Data Structures

Course 3,
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- Interface
  - List-Insert(L, x) adds element x at beginning of a list L
  - List-Delete(L, x) removes element x from a list L
  - List-Search(L, k) finds an element whose key is k in a list L

Linked Lists

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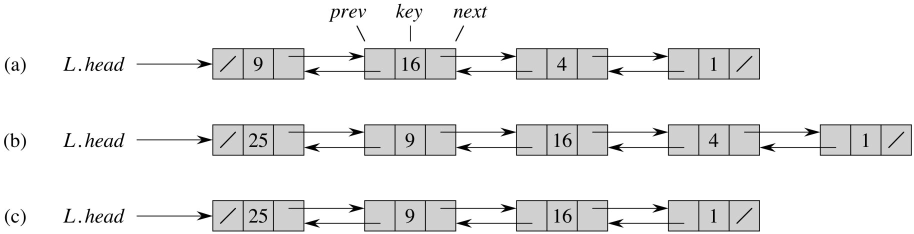
- Implementation
  - a doubly-linked list
  - each element x: two "links" $x.\mathrm{prev}$ and $x.\mathrm{next}$ to the previous and next elements, respectively
  - each element x: key $x.\mathrm{key}$

Linked List: Implementation



- (a). Linked list representing set $S = \{1, 4, 9, 16\}$.
- (b). After LIST-INSERT(S,25).
- (c). After LIST-DELETE(S,4).

Linked List: Implementation

List-Init(L)

$$\text{L.head} = \text{NIL}$$

List-Insert(L, x)

```
x.next = L.head
if L.head ≠ NIL
    L.head.prev = x
    L.head = x
    x.prev = NIL
```

List-Search(L, k)

```
x = L.head.next
while x ≠ NIL ∧ x.key ≠ k
    x = x.next
return x
```

Linked List: Implementation (II)

```
List-Delete(L, x)
if x.prev ≠ NIL
        x.prev.next = x.next
else L.head = x.next
if x.next ≠ NIL
        x.next.prev = x.prev
```

Linked List With a “Sentinel”

- instead of NIL sometimes convenient to have a dummy “sentinel” element L.nil
- Simplifies LIST-DELETE .
- Adds more memory $\times$.

Linked List With a “Sentinel”

```
List-Init(L)
L.nil.prev = L.nil
L.nil.next = L.nil
```

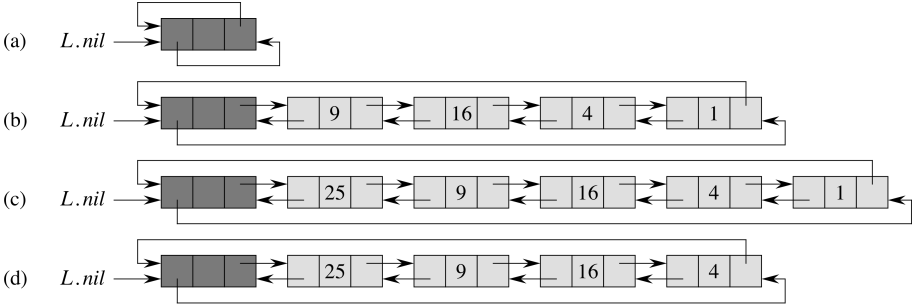
```
List-Insert(L, x)
x.next = L.nil.next
L.nil.next.prev = x
L.nil.next = x
x.prev = L.nil
```

```
List-Search(L, k)
x = L.nil.next
while x ≠ L.nil ∧ x.key ≠ k
        x = x.next
return x
```

Linked Lists: Observations on Implementation

- Insert: at the head of the list.
- Possible: insert arbitrary position.

Circular Linked Lists



- Can use nil sentinel as head of the list.
- (a): empty circular list.
- (b): Linked list representing set $S = \{1, 4, 9, 16\}$.
- (c): After LIST-INSERT(S,25).
- (d): After LIST-DELETE(S,4).

