Seminar 12

Geometrie analitică

(1) $(\mathbb{R}^3, (\mathbb{R}^3, g_0), \varphi)$

Fie $A(1,2,1)$, $B(2,1,3)$, $C(-2,1,3)$, $D(0,2,0)$

a) V_{ABCD}

b) $S_{\Delta BCD}$

c) $dist(A,(BCD))$

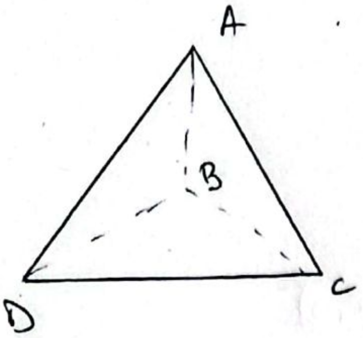
a) $$V_{ABCD} = \frac{1}{6}\left|\det\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ -2 & 1 & 3 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix}\right| = \frac{1}{6}\left|\det\begin{pmatrix} 1 & -1 & 2 \\ -3 & -1 & 2 \\ -1 & 0 & -1 \end{pmatrix}\right| =$$

$$= \frac{1}{2}\left|\det\begin{pmatrix} 1 & -1 & 1 \\ -3 & -1 & 5 \\ -1 & 0 & 0 \end{pmatrix}\right| = \frac{1}{6}\left|\det\begin{pmatrix} -1 & 1 \\ -1 & 5 \end{pmatrix}\right| = \frac{2}{3}$$

b) $$A_{\Delta BCD} = \frac{1}{2}\|\vec{BC}\times\vec{BD}\| = \frac{1}{2}\sqrt{12^2+4^2} = \frac{4}{2}\sqrt{10} = 2\sqrt{10}$$

$$\vec{BC}\times\vec{BD} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -4 & 0 & 0 \\ -2 & 1 & -3 \end{vmatrix} = e_1\begin{vmatrix} 0 & 0 \\ 1 & -3 \end{vmatrix} - e_2\begin{vmatrix} -4 & 0 \\ -2 & -3 \end{vmatrix} +$$

$$+ e_3\begin{vmatrix} -4 & 0 \\ -2 & 1 \end{vmatrix} = (0,-12,-4)$$



c) $V_{ABCD} = \dfrac{S_{\Delta BCD} \cdot h}{3} \Rightarrow \dfrac{2}{3} = \dfrac{2\sqrt{10} \cdot h}{3}$

$\Rightarrow 2 = 2\sqrt{10}\,h \Rightarrow h = \dfrac{1}{\sqrt{10}} = \dfrac{\sqrt{10}}{10}$

(2) $A(1,1,1)$ $\quad \mathcal{D}: \begin{cases} x_1 + x_2 - x_3 + 1 = 0 \\ 2x_1 + x_2 - 3x_3 + 2 = 0 \end{cases}$;

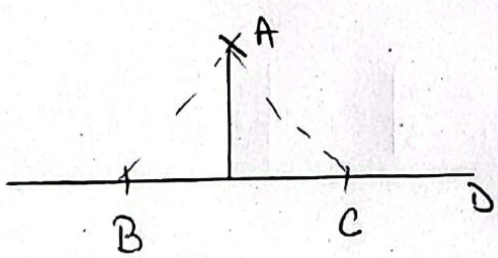
$\pi: x_1 + x_2 + x_3 = 0$

a) $dist(A, \mathcal{D}) = ?$

b) $dist(A, \pi) = ?$

$\begin{cases} x_1 + x_2 = t - 1 \\ 2x_1 + x_2 = 3t - 2 \end{cases} \ominus$ $\qquad \mathcal{D}: \begin{cases} x_1 = 2t - 1 \\ x_2 = -t \\ x_3 = t \end{cases}, t \in \mathbb{R}$

$x_1 = 2t - 1 \Rightarrow x_2 = -t$



a) Metoda 1

$t = 0 \Rightarrow B(-1,0,0)$

$t = 1 \Rightarrow C(1,-1,1)$

$S_{\Delta ABC} = \frac{1}{2} \| \vec{AB} \times \vec{AC} \| = \dfrac{h \cdot \| \vec{BC} \|}{2} \Rightarrow$

