Seminar 8 | 13.04.2023

Forme biliniare. Forme pătratice. Formă canonică
Metoda Gauss. Metoda Jacobi

① $Q:\mathbb{R}^3\to\mathbb{R}$, $Q(x)=x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3$

a) G matricea asociată în raport cu $R_0=\{e_1,e_2,e_3\}$

b) $g:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ forma polară asociată

c) Să se aducă Q la o formă canonică — metoda Gauss / metoda Jacobi

Este Q poz. definită? Generalizare

$$Q(x)=\sum_{i=1}^{3} g_{ii}x_i^2+2\sum_{i<j}^{3} g_{ij}x_ix_j$$

a) $$G=\begin{pmatrix}1 & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & 1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & 1\end{pmatrix}$$

b) $g(x,y)=x_1y_1+\frac{1}{2}(x_1y_2+x_1y_3+x_2y_1+x_2y_3+x_3y_1+x_3y_2)+$

c) Gauss $\quad Q(x)=x_1^2+x_2^2+x_3^2+x_1x_2+x_2x_3+x_1x_3=(x_1+\frac{1}{2}x_2+$

$-\frac{1}{4}x_2^2-\frac{1}{4}x_3^2-\frac{1}{2}x_2x_3$

$+x_2^2+x_3^2+x_2x_3$

$$Q(x)=(x_1+\tfrac{1}{2}x_2+\tfrac{1}{2}x_3)^2+\tfrac{3}{4}x_2^2+\tfrac{3}{4}x_3^2+\tfrac{1}{2}x_2x_3$$

$$=(x_1+\tfrac{1}{2}x_2+\tfrac{1}{2}x_3)^2+\tfrac{3}{4}\left(x_2^2+\tfrac{2}{3}x_2x_3\right)+\tfrac{3}{4}x_3^2$$

$$=(x_1+\tfrac{1}{2}x_2+\tfrac{1}{2}x_3)^2+\tfrac{3}{4}\left(x_2+\tfrac{1}{3}x_3\right)^2+\underbrace{\tfrac{3}{4}x_3^2-\tfrac{3}{4}\cdot\tfrac{1}{9}x_3^2}_{\frac{2}{3}x_3}$$

$$=(x_1+\tfrac{1}{2}x_2+\tfrac{1}{2}x_3)^2+\tfrac{3}{4}(x_2+\tfrac{1}{3}x_3)^2+\tfrac{2}{3}x_3^2$$

Schimbare de reper:

$$\begin{cases}y_1=x_1+\frac{1}{2}x_2+\frac{1}{2}x_3\\ y_2=x_2+\frac{1}{3}x_3\\ y_3=x_3\end{cases}\Rightarrow$$

$Q(x)=y_1^2+\frac{3}{4}y_2^2+\frac{2}{3}y_3^2$

Signatura $(3,0)$

Este pozitiv definită.

Jacobi

$$\Delta_2 = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = 1 - \frac{1}{4} = \frac{3}{4} \neq 0$$

$$\Delta_3 = \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix} = \begin{vmatrix} 2 & \frac{1}{2} & \frac{1}{2} \\ 2 & 1 & \frac{1}{2} \\ 2 & \frac{1}{2} & 1 \end{vmatrix} = \begin{vmatrix} 2 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = 2 \cdot \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = 2 \cdot \frac{1}{4} = \frac{1}{2} \neq 0$$

$\exists R$ reper a.î. $Q(x) = \frac{1}{\Delta_1}(x_1')^2 + \frac{\Delta_1}{\Delta_2}(x_2')^2 + \frac{\Delta_2}{\Delta_3}(x_3')^2$

$$= 1 \cdot (x_1')^2 + \frac{4}{3}(x_2')^2 + \frac{3}{2}(x_3')^2$$

$\left.\begin{matrix} \Delta_1 > 0 \\ \Delta_2 > 0 \\ \Delta_3 > 0 \end{matrix}\right| \Rightarrow Q$ este pozitiv definită

$\boxed{\text{Generalizare}}$

$Q(x) = \sum_{i=1}^{n} x_i^2 + \sum_{i<j}^{n} x_i x_j$ e poz. definită

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2) $Q: \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = 2x_1x_2 \underbrace{- 6x_1x_3 - 6x_2x_3}_{-6x_3(x_1+x_2)}$

Să se aducă la o formă canonică Gauss / Jacobi, signatura.

