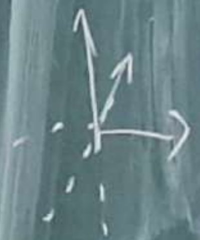
Criteriul de liniar independență

Pentru a avea un SLI, scriem vectorii pe coloană și verificăm că rg A = maxim!

(1, 2, 7)
(1, 4, 3)



$$A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 7 & 3 \end{pmatrix}$$

nr. vect SLI ≤ dim spațiului ≤ nr. vect generator

nr vectori SLI <=
dimensiunea spațiului
<= nr vectori din SG

Ex 1 $S=\{(1,1,0),(1,-1,-1),(2,0,-1)\}$, $(\mathbb{R}^3,+,\cdot)|_{\mathbb{R}}$

a) extragem din S un S_1 SLI maximal

b) extindere la o bază

$$A=\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

$$\det A = 1+(-2)+0-(0+0-1)=0 \Rightarrow rg A \neq 3 \xRightarrow{CLI} S \text{ SLD}$$

$$B=\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & -1 \end{pmatrix} \quad \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -1-1=-2\neq 0 \Rightarrow rg B = 2 \xRightarrow{CLI} \{(1,1,0),(1,-1,-1)\}=S_1$$

S_1 SLI maximal din S

$B_0=\{(1,0,0),(0,1,0),(0,0,1)\}$ baza canonică

$$C=\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad \det C = -1-0=-1\neq 0 \Rightarrow rg C = 3 \xRightarrow{CLI} S_1\cup\{(1,0,0)\} \text{ SLI}$$

$$\operatorname{card}(S_1\cup\{(1,0,0)\})=3=\dim_{\mathbb{R}}\mathbb{R}^3$$

$\Rightarrow$ Bază

Repere Schimbare de reper

Bază ordonată

Reper —

$\Rightarrow$ Putem defini coordonate unice

$(1,0), (0,1)$

$a(1,0)+b(0,1)=(x,y)$

$7(1,0)+(-2)(0,1)=(7,-2) \Rightarrow \text{coord}=(7,-2)$

$a(1,1)+b(1,-1)=(x,y)$

$(a+b,\ a-b)=(7,-2)$

$$\begin{cases} a+b=7 \\ a-b=-2 \end{cases}$$

$2a=5$

$a=5/2$

$b=9/2 \Rightarrow \text{coord}=\left(\frac{5}{2},\frac{9}{2}\right)$

$\Rightarrow \forall x \in V \ \exists! \ (x_1, x_2, \dots, x_n) \in \mathbb{K}^n$ a.î. $x = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$

Modificarea coordonatelor la schimbarea de reper

$$\mathcal{R} = \{e_1, \dots, e_n\} \xrightarrow{A} \mathcal{R}' = \{e_1', \dots, e_n'\}$$

$$X = A \cdot X'$$

X = coord inițiale (în raport cu $\mathcal{R}$)

A = matricea de schimbare a reperului

X' = coord finale (în raport cu $\mathcal{R}'$)

$$\Rightarrow A^{-1} X = X'$$

ex 2 $(\mathbb{R}^2, +, \cdot)|_{\mathbb{R}}$ $R_0 = \{e_1 = (1,0), e_2 = (0,1)\}$ rep. canonic

$R' = \{e_1' = (2,1), e_2' = (3,0)\}$

a) R' reper

b) $R_0 \xrightarrow[A]{} R'$

c) $x = (1,2)$

coord lui x în rep cu R_0 și R'

a) $A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$ $\det A = -3 \neq 0 \Rightarrow \operatorname{rg} A = 2$ maxim $\overset{\text{cLi}}{\Longrightarrow} R'$ SLi

