$\boxed{C_7 - GA}$

Vectori proprii. Valori proprii. Diagonalizare.

Proiecții și simetrii (cont. C_6)

Def $(V,+,\cdot)/\mathbb{K}$.

$p \in End(V)$ s.n. proiecție pe V_1 de-a lungul lui V_2, $V = V_1 \oplus V_2 \Leftrightarrow p(v) = p(\underset{V_1}{\underset{\cap}{v_1}} + \underset{V_2}{\underset{\cap}{v_2}}) = v_1$

Prop

$p \in End(V)$

p proiecție $\Leftrightarrow p \circ p = p$.

Dem

"$\Rightarrow$" Ip p proiecție.

$p \circ p(v) = p \circ p(v_1 + v_2) = p(v_1) = p(v_1 + 0) = v_1 = p(v)$

$\forall v \in V \Rightarrow p \circ p = p$.

"$\Leftarrow$" Ip: $p \in End(V)$ a.î. $p \circ p = p$.

"$\supset$" (din constr.)

$V = Imp \oplus Ker p$ (dem) $\quad p(v) = p(\underset{p(v)}{\underset{\shortparallel}{p(v)}}) + \underset{0}{\underset{\shortparallel}{p(v - p(v))}}$

"$\subset$" (dem)

$v = \underset{Im p}{\underset{\cap}{p(v)}} + \underset{Ker p}{\underset{\cap}{\overbrace{v - p(v)}^{u}}}$ (dem)

$p(u) = p(v - p(v)) = p(v) - \underset{p(v)}{\underset{\shortparallel}{p(p(v))}} = 0 \Rightarrow u \in Ker p$

"$\oplus$": Fie $v \in Imp \cap Ker p$

$\Rightarrow \exists w \in V$ a.î. $p(w) = v \mid p \Rightarrow \underset{\underset{v}{\shortparallel}}{\underset{p(w)}{\underset{\shortparallel}{p(p(w))}}} = \underset{0}{\underset{\shortparallel}{p(v)}}$

și

$p(v) = 0$

Def $s \in End(V)$ s.n. simetrie sau involuție $\Leftrightarrow$
$s \circ s = id_V$.

Prop $p \in End(V)$, $ch K \neq 2$ (*)
p = proiecție $\Leftrightarrow$ $s = 2p - id_V$ e simetrie.

Dem
"$\Rightarrow$" $V = V_1 \oplus V_2$. proiecție pe V_1, de-a lungul lui V_2.
$p(v) = p(v_1 + v_2) = v_1$.
$s = 2p - id_V$. Dem că $s \circ s = id_V$.
$s \circ s(v) = (2p - id_V) \circ (2p - id_V)(v) =$
$= 4 p \circ p(v) - 4 p \circ id_V(v) + id_V \circ id_V(v) =$
$= 4 p(v) - 4p(v) + id_V(v) = v$, $\forall v \in V$
"$\Leftarrow$" $s \in End(V)$ $s = 2p - id_V$ și $s \circ s = id_V$
$\overset{?}{\Rightarrow}$ p = proiecție ie $p \circ p = p$?
$s \circ s = id_V \Rightarrow 4 p \circ p - 4p + id_V = id_V$
$p \circ p = p$.

$\boxed{OBS}$ $p, s : V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2$
p = proiecție pe V_1, de-a lungul lui V_2
s = simetric față de V_1
$p(v) = p(v_1 + v_2) = v_1$, $s(v) = s(v_1 + v_2) =$
$2p(v_1 + v_2) - id_V(v_1 + v_2) = 2v_1 - (v_1 + v_2) = v_1 - v_2$.

OBS $V = V_1 \oplus V_2$ $\quad$ $R_1 = \{e_1, \dots, e_k\}$ reper în V_1
$R_2 = \{e_{k+1}, \dots, e_n\}$ reper în V_2.
$R = R_1 \cup R_2$ reper în V

p proiecție pe V_1 $\quad p(e_i) = e_i, \ \forall i = \overline{1,k}$

$p(e_j) = 0 \quad \forall j = \overline{k+1,n}$

$$[p]_{R,R} = A_p = \left(\begin{array}{c|c} I_k & O_{k,n-k} \\ \hline O_{n-k,k} & O_{n-k,n-k} \end{array}\right)$$

s simetrie față de V_1 $\quad s(v_1 + v_2) = v_1 - v_2$

$s(e_i) = e_i, \quad \forall i = \overline{1,k}$

$s(e_j) = -e_j, \quad \forall j = \overline{k+1,n}$ $\quad [s]_{R,R} = A_s = \left(\begin{array}{c|c} I_k & O \\ \hline O & -I_{n-k} \end{array}\right)$

$A_s \in O(n), \quad A_s \cdot A_s^T = I_n.$

Vectori proprii. Valori proprii. Diagonalizare.

