Seminarul 12

① $(\mathbb{R}^3(\mathbb{R}^3, g_0), \varphi)$

Fie $A(1,2,1)$, $B(2,1,3)$, $C(-2,1,3)$, $D(0,2,0)$

a) V_{ABCD}

$$V_{ABCD} = \frac{1}{6}|\Delta|$$

$$\Delta = \begin{vmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ -2 & 1 & 3 & 1 \\ 0 & 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 & 1 \\ 1 & -1 & 2 & 0 \\ -3 & -1 & 2 & 0 \\ -1 & 0 & -1 & 0 \end{vmatrix} =$$

$$= -\begin{vmatrix} 1 & -1 & 2 \\ -3 & -1 & 2 \\ -1 & 0 & -1 \end{vmatrix} = -4$$

$$V = \frac{4}{6} = \frac{2}{3}$$

b) $S_{\Delta BCD}$

$$S_{\Delta BCD} = \frac{1}{2} \cdot \|\overrightarrow{BC} \times \overrightarrow{BD}\|$$

$\overrightarrow{BC} = (-4, 0, 0)$

$\overrightarrow{BD} = (-2, 1, -3)$

$$\overrightarrow{BC} \times \overrightarrow{BD} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -4 & 0 & 0 \\ -2 & 1 & -3 \end{vmatrix} = +4 \begin{vmatrix} e_2 & e_3 \\ 1 & -3 \end{vmatrix} =$$

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$= 4(0,-3,-1) = (0,-12,-4)$

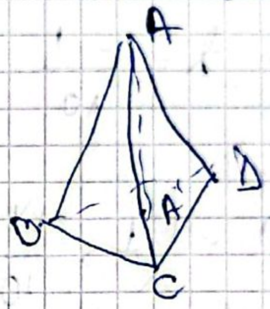
$\|\vec{BC}\times\vec{BD}\| = \sqrt{12^2+4^2} = \sqrt{4^2(9+1)} = 4\sqrt{10}$

$S_{\Delta BCD} = 2\sqrt{10}$

c) dist(A, (BCD))

Metoda 1

$V_{ABCD} = \frac{1}{3} S_{BCD}\cdot h$, unde $h = dist(A,(BCD))$



$h = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$

Metoda 2

OBS: Ec. lui BCD: $\begin{vmatrix} x_1 & x_2 & x_3 & 1 \\ 2 & 1 & 3 & 1 \\ -2 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{vmatrix} = 0$

$N_{BCD} = \vec{BC}\times\vec{BD} = (0,-12,-4) = -4(0,3,1)$

$AA': \frac{x_1-1}{0} = \frac{x_2-2}{3} = \frac{x_3-1}{1} = t \Rightarrow$

$\Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 3t+2 \\ x_3 = t+1 \end{cases}$

$$= \begin{vmatrix} x_1 & x_2 & x_3 & 1 \\ 4 & 0 & 0 & 0 \\ 2 & 1 & 3 & 1 \\ 0 & 2 & 0 & 1 \end{vmatrix} = -4 \cdot \begin{vmatrix} x_2 & x_3 & 1 \\ 1 & 3 & 1 \\ 2 & 0 & 1 \end{vmatrix} =$$

$$= 3x_2 + x_3 - 6 = 0$$

$AA' \cap (BCD): 3(3t+2) + t + 1 - 6 = 0 \Rightarrow$

$\Rightarrow t = \frac{-1}{10}$

$A' = \left(1, \frac{17}{10}, \frac{9}{10}\right)$

$dist(A, (BCD)) = dist(A, A')$

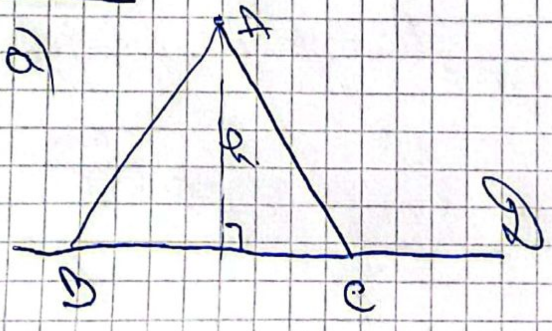
2) Fie $A(1,1,1)$

$\mathcal{D}: \begin{cases} x_1 + x_2 - x_3 + 1 = 0 \\ 2x_1 + x_2 - 3x_3 + 2 = 0 \end{cases}$, $\pi: x_1 + x_2 + x_3 = 0$

a) $dist(A, \mathcal{D}) = ?$

b) $dist(A, \pi) = ?$

a)



$$\begin{cases} x_3 = t \\ x_1 + x_2 = t - 1 \\ 2x_1 + x_2 = 3t - 2 \end{cases} \ominus$$

$x_1 = 2t - 1$

$x_2 = t - 1 - 2t + 1 = -t$ $\Rightarrow D: \begin{cases} x_1 = 2t - 1 \\ x_2 = -t \\ x_3 = t \end{cases}$

$x_3 = t$

$A(1, 1, 1)$

$t = 0 \Rightarrow B(-1, 0, 0)$

$t = 1 \Rightarrow C(1, -1, 1)$

$$S_{\Delta ABC} = \frac{1}{2} \| \overrightarrow{AB} \times \overrightarrow{AC} \|$$