$\operatorname{card} R' = 2 = \dim_{\mathbb{R}} \mathbb{R}^2$ $\Big| \Rightarrow R'$ reper

b) $e_1' = (2,1) = a(1,0) + b(0,1) \Rightarrow a = 2 \quad b = 1$

$e_2' = (3,0) = \quad a = 3, \ b = 0$

$A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$

c) $x = (1,2) = a(1,0) + b(0,1) \Rightarrow a = 1,\ b = 2 \Rightarrow (1,2)$ coord. în raport cu R_0

$$X = AX' = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 2x_1' + 3x_2' \\ 1x_1' + 0x_2' \end{pmatrix}$$

$$\begin{cases} 2x_1' + 3x_2' = 1 \\ x_1' = 2 \end{cases} \Leftrightarrow \begin{cases} x_2' = -1 \\ x_1' = 2 \end{cases} \Rightarrow (2,-1) \text{ coord. în raport cu } R'$$

ex3 $(\mathbb{R}_2[X], +, \cdot)_{\mathbb{R}}$, $R_0 = \{e_1 = 1, e_2 = X, e_3 = X^2\}$ – rep canonic

$R_1 = \{-1+2X+3X^2, X-X^2, X-2X^2\}$

$R_1' = \{(-1,2,3), (0,1,-1), (0,1,-2)\}$

a) R^1 reper in $\mathbb{R}_2[X]$

b) $R_0 \xrightarrow{A} R$, $A = ?$

c) coord $P = 3 - X + X^2$ in rep cu R^1

a) $B = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & -2 \end{pmatrix}$ $\det B = 1 \neq 0 \Rightarrow \operatorname{rg} B = 3 = \text{maxim} \xrightarrow{CLi} R_1$ SLi

card $R_1 = 3 = \dim_{\mathbb{R}} \mathbb{R}_2[X]$ $\Rightarrow R_1$ reper

$\det B = (-1)\cdot(-1)^{1+1}\cdot\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} + 0\cdot(-1)^{1+2}\cdot\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} + 0\cdot(-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}$

$= (-1)\cdot 1\cdot(-1) = 1$

b) $e_1' = -1+2X+3X^2 = a\cdot 1 + b\cdot X + cX^2 \Rightarrow (-1,2,3)$ coord

$e_2' = X - X^2 \Rightarrow (0,1,-1)$ coord

$e_3' = X - 2X^2 \Rightarrow (0,1,-2)$ coord

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & -2 \end{pmatrix}$$

c) $P = 3 - x + x^2$

V1 $\quad X = AX' \Rightarrow \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} -x_1' \\ 2x_1' + x_2' + x_3' \\ 3x_1' - x_2' - 2x_3' \end{pmatrix}$

$$\begin{cases} -x_1' = 3 \\ 2x_1' + x_2' + x_3' = -1 \\ 3x_1' - x_2' - 2x_3' = 1 \end{cases}$$

$x_1' = -3 \Rightarrow \begin{cases} x_2' = 20 \\ x_3' = -15 \end{cases} \Rightarrow (x_1', x_2', x_3') = (-3, 20, -15)$ coord. în raport cu R'

V2 $\quad A^{-1} X = X'$

$$\begin{pmatrix} -1 & 0 & 0 \\ 7 & 2 & 1 \\ -5 & -1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 20 \\ -15 \end{pmatrix}$$

Operații cu subspații vectoriale

① $V_1, V_2 \subset V$ subspații vectoriale.

$$\langle V_1 \cup V_2 \rangle = V_1 + V_2 = \{ x \in V \mid x \in V_1 + V_2,\ V_1 \in V,\ V_2 \in V \}$$

④ $V = \{(1,2), (-2,5), (4,2)\}$

$V_1 = \{(1,2), (4,2)\}$

$V_2 = \{(-2,5), (4,2)\}$ $\Big| \Rightarrow$ se pot reprezenta

② Teorema Grossmann

$(V, +, \cdot)_{/\mathbb{K}}$ sp. vectorial

V_1, V_2 subsp. vectoriale

$$\dim_{\mathbb{K}}(V_1 + V_2) = \dim_{\mathbb{K}} V_1 + \dim_{\mathbb{K}} V_2 - \dim_{\mathbb{K}}(V_1 \cap V_2)$$

(3) Suma directă

$V_1 \oplus V_2$ dacă $V_1 \cap V_2 = \{0_V\}$

! Subspațiul complementar NU e unic.

