6.05.2021

Seminar 11

Spații vectoriale euclidiene
Endomorfisme simetrice

OBS $(E, \langle \cdot, \cdot \rangle)$

$f \in End(E)$

- $f \in Sim(E) \iff \langle x, f(y) \rangle = \langle f(x), y \rangle, \ \forall x, y \in E$

$\iff A = [f]_{R,R}$ este simetrica $(A = A^T)$

$\forall R =$ reper ortonormat în E.

(T) $f \in Sim(E) \Rightarrow \exists R$ reper ortonormat aî $[f]_{R,R}$ diagonală

- $f \in Sim(E) \Rightarrow$ toate răd pol caract sunt reale

și

$\dim V_{\lambda_i} = m_i, \ i = \overline{1,r}$

$\lambda_1, \dots, \lambda_r$ val pr dist, $m_1 + \dots + m_r = n$.

- $A = A^T \to f \in Sim(E)$

$\quad \searrow Q : E \to \mathbb{R}$ formă pătratica

$\langle x, f(x) \rangle = Q(x) = X^T A X$

$f(x) = y \iff Y = AX$.

Ex1 $(\mathbb{R}^3, g_0)$, $f \in End(\mathbb{R}^3)$

$$A = [f]_{R_0 R_0} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

a) Dem că $f \in Sim(\mathbb{R}^3)$. Det f.

b) Să se afle $Q: \mathbb{R}^3 \to \mathbb{R}$ forma pătratică asociată

c) Să se aducă Q la o formă canonică, efectuând o transformare ortogonală h (schimbare de repere ortonormate)

Sol

a) $\left.\begin{array}{l} A = A^T \\ f \in End(\mathbb{R}^3) \end{array}\right\} \Rightarrow f \in Sim(\mathbb{R}^3)$ $\quad f(x) = y \Leftrightarrow Y = AX$

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1 + x_3, x_2, x_1 + x_3)$

b) $Q: \mathbb{R}^3 \to \mathbb{R}$ forma pătratică asociată lui f

$$Q(x) = \sum_{i,j=1}^{3} a_{ij} x_i x_j = x_1^2 + 2x_1x_3 + x_2^2 + x_3^2$$

Aplicăm metoda valorilor proprii

$P(\lambda) = \det(A - \lambda I_3) = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(1-\lambda)^2 - 1\right] = 0 \Rightarrow (1-\lambda)(1-\lambda-1)(1-\lambda+1) = 0$$

$$\Rightarrow (1-\lambda)(-\lambda)(2-\lambda) = 0$$

$\lambda_1 = 0$, $m_1 = 1$

$\lambda_2 = 1$, $m_2 = 1$

$\lambda_3 = 2$, $m_3 = 1$.

- $V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = 0\} = \ker f = \left\{x \in \mathbb{R}^3 \,\middle|\, \begin{cases} x_1 + x_3 = 0 \\ x_2 = 0 \end{cases}\right\}$
$= \{(x_1, 0, -x_1) \mid x_1 \in \mathbb{R}\} = \langle\{(1,0,-1)\}\rangle.$

- $V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid f(x) = x\} = \left\{x \in \mathbb{R}^3 \,\middle|\, \begin{cases} x_1 + x_3 = x_1 \\ x_2 = x_2 \\ x_1 + x_3 = x_3 \end{cases}\right\}$
$= \{(0, x_2, 0) \mid x_2 \in \mathbb{R}\} = \langle\{(0,1,0)\}\rangle.$

- $V_{\lambda_3} = \{x \in \mathbb{R}^3 \mid f(x) = 2x\} = \left\{x \in \mathbb{R}^3 \,\middle|\, \begin{cases} x_1 + x_3 = 2x_1 \\ x_2 = 2x_2 \\ x_1 + x_3 = 2x_3 \end{cases}\right\}$
$= \{(x_1, 0, x_1) \mid x_1 \in \mathbb{R}\} = \langle\{(1,0,1)\}\rangle.$

$R = \{\underbrace{\tfrac{1}{\sqrt{2}}(1,0,-1)}_{e_1}, \underbrace{(0,1,0)}_{e_2}, \underbrace{\tfrac{1}{\sqrt{2}}(1,0,1)}_{e_3}\}$ reper ortonormat în $\mathbb{R}^3$ a.î. $[f]_{R,R} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$Q(x) = x_2'^2 + 2x_3'^2$ $\quad$ $(2,0)$ signatura

$R_0 = \{e_1^0, e_2^0, e_3^0\}$ reperul canonic în $\mathbb{R}^3$.

