

# Geometrie analitică euclidiană

## §1 Arie, volume, distante, unghiuri

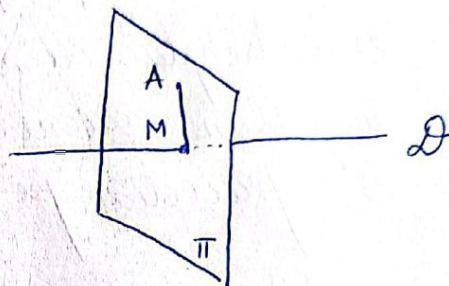
$$S_{\Delta ABC} = \frac{1}{2} \|\vec{AB} \times \vec{AC}\|$$

$$\text{dist}(A, \mathcal{D}) = \frac{\|\vec{AB} \times \vec{AC}\|}{\|\vec{BC}\|}, \quad B, C \in \mathcal{D} \text{ sau}$$

$$\text{dist}(A, \mathcal{D}) = \text{dist}(A, M),$$

$$\text{unde } \pi \perp \mathcal{D}, A \in \pi$$

$$\mathcal{D} \cap \pi = \{M\}$$



$$V_{ABCD} = \frac{1}{6} |\Delta|, \quad \Delta = \begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix}$$

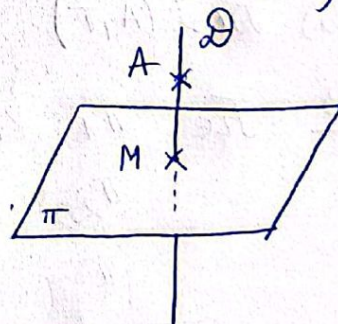
$$A, B, C, D \text{ coplanare} \Leftrightarrow \Delta = 0.$$

$$\text{dist}(A, \pi) = \frac{|a a_1 + b a_2 + c a_3 + d|}{\sqrt{a^2 + b^2 + c^2}}, \quad \pi: ax_1 + bx_2 + cx_3 + d = 0$$

$$A(a_1, a_2, a_3)$$

$$\text{sau } \text{dist}(A, \pi) = \text{dist}(A, M),$$

$$\text{unde } A \in \mathcal{D}, \mathcal{D} \perp \pi, \mathcal{D} \cap \pi = \{M\}.$$



$$\text{dist}(\mathcal{D}_1, \mathcal{D}_2) = \frac{|\langle \vec{AB}, N \rangle|}{\|N\|},$$

$$\text{unde } \mathcal{D}_1, \mathcal{D}_2 \text{ drepte necoplanare, } A \in \mathcal{D}_1, u = u_{\mathcal{D}_1}$$

$$B \in \mathcal{D}_2, v = u_{\mathcal{D}_2}$$

$$N = u \times v$$



•  $\angle(D_1, D_2) = \angle(u_1, u_2) = \angle \varphi \in [0, \pi]$

$\cos \varphi = \frac{\langle u_1, u_2 \rangle}{\|u_1\| \cdot \|u_2\|}$ , unde  $D_k = \text{dreaptă orientată de } u_k, k=1,2$

•  $\angle(\pi_1, \pi_2) = \angle(N_1, N_2) = \angle \varphi$

$\cos \varphi = \frac{\langle N_1, N_2 \rangle}{\|N_1\| \cdot \|N_2\|}$ , unde  $\pi_k = \text{plan orientat de } N_k, k=1,2$

•  $\angle(D, \pi) = \angle(D, D') = \angle \varphi$

$D = \text{dreaptă orientată de } u$

$\pi = \text{plan orientat de } N$

$D' = \text{proiecția pe } \pi \text{ a lui } D$

$(\mathbb{R}^3, \langle \cdot, \cdot \rangle, \varphi)$

Ex1 Fie  $A(1,2,1), B(2,1,3), C(-2,1,3), D(0,2,0)$

a)  $V_{ABCD}$ ; b)  $S_{\triangle BCD}$ ; c)  $\text{dist}(A, (BCD))$

Ex2 Fie  $A(1,1,1)$ ,  $D: \begin{cases} x_1 + x_2 - x_3 + 1 = 0 \\ 2x_1 + x_2 - 3x_3 + 2 = 0 \end{cases}$ ;  $\pi: x_1 + x_2 + x_3 = 0$