<u>Problemă</u> $(V,+,\cdot)/K, \ f \in End(V)$

$\exists$ un reper $R = \{e_1, \dots, e_n\}$ în V a.î. $[f]_{R,R} = A =$ diagonală $= \begin{pmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_n \end{pmatrix}$?

$$\begin{cases} f(e_1) = \lambda_1 e_1 \\ \vdots \\ f(e_n) = \lambda_n e_n. \end{cases}$$

<u>Def</u> $f \in End(V)$

$x \in V$, $x \neq 0_V$ s.n. <u>vector propriu</u> $\Leftrightarrow \exists \lambda \in K$ a.î. $f(x) = \lambda x$ (valoare proprie)

<u>Obs</u> $f(0_V) = 0_V = \lambda \cdot 0_V$

<u>Def</u> $V_\lambda = \{x \in V \mid f(x) = \lambda x\}$ subspațiul propriu corespunzător valorii proprii λ.

Prop a) $V_\lambda \subseteq V$ subspațiu vectorial

b) V_λ = subspațiu invariant al lui f i.e $f(V_\lambda) \subseteq V_\lambda$

Dem

a) $\forall v, w \in V_\lambda$, $\forall a, b \in \mathbb{K}$ $\overset{?}{\Rightarrow} av + bw \in V_\lambda$

$f(av+bw) \overset{f\ lin}{=} a \underset{\lambda v}{\underset{\parallel}{f(v)}} + b \underset{\lambda w}{\underset{\parallel}{f(w)}} = \lambda(av+bw) \Rightarrow V_\lambda \subseteq V$ sspvect

b) Fie $v \in V_\lambda \Rightarrow f(v) = \lambda v \in V_\lambda \Rightarrow f(V_\lambda) \subseteq V_\lambda$.

Polinomul caracteristic

$R = \{e_1, .., e_n\}$ reper în V, $[f]_{R,R} = A$

x = vector propriu coresp. valorii proprii λ $\quad f(e_i) = \sum_{j=1}^{n} a_{ji} e_j, \forall i = \overline{1,n}$

$f(x) = \lambda x$, $\quad f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i f(e_i)$

$$= \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} a_{ji} e_j\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ji} x_i\right) e_j$$

$$\lambda x = \lambda \sum_{j=1}^{n} x_j e_j \qquad \delta_{ji} = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}$$

R SLI $\Rightarrow \sum_{i=1}^{n} a_{ji} x_i = \lambda x_j = \lambda \sum_{i=1}^{n} \delta_{ji} x_i$

(*) $\sum_{i=1}^{n} (a_{ji} - \lambda \delta_{ji}) x_i = 0, \ \forall j = \overline{1,n}$

(*) SLO care are și sol nenule $\Leftrightarrow$

$= \det(A - \lambda I_n) = 0$

$$p(\lambda) \overset{\parallel}{} (-1)^n \left[\lambda^n - \sigma_1 \lambda^{n-1} + \ldots + (-1)^n \sigma_n\right]$$

σ_k = suma minorilor diagonali de ordin k, $k = \overline{1,n}$

Prop Polinomul caracteristic este un invariant la schimbarea reperului.

Dem

$$\begin{array}{ccc} R & \longrightarrow & R \\ C\downarrow & & \downarrow \\ R' & \longrightarrow & R' \end{array}$$

$R=\{e_1,..,e_n\}$, $R'=\{e'_1,..,e'_n\}$ repere în V

$[f]_{R,R}=A$, $[f]_{R',R'}=A'$

$A'=C^{-1}AC$, $e'_i=\sum_{j=1}^{n} c_{ji}e_j$, $\forall i=\overline{1,n}$

$$\det(A'-\lambda I_n)=\det(C^{-1}AC-\lambda C^{-1}C)=$$
$$=\det\left(C^{-1}(A-\lambda I_n)\cdot C\right)=\det(A-\lambda I_n)$$

! OBS valorile proprii = rădăcinile din $\mathbb{K}$ ale polinomului caracteristic.

Exemplu $(\mathbb{R}^2,+,\cdot)/\mathbb{R}$, $J:\mathbb{R}^2\to\mathbb{R}^2$, $J(x_1,x_2)=(-x_2,x_1)$

$$[J]_{R_0,R_0}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=A$$

$$\det(A-\lambda I_2)=0\Rightarrow\begin{vmatrix}-\lambda & -1\\ 1 & -\lambda\end{vmatrix}=0\Rightarrow\lambda^2+1=0$$

$\lambda=\pm i\in\mathbb{C}\setminus\mathbb{R}$ J nu are valori proprii.