$R_0 \xrightarrow{C} R$, $C \in O(3)$

repere orton.

$e_1 = \frac{1}{\sqrt{2}}(1,0,-1) = \frac{1}{\sqrt{2}} e_1^0 + 0 \cdot e_2^0 - \frac{1}{\sqrt{2}} e_3^0$

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$h \in O(\mathbb{R}^3)$, $h(e_i^0) = e_i$, $i = \overline{1,3}$

$h(x) = \frac{1}{\sqrt{2}}(x_1 + x_3, \sqrt{2}x_2, -x_1 + x_3)$

$h(x) = y$, $Y = CX$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

OBS

a) $Q(x) = \underline{x_1^2 + 2x_1x_3} + x_2^2 + x_3^2$

$= (x_1+x_3)^2 + x_2^2$

$$\begin{cases} x_1' = x_1 + x_3 \\ x_2' = x_2 \\ x_3' = x_3 \end{cases} \Rightarrow Q(x) = x_1'^2 + x_2'^2$$

b) Este $(\mathbb{R}^3, g)$ sp. vect euclidian ? NU

g = forma polară asoc. lui Q ; $Q(x) = g(x,x)$

Q nu e poz def.

EX2 $(\mathbb{R}^3, g_0)$, $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = g_0(x,u) \cdot u$,

unde $u = (1,-1,2)$.

a) Să se arate că $f \in Sim(\mathbb{R}^3)$; $f = ?$

b) Să se afle $Q: \mathbb{R}^3 \to \mathbb{R}$ formă pătratică asociată. Să se aducă la o formă canonică, efectuând o transformare ortogonală h.

SOL

a) $f(x) = g_0(x,u)\, u = (x_1 - x_2 + 2x_3)(1,-1,2)$

$f(x) = (x_1 - x_2 + 2x_3, -x_1 + x_2 - 2x_3, 2x_1 - 2x_2 + 4x_3)$

$$\left.\begin{aligned} &A = [f]_{R_0,R_0} = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix} = A^T \\ &f(x) = y \Leftrightarrow Y = AX \Leftrightarrow f \in End(\mathbb{R}^3) \end{aligned}\right\}$$

$\Rightarrow f \in Sim(\mathbb{R}^3)$

b) $Q: \mathbb{R}^3 \longrightarrow \mathbb{R}$,

$Q(x) = x_1^2 + x_2^2 + 4x_3^2 - 2x_1x_2 + 4x_1x_3 - 4x_2x_3$

$P(\lambda) = \det(A - \lambda I_3) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & -2 \\ 2 & -2 & 4-\lambda \end{vmatrix} = 0$

$\lambda^3 - \sigma_1\lambda^2 + \sigma_2\lambda - \sigma_3 = 0$

$\sigma_1 = Tr A = 6$; $\sigma_2 = \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0$

$\sigma_3 = \det A = 0$

$\lambda^3 - 6\lambda^2 = 0 \Rightarrow \lambda^2(\lambda - 6) = 0$

$\begin{cases} \lambda_1 = 0, \ m_1 = 2 \\ \lambda_2 = 6, \ m_2 = 1 \end{cases}$

• $V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid f(x) = 0 \} = \{ x \in \mathbb{R}^3 \mid x_1 - x_2 + 2x_3 = 0 \}$