$\overrightarrow{AB} : (-2, -1, -1)$

$\overrightarrow{AC} : (0, -2, 0)$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -2 & -1 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 2 \cdot \begin{vmatrix} e_1 & e_3 \\ -2 & -1 \end{vmatrix} =$$

$$= 2(-e_1 + 2e_3) = 2(-1, 0, 2) = (-2, 0, 4)$$

$$S_{\Delta ABC} = \frac{1}{2} \sqrt{4 + 0 + 16} = \frac{\sqrt{20}}{2} = \sqrt{5}$$

$$S_{\Delta ABC} = \frac{h \cdot \|\overrightarrow{BC}\|}{2}$$

$\overrightarrow{BC} = (2, -1, 1)$

$\|\overrightarrow{BC}\| = \sqrt{6}$

$$S_{\Delta ABD} = \frac{h \cdot \sqrt{6}}{2} = \sqrt{5} \Rightarrow h = \frac{2\sqrt{5}}{\sqrt{6}}$$

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b) $$dist(A, \pi) = \frac{|1+1+1|}{\sqrt{1+1+1}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

Formulă generală: $$\frac{|ax_1^0 + bx_2^0 + cx_3^0 + d|}{\sqrt{a^2+b^2+c^2}}$$

Conice. Formă canonică

5) Fie conica Γ: $f(x_1, x_2) = 5x_1^2 + 8x_1x_2 + 5x_2^2 - 18x_1 - 18x_2 + 9 = 0$

Să se aducă la o formă canonică, efectuând izometrii. Reprezentare grafică

$f(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2b_1x_1 + 2b_2x_2 + c = 0$

$$A = \begin{pmatrix} 5 & 4 & 4 \\ & 5 & \\ 4 & & \end{pmatrix} \qquad \tilde{A} = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$$

$$B(b_1, b_2)$$

$$A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} \boxed{\begin{matrix} 5 & 4 \\ 4 & 5 \end{matrix}} & \begin{matrix} -9 \\ -9 \end{matrix} \\ \begin{matrix} -9 & -9 \end{matrix} & 9 \end{pmatrix}$$

$\delta = \det(A) = 25 - 16 = 9 \neq 0$

Γ are centru unic

$$\begin{cases} \dfrac{\partial f}{\partial x_1} = 0 \\ \dfrac{\partial f}{\partial x_2} = 0 \end{cases} \Leftrightarrow \begin{cases} 10x_1 + 8x_2 - 18 = 0 \\ 8x_1 + 10x_2 - 18 = 0 \end{cases} \Big| : 2$$

$$= \begin{cases} 5x_1 + 4x_2 = 9 \\ 4x_1 + 5x_2 = 9 \end{cases} \Rightarrow x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$$

$$-5x_2 + 4x_2 = 9 \Rightarrow -x_2 = 9 \Rightarrow x_1 = 9$$

$P_0(1, 1) =$

$$\Delta = \det \tilde{A} = \begin{vmatrix} 0 & 0 & -9 \\ 4 & 5 & -9 \\ -9 & -9 & 9 \end{vmatrix} = -9 \begin{vmatrix} 4 & 5 \\ -9 & -9 \end{vmatrix} =$$

$$= -9^2(-4+5) = -81 \neq 0 \Rightarrow \text{conică nedegenerată}$$

OBS:

$\mathcal{R} = \{O; e_1, e_2\} \to \mathcal{R}' = \{P_0; e_1, e_2\} \to$

$\to \mathcal{R}' = \{P_0; e_1', e_2'\}$

rotație

translație

$\theta: X = X' + X_0$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow$ translație

$\Rightarrow \theta(\Gamma) = \underbrace{5x_1'^2 + 8x_1'x_2' + 8x_1'x_2'}_{Q'(x)} + \frac{\Delta}{\delta} = 0$

(metoda valorilor proprii)

$A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$

$\begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)^2 - 16 = 0$

$\lambda^2 - Tr(A)\lambda + \det A = 0 \Rightarrow$

$\Rightarrow \lambda^2 - 10\lambda + 9 = 0 \Leftrightarrow (\lambda - 1)(\lambda - 9) = 0$

$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = 9 \end{cases}$

$Q(x) = \cancel{1 + x_1''^2} \quad 1 \cdot x_1'^2 + 9 x_2'^2$

$V_{\lambda_1} = \{ x \in \mathbb{R}^2 \mid Ax = 1 \cdot x \}$

$(A - I_2) \cdot x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2.$

$V_{\lambda_1} = \{ (x_1, -x_1) \mid x_1 \in \mathbb{R} \}$

$e_1' = \dfrac{(1,-1)}{\sqrt{2}}$

$V_{\lambda_2} = \{ x \in \mathbb{R}^2 \mid A \cdot x = 9x \}$

$(A - 9 I_2) \cdot x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -4 & 4 \\ +4 & -4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 = x_2.$

$V_{\lambda_2} = \{ (x_1, x_1) \mid x_1 \in \mathbb{R} \}$

$e_2' = \dfrac{(1,1)}{\sqrt{2}}$

~~§~~ 2) Rotația

$$\sigma : X' = R X'' \qquad R = \frac{1}{\sqrt{2}} \begin{array}{cc} e_1' & e_2' \\ \left(\begin{matrix} 1 \\ -1 \end{matrix}\right. & \left.\begin{matrix} 1 \\ 1 \end{matrix}\right) \end{array} \in SO(2)$$

$$\sigma(\theta(T)) : x_1''^2 + 9x_2''^2 + \underbrace{\frac{\Delta}{\delta}}_{-9} = 0$$

$$\mathcal{E} : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

$$x_1''^2 + 9x_2''^2 = 9 \;\Big|\; :9$$

$$\frac{1}{\underset{3^2}{9}} x_1''^2 + \frac{x_2''^2}{1} = 1 \iff a = 3$$

$$b = 1$$

$$e_1' : \frac{1}{\sqrt{2}}(1, -1)$$

$$e_2' : \frac{(1, 1)}{\sqrt{2}}$$

(2)