Seminar 10

1. Determinați punctele de extrem local, precizând natura lor, pentru funcțiile:

a) $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 + 8y^3 - 2xy$.

<u>Soluție</u>. $\mathbb{R}^2$ deschisă.

Determinăm punctele critice ale lui f.

f continuă.

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 - 2y \quad \forall (x,y) \in \mathbb{R}^2.$$

$$\frac{\partial f}{\partial y}(x,y) = 24y^2 - 2x \quad \forall (x,y) \in \mathbb{R}^2.$$

$$\left.\begin{array}{l} \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ continue pe } \mathbb{R}^2 \\ \mathbb{R}^2 \text{ deschisă} \end{array}\right| \Rightarrow f \text{ diferențiabilă pe } \mathbb{R}^2.$$

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 - 2y = 0 \\ 24y^2 - 2x = 0 \,|:2 \end{cases} \Leftrightarrow \begin{cases} 3x^2 - 2y = 0 \\ 12y^2 - x = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 3\cdot 144 \cdot y^4 - 2y = 0 \,|:2 \\ x = 12y^2 \end{cases} \Leftrightarrow \begin{cases} 216y^4 - y = 0 \\ x = 12y^2 \end{cases} \Leftrightarrow \begin{cases} y(216y^3 - 1) = 0 \\ x = 12y^2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = 0 \\ x = 0 \end{cases} \text{ sau } \begin{cases} 216y^3 - 1 = 0 \\ x = 12y^2 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = 0 \end{cases} \text{ sau } \begin{cases} y^3 = \frac{1}{216} \\ x = 12y^2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y=0 \\ x=0 \end{cases} \text{ sau } \begin{cases} y=\frac{1}{6} \\ x=\frac{1}{3}. \end{cases}$$

Punctele critice ale lui f sunt: $(0,0)$ și $\left(\frac{1}{3}, \frac{1}{6}\right)$.

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) = 6x \quad \forall (x,y)\in \mathbb{R}^2.$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y) = 48y \quad \forall (x,y)\in \mathbb{R}^2.$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) = -2 = \frac{\partial^2 f}{\partial y \partial x}(x,y) \quad \forall (x,y)\in \mathbb{R}^2.$$

Toate derivatele parțiale de ordinul doi ale lui f sunt continue.

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 6x & -2 \\ -2 & 48y \end{pmatrix} \quad \forall (x,y)\in \mathbb{R}^2.$$

$$H_f(0,0) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

$$\left.\begin{array}{l} \Delta_1 = 0 \\ \Delta_2 = \begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix} = 0-4 = -4 < 0 \end{array}\right\} \Rightarrow (0,0) \text{ nu este punct de extrem local al lui } f.$$

$$H_f\left(\frac{1}{3}, \frac{1}{6}\right) = \begin{pmatrix} 2 & -2 \\ -2 & 8 \end{pmatrix}$$

$\Delta_1 = 2 > 0$

$\Delta_2 = \begin{vmatrix} 2 & -2 \\ -2 & 8 \end{vmatrix} = 16 - 4 = 12 > 0$ $\Rightarrow \left(\frac{1}{3}, \frac{1}{6}\right)$ punct de minim local al lui f. $\square$

b) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = x^2 + y^2 + z^2 - xy + x - 2z$.

Soluție. $\mathbb{R}^3$ deschisă

Determinăm punctele critice ale lui f.

f continuă

$\frac{\partial f}{\partial x}(x, y, z) = 2x - y + 1 \quad \forall (x, y, z) \in \mathbb{R}^3$

$\frac{\partial f}{\partial y}(x, y, z) = 2y - x \quad \forall (x, y, z) \in \mathbb{R}^3$.

$\frac{\partial f}{\partial z}(x, y, z) = 2z - 2 \quad \forall (x, y, z) \in \mathbb{R}^3$.