$\boxed{\text{OBS}}$

$$P(\lambda)=(\lambda-\lambda_1)^{m_1}\cdot\ldots(\lambda-\lambda_r)^{m_r}$$

$\lambda_1,..,\lambda_r$ răd. distincte

$m_1,..,m_r$ = multiplicități a.î. $m_1+\ldots+m_r=n$.

$\sigma(f)=\{\lambda_1,..,\lambda_r\}$

$$\mathrm{Spec}(f)=\{\underbrace{\lambda_1=..=\lambda_1}_{m_1}<\underbrace{\lambda_2=..=\lambda_2}_{m_2}<\ldots<\underbrace{\lambda_r=..\lambda_r}_{m_r}\}$$

spectrul lui f.

Prop Vectorii proprii corespunzători la valori proprii distincte formează un SLI.

Dem. Dem. prin ind. după nr. de vectori proprii

- $\underset{\neq 0}{x}$ vector propriu $\Rightarrow \{x\}$ SLI
- Pp adev P_{n-1}: $\{v_1, .., v_{n-1}\}$ SLI

vect. proprii coresp la valori proprii dist.

Dem P_n. Fie $S = \{v_1, .., v_n\}$ vect. proprii coresp la valori proprii dist i.e $f(v_k) = \lambda_k v_k$, $k = \overline{1,n}$

$\lambda_1, .., \lambda_n$ valori proprii dist.

Dem că S este SLI.

Fie $a_1, .., a_n \in \mathbb{K}$ aî $a_1 v_1 + \ldots + a_n v_n = 0_V$ (1) $| f$

$a_1 \underset{\lambda_1 v_1}{\underbrace{f(v_1)}} + \ldots + a_n \underset{\lambda_n v_n}{\underbrace{f(v_n)}} = 0_V$ (*)

Fie $\lambda_n \neq 0$ (eventual renumerotăm indicii)

$(1) | \cdot \lambda_n \Rightarrow a_1 \lambda_n v_1 + \ldots + a_{n-1} \lambda_n v_{n-1} + a_n \lambda_n v_n = 0_V$ (**)

$(*) - (**)$ $a_1 \overbrace{(\lambda_1 - \lambda_n)}^{\neq 0} v_1 + \ldots + a_{n-1} \overbrace{(\lambda_{n-1} - \lambda_n)}^{\neq 0} v_{n-1} = 0$.

$\overset{P_{n-1}}{\Longrightarrow} a_1 = .. = a_{n-1} = 0 \overset{(1)}{\Longrightarrow} a_n v_n = 0_V \Rightarrow a_n = 0$

$\Rightarrow S = \{v_1, .., v_n\}$ SLI

Prop $f \in End(V)$, λ = valoare proprie

V_λ = subspațiul propriu coresp. valorii proprii λ

$\overset{?}{\Rightarrow} \dim V_\lambda \leq m_\lambda$

Dem Not $\dim V_\lambda = n_\lambda$ $\quad R_0 = \{e_1, .., e_{n_\lambda}\}$ reper în V_λ.

Il extindem la $R = \{e_1, .., e_{n_\lambda}, e_{n_\lambda + 1}, .., e_n\}$ reper în V.

- $f(e_1) = \lambda e_1, .., f(e_{n_\lambda}) = \lambda e_{n_\lambda}$, $f(e_j) = \sum_{k=1}^{n} a_{kj} e_k$ $\quad j = \overline{n_\lambda + 1, n}$

$$[f]_{R,R} = A = \begin{pmatrix} \begin{matrix}\lambda & & 0 \\ & \ddots & \\ 0 & & \lambda\end{matrix} & * \\ 0 & * \end{pmatrix}$$

$$P(X) = \det(A - XI_n) = \begin{vmatrix} \begin{matrix}\lambda - X & & 0 \\ & \ddots & \\ 0 & & \lambda - X\end{matrix} & * \\ 0 & * \end{vmatrix} = (\lambda - X)^{n_\lambda} g(X)$$

$\Rightarrow \underset{\dim V_\lambda}{\overset{\shortparallel}{n_\lambda}} \leq \underset{\text{multiplicitatea lui } \lambda}{\overset{\shortparallel}{m_\lambda}}$

$\boxed{\text{Teoremă}}$ $f \in End(V)$.

