DIMA OANA
GR 141

TEMA 3 (CURS 6)

① $f: \mathbb{R}^3 \to \mathbb{R}^3$
$f(x) = (x_1 - x_2 + x_3, x_1 - x_2 + x_3, x_3)$

a) $A = [f]_{R_0, R_0}$

b) valorile proprii
subspațiile proprii
repere în fiecare

c) $\operatorname{Ker} f$, $\operatorname{Im} f$, reper în fiecare

d) $\mathbb{R}^3 = \operatorname{Ker} f \oplus W$; $W = ?$
$p : \operatorname{Ker} f \oplus W \to \operatorname{Ker} f$
$p(1,0,3) = ?$
$s(1,0,3) = ?$

s = simetria față de $\operatorname{Ker} f$

$Y = AX \Rightarrow$
a) $\Rightarrow A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\det(A) = \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = 0 \Rightarrow f$ nu e injectivă

b) $P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & -1-\lambda & 1 \\ 0 & 0 & (1-\lambda) \end{vmatrix} =$
$P(\lambda) = 0$

$= (+1-\lambda) \cdot (-1)^6 \begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)[(1-\lambda)(-1-\lambda)+1] =$

$= (1-\lambda)(\cancel{-1} \cancel{-\lambda} \cancel{+\lambda} + \lambda^2 \cancel{+1}) = +\lambda^2(1-\lambda) = 0$

$\lambda_1 = 0 \quad m_1 = 2$
$\lambda_2 = 1 \quad m_2 = 1$

$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid f(x) = 0 \}$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \Rightarrow \boxed{x_1 = x_2} \\ \boxed{x_3 = 0} \end{cases}$$

$V_{\lambda_1} = \{ (x_1, x_1, 0) \mid x_1 \in \mathbb{R}^3 \}$

$V_{\lambda_1} = <\{ (1,1,0) \}> \Rightarrow R_1 = \{(1,1,0)\}$ reper în V_{λ_1}

$V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid f(x) = x \}$

$$\begin{cases} x_1 - x_2 + x_3 = x_1 \rightarrow -x_2 = -x_3 \Rightarrow \boxed{x_2 = x_3} \\ x_1 - x_2 + x_3 = x_2 \Rightarrow x_1 + x_3 = 2x_2 \\ \boxed{x_3 = x_3} \end{cases}$$

$x_1 = 2x_2 - x_2$

$\boxed{x_1 = x_2 = x_3}$

$V_{\lambda_2} = \{ (x_1, x_1, x_1) \mid x_1 \in \mathbb{R} \}$

$V_{\lambda_2} = <\{ (1,1,1) \}> \Rightarrow R_2 = \{(1,1,1)\}$ reper în V_{λ_2}

c) $\operatorname{Ker} f = \{ x \in \mathbb{R}^3 \mid AX = 0 \} = V_{\lambda_1} = <\{(1,1,0)\}>$

$\dim \mathbb{R}^3 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$ $\quad R_1 = \{(1,1,0)\}$

$\dim \operatorname{Im} f = 3 - 1 = 2$

$\operatorname{Im} f = \{ y \in \mathbb{R}^3 \mid \exists\, x \in \mathbb{R}^3$ a.î. $f(x) = y \}$

$$\begin{cases} x_1 - x_2 + x_3 = y_1 \\ x_1 - x_2 + x_3 = y_2 \\ x_3 = y_3 \end{cases}$$

$\Delta p = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1 \neq 0$

$\Delta c = \begin{vmatrix} -1 & 1 & y_1 \\ -1 & 1 & y_2 \\ 0 & 1 & y_3 \end{vmatrix} =$

$= -y_3 + 0 - y_1 - 0 + y_3 + y_2 = y_2 - y_1 = 0$

$\Rightarrow y_2 = y_1$

$\operatorname{Im} f = \{ y \in \mathbb{R}^3 \mid y_2 - y_1 = 0 \}$

$\operatorname{Im} f = \{ (y_1, y_1, y_3) \mid y_1, y_3 \in \mathbb{R} \}$

$Im f = < \{ (1,1,0), (0,0,1) \} >$

$\dim Im f = 2$

$R_2 = \{ (1,1,0), (0,0,1) \}$ reper în $Im f$

d) $\dim \mathbb{R}^3 = \dim Ker f + \dim V'$

$\Rightarrow \dim V' = 2$

$rg \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 3$

$V' = \{ (1,0,1), (0,1,1) \}$

$rg \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = 2$ (maxim) $\Rightarrow R = \{ (1,1,0), (1,0,1), (0,1,1) \}$ reper în $\mathbb{R}^3$

$(1,0,3) = a(1,1,0) + b(1,0,1) + c(0,1,1) =$

$= (a+b, a+c, b+c)$

$\left.\begin{array}{l} a+b=1 \\ a+c=0 \\ b+c=3 \end{array}\right\} \Rightarrow$ $\begin{array}{l} b-c=1 \\ b+c=3 \\ \hline 2b=4 \Rightarrow \boxed{b=2} \\ 2-c=1 \Rightarrow \boxed{c=1} \\ \boxed{a=-1} \end{array}$

$(1,0,3) = v + w$

$v \in Ker f$

$w \in V'$

$v = -1(1,1,0) = (-1,-1,0)$

$w = 2(1,0,1) + (+1)(0,1,1) = (2,1,3)$

$p(1,0,3) = (-1,-1,0)$

$s(1,0,3) = 2p - id_{\mathbb{R}^3} = 2(-1,-1,0) - (1,0,3) =$

$= (-2,-2,0) - (1,0,3) = (-3,-2,-3)$

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③ $f: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad A = \begin{pmatrix} \alpha & 1-\alpha \\ 1 & 2 \end{pmatrix}$

$A = [f]_{R_0,R_0}$

$\alpha = ?$ a.î. a) $\lambda = 1$ valoare proprie c) $0 \notin \Gamma(f)$

b) $\lambda = -1$ —"—

$$P(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} \alpha-\lambda & 1-\alpha \\ 1 & 2-\lambda \end{vmatrix} =$$

$$= (\alpha-\lambda)(2-\lambda) - (1-\alpha) =$$

$$= 2\alpha - \alpha\lambda - 2\lambda + \lambda^2 - 1 + \alpha =$$

$$= \lambda^2 - \lambda(\alpha+2) + 3\alpha - 1 = 0$$

a) $\lambda = 1 \Rightarrow 1 - 1(\alpha+2) + 3\alpha - 1 = 0$

$\cancel{1} - \alpha - 2 + 3\alpha - \cancel{1} = 0$

$2\alpha - 2 = 0 \Rightarrow \boxed{\alpha = 1}$

b) $\lambda = -1$ $\quad \cancel{1} + (\alpha+2) + 3\alpha - \cancel{1} = 0$

$2 + \alpha + 3\alpha + 1 = 0$

$4\alpha + 2 = 0 \Rightarrow \boxed{\alpha = -\frac{1}{2}}$

c) Presupun că $\lambda = 0$ valoare proprie

$\det A = \begin{vmatrix} \alpha & 1-\alpha \\ 1 & 2 \end{vmatrix} = 2\alpha - 1 + \alpha = 3\alpha - 1$

$\lambda = 0 \in \Gamma(f) \Rightarrow 3\alpha - 1 = 0 \Rightarrow \alpha = \frac{1}{3}$

$\alpha = \frac{1}{3} \Rightarrow \det A = 0$ (nu convine) $\Rightarrow$ α

Deci $\lambda = 0 \notin \Gamma(f)$

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(2) $S : V \to W$ liniară

$S^* : W^* \to V^*$, $S^*(f) = f \circ S$, $\forall f \in W^*$

(pull - back)

a) S^* liniară

b) S surj $\Rightarrow S^*$ injectivă

DIMA OANA
GR 141

TEMA NR 3 (SEMINAR 6)