(4) $V' \subseteq V$ subesp vect.

$\dim_{\mathbb{K}} V' = n = \dim_{\mathbb{K}} V \Rightarrow V' = V$

(5) $\dim_{\mathbb{R}} S(A) = n - rg(A)$

dim subspațiului | dim spațiului | A - mat sist

Ex 4) $(\mathbb{R}^3,+,\cdot)/\mathbb{R}$, $V' = \{x \in \mathbb{R}^3 \mid \begin{cases} 2x_1 + x_2 = 0 \\ x_1 + 4x_3 = 0 \end{cases}\}$

a) bază în V'

b) subsp. complementar V'' a lui V' a.î. $\mathbb{R}^3 = V' \oplus V''$

c) descompunem $x = (1,1,2)$ în raport cu $\mathbb{R}^3 = V' \oplus V''$

a) $A = \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 1 & 0 & 4 \end{array}\right) \Big| \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \det\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = -1 \neq 0 \Rightarrow rg A = 2$

$\dim_{\mathbb{R}} V' = n - rg A = 3 - 2 = 1$

$\left.\begin{array}{l} x_1 = -4x_3 \\ x_2 = -2x_1 = (-2)\cdot(-4x_3) = 8x_3 \end{array}\right| \Rightarrow V' = \{(-4x_3, 8x_3, x_3) \mid x_3 \in \mathbb{R}\} =$

$= \{x_3(-4,8,1) \mid x_3 \in \mathbb{R}\}$

$= \langle\{(-4,8,1)\}\rangle \Rightarrow R' = \{(-4,8,1)\}$ SG

$B = \begin{pmatrix} -4 \\ 8 \\ 1 \end{pmatrix}$ $rg B = 1$ maxim $\Rightarrow R'$ SLI $\Rightarrow R'$ Bază

b) $\mathbb{R}^3 = V' \oplus V''$

Aplicăm T. Grossmann $\Rightarrow \underbrace{\dim_{\mathbb{R}} \mathbb{R}^3}_{3} = \underbrace{\dim_{\mathbb{R}} V'}_{1} + \dim_{\mathbb{R}} V'' - \underbrace{\dim_{\mathbb{R}}(V' \cap V'')}_{0}$

$\Rightarrow \dim_{\mathbb{R}} V'' = 2$

luăm $V'' = <\{e_2, e_3\}>$, $R'' = \{e_2, e_3\}$

$\det \underbrace{\begin{pmatrix} -4 & 0 & 0 \\ 8 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{C} = -4 \neq 0 \Rightarrow \operatorname{Rg} C = 3$ maximal SLI $\Big| \Rightarrow R' \cup R''$ bază în $\mathbb{R}^3$

$\operatorname{card}(R' \cup R'') = 3 = \dim_{\mathbb{R}} \mathbb{R}^3$

$V'' = < \underbrace{\{(0,1,0), (0,0,1)\}}_{R''} >$

c) $R = R' \cup R'' = \{\underbrace{(-4,8,1)}_{R'}, (0,1,0), (0,0,1)\}$

$x = (1,1,2) = a(-4,8,1) + b(0,1,0) + c(0,0,1) = (-4a, 8a+b, a+c)$

$\begin{cases} -4a = 1 \\ 8a+b = 1 \\ a+c = 2 \end{cases} \Rightarrow \begin{cases} a = -1/4 \\ b = 3 \\ c = 9/4 \end{cases} \Rightarrow$ $u = -\frac{1}{4}(-4,8,1) \in V'$

$v = 3(0,1,0) + \frac{9}{4}(0,0,1) = (0,3,\frac{9}{4}) \in V''$

$x = u + v = (1,1,2) \in \mathbb{R}^3$