Probleme lucrare

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Nr 1

Ex 1 $(\mathbb{R}^3, g_0)$, $f:\mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (3x_1+x_2+x_3, x_1+3x_2+x_3, x_1+x_2+3x_3)$

a) $f \in Sim(\mathbb{R}^3)$

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \Rightarrow A^t = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \Rightarrow A = A^t \Rightarrow f \in Sim(\mathbb{R}^3)$$

b) Det $Q: \mathbb{R}^3 \to \mathbb{R}$ forma pătratică asociată lui f.
Să se aducă la o formă pătratică prin transf. ortogonale.

$$Q(x) = 3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_3$$

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^3 + 2 - 3(3-\lambda) =$$

$$= (3-\lambda)(9-6\lambda+\lambda^2-3)+2 = 18-18\lambda+3\lambda^2-6\lambda+6\lambda^2 - \lambda^3 = -\lambda^3+9\lambda^2-24\lambda+20 = -\lambda^2(\lambda-5)+4\lambda(\lambda-5) - 4(\lambda-5) = -(\lambda-2)^2(\lambda-5)$$

$$\begin{cases} \lambda_1 = 5, m_{\lambda_1} = 1 \\ \lambda_2 = 2, m_{\lambda_2} = 2 \end{cases}$$

$V_{\lambda_1} = \{X \in \mathbb{R}^3 \mid AX = \lambda_1 X\}$.

$(A - \lambda_1 I_3)X = O_{3,1} \to (A - 5I_3)X = O_{3,1}$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = x_2 = x_3$$

$V_{\lambda_1} = <\{(1,1,1)\}> \Rightarrow e_1 = \frac{1}{\sqrt{3}}(1,1,1) \Rightarrow \dim V_{\lambda_1} = m_{\lambda_1}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_3 = -x_1 - x_2$$

$\Rightarrow V_{\lambda_2} = <\{\underbrace{(1,0,-1)}_{f_2}, \underbrace{(0,1,-1)}_{f_3}\}>$

$\overline{e_2} = f_2 = (1,0,-1)$

$\overline{e_3} = f_3 - \frac{<f_3, \overline{e_2}>}{<\overline{e_2}, \overline{e_2}>} \cdot \overline{e_2} = (0,1,-1) - \frac{1}{2} \cdot (1,0,-1) =$

$= (0,1,-1) - \left(\frac{1}{2}, 0, -\frac{1}{2}\right) = \left(-\frac{1}{2}, 1, -\frac{1}{2}\right)$

$\Rightarrow \begin{cases} e_2 = \frac{1}{\sqrt{2}}(1,0,-1) \\ e_3 = \frac{\sqrt{2}}{\sqrt{3}}\left(-\frac{1}{2}, 1, -\frac{1}{2}\right) \end{cases} \Rightarrow \dim V_{\lambda_2} = m_{\lambda_2}$

$R = \left\{\frac{1}{\sqrt{3}}(1,1,1);\ \frac{1}{\sqrt{2}}(1,0,-1);\ \frac{\sqrt{2}}{\sqrt{3}}\left(-\frac{1}{2}, 1, -\frac{1}{2}\right)\right\}$ reper ortonormat

$[f]_{R,R} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow Q(x) = 5x_1^2 + 2x_2^2 + 2x_3^2$

$R_0 \xrightarrow{C} R$, $C = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix}$ $C \in O(3)$

$h \in O(R^3)$ $h(e_i^0) = e_i$, $i = \overline{1,3}$

$R_0 = \{e_1^0, e_2^0, e_3^0\}$

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c) $g: \mathbb{R}^3 \to \mathbb{R}^3$ forma polară
Este $(\mathbb{R}^3, g)$ sp. euclidian?

$(\mathbb{R}^3, g)$ sp. euclidian $\Leftrightarrow \begin{cases} \text{sim.} \\ \text{poz. definită} \\ g \text{ liniară} \end{cases}$

- Am dem. la punctul a) că matricea este simetrică.
- $Q(x) = 5x_1^2 + 2x_2^2 + 2x_3^2 \Rightarrow$ Signatura: $(3, 0) \Rightarrow$ poz. def.
- $g(x, y) = \sum_{i,j=1}^{3} a_{ij} x_i y_j \Rightarrow g \in L(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

