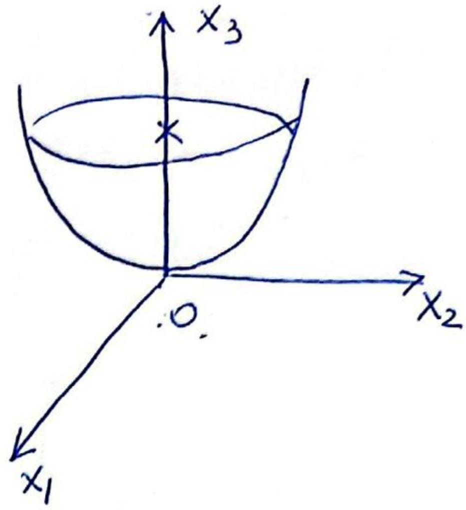
§1. Cuadrice studiate pe ec. reduse (II)



- **Paraboloidul eliptic**

$$\mathcal{P}_e : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 2x_3 \quad , a>0, \; b>0$$

Ec. parametrice

$x_1 = a\cos\theta \cdot t$

$x_2 = b \sin\theta \cdot t$

$x_3 = \frac{1}{2}(t^2) \quad , t \in \mathbb{R} , \theta \in [0, 2\pi)$

$\cap$ cu plane // cu planele de coord.

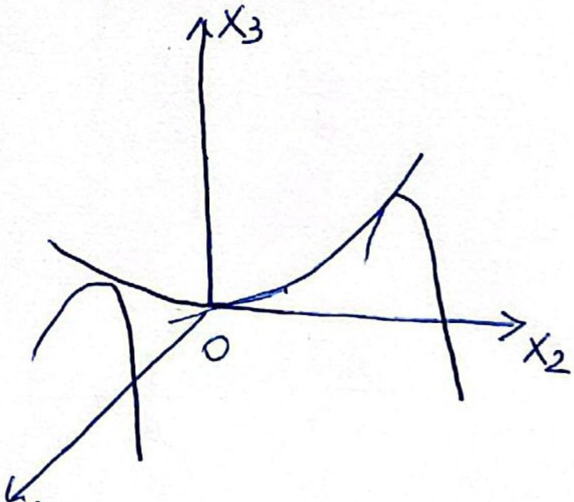
1) $x_3 = \gamma \in [0, \infty) \quad \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 2\gamma$

a) $\gamma > 0 \Rightarrow$ Elipsă

b) $\gamma = 0 \Rightarrow O(0,0,0)$

2) $x_2 = \beta \in \mathbb{R} \quad \frac{x_1^2}{a^2} = 2x_3 - \frac{\beta^2}{b^2} = 2\left(x_3 - \frac{\beta^2}{2b^2}\right)$ parabolă

3) $x_1 = \alpha \in \mathbb{R}$ Analog cu 2)



- **Paraboloidul hiperbolic**

$$\mathcal{P}_h : \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 2x_3 \quad , a>0, \; b>0$$

Ec. param.

$$\begin{cases} x_1 = at \\ x_2 = bs \\ x_3 = \frac{1}{2}(t^2 - s^2) , \; t, s \in \mathbb{R}. \end{cases}$$

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$\cap$ cu plane // cu planele de coord.

1) $x_3 = \gamma$ $\quad \dfrac{x_1^2}{a^2} - \dfrac{x_2^2}{b^2} = 2\gamma$

- $\gamma > 0 \Rightarrow$ Hiperbolă
- $\gamma = 0 \Rightarrow x_2 = \pm \dfrac{b}{a} x_1$ drepte concurente (în origine)

2) $x_2 = \beta \in \mathbb{R}$ $\quad \dfrac{x_1^2}{a^2} = 2x_3 + \dfrac{\beta^2}{b^2}$ parabolă

1) $x_1 = \alpha \in \mathbb{R}$ $\quad \dfrac{x_2^2}{b^2} = -2x_3 + \dfrac{\alpha^2}{a^2}$ parabolă.

<u>Teoremă</u> $\mathcal{P}_h$ = cuadrică dublu riglată și prin fiecare pct $\in \mathcal{P}_h$ trece câte o dreaptă din fiecare familie de generatoare.