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ continue pe $\mathbb{R}^3$

$\mathbb{R}^3$ deschisă

$\Rightarrow f$ diferențiabilă pe $\mathbb{R}^3$.

$$\begin{cases} \frac{\partial f}{\partial x}(x, y, z) = 0 \\ \frac{\partial f}{\partial y}(x, y, z) = 0 \\ \frac{\partial f}{\partial z}(x, y, z) = 0 \end{cases} \Leftrightarrow \begin{cases} 2x - y + 1 = 0 \\ 2y - x = 0 \\ 2z - 2 = 0 \end{cases} \Leftrightarrow \begin{cases} 2x - y + 1 = 0 \\ 2y - x = 0 \\ z = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 4y-y+1=0 \\ x=2y \\ z=1 \end{cases} \Leftrightarrow \begin{cases} y=-\frac{1}{3} \\ x=-\frac{2}{3} \\ z=1 \end{cases}.$$

Singurul punct critic al funcției f este $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$.

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y,z) = 2 \quad \forall (x,y,z) \in \mathbb{R}^3.$$

$$\frac{\partial^2 f}{\partial y^2}(x,y,z) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y,z) = 2 \quad \forall (x,y,z) \in \mathbb{R}^3.$$

$$\frac{\partial^2 f}{\partial z^2}(x,y,z) = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)(x,y,z) = 2 \quad \forall (x,y,z) \in \mathbb{R}^3.$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y,z) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y,z) = -1 = \frac{\partial^2 f}{\partial y \partial x}(x,y,z) \quad \forall (x,y,z) \in \mathbb{R}^3.$$

$$\frac{\partial^2 f}{\partial x \partial z}(x,y,z) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)(x,y,z) = 0 = \frac{\partial^2 f}{\partial z \partial x}(x,y,z) \quad \forall (x,y,z) \in \mathbb{R}^3.$$

$$\frac{\partial^2 f}{\partial y \partial z}(x,y,z) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)(x,y,z) = 0 = \frac{\partial^2 f}{\partial z \partial y}(x,y,z) \quad \forall (x,y,z) \in \mathbb{R}^3.$$

Toate derivatele parțiale de ordinul doi ale lui f sunt continue.

$$H_f(x,y,z)=\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y,z) & \frac{\partial^2 f}{\partial x\partial y}(x,y,z) & \frac{\partial^2 f}{\partial x\partial z}(x,y,z) \\ \frac{\partial^2 f}{\partial y\partial x}(x,y,z) & \frac{\partial^2 f}{\partial y^2}(x,y,z) & \frac{\partial^2 f}{\partial y\partial z}(x,y,z) \\ \frac{\partial^2 f}{\partial z\partial x}(x,y,z) & \frac{\partial^2 f}{\partial z\partial y}(x,y,z) & \frac{\partial^2 f}{\partial z^2}(x,y,z) \end{pmatrix}=$$

$$=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \forall (x,y,z)\in\mathbb{R}^3.$$

$$H_f\left(-\frac{2}{3},-\frac{1}{3},1\right)=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

$\Delta_1 = 2>0$

$\Delta_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1=3>0$

$\Delta_3 = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 6>0$

$\Longrightarrow \left(-\frac{2}{3},-\frac{1}{3},1\right)$ punct de minim local al lui f. $\square$

c) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=4xy-x^4-y^4$.

Soluție. $\mathbb{R}^2$ deschisă

Determinăm punctele critice ale funcției f.

$$\frac{\partial f}{\partial x}(x,y)=4y-4x^3 \quad \forall (x,y)\in\mathbb{R}^2.$$

$$\frac{\partial f}{\partial y}(x,y)=4x-4y^3 \quad \forall (x,y)\in\mathbb{R}^2.$$

$$\left.\begin{array}{l}\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ continue pe } \mathbb{R}^2 \\ \mathbb{R}^2 \text{ deschisă}\end{array}\right| \Longrightarrow f \text{ diferențiabilă pe } \mathbb{R}^2.$$

$$\begin{cases}\frac{\partial f}{\partial x}(x,y)=0 \\ \frac{\partial f}{\partial y}(x,y)=0\end{cases} \Longleftrightarrow \begin{cases}4y-4x^3=0 \mid :4 \\ 4x-4y^3=0 \mid :4\end{cases} \Longleftrightarrow \begin{cases}y-x^3=0 \\ x-y^3=0\end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases}y=x^3 \\ x-x^9=0\end{cases} \Longleftrightarrow \begin{cases}y=x^3 \\ x(1-x^8)=0\end{cases} \Longleftrightarrow \begin{cases}y=0 \\ x=0\end{cases} \text{ sau } \begin{cases}y=1 \\ x=1\end{cases} \text{ sau }$$

$$\begin{cases}y=-1 \\ x=-1.\end{cases}$$

Punctele critice ale lui f sunt: $(0,0)$, $(1,1)$ și $(-1,-1)$.

