CURS 14

25.05.2021

Hipercuadrice.
Conice. Aducerea la formă canonică
Cuadrice studiate pe ecuații reduse.

Def. $(\mathbb{R}^n, \mathbb{R}^n/_{\mathbb{R}}, \varphi)$ sp. afin.

$\tau: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ s.n. transformare afină $\Leftrightarrow$

1) τ aplicație afină: $\tau(aP+bQ) = a\tau(P)+b\,\tau(Q)$

$a+b=1$, $a, b \in \mathbb{R}$, $P, Q \in \mathbb{R}^n$

2) τ bijecție.

Prop $\tau: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ transf. afină $\Leftrightarrow$

$\tau: X' = AX + B$, $A \in GL(n, \mathbb{R})$

OBS a) $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $T: X' = AX$ urma lui τ, apl. liniară (izom. de sp. vect)

b) $\tau: X' = X + B$ translație

$\tau: X' = AX$ centro-afinitate

Def $(\mathcal{E}_n, (\mathbb{E}_n, \langle\,;\,\rangle), \varphi)$ sp. afin euclidian.

$\tau: \mathcal{E}_n \longrightarrow \mathcal{E}_n$ izometrie $\Leftrightarrow d(P, Q) = d(\tau(P), \tau(Q))$

$\forall P, Q \in \mathcal{E}_n$. (păstrează distanța)

Prop $\tau: \mathcal{E}_n \to \mathcal{E}_n$ izometrie $\Leftrightarrow \tau: X' = AX+B$,

$A \in O(n)$ i.e. $T: E_n \to E_n$, $T: X' = AX$

transformare liniară ortogonală.

Not $(\mathcal{I}so(\mathcal{E}_n), \circ)$ grupul izometriilor

$\tau \in \mathcal{I}zo(\mathcal{E}_n)$ s.n. de speță 1 (resp. speța 2) dacă

$T \in O(E_n)$ transf. ortog de speță 1 (resp 2)

Def $(\mathbb{R}^n, \mathbb{R}^n/_{\mathbb{R}}, \varphi)$ (sau $(\mathcal{E}_n, (E_n \langle ; \rangle), \varphi)$)

$\mathcal{R} = \{O; e_1, .., e_n\}$ reper cartezian.

S.n. hipercuadrică în $\mathbb{R}^n$ L.G. al punctelor

$P(x_1, .., x_n)$ aî

$$\Gamma : f(x) = a_{11} x_1^2 + \ldots + a_{nn} x_n^2 + 2a_{12} x_1 x_2 + \ldots + 2 a_{n-1 n} x_{n-1} x_n$$

$$+ 2b_1 x_1 + \ldots + 2b_n x_n + c = 0$$

$$\Gamma \quad X^T A X + 2BX + c = 0$$

$A = A^T$, $rg A \geq 1$, $A = (a_{ij})_{i,j = \overline{1,n}}$

$$\tilde{A} = \left(\begin{array}{c|c} A & B^T \\ \hline B & c \end{array}\right) \qquad B = (b_1 \ldots b_n)$$

$r = rg A$, $r' = rg \tilde{A}$, $r \leq r' \leq r + 2$

$\delta = \det A$, $\Delta = \det \tilde{A}$

Dacă $\Delta = 0$, at Γ s.n. hipercuadrică degenerată

$\Delta \neq 0$ — // — nedegenerată.

OBS a) $(\mathbb{R}^n, \mathbb{R}^n/_{\mathbb{R}}, \varphi)$ sp. afin.

$\Gamma_1 \sim \Gamma_2$ afin echivalente $\Leftrightarrow \exists\, \tau : \mathbb{R}^n \to \mathbb{R}^n$

transf. afină aî $\Gamma_2 = \tau(\Gamma_1)$

$\tau : X' = CX + D$, $C \in GL(n, \mathbb{R})$

Invarianți afini Δ / δ ; r, r'

b) $(\mathcal{E}_n, (E_n, \langle , \rangle), \varphi)$ sp. punctual euclidian

$\Gamma_1 \sim \Gamma_2$ congruente metric $\Leftrightarrow \exists\, \tau \in Iso(\mathcal{E}_n)$ a.î. $\Gamma_2 = \tau(\Gamma_1)$ $\quad \tau: X' = CX + D,\ C \in O(n)$

Invarianți metrici: $\frac{\Delta}{\delta}, r, r', \Delta, \delta$

2

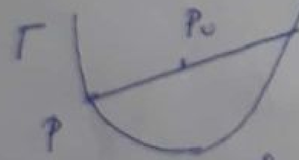
- $n = 2 \Rightarrow \Gamma =$ conică
- $n = 3 \Rightarrow \Gamma =$ cuadrice.