① $f \in END(\mathbb{R}^3)$

$A = [f]_{R_0,R_0}$ $A = \begin{pmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix}$

a) valorile proprii
b) subspațiile proprii + repere

$f(x) = (-3x_1 - 7x_2 - 5x_3, 2x_1 + 4x_2 + 3x_3, x_1 + 2x_2 + 2x_3)$

a) $P(\lambda) = \det(A - \lambda I_3) = 0$

$$\begin{vmatrix} -3-\lambda & -7 & -5 \\ 2 & 4-\lambda & 3 \\ 1 & 2 & 2-\lambda \end{vmatrix} \overset{l_1' = l_1 + l_2 + l_3}{=} \begin{vmatrix} -\lambda & -1-\lambda & -\lambda \\ 2 & 4-\lambda & 3 \\ 1 & 2 & 2-\lambda \end{vmatrix} =$$

$$= (-\lambda) \begin{vmatrix} 4-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} + (1+\lambda) \begin{vmatrix} 2 & 3 \\ 1 & 2-\lambda \end{vmatrix} + (-\lambda) \begin{vmatrix} 2 & 4-\lambda \\ 1 & 2 \end{vmatrix} =$$

$$= (-\lambda)(\lambda^2 - 6\lambda + 2) + (1+\lambda)(1-2\lambda) + (-\lambda)(+\lambda) =$$

$$= -\lambda^3 + 6\lambda^2 - 2\lambda + 1 - 2\lambda + \lambda - 2\lambda^2 - \lambda^2 =$$

$$= -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = 0$$

$$= (\lambda - 1)^3(-1) = -(\lambda-1)^3 = 0$$

$\lambda_1 = 1 \quad m_1 = 3$

$M = \begin{pmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{pmatrix}$

$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid f(x) = x \}$

$$\begin{cases} -3x_1 - 7x_2 - 5x_3 = x_1 \\ 2x_1 + 4x_2 + 3x_3 = x_2 \\ x_1 + 2x_2 + 2x_3 = x_3 \end{cases} \Rightarrow \begin{cases} -4x_1 - 7x_2 - 5x_3 = 0 \\ 2x_1 + 3x_2 + 3x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \end{cases}$$

$\det M = 0$; $\Delta p = \begin{vmatrix} -4 & -7 \\ 2 & 3 \end{vmatrix} = -12 + 14 = 2 \neq 0$

Sistem omogen $\Rightarrow$ întotdeauna compatibil

$\Rightarrow rg\, A = rg\, \tilde{A} = 2$

$\cancel{x_1 + x_2 = -x_3 \mid (-2)} \Rightarrow$

$\cancel{2x_1 + 3x_2 = -3x_3}$

$\cancel{\Rightarrow -2x_1 - 2x_2 = 2x_3}$

$\cancel{\boxed{x_2 = -x_3}}$

$\cancel{x_1 - x_3 = -x_3} \Rightarrow \cancel{\boxed{x_1 = 0}}$

$$\begin{cases} -4x_1 - 7x_2 = 5x_3 \\ 2x_1 + 3x_2 = -3x_3 \end{cases}$$

$-2 \cdot 2x_1 - 7x_2 = 5x_3$

$2x_1 = -3x_3 - 3x_2$

$+6x_3 + 6x_2 - 7x_2 = 5x_3$

$-x_2 = -x_3$

$\boxed{x_2 = x_3}$

$2x_1 = -3x_3 - 3x_3 \Rightarrow 2x_1 = -6x_3 \mid :2$

$\boxed{x_1 = -3x_3}$

$V_{\lambda_1} = \{ (-3x_3, x_3, x_3) \mid x_3 \in \mathbb{R} \}$

$V_{\lambda_1} = < \{ (-3, 1, 1) \} > \Rightarrow R_1 = \{ (-3, 1, 1) \}$ reper în V_{λ_1}

DIMA OANA TEMA 4 (CURS 8)

GR. 141

① Fie $f: \mathbb{R}^3 \to \mathbb{R}^3$ endomorfism

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} = [f]_{R_0,R_0}$$

f e diagonalizabil

$f(x) = (x_1 - 3x_2 + 3x_3,\ 3x_1 - 5x_2 + 3x_3,\ 6x_1 - 6x_2 + 4x_3)$

$P(\lambda) = 0$

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = (l_1' = l_1 - l_2)$$

$$= \begin{vmatrix} -2-\lambda & 2+\lambda & 0 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} -5-\lambda & 3 \\ -6 & 4-\lambda \end{vmatrix} +$$

$$+(-1)(2+\lambda)\begin{vmatrix} 3 & 3 \\ 6 & 4-\lambda \end{vmatrix} = (-2-\lambda)(-20+5\lambda-4\lambda+\lambda^2+18) +$$

$$+(-1)(2+\lambda)(12-3\lambda-18) =$$

$$= (-2-\lambda)(\lambda^2 + \lambda - 2) - (2+\lambda)(-6-3\lambda) =$$

$$= (-2-\lambda)(\lambda^2 + \lambda - 2 - 6 - 3\lambda)$$

$$= (-2-\lambda)(\underline{\lambda^2 - 2\lambda - 8})$$

$\Delta = 4 + 4 \cdot 1 \cdot 8 = 36$

$\lambda_{1,2} = \frac{2 \pm 6}{2} = 1 \pm 3$; $\lambda_1 = 4$, $\lambda_2 = -2$

$$= (-2-\lambda)(\lambda+2)(\lambda-4) = (-1)(\lambda+2)(\lambda+2)(\lambda-4) =$$

$$= -(\lambda+2)^2(\lambda-4) = 0$$

$\lambda_1 = -2 \quad m_1 = 2$

$\lambda_2 = 4 \quad m_2 = 1$

$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid f(x) = -2x \}$

$$\begin{cases} x_1 - 3x_2 + 3x_3 = -2x_1 \\ 3x_1 - 5x_2 + 3x_3 = -2x_2 \\ 6x_1 - 6x_2 + 4x_3 = -2x_3 \end{cases} \Rightarrow \begin{cases} 3x_1 - 3x_2 + 3x_3 = 0 \\ 3x_1 - 3x_2 + 3x_3 = 0 \\ 6x_1 - 6x_2 + 6x_3 = 0 \end{cases}$$

$x_1 - x_2 + x_3 = 0 \Rightarrow$

$\Rightarrow x_1 = x_2 - x_3$

$V_{\lambda_1} = \{ (x_2 - x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R} \}$

$V_{\lambda_1} = < \{ (1, 1, 0), (-1, 0, 1) \} >$ dim $V_{\lambda_1} = m_1 = 2$

$\mathcal{R}_1 = \{ (1, 1, 0), (-1, 0, 1) \}$ reper în V_{λ_1}

$V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid f(x) = 4x \}$

$$\begin{cases} x_1 - 3x_2 + 3x_3 = 4x_1 \\ 3x_1 - 5x_2 + 3x_3 = 4x_2 \\ 6x_1 - 6x_2 + 4x_3 = 4x_3 \end{cases} \Rightarrow \begin{cases} -3x_1 - 3x_2 + 3x_3 = 0 \\ 3x_1 - 9x_2 + 3x_3 = 0 \\ 6x_1 - 6x_2 = 0 \Rightarrow \end{cases}$$

$\Rightarrow \boxed{x_1 = x_2}$

$V_{\lambda_2} = \{ (x_1, x_1, +2x_1) \mid x_1 \in \mathbb{R} \}$ $\quad -6x_1 = -3x_3 \Rightarrow$ $\boxed{x_3 = +2x_1}$

