CURS 9

13.04.2021

Forme pătratice. Metoda Jacobi
Spații vectoriale euclidiene reale

- $Q: V \to \mathbb{K}$ formă pătratică $\Leftrightarrow$

$\exists\, g: V \times V \to \mathbb{K}$ formă biliniară simetrică a.î.

$Q(x) = g(x,x),\ \forall x \in V$

$Q(x) = X^T G X = \sum_{i,j=1}^{n} g_{ij} x_i x_j\ ,\ g_{ij} = g(e_i, e_j)\ \forall i,j = \overline{1,n}$

$Q(x) = a_1 x_1^2 + \dots + a_r x_r^2\ ,\ r = rg\, Q$

formă canonică

Teorema Gauss Fie $Q: V \to \mathbb{K}$ f. pătratică

$\Rightarrow \exists$ un reper $R = \{e_1, .., e_n\}$ în V a.î. Q are o formă canonică

Teoremă $Q: V \to \mathbb{R}$ f. pătratică reală

$\Rightarrow \exists\, R$ reper în V a.î. $Q(x) = x_1^2 + .. + x_p^2 - x_{p+1}^2 - \dots - x_r^2$

(formă normală)

$(p, r-p)$ = signatura (invar. la sch. reperelor)

Q poz. definită $\Leftrightarrow (n,0)$ signatura

Metoda Jacobi

Fie $Q: V \to \mathbb{R}$ f. pătratică reală.

Fie $R = \{e_1, .., e_n\}$ un reper în V. Dacă matricea G asociată lui Q în raport cu R are minorii diagonali $\Delta_1 = \det(g_{11})$, $\Delta_2 = \det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \dots, \Delta_n = \det G$ nenuli, atunci $\exists$ un reper $\mathcal{R}' = \{e_1', \dots, e_n'\}$ în V aî

Q are o formă canonică

$$Q(x) = \frac{1}{\Delta_1} x_1'^2 + \frac{\Delta_1}{\Delta_2} x_2'^2 + \dots + \frac{\Delta_{n-1}}{\Delta_n} x_n'^2$$

Mai mult, Q pozitiv definită $\iff \Delta_i > 0, \forall i = \overline{1,n}$

$$G = \begin{pmatrix} \boxed{g_{11}} & g_{12} & \dots & g_{1n} \\ g_{21} & g_{22} & \dots & g_{2n} \\ \vdots & & & \\ g_{n1} & g_{n2} & \dots & g_{nn} \end{pmatrix}$$

OBS a) Metoda Jacobi este restrictivă (tb $\Delta_i \neq 0, \forall i = \overline{1,n}$)

b) Metoda Gauss se poate aplica totdeauna

Aplicație

Fie $Q: \mathbb{R}^4 \to \mathbb{R}$, $Q(x) = x_1^2 + x_3^2 + x_1 x_2 + x_3 x_4$

Să se aducă la o formă canonică, utilizând metoda Jacobi și metoda Gauss.

SOL

$\mathcal{R}_0 = \{e_1, e_2, e_3, e_4\}$ reperul canonic.

$$Q(x) = g_{11} x_1^2 + \dots + g_{44} x_4^2 + 2g_{12} x_1 x_2 + \dots + 2g_{34} x_3 x_4.$$

$$G = \begin{pmatrix} \boxed{1} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Met Jacobi

$\Delta_1 = \det(1) = 1$

$\Delta_2 = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} = -\frac{1}{4}$

$\Delta_3 = \begin{vmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\frac{1}{4}$

$\Delta_4 = \det G = -\frac{1}{2} \begin{vmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{vmatrix} = \frac{1}{16}$

$\frac{1}{\Delta_1} = 1$; $\frac{\Delta_1}{\Delta_2} = -4$, $\frac{\Delta_2}{\Delta_3} = 1$, $\frac{\Delta_3}{\Delta_4} = -4$.

$\exists\ \mathcal{R}' = \{e_1', .., e_4'\}$ reper în $\mathbb{R}^4$ ai

$Q(x) = x_1'^2 - 4x_2'^2 + x_3'^2 - 4x_4'^2$

Fie $\mathcal{R}'' = \{e_1', e_3', e_2', e_4'\}$ reper în $\mathbb{R}^4$

$Q(x) = x_1''^2 + x_2''^2 - 4x_3''^2 - 4x_4''^2$

$(2,2)$ signatura

Met Gauss

$Q(x) = x_1^2 + x_3^2 + x_1x_2 + x_3x_4 =$

$= \left(x_1 + \frac{1}{2}x_2\right)^2 - \frac{x_2^2}{4} + x_3^2 + x_3x_4 =$

$= \left(x_1 + \frac{1}{2}x_2\right)^2 - \frac{x_2^2}{4} + \left(x_3 + \frac{1}{2}x_4\right)^2 - \frac{1}{4}x_4^2$

$$\begin{cases} x_1' = x_1 + \frac{1}{2}x_2 \\ x_2' = x_3 + \frac{1}{2}x_4 \\ x_3' = x_2 \\ x_4' = x_4 \end{cases} \Rightarrow Q(x) = x_1'^2 + x_2'^2 - \frac{x_3'^2}{4} - \frac{x_4'^2}{4}$$

