Seminar 1

Exercise 1. Determine which of the following claims are true for any two nonempty finite sets $A, B \subseteq \mathbb{N}$, and prove your claim.

(i) $A \cap B \subseteq A \cup B$;

(ii) $A \cap B \subsetneq A \cup B$;

(iii) $|A \cup B| = |A| + |B|$;

(iv) $|A \times B| = |A| \cdot |B|$;

(v) $|A^n| = |A|^n$, for any $n \geq 1$, where $A^n = A \underbrace{\times \ldots \times}_{n \text{ times}} A$.

Exercise 2. Is the proof given for the claim below correct? If not, why?

Claim. *For any set of n horses, where $n \geq 1$, all horses are of the same color.*

Proof. We argue by induction on n.
$\underline{n = 1}$: Obvious.
$\underline{n \to n + 1}$:
Let $n \in \mathbb{N}$ and assume that in any set of n horses, all horses have the same color. Let H be a set of $n + 1$ horses. We show that all horses in H have the same color. Pick some horse $h \in H$. Clearly, the set $H \setminus \{h\}$ has n elements, and thus, by the induction hypothesis, any two horses $H \setminus \{h\}$ have the same color. Pick some other $h' \neq h \in H$. By the same argument, all horses $H \setminus \{h'\}$ (a set which clearly contains h) have the same color. It follows that h has to have the same color as all the other horses in H, concluding the proof. $\square$

Exercise 3. Give examples of relations R between elements of a set A such that:

(i) R is reflexive and symmetric but not transitive;

(ii) R is reflexive and transitive but not symmetric;

(iii) R is symmetric and transitive but not reflexive.

Exercise 4. Prove or disprove the following claim: if G is a graph with n nodes, where $n \geq 2$, then it has at least two nodes that have the same degree.