<u>Dem</u>

$$\mathcal{P}_h: \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 2x_3 \Rightarrow \left(\frac{x_1}{a} - \frac{x_2}{b}\right)\left(\frac{x_1}{a} + \frac{x_2}{b}\right) = x_3 \cdot 2$$

$$G_1\, d_\lambda: \begin{cases} \dfrac{x_1}{a} + \dfrac{x_2}{b} = \lambda \cdot x_3 \\ \lambda\left(\dfrac{x_1}{a} - \dfrac{x_2}{b}\right) = 2. \end{cases} \qquad d_\infty: \begin{cases} x_3 = 0 \\ \dfrac{x_1}{a} - \dfrac{x_2}{b} = 0 \end{cases}$$

$$G_2: \bar{d}_\mu: \begin{cases} \dfrac{x_1}{a} - \dfrac{x_2}{b} = \mu x_3 \\ \mu\left(\dfrac{x_1}{a} + \dfrac{x_2}{b}\right) = 2 \end{cases} \qquad \bar{d}_\infty: \begin{cases} x_3 = 0 \\ \dfrac{x_1}{a} + \dfrac{x_2}{b} = 0. \end{cases}$$

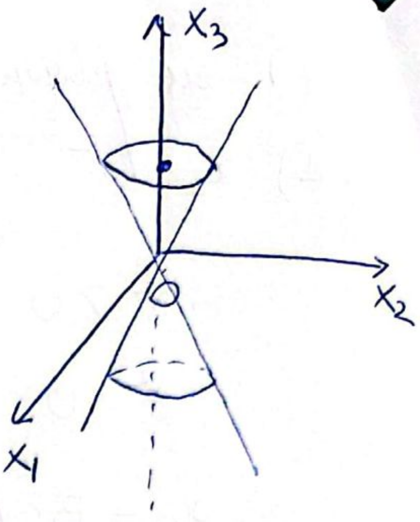
$\lambda, \mu \in \mathbb{R}^*$

$$d_\lambda \cap \bar{d}_\mu: \quad \lambda x_3 = \frac{2}{\mu} \Rightarrow x_3 = \frac{2}{\lambda\mu}.$$

$$\begin{cases} \dfrac{x_1}{a} + \dfrac{x_2}{b} = \dfrac{2}{\mu} \\ \dfrac{x_1}{a} - \dfrac{x_2}{b} = \dfrac{2}{\lambda} \end{cases} \Rightarrow \begin{array}{l} \dfrac{x_1}{a} = \dfrac{1}{\mu} + \dfrac{1}{\lambda} = \dfrac{\lambda+\mu}{\lambda\mu} \\ \dfrac{x_2}{b} = \dfrac{1}{\mu} - \dfrac{1}{\lambda} = \dfrac{\lambda-\mu}{\lambda\mu}. \end{array}$$

$$\underline{P}\left(a \cdot \frac{\lambda+\mu}{\lambda\mu},\ b \cdot \frac{\lambda-\mu}{\lambda\mu},\ \frac{2}{\lambda\mu}\right)$$

- Conul pătratic



Con : $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 0$

$a>0,\ b>0,\ c>0$

$\cap$ plane // cu planele de coord.

1) $x_3 = \gamma \in \mathbb{R} \Rightarrow \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = \frac{\gamma^2}{c^2}$

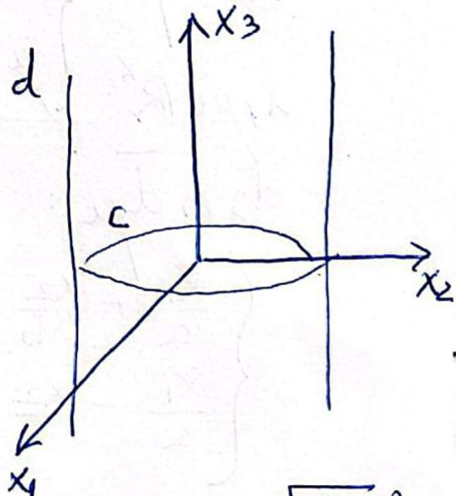
- $\gamma \neq 0 \Rightarrow$ Elipsă
- $\gamma = 0 \Rightarrow O(0,0,0)$

2) $x_2 = \beta$ $\quad \frac{x_3^2}{c^2} - \frac{x_1^2}{a^2} = \frac{\beta^2}{b^2}$

- $\beta \neq 0 \Rightarrow$ Hiperbolă
- $\beta = 0 \Rightarrow x_1 = \pm \frac{a}{c} x_3$ (drepte concurente)

3) $x_1 = \alpha$. Analog cu 2)

- <u>Cilindrul</u> = cuadrică riglată generată de o dreaptă d (de direcție dată), numită generatoare, care mai este supusă unei cond: de ex: intersectează o curbă dată c, numită curbă directoare.