$= \{ (\underbrace{x_2 - 2x_3, x_2, x_3}_{x_2(1,1,0) + x_3(-2,0,1)}) \mid x_2, x_3 \in \mathbb{R} \} = \langle \{ \underbrace{(1,1,0)}_{f_1}, \underbrace{(-2,0,1)}_{f_2} \} \rangle$

$\dim V_{\lambda_1} = 2$

$\Rightarrow \{f_1, f_2\}$ reper $\forall$ în V_{λ_1}

Aplicăm procedeul Gram - Schmidt

$\begin{cases} e_1 = f_1 = (1,1,0) \\ e_2 = f_2 - \dfrac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 = (-2,0,1) - \dfrac{-2}{2}(1,1,0) = \end{cases}$

$= (-2,0,1) + (1,1,0) = (-1,1,1)$

$\{e_1, e_2\}$ reper orthogonal în V_{λ_1}

$R_1 = \{ e_1' = \frac{1}{\sqrt{2}}(1,1,0), \ e_2' = \frac{1}{\sqrt{3}}(-1,1,1) \}$ reper orton în V_{λ_1}

$$\bullet\ V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid f(x) = 6x\} = \left\{x \in \mathbb{R}^3 \;\middle|\; \begin{cases} x_1 - x_2 + 2x_3 = 6x_1 \\ -x_1 + x_2 - 2x_3 = 6x_2 \\ 2x_1 - 2x_2 + 4x_3 = 6x_3 \end{cases}\right\}$$

$$= \left\{x \in \mathbb{R}^3 \;\middle|\; \begin{cases} -5x_1 - x_2 + 2x_3 = 0 \\ -x_1 - 5x_2 - 2x_3 = 0 \\ 2x_1 - 2x_2 - 2x_3 = 0 \end{cases}\right\}$$

$$\det\begin{pmatrix} -5 & -1 & 2 \\ -1 & -5 & -2 \\ 2 & -2 & -2 \end{pmatrix} = \begin{vmatrix} -5 & -1 & 2 \\ -6 & -6 & 0 \\ -3 & -3 & 0 \end{vmatrix} = 0$$

$$\begin{cases} -5x_1 - x_2 = -2x_3 \\ -x_1 - 5x_2 = 2x_3 \end{cases} \Rightarrow \quad x_1 = \frac{x_3}{2}, \; x_2 = -\frac{x_3}{2}$$

$$V_{\lambda_2} = \left\{\frac{x_3}{2}(1,-1,2) \mid x_3 \in \mathbb{R}\right\} = \langle \{(1,-1,2)\} \rangle$$

$R_2 = \left\{\frac{1}{\sqrt{6}}(1,-1,2)\right\}$ reper orton în V_{λ_2}.

$\quad\quad\quad$ "e_3'"

$R = R_1 \cup R_2$

reper orton în $\mathbb{R}^3$ aî $A' = [f]_{R,R} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_1 & \\ 0 & & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

$Q(x) = 6x_3'^2 \qquad (1,0)$ signatura

$$R_0 = \{e_1^0, e_2^0, e_3^0\} \xrightarrow{C} R = \left\{e_1' = \frac{1}{\sqrt{2}}(1,1,0),\ e_2' = \frac{1}{\sqrt{3}}(-1,1,1),\ e_3' = \frac{1}{\sqrt{6}}(1,-1,2)\right\}$$

repere ortonormate

$$C = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & -\sqrt{2} & 1 \\ \sqrt{3} & \sqrt{2} & -1 \\ 0 & \sqrt{2} & 2 \end{pmatrix} \in O(3),$$

$h \in O(\mathbb{R}^3)$, $h(e_i^0) = e_i'$, $\forall\, i = \overline{1,3}$

$$h(x) = \frac{1}{\sqrt{6}}\left(\sqrt{3}x_1 - \sqrt{2}x_2 + x_3,\ \sqrt{3}x_1 + \sqrt{2}x_2 - x_3,\ \sqrt{2}x_2 + 2x_3\right)$$

EX3

$(E, <,>)$ s.v.e.r, $\dim E = 2$

$f \in Sim(E)$, $Q_k : E \longrightarrow \mathbb{R}$, $k = \overline{1,3}$

$Q_1(x) = <x, x>$, $Q_2(x) = <f(x), x>$, $Q_3(x) = <f(x), f(x)>$,

$\forall x \in E$ (forme fundamentale)

Să se arate că

$Q_3(x) - Tr(A_f) Q_2(x) + \det(A_f) Q_1(x) = 0, \forall x \in E$ (*)

SOL

$R = \{e_1, e_2\}$ reper orton. in E.