Linked Lists: Scorecard

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Algorithm	Complexity
List-Insert	

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List-Insert	O(1) ✓
List-Delete (with pointer)	

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List-Insert	O(1) ✓
List-Delete (with pointer)	O(1) ✓
List-Search	Θ(n) $\times$

Linked Lists: to conclude

- Can reimplement Stacks/Queues using Linked Lists.
- Implementation with pointers: will not pass the class if you don't know it !

Advanced topic - Skip lists

Caution

Topic not in Cormen. See Drozdek for details/C++ implementation.

- Problem with linked list: search is slow !... even when elements sorted.
- Solution: lists of ordered elements that allow skipping some elements to speed up search.
- Skip lists: variant of ordered linked lists that makes such search possible.

More advanced data structure (W. Pugh ”Skip lists: a Probabilistic Alternative to Balanced Trees”, Communication of the ACM 33(1990), pp. 668-676.) If anyone curious/interested in data structures/algorithms, can give paper to read; taste how a research article looks like.

Skip lists

Where does this ever get applied ?

...

Skip lists in real life

According to Wikipedia:

- MemSQL - skip lists as prime indexing structure for its database technology.
- Cyrus IMAP server - ”skiplist” backend DB implementation
- Lucene uses skip lists to search delta-encoded posting lists in logarithmic time.
- QMap (up to Qt 4) template class of Qt that provides a dictionary.
- Redis, ANSI-C open-source persistent key/value store for Posix systems, skip lists in implementation of ordered sets.
- nessDB, a very fast key-value embedded Database Storage Engine.
- skipdb: open-source DB format using ordered key/value pairs.
- ConcurrentSkipListSet and ConcurrentSkipListMap in the Java 1.6 API.

Skip lists in real life (II)

According to Wikipedia:

- Speed Tables: fast key-value datastore for Tcl that use skiplists for indexes and lockless shared memory.
- leveldb, a fast key-value storage library written at Google that provides an ordered mapping from string keys to string values
- MuQSS Scheduler for the Linux kernel uses skip lists
- SkipMap uses skip lists as base data structure to build a more complex 3D Sparse Grid for Robot Mapping systems.

Skip lists: implementation

What we want

$k = 1, \ldots, \lfloor \log_2(n) \rfloor$, $1 \leq i \leq \lfloor n/2^{k-1} \rfloor - 1$.

- Item $2^{k-1} \cdot i$ points to item $2^{k-1} \cdot (i+1)$.
- every second node points to positions two node ahead,
- every fourth node points to positions four nodes ahead,
- every eigth node points to positions eigth nodes ahead,
- ......, and so on.

- Different number of pointers in different nodes in the list !
- half the nodes only one pointer.
- a quarter of the nodes two pointers,
- an eigth of the nodes four pointers,
- ......, and so on.
- $n \log_2(n)/2$ pointers.

Search Algorithm

1. First follow pointers on the highest level until a larger element is found or the list is exhausted.
2. If a larger element is found, restart search from its predecessor, this time on a lower level.
3. Continue doing this until element found, or you reach the first level and a larger element or the end of the list.

Inserting and deleting nodes

Major problem

- When inserting/deleting a node, pointers of prev/next nodes have to be restructured.
- Solution: rather than equal spacing, random spacing on a level.
- Invariant: Number of nodes on each level: equal, in expectation to what it would be under equal spacing

Principle

If you're traveling 10 meters in 10 steps, a step is on average one meter.

Inserting and deleting nodes (II)

- Level numbering: start with zero.
- New node inserted: probability 1/2 on first level, 1/4 second level, 1/8 third level, ..., etc.
- Function chooseLevel: chooses randomly the level of the new node.
- Generate random number. If in [0,1/2] level 1, [1/2,3/4] level 2, etc.
- To delete node: have to update all links.

