

SEMINAR 6

Aplicații liniare. Matricea asociată

(OBS) • $f: V_1 \rightarrow V_2$ liniară

$$R_1 = \{e_1, \dots, e_n\} \xrightarrow{A} R_2 = \{\bar{e}_1, \dots, \bar{e}_m\}$$

reper în V_1 reper în V_2

$$f(e_i) = \sum_{j=1}^m a_{ji} \bar{e}_j, \quad \forall i = \overline{1, n}, \quad A = (a_{ji})_{\substack{j=\overline{1, m} \\ i=\overline{1, n}}} \in M_{m,n}(\mathbb{K})$$

• f aplicație liniară $\Leftrightarrow \exists A \in M_{m,n}(\mathbb{K})$ aî $Y = AX$

$$f(x) = y, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}, \quad x = \sum_{i=1}^n x_i e_i$$

$$y = \sum_{j=1}^m y_j \bar{e}_j$$

T. dim $f: V_1 \rightarrow V_2$ liniară

$$\dim V_1 = \dim \ker f + \dim \operatorname{Im} f$$

OBS $f(V_1) \subseteq V_2$ subsp. vect., $\dim f(V_1) \leq \dim V_1$

• f inj $\Leftrightarrow \dim V_1 = \operatorname{rg} A$

• f surj $\Leftrightarrow \dim V_2 = \operatorname{rg} A$

• f bij $\Leftrightarrow \dim V_1 = \dim V_2 = \operatorname{rg} A \Leftrightarrow A \in GL(m, \mathbb{K})$

$$\Leftrightarrow [\forall R \text{ reper în } V_1 \Rightarrow f(R) \text{ reper în } V_2]$$

Ex1 $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$f(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, -x_1 - 2x_2 - x_3, x_1 + x_2 + x_3)$$

a) $[f]_{R_0, R_0} = A$, $R_0 = \{e_1, e_2, e_3\}$ reperul canonic din \mathbb{R}^3

b) $\dim \ker f$, $\dim \text{Im } f$

c) $f(V')$, $V' = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 + 2x_2 - x_3 = 0 \end{cases}\}$

SOL

a) $f(e_1) = f(1, 0, 0) = (1, -1, 1) = e_1 - e_2 + e_3$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$f(e_2) = f(0, 1, 0) = (2, -2, 1) = 2e_1 - 2e_2 + e_3$

$f(e_3) = f(0, 0, 1) = (1, -1, 1) = e_1 - e_2 + e_3$

$$f(x) = y \Leftrightarrow Y = AX \Leftrightarrow \begin{pmatrix} x_1 + 2x_2 + x_3 \\ -x_1 - 2x_2 - x_3 \\ x_1 + x_2 + x_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\Leftrightarrow f$ liniară

b) $\ker f = \{x \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3}\} = \{x \in \mathbb{R}^3 \mid AX = 0\} = S(A)$

$\dim \ker f = 3 - \text{rg } A = 3 - 2 = 1$

$\det A = 0$

$\dim \mathbb{R}^3 = \dim \ker f + \dim \text{Im } f \Rightarrow \dim \text{Im } f = 2$

c) $V' = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 + 2x_2 - x_3 = 0 \end{cases}\}$

$$A_{V'} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

$x_2 = \frac{2}{3}x_3$

$x_1 = \frac{2}{3}x_3 - x_3 = -\frac{x_3}{3}$

$$\begin{cases} x_1 - x_2 = -x_3 \\ x_1 + 2x_2 = x_3 \end{cases}$$

$/ \quad 3x_2 = 2x_3$

$V' = \left\{ \left(-\frac{x_3}{3}, \frac{2}{3}x_3, x_3 \right) = \frac{x_3}{3}(-1, 2, 3), x_3 \in \mathbb{R} \right\} = \langle \{(-1, 2, 3)\} \rangle$

$R' = \{(-1, 2, 3)\}$ reper în V'

(dreapta)