$(2,2)$

Spatii vectoriale euclidiene reale

Def $(V, +, \cdot)_{/\mathbb{R}}$ sp. vect. real și $g: V\times V \to \mathbb{R}$

g s.n. produs scalar $\Leftrightarrow$

1) g formă biliniară simetrică

2) g este poz def

(i.e $Q: V \to \mathbb{R}$ f. pătratica asociată, $Q(x) = g(x,x)$, $\forall x \in V$

poz definită $\Leftrightarrow$ $Q(x) > 0$, $\forall x \in V \setminus \{0_V\}$; $Q(x) = 0 \Leftrightarrow x = 0$)

<u>Not</u> (V, g) ; $(E, \langle \cdot, \cdot \rangle)$; $(E, (\cdot, \cdot))$

spațiu vect. euclidian real

<u>Def</u> $\|x\| = \sqrt{g(x,x)} = \sqrt{Q(x)}, \forall x \in V$

<u>norma</u> lui x.

<u>Def</u> $(E, \langle \cdot, \cdot \rangle)$ s.v.e.r, $R = \{e_1, \dots, e_n\}$ reper in V

a) R s.n. <u>reper ortogonal</u> $\Leftrightarrow \langle e_i, e_j \rangle = 0, \forall i,j = \overline{1,n}; i \neq j$

b) R s.n. <u>reper ortonormat</u> $\Leftrightarrow \langle e_i, e_j \rangle = \delta_{ij}, \forall i,j = \overline{1,n}$

(vectorii sunt mutual $\perp$ și versori)

<u>OBS</u>

$R = \{e_1, \dots, e_n\} \xrightarrow{A} R' = \{e_1', \dots, e_n'\}$ repere ortonormate

$\Rightarrow A \in O(n)$ i.e. $A A^T = A^T \cdot A = I_n$.

$$e_r' = \sum_{i=1}^{n} a_{ir} e_i$$

$$\underbrace{\langle e_r', e_s' \rangle}_{= \delta_{rs}} = \langle \sum_{i=1}^{n} a_{ir} e_i, \sum_{j=1}^{n} a_{js} e_j \rangle = \sum_{i,j=1}^{n} a_{ir} a_{js} \underbrace{\langle e_i, e_j \rangle}_{= \delta_{ij}}$$

$$\Rightarrow \delta_{rs} = \sum_{i=1}^{n} a_{ir} a_{is} \Rightarrow I_n = A^T A$$

Dacă R, R' sunt la fel orientate $(\det A > 0)$

$\Rightarrow \det A = 1$ și $A \in SO(n)$

<u>Prop</u> $(E, \langle \cdot, \cdot \rangle)$ s.v.e.r și $S = \{x_1, \dots, x_k\}$, $k \leq n = \dim V$

Dacă S este un sist. de vect. nenuli, mutual $\perp$,

atunci S este un SLI.

Dem

Fie $a_1, \dots, a_k \in \mathbb{R}$ ai $\sum_{i=1}^{k} a_i x_i = 0_V \quad | <\cdot, x_1>$

$< \sum_{i=1}^{k} a_i x_i , x_1 > = 0_{\mathbb{R}} \Rightarrow$

$a_1 \underbrace{<x_1, x_1>}_{\|x_1\|^2} + a_2 \underbrace{<x_2, x_1>}_{0} + \dots + a_k \underbrace{<x_k, x_1>}_{0} = 0_{\mathbb{R}}$

$a_1 \|x_1\|^2 = 0_{\mathbb{R}} \Rightarrow a_1 = 0_{\mathbb{R}}$

Analog $<\cdot, x_j>, \forall j = \overline{2,k} \Rightarrow a_j = 0, \forall j = \overline{2,k}$

$\Rightarrow a_i = 0, \forall i = \overline{1,k}$ și S este un SLI

Exemplu $(\mathbb{R}^n, g_0)$, $g_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $g_0(x,y) = x_1 y_1 + \dots + x_n y_n$

g_0 produs scalar canonic

- g_0 formă bil, sim. $g_0(x,y) = X^T I_n Y$

$I_n = I_n^T$

- g_0 este poz. def : $Q_0(x) = g_0(x,x) = x_1^2 + \dots + x_n^2$

$(n, 0)$ = signatura

Def (produs vectorial)