$\Rightarrow h = \dfrac{\| \vec{AB} \times \vec{AC} \|}{\| \vec{BC} \|} = \dfrac{\sqrt{20}}{\sqrt{6}} = \sqrt{\dfrac{10}{3}}$

$\overrightarrow{AB} = (-2, -1, -1)$

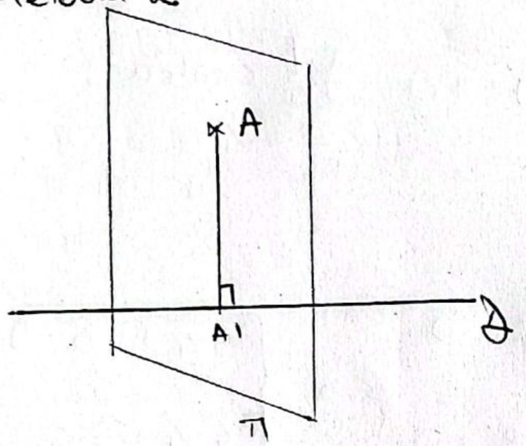
$\overrightarrow{AC} = (0, -2, 0)$

$\overrightarrow{BC} = (2, -1, 1)$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -2 & -1 & -1 \\ 0 & -2 & 0 \end{vmatrix} = e_1 \begin{vmatrix} -1 & -1 \\ -2 & 0 \end{vmatrix} - e_2 \begin{vmatrix} -2 & -1 \\ 0 & 0 \end{vmatrix} + e_3 \begin{vmatrix} -2 & -1 \\ 0 & -2 \end{vmatrix} =$$

$$= (-2, 0, 4)$$

Metoda 2



$A \in \pi,\ \pi \perp D$

$u_D = (2, -1, 1) = N_\pi$

$\pi : 2x_1 - x_2 + x_3 + \alpha = 0$

$A \in \pi \Rightarrow 2 - 1 + 1 + \alpha = 0 \Rightarrow \alpha = -2$

$\pi : 2x_1 - x_2 + x_3 - 2 = 0$

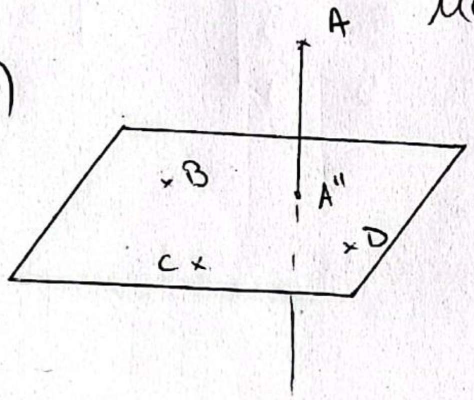
$\pi \cap D : 4t - 2 + t + t - 2 = 0$

$t = \frac{2}{3} \Rightarrow A'\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$

$$dist(A, D) = d(A, A') = \| \overrightarrow{AA'} \| = \left\| \left(-\frac{2}{3}, -\frac{5}{3}, -\frac{1}{3}\right) \right\| =$$

$$= \sqrt{\frac{10}{3}}$$

b)



Metoda I

$$d(A, \pi') = \frac{|1+1+1|}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

Metoda II

$\left.\begin{array}{l} B = (0, 0, 0) \\ C = (1, -1, 0) \\ D = (0, 1, -1) \end{array}\right\}$ la alegere

$$V_{ABCD} = \frac{1}{6} |\det| = \frac{A_{BCD} \cdot h}{3} = \frac{\| \overrightarrow{BC} \times \overrightarrow{BD} \| \cdot h}{6} \Rightarrow h = \frac{|\Delta|}{\| \overrightarrow{BC} \times \overrightarrow{BD} \|}$$