Indexing

Computing the i'th element faster than in O(i)

- If we record "step sizes" in our lists we can even mimic indexing !
- Start on highest level.
- If step too big, restart search from predecessor, this time on a lower level.
- Continue doing this until element found.

Update "step sizes" by insertion/deletion

Easy if you have doubly linked lists.

- On deletion: pred[i].size+ = deleted.size on all levels i.
- On insertion: Simply keep track of predecessors and index of the inserteed sequence.

Skip Lists: Scorecard

Method	Average	Worst-Case
SPACE: ✓	O(n)	O(nlog(n))
SEARCH: ✓	O(log(n))	O(n)
INSERT: ✓	O(log(n))	O(n)
DELETE: ✓	O(log(n))	O(n)

- quite practical ! ✓
- Probabilistic, worst-case still bad. $\times$
- Not completely easy to implement. $\times$.

Compared to what ?

Binary search trees. Will learn about them later.

Hash Tables

- Idea
  - use a table T with $|T| \ll |U|$
  - map each key $k \in U$ to a position in T, using a hash function

$$h : U \to \{1, \ldots, |T|\}$$

  - $h(k)$ easy ($O(1)$) to compute given k

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Are these algorithms always correct? No! What if two distinct keys $k_1 \neq k_2$ collide? (I.e., $h(k_1) = h(k_2)$)

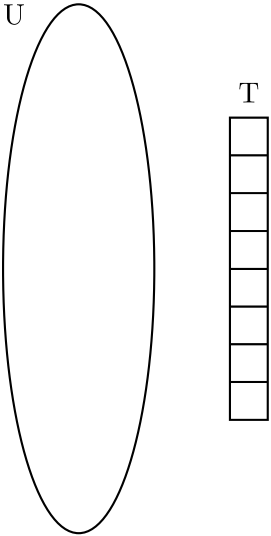
Hash tables

- Work well "on the average"
- Analogy: throw T balls at random into N bins.
- If T << N (in fact $T = o(\sqrt{N})$) then with high-probability no two balls land in the same bin.
- On the average: T/N balls in each bin.
- Want our hash-function to be "random-like": elements of U "thrown out uniformly" by h onto elements of T.

Hashing With Chaining

- Store all objects that map to the same bucket in a linked list.
- "Hope" that hash function is "uniform enough", linked lists are not too large, set operations are efficient.

Hash Table: Chaining



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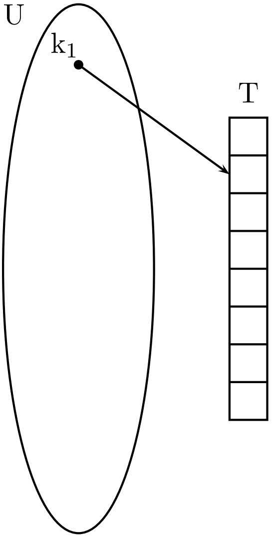

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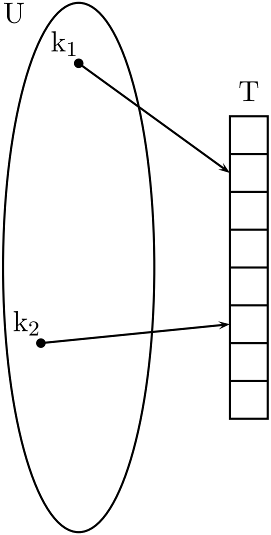

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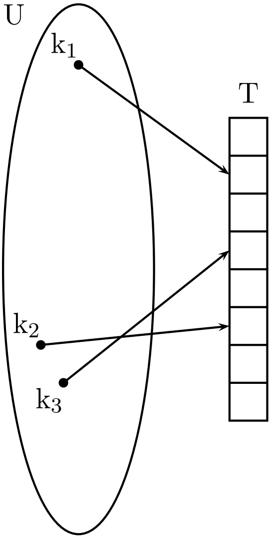

