}{f(x_2)}} = 0_{V_2}.$$

$\text{Im} f = f(V_1) \subset V_2$ subsp. vect

b) "$\Rightarrow$" Ip: f inj

Fie $x \in \text{Ker} f \Rightarrow \left.\begin{array}{l} f(x) = 0_{V_2} \\ \text{dar } f(0_{V_1}) = 0_{V_2} \end{array}\right\} \xRightarrow{f\, inj} x = 0_{V_1} \Rightarrow \text{Ker} f = \{0_{V_1}\}$

"$\Leftarrow$" Ip: $\mathrm{Ker} f = \{0_{V_1}\}$.

Fie $x_1, x_2 \in V_1$ aî $f(x_1) = f(x_2) \xrightarrow{f\ lin} f(x_1) - f(x_2) = 0_{V_2}$

$f(x_1) - f(x_2) = f(x_1 - x_2)$

$\Rightarrow x_1 - x_2 \in \mathrm{Ker} f \Rightarrow x_1 - x_2 = 0_{V_1} \Rightarrow x_1 = x_2 \Rightarrow f$ inj.

c) "$\Rightarrow$" Ip: f surj $\Leftrightarrow \mathrm{Im} f = V_2 \Leftrightarrow \dim \mathrm{Im} f = \dim V_2$

"$\Leftarrow$" Ip: $\left.\begin{array}{l}\dim \mathrm{Im} f = \dim V_2 \\ \mathrm{Im} f \subseteq V_2 \text{ subsp v.}\end{array}\right\} \xrightarrow{OBS} \mathrm{Im} f = V_2 \Rightarrow f$ surj.

<u>OBS</u>. $f: V_1 \longrightarrow V_2$ lin

f izom. sp. vect $\Leftrightarrow \begin{cases} \mathrm{Ker} f = \{0_{V_1}\} \\ \dim \mathrm{Im} f = \dim V_2 \end{cases}$

<u>Aplicatie</u>

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $f(x_1, x_2, x_3) = (x_1 + x_2 - x_3, x_1 + x_2, x_1 + x_2 + x_3)$

a) f lin.

b) $\dim \ker f$, $\dim \mathrm{Im} f = ?$

<u>SOL</u>

a) $f(ax + by) = af(x) + bf(y)$

b) $\ker f = \{x \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3}\}$

$= \left\{x \in \mathbb{R}^3 \,\middle|\, \begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 + x_2 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases}\right\} = S(A)$, $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$\det A = 0 \Rightarrow \dim \ker f = 3 - rg A = 3 - 2 = 1$

$\mathrm{Im} f = \{y \in \mathbb{R}^3 \mid \exists x \in \mathbb{R}^3 \text{ aî } f(x) = y\} = \{y \in \mathbb{R}^3 \mid y_1 - 2y_2 + y_3 = 0\}$

$\begin{cases} x_1 + x_2 - x_3 = y_1 \\ x_1 + x_2 = y_2 \\ x_1 + x_2 + x_3 = y_3 \end{cases}$ $\Delta_c = \begin{vmatrix} 1 & -1 & y_1 \\ 1 & 0 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = 0 \Rightarrow y_1 - 2y_2 + y_3 = 0$

$\dim \mathrm{Im} f = 2$