Aducerea la formă canonică a conicelor

$\Gamma: X^T A X + 2BX + c = 0$

$\Gamma: \underbrace{a_{11} x_1^2 + a_{22} x_2^2 + 2a_{12} x_1 x_2 + 2b_1 x_1 + 2b_2 x_2 + c}_{f(x)=} = 0$

Def P_0 s.n. centru pentru $\Gamma \Leftrightarrow [\forall P \in \Gamma \Leftrightarrow S_{P_0}(P) \in \Gamma]$



$P' = S_{P_0}(P)$

Obs $P_0: \begin{cases} \frac{\partial f}{\partial x_1}(x) = 0 \\ \frac{\partial f}{\partial x_2}(x) = 0 \end{cases} \Rightarrow \begin{cases} 2a_{11}x_1 + 2a_{12}x_2 + 2b_1 = 0 \\ 2a_{12}x_1 + 2a_{22}x_2 + 2b_2 = 0 \end{cases}$

$AX + B^T = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad (*) \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix}$

- $P_0(x_1^0, x_2^0)$ centrul

$(*)$ are sol unică (centru unic) $\Leftrightarrow \delta = \det A \neq 0$

Prop Dacă $\delta \neq 0$, atunci $f(x_1^0, x_2^0) = \frac{\Delta}{\delta}$ unde $P_0(x_1^0, x_2^0)$ este centrul conicei.

I) $\delta \neq 0$ (centru unic)

a) $(\mathbb{R}^2, \mathbb{R}^2_{/\mathbb{R}}, \varphi)$ sp. afin

$$\mathcal{R}=\{O; e_1, e_2\} \xrightarrow[\text{translatie}]{\theta} \mathcal{R}'=\{P_0; e_1, e_2\} \xrightarrow[\substack{\text{transf. afină} \\ \text{(centro-afinitate)}}]{\tau} \mathcal{R}''=\{P_0; e_1', e_2'\}$$

$\theta: X = X' + X_0$

$\theta(\Gamma): \quad X'^T A X' + \dfrac{\Delta}{\delta} = 0.$

$Q: \mathbb{R}^2 \longrightarrow \mathbb{R}, \ Q(x) = X'^T A X'$ formă pătratică.

Aducem Q la o formă canonică (met. Gauss)

$Q(x) = \lambda_1 x_1''^2 + \lambda_2 x_2''^2$

$\tau: X' = CX'', \quad C \in GL(2, \mathbb{R})$

$\tau(\theta(\Gamma)): \quad \lambda_1 x_1''^2 + \lambda_2 x_2''^2 + \dfrac{\Delta}{\delta} = 0.$

$\overset{\shortparallel}{\Gamma'}$

Γ, Γ' conice afin echivalente

$X \longrightarrow X' + X_0 \longrightarrow CX'' + X_0$ transf. afină

$\underbrace{\hspace{6em}}_{\tau \circ \theta}$

b) $(\mathcal{E}_2, (E_2, \langle , \rangle), \varphi)$ sp. afin euclidian.

$Q: \mathbb{R}^2 \longrightarrow \mathbb{R}, \ Q(x) = X'^T A X'$

$\exists$ un reper ortonormat format din vectori proprii ai A = diagonală

$P(\lambda) = \det(A - \lambda I_2) = 0_2$

1) $\lambda_1 \neq \lambda_2$, $m_1 = m_2 = 1$.

$V_{\lambda_i} = \langle \{e_i'\} \rangle$, $i = \overline{1,2}$ $\langle e_i', e_j' \rangle = \delta_{ij}$

$e_1' = (l_1, m_1)$, $e_2' = (l_2, m_2)$

$R = \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \in SO(2)$

(dacă $\det R = -1$, se schimbă col)

$\tau : X' = RX''$ izometrie de speța 1 (rotație)

$\tau \circ \theta(\Gamma) : \lambda_1 x_1''^2 + \lambda_2 x_2''^2 + \frac{\Delta}{\delta} = 0$

$\overset{\shortparallel}{\Gamma'}$

Γ, Γ' = conice congruente metric

$X \longrightarrow X' + X_0 \longrightarrow RX'' + X_0$

2) $\lambda_1 = \lambda_2$, $m_1 = 2$

$V_{\lambda_1} = \langle \{f_1, f_2\} \rangle$. Aplicăm G-S $\Rightarrow \{e_1', e_2'\}$ reper ortonormat

Analog.