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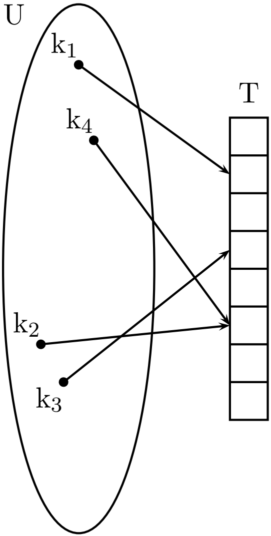

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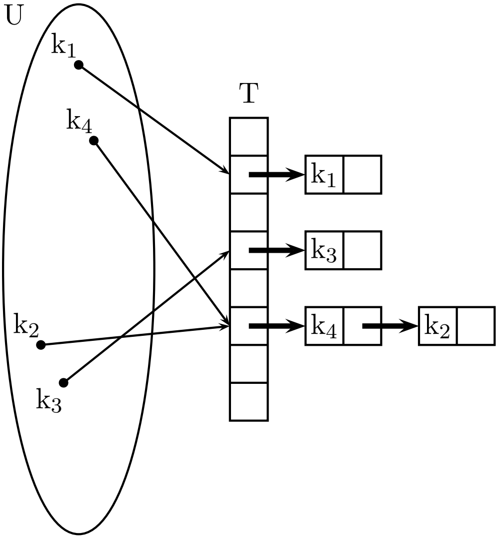

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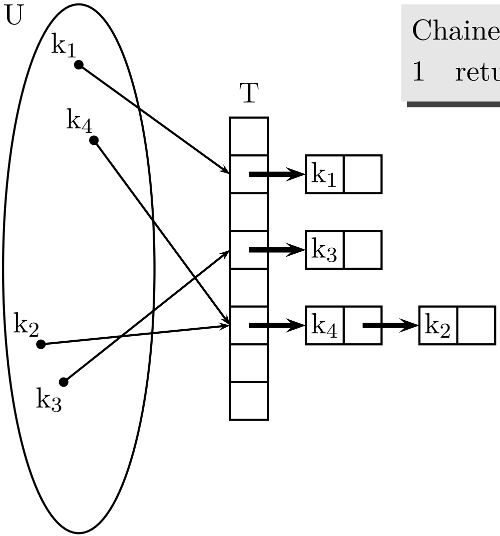

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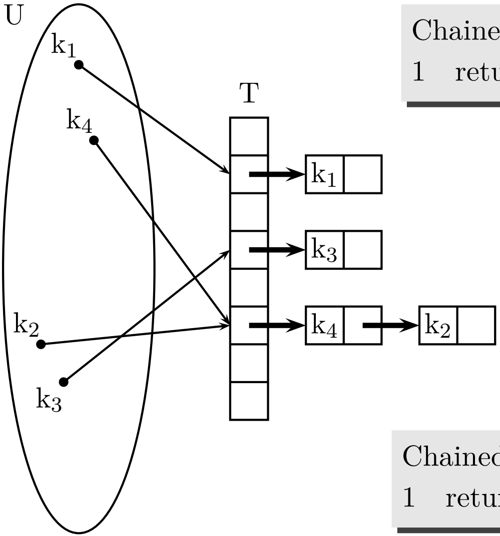

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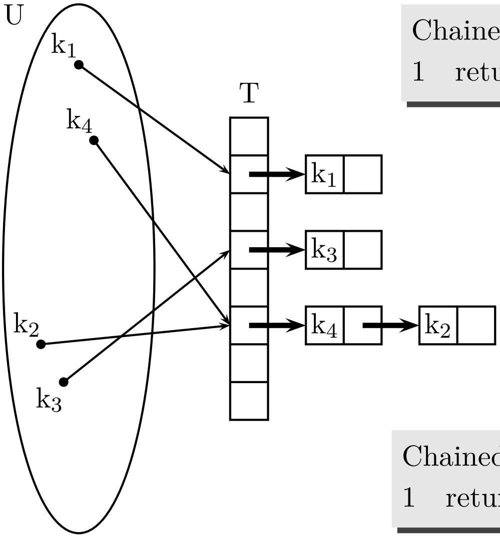

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load factor
$\alpha = \frac{n}{|T|}$

Chained-Hash-Search(T, k)
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Hashing With Chaining: Analysis

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(The formalism is actually a bit more complicated.)