a)  $\text{dist}(A, D) = ?$

b)  $\text{dist}(A, \pi)$

Ex3 Fie  $\pi_1: x_1 - 3x_2 - 1 = 0$

$\pi_2: 2x_2 + x_3 - 2 = 0$

$\pi: x_2 - x_3 - 1 = 0$

$D_1: \begin{cases} x_1 - x_2 = 2 = 0 \\ x_1 + x_3 - 3 = 0 \end{cases}$

$D_2: \frac{x_1 - 1}{3} = \frac{x_2 + 1}{0} = \frac{x_3 - 1}{-1}$

$D: \frac{x_1 - 1}{-1} = \frac{x_2}{2} = \frac{x_3 + 1}{-5}$

a)  $\angle(D_1, D_2)$ ; b)  $\angle(D, \pi)$ ; c)  $\angle(\pi_1, \pi_2)$

Ex4 Fie  $D_1: \begin{cases} x_1 - x_2 = 2 \\ x_1 + x_3 = 3 \end{cases}$ ,  $D_2: \frac{x_1 - 1}{3} = \frac{x_2 + 1}{0} = 1 - x_3$

$\text{dist}(D_1, D_2) = ?$



## §2 Conice. Formă canonică

S.n. conică în  $\mathbb{R}^2$  LG al punctelor  $P(x_1, x_2)$  ai

$$\Gamma: f(x) = a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2 + 2b_1x_1 + 2b_2x_2 + c = 0$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = A^T, \quad \tilde{A} = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{12} & a_{22} & b_2 \\ b_1 & b_2 & c \end{pmatrix} = \begin{pmatrix} A & B^T \\ B & c \end{pmatrix},$$

$$B = (b_1, b_2)$$

$$\delta = \det A, \quad \Delta = \det \tilde{A}$$

$$r = \operatorname{rg} A, \quad r' = \operatorname{rg} \tilde{A}, \quad r \leq r' \leq r+2$$

$$\Delta = 0 \quad \Gamma = \text{conică degenerată}$$

$$\Delta \neq 0 \quad \Gamma = \text{conică nedegenerată}$$

1)  $(\mathbb{R}^2, \mathbb{R}/\mathbb{R}, \varphi)$  sp. afin

$$\Gamma_1 \sim \Gamma_2 \text{ conice afin echivalente} \Leftrightarrow$$

$$\exists \mathcal{C}: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ transformare afină ai } \Gamma_2 = \mathcal{C}(\Gamma_1),$$

$$\text{unde } \mathcal{C}: X' = CX + D, \quad C \in GL(n, \mathbb{R})$$

$$\text{Invarianti afini: } \frac{\Delta}{\delta}, r, r'$$

2)  $(\mathbb{R}^2, (\mathbb{R}^2, g_0), \varphi)$  sp. punctual euclidian.

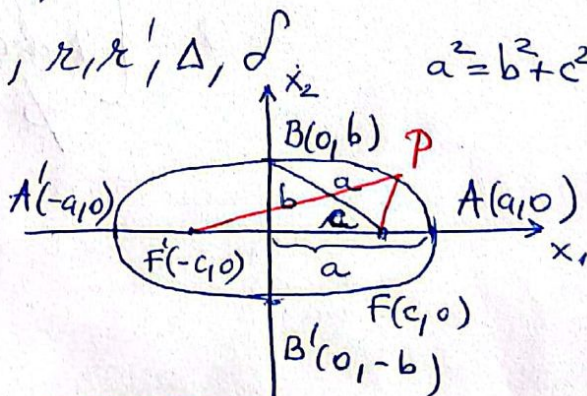
$$\Gamma_1 \equiv \Gamma_2 \text{ conice congruente metric} \Leftrightarrow$$

$$\exists \mathcal{C}: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ izometrie ai } \Gamma_2 = \mathcal{C}(\Gamma_1), \text{ unde}$$

$$\mathcal{C}: X' = CX + D, \quad C \in O(n)$$

$$\text{Invarianti metrici: } \frac{\Delta}{\delta}, r, r', \Delta, \delta, a^2 = b^2 + c^2$$

① Elipsa:  $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$   
 $a, b > 0$



$A, A', B, B' = \text{vârfuri}$

$F, F' = \text{focare}$

$$PF + PF' = 2a, \quad a > 0$$



② Hiperbola :  $\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1$   
 $a, b > 0$