(II) $\delta = 0$ (centrul nu e unic)

a) $(\mathbb{R}^2, \mathbb{R}^2/_\mathbb{R}, \varphi)$ sp. afin.

$\mathcal{R} = \{O; e_1, e_2\} \xrightarrow{\theta} \mathcal{R}' = \{O; e_1', e_2'\} \longrightarrow \mathcal{R}'' = \{P; e_1', e_2'\}$

centro-afinitate translație

$Q : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $Q(x) = X^T A X$ (met. Gauss)

$\theta : X = CX'$ $Q(x) = \lambda_1 x_1'^2$, $\lambda_1 \neq 0$

$\begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix}$

$\theta(\Gamma) : \lambda_1 x_1'^2 + 2b_1' x_1' + 2b_2' x_2' + c = 0$

$\lambda_1 x_1'^2 + \underline{2b_1' x_1' + 2b_2' x_2'} + c = 0$

$$\Delta = \begin{vmatrix} \lambda_1 & 0 & b_1' \\ 0 & 0 & b_2' \\ b_1' & b_2' & c \end{vmatrix} = -\lambda_1 b_2'^2$$

1. $\Delta = 0 \Rightarrow b_2' = 0 \Rightarrow$

$\theta(\Gamma) \quad \lambda_1 x_1'^2 + 2b_1' x_1' + c = 0.$

$$\lambda_1 \underbrace{\left(x_1'^2 + 2\frac{b_1'}{\lambda_1} x_1' + \left(\frac{b_1'}{\lambda_1}\right)^2\right)}_{\left(x_1' + \frac{b_1'}{\lambda_1}\right)^2} \underbrace{- \frac{b_1'^2}{\lambda_1} + c}_{c'} = 0$$

$$\tau : \begin{cases} x_1'' = x_1' + \frac{b_1'}{\lambda_1} \\ x_2'' = x_2' \end{cases} \Rightarrow \tau(\theta(\Gamma)) : \lambda_1 x_1''^2 + c' = 0$$

(dreaptă dublă drepte //)

$$\tau : X' = X'' + X_0 , \quad X_0 = \begin{pmatrix} -\frac{b_1'}{\lambda_1} \\ 0 \end{pmatrix}$$

$X \rightarrow CX' \longrightarrow C(X'' + X_0) = CX'' + CX_0$

$CX_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $\quad P(\alpha, \beta)$ în rap cu $\mathcal{R}$.

2. $\Delta \neq 0 \Rightarrow b_2' \neq 0.$

$\lambda_1 x_1'^2 + 2b_1' x_1' + 2b_2' x_2' + c = 0$

$$\lambda_1 \left(x_1' + \frac{b_1'}{\lambda_1}\right)^2 + 2b_2'\left(x_2' + \frac{c'}{2b_2'}\right) = 0, \quad c' = c - \frac{b_1'^2}{\lambda_1}$$

$$\tau : \begin{cases} x_1'' = x_1' + \frac{b_1'}{\lambda_1} \\ x_2'' = x_2' + \frac{c'}{2b_2'} \end{cases} \quad \tau(\theta(\Gamma)) : \lambda_1 x_1''^2 + 2b_2' x_2'' = 0$$

parabolă.

$\tau : X' = X'' + X_0. \quad ; \quad X \rightarrow CX' \rightarrow CX'' + CX_0$

b) $(\mathcal{E}_2, (E_2, \langle\cdot,\cdot\rangle), \varphi)$ sp. punctual euclidian.

$Q : \mathbb{R}^2 \to \mathbb{R}$, $Q(x) = X^T A X$

Aducem Q la o formă canonică, utilizând metoda valorilor proprii

$$\lambda^2 - Tr(A)\lambda + \underset{\underset{\underset{0}{\parallel}}{\delta}}{\overset{}{\det A}} = 0 \Rightarrow \begin{array}{l}\lambda_1 \neq 0 \\ \lambda_2 = 0.\end{array}$$

$$\mathcal{R} = \{O; e_1, e_2\} \xrightarrow[\text{rotație}]{\theta} \mathcal{R}' = \{O; e_1', e_2'\} \xrightarrow[\text{translație}]{\tau} \mathcal{R}'' = \{P; e_1', e_2'\}$$

$e_k' =$ versor propriu al val. proprii λ_k, $k = \overline{1,2}$

$e_1' = (l_1, m_1)$, $e_2' = (l_2, m_2)$

$$\theta : X = RX', \quad R = \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix}$$

Alegem $R \in SO(2)$ (θ = rotație)

Discuția este analoagă cazului a)

$\tau : X' = X'' + X_0$ translație.