$\Rightarrow (\mathbb{R}^3, g)$ sp. euclidian

d) Fie $A = [f]_{R_0, R_0}$. Calc. A^n, $n \in \mathbb{N}^*$

$$A' = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A^n = C (A')^n C^t = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 5^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{5^n}{\sqrt{3}} & \frac{2^n}{\sqrt{2}} & -\frac{2^n}{\sqrt{6}} \\ \frac{5^n}{\sqrt{3}} & 0 & \frac{2^n\sqrt{2}}{\sqrt{3}} \\ \frac{5^n}{\sqrt{3}} & -\frac{2^n}{\sqrt{2}} & -\frac{2^n}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix}$$

Ex2 $(\mathbb{R}^3, g_0)$, $U = \{x \in \mathbb{R}^3 \mid 2x_1 + x_2 - x_3 = 0\}$

a) Det. un rep. ortonormat $R = R_1 \cup R_2$ în $\mathbb{R}^3$, unde R_1, resp R_2 repor ortonormat în $U^\perp$, resp în U.

b) Det $f = R_\varphi$ rotația de $\measuredangle \varphi = \frac{\pi}{2}$ și axa $U^\perp$

a) $g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g_0 = x_1 y_1 + x_2 y_2 + x_3 y_3$

$U = \{x \in \mathbb{R}^3 \mid g_0(x, (2,1,-1)) = 0\} \Rightarrow U^\perp = \langle \underbrace{\{(2,1,-1)\}}_{f_1} \rangle$

$\Rightarrow R_1 = \{\frac{1}{\sqrt{6}}(2,1,-1)\}$ repor ortonormat în U

$x_3 = 2x_1 + x_2 \Rightarrow U = \{(x_1, x_2, 2x_1 + x_2) \mid x_1, x_2 \in \mathbb{R}\} =$

$= \langle \{\underbrace{(1,0,2)}_{f_2}, \underbrace{(0,1,1)}_{f_3}\} \rangle$

$\bar{e}_2 = f_2$

$\bar{e}_3 = f_3 - \frac{\langle f_3, \bar{e}_2 \rangle}{\langle \bar{e}_2, \bar{e}_2 \rangle} \cdot \bar{e}_2 = (0,1,1) - \frac{2}{5}(1,0,2) =$

$= (0,1,1) - (\frac{2}{5}, 0, \frac{4}{5}) = (-\frac{2}{5}, 1, \frac{1}{5})$

$e_2 = \frac{1}{\sqrt{5}}(1,0,2)$

$e_3 = \frac{\sqrt{5}}{\sqrt{6}}(-\frac{2}{5}, 1, \frac{1}{5})$

$R_2 = \{\frac{1}{\sqrt{5}}(1,0,2), \frac{\sqrt{5}}{\sqrt{6}}(-\frac{2}{5}, 1, \frac{1}{5})\}$

$R = \{\frac{1}{\sqrt{6}}(2,1,-1), \frac{1}{\sqrt{5}}(1,0,2), \frac{\sqrt{5}}{\sqrt{6}}(-\frac{2}{5}, 1, \frac{1}{5})\}$

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EX 3 $f \in End(\mathbb{R}^3)$, $A = [f]_{R_0, R_0} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}$

dim Ker f = ?, dim Im f = ?

$f(x) = f(x_1, x_2, x_3) = (4x_1 + 2x_2, 2x_1 + x_2, 2x_1 + 3x_2 + x_3)$

$2x_1 + x_2 = 0 \Rightarrow x_2 = -2x_1$

$2x_2 + x_3 = 0 \Rightarrow x_3 = -2x_2 = 4x_1$

$\operatorname{Ker} f = \langle \{(1, -2, 4)\} \rangle \Rightarrow \dim \operatorname{Ker} f = 1$

$\Rightarrow \dim \operatorname{Im} f = 3 - 1 = 2$

Nr 2

EX 1 (2) Det $\alpha \in \mathbb{R}$ a.î. $R = \{-x + \alpha x^2, \alpha x + 2x^2, 1 + x^2\}$ reper în $(\mathbb{R}_2[x], +, \cdot)$

Reper $\Leftrightarrow \begin{cases} SLi \\ \dim R = \dim \mathbb{R}_2[X] \end{cases}$

$\dim \mathbb{R}_2[X] = \dim \mathbb{R}^3 = 3$

$A = \begin{vmatrix} 0 & 0 & 1 \\ -1 & \alpha & 0 \\ \alpha & 2 & 1 \end{vmatrix}$