$V_{\lambda_2} = < \{ (1, 1, 2) \} >$ dim $V_{\lambda_2} = 1 = m_2$

$\mathcal{R} = \{ (1,1,0), (-1,0,1), (1,1,2) \}$ reper în $\mathbb{R}^3$

$A' = [f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

$f(1,1,0) = -2(1,1,0)$

$f(-1,0,1) = -2(-1,0,1)$

$f(1,1,2) = 4(1,1,2)$ $\Rightarrow f$ end diagonalizabil

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(2) Fie $Q : \mathbb{R}^3 \to \mathbb{R}$

$Q(x) = 5x_1^2 + 6x_2^2 + 4x_3^2 - 4x_1x_2 - 4x_1x_3$

a) forma polară asociată $g = ?$

Ker $g = ?$; g este nedegenerată?

b) Să se aducă Q la o formă canonică utilizând metoda Gauss, metoda Jacobi resp metoda valorilor proprii

a) $G = \begin{pmatrix} 5 & -2 & -2 \\ -2 & 6 & 0 \\ -2 & 0 & 4 \end{pmatrix}$

$g(x,y) = 5x_1y_1 + 6x_2y_2 + 4x_3y_3 - 2x_1y_2 - 2x_2y_1$
$- 2x_1y_3 - 2x_3y_1$

$\text{Ker}\, g = \{ x \in \mathbb{R}^2 \mid g(x,y) = 0, \forall y \in \mathbb{R}^3 \}$

$$\begin{cases} 5x_1 - 2x_2 - 2x_3 = 0 \\ -2x_1 + 6x_2 = 0 \Rightarrow 2x_1 = 6x_2 \Rightarrow \boxed{x_1 = 3x_2} \\ -2x_1 + 4x_3 = 0 \Rightarrow 2x_1 = 4x_3 \Rightarrow \boxed{x_1 = 2x_3} \end{cases}$$

$x_2 = \frac{1}{3}x_1$

$x_3 = \frac{1}{2}x_1$

$\text{Ker}\, g = \{ (x_1, \frac{1}{3}x_1, \frac{1}{2}x_1) \mid x_1 \in \mathbb{R} \}$

$\text{Ker}\, g = < \{ (6,2,3) \} > \neq \{ 0_{\mathbb{R}_3} \} \Rightarrow$

$\Rightarrow$ g nu e ~~d~~ nedegenerată

b) Metoda Jacobi:

$\Delta_1 = 5 \neq 0$

$\Delta_2 = \begin{vmatrix} 5 & -2 \\ -2 & 6 \end{vmatrix} = 30 - 4 = 26 \neq 0$

$\Delta_3 = \det G = 120 + 0 + 0 - 24 - 16 - 0 = 80 \neq 0$

$\frac{1}{\Delta_1} = \frac{1}{5}$; $\frac{\Delta_1}{\Delta_2} = \frac{5}{26}$; $\frac{\Delta_2}{\Delta_3} = \frac{26}{80} = \frac{13}{40}$

$Q(x) = \frac{1}{5}x_1^2 + \frac{5}{26}x_2^2 + \frac{13}{40}x_3^2$ (3,0) signatura

$\Rightarrow$ Q poz definită

Metoda valorilor proprii:

$$P(\lambda) = \det(G - \lambda I_3) = \begin{vmatrix} 5-\lambda & -2 & -2 \\ -2 & 6-\lambda & 0 \\ -2 & 0 & 4-\lambda \end{vmatrix} =$$

$$= (-2)\cdot(-1)^4 \begin{vmatrix} -2 & -2 \\ 6-\lambda & 0 \end{vmatrix} + (4-\lambda)(-1)^6 \begin{vmatrix} 5-\lambda & -2 \\ -2 & 6-\lambda \end{vmatrix} =$$

$$= (-2)(+2)(6-\lambda) + (4-\lambda)[(6-\lambda)(5-\lambda) - 4] =$$

$$= -24 + 4\lambda + (4-\lambda)(30 - 6\lambda - 5\lambda + \lambda^2 - 4) =$$

$$= -24 + 4\lambda + (4-\lambda)(\lambda^2 - 11\lambda + 26) =$$

$$= -24 + 4\lambda + 4\lambda^2 - 44\lambda + 104 - \lambda^3 + 11\lambda^2 - 26\lambda =$$

$$= -\lambda^3 + 15\lambda^2 - 66\lambda + 80 = (\lambda-2)(-\lambda^2 + 13\lambda - 40)$$

	λ^3	λ^2	λ	λ^0
	-1	15	-66	80
2	-1	13	-40	0

$\Delta = 169 - 4\cdot 1\cdot 40$

$\Delta = 9$

$\lambda_{2,3} = \dfrac{-13 \pm 3}{-2}$

$\lambda_2 = \dfrac{-16}{-2} = 8$

$\lambda_3 = \dfrac{-10}{-2} = 5$

$$= (\lambda-2)(\lambda-8)(\lambda-5) = 0$$

$\lambda_1 = 2 \quad m_1 = 1$

$\lambda_2 = 8 \quad m_2 = 1$

$\lambda_3 = 5 \quad m_3 = 1$

$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid g(x) = 2x \}$

$$\begin{cases} 5x_1 - 2x_2 - 2x_3 = 2x_1 \\ -2x_1 + 6x_2 = 2x_2 \\ -2x_1 + 4x_3 = 2x_3 \end{cases} \Rightarrow \begin{cases} 3x_1 - 2x_2 - 2x_3 = 0 \\ -2x_1 + 4x_2 = 0 \\ -2x_1 + 2x_3 = 0 \end{cases}$$

$4x_2 = 2x_1 \Rightarrow x_1 = 2x_2$

$2x_3 = 2x_1 \Rightarrow x_1 = x_3$

$x_2 = \frac{1}{2}x_1$

$x_3 = x_1$

$V_{\lambda_1} = \{ (x_1, \frac{1}{2}x_1, x_1) \mid x_1 \in \mathbb{R} \}$

$V_{\lambda_1} = < \{ (2, 1, 2) \} >$

$f(2,1,2) = 2(2,1,2)$

$V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid f(x) = 8x \}$

$$\begin{cases} 5x_1 - 2x_2 - 2x_3 = 8x_1 \\ -2x_1 + 6x_2 = 8x_2 \\ -2x_1 + 4x_3 = 8x_3 \end{cases} \Rightarrow \begin{cases} -3x_1 - 2x_2 - 2x_3 = 0 \\ -2x_1 - 2x_2 = 0 \\ -2x_1 - 4x_3 = 0 \end{cases}$$

$x_2 = -x_1$

$2x_3 = -x_1 \Rightarrow x_3 - \frac{1}{2}x_1$

$x_2 = -x_3$

$\boxed{x_2 = 2x_1}$

$-x_3 = 2x_1$

$\boxed{x_3 = -2x_1}$

$V_{\lambda_2} = \{ (x_1, -x_1, -\frac{1}{2}x_1) \mid x_1 \in \mathbb{R} \}$

$V_{\lambda_2} = < \{ (2, -2, -1) \} >$

$f(2,-2,-1) = 8(2,-2,-1)$

$-x_3 = 2x_1$

$x_1 = -\frac{1}{2}x_3$

$V_{\lambda_3} = \{ x \in \mathbb{R}^3 \mid f(x) = 5x \}$

$$\begin{cases} \cancel{5x_1} - 2x_2 - 2x_3 = \cancel{5x_1} \\ -2x_1 + 6x_2 = 5x_2 \\ -2x_1 + 4x_3 = 5x_3 \end{cases}$$