$\exists$ un reper $R = \{e_1, .., e_n\}$ în V a.î. $[f]_{R,R}$ = diagonală

$\Leftrightarrow$ 1) toate răd. polinomului caracteristic $\in \mathbb{K}$.

($\lambda_1, .., \lambda_r \in \mathbb{K}$, răd. dist. ale polin. caract)

2) dimensiunile subsp. proprii = multiplicitățile valorilor proprii

(i.e $\dim V_{\lambda_i} = m_{\lambda_i}$, $i = \overline{1,r}$, $m_1 + ... + m_r = n$)

<u>Dem</u>

"$\Rightarrow$" <u>Ip</u>: $\exists R = \{e_1, .., e_n\}$ reper în V a.î.

$$A = [f]_{R,R} = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix} \in \mathcal{M}_n(\mathbb{K})$$

Eventual renumerotăm și considerăm. (schimbare de reper) $\lambda_1, .., \lambda_r \in \mathbb{K}$. (distincte)

$$A = \begin{pmatrix} \overbrace{\lambda_1 \cdots \lambda_1}^{m_{\lambda_1}} & & 0 \\ & \ddots & \\ 0 & & \overbrace{\lambda_r \cdots \lambda_r}^{m_{\lambda_r}} \end{pmatrix} \in \mathcal{M}_n(\mathbb{K})$$

$$P(\lambda) = \det(A - \lambda I_n) = \begin{vmatrix} \lambda_1 - \lambda & & & & 0 \\ & \lambda_1 - \lambda & & & \\ & & \ddots & & \\ & & & \lambda_r - \lambda & \\ 0 & & & & \lambda_r - \lambda \end{vmatrix}$$

$= (\lambda_1 - \lambda)^{m_1} \dots (\lambda_r - \lambda)^{m_r}$

$\lambda_1, .., \lambda_r$ răd pol caract $\in \mathbb{K}$.

Prop $\Rightarrow \dim V_{\lambda_i} \le m_{\lambda_i}$, $i=\overline{1,r}$ (1)

$\{e_1, \dots, e_{m_{\lambda_1}}\} \subset V_{\lambda_1} \Rightarrow \dim V_{\lambda_1} \geq m_{\lambda_1}$

Analog $\dim V_{\lambda_j} \geq m_{\lambda_j}$, $j=\overline{1,r}$ (2) $\Rightarrow$

$\dim V_{\lambda_j} = m_{\lambda_j}$, $j=\overline{1,r}$

„$\Leftarrow$" $f \in End(V)$ 1) $\lambda_1, \dots, \lambda_r \in \mathbb{K}$ răd dist pol caract

2) $\dim V_{\lambda_i} = m_{\lambda_i}$, $\forall i=\overline{1,r}$

$m_{\lambda_1} + \dots + m_{\lambda_r} = n$.

Construim R reper în V ai $[f]_{R,R}$ diag.

$R_1, \dots, R_r$ repere în $V_{\lambda_1}, \dots, V_{\lambda_r}$

$R = R_1 \cup \dots \cup R_r$ (dem că R reper în V)

$|R| = n = \dim V$. Este suficient să dem că R e SLI

$$\underbrace{\sum_{i=1}^{m_{\lambda_1}} a_i e_i}_{f_1 \in V_{\lambda_1}} + \dots + \underbrace{\sum_{j=m_{\lambda_1}+\dots+m_{\lambda_{r-1}}+1}^{n} a_j e_j}_{f_r \in V_{\lambda_r}} = 0_V$$

Pp. abs. că $\exists f_{i_1}, \dots, f_{i_s}$ nenuli $\{i_1, \dots i_s\} \subset \{1, \dots, r\}$ (restul sunt nule) valori proprii dist $\Rightarrow$ SLI

vectori proprii coresp la

$f_{i_1} + \dots + f_{i_s} = 0_V$ ↯.

Pp. este falsă $\Rightarrow f_1 = 0, \dots, f_r = 0$.

$\Downarrow R_1$ SLI $\qquad \Downarrow R_r$ SLI

$a_1 = \dots = a_{m_{\lambda_1}} = 0 \qquad a_{m_{\lambda_1}+\dots+m_{\lambda_{r-1}}+1} = \dots = a_n =$

$\Rightarrow a_i = 0$, $i=\overline{1,n} \Rightarrow R$ este reper în V

$V = V_{\lambda_1} \oplus \dots \oplus V_{\lambda_r}$ $\quad [f]_{R,R} = \begin{pmatrix} \underbrace{\lambda_1 \dots \lambda_1}_{m_{\lambda_1} \text{ ori}} & & & 0 \\ & \ddots & & \\ & & \underbrace{\lambda_r \dots \lambda_r}_{m_{\lambda_r} \text{ ori}} & \\ 0 & & & \end{pmatrix}$

Ex $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1, x_2 + x_3, 2x_3)$.