$\det A = 0 + 0 - 2 - \alpha^2 - 0 - 0 = -2 - \alpha^2 < 0, \forall \alpha \in \mathbb{R}$

$\Rightarrow \det A \neq 0, \forall \alpha \in \mathbb{R} \Rightarrow R$ reper pt $\forall \alpha \in \mathbb{R}$

EX2 (2) $(\mathbb{R}^3, g_0)$ $\quad U = <\{(1,1,1)\}>$

a) Precizați un reper ortonormat $R = R_1 \cup R_2$ în $\mathbb{R}^3$, unde R_1, R_2 repere ortonormate în U, resp $U^\perp$

$R_1 = \{e_1 = \frac{1}{\sqrt{3}}(1,1,1)\}$

$U^\perp = \{x \in \mathbb{R}^3 \mid g_0(x,(1,1,1)) = 0\}$

$x_1 + x_2 + x_3 = 0 \Rightarrow x_3 = -x_1 - x_2$

$U^\perp = <\{\underbrace{(1,0,-1)}_{f_2}, \underbrace{(0,1,-1)}_{f_3}\}>$

$\bar{e}_2 = f_2$

$\bar{e}_3 = f_3 - \frac{<f_3, \bar{e}_2>}{<\bar{e}_2, \bar{e}_2>} \cdot \bar{e}_2 = (0,1,-1) - \frac{1}{2}(1,0,-1) =$

$= (0,1,-1) - (\frac{1}{2}, 0, -\frac{1}{2}) = (-\frac{1}{2}, 1, \frac{1}{2})$

$e_2 = \frac{1}{\sqrt{2}}(1,0,-1)$

$e_3 = \frac{\sqrt{2}}{\sqrt{3}}(-\frac{1}{2}, 1, \frac{1}{2})$

$R_2 = \{\frac{1}{\sqrt{2}}(1,0,-1), \frac{\sqrt{2}}{2\sqrt{3}}(-1,2,1)\}$

$R = \{\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(1,0,-1), \frac{1}{\sqrt{6}}(-1,2,1)\}$

b) $f = s_0 R_\varphi \rightarrow$ speța 2, $\varphi = \frac{\pi}{2}$ și axa de rotație U

$$A' = [f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ 0 & \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

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$$R_0 \xrightarrow{C} R$$

$$C = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{3} & -1 \\ \sqrt{2} & 0 & 2 \\ \sqrt{2} & -\sqrt{3} & 1 \end{pmatrix}$$

$A = [f]_{R_0, R_0}$; $A' = C^{-1}AC = C^{t}AC$

$$\Rightarrow A = CA'C^{t} = \frac{1}{6} \begin{pmatrix} \sqrt{2} & \sqrt{3} & -1 \\ \sqrt{2} & 0 & 2 \\ \sqrt{2} & -\sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{3} & 0 & -\sqrt{3} \\ -1 & 2 & 1 \end{pmatrix}$$

<u>EX 3</u> (2) $(\mathbb{R}^3, g_0)$, $A = [f]_{R_0, R_0} = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 16 & 4 \\ 0 & 4 & 2 \end{pmatrix}$, $f \in End(\mathbb{R}^3)$

a) Se poate diagonaliza f?

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 2-\lambda & -4 & 0 \\ -4 & 16-\lambda & 4 \\ 0 & 4 & 2-\lambda \end{vmatrix} =$$

$$= \begin{vmatrix} 2-\lambda & -4 & 0 \\ 0 & 16-\lambda & 4 \\ 2-\lambda & 4 & 2-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 1 & -4 & 0 \\ 0 & 16-\lambda & 4 \\ 1 & 4 & 2-\lambda \end{vmatrix} =$$

$$= (2-\lambda) \cdot \begin{vmatrix} 1 & -4 & 0 \\ 0 & 16-\lambda & 4 \\ 2 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\left((16-\lambda)(2-\lambda) - 32\right) =$$

$$= (2-\lambda)(\cancel{32} - 16\lambda - 2\lambda + \lambda^2 - 32) = (2-\lambda)(\lambda^2 - 18\lambda) =$$

$$= \lambda(2-\lambda)(\lambda - 18)$$

$$\begin{cases} \lambda_1 = 0, \ m_{\lambda_1} = 1 \\ \lambda_2 = 2, \ m_{\lambda_2} = 1 \\ \lambda_3 = 18, \ m_{\lambda_3} = 1 \end{cases}$$