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

$-2x_1 + x_2 = 0 \Rightarrow x_2 = 2x_1$

$x_1 > 2x_1, -2x_1$

$x_3 = -x_2$

$x_1 = \frac{1}{2}x_2$

$V_{\lambda_3} = \{ (\frac{1}{2}x_2, x_2, -x_2) \mid x_2 \in \mathbb{R} \}$

$x_1 = \frac{1}{2} 2x_1$

$V_{\lambda_3} = < \{ (1, 2, -2) \} >$

$f(1,2,-2) = 5(1,2,-2)$

$R = \{ (2,1,2), (2,-2,-1), (1,2,-2) \}$ reper în $\mathbb{R}^3$

$$A' = [f]_{R,R} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$Q(x) = 2x_1^2 + 8x_2^2 + 5x_3^2 \Rightarrow$ signatura $(3,0) \Rightarrow$

$\Rightarrow Q$ pozitiv definită

Metoda Gauss:

$$Q(x) = 5x_2^2 + (x_1 - 2x_3)^2 + (2x_1 - x_2)^2$$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_1 - 2x_3 \\ x_3' = 2x_1 - x_2 \end{cases} \Rightarrow \quad x_1 = \frac{x_3' + x_1'}{2}, \quad x_2 = x_1', \quad x_3 = \frac{x_3' + x_1' - x_2'}{4}$$

$$Q(x) = 5x_1'^2 + x_2'^2 + x_3'^2$$

signatura $(3,0) \Rightarrow Q$ e pozitiv definită

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③ $Q: \mathbb{R}^n \to \mathbb{R}$, $Q(x) = \frac{1}{2} \sum_{i \neq j} x_i x_j$

Să se aducă la o formă canonică, utilizând metodele de la ex 2b)

Iau un exemplu:

pt. $n=3 \Rightarrow j = \overline{1,3}$

$$Q(x) = \frac{1}{2}(x_1x_2 + x_2x_1 + x_2x_3 + x_3x_2 + x_1x_3 + x_3x_1)$$

$$= \frac{2}{2}(x_1x_2 + x_2x_3 + x_1x_3)$$

$$= x_1x_2 + x_2x_3 + x_1x_3$$

$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ Se poate observa că metoda Jacobi nu merge deoarece $\Delta_1 = 0$

Metoda Gauss:

$$\begin{cases} x_1' = x_1 + x_2 \\ x_2' = x_1 - x_2 \\ x_3' = x_3 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2}(x_1' + x_2') \\ x_2 = \frac{1}{2}(x_1' - x_2') \\ x_3 = x_3' \end{cases}$$

$$Q(x') = \frac{1}{4}(x_1'^2 - x_2'^2) + \frac{1}{2} x_3'(x_1' - x_2') + \frac{1}{2} x_3'(x_1' + x_2')$$

$$Q(x') = \frac{1}{4}\left(x_1'^2 - x_2'^2 + 2x_3'x_1' - \cancel{2x_3'x_2'} + 2x_3'x_1' + \cancel{2x_3'x_2'}\right)$$

$$Q(x) = \frac{1}{4}\left(x_1'^2 - x_2'^2 + 4x_3'x_1'\right) =$$

$$= \frac{1}{4}\left(x_1'^2 + 4x_3'^2 + 4x_3'x_1' - 4x_3'^2 - x_2'^2\right) =$$

$$= \frac{1}{4}\left[(x_1' + 2x_3')^2 - x_2'^2 - 4x_3'^2\right] =$$

$$Q(x) = \frac{x_1''^2}{4} - \frac{x_2''^2}{4} - x_3''^2$$

$x_1'' = x_1' + 2x_3'$

$x_2'' = x_2'$

$x_3'' = x_3'$

Pt $n = 2 \Rightarrow \begin{cases} x_1' = x_1 + x_2 \\ x_2' = x_2 \end{cases} \Rightarrow \begin{cases} x_1 = x_1' - x_2' \\ x_2 = x_2' \end{cases}$

$$Q(x) = (x_1' - x_2')\, x_2' =$$

$$= x_2'x_1' - x_2'^2 =$$

$$= -\left(x_2'^2 - 2\cdot\frac{1}{2}x_2'x_1' + \frac{1}{4}x_1'^2\right) + \frac{1}{4}x_1'^2$$

$$= -\left(x_2' - \frac{1}{2}x_1'\right)^2 + \frac{1}{4}x_1'^2$$

$$= \frac{1}{4}x_1''^2 - x_2''^2$$

$x_1'' = x_1'$

$x_2'' = x_2' - \frac{1}{2}x_1'$

Dem prin inducție că pt $\forall\, n \in \mathbb{N}$ că:

$$Q(x) = \frac{x_1''^2}{4} - \frac{x_2''^2}{4} - \dots x_n''^2$$

Etapa de Verificare a fost făcută pt $n=2$ și $n=3$ 7.

II Etapa de demonstrație: $p(k) \rightarrow p(k+1)$

Pp că $p(k)$: $Q(x) = \frac{x_1''^2}{4} - \frac{x_2''^2}{4} - \dots - x_k''^2$ este A.

Dem că $p(k+1) = Q(x) = \frac{x_1''^2}{4} - \frac{x_2''^2}{4} - \dots - \frac{x_k''^2}{4} - x_{k+1}''^2$ este A

Concluzia este evidentă.

Metoda valorilor proprii:

pt $n=3$: $G = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$P(\lambda) = 0$; $\det(G - \lambda I_3) = P(\lambda)$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2-\lambda & 2-\lambda \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} =$$

$C_1' = C_1 - C_3$

$C_2' = C_2 - C_3$

$$= (2-\lambda)\begin{vmatrix} 0 & 0 & 1 \\ 0 & -\lambda-1 & 1 \\ 1+\lambda & 1+\lambda & -\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 0 & -\lambda-1 \\ 1+\lambda & 1+\lambda \end{vmatrix} =$$

$$= (2-\lambda)(\lambda+1)\begin{vmatrix} 0 & -\lambda-1 \\ 1 & 1 \end{vmatrix} =$$

$$= (2-\lambda)(\lambda+1)(0 - (-\lambda-1)) = (2-\lambda)(\lambda+1)^2$$

$\lambda_1 = 2 \quad m_1 = 1$

$\lambda_2 = -1 \quad m_2 = 2$

$Q(x) = \begin{pmatrix} n-1 & 0 & 0 & \dots \\ 0 & -1 & 0 & \dots \\ 0 & 0 & -1 & \dots \\ 0 & 0 & 0 & \dots \end{pmatrix}$

pt $n=4$:
$$\begin{vmatrix} -\lambda & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 1 \\ 1 & 1 & -\lambda & 1 \\ 1 & 1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 3-\lambda & 3-\lambda & 3-\lambda \\ 1 & -\lambda & 1 & 1 \\ 1 & 1 & -\lambda & 1 \\ 1 & 1 & 1 & -\lambda \end{vmatrix} =$$

$C_1' = C_1 - C_4$

$C_2' = C_2 - C_4$

$C_3' = C_3 - C_4$

$$= (3-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 1 \\ 1 & 1 & -\lambda & 1 \\ 1 & 1 & 1 & -\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & -\lambda-1 & 0 & 1 \\ 0 & 0 & -\lambda-1 & 1 \\ 1+\lambda & 1+\lambda & 1+\lambda & -\lambda \end{vmatrix} =$$

$$= (-1)(3-\lambda)\begin{vmatrix} 0 & -\lambda-1 & 0 \\ 0 & 0 & -\lambda-1 \\ 1+\lambda & 1+\lambda & 1+\lambda \end{vmatrix} =$$

$$= (-1)(3-\lambda)(1+\lambda) \begin{vmatrix} 0 & -\lambda-1 & 0 \\ 0 & 0 & -\lambda-1 \\ 1 & 1 & 1 \end{vmatrix} = (\lambda-3)(-1)(-\lambda-1) \begin{vmatrix} 0 & -\lambda-1 \\ 1 & 1 \end{vmatrix} =$$

$$= (\lambda-3)(\lambda+1)^3$$

$\lambda_1 = 3 \quad m_1 = 1$

$\lambda_2 = -1 \quad m_2 = 3$

Deci că pt $\forall n \in \mathbb{N}$: ~~forma canonică a lui $Q(x)$ este:~~

$$\begin{cases} n = \text{par} \Rightarrow \cancel{(-1)} \; P(\lambda) = (-1)^{n-1}((n-1)-\lambda)(\lambda+1)^{(n-1)} \\ n \text{ impar} \Rightarrow P(\lambda) = ((n-1)-\lambda)(\lambda+1)^{(n-1)} \end{cases}$$