$f(-1, 2, 3) = (6, -6, 4) \neq 0_{\mathbb{R}^3}$, $f(V') = \langle \{(6, -6, 4)\} \rangle$

Ex 2

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}^3,$$

$$f(x_1, x_2, x_3) = (2x_1 + 2x_2, x_1 + x_3, x_1 + 3x_2 - 2x_3)$$

a) f liniară, dar nu e izomorfism

b) $f|_{V'}: V' \rightarrow V''$ izomorfism,

$$V' = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\}$$

$$V'' = \{x \in \mathbb{R}^3 \mid 3x_1 - 4x_2 - 2x_3 = 0\}$$

c) $f(V' \cap V'')$

SOL

a) $f(x) = y \Leftrightarrow Y = AX, A = [f]_{R_0, R_0} \Leftrightarrow f$ liniară

$$f(e_1) = f(1, 0, 0) = (2, 1, 1) = 2e_1 + e_2 + e_3$$

$$f(e_2) = f(0, 1, 0) = (2, 0, 3) = 2e_1 + 0e_2 + 3e_3$$

$$f(e_3) = f(0, 0, 1) = (0, 1, -2) = 0e_1 + e_2 - 2e_3$$

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix}$$

(SAU) $y = f(x) \Leftrightarrow Y = AX$

$$\begin{pmatrix} 2x_1 + 2x_2 \\ x_1 + x_3 \\ x_1 + 3x_2 - 2x_3 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & -2 \end{vmatrix} = 0 \Rightarrow A \notin GL(3, \mathbb{R})$$

$\Rightarrow f$ nu e bijectivă

OBS. $\text{Ker } f = \{x \in \mathbb{R}^3 \mid AX = 0\}$

$$\dim \text{Ker } f = 3 - \text{rg } A = 3 - 2 = 1 \Rightarrow f \text{ nu e iny.}$$

$$\dim \mathbb{R}^3 = \dim \text{Ker } f + \dim \text{Im } f \Rightarrow \dim \text{Im } f = 2$$

$\Rightarrow f$ nu e surj

b) $f|_{V'} : V' \rightarrow V''$ ⁻⁴⁻ liniară

$$V' = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\} \quad A_{V'} = (1 \ 1 \ -1)$$

$$V'' = \{x \in \mathbb{R}^3 \mid 3x_1 - 4x_2 - 2x_3 = 0\}$$

$$f \text{ bij} \Leftrightarrow \left[\forall R' \text{ din } V' \Rightarrow f(R') \text{ reper în } V'' \right]$$

$$V' : x_3 = x_1 + x_2$$

$$V' = \{ (x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R} \} = \langle \underbrace{\{(1, 0, 1), (0, 1, 1)\}}_{R'} \rangle$$

$$\Rightarrow R' \text{ este SG pt } V'$$

$$\dim V' = 3 - \text{rg } A_{V'} = 3 - 1 = 2 = |R'| \quad \left. \vphantom{\dim V'} \right\} \Rightarrow R' \text{ reper în } V'$$

$$f(R') = \{ f(1, 0, 1), f(0, 1, 1) \}$$

$$f(1, 0, 1) = (2, 2, -1) \in V'' \quad ; \quad 3 \cdot 2 - 4 \cdot 2 - 2(-1) = 8 - 8 = 0$$

$$f(0, 1, 1) = (2, 1, 1) \in V'' \quad ; \quad 3 \cdot 2 - 4 \cdot 1 - 2 \cdot 1 = 6 - 6 = 0$$

$$V'' : 3x_1 - 4x_2 - 2x_3 = 0$$

Deci $f(R')$ este reper în V''

$$\text{rg} \begin{pmatrix} 2 & 2 \\ 2 & 1 \\ -1 & 1 \end{pmatrix} = 2 = \max \Rightarrow f(R') \text{ este SLI} \quad \left. \vphantom{\text{rg}} \right\} \Rightarrow f(R') \text{ reper în } V''$$

$$\dim V'' = 3 - \text{rg } A_{V''} = 3 - 1 = 2 = |f(R')|$$

Deci f bij dar f lin $\Rightarrow f|_{V'} : V' \rightarrow V''$ izom de spect.