$$\frac{\partial^2 f}{\partial x^2}(x,y)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y)=-12x^2 \quad \forall (x,y)\in\mathbb{R}^2.$$

$$\frac{\partial^2 f}{\partial y^2}(x,y)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y)=-12y^2 \quad \forall (x,y)\in\mathbb{R}^2.$$

$$\frac{\partial^2 f}{\partial x\partial y}(x,y)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y)=4=\frac{\partial^2 f}{\partial y\partial x}(x,y) \quad \forall (x,y)\in\mathbb{R}^2.$$

Toate derivatele parțiale de ordinul doi ale lui f sunt continue.

$$H_f(x,y)=\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix}=\begin{pmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{pmatrix}$$

$\forall (x,y)\in \mathbb{R}^2$.

$$H_f(0,0)=\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$$

$\Delta_1=0$

$\Delta_2=\begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix}=-16<0$ $\Rightarrow$ $(0,0)$ nu este punct de extrem local al lui f.

$$H_f(1,1)=\begin{pmatrix} -12 & 4 \\ 4 & -12 \end{pmatrix}$$

$\Delta_1=-12<0$

$\Delta_2=\begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix}=144-16=108>0$ $\Rightarrow$ $(1,1)$ punct de maxim local al lui f.

$$H_f(-1,-1)=\begin{pmatrix} -12 & 4 \\ 4 & -12 \end{pmatrix}$$

$\Delta_1=-12<0$

$\Delta_2=\begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix}=144-16=108>0$ $\Rightarrow$ $(-1,-1)$ punct de maxim local al lui f. □

d) $f:\underset{\substack{\| \\ (0,\infty)\times(0,\infty)\times(0,\infty)}}{(0,\infty)^3} \to \mathbb{R}$, $f(x,y,z)=\frac{1}{x}+\frac{x}{y}+\frac{y}{z}+z$.

<u>Soluție.</u> $(0,\infty)^3$ deschisă.

Determinăm punctele critice ale lui f.

f continuă.

$$\frac{\partial f}{\partial x}(x,y,z)=-\frac{1}{x^2}+\frac{1}{y} \quad \forall (x,y,z)\in(0,\infty)^3.$$

$$\frac{\partial f}{\partial y}(x,y,z)=-\frac{x}{y^2}+\frac{1}{z} \quad \forall (x,y,z)\in(0,\infty)^3.$$

$$\frac{\partial f}{\partial z}(x,y,z)=-\frac{y}{z^2}+1 \quad \forall (x,y,z)\in(0,\infty)^3.$$

$\left.\begin{array}{l}\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \text{ continue pe } (0,\infty)^3 \\ (0,\infty)^3 \text{ deschisă}\end{array}\right\} \Rightarrow$ f diferențiabilă pe $(0,\infty)^3$.

$$\begin{cases}\frac{\partial f}{\partial x}(x,y,z)=0\\ \frac{\partial f}{\partial y}(x,y,z)=0\\ \frac{\partial f}{\partial z}(x,y,z)=0\end{cases} \Leftrightarrow \begin{cases}-\frac{1}{x^2}+\frac{1}{y}=0\\ -\frac{x}{y^2}+\frac{1}{z}=0\\ -\frac{y}{z^2}+1=0\end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases}y=x^2\\ -\frac{x}{y^2}+\frac{1}{z}=0\\ y=z^2\end{cases} \underset{x,y,z\in(0,\infty)}{\Leftrightarrow} \begin{cases}y=x^2\\ -\frac{x}{x^4}+\frac{1}{x}=0\\ z=x\end{cases} \Leftrightarrow \begin{cases}y=x^2\\ \frac{1}{x}=\frac{1}{x^3}\\ z=x\end{cases} \Leftrightarrow$$

$$\underset{\substack{x,y,z\in\\ \in(0,\infty)}}{\iff} \begin{cases} y=1 \\ x=1 \\ z=1. \end{cases}$$

Singurul punct critic al lui f este $(1,1,1)$.