Conice. Formă canonică

(5) $(\mathbb{R}^2, (\mathbb{R}^2, g_0), \varphi)$

Fie conica:

$$f(x_1, x_2) = 5x_1^2 + 8x_1x_2 + 5x_2^2 - 18x_1 - 18x_2 + 9 = 0$$

Să se aducă la o formă canonică, efectuând izometrii. Reprezentare grafică

$$A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}, \quad \delta = \det A = 9 \neq 0 \quad \exists!\ \text{centru}:$$

$$: \begin{cases} \frac{\partial f}{\partial x_1} = 0 \\ \frac{\partial f}{\partial x_2} = 0 \end{cases} \Leftrightarrow \begin{cases} 10x_1 + 8x_2 - 18 = 0 \\ 10x_2 + 8x_1 - 18 = 0 \end{cases} \Leftrightarrow \begin{cases} 5x_1 + 4x_2 - 9 = 0 \mid \cdot 3 \\ 4x_1 + 5x_2 - 9 = 0 \mid \cdot (-3) \end{cases} \oplus$$

$$-9x_2 = -9 \Rightarrow$$

$$\Rightarrow x_2 = 1 \Rightarrow x_1 = 1 \quad \Rightarrow P_0(1,1)$$

$$\tilde{A} = \left(\begin{array}{cc|c} 5 & 4 & -9 \\ 4 & 5 & -9 \\ \hline -9 & -9 & 9 \end{array}\right) \quad \Delta = \begin{vmatrix} 0 & 0 & -9 \\ 4 & 5 & -9 \\ -9 & -9 & 9 \end{vmatrix} = -9 \begin{vmatrix} 4 & 5 \\ -9 & -9 \end{vmatrix}$$

$$= 9^2 \begin{vmatrix} 4 & 5 \\ 1 & 1 \end{vmatrix} = -9^2 \neq 0 \Rightarrow \text{Conică nedegenerată.}$$

$R^{*} = \{O, e_1, e_2\} \longrightarrow R' = \{P_0, e_1, e_2\} \longrightarrow R'' = \{P_0, e_1', e_2'\}$

$P_0(1,1)$, $\delta = 9$, $A = 9^2$

$\cancel{\theta}: X = X' + X_0$ (translație , mut O în P_0)

$\cancel{\theta}(\Gamma): X'^{T} A X' + \frac{\Delta}{\delta} = 0$

$$\underbrace{5x_1'^2 + 8x_1'x_2' + 5x_2'^2}_{Q(x)} - 9 = 0$$

$Q(x)$ f canonică (metoda val proprii)

$A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ $\quad \lambda^2 - Tr(A)\lambda + \det A = 0$

$\lambda^2 - 10\lambda + 9 = 0$

$\begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = 0$ $\quad (\lambda - 1)(\lambda - 9) = 0 \Rightarrow \lambda_1 = 1$

$\lambda_2 = 9$

$Q(x) = \lambda_1 x_1''^2 + \lambda_2 x_2''^2 = x_1''^2 + 9x_2''^2$

$V_{\lambda_1} = \{ X \in \mathbb{R}^2 \mid AX = X \}$

$(A - I_2)X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad x_1 + x_2 = 0$

$x_2 = -x_1$

$= \{ (x_1, -x_1) \mid x_1 \in \mathbb{R} \} \Rightarrow \boxed{e_1' = \frac{1}{\sqrt{2}}(1, -1)}$

$$V_{\lambda_2} = \{ x \in \mathbb{R}^2 \mid Ax = 9x \}$$

$$\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = x_2$$

$$= \{ (x_1, x_1) \mid x_1 \in \mathbb{R} \} \Rightarrow \boxed{e_2' = \frac{1}{\sqrt{2}}(1,1)}$$

$$\tau: \ x' = R x'' \qquad R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\tau(\theta(T)): \ x_1''^2 + 9x_2''^2 - 9 = 0 \Rightarrow \frac{x_1''^2}{9} + x_2''^2 = 1$$