$V_{\lambda_1} = \{ X \in \mathbb{R}^3 \mid AX = \lambda_1 X \}$

$AX = O_{3,1}$

$$\begin{pmatrix} 2 & -4 & 0 \\ -4 & 16 & 4 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$2x_1 - 4x_2 = 0 \Rightarrow x_1 = 2x_2$

$4x_2 + 2x_3 = 0 \Rightarrow x_3 = -2x_2$

$V_{\lambda_1} = \langle \{(2,1,-2)\} \rangle$, $\dim V_{\lambda_1} = 1 = m_{\lambda_1}$

$e_1 = \frac{1}{\sqrt{3}}(2,1,-2)$

$V_{\lambda_2} = \{ X \in \mathbb{R}^3 \mid AX = \lambda_2 X \}$

$(A - 2I_3)X = O_{3,1} \Leftrightarrow \begin{pmatrix} 0 & -4 & 0 \\ -4 & 14 & 4 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow x_2 = 0 \Rightarrow x_1 = x_3 \Rightarrow V_{\lambda_2} = \langle \{(1,0,1)\} \rangle$

$\Rightarrow \dim V_{\lambda_2} = 1 = m_{\lambda_2}$

$e_2 = \frac{1}{\sqrt{2}}(1,0,1)$

$V_{\lambda_3} = \{ X \in \mathbb{R}^3 / AX = \lambda_3 X \}$

$(A - 18I_3)X = O_{3,1}$

$$\begin{pmatrix} -16 & -4 & 0 \\ -4 & -2 & 4 \\ 0 & 4 & -16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -16x_1 - 4x_2 = 0 \\ 4x_2 - 16x_3 = 0 \end{cases}$$

$\Rightarrow x_2 = 4x_3$
$x_1 = -\frac{1}{4}x_2$

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~~$V_{\lambda_3} = < \frac{1}{4}, -\frac{1}{4}, 1$~~

~~$V_{\lambda_3} = < \{(-\frac{1}{4}, 1, 4)\} > \Rightarrow e_3 =$~~

~~$m_{\lambda_3} = 1 = \dim V_{\lambda_3}$~~

$V_{\lambda_3} = < \{(1, -4, -1)\} > \rightarrow \dim V_{\lambda_3} = 1 = m_{\lambda_3}$

$e_3 = \frac{1}{\sqrt{18}}(1, -4, -1)$

$R = \{\frac{1}{\sqrt{5}}(2,1,-2), \frac{1}{\sqrt{2}}(1,0,1), \frac{1}{\sqrt{18}}(1,-4,-1)\}$

$\left.\begin{array}{l} \dim V_{\lambda_i} = m_{\lambda_i} \\ \lambda_i \in \mathbb{R} \end{array}\right| \Rightarrow f$ diag.

b) Det $Q: \mathbb{R}^3 \to \mathbb{R}$ forma pătratică.
Să se aducă la o formă canonică prin met. val. proprii.

$Q(x) = 2x_1^2 + 16x_2^2 + 2x_3^2 - 8x_1x_2 + 8x_2x_3$

$A' = [f]_{R,R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 18 \end{pmatrix} \Rightarrow Q(x) = 2x_2^2 + 18x_3^2$

c) $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ forma polară
$(\mathbb{R}^3, g)$ sp euclidian?

$\left\{\begin{array}{l} \text{Sim.} \\ \text{Poz def.} \\ g \text{ biliniară} \end{array}\right.$

$g(x,y) = \sum_{i,j=1}^{3} g_{ij} x_i y_j \Rightarrow g \in L(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

$A = A^t \Rightarrow$ Simetrică

Signatura: $(2,0) \Rightarrow (\mathbb{R}^3, g)$ nu e sp. euclidian
(g nu e poz. def)

d) Calculați A^n, $n \in \mathbb{N}^*$

$$C = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ \frac{1}{\sqrt{5}} & 0 & -\frac{4}{\sqrt{18}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} \end{pmatrix} \Rightarrow C^t = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{18}} & -\frac{4}{\sqrt{18}} & -\frac{1}{\sqrt{18}} \end{pmatrix}$$

$A' = C^{-1} A C = C^t A C$

$A = C A' C^t$

$$A^n = C (A')^n C^t = C \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 18^n \end{pmatrix} C^t$$