$\rightarrow \lambda_1 = 3 \quad m_1 = 1$

$\lambda_2 = -1 \quad m_2 = n-1$

$$A' = [f]_{R,R} = \begin{pmatrix} n-1 & 0 & 0 & \dots & \dots \\ 0 & -1 & 0 & \dots & \dots \\ 0 & 0 & -1 & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$Q(x) = (n-1)x_1'^2 - x_2'^2 - x_3'^2 - \dots - x_n'^2$$

Inducție matematică:

I Et de verificare a fost făcută pt $n = 3$ și $n = 4$.

II Et de demonstrație: $p(k) \rightarrow p(k+1)$

Pp că $p(k)$: $Q(x) = (k-1)x_1'^2 - x_2'^2 - x_3'^2 - \dots - x_k'^2$ este A

Deci că $p(k+1)$: $Q(x) = (k)x_1'^2 - x_2'^2 - \dots - x_{k+1}'^2$ este A.

Concluzia este evidentă.

TEMA 5 (SEMINAR 10)

DIMA OANA TEMA 4 (SEMINAR 8)
GR 141

① $f: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x) = (x_1 - 2x_2, -2x_1 + 2x_2 - 2x_3, -2x_2 + 3x_3)$

Este f endomorfism diagonalizabil?

$P(\lambda) = \det(A - \lambda I_3) = 0$

$$A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix} = [f]_{R_0, R_0}$$

$$\begin{vmatrix} 1-\lambda & -2 & 0 \\ -2 & 2-\lambda & -2 \\ 0 & -2 & 3-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & -2 \\ -2 & 3-\lambda \end{vmatrix} + (-2)(-1)^3 \begin{vmatrix} -2 & -2 \\ 0 & 3-\lambda \end{vmatrix}$$

$= (1-\lambda)[(2-\lambda)(3-\lambda) - 4] - 2[(-2)(3-\lambda) - 0] =$

$= (1-\lambda)(6 - 2\lambda - 3\lambda + \lambda^2 - 4) - 2(-6 + 2\lambda) =$

$= (1-\lambda)(\lambda^2 - 5\lambda + 2) + 12 - 4\lambda =$

$= \cancel{\lambda^2} - 5\lambda + 2 - \cancel{\lambda^3} + 5\lambda^2 - 2\lambda + 12 + 4\lambda =$

$= 6\lambda^2 - 3\lambda + 10 - \lambda^3 = (-1)(\lambda - 2)(\lambda - 5)(\lambda + 1) = 0$

$\lambda_1 = 2 \quad m_1 = 1$

$\lambda_2 = 5 \quad m_2 = 1$

$\lambda_3 = -1 \quad m_3 = 1$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = 2x\}$

$$\begin{cases} x_1 - 2x_2 = 2x_1 \\ -2x_1 + \cancel{2x_2} - 2x_3 = \cancel{2x_2} \\ -2x_2 + 3x_3 = 2x_3 \end{cases} \Rightarrow \begin{cases} -x_1 - 2x_2 = 0 \\ -2x_1 - 2x_3 = 0 \Rightarrow \\ \Rightarrow \boxed{x_1 = -x_3} \\ -2x_2 + x_3 = 0 \end{cases}$$

$\boxed{x_3 = 2x_2}$ $\boxed{x_1 = -2x_2}$

TEMA 5 (SEMINAR 10)

$V_{\lambda_1} = \{ (-2x_2, x_2, 2x_2) \mid x_2 \in \mathbb{R} \}$

$V_{\lambda_1} = < \{ (-2,1,2) \} >$ $\dim V_{\lambda_1} = m_1 = 1$

$V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid f(x) = 5x \}$

$\begin{cases} x_1 - 2x_2 = 5x_1 \\ -2x_1 + 2x_2 - 2x_3 = 5x_2 \\ -2x_2 + 3x_3 = 5x_3 \end{cases} \Rightarrow \begin{cases} -4x_1 - 2x_2 = 0 \\ -2x_1 - 3x_2 - 2x_3 = 0 \\ -2x_2 - 2x_3 = 0 \end{cases}$

$-4x_1 = 2x_2 \Rightarrow \boxed{x_2 = -2x_1}$

$2x_2 = -2x_3 \mid :2$

$\boxed{x_2 = -x_3}$ $\boxed{x_3 = +2x_1}$

$V_{\lambda_2} = \{ (x_1, -2x_1, +2x_1) \mid x_1 \in \mathbb{R} \}$

$V_{\lambda_2} = < \{ (1,-2,+2) \} >$ $\dim V_{\lambda_2} = m_2 = 1$

$V_{\lambda_3} = \{ x \in \mathbb{R}^3 \mid f(x) = -x \}$

$\begin{cases} x_1 - 2x_2 = -x_1 \\ -2x_1 + 2x_2 - 2x_3 = -x_2 \\ -2x_2 + 3x_3 = -x_3 \end{cases} \Rightarrow \begin{cases} 2x_1 - 2x_2 = 0 \Rightarrow \boxed{x_1 = x_2} \\ -2x_1 + 3x_2 - 2x_3 = 0 \\ -2x_2 + 4x_3 = 0 \end{cases}$

$4x_3 = 2x_2$

$2x_3 = x_2$

$2x_3 = x_1 \Rightarrow \boxed{x_3 = \frac{1}{2}x_1}$

$V_{\lambda_3} = \{ (x_1, x_1, \frac{1}{2}x_1) \mid x_1 \in \mathbb{R} \}$

$V_{\lambda_3} = < \{ (2,2,1) \} >$ $\dim V_{\lambda_3} = m_3 = 1$

$R = \{ (-2,1,2), (1,-2,2), (2,2,1) \}$ reper în $\mathbb{R}^3$

$f(-2,1,2) = 2(-2,1,2)$

$f(1,-2,2) = (5,-10,10) = 5(1,-2,2)$

TEMA 5 SEMINAR

$$\begin{pmatrix} 1+4 & -2 & -4 \\ \dots & 7 & \dots \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \dots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$f(2,2,1) = (-2,-2,-1) = -(2,2,1)$

$$A' = [f]_{R,R} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\Rightarrow f$ endomorfism diagonalizabil

② Fie $G = \begin{pmatrix} 3 & -2 & -4 \\ -2 & 6 & 2 \\ -4 & 2 & 3 \end{pmatrix}$

matricea asociată formei pătratice $Q : \mathbb{R}^3 \to \mathbb{R}$ în raport cu reperul canonic

a) $Q = ?$

b) $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ forma polară asociată

$Ker(g)$

c) Să se aducă Q la o formă canonică, utilizând metoda Gauss, Jacobi, val. proprii.