$Q(x) = \underbrace{\sum_{i=1}^{3} g_{ii} x_i^2}_{0} + 2\sum_{i<j} g_{ij} x_i x_j$

$$G = \begin{pmatrix} 0 & \boxed{1} & -3 \\ 1 & 0 & -3 \\ -3 & -3 & 0 \end{pmatrix}$$

Fie $g_{12} \neq 0$. Facem sch de reper:

$$\begin{cases} y_1 = x_1 + x_2 \\ y_2 = x_1 - x_2 \\ y_3 = x_3 \end{cases} \longrightarrow \begin{cases} x_1 = \frac{1}{2}(y_1 + y_2) \\ x_2 = \frac{1}{2}(y_1 - y_2) \\ x_3 = y_3 \end{cases}$$

* nu se poate aplica metoda Jacobi (e 0 pe diag)

$$Q(x) = 2 \cdot \frac{1}{4}(y_1^2 - y_2^2) - 6y_3(y_1) = \frac{1}{2}y_1^2 - \frac{1}{2}y_2^2 - 6y_1y_3$$

$$= \frac{1}{2}(y_1^2 - 12y_1y_3) - \frac{1}{2}y_2^2$$

$$= \frac{1}{2}(y_1 - 6y_3)^2 - \cancel{\frac{1}{2}} \cdot \overset{18}{\cancel{36}} y_3^2 - \frac{1}{2}y_2^2$$

$$= \frac{1}{2}(y_1 - 6y_3)^2 - 18y_3^2 - \frac{1}{2}y_2^2$$

Fie sch. de reper:

$$\begin{cases} z_1 = y_1 - 6y_3 \\ z_2 = y_2 \\ z_3 = y_3 \end{cases} \Rightarrow Q(x) = \frac{1}{2}z_1^2 - \frac{1}{2}z_2^2 - 18z_3^2$$

Signatura este $(1, 2) \Rightarrow$ nu e poz def

⑧ Fie $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g(x,y) = x_1y_1 - x_2y_2 - x_1y_3 - x_3y_1 + 2x_2y_3 + 2x_3y_2$

a) $g \in L^s(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

b) Precizați matricea G asociată lui g în raport cu $R_0 = \{e_1, e_2, e_3\}$

c) $\operatorname{Ker} g = ?$ Este g nedegenerată?

d) Să se afle matricea G' asociată lui g în raport cu reperul $R' = \{e_1' = (1,1,1),\ e_2' = (1,2,1),\ e_3' = (0,0,1)\}$

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a) $g(x,y) = X^T G Y = \sum_{i,j=1}^{3} g_{ij} x_i y_j$

$\hookrightarrow g$ formă biliniară $\Big| \Rightarrow g \in L^s(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

b) $G = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix} = G^T \Rightarrow g$ e simetrică

c) $\operatorname{Ker} g = \{x \in \mathbb{R}^3 \mid g(x,y) = 0,\ \forall y \in \mathbb{R}^3\}$

g nedegenerată $\Leftrightarrow \operatorname{Ker} g = \{0_{\mathbb{R}^3}\} \Leftrightarrow \det G \neq 0$

$x \in \operatorname{Ker} g \Leftrightarrow \begin{cases} g(x, e_1) = 0 \\ g(x, e_2) = 0 \\ g(x, e_3) = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 - x_3 = 0 \\ -x_2 + 2x_3 = 0 \\ -x_1 + 2x_2 = 0 \end{cases}$

$\det G = \begin{vmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ -1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = 1 - 4 = -3 \neq 0 \Rightarrow$ S.C. soluție unică nulă

$\Rightarrow g$ este nedegenerat

d) $R_0 \xrightarrow{C} R'$

$G \qquad G'$

$C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$G' = C^T G C$

$G' = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 3 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 3 & 1 \\ 3 & 3 & 3 \\ 1 & 3 & 0 \end{pmatrix}$

În R': $g(x,y) = 2x_1'y_1' + 3x_1'y_2' + x_1'y_3' + 3(x_3'y_1' + x_3'y_2' + x_3'y_3') + \ldots$

7) Fie $f \in End(\mathbb{R}^3)$, $g \in L(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

Fie $g_f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g_f(x,y) = g(f(x), y)$, $\forall x, y \in \mathbb{R}^3$

a) $g_f \in L(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

b) Dacă $G = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ -2 & -1 & -1 \end{pmatrix}$ și $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$ sunt matricele asociate lui g și f, în raport cu reperul canonic R_0, să se afle $\tilde{G}$ matricea asociată lui g_f în raport cu R_0.