$A_f = [f]_{R,R} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = A_f^T$ ($f \in Sim(E)$)

$Tr(A_f) = a + c$, $\det A_f = ac - b^2$

$f : E \to E$, $f(x) = (ax_1 + bx_2, bx_1 + cx_2)$

$f(e_1) = (a, b) = ae_1 + be_2$

$f(e_2) = (b, c) = be_1 + ce_2$.

1) $Q_3(e_1) = <f(e_1), f(e_1)> = <ae_1 + be_2, ae_1 + be_2> = \underline{a^2 + b^2}$

$-Tr(A_f) Q_2(e_1) = -(a+c) <f(e_1), e_1> =$

$= -(a+c) <ae_1 + be_2, e_1> = \underline{-a(a+c)}$

$\det(A_f) Q_1(e_1) = (ac - b^2) <e_1, e_1> = \underline{ac - b^2}$

2) $Q_3(e_2) = <f(e_2), f(e_2)> = <be_1 + ce_2, be_1 + ce_2> = \underline{b^2 + c^2}$

$-Tr(A_f) Q_2(e_2) = -(a+c) <f(e_2), e_2> =$

$= -(a+c) <be_1 + ce_2, e_2> = \underline{-c(a+c)}$

$\det(A_f) Q_1(e_2) = (ac - b^2) <e_2, e_2> = \underline{ac - b^2}$

$Q_3(e_K) - Tr(A_f) Q_2(e_K) + \det(A_f) Q_1(e_K) = 0, \forall K = \overline{1,2} \Rightarrow$ (*)

Ex4 $(E, <\cdot,\cdot>)$, $u \in E$, $u \neq 0_E$

Fie $s \in End(E)$, s = simetria ortogonală față de hiperplanul $<\{u\}>^{\perp}$

$p \in End(E)$, p = proiecția ortogonală pe $-\|-$

a) $p(x) = x - \dfrac{<x,u>}{<u,u>} \cdot u, \ \forall x \in E$

b) $s(x) = s_{u^{\perp}}(x) = x - 2\dfrac{<x,u>}{<u,u>} u.$

$(s = 2p - id_E)$

SOL

a) $E = <\{u\}> \oplus <\{u\}>^{\perp}$

$\left\{\dfrac{u}{\|u\|}\right\}$ versor în $<\{u\}>$.

reper ortonormat în $<\{u\}>$.

- $x' = x - <x, \dfrac{u}{\|u\|}> \dfrac{u}{\|u\|}, \quad x' \in <\{u\}>^{\perp}$

$<x', u> = <x,u> - <x, \dfrac{u}{\|u\|}> < \dfrac{u}{\|u\|}, u> =$

$= <x,u> - \dfrac{1}{\cancel{\|u\|^2}} <x,u> \cancel{\|u\|^2} = 0$

$x = x'' + x'$

$x'' = <x, \dfrac{u}{\|u\|}> \cdot \dfrac{u}{\|u\|} = \dfrac{<x,u>}{<u,u>} \cdot u \in <\{u\}>$