Este Q poz. definită?

a) $Q(x) = 3x_1^2 + 6x_2^2 + 3x_3^2 - 4x_1x_2 - 8x_1x_3 + 4x_2x_3$

b) $g(x,y) = 3x_1y_1 + 6x_2y_2 + 3x_3y_3 - 2x_1y_2 - 2x_2y_1$

$-4x_1y_3 - 4x_3y_1 + 2x_2y_3 + 2x_3y_2$

$Ker\, g = \{x \in \mathbb{R}^3 \mid g(x,y) = 0, \forall y \in \mathbb{R}^3\}$

$$\begin{cases} 3x_1 - 2x_2 - 4x_3 = 0 \\ -2x_1 + 6x_2 + 2x_3 = 0 \\ -4x_1 + 2x_2 + 3x_3 = 0 \end{cases}$$

$\det G = 3\cdot 6\cdot 3 + (-2)\cdot 2\cdot(-4) + 2\cdot(-2)\cdot(-4) - 6\cdot(-4)\cdot(-4)$

$-(-2)(-2)\cdot 3 - 2\cdot 2\cdot 3 =$

$= 54 + 16 + 16 - 96 - 12 - 12 =$

$= -34 \neq 0$

Un sistem omogen este întotdeauna compatibil

$\det G \neq 0 \Rightarrow x_1 = x_2 = x_3 = 0$ (sol. banală)

$\operatorname{Ker} g = \{0_{\mathbb{R}_3}\} \Rightarrow g$ nedegenerată

c) Metoda Jacobi:

$\Delta_1 = 3 \neq 0$

$\Delta_2 = \begin{vmatrix} 3 & -2 \\ -2 & 6 \end{vmatrix} = 18 - 4 = 14 \neq 0$

$\Delta_3 = \det G = -34 \neq 0$

$\frac{1}{\Delta_1} = \frac{1}{3}$; $\frac{\Delta_1}{\Delta_2} = \frac{3}{14}$; $\frac{\Delta_2}{\Delta_3} = \frac{14}{-34} = \frac{7}{-17}$

$Q(x) = \frac{1}{\Delta_1} x_1^2 + \frac{\Delta_1}{\Delta_2} x_2^2 + \frac{\Delta_2}{\Delta_3} x_3^2 = \frac{1}{3} x_1^2 + \frac{3}{14} x_2^2 - \frac{7}{14} x_3^2$

Signatura $(2,1) \Rightarrow Q$ nu e pozitiv definită

Metoda valorilor proprii:

$P(\lambda) = \det(G - \lambda I_3) = 0$

$\begin{vmatrix} 3-\lambda & -2 & -4 \\ -2 & 6-\lambda & 2 \\ -4 & 2 & 3-\lambda \end{vmatrix} \xlongequal{\ell_1' = \ell_1 + \ell_3} \begin{vmatrix} -1-\lambda & 0 & -1-\lambda \\ -2 & 6-\lambda & 2 \\ -4 & 2 & 3-\lambda \end{vmatrix}$

$= (-1-\lambda) \begin{vmatrix} 6-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} + (-1-\lambda) \begin{vmatrix} -2 & 6-\lambda \\ -4 & 2 \end{vmatrix}$

$= (-1-\lambda)(18 - 6\lambda - 3\lambda + \lambda^2 - 4) + (-1-\lambda)(-4 + 24 - 4\lambda)$

$= (-1-\lambda)(\lambda^2 - 9\lambda + 14 - 4 + 24 - 4\lambda)$

$= (-1-\lambda)(\lambda^2 - 13\lambda + 34) = (1+\lambda)(\lambda^2 - 13\lambda + 34)$

$\Delta = 169 - 136 = 33$

$\lambda_1 = -1 \quad m_1 = 1$

$\lambda_2 = \frac{13 - \sqrt{33}}{2} > 0 \quad m_2 = 1$

$\lambda_3 = \frac{13 + \sqrt{33}}{2} > 0 \quad m_3 = 1$

$169 > 33 \Rightarrow 13 > \sqrt{33}$

6.

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid GX = -X\}$ $\begin{pmatrix} +4 & -2 & -4 \\ -2 & 7 & 2 \\ -4 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\dim = 3 - 2 = 1$

$\hookrightarrow$ rang matrice

$V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid GX = \lambda_2 X\}$ $\begin{pmatrix} 3-\frac{13-\sqrt{33}}{2} & -2 & -4 \\ -2 & 7-\frac{13-\sqrt{33}}{2} & 2 \\ -4 & 2 & 3-\frac{13-\sqrt{33}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\dim V_{\lambda_2} = 3 - 2 = 1$

analog pt $\dim V_{\lambda_3} = 3 - 2 = 1$

$A = [f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{13-\sqrt{33}}{2} & 0 \\ 0 & 0 & \frac{13+\sqrt{33}}{2} \end{pmatrix}$

$Q(x) = \lambda_1 x_1'^2 + \lambda_2 x_2'^2 + \lambda_3 x_3'^2$

$\lambda_1 < 0,\ \lambda_2, \lambda_3 > 0 \Rightarrow$ signatura $(2,1) \Rightarrow$

$\Rightarrow Q$ nu este poz def

Metoda Gauss:

$g(x,y) = 3x_1y_1 + 6x_2y_2 + 3x_3y_3 - 2x_1y_2 - 2x_2y_1 - 4x_1y_3 -$
$- 4x_3y_1 + 2x_2y_3 + 2x_3y_2$

$Q(x) = (\sqrt{3}x_1)^2 + \left(\frac{2}{\sqrt{3}}x_2\right)^2 + \left(\frac{4}{\sqrt{3}}\right)^2 - 2\sqrt{3}\cdot\frac{2}{\sqrt{3}}x_1x_2 - 2\sqrt{3}\frac{4}{\sqrt{3}}x_1x_3 +$
$+ 2\frac{8}{3}x_2x_3 + \frac{14}{3}x_2^2 - \frac{7}{3}x_3^2 - \frac{4}{7}x_2x_3 =$

$= 3\left(x_1 - \frac{2}{3}x_2 - \frac{4}{3}x_3\right)^2 + \frac{2}{3}\left[(\sqrt{7}x_2)^2 - 2\sqrt{7}\frac{x_2x_3}{\sqrt{7}} +\right.$
$\left. + \left(\frac{1}{\sqrt{7}}x_3\right)^2\right] - \frac{51}{21}x_3^2$

$\begin{cases} x_1' = x_1 - \frac{2}{3}x_2 - \frac{4}{3}x_3 \quad x_3' = x_3 \\ x_2' = x_2 + \frac{1}{7}x_3 \end{cases}$

$Q(x) = 3x_1'^2 + \frac{14}{3}x_2'^2 - \frac{51}{21}x_3'^2$

$\Rightarrow (2,1)$ signatura $\Rightarrow Q$ nu e poz def

DIMA OANA TEMA 5 (CURS 10)
GR 141

① Fie $(\mathbb{R}^3, g_0)$ s.v.e.r, cu str canonică

~~a) Ar că~~ $f : \mathbb{R}^3 \to \mathbb{R}$
$f(x) = (x_3, x_2, x_1)$

---

a) Ar că f e transf ortogonală de speță 2

(b) Să se det un reper ortonormat (pozitiv orientat) $R = \{e_1, e_2, e_3\}$ a.î.

$$[f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

unde $f = s \circ R_\varphi$, R_φ = rotație de unghi orientat φ și s = simetrie ortogonală față de $<\{e_1\}>^\perp$

---

② Fie $u = (1, 1, -1)$
$v = (0, 1, 2)$
$w = (0, 0, 1)$

---

a) Să se afle volumul paralelipipedului determinat de u, v, w

b) Să se arate că $\{u, v, w\}$ este un reper în $\mathbb{R}^3$. Să se ortonormeze, utilizând Gram - Schmidt.