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a) $g(x,y) = X^T G Y$ $\qquad$ Dem că: $\boxed{g_f(x,y) = X^T \tilde{G} Y.}$

$f(x) = z \iff AX = Z$

$g_f(x,y) = g(f(x), y) = g(z, y) = Z^T G Y \iff g_f(x,y) = (AX)^T G Y$

$\iff g_f(x,y) = X^T A^T G Y = X^T (A^T G) Y = X^T \tilde{G} Y$

b) $\tilde{G} = A^T G = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ -2 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ -2 & -2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

$\boxed{g_f(x,y) = -x_1 y_3 - 2x_2 y_1 - 2x_2 y_2 + x_3 y_2 - x_3 y_2}$

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8) Fie $g : \mathcal{M}_2(\mathbb{R}) \times \mathcal{M}_2(\mathbb{R}) \to \mathbb{R}$

$g(X, Y) = 2Tr(X \cdot Y) - Tr(X)Tr(Y)$, $\forall X, Y \in \mathcal{M}_2(\mathbb{R})$

a) $g \in L^s(\mathcal{M}_2(\mathbb{R}), \mathcal{M}_2(\mathbb{R}); \mathbb{R})$

b) $G = ?$ matricea în rap cu $R_0 = \{E_{ij}\}_{i,j=\overline{1,2}}$

c) Să se afle expresia analitică a lui $Q : \mathcal{M}_2(\mathbb{R}) \to \mathbb{R}$ forma pătratică asociată.

d) Să se aducă Q la o formă canonică

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a) $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$; $\quad XY = \begin{pmatrix} x_1 y_1 + x_2 y_3 & x_1 y_2 + x_2 y_4 \\ x_3 y_1 + x_4 y_3 & x_3 y_2 + x_4 y_4 \end{pmatrix}$

$G(X,Y) = 2(x_1 y_1 + x_2 y_3 + x_3 y_2 + x_4 y_4) - \underbrace{(x_1 + x_4)(y_1 + y_4)}_{x_1 y_1 + x_1 y_4 + x_4 y_1 + x_4 y_4}$

$= x_1 y_1 + 2x_2 y_3 + 2x_3 y_2 + x_4 y_4 - x_1 y_4 - x_4 y_1$

$$G = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad G = G^T, \; g \text{ liniară} \Big| \Rightarrow g \in L^s(\mathcal{M}_2(\mathbb{R}), \mathcal{M}_2(\mathbb{R}),$$

c) $Q(X) = \sum_{i=1}^{4} g_{ii} x_i^2 + 2 \sum_{i<j}^{4} g_{ij} x_i x_j$

$= x_1^2 + x_4^2 + 2(-x_1 x_4) + 2(2x_2 x_3) = \underline{x_1^2} + x_4^2 - \underline{2x_1x_4} + 4x_2$

$= (x_1 - x_4)^2 + \cancel{x_4^2} - \cancel{x_4^2} + \underline{4x_2x_3} = (x_1 - x_4)^2 + 4x_2x_3$

Fie schimbarea de reper:

$$\begin{cases} y_1 = x_1 - x_4 \\ y_2 = x_2 + x_3 \Rightarrow x_2 = \frac{1}{2}(y_2 + y_3) \\ y_3 = x_2 - x_3 \Rightarrow x_3 = \frac{1}{2}(y_2 - y_3) \\ y_4 = x_4 \end{cases}$$

$Q(x) = y_1^2 + 4 \cdot \frac{1}{2} \cdot \frac{1}{2}(y_2 + y_3)(y_2 - y_3)$

$Q(x) = y_1^2 + y_2^2 - y_3^2$ (forma canonică

Signatura $(2,1) \Rightarrow$ nu e poz. def.

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(10) Fie $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g(x,y) = x_1y_1 + x_1y_3 + 3x_2y_1 + x_2y_2 + 2x_2y_3 + 2x_3y_1 - x_3y_2 + x_3y_3$

G matr. asoc. în raport cu R_0.

Fie $G^s = \frac{1}{2}(G + G^T)$,

$G^a = \frac{1}{2}(G - G^T)$.

$$G = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}; \quad G^s = \frac{1}{2}(G + G^T)$$

Să se determine:

$g^s : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ a.î. G^s sunt matricele asoc.

$g^a : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ G^a în raport cu R_0

$g = g^s + g^a$

$g^s \in L^s(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

$g^a \in L^a(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

$$G^s = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}\right]$$

$$G^s = \begin{pmatrix} 1 & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

$g^s(x,y) = x_1y_1 + x_2y_2 + x_3y_3$
$+ \frac{3}{2}(x_2y_1 + x_1y_2 + x_1y_3) + \frac{1}{2}(3\ldots$
$+ \frac{1}{2}(3x_3y_1 + x_3y_2)$

$$G^a = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & \frac{-3}{2} & \frac{-1}{2} \\ \frac{3}{2} & 0 & \frac{3}{2} \\ \frac{1}{2} & \frac{-3}{2} & 0 \end{pmatrix}$$

$g^a(x,y) = \frac{1}{2}(-3x_1y_2 - x_1y_3 + 3x_2y_1 + x_2y_3 + x_3y_1 - 3x_3y_2)$