1) C_e : $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$

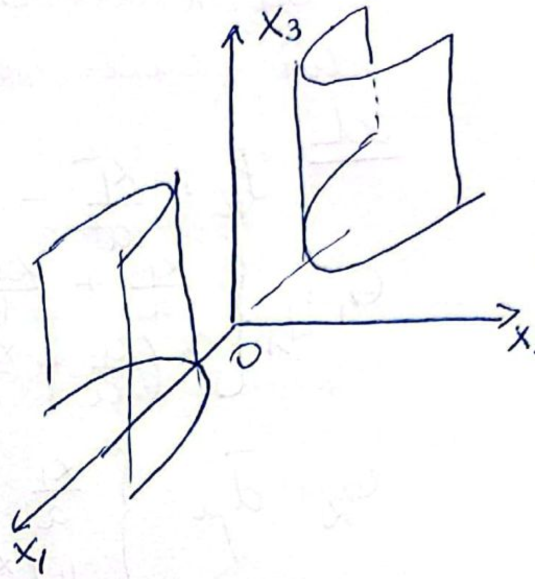
<u>Cilindrul eliptic</u>

$d \parallel Ox_3$ $\quad C : \begin{cases} \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \\ x_3 = 0 \end{cases}$

- $x_3 = \gamma$ $\quad \begin{cases} \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \\ x_3 = \gamma \end{cases}$ Elipsă
- $\underset{[-b,b]}{x_2 = \beta}$ $\quad \frac{x_1^2}{a^2} = 1 - \frac{\beta^2}{b^2} \geq 0 \Rightarrow x_1 = \pm \frac{a}{b}\sqrt{b^2-\beta^2}$ drepte //

- $x_1 = \alpha$ analog cu cazul precedent

2) Cilindrul hiperbolic



$C_h : \dfrac{x_1^2}{a^2} - \dfrac{x_2^2}{b^2} = 1.$

$d \parallel Ox_3 \qquad c : \begin{cases} \dfrac{x_1^2}{a^2} - \dfrac{x_2^2}{b^2} = 1 \\ x_3 = 0 \end{cases}$

- $x_3 = \gamma \in \mathbb{R} \quad \begin{cases} \dfrac{x_1^2}{a^2} - \dfrac{x_3^2}{b^2} = 1 \\ x_3 = \gamma \end{cases}$ Hiperbola

- $x_2 = \beta \in \mathbb{R} \quad \dfrac{x_1^2}{a^2} = 1 + \dfrac{\beta^2}{b^2} \Rightarrow x_1 = \pm \dfrac{a}{b}\sqrt{b^2+\beta^2}$ drepte $\parallel$.

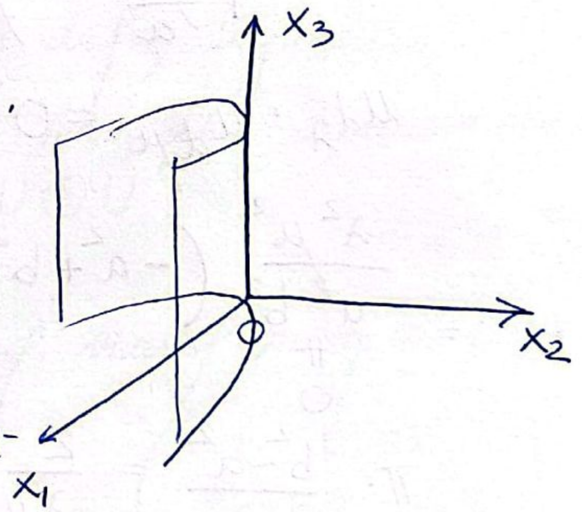
- $x_1 = \alpha \quad \dfrac{x_2^2}{b^2} = \dfrac{\alpha^2}{a^2} - 1 \geq 0 \Rightarrow x_2 = \pm \dfrac{b}{a}\sqrt{\alpha^2 - a^2}$

$\alpha \in (-\infty, -a] \cup [a, \infty)$

$\alpha \neq \pm a$ 2 drepte $\parallel$

$\alpha = \pm a$ câte 1 dr.

