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CURS 1

Determinanți. Teorema Laplace. Teorema Hamilton-Cayley.

Def $\det : \mathcal{M}_n(\mathbb{K}) \longrightarrow \mathbb{K}$, $(\mathbb{K}, +, \cdot)$ corp com.
$\mathbb{K} = \mathbb{R}$ sau $\mathbb{K} = \mathbb{C}$

$$\det(A) = \sum_{\sigma \in S_n} \varepsilon_\sigma a_{1\sigma(1)} \cdot \ldots \cdot a_{n\sigma(n)}$$

$(S_n, \circ)$ grupul permutărilor, $\sigma = \begin{pmatrix} 1 & \ldots & n \\ \sigma(1), & \ldots & \sigma(n) \end{pmatrix} \in S_n$

$(\sigma : \{1, .., n\} \longrightarrow \{1, .., n\}$ bijecție$)$

Prop.

a) $\det(AB) = \det(A)\det(B)$, $\det(A^k) = (\det A)^k$, $k \in \mathbb{N}^*$

b) $\det(A^T) = \det A$, $A^T = B$, $b_{ij} = a_{ji}$, $\forall i, j = \overline{1,n}$

c) $\det(\alpha A) = \alpha^n \det A$, $\alpha \in \mathbb{K}$, $\forall A, B \in \mathcal{M}_n(\mathbb{K})$

OBS a) Dacă $l_i \leftrightarrow l_j$ (resp $c_i \leftrightarrow c_j$), at $\Delta' = -\Delta$

b) Dacă $l_i = \sum_{\substack{j \neq i \\ j = \overline{1,n}}} \alpha_j l_j$ (resp $c_i = \sum_{\substack{j \neq i \\ j = \overline{1,n}}} \alpha_j c_j$), at $\Delta = 0$

(l_i = combinație liniară a celorlalte linii)

În particular, $l_i = \alpha l_j$ (resp $c_i = \alpha c_j$) $\Rightarrow \Delta = 0$

$l_i = 0$ $(c_i = 0)$ $\Rightarrow \Delta = 0$

c) Dc $l_i = l_i' + l_i''$, at $\Delta = \Delta' + \Delta''$

(resp $c_i = c_i' + c_i''$)

d) Dc $l_i' = \alpha l_i$, at $\Delta' = \alpha \Delta$

Exemple

Ex1 Fie $A \in \mathcal{M}_{2n+1}(\mathbb{R})$, $B = A - A^T \Rightarrow \det B = 0$

SOL $\det B = \det B^T = \det(A^T - A) = \det((-1)(A - A^T)) =$
$= (-1)^{2n+1} \det(A - A^T) = -\det B \Rightarrow$
$2\det B = 0 \Rightarrow \det B = 0$

Ex2
$$\Delta = \begin{vmatrix} \sin^2 a & \cos^2 a & \cos 2a \\ \sin^2 b & \cos^2 b & \cos 2b \\ \sin^2 c & \cos^2 c & \cos 2c \end{vmatrix} = ?$$

SOL
$\cos 2a = \cos^2 a - \sin^2 a \Rightarrow c_3 = c_2 - c_1 \Rightarrow \Delta = 0$

Def $A \in \mathcal{M}_n(\mathbb{K})$

a) Minor de ordin p
$$\Delta_p = \begin{vmatrix} a_{i_1 j_1} & \dots & a_{i_1 j_p} \\ a_{i_p j_1} & \dots & a_{i_p j_p} \end{vmatrix} \quad \begin{matrix} 1 \le i_1 < \dots < i_p \le n \\ 1 \le j_1 < \dots < j_p \le n. \end{matrix}$$

b) Minor complementar lui Δ_p
Δ_c obținut din A, suprimând liniile $l_{i_1}, \dots, l_{i_p}$
coloanele $c_{j_1}, \dots, c_{j_p}$.

c) Complement algebric pt Δ_p
$c = (-1)^{i_1 + \dots + i_p + j_1 + \dots + j_p} \Delta_c$

Caz particular pt $p = 1$.
1 element
$\Delta_1 = \det(a_{ij})$
$$A = \begin{array}{c} \\ i \end{array}\!\!\begin{array}{c} j \\ \begin{pmatrix} \dots & a_{ij} & \dots \end{pmatrix} \end{array}$$
$\Delta_c = \Delta_{ij}$
$c_{ij} = (-1)^{i+j} \Delta_{ij}$ (complementul algebric pt a_{ij})

Teorema Laplace

$\Delta_A = \det(A)$ = suma produselor minorilor de ordinul p, pt p linii fixate (respectiv p coloane fixate) cu complemenții algebrici corespunzători.

Caz particular

$p = 1$. Fie $i \in \{1, .., n\}$ fixat.