$\tau \circ \theta : X \to RX' = RX'' + RX_0$, $RX_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$P(\alpha, \beta)$ în rap. cu $\mathcal{R}$.

$\boxed{\text{OBS}}$

1) $\Delta \neq 0$ (conică nedeg).

$\delta > 0 \Rightarrow$ Elipsă sau ϕ

$\delta < 0 \Rightarrow$ Hiperbola

$\delta = 0 \Rightarrow$ Parabola.

2) $\Delta = 0$ $\begin{cases} \delta > 0 & \text{Punct dublu} \\ \delta < 0 & \text{Drepte concurente} \\ \delta = 0 & \text{Drepte } \parallel\text{, confundate, } \phi \end{cases}$

Aplicatii

Ex1 ($\delta \neq 0$)

În sp. euclidian $\mathcal{E}_2$ se consideră conica

$$\Gamma: f(x) = \underline{7x_1^2 - 8x_1x_2 + x_2^2} - \underline{6x_1 - 12x_2} - 9 = 0$$

Să se aducă la o formă canonică, utilizând izometrii. Reprez. grafică

SOL

$$A = \begin{pmatrix} 7 & -4 \\ -4 & 1 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 7 & -4 & -3 \\ -4 & 1 & -6 \\ -3 & -6 & -9 \end{pmatrix}$$

$\delta = \det A = 7 - 16 = -9 \neq 0$ (Γ are centru unic)

$\Delta = \det \tilde{A} = -9 \cdot 36 \neq 0$ (Γ nedegenerată)

Det. centrul conicei

$$P_0: \begin{cases} \frac{\partial f}{\partial x_1} = 0 \\ \frac{\partial f}{\partial x_2} = 0 \end{cases} \Rightarrow \begin{cases} 14x_1 - 8x_2 - 6 = 0 \\ -8x_1 + 2x_2 - 12 = 0 \end{cases}$$

$$\begin{cases} 7x_1 - 4x_2 = 3 \\ -4x_1 + x_2 = 6 \quad | \cdot 4 \end{cases}$$

$$-9x_1 \quad / = 27$$

$x_1 = -3$

$x_2 = 6 - 12 = -6$

$P_0(-3, -6)$.

$$\mathcal{R} = \{O; e_1, e_2\} \xrightarrow[\text{translatie}]{\theta} \mathcal{R}' = \{P_0; e_1, e_2\} \xrightarrow[\text{rotatie}]{\tau} \mathcal{R}'' = \{P_0; e_1', e_2'\}$$

$\theta: X = X' + X_0, \quad X_0 = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$

$$\theta(\Gamma): X'^T A X' + \frac{\Delta}{\delta_{11}} = 0$$

36

$\theta(\Gamma)$: $\underbrace{7x_1'^2 - 8x_1'x_2' + x_2'^2} + 36 = 0$.

$Q: \mathbb{R}^2 \to \mathbb{R}$, $Q(x) = 7x_1'^2 - 8x_1'x_2' + x_2'$

Aplicăm met val. proprii

$\lambda^2 - Tr(A)\lambda + \det(A) = 0 \Rightarrow \lambda^2 - 8\lambda - 9 = 0$

$(\lambda+1)(\lambda-9) = 0$

1. $\lambda_1 = -1$, 2. $\lambda_2 = 9$.

$V_{\lambda_1} = \{x \in \mathbb{R}^2 \mid AX = -X\}$

$(A + I_2)X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-4x_1 + 2X_2 = 0 \Rightarrow X_2 = 2X_1$

$V_{\lambda_1} = \{(X_1, 2X_1) = X_1\underline{(1,2)},\ X_1 \in \mathbb{R}\}$

$e_1' = \frac{1}{\sqrt{5}}(1,2)$

$V_{\lambda_2} = \{x \in \mathbb{R}^2 \mid AX = 9X\}$

$(A - 9I_2)X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & -4 \\ -4 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-2X_1 - 4X_2 = 0 \Rightarrow X_1 = -2X_2$

$V_{\lambda_2} = \{(-2X_2, X_2) = X_2(-2,1),\ X_2 \in \mathbb{R}\}$

$e_2' = \frac{1}{\sqrt{5}}(-2,1)$.