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$$E[n_i] = \frac{n}{|T|} = \alpha$$

- We further assume that h(k) can be computed in O(1) time
- Therefore, the complexity of Chained-Hash-Search is

$$\boxed{\Theta(1 + \alpha)}$$

Hashing with Open Addressing

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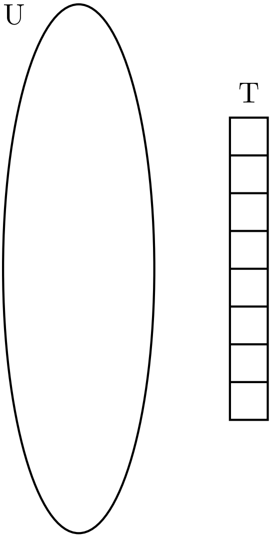
  - this implies $\alpha \leq 1$
- When a collision occurs, simply find another free cell in T
- A sequential "probing" method may not be optimal
  - can you imagine why?

Open-Address Hash Table



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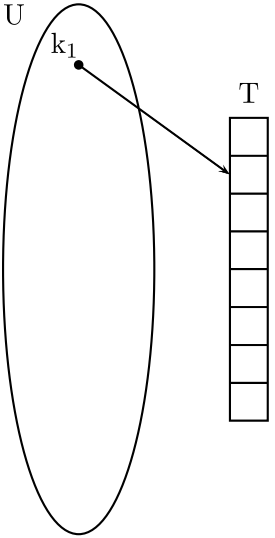

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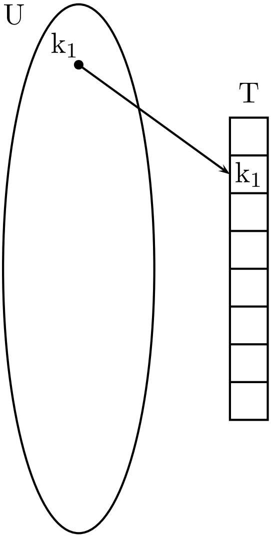

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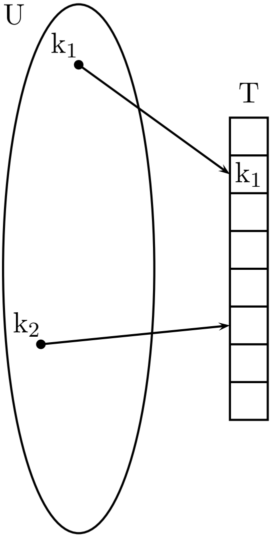

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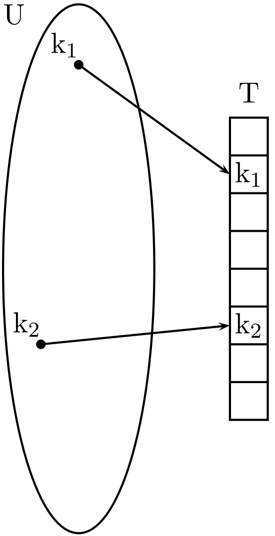

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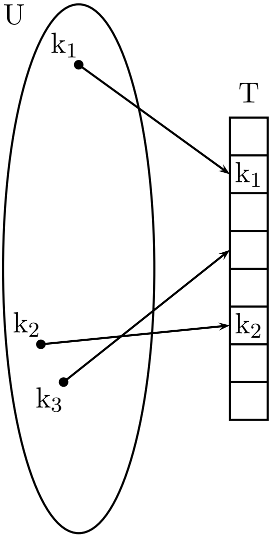