$\det A = a_{i1} c_{i1} + \dots + a_{in} c_{in}$ (dezvoltarea după linia i)

(Analog pt dezv. după coloana j)

Exemple

Ex1

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{pmatrix} \Rightarrow \Delta_A = ?$$

SOL

M_1 (Th. Laplace)

Considerăm $p = 2$ și l_1, l_2 fixate.

$$\Delta_A = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} (-1)^{1+2+1+2} \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} (-1)^{1+2+1+3} \begin{vmatrix} 5 & -1 \\ -2 & 4 \end{vmatrix} +$$

$$+ \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} (-1)^{1+2+1+4} \begin{vmatrix} 5 & 1 \\ -2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} (-1)^{1+2+2+3} \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} (-1)^{1+2+2+4} \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} (-1)^{1+2+3+4} \begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix}$$

$= -5$

M_2

$$\begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 3 & -3 & -7 \\ 0 & -1 & 4 & 7 \end{vmatrix} \quad \begin{aligned} l_2' &= l_2 - l_1 \\ l_3' &= l_3 - 2l_1 \\ l_4' &= l_4 + l_1 \end{aligned}$$

$$c_2' = c_2 - c_3$$

$$= 1(-1)^{1+1} \begin{vmatrix} 0 & 1 & 1 \\ 3 & -3 & -7 \\ -1 & 4 & 7 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 3 & 4 & -7 \\ -1 & -3 & 7 \end{vmatrix}$$

$$= 1\cdot(-1)^{1+3} \begin{vmatrix} 3 & 4 \\ -1 & -3 \end{vmatrix} = -9+4 = -5$$

Ex 2.

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = ?$$

SOL

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

$l_1' = l_1 + l_2 + l_3$

$$= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix} = (a+b+c)\cdot 1(-1)^{1+1} \begin{vmatrix} c-b & a-b \\ a-c & b-c \end{vmatrix}$$

$c_2' = c_2 - c_1$

$c_3' = c_3 - c_1$

$$\Delta = (a+b+c)\left[(c-b)(b-c) - (a-b)(a-c)\right] =$$

$$= -(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$$

$$= -\frac{1}{2}(a+b+c)\left[(a^2-2ab+b^2) + (a^2-2ac+c^2) + (b^2-2bc+c^2)\right]$$

$$\Delta = -\frac{1}{2}(a+b+c)\left[(a-b)^2 + (a-c)^2 + (b-c)^2\right]$$

Ex 3 (det. Vandermonde)

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = ?$$

SOL

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \end{vmatrix}$$

$$= 1(-1)^{1+1}(b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} =$$

$$\Rightarrow \Delta = (b-a)(c-a)(c-b)$$

Def $A \in \mathcal{M}_n(\mathbb{K})$ nesingulară $\Leftrightarrow \det A \neq 0$

$A \in \mathcal{M}_n(\mathbb{K})$ inversabilă $\Leftrightarrow \exists A^{-1} \in \mathcal{M}_n(\mathbb{K})$ aî

$$A \cdot A^{-1} = A^{-1} A = I_n.$$

OBS A inversabilă $\Leftrightarrow$ nesingulară

$$A^{-1} = \frac{1}{\det A} \cdot A^*, \quad A^*_{ij} = \text{complementul algebric pt } a_{ji}$$

Prop a) $(AB)^{-1} = B^{-1} A^{-1}$

b) $\det(A^{-1}) = \dfrac{1}{\det A}$

c) $\det(A^*) = (\det A)^{n-1}$, $\forall A, B \in \mathcal{M}_n(\mathbb{K})$, $n \geq 2$.

Dem

b) $A \cdot A^{-1} = I_n \mid \det \Rightarrow \det(A \cdot A^{-1}) = \det I_n \Rightarrow \det A \cdot \det A^{-1} = 1$

$$\det(A^{-1}) = \frac{1}{\det A}$$

c) $A \cdot A^{-1} = \dfrac{1}{\det A} \cdot A \cdot A^* \Rightarrow A \cdot A^* = I_n \cdot \det A \quad \mid \det$

$$\det(A \cdot A^*) = \det(\underbrace{\det(A)}\, I_n) \Rightarrow \det(A) \cdot \det(A^*) = (\det A)^n \underbrace{\det I_n}_{1}$$

$$\det(A^*) = (\det A)^{n-1}$$

Exemple

Ex1
$$A = \begin{pmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{Z})$$

$m = ?$ aî $A^{-1} \in \mathcal{M}_3(\mathbb{Z})$

SOL

$$\left.\begin{array}{l} \det A, \det A^{-1} \in \mathbb{Z} \\ \det(A^{-1}) = \dfrac{1}{\det A} \end{array}\right\} \Rightarrow \det A = \pm 1$$

$$\det A = \begin{vmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -3 & 3m+4 \\ 1 & m-1 & 1 \\ -1 & 0 & 0 \end{vmatrix}$$

$$c_2' = c_2 - c_1$$

$$= (-1)(-1)^{3+1} \begin{vmatrix} -3 & 3m+4 \\ m-1 & 1 \end{vmatrix} = -\left(-3-3m^2+3m-4m+4\right)$$

$$= 3m^2+m-1$$

① $3m^2+m-1=1 \Rightarrow 3m^2+m-2=0$

$$\left.\begin{array}{l} m_1=-1\in\mathbb{Z} \\ m_1 m_2 = -\frac{2}{3} \end{array}\right\} \Rightarrow m_2 = +\frac{2}{3} \notin \mathbb{Z}.$$