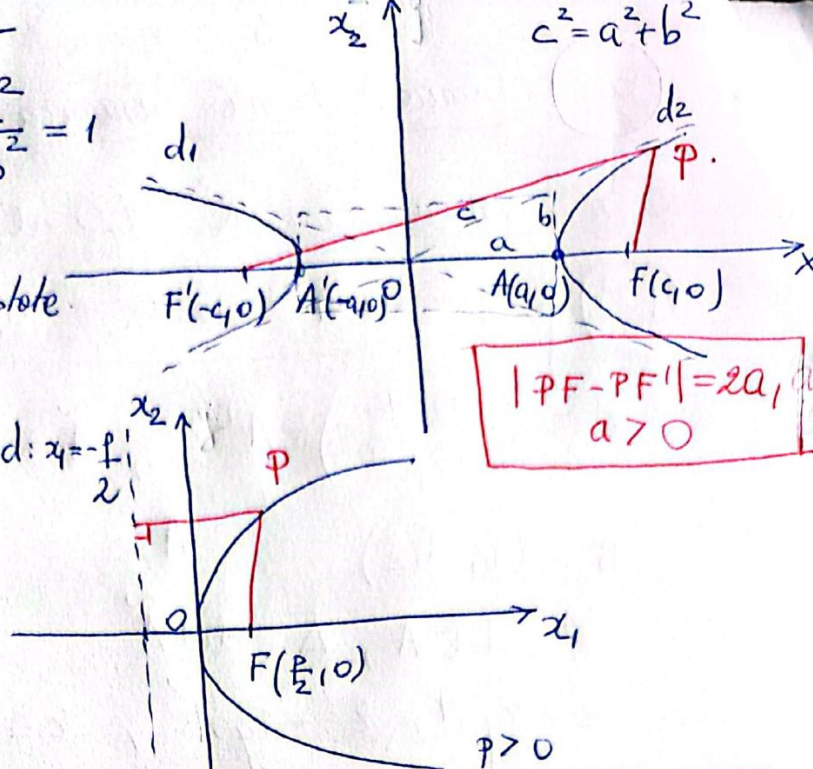
$d_1 \cup d_2 : x_2 = \pm \frac{b}{a} x_1$  asimptote

$A, A' = \text{vârfuluri}$ ,  $F, F' = \text{focar}$

③ Parabola :  $x_2^2 = 2px_1$

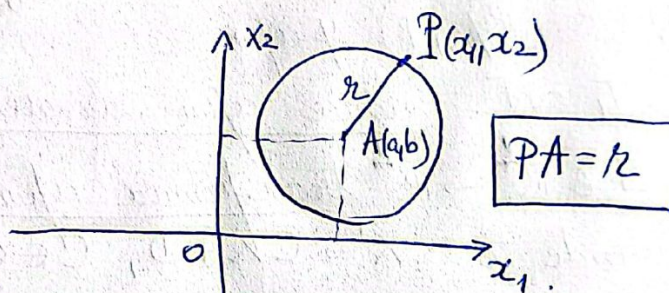
$F = \text{focar}$

$d = \text{dreaptă directoare}$



Cercul

$C(A(a,b), r) : (x_1 - a)^2 + (x_2 - b)^2 = r^2$



Conice nedegenerate: elipsa, hiperbola, parabola

Conice degenerate

{  
 punct dublu  
 drepte concurente  
 drepte paralele  
 dreaptă dublă  
 $\emptyset$



$$P_0 \text{ centrul pt } \Gamma \Leftrightarrow [\forall P \in \Gamma \Rightarrow \mathcal{I}_{P_0}(P) \in \Gamma]$$

$$\delta \neq 0 \Rightarrow \Gamma \text{ are centru unic. } P_0(x_1^0, x_2^0)$$

$$\delta \neq 0 \Rightarrow f(x_1^0, x_2^0) = \frac{\Delta}{\delta}$$

① Aducerea la f. canonică pt  $\delta \neq 0$ .