Obs: $\mathcal{E}: \ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$

$$x \xrightarrow{\theta} x = x' + x_0 \xrightarrow{\tau} \boxed{x = Rx'' + x_0}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} x_1 = \frac{1}{\sqrt{2}}(x_1'' + x_2'') + 1 \\ x_2 = \frac{1}{\sqrt{2}}(-x_1'' + x_2'') + 1 \end{cases}$$

$$x - x_0 = R x''$$

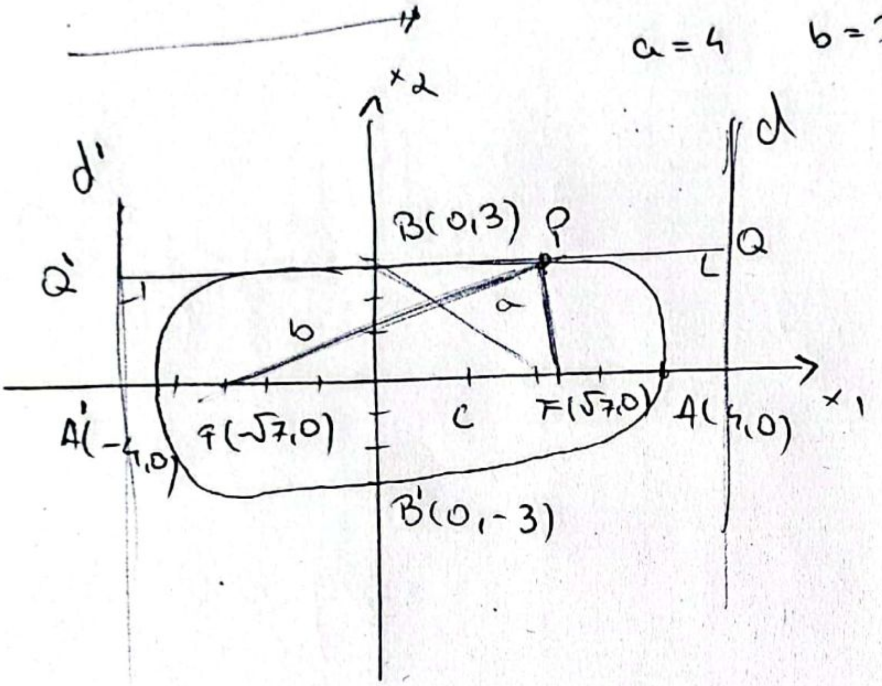
$$x'' = R^T (x - x_0)$$

Ex2 - listă - continuare

Fie eclipsa $\mathcal{E}$ $\frac{x_1^2}{16} + \frac{x_2^2}{9} = 1$

a) Precizați coord. vârfurilor, focarelor, excentricitatea, și ec. directoarelor.

$a = 4 \quad b = 3$



$c = \sqrt{a^2 - b^2} = \sqrt{16-9} = \sqrt{7}$

$e = \frac{c}{a} = \frac{\sqrt{7}}{4}$

$d: x_1 = \frac{a^2}{c} = \frac{16}{\sqrt{7}}$

$d': x_1 = \frac{-a^2}{c} = -\frac{16}{\sqrt{7}}$

Ex3

$\mathcal{H} = ?$

$A(9,0) \in \mathcal{H}$ are asimptotele $d_1 \cup d_2 : x_2 = \pm \frac{2}{3} x_1$

coord vârfurilor, focare, ~~ext~~ excentricitate, directoare

$\mathcal{H}$ $\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1$

$A(9,0) \in \mathcal{H} \Rightarrow \frac{9^2}{a^2} = 1 \Rightarrow a = 9$

$\Rightarrow b = 6$

$D_1 \cup D_2 : \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 0$

$x_2 = \pm \frac{b}{a} x_1 \Rightarrow \boxed{\frac{b}{a} = \frac{2}{3}}$

$\mathcal{H}: \frac{x_1^2}{81} - \frac{x_2^2}{36} = 1$

$c = \sqrt{a^2+b^2} = \sqrt{117}$

$F(\sqrt{117},0)$; $F'(-\sqrt{117},0)$

$A(9,0)$ $A'(-9,0)$

$e = \frac{c}{a} = \frac{\sqrt{117}}{9}$

$D' \cup D$: $x_1 = \pm \frac{81}{\sqrt{117}}$