$$\frac{\partial^2 f}{\partial x^2}(x,y,z)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y,z)=\frac{2}{x^3}\quad \forall\,(x,y,z)\in(0,\infty)^3.$$

$$\frac{\partial^2 f}{\partial y^2}(x,y,z)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y,z)=\frac{2x}{y^3}\quad \forall\,(x,y,z)\in(0,\infty)^3.$$

$$\frac{\partial^2 f}{\partial z^2}(x,y,z)=\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)(x,y,z)=\frac{2y}{z^3}\quad \forall\,(x,y,z)\in(0,\infty)^3.$$

$$\frac{\partial^2 f}{\partial x\partial y}(x,y,z)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y,z)=-\frac{1}{y^2}=\frac{\partial^2 f}{\partial y\partial x}(x,y,z)$$
$$\forall\,(x,y,z)\in(0,\infty)^3.$$

$$\frac{\partial^2 f}{\partial x\partial z}(x,y,z)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)(x,y,z)=0=\frac{\partial^2 f}{\partial z\partial x}(x,y,z)$$
$$\forall\,(x,y,z)\in(0,\infty)^3.$$

$$\frac{\partial^2 f}{\partial y\partial z}(x,y,z)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)(x,y,z)=-\frac{1}{z^2}=\frac{\partial^2 f}{\partial z\partial y}(x,y,z)$$
$$\forall\,(x,y,z)\in(0,\infty)^3.$$

Toate derivatele parțiale de ordinul doi ale lui f sunt continue.

$$H_f(x,y,z)=\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y,z) & \frac{\partial^2 f}{\partial x \partial y}(x,y,z) & \frac{\partial^2 f}{\partial x \partial z}(x,y,z) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y,z) & \frac{\partial^2 f}{\partial y^2}(x,y,z) & \frac{\partial^2 f}{\partial y \partial z}(x,y,z) \\ \frac{\partial^2 f}{\partial z \partial x}(x,y,z) & \frac{\partial^2 f}{\partial z \partial y}(x,y,z) & \frac{\partial^2 f}{\partial z^2}(x,y,z) \end{pmatrix}=$$

$$=\begin{pmatrix} \frac{2}{x^3} & -\frac{1}{y^2} & 0 \\ -\frac{1}{y^2} & \frac{2x}{y^3} & -\frac{1}{z^2} \\ 0 & -\frac{1}{z^2} & \frac{2y}{z^3} \end{pmatrix} \quad \forall (x,y,z)\in(0,\infty)^3.$$

$$H_f(1,1,1)=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\left.\begin{array}{l} \Delta_1 = 2>0 \\ \Delta_2=\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}=3>0 \\ \Delta_3=\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}=8+0+0-0-2-2=4>0 \end{array}\right| \Rightarrow (1,1,1)$$

punct de minim local al lui f. $\square$

e) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^4 + y^4$.

<u>Soluție.</u> $\mathbb{R}^2$ deschisă

Determinăm punctele critice ale lui f.

f continuă

$$\frac{\partial f}{\partial x}(x,y) = 4x^3 \quad \forall (x,y) \in \mathbb{R}^2.$$

$$\frac{\partial f}{\partial y}(x,y) = 4y^3 \quad \forall (x,y) \in \mathbb{R}^2.$$

$$\left.\begin{array}{l} \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ continue pe } \mathbb{R}^2 \\ \mathbb{R}^2 \text{ deschisă} \end{array}\right\} \Rightarrow f \text{ diferențiabilă pe } \mathbb{R}^2.$$

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} 4x^3 = 0 \\ 4y^3 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0. \end{cases}$$

Singurul punct critic al lui f este $(0,0)$.

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) = 12x^2 \quad \forall (x,y) \in \mathbb{R}^2.$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y) = 12y^2 \quad \forall (x,y) \in \mathbb{R}^2.$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) = 0 = \frac{\partial^2 f}{\partial y \partial x}(x,y) \quad \forall (x,y) \in \mathbb{R}^2.$$

Toate derivatele parțiale de ordinul doi ale lui f sunt continue.

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{pmatrix}$$

$\forall (x,y) \in \mathbb{R}^2.$

$$H_f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$\Delta_1 = 0$

$\Delta_2 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$

$\Rightarrow$ Criteriul nu decide.