①  $(\mathbb{R}^2, \mathbb{R}^2/\mathbb{R}, \varphi)$  sp. afin.

$$\mathcal{R} = \{0; e_1, e_2\} \xrightarrow[\text{translatie}]{\theta} \mathcal{R}' = \{P_0; e_1, e_2\} \xrightarrow[\text{transf. afină}]{\zeta} \mathcal{R}'' = \{P_0; e'_1, e'_2\}$$

$$a) \theta = X = X' + X_0. \quad X_0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}, P_0 = \text{centru.} \quad \begin{cases} \frac{\partial f}{\partial x_1} = 0 \\ \frac{\partial f}{\partial x_2} = 0 \end{cases}$$

$$\theta(\Gamma): X'^T A X' + \frac{\Delta}{\delta} = 0$$

b)  $Q: \mathbb{R}^2 \rightarrow \mathbb{R}, Q(x) = X'^T A X'$  formă pătratică  
Aducem  $Q$  la o formă canonică (met Gauss/Jacobi)

$$Q(x) = \lambda_1 x_1''^2 + \lambda_2 x_2''^2$$

$$\zeta: X' = C X'', \quad C \in GL(2, \mathbb{R})$$

$$\text{Deci: } \zeta \circ \theta(\Gamma): X = C X'' + X_0 \quad (\text{transf. afină})$$

$\Gamma \sim \Gamma'$  afin echivalente

$$\zeta \circ \theta(\Gamma): \lambda_1 x_1''^2 + \lambda_2 x_2''^2 + \frac{\Delta}{\delta} = 0$$

②  $(\mathbb{R}^2, (\mathbb{R}^2, g_0), \varphi)$  sp. punctual euclidian.

a) Analog cu cazul ①

$$b) Q: \mathbb{R}^2 \rightarrow \mathbb{R}, Q(x) = X'^T A X'$$

$\exists$  un reper ortonormat format din versori proprii ai  $A$  diag.

$$P(\lambda) = \det(A - \lambda I_2) = 0 \Rightarrow \lambda_1, \lambda_2 \Rightarrow e'_1, e'_2 \text{ versori proprii ortogonali}$$

$$e'_k = (l_k, m_k)$$

$$R = \begin{pmatrix} e_1 & e_2 \\ m_1 & m_2 \end{pmatrix}$$

Pt  $\det R = 1 \Rightarrow$  reper pozitiv orientat



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$\tau: X' = RX''$  izometrie ( $\det R = 1 \Rightarrow \text{pe la } 1$ )

$$\tau \circ \theta(\Gamma) = \Gamma': \lambda_1 x_1''^2 + \lambda_2 x_2''^2 + \frac{\Delta}{\delta} = 0$$

$\Gamma \equiv \Gamma'$  congruente metric.

$$\tau \circ \theta: X = RX'' + X_0, \quad R \in SO(2), (\det R = 1)$$

$$R \xrightarrow{\theta} R' \xrightarrow{\tau} R'' \quad \begin{array}{l} \text{translatie} \\ \text{rotatie} \end{array} \quad \text{repere orthonormate}$$

EX5  $(\mathbb{R}^2, (\mathbb{R}_1^2, g_0), \varphi)$

Fie conica:

$$\Gamma: f(x_1, x_2) = 5x_1^2 + 8x_1x_2 + 5x_2^2 - 18x_1 - 18x_2 + 9 = 0$$

Să se aducă la o formă canonică, efectuând izometrie.

Reprezentare grafică.

EX6 Fie conica:

$$\Gamma: f(x_1, x_2) = 3x_1^2 - 8x_1x_2 + 3x_2^2 + 2x_1 + 2x_2 + 2 = 0$$

Să se aducă la o formă canonică, cf. izometrie.

Reprez. grafică.

EX7  $(\mathbb{R}^2, (\mathbb{R}_1^2, g_0), \varphi)$  Fie conica  $\Gamma$ :

a)  $f(x_1, x_2) = 3x_1^2 - 4x_1x_2 - 2x_1 + 4x_2 - 3 = 0$

b)  $f(x_1, x_2) = 4x_1x_2 - 3x_2^2 + 4x_1 - 14x_2 - 7 = 0$

c)  $f(x_1, x_2) = 3x_1^2 - 4x_1x_2 + 3x_2^2 - 4x_1 + 6x_2 - 4 = 0$

d)  $f(x_1, x_2) = 16x_1^2 + 4x_1x_2 + 19x_2^2 + 80x_1 + 10x_2 + 40 = 0$

- 1) Să se det centrul conicei.
- 2) Să se aducă conica  $\Gamma$  la o formă canonică, efectuând o izometrie de pe la 1. Precizați schimbările de repere orthonormate.
- 3) Reprez. grafică.