$$c) f(V' \cap V'') \quad V' \cap V'' = \left\{ x \in \mathbb{R}^3 \mid \begin{cases} x_1 + x_2 - x_3 = 0 \\ 3x_1 - 4x_2 - 2x_3 = 0 \end{cases} \right\}$$

$$A_{V' \cap V''} = \begin{pmatrix} 1 & 1 & -1 \\ 3 & -4 & -2 \end{pmatrix}$$

$$x_1 = \frac{6}{7} x_3$$

$$x_2 = x_3 - \frac{6}{7} x_3 = \frac{1}{7} x_3$$

$$\begin{cases} x_1 + x_2 = x_3 \\ 3x_1 - 4x_2 = 2x_3 \end{cases} \cdot 4$$

$$\frac{7x_1}{7x_1} = 6x_3 \quad (4)$$

$$V' \cap V'' = \left\{ \left(\frac{6}{7}x_3, \frac{1}{7}x_3, \frac{-5}{7}x_3 \right) = \frac{x_3}{7}(6, 1, -5), x_3 \in \mathbb{R} \right\} = \langle \{6, 1, -5\} \rangle$$

$$f(6, 1, -5) = (14, 13, -5) \neq 0_{\mathbb{R}^3}$$

$$f(V' \cap V'') = \langle \{(14, 13, -5)\} \rangle \text{ (o dreapta)}.$$

Ex

$$f: \mathbb{R}_3[X] \rightarrow \mathbb{R}_2[X], f(P) = P'$$

a) $[f]_{\mathcal{R}_0, \mathcal{R}_0'}$, $\mathcal{R}_0 = \{1, x, x^2, x^3\}$ reper canonic în $\mathbb{R}_3[X]$
 $\mathcal{R}_0' = \{1, x, x^2\}$ —||— $\mathbb{R}_2[X]$

b) $\dim \ker f$, $\dim \operatorname{Im} f$.

Sol

a) $f(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$, $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

$$f(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$

$$f(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$$

$$f(x^3) = 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2$$

b) $\ker f = \{P \in \mathbb{R}_3[X] \mid P' = 0\} = \mathbb{R}$

$$\dim \ker f = 4 - 3 = 1$$

$$\dim \mathbb{R}_3[X] = \dim \ker f + \dim \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 3$$

f nu e inj; este surj.

Q35

$$\ker f = \langle \{1\} \rangle$$

Extindem la $\{1, x, x^2, x^3\}$ reper în $\mathbb{R}_3[X]$

$\{f(x), f(x^2), f(x^3)\}$ reper în $\operatorname{Im} f$

(din dem Th dimensiunii)

T₃ (suminar)

① $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ liniară

$$f(1,1) = (3,5) \quad ; \quad f(-1,2) = (0,1)$$

a) $f(x) = ?$

b) Este izomorfism?

② $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $f(x_1, x_2, x_3) = (x_1 + x_3, 0, 2x_1 + x_2)$

a) $[f]_{R_0, R_0}$, $R_0 = \{e_1, e_2, e_3\}$ reper canonic în \mathbb{R}^3

b) $\text{Ker } f$, $\text{Im } f$, \dim și câte un reper

③ $V' = \langle \{(1,1,1,0), (1,2,3,4)\} \rangle \subset \mathbb{R}^4$

a) Scrieți un sistem de ec. liniare care îl definesc pe V'

b) $\mathbb{R}^4 = V' \oplus V''$, $V'' = ?$

c) $p: \mathbb{R}^4 \rightarrow \mathbb{R}^4$, $p =$ proiecția pe V' de-a lungul lui V''

$$p(1,2,0,-1) = ?$$

$$s(1,2,0,-1) = ?$$

$$s = 2p - \text{id}_{\mathbb{R}^4}.$$