$p(x) = p(x''+x') = x' = x - \dfrac{<x,u>}{<u,u>} \cdot u.$

b) $s(x) = 2p(x) - x = 2x - 2\dfrac{<x,u>}{<u,u>} \cdot u - x =$

$= x - 2\dfrac{<x,u>}{<u,u>} \cdot u.$

Caz particular

$(\mathbb{R}^3, g_0)$, $u = (1, -1, 0)$

a) Să se scrie s = simetria ortogonală față de planul $\{u\}^{\perp}$

$$s(x) = x - 2\frac{\langle x, u\rangle}{\langle u, u\rangle}\cdot u =$$

$$= (x_1, x_2, x_3) - \cancel{2}\,\frac{x_1 - x_2}{\cancel{2}}(1, -1, 0)$$

$$= (x_1 - x_1 + x_2,\ x_2 + x_1 - x_2,\ x_3) = (x_2, x_1, x_3)$$

b) Să se determine p = proiecția ortogonală pe $\langle\{u\}\rangle^{\perp}$

$$p(x) = x - \frac{\langle x, u\rangle}{\langle u, u\rangle}\,u = (x_1, x_2, x_3) - \frac{x_1 - x_2}{2}(1, -1, 0)$$

$$= \left(x_1 - \frac{x_1 - x_2}{2},\ x_2 + \frac{x_1 - x_2}{2},\ x_3\right) = \left(\frac{x_1 + x_2}{2},\ \frac{x_1 + x_2}{2},\ x_3\right)$$

EX5 $(\mathbb{R}^3, g_0)$

$Q: \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = 4x_1^2 + 2x_1x_2 + 2x_1x_3 + 4x_2^2 + 2x_2x_3 + 4x_3^2$

Să se aducă la o formă canonică, utilizând metoda Jacobi, metoda Gauss și metoda valorilor proprii.

SOL

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix} = A^T$$

1) Metoda Jacobi

$\Delta_1 = 4$; $\Delta_2 = \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = 15$, $\Delta_3 = \det A = 6\begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} =$

$Q(x) = \frac{1}{\Delta_1}x_1'^2 + \frac{\Delta_1}{\Delta_2}x_2'^2 + \frac{\Delta_2}{\Delta_3}x_3'^2$; $\quad = 6\begin{vmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 6\cdot 9$

$Q(x) = \frac{1}{4}x_1'^2 + \frac{4}{15}x_2'^2 + \frac{5}{18}x_3'^2$.

2) Met. Gauss

$$Q(x) = 4x_1^2 + 2x_1x_2 + 2x_1x_3 + 4x_2^2 + 2x_2x_3 + 4x_3^2$$

$$= \left(2x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3\right)^2 \underbrace{- \frac{1}{4}x_2^2 - \frac{1}{4}x_3^2 - \frac{1}{2}x_2x_3 + 4x_2^2 + 2x_2x_3 + 4x_3^2}$$

$$\frac{15}{4}x_2^2 + \frac{3}{2}x_2x_3 + \frac{15}{4}x_3^2$$

$$\frac{15}{4}\left(x_2^2 + \frac{2}{5}x_2x_3\right) + \frac{15}{4}x_3^2$$

$$\frac{15}{4}\left(x_2 + \frac{1}{5}x_3\right)^2 - \frac{3}{20}x_3^2 + \frac{15}{4}x_3^2$$

$$Q(x) = \left(2x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3\right)^2 + \frac{15}{4}\left(x_2 + \frac{1}{5}x_3\right)^2 + \frac{18}{5}x_3^2$$

$$\begin{cases} x_1' = 2x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ x_2' = x_2 + \frac{1}{5}x_3 \\ x_3' = x_3 \end{cases} \Rightarrow Q(x) = x_1'^2 + \frac{15}{4}x_2'^2 + \frac{18}{5}x_3'^2$$

3) Met. valorilor proprii

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = (6-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 1 & 3-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(3-\lambda)^2 = 0$$

$\lambda_1 = 6$, $m_1 = 1$

$\lambda_2 = 3$, $m_2 = 2$

$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ reper ortonormat în $\mathbb{R}^3$,

unde $\mathcal{R}_1$ —〃— în V_{λ_1}

$\mathcal{R}_2$ —〃— în V_{λ_2}

Q formă pătratică $\longrightarrow f \in Sim(\mathbb{R}^3)$

$<x, f(x)> = Q(x)$

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3, f(x) = (4x_1 + x_2 + x_3, x_1 + 4x_2 + x_3, x_1 + x_2 + 4x_3)$

$[f]_{R,R} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$Q(x) = 6x_1'^2 + 3x_2'^2 + 3x_3'^2$

OBS $(3,0)$ signatura $\Rightarrow Q$ poz. def

$g =$ forma polară asociată

$(\mathbb{R}^3, g)$ sp. vect. euclidian.