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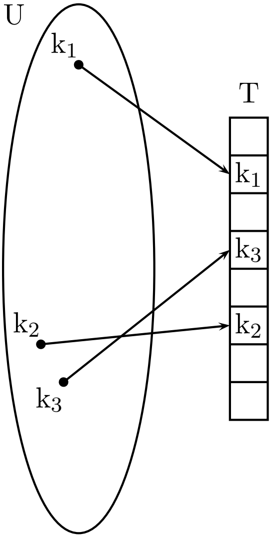

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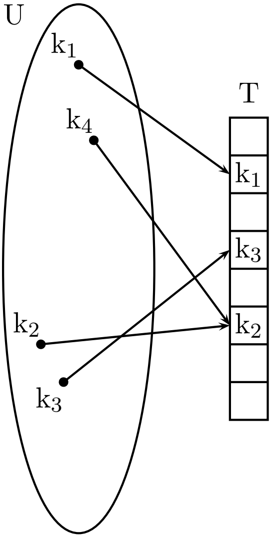

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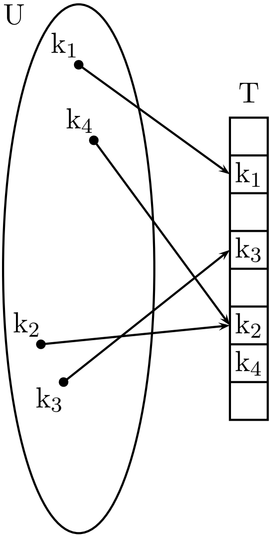

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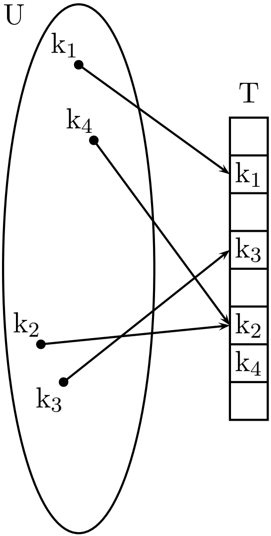


```
Hash-Insert(T, k)
j = h(k)
for i = 1 to T.length
    if T[j] == nil
        T[j] = k
        return j
    elseif j < T.length
        j = j + 1
    else j = 1
error "overflow"
```

Open-Addressing (3)

```
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```

- Notice that $h(k, \cdot)$ must be a permutation
  - i.e., $h(k, 1), h(k, 2), \ldots, h(k, |T|)$ must cover the entire table T

Procedure HASH-SEARCH