3) Cilindrul parabolic



$C_p : x_2^2 = 2px_1, \; p > 0.$

$d \parallel Ox_3 \qquad c : \begin{cases} x_2^2 = 2px_1 \\ x_3 = 0. \end{cases}$

- $x_3 = \gamma \in \mathbb{R} \quad \begin{cases} x_2^2 = 2px_1 \\ x_3 = \gamma \end{cases}$ Parabola

- $x_2 = \beta \quad x_1 = \dfrac{\beta^2}{2p}.$ 1 dreaptă

- $x_1 = \alpha \geq 0 \quad x_2^2 = 2p\alpha$

$\alpha > 0 \quad x_2 = \pm\sqrt{2p\alpha}$. drepte $\parallel$.

$\alpha = 0 \quad x_2 = 0$

Teoremă $\forall$ suprafață dublu riglată = planul, $\mathcal{H}_1$, $\mathcal{P}_h$

Aplicatie

Să se determine LG al punctelor $\in \mathcal{P}_h$ din care se pot duce generatoare $\perp$.

SOL. $\mathcal{P}_h : \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 2x_3$.

$$G_1 : d_\lambda \begin{cases} \frac{x_1}{a} + \frac{x_2}{b} = \lambda x_3 \\ \lambda\left(\frac{x_1}{a} - \frac{x_2}{b}\right) = 2 \end{cases} ; \quad d_\infty : \begin{cases} x_3 = 0 \\ \frac{x_1}{a} - \frac{x_2}{b} = 0 \end{cases}$$

$$G_2 : \bar{d}_\mu : \begin{cases} \frac{x_1}{a} - \frac{x_2}{b} = \mu x_3 \\ \mu\left(\frac{x_1}{a} + \frac{x_3}{b}\right) = 2 \end{cases} ; \quad \bar{d}_\infty : \begin{cases} x_3 = 0 \\ \frac{x_1}{a} + \frac{x_2}{b} = 0 . \end{cases}$$

$$u_{d_\lambda} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{1}{a} & \frac{1}{b} & -\lambda \\ \frac{\lambda}{a} & -\frac{\lambda}{b} & 0 \end{vmatrix} = \left(-\frac{\lambda^2}{b}, -\frac{\lambda^2}{a}, \frac{-2\lambda}{ab}\right)$$

$$u_{\bar{d}_\mu} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{1}{a} & -\frac{1}{b} & -\mu \\ \frac{\mu}{a} & \frac{\mu}{b} & 0 \end{vmatrix} = \left(\frac{\mu^2}{b}, -\frac{\mu^2}{a}, \frac{2\mu}{ab}\right)$$

$$u_{d_\lambda} \cdot u_{\bar{d}_\mu} = 0 \Rightarrow \frac{-\lambda^2\mu^2}{b^2} + \frac{\lambda^2\mu^2}{a^2} - \frac{4\lambda\mu}{a^2b^2} = 0 .$$

$$\underset{\substack{\neq \\ 0}}{\frac{\lambda^2\mu^2}{a^2b^2}}\left(-a^2 + b^2 - \frac{4}{\lambda\mu}\right) = 0 \Rightarrow -a^2 + b^2 = \frac{4}{\lambda\mu}.$$

$$\Pi : \frac{b^2 - a^2}{2} = \frac{2}{\lambda\mu} = x_3 ; \quad P\left(a\frac{\lambda+\mu}{\lambda\mu}, b\frac{\lambda-\mu}{\lambda\mu}, \frac{2}{\lambda\mu}\right)$$

(Π = planul Monge) $\quad d_\lambda \cap \bar{d}_\mu = \{P\}$.

$$\Pi \cap \mathcal{P}_h \quad \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = \frac{b^2 - a^2}{\cancel{2}} \cdot \cancel{2} = b^2 - a^2$$

- $b \neq a$ Hiperbolă.
- $b = a$. $\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 0 \Rightarrow \begin{matrix} x_2 = \pm \frac{b}{a} x_1 \\ x_3 = 0 \end{matrix}$ drepte concurente