② $3m^2+m-1=-1 \Rightarrow m(3m+1)=0 < \begin{array}{l} m_1'=0\in\mathbb{Z} \\ m_2' = -\frac{1}{3}\notin\mathbb{Z} \end{array}$

Deci $m\in\{-1,0\}$.

__Ex2__ $A = \begin{pmatrix} 1+a^2 & ba & ca \\ ab & 1+b^2 & cb \\ ac & bc & 1+c^2 \end{pmatrix} \Rightarrow \det(A^*) = ?$

$a,b,c\in\mathbb{R}$

__Sol__

$n=3 \Rightarrow \det(A^*) = (\det A)^2.$

$$\Delta_A = \begin{array}{c} \begin{array}{cccccc} 1 & 1' & 2 & 2' & 3 & 3' \end{array} \\ \begin{vmatrix} 1+a^2 & 0+ba & 0+ca \\ 0+ab & 1+b^2 & 0+cb \\ 0+ac & 0+bc & 1+c^2 \end{vmatrix} \end{array} = |1\ 2\ 3| + |1\ 2\ 3'| + |1\ 2'\ 3| + |1\ 2'\ 3'|$$

$$+ |1'\ 2\ 3| + |1'\ 2\ 3'| + |1'\ 2'\ 3| + |1'\ 2'\ 3'|$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + a\begin{vmatrix} a & 0 & 0 \\ b & 1 & 0 \\ c & 0 & 1 \end{vmatrix} + b\begin{vmatrix} 1 & a & 0 \\ 0 & b & 0 \\ 0 & c & 1 \end{vmatrix} + c\begin{vmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{vmatrix}$$

$$\Delta_A = 1+a^2+b^2+c^2 \Rightarrow \det(A^*) = (1+a^2+b^2+c^2)^2$$

$$|1\ 2'\ 3'| = \begin{vmatrix} 1 & ba & ca \\ 0 & b^2 & cb \\ 0 & bc & c^2 \end{vmatrix} = bc\begin{vmatrix} 1 & a & a \\ 0 & b & b \\ 0 & c & c \end{vmatrix} = 0$$

Ex a) $(GL(n,\mathbb{K}) = \{A \in \mathcal{M}_n(\mathbb{K}) \mid \det A \neq 0\}, \cdot)$

grupul general liniar (real pt $\mathbb{K} = \mathbb{R}$)

(complex pt $\mathbb{K} = \mathbb{C}$)

b) $O(n) = \{A \in \mathcal{M}_n(\mathbb{K}) \mid A \cdot A^T = I_n\} \subset GL(n,\mathbb{K})$

subgrupul matricelor ortogonale.

$A \cdot A^T = I_n \mid \det \Rightarrow (\det A)^2 = 1 \Rightarrow \det A \in \{-1, 1\}$

c) $SO(n) = \{A \in O(n) \mid \det A = 1\} \subset O(n)$

subgrupul matricelor special ortogonale.

Def $Tr: \mathcal{M}_n(\mathbb{K}) \longrightarrow \mathbb{K}$, $Tr(A) = \sum_{i=1}^{n} a_{ii}$

urma matricei A

Prop a) $Tr(A+B) = Tr(A) + Tr(B)$

b) $Tr(\alpha A) = \alpha Tr(A)$

c) $Tr(AB) = Tr(BA)$

d) $Tr(A) = Tr(A^T)$, $\forall \alpha \in \mathbb{K}$, $\forall A, B \in \mathcal{M}_n(\mathbb{K})$

Def Fie $A \in \mathcal{M}_n(\mathbb{K})$.

$P_A(X) = \det(A - XI_n) = (-1)^n \left[X^n - \sigma_1 X^{n-1} + \ldots + (-1)^n \sigma_n\right]$

s.n. polinomul caracteristic asociat lui A,

unde σ_k = suma minorilor diagonali de ordinul k, $k = \overline{1,n}$

$\sigma_1 = \sum_{i=1}^{n} a_{ii} = Tr(A)$

$\sigma_2 = \sum_{i<j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$

$\vdots$

$\sigma_n = \det(A)$

Teorema Hamilton-Cayley

$\forall A \in \mathcal{M}_n(\mathbb{K})$ își anulează polinomul caracteristic

i.e. $P_A(A) = O_n \iff A^n - \sigma_1 A^{n-1} + \ldots + (-1)^n I_n \sigma_n = O_n$.