```
HASH-SEARCH(T, k)
i ← 0
repeat  j ← h(k, i)
        if T[j] = k
           then return j
        i ← i + 1
   until T[j] = NIL or i = m
return NIL
```

Open-address hashing

- Deletion: difficult. Marking NIL does not work.
- Doing so might make it impossible to retrieve any key during whose insertion probed slot i and found it occupied.
- One solution: DELETED instead of NIL. Problem: search time no longer dependent on load factor.
- Techniques for probing: linear probing, quadratic probing and double hashing.
- Linear probing: given auxiliary hash function $h' : U \rightarrow \{0, \ldots, m-1\}$, use hash function

$$h(k, i) = (h'(k) + i) \bmod m.$$

- Easy to implement but suffers from problem called primary clustering.
- Long runs of occupied slots build up, increasing average search time.

Open-address hashing

- Quadratic probing

$$h(k, i) = (h'(k) + c_1 i + c_2 i^2) \bmod m.$$

- Works much better than linear probing, but to make use of full hash table the values of c_1, c_2, m are constrained.
- Suffers from secondary clustering: if two keys have the same initial probe position then their probe sequences are the same.

Open-address hashing

- Double hashing

$$h(k, i) = (h_1(k) + ih_2(k)) \bmod m.$$

- Among the best methods for open addressing.
- $h_2(k)$ must be relative prime to m. One solution is m a power of two and $h_2(k)$ odd.
- Another one: m prime, $h_2(k) < m$.
- Given an open address hash table with load factor $\alpha = n/m < 1$ the expected number of probes in an unsuccessful search is at most $1/(1 - \alpha)$, assuming uniform hashing.

Double hashing

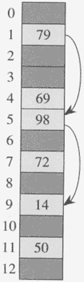

Figure 11.5 Insertion by double hashing. Here we have a hash table of size 13 with $h_1(k) = k \bmod 13$ and $h_2(k) = 1 + (k \bmod 11)$. Since $14 \equiv 1 \pmod{13}$ and $14 \equiv 3 \pmod{11}$, the key 14 is inserted into empty slot 9, after slots 1 and 5 are examined and found to be occupied.

Good hash functions

Caution

- The area of designing good hash function huge.
- Theoreticians and practitioners as well.
- Many hash functions good for specific goal.
- Appearing next does not mean you should blindly use them !
- Drozdek: discusses some more "practical" examples.
- Here: we follow CORMEN, concentrate on general ideas.

Requirements

- A good hash function satisfies (approximately) the assumption of uniform hashing.
- Unfortunately, usually we don't know probability distribution of the keys, and keys might not be drawn independently.

Good hash functions

- Good case: if items are random real numbers k uniformly distributed in $[0,1)$, $h(k) = \lfloor km \rfloor$ satisfies simple uniform hashing conditions.
- Most hash functions assume universe of keys are the natural numbers.
- E.g. character string = integer in base 128 notation.
- Identifier pt. ASCII p = 112, t = 116, becomes $112 \cdot 128 + 116 = 14452$.
- Division method: h(k) = k mod m. Avoid some values of m,e.g. powers of two. Indeed, if $m = 2^p$ then h(k) = the p lowest bits of k. Unless we know that p lowest bits of keys are uniform not a good idea.
- Prime not too close to an exact power of two = often a good choice.

Folding

- Input: broken into pieces.
- Combined in some way.

Example

- SSN (American CNP): 123456789.
- Divide into three parts: 123, 456, 789.
- Add these: 1368.
- Reduced modulo table-size (1000): 368

Good hash functions: Multiplication method

- Multiplication method: Two stage procedure
- First, multiply key by constant A in range $0 < A < 1$, extract fractional part.
- Then multiply by m, extract floor.
- $h(k) = \lfloor m(kA \bmod 1) \rfloor$.
- Value of m not critical. $m = 2^p$.
- Easy implementation. Restrict $A = s/2^w$, w=machine word size.
- Better with some values of A than other. Knuth suggests $A \sim (\sqrt{5} - 1)/2$ will work well.

Multiplication-method of hashing

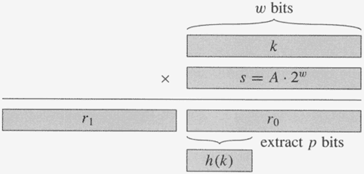

Figure 11.4 The multiplication method of hashing. The w-bit representation of the key k is multiplied by the w-bit value $s = A \cdot 2^w$. The p highest-order bits of the lower w-bit half of the product form the desired hash value $h(k)$.

Universal hashing

- Any fixed hash function vulnerable to worst case behavior:
- if "adversary" chooses n keys that all hash to the same slot, this yields average search time of $\theta(n)$.
- Practical example: Crosby and Wallach (USENIX'03) have shown that one can slow down to a halt systems by attacking implementations of hash tables in Perl, squid web proxy, Bro intrusion detection.
- Cause: hashing mechanism known (due to, e.g. publicly available implementation).
- Solution: choose hash function randomly, independent of the keys that are going to be stored.

Attack when hashing mechanism known

Universal hashing

- $\mathcal{H}$ finite collection of hash functions that map universe U into $\{0, 1, \ldots, m-1\}$.
- Such a collection is called universal if for every keys $k \neq l \in U$, the number of hash functions $h \in \mathcal{H}$ for which $h(k) = h(l)$ is at most $|\mathcal{H}/m|$.
- Suppose a hash function h is chosen from a universal collection of hash functions, and is used to hash n keys into a table T of size m (using chaining).
- If key is not in the table expected length of the list that k hashes to is at most α.
- If key is not in the table expected length of the list that k hashes to is at most $1 + \alpha$.

A universal class of hash functions

- Due to Carter and Wegman.
- $a \equiv b \pmod{p}$ if $p|(a-b)$.
- Z_p: integers modulo p. p prime.
- How do we choose p ? So that all keys are in the range 0 to $p-1$.
- m: number of slots in the hash table.
- $a \in Z_p^*$, $b \in Z_p$.
- $h_{a,b}(k) = ((ak+b) \pmod{p}) \pmod{m}$.
- $\mathcal{H}_{p,m} = \{h_{a,b} : a \in Z_p^*, b \in Z_p\}$.
- Other applications of this set of hash functions: pseudo-random generators.

Perfect hashing

- Hashing can provide worst-case performance when the set of keys is static: once stored in the table, the set of keys never changes.
- Example: set of files on a DVD-R (finished).
- Perfect hashing: the worst-case number of accesses to perform a search is O(1).
- Idea: two-level hashing with universal hashing at each level.
- First level: the n keys are hashed into m slots using a hashing function chosen from a family of universal hash functions.
- Instead of chaining: Use (small) secondary table S_j with an associated hash function h_j.
- Choosing h_j carefully guarantees no collisions.

Perfect hashing with chaining

Perfect hashing: design

- n_j = number of elements that hash to slot j.
- We let $m_j = |S_j| = n_j^2$.
- Idea: if $m = n^2$ and we store n keys in a table of size $m = n^2$ using a hash function randomly chosen from a set of universal hash function then the collision probability is at most 1/2.
- Find a good hash function using O(1) trials.
- Expected amount of memory O(n).
- Why this works: proof omitted (see Cormen if curious).

Hashing: there is more

- Cryptographic hash functions: hash functions with good security properties.
- Most well-known cryptographic hash function: md5 (Rabin). You probably have encountered it if you downloaded anything large from the web.
- (sha-1), sha-2, sha-3.
- U.S. Government standards.
- (Some) attacks on sha-1 (CWI Amsterdam, 2017)

SHA1("The quick brown fox jumps over the lazy dog") gives
hexadecimal: 2fd4e1c67a2d28fced849ee1bb76e7391b93eb12 gives
Base64 binary to ASCII text encoding:
L9ThxnotKPzthJ7hu3bnORuT6xI=

Bloom filters

- Probabilistic data structure. Used to test membership of an element in a dataset.
- NO answer: correct
- YES anwer: possibly false positive.

Hashing: where to go from here

- MapReduce model for grid computing: programming with hash functions.
- Data: key-value pairs.
- Map: applied in parallel to every pair (keyed by k1). Produces a list of pairs (keyed by k2) for each call.
- Mapreduce collects all pairs with the same k2 and groups them together.
- Reduce: applied in parallel to each group, which in turn produces a collection of values in the same domain.

Hashing: where to go from here

- Locality-sensitive hashing: reduces the dimensionality of high-dimensional data. LSH hashes input items so that similar items map to the same 'buckets' with high probability.

Hashing in programming languages

- Python: dictionaries.
- Hash tables in STL: some implementations (e.g. SGI). Most functionality provided by associative container map (implemented using red-black trees).
- However: C++-11: <u>two</u> implementations, std::unordered_map and std::unordered_set.