$f(x,y) \geq f(0,0)$

$x^4 + y^4 \geq 0^4 + 0^4 = 0$

$\forall (x,y) \in \mathbb{R}^2 \Rightarrow (0,0)$ punct de minim global al lui $f \Rightarrow (0,0)$ punct de minim local al lui f. $\square$

f) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = -x^4 - y^4$.

<u>Soluție.</u> Rezolvați-l voi!

g) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x^4-y^4$.

<u>Soluție</u>. $\mathbb{R}^2$ deschisă.

Determinăm punctele critice ale lui f.

f continuă.

$$\frac{\partial f}{\partial x}(x,y)=4x^3 \quad \forall (x,y)\in\mathbb{R}^2.$$

$$\frac{\partial f}{\partial y}(x,y)=-4y^3 \quad \forall (x,y)\in\mathbb{R}^2.$$

$\left.\begin{array}{l}\dfrac{\partial f}{\partial x},\ \dfrac{\partial f}{\partial y} \text{ continue pe } \mathbb{R}^2\\ \mathbb{R}^2 \text{ deschisă}\end{array}\right| \Longrightarrow f$ diferențială pe $\mathbb{R}^2$.

$$\begin{cases}\dfrac{\partial f}{\partial x}(x,y)=0\\ \dfrac{\partial f}{\partial y}(x,y)=0\end{cases} \Leftrightarrow \begin{cases}4x^3=0\\ -4y^3=0\end{cases} \Leftrightarrow \begin{cases}x=0\\ y=0.\end{cases}$$

Singurul punct critic al lui f este $(0,0)$.

$$\frac{\partial^2 f}{\partial x^2}(x,y)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y)=12x^2 \quad \forall (x,y)\in\mathbb{R}^2.$$

$$\frac{\partial^2 f}{\partial y^2}(x,y)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y)=-12y^2 \quad \forall (x,y)\in\mathbb{R}^2.$$

$$\frac{\partial^2 f}{\partial x\partial y}(x,y)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y)=0=\frac{\partial^2 f}{\partial y\partial x}(x,y) \quad \forall (x,y)\in\mathbb{R}^2.$$

Toate derivatele parțiale de ordinul doi ale lui f sunt continue.

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 12x^2 & 0 \\ 0 & -12y^2 \end{pmatrix}$$

$\forall (x,y) \in \mathbb{R}^2$.

$$H_f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$\left.\begin{array}{l} \Delta_1 = 0 \\ \Delta_2 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0 \end{array}\right\} \Rightarrow \text{Criteriul nu decide.}$$

$$\underset{x^4 - y^4}{\overset{f(x,y)}{\|}} \qquad \underset{0}{\overset{f(0,0)}{\|}}$$

$$\left.\begin{array}{l} f(1,0) = 1 > 0 = f(0,0) \\ f(0,1) = -1 < 0 = f(0,0) \end{array}\right\} \Rightarrow (0,0) \text{ nu este punct de extrem global al lui } f.$$