$\tau: X' = RX''$, $R = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \in SO(2)$.

$\tau \circ \theta: X = RX'' + X_0$ (izometrie).

$\tau \circ \theta(\Gamma)$: $-X_1''^2 + 9X_2''^2 + 36 = 0$

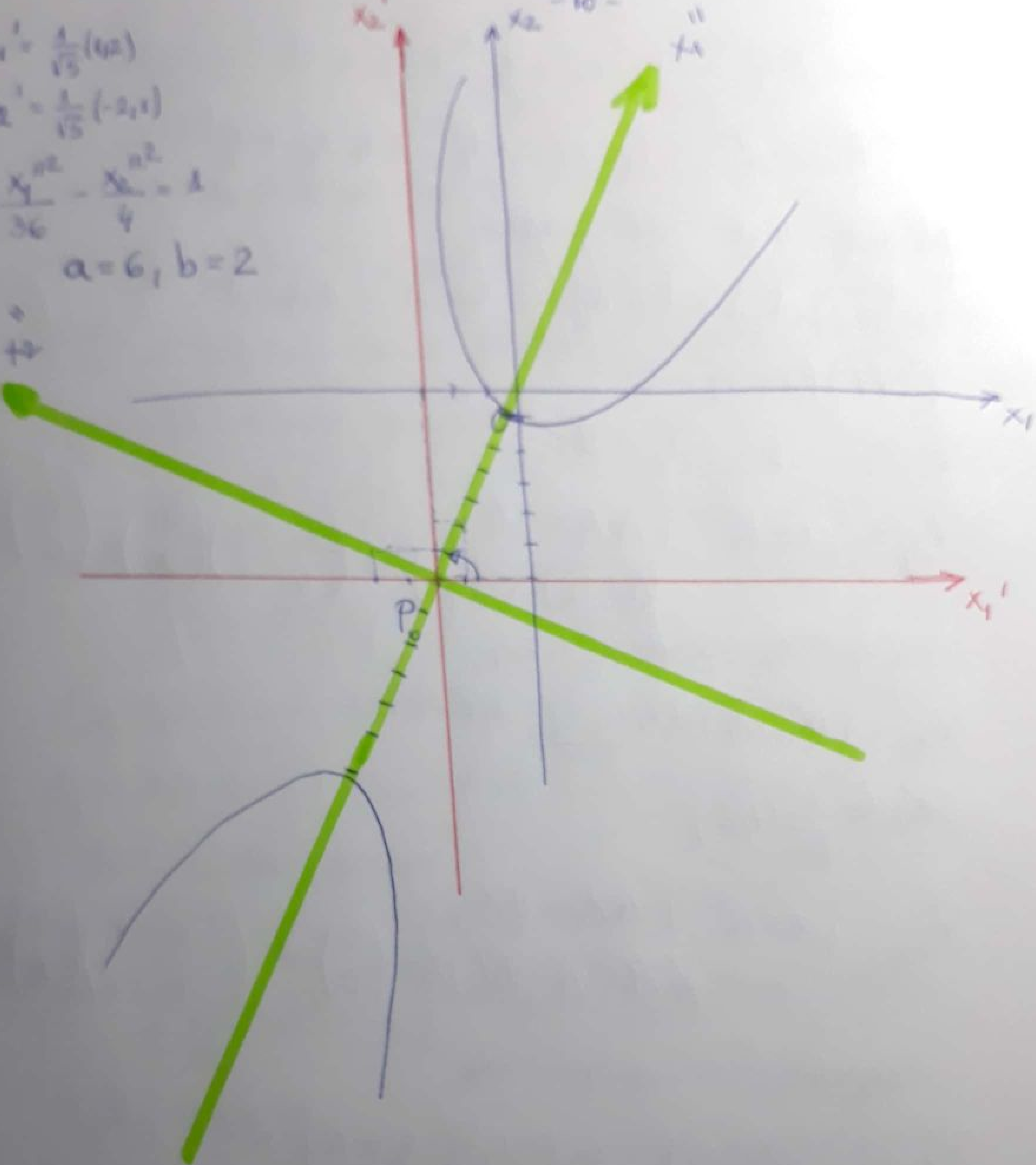
$\frac{X_1''^2}{36} - \frac{X_2''^2}{4} = 1$ (hiperbolă)

$e_1' = \frac{1}{\sqrt{5}}(1,2)$

$e_2' = \frac{1}{\sqrt{5}}(-2,1)$

$\frac{x_1''^2}{36} - \frac{x_2''^2}{4} = 1$

$a = 6,\ b = 2$



Ex2 $(\delta \neq 0)$ În planul euclidian $\mathcal{E}_2$ se consideră conica

$\Gamma\ f(x) = x_1^2 - 4x_1x_2 + 4x_2^2 - 6x_1 + 2x_2 + 1 = 0$

Să se aducă la o formă canonică, utilizând izometrii.