Ex 6. $(\mathbb{R}^3, g_0)$

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = \left(\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{3}} + cx_3, ax_1 + \frac{1}{\sqrt{3}}x_2 + dx_3, \frac{x_1}{\sqrt{2}} + bx_2 + ex_3\right)$

a) Să se det $a, b, c, d, e \in \mathbb{R}$ a.î. $f \in O(\mathbb{R}^3)$

b) —— // —— $f \in SO(\mathbb{R}^3)$

Sol

a) $\mathcal{R}_0$ = reperul canonic

$$A = [f]_{\mathcal{R}_0, \mathcal{R}_0} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & c \\ a & \frac{1}{\sqrt{3}} & d \\ \frac{1}{\sqrt{2}} & b & e \end{pmatrix}$$

$A \in O(3) \iff \mathcal{R} = \left\{ e_1 = \left(\frac{1}{\sqrt{2}}, a, \frac{1}{\sqrt{2}}\right), e_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, b\right), e_3 = (c, d, e) \right\}$ reper ortonormat în $\mathbb{R}^3$

$$\mathcal{R}_0 \xrightarrow{A} \mathcal{R}$$

1) $\|e_1\| = 1 \Rightarrow \frac{1}{2} + a^2 + \frac{1}{2} = 1 \Rightarrow \boxed{a = 0}$

2) $\|e_2\| = 1 \Rightarrow \frac{1}{3} + \frac{1}{3} + b^2 = 1 \Rightarrow b^2 = \frac{1}{3}$

3) $\|e_3\| = 1 \Rightarrow c^2 + d^2 + e^2 = 1$ (*)

4) $\langle e_1, e_2 \rangle = 0 \Rightarrow \frac{1}{\sqrt{6}} + \frac{a}{\sqrt{3}} + \frac{b}{\sqrt{2}} = 0 \Rightarrow \boxed{b = -\frac{1}{\sqrt{3}}}$

5) $\langle e_1, e_3 \rangle = 0 \Rightarrow \frac{c}{\sqrt{2}} + ad + \frac{e}{\sqrt{2}} = 0 \Rightarrow c = -e$ (**)

6) $\langle e_2, e_3 \rangle = 0 \Rightarrow \frac{c}{\sqrt{3}} + \frac{d}{\sqrt{3}} + eb = 0 \Rightarrow$

$-\frac{e}{\sqrt{3}} + \frac{d}{\sqrt{3}} - \frac{e}{\sqrt{3}} = 0 \Rightarrow d = 2e$ (***)

(*), (**), (***) $\Rightarrow e^2 + 4e^2 + e^2 = 1 \Rightarrow e^2 = \frac{1}{6} \Rightarrow e = \pm\frac{1}{\sqrt{6}}$

(I) $a = 0, b = -\frac{1}{\sqrt{3}}, c = -\frac{1}{\sqrt{6}}, d = \frac{2}{\sqrt{6}}, e = \frac{1}{\sqrt{6}}$

(II) $a = 0, b = -\frac{1}{\sqrt{3}}, c = \frac{1}{\sqrt{6}}, d = -\frac{2}{\sqrt{6}}, e = -\frac{1}{\sqrt{6}}$

I

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\det A = \frac{1}{6} \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = \frac{1}{6} \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & -2 & 2 \end{vmatrix}$$

$= \frac{1}{6}(2+4) = 1 \Rightarrow A \in SO(3)$ și $f \in SO(\mathbb{R}^3)$ pt b)

II

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\det A = \frac{1}{6} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{vmatrix} = -1 \Rightarrow$$

$f \in O(\mathbb{R}^3)$ de speță 2.