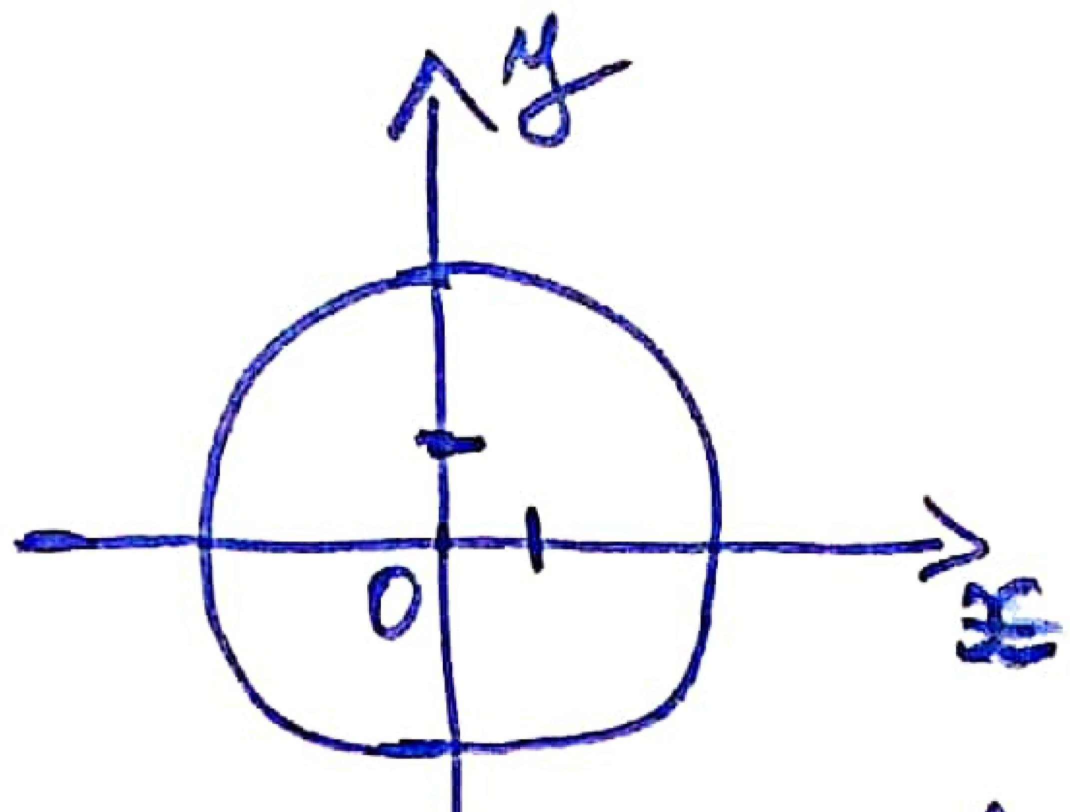
Totuși $(0,0)$ poate fi punct de extrem local al lui f.

$? \ \exists V \in \mathcal{V}_{(0,0)}$ a.î. $f(x,y) \geq f(0,0) \ \forall (x,y) \in V \cap \mathbb{R}^2 = V$ sau

$\exists V \in \mathcal{V}_{(0,0)}$ a.î. $f(x,y) \leq f(0,0) \ \forall (x,y) \in V \cap \mathbb{R}^2 = V$?

$f(x,0)=x^4-0^4=x^4>0=f(0,0)$
$\forall\, x\in\mathbb{R}^*.$

$f(0,y)=0^4-y^4=-y^4<0=f(0,0)$
$\forall\, y\in\mathbb{R}^*.$



Deci $(0,0)$ nu este punct de extrem local al lui f. $\square$

2. Să se arate că ecuația $x\cos y+y\cos z+z\cos x=1$ definește într-o vecinătate a punctului $(1,0,0)$ unica funcție implicită $z=z(x,y)$ și determinați $\frac{\partial z}{\partial x}(1,0)$, $\frac{\partial z}{\partial y}(1,0)$ și $dz(1,0)$.

<u>Soluție</u>. Fie $D=\mathbb{R}^3$ și $F:D\to\mathbb{R}$, $F(x,y,z)=$

$=x\cos y+y\cos z+z\cos x-1$.

D deschisă.

1) $F(1,0,0)=0$.

2) $\frac{\partial F}{\partial x}(x,y,z)=\cos y-z\sin x \quad \forall\,(x,y,z)\in\mathbb{R}^3.$

$\frac{\partial F}{\partial y}(x,y,z)=-x\sin y+\cos z \quad \forall\,(x,y,z)\in\mathbb{R}^3.$

$\frac{\partial F}{\partial z}(x,y,z)=-y\sin z+\cos x \quad \forall\,(x,y,z)\in\mathbb{R}^3.$

$\left.\begin{array}{l}\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \text{ continue pe } \mathbb{R}^3 \\ \mathbb{R}^3 \text{ deschisă}\end{array}\right\} \Rightarrow F$ de clasă C^1.

3) $\frac{\partial F}{\partial z}(1,0,0) = \cos 1 \neq 0$.

Conform T.F.i. $\exists U \in \mathcal{V}_{(1,0)}$, U deschisă, $\exists V \in \mathcal{V}_0$, V deschisă, $\exists!\ z : U \to V$ a.î.:

a) $z(1,0) = 0$.

b) $F(x,y,z(x,y)) = 0 \ \forall (x,y) \in U$.

c) z este de clasă C^1 și

$$\frac{\partial z}{\partial x}(x,y) = -\frac{\frac{\partial F}{\partial x}(x,y,z(x,y))}{\frac{\partial F}{\partial z}(x,y,z(x,y))} \quad \forall (x,y) \in U,$$

$$\frac{\partial z}{\partial y}(x,y) = -\frac{\frac{\partial F}{\partial y}(x,y,z(x,y))}{\frac{\partial F}{\partial z}(x,y,z(x,y))} \quad \forall (x,y) \in U.$$

Pentru a determina $\frac{\partial z}{\partial x}(1,0)$ și $\frac{\partial z}{\partial y}(1,0)$ avem două variante.