Dem

Notăm $M = A - XI_n$

$$P_A(X) = \det M = (-1)^n (X^n - \sigma_1 X^{n-1} + \ldots + \sigma_n (-1)^n)$$

$$M \cdot M^* = I_n \cdot \det M = (-1)^n (X^n - \sigma_1 X^{n-1} + \ldots + \sigma_n (-1)^n) I_n = (*)$$

$$M^* = X^{n-1} B_{n-1} + X^{n-2} B_{n-2} + \ldots + X B_1 + B_0$$

$$(A - XI_n)(X^{n-1} B_{n-1} + X^{n-2} B_{n-2} + \ldots + X B_1 + B_0) = (*)$$

$$\left\{\begin{array}{rcl|l} A B_0 & = & (-1)^{2n} \sigma_n I_n. & \cdot \\ -B_0 + A B_1 & = & (-1)^{2n-1} \sigma_{n-1} I_n & A \\ -B_1 + A B_2 & = & (-1)^{2n-2} \sigma_{n-2} I_n & A^2 \\ & \vdots & & \vdots \\ -B_{n-2} + A B_{n-1} & = & (-1)^{n+1} \sigma_1 I_n. & \vdots \\ -B_{n-1} & = & (-1)^n I_n. & A^n \end{array}\right. \quad \oplus$$

$$O_n = (-1)^n \left[A^n - \sigma_1 A^{n-1} + \ldots + (-1)^n \sigma_n I_n \right]$$

$$\Rightarrow A^n - \sigma_1 A^{n-1} + \ldots + (-1)^n \sigma_n I_n = O_n$$

În particular, pt $n = 2$: $A^2 - \sigma_1 A + \sigma_2 I_2 = O_2$

$\sigma_1 = \operatorname{Tr} A$

$\sigma_2 = \det A$

xemple

Ex1. Fie $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Calculați A^{-1}, utilizând Th H-C

SOL

$\sigma_1 = Tr A = 4$

$\sigma_2 = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 2+2+1=5$

$\sigma_3 = \det A = 1 \cdot 1 \cdot 2 = 2$

H-C: $A^3 - 4A^2 + 5A - 2I_3 = O_3$

$A^3 - 4A^2 + 5A = 2I_3 \Rightarrow A \cdot \frac{1}{2}(A^2 - 4A + 5I_3) = I_3$

$\frac{1}{2}(A^2 - 4A + 5I_3) \cdot A$

$A^{-1} = \frac{1}{2}(A^2 - 4A + 5I_3) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix}$

Ex2 Fie $A \in SO(3)$. Not. $A^{100} = aA^2 + bA + cI_3$, $a, b, c \in \mathbb{R}$

Dacă $\varepsilon = \frac{-1+i\sqrt{3}}{2}$ este răd a polinomului caracteristic asociat lui A, atunci $a, b, c = ?$

SOL

$A \in SO(3) \Rightarrow A \cdot A^T = I_3$ și $\det A = 1$

$P = X^3 - \sigma_1 X^2 + \sigma_2 X - \sigma_3 \in \mathbb{R}[X]$, $\sigma_3 = \det A$

$x_1 = \varepsilon \in \mathbb{C} \setminus \mathbb{R}$ răd $\Rightarrow \overline{x_1} = \overline{\varepsilon} = x_2$ răd.

$x^3 - \sigma_1 x^2 + \sigma_2 x - 1 = 0$

$s_3 = x_1 x_2 x_3 = 1 \Rightarrow \varepsilon \cdot \overline{\varepsilon} \cdot x_3 = 1 \Rightarrow x_3 = 1$

(a 3-a relație Viete)

$\varepsilon \cdot \overline{\varepsilon} = |\varepsilon|^2 = 1$

OBS

$ax^3 + bx^2 + cx + d = 0$

$s_1 = x_1 + x_2 + x_3 = -\frac{b}{a}$

$s_2 = x_1x_2 + x_1x_3 + x_2x_3 = \frac{c}{a}$

$s_3 = x_1x_2x_3 = -\frac{d}{a}$

P are $1, \varepsilon, \overline{\varepsilon}$ răd

$P = X^3 - 1 \overset{TH-C}{\Rightarrow} A^3 - I_3 = O_3$

$A^3 = I_3 \Longrightarrow A^{100} = (A^3)^{33} \cdot A = \underline{A}$

$A^{100} = a \cdot A^2 + b A + c I_3 = 0 \cdot A^2 + 1 A + 0 I_3$

$\Longrightarrow \begin{cases} a = 0 \\ b = 1 \\ c = 0 \end{cases}$

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