SOL

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 1 & -2 & -3 \\ -2 & 4 & 1 \\ -3 & 1 & 1 \end{pmatrix}$$

$\delta = 0$ ($\nexists!$ centru); $\Delta = -25 \neq 0$ (Γ nedegenerată).

$\mathcal{R}=\{O; e_1, e_2\} \xrightarrow[\text{rotație}]{\tau} \mathcal{R}'=\{O; e_1', e_2'\} \xrightarrow[\text{translație}]{\theta} \mathcal{R}''=\{P; e_1', e_2'\}$

$Q: \mathbb{R}^2 \to \mathbb{R}, \ Q(x) = x_1^2 - 4x_1x_2 + 4x_2^2$

$\lambda^2 - 5\lambda = 0 \Rightarrow \lambda(\lambda-5) = 0$

$\lambda_1 = 5, \quad \lambda_2 = 0.$ $\qquad \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$

$V_{\lambda_1} = \{ x \in \mathbb{R}^2 \mid AX = 5X \}$

$(A - 5I_2)X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-2x_1 - x_2 = 0 \Rightarrow x_2 = -2x_1$

$V_{\lambda_1} = \{ (x_1, -2x_1) = x_1(1,-2), \ x_1 \in \mathbb{R} \}$

$e_1' = \frac{1}{\sqrt{5}}(1,-2)$

$V_{\lambda_2} = \{ x \in \mathbb{R}^2 \mid AX = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$

$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 - 2x_2 = 0 \Rightarrow x_1 = 2x_2$

$V_{\lambda_2} = \{ \underset{x_2(2,1)}{\underset{\parallel}{(2x_2, x_2)}}, \ x_2 \in \mathbb{R} \}$

$e_2' = \frac{1}{\sqrt{5}}(2,1).$

$\theta: X = RX', \quad R = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \in SO(2).$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} x_1' + 2x_2' \\ -2x_1' + x_2' \end{pmatrix} \qquad \begin{cases} x_1 = \frac{1}{\sqrt{5}}(x_1' + 2x_2') \\ x_2 = \frac{1}{\sqrt{5}}(-2x_1' + x_2') \end{cases}$

$\theta(\Gamma): 5x_1'^2 - \frac{6}{\sqrt{5}}(\underline{x_1'} + \underline{\underline{2x_2'}}) + \frac{2}{\sqrt{5}}(\underline{-2x_1'} + \underline{\underline{x_2'}}) + 1 = 0$

$5x_1'^2 - \frac{10}{\sqrt{5}}x_1' - \frac{10}{\sqrt{5}}x_2' + 1 = 0 \quad | : 5$

Scanned with CamScanner

$$\theta(\Gamma)\quad \underbrace{x_1'^2-\frac{2}{\sqrt5}x_1'+\frac15}_{\left(x_1'-\frac{1}{\sqrt5}\right)^2}-\frac{2}{\sqrt5}x_2'=0$$

$$\tau\ \begin{cases} x_1''=x_1'-\frac{1}{\sqrt5} \\ x_2''=x_2' \end{cases} \Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}=\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix}+\begin{pmatrix} \frac{1}{\sqrt5} \\ 0 \end{pmatrix}$$

$\tau: X'=X''+X_0,\ X_0=\begin{pmatrix} \frac{1}{\sqrt5} \\ 0 \end{pmatrix}$ translația.

$$\tau(\theta(\Gamma))\quad x_1''^2=\frac{2}{\sqrt5}x_2''$$

$$\tau\circ\theta:\ X=RX'=R(X''+X_0)=RX''+RX_0.$$

$$RX_0=\frac{1}{\sqrt5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt5} \\ 0 \end{pmatrix}=\frac15\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=$$

$$=\frac15\begin{pmatrix} 1 \\ -2 \end{pmatrix}\qquad \underline{P}\left(\frac15,-\frac25\right)$$

în rap cu $\mathcal{R}$.

$e_1'=\frac{1}{\sqrt5}(1,-2)$

$e_2'=\frac{1}{\sqrt5}(2,1)$