<u>Variantа 1 (Folosim c))</u>

$$\frac{\partial z}{\partial x}(x,y) = -\frac{\frac{\partial F}{\partial x}(x,y,z(x,y))}{\frac{\partial F}{\partial z}(x,y,z(x,y))} = -\frac{\cos y - z(x,y)\sin x}{-y\sin z(x,y) + \cos x}$$

$\forall (x,y) \in U \Rightarrow$

$$\underset{z(1,0)=0}{\overset{}{\Longrightarrow}} \frac{\partial z}{\partial x}(1,0) = -\frac{1}{\cos 1}.$$

$$\frac{\partial z}{\partial y}(x,y) = -\frac{\frac{\partial F}{\partial y}(x,y,z(x,y))}{\frac{\partial F}{\partial z}(x,y,z(x,y))} =$$

$$= -\frac{-x\sin y + \cos z(x,y)}{-y\sin z(x,y) + \cos x} \quad \forall (x,y) \in U \Longrightarrow$$

$$\underset{z(1,0)=0}{\Longrightarrow} \frac{\partial z}{\partial y}(1,0) = -\frac{1}{\cos 1}.$$

<u>Varianta 2 (derivare directă; folosim b))</u>

$F(x,y,z(x,y)) = 0 \Longrightarrow x\cos y + y\cos z(x,y) + z(x,y)\cos x - 1 = 0.$

Derivăm parțial relația de mai sus în raport cu x și obținem:

$\cos y - y\left(\sin z(x,y)\right)\frac{\partial z}{\partial x}(x,y) + \frac{\partial z}{\partial x}(x,y)\cos x - z(x,y)\sin x = 0 \Rightarrow \frac{\partial z}{\partial x}(x,y)\left(-y\sin z(x,y) + \cos x\right) =$

$$= -\cos y + z(x,y)\sin x \Rightarrow \frac{\partial z}{\partial x}(x,y) = \frac{-\cos y + z(x,y)\sin x}{-y\sin z(x,y) + \cos x}$$

$$\forall (x,y)\in U \underset{\substack{\uparrow \\ z(1,0)=0}}{\Rightarrow} \frac{\partial z}{\partial x}(1,0) = \frac{-1}{\cos 1} = -\frac{1}{\cos 1}.$$

$F(x,y,z(x,y)) = 0 \Rightarrow x\cos y + y\cos z(x,y) + z(x,y)\cos x - 1 = 0.$

Derivăm parțial relația de mai sus în raport cu y și obținem:

$$-x\sin y + \cos z(x,y) - y(\sin z(x,y))\frac{\partial z}{\partial y}(x,y) + \frac{\partial z}{\partial y}(x,y)\cos x = 0 \Rightarrow \frac{\partial z}{\partial y}(x,y)\left(-y\sin z(x,y) + \cos x\right) = x\sin y - \cos z(x,y) \Rightarrow \frac{\partial z}{\partial y}(x,y) = \frac{x\sin y - \cos z(x,y)}{-y\sin z(x,y) + \cos x}$$

$$\forall (x,y)\in U \underset{\substack{\uparrow \\ z(1,0)=0}}{\Rightarrow} \frac{\partial z}{\partial y}(1,0) = \frac{-1}{\cos 1} = -\frac{1}{\cos 1}.$$

Conform c), z este de clasă C^1, deci z este diferențiabilă. Avem $dz(1,0): \mathbb{R}^2 \to \mathbb{R}$, $dz(1,0)(u,v) = \frac{\partial z}{\partial x}(1,0)\cdot u + \frac{\partial z}{\partial y}(1,0)\cdot v = -\frac{1}{\cos 1}u - \frac{1}{\cos 1}v$, i.e. $dz(1,0) = -\frac{1}{\cos 1}dx - \frac{1}{\cos 1}dy$. $\square$