---

① a) $A = [f]_{R_0, R_0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Trebuie să arăt că:

$A \in O(3) \Leftrightarrow A \cdot A^T = I_3$ şi $\det A = -1$

$$A \cdot A^T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \Big\} \Rightarrow$$

$\det A = 0 + 0 + 0 - 1 - 0 - 0 - 0 = -1$

$\Rightarrow f$ transf ortog de speţă $2 \Leftrightarrow f \in O(\mathbb{R}^3)$ de speţă 2

b) $\operatorname{Tr} A = 1$

$\operatorname{Tr} A = -1 + 2\cos\varphi$ $\Big\} \Rightarrow 1 = -1 + 2\cos\varphi \Rightarrow 2\cos\varphi = 2 \Rightarrow$

$\Rightarrow \cos\varphi = 1 \Rightarrow \varphi = 0$ unghi de rotaţie

Trebuie să aflu axa de rotaţie:

$f(x) = x \Rightarrow \begin{cases} x_3 = x_1 \\ x_2 = x_2 \\ x_1 = x_2 \end{cases} \Rightarrow \langle (1,1,1) \rangle$ axa de rotaţie

$\cancel{\langle u \rangle}$

$\langle \{u\} \rangle^{\perp} = \{ x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0 \}$

$\langle \{u\} \rangle^{\perp} = \{ (-x_2 - x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R} \}$

$= \langle \{ \underbrace{(-1,1,0)}_{f_2}, \underbrace{(-1,0,1)}_{f_3} \} \rangle$

Aplic Gram Schmidt:

$e_2 = f_1 = (-1, 1, 0)$

$e_3 = f_3 - \dfrac{\langle f_3, e_2 \rangle}{\langle e_2, e_2 \rangle} \cdot e_2 = (-1,0,1) - \frac{1}{2}(-1,1,0) =$

$= \left(-\frac{1}{2}, -\frac{1}{2}, 1\right) = \frac{1}{2}(-1,-1,2)$

$$e_2 \times e_3 = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{vmatrix} = i \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} - j \begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} +$$

$$+ k \begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} = (2, +2, 2)$$

$e_1 = \frac{1}{2\sqrt{3}}(2,2,2)$ versorul axei

$\mathcal{R} = \{ e_1 = \frac{1}{2\sqrt{3}}(2,2,2),\ e_2 = \frac{1}{\sqrt{2}}(-1,1,0),\ e_3 = \frac{1}{\sqrt{6}}(-1,-1,2) \}$

reper ortonormat în $\mathbb{R}^3$ poz. orientat

$[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ f = simetrie față de $<\{e_1\}>^{\perp}$

---

② a) $V_p = |u \wedge v \wedge w| = |w \wedge u \wedge v| =$

$= |<w, u \times v>|$

$u \times v = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{vmatrix} = i \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - j \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} +$

$+ k \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = (3, -2, 1)$

$<w, u \times v> = \cancel{3 - 2 - 1} = 0 + 0 + 1 = 1$

$V_p = |1| = 1$

b) $\mathcal{R} = \{ \underbrace{(1,1,-1)}_{f_1}, \underbrace{(0,1,2)}_{f_2}, \underbrace{(0,0,1)}_{f_3} \}$

$rg \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} = 3$ (maxim) $\Rightarrow rg\, \mathcal{R} = |\mathcal{R}| = 3 \Rightarrow \mathcal{R}$ SLI ; $|\mathcal{R}| = \dim \mathbb{R}^3 = 3 \Rightarrow \mathcal{R}$ bază

$\Rightarrow \mathcal{R}$ reper

Aplic Gram Schmidt :

$e_1 = f_1 = (1,1,-1)$

$e_2 = f_2 - \frac{<f_2, e_1>}{<e_1, e_1>} \cdot e_1 = (0,1,2) - \frac{(-1)}{3}(1,1,-1) =$

$= (0,1,2) + \frac{1}{3}(1,1,-1) =$

$= \left(\frac{1}{3}, \frac{4}{3}, \frac{5}{3}\right) = \frac{1}{3}(1,4,5)$

$$e_3 = f_3 - \frac{\langle f_3, e_2\rangle}{\langle e_2, e_2\rangle} \cdot e_2 - \frac{\langle f_3, e_1\rangle}{\langle e_1, e_1\rangle} \cdot e_1 =$$

$$= (0,0,1) - \cancel{\frac{\frac{1}{3}\cdot 5}{\frac{1}{9}\cdot 42}} \; \frac{1}{3}\cdot\frac{5}{14}(1,4,5) + \frac{1}{3}(1,1,-1)$$

$$\langle f_3, e_2\rangle = \langle (0,0,1), \tfrac{1}{3}(1,4,5)\rangle = \frac{1}{3}\cdot 5 = \frac{5}{3}$$

$$\langle e_2, e_2\rangle = \langle \tfrac{1}{3}(1,4,5), \tfrac{1}{3}(1,4,5)\rangle = \frac{1}{9}(1+16+25) = \frac{42}{9} = \frac{14}{3}$$

$$\frac{\langle f_3, e_2\rangle}{\langle e_2, e_2\rangle} = \frac{5}{3}\cdot\frac{3}{14} = \frac{5}{14}.$$

$$\langle f_3, e_1\rangle = \langle (0,0,1), (1,1,-1)\rangle = -1$$

$$\langle e_1, e_1\rangle = \langle (1,1,-1), (1,1,-1)\rangle = 3$$

$$e_3 = {}^{42)}(0,0,1) - \frac{5}{42}(1,4,5) \overset{14)}{+} \frac{1}{3}(1,1,-1)$$

$$= \frac{1}{42}\left[(0,0,42) - \underset{(5,20,25)}{(\cancel{\quad})} + (14,14,-14)\right] =$$

$$= \frac{1}{42}(9,-6,-39)$$

$$R = \{ e_1 = \frac{1}{\sqrt{3}}(1,1,-1), e_2 = \frac{1}{\sqrt{42}}(1,4,5), e_3 = \frac{1}{\sqrt{1638}}(9,-6,-39)\}$$

reper ortonormat în $\mathbb{R}^3$

< e

Apl

-

$f_2 \times 7$

$R = \{$

$(-x)$

(f_2)

DIMA OANA TEMA 5 (SEMINAR 10)

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① $(\mathbb{R}^3, g_0)$

$x = (0, 1, -1)$

a) Să se scrie ec. rotației de $\sphericalangle \varphi = \pi$ și axă $\langle\{x\}\rangle$

b) Să se det ec. simetriei ortog față de $\langle\{x\}\rangle^{\perp}$

b) $s(x) = (x_1, x_2, x_3) - 2 \frac{\langle x, x \rangle}{\langle x, x \rangle} \cdot x =$

$= (x_1, x_2, x_3) - \cancel{2} \cdot \frac{x_2 - x_3}{\cancel{2}} \cdot (0, 1, -1) =$

$= (x_1, x_2, x_3) - x_2 (0, 1, -1) + x_3 (0, 1, -1) =$

$= (x_1, x_2, x_3) + (0, -x_2, x_2) + (0, x_3, -x_3) =$

$= (x_1, x_3, x_2)$

$\langle u \rangle^{\perp} = \{ x \in \mathbb{R}^3 \mid x_2 - x_3 = 0 \}$ $x_2 = x_3$

$= \langle \{ \underset{f_2}{\underset{\parallel}{(1,0,0)}}, \underset{f_3}{\underset{\parallel}{(0,1,1)}} \} \rangle$

Aplic Gram-Schmidt:

$e_2 = f_2 = (1, 0, 0)$

$e_3 = f_3 - \frac{\langle f_3, e_2 \rangle}{\langle e_2, e_2 \rangle} \cdot e_2 = (0, 1, 1) - 0 = (0, 1, 1) = f_3$

$f_2 \times f_3 = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = i \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} - j \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + k \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} =$