Determinați un reper $\mathcal{R}$ în $\mathbb{R}^3$ a.î. $[f]_{\mathcal{R},\mathcal{R}}$ diagonală.

SOL $\mathcal{R}_0 = \{e_1, e_2, e_3\}$ reperul canonic

$$[f]_{\mathcal{R}_0,\mathcal{R}_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + x_3 \\ 2x_3 \end{pmatrix}$$

$$\begin{cases} f(e_1) = f(1,0,0) = (1,0,0) = e_1 + 0e_2 + 0e_3 \\ f(e_2) = f(0,1,0) = (0,1,0) = e_2 \\ f(e_3) = f(0,0,1) = (0,1,2) \\ \qquad = e_2 + 2e_3 \end{cases}$$

<u>Polinomul caracteristic</u>

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda) = 0$$

$\lambda_1 = 1$, $m_1 = 2$

$\lambda_2 = 2$, $m_2 = 1$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_1 x\}$

$$AX = X \Rightarrow (A - I_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_3 = 0$

$$V_{\lambda_1} = \{\underbrace{(x_1, x_2, 0)}_{= x_1(1,0,0) + x_2(0,1,0)} \mid x_1, x_2 \in \mathbb{R}\} = \langle \{e_1, e_2\} \rangle \Rightarrow \dim V_{\lambda_1} = 2 = m_1$$

$V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_2 x\}$

$$AX = 2X \Rightarrow (A - 2I_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x_1 = 0 \\ -x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = x_3 \end{cases}$$

$$V_{\lambda_2} = \{(0, x_2, x_2) = x_2(0,1,1)\} = \langle \{(0,1,1)\} \rangle \quad \dim V_{\lambda_2} = 1 = m_2$$

$$\mathcal{R} = \{(1,0,0), (0,1,0), (0,1,1)\}.$$

$$[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 2 \end{pmatrix}$$

OBS

$$\begin{tikzcd} \mathcal{R}_0 \arrow[r, "A"] \arrow[d, "C"'] & \mathcal{R}_0 \arrow[d, "C"] \\ \mathcal{R} \arrow[r, "A'"] & \mathcal{R} \end{tikzcd}$$

$A' = C^{-1}AC \Rightarrow A = CA'C^{-1}$

$A' =$ matrice diag.

$A^n = CA'^nC^{-1}$

Ex $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x) = (3x_1 + 2x_2, -x_1)$

$A = [f]_{\mathcal{R}_0, \mathcal{R}_0}$. Calculați A^n.

SOL $A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$ $\quad P(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = 0$

$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda_1 = 1, \ m_1 = 1; \quad \lambda_2 = 2, \ m_2 = 1$

$V_{\lambda_1} = \{x \in \mathbb{R}^2 \mid f(x) = x\}$

$AX = X \Leftrightarrow (A - I_2)X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-x_1 - x_2 = 0 \Rightarrow x_2 = -x_1$

$V_{\lambda_1} = \{(x_1, -x_1) = x_1(1,-1)\} = \langle\{(1,-1)\}\rangle$

$V_{\lambda_2} = \{x \in \mathbb{R}^2 \mid f(x) = 2x\}$

$AX = 2X \Rightarrow (A - 2I_2)X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$

$V_{\lambda_2} = \{(-2x_2, x_2) \mid x_2 \in \mathbb{R}\} = \langle\{(-2,1)\}\rangle$

$\mathcal{R} = \{(1,-1), (-2,1)\} = \mathcal{R}_1 \cup \mathcal{R}_2$; $A' = [f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$\mathcal{R}_0 = \{e_1 = (1,0), e_2 = (0,1)\} \xrightarrow{C} \mathcal{R} = \{e_1' = (1,-1), e_2' = (-2,1)\}$

$e_1' = (1,-1) = e_1 - e_2$ $\qquad C = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$, $\det C = 1 - 2 = -1$

$e_2' = (-2,1) = -2e_1 + e_2$

$C^T = \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$, $C^* = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

$A^n = CA'^nC^{-1}$

$C^{-1} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$

$$A^n = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}}_{\begin{pmatrix} -1 & -2 \\ -2^n & -2^n \end{pmatrix}} = \begin{pmatrix} -1 + 2^{n+1} & -2 + 2^{n+1} \\ 1 - 2^n & 2 - 2^n \end{pmatrix}$$