$= (0, -1, 1) = -x$

$R = \{ -x, f_2, f_3 \}$ reper ortogonal

$s(-x) = (0, 1, -1) = x = -(-x)$

$s(f_2) = (1, 0, 0) = f_2$

$s(f_3) = (0,1,1) = f_3$

$[s]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ~~Ecuația $-x_1+x_2+x_3=0$~~

a) $<\{x\}^{\perp}> = \{x \in \mathbb{R}^3 \mid x_2 - x_3 = 0\} = <\{\underbrace{(1,0,0)}_{f_2}, \underbrace{(0,1,1)}_{f_3}\}$

$e_2 = f_2 = (1,0,0)$

$e_3 = f_3 = f_3 - \frac{<f_3, e_2>}{<e_2, e_2>} \cdot e_2 = (0,1,1)$

$e_2 \times e_3 = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = (0,-1,1)$

$\{e_1 = \frac{1}{\sqrt{2}}(0,1,-1),\ e_2 = (1,0,0),\ e_3 = \frac{1}{\sqrt{2}}(0,1,1)\}$ reper ortonormat

$e_1 = \frac{1}{\sqrt{2}}(0,1,-1)$ versorul axei de rotație

$A' = [f]_{R,R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\pi & -\sin\pi \\ 0 & \sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$A = [f]_{R_0,R_0}$

$R_0 \xrightarrow{C} R \quad C = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}$

$A' = C^{-1}AC$ $A' \quad C^{-1} = C^T$

$\Longleftrightarrow A = CA'C^T$

$A = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} =$

$= \frac{1}{2} \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 1 & -1 & 0 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} & \sqrt{2} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} & -1 \\ -\frac{\sqrt{2}}{2} & -\frac{1}{2} & 1 \end{pmatrix}$

$f: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x) = \frac{1}{2}(\sqrt{2}x_2 + \sqrt{2}x_3, -\sqrt{2}x_1 + x_2 - x_3, -\sqrt{2}x_1 - x_2 + x_3)$

2.

② $(\mathbb{R}^3, g_0)$, $f \in End(\mathbb{R}^3)$

$f(x) = (x_1+x_2-x_3,\ x_1+x_2-x_3,\ -x_1-x_2+x_3)$

a) Să se arate că $f \in Sim(\mathbb{R}^3)$

b) Să se scrie forma pătratică Q asociată. Să se aducă Q la o formă canonică, efectuând o transformare ortogonală R. Să se precizeze R.

c) Fie $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ forma polară asociată lui Q. Este $(\mathbb{R}^3, g)$ un sp. vect euclidian?

a) $f(x) = (x_1+x_2-x_3,\ x_1+x_2-x_3,\ -x_1-x_2+x_3)$

$A = A^T \Rightarrow f \in Sim(\mathbb{R}^3)$

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \Rightarrow A = A^T \Rightarrow f \in Sim(\mathbb{R}^3)$$

b) $Q: \mathbb{R}^3 \to \mathbb{R}$

$Q(x) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 2x_2x_3$

forma pătratică asociată

Aplic metoda valorilor proprii: $\quad C_1' = C_1 + C_3$

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & 0 & -\lambda \\ 1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{vmatrix} = (-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} + (-\lambda)\begin{vmatrix} 1 & 1-\lambda \\ -1 & -1 \end{vmatrix} = (-\lambda)\left[(1-\lambda)^2 - 1\right] + (-\lambda)(-\cancel{1}+\cancel{1}-\lambda) =$$

$$= (-\lambda)(\cancel{1}-\lambda-\cancel{1})(1-\lambda+1) + \lambda^2 =$$

$$= \lambda^2(2-\lambda)+\lambda^2 = \lambda^2(2-\lambda+1) = \lambda^2(3-\lambda) = 0$$

$\lambda_1 = 0 \quad m_1 = 2$

$\lambda_2 = 3 \quad m_2 = 1$

$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid AX = 0_3 \}$

$$\begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 + x_2 - x_3 = 0 \\ -x_1 - x_2 + x_3 = 0 \end{cases} \Rightarrow \boxed{x_3 = x_1 + x_2}$$

$V_{\lambda_1} = \{ (x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R} \}$

$V_{\lambda_1} = \langle \{ (1,0,1), (0,1,1) \} \rangle$

$V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid AX = 3X \}$

$$\begin{cases} x_1 + x_2 - x_3 = 3x_1 \\ x_1 + x_2 - x_3 = 3x_2 \\ -x_1 - x_2 + x_3 = 3x_3 \end{cases} \Rightarrow \begin{cases} -2x_1 + x_2 - x_3 = 0 \\ x_1 - 2x_2 - x_3 = 0 \\ -x_1 - x_2 - 2x_3 = 0 \end{cases} \Rightarrow$$

$\Rightarrow \quad -3x_2 - 3x_3 = 0 \Rightarrow \boxed{x_2 = -x_3}$

$x_2 - x_3 = 2x_1 \Rightarrow \quad -2x_3 = 2x_1 \Rightarrow \boxed{x_3 = -x_1}$

$V_{\lambda_3} = \{ (-x_3, -x_3, x_3) \mid x_3 \in \mathbb{R} \}$

$V_{\lambda_3} = \langle \{ (-1,-1,1) \} \rangle$

Aplic Gram Schmidt:

$f_1 = (1,0,1)$; $f_2 = (0,1,1)$; $f_3 = (-1,-1,1)$

$e_1 = f_1 = (1,0,1)$

$$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 = (0,1,1) - \frac{1}{2}(1,0,1) = \left(-\frac{1}{2}, 1, \frac{1}{2}\right) = \frac{1}{2}(-1,1,1)$$

$$e_3 = f_3 - \frac{\langle f_3, e_1\rangle}{\langle e_1, e_1\rangle} \cdot e_1 - \frac{\langle f_3, e_2\rangle}{\langle e_2, e_2\rangle} \cdot e_2 =$$

$$= f_3 - 0 - 0 = (-1, -1, 1)$$

$$R = \left\{ e_1 = \frac{1}{\sqrt{2}}(1,0,1),\ e_2 = \frac{1}{2\sqrt{3}}(-1,1,1),\ e_3 = \frac{1}{\sqrt{3}}(-1,-1,1) \right\}$$

reper ortonormat

$$A' = [f]_{R,R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad Q(x) = 3x_3'^2$$

Signatura $(1,0) \Rightarrow$

$\Rightarrow Q$ nu e pozitiv def.

$$R_0 = \{e_1^0, e_2^0, e_3^0\} \xrightarrow{C} R$$

$$C = [h]_{R,R} \in O(3)$$

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$h(x) = \left(\frac{1}{\sqrt{2}}x_1 - \frac{1}{2\sqrt{3}}x_2 - \frac{1}{\sqrt{3}}x_3,\ \frac{1}{2\sqrt{3}}x_2 - \frac{1}{\sqrt{3}}x_3,\ \frac{1}{\sqrt{2}}x_1 + \frac{1}{2\sqrt{3}}x_2 + \frac{1}{\sqrt{3}}x_3 \right)$$

c) $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$$g(x,y) = x_1y_1 + x_2y_2 + x_3y_3 + x_1y_2 + x_2y_1 - x_1y_3 - x_3y_1 - x_2y_3 - x_3y_2$$

Dem că g formă biliniară simetrică:

$\cancel{g(x,y) = g(y,x)}$

$\cancel{g(ax+by,z)} =$ $G = G^T$ $\quad G = A$

$\cancel{G = (1}$ Dem a) $\Rightarrow A = A^T \Leftrightarrow G = G^T \Rightarrow$

5.

g formă biliniară simetrică

~~Dem că~~ ~~Veri~~

Caut să văd dacă e pozitiv definită:

Conf pct b) $Q(x) = {x_3'}^2$ $\Rightarrow$ signatura $(1,0) \Rightarrow$

$\Rightarrow$ Q nu e pozitiv def $\Rightarrow$ g nu e produs scalar $\Rightarrow$ g nu e sp. vect. euclidian