Fie $(\mathbb{R}^3, g_0)$ s.v.e.r, cu str. euclidiană canonică

Fie $S = \{x, y\} \subset \mathbb{R}^3$ și $z = x \times y$ numit produs vectorial def astfel:

① Dc S este SLD, atunci $z = 0_{\mathbb{R}^3}$

② Dc S este SLI, atunci

a) $\|z\|^2 = \begin{vmatrix} <x,x> & <x,y> \\ <y,x> & <y,y> \end{vmatrix}$

b) $z \perp x$, $z \perp y$ i.e $<z, x> = <z, y> = 0$

c) $\{x, y, z\}$ reper pozitiv orientat (la fel orientat ca și reperul canonic)

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OBS „×" este un „determinant formal"

$$z = x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & e_1 \\ x_2 & y_2 & e_2 \\ x_3 & y_3 & e_3 \end{vmatrix}$$

$$= e_1 \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - e_2 \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + e_3 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

$$= (x_2 y_3 - x_3 y_2,\ x_3 y_1 - x_1 y_3,\ x_1 y_2 - x_2 y_1)$$

Prop $(\mathbb{R}^3, g_0)$, $S = \{x, y\}$ un SLI

a) $x \times y = - y \times x$

b) $(x \times y) \times z = \langle x, z\rangle y - \langle y, z \rangle x$ (nu e asociativ)

c) $\sum_{x,y,z} (x \times y) \times z = 0$ (ident. Jacobi)

Dem

a) $x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = - \begin{vmatrix} e_1 & e_2 & e_3 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = - y \times x.$

b) $(x \times y) \times z = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \\ z_1 & z_2 & z_3 \end{vmatrix}$

$= e_1 \cdot \alpha - e_2 \beta + e_3 \gamma = (\alpha, -\beta, \gamma)$

$\langle x, z \rangle y - \langle y, z \rangle x =$

$= (\underbrace{x_1 z_1 + x_2 z_2 + x_3 z_3}_{\in \mathbb{R}})(y_1 \underline{e_1} + y_2 \underline{\underline{e_2}} + y_3 e_3) -$

$- (\underbrace{y_1 z_1 + y_2 z_2 + y_3 z_3}_{\in \mathbb{R}})(x_1 \underline{e_1} + x_2 \underline{\underline{e_2}} + x_3 e_3)$

$= e_1 \cdot \alpha - e_2 \beta + e_3 \gamma = (\alpha, -\beta, \gamma)$

c) $\sum\limits_{x,y,z}^{c} (x\times y)\times z = (x\times y)\times z + (y\times z)\times x + (z\times x)\times y$

$= \cancel{\langle x,z\rangle y} - \langle y,z\rangle x +$
$\langle y,x\rangle z - \cancel{\langle z,x\rangle y} +$
$\langle z,y\rangle x - \langle x,y\rangle z \quad = 0$

Def (produs mixt)

$(\mathbb{R}^3, g_0)$, $S = \{x,y\}$ un SLI. Fie $z \in \mathbb{R}^3$

$z \wedge x \wedge y = \langle z, x\times y\rangle$ (produsul mixt)

OBS

$z\wedge x\wedge y = x\wedge y\wedge z$

$$z\wedge x\wedge y = \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = x\wedge y\wedge z$$

Aplicatie

$(\mathbb{R}^3, g_0)$ $\quad g_0 : \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}$, $g_0(x,y) = x_1y_1 + x_2y_2 + x_3y_3$

$u = (1,-1,2)$
$v = (0,1,3)$
$w = (1,4,0)$

$\{u,v\}$ SLI $\Leftrightarrow \operatorname{rg}\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 3 \end{pmatrix} = 2$

a) $u\times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{vmatrix} = e_1 \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} - e_2 \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} + e_3 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$

$= \underline{(-5,-3,1)}$

b) $w\wedge u\wedge v = \langle w, u\times v\rangle = \begin{vmatrix} 1 & 4 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{vmatrix}$

$= 1(-5) + 4(-3) + 0\cdot 1 = -5-12 = -17.$

c) $\{u, v, u\times v\}$ reper poz. orientat în $\mathbb{R}^3$

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Teoremă (procedeul Gram-Schmidt)

Fie $(E, \langle \cdot, \cdot \rangle)$ s.v.e.r, $R = \{f_1, \dots, f_n\}$ reper în E.

$\Rightarrow \exists R' = \{e_1, \dots, e_n\}$ reper ortogonal în E a.î.

$$Sp\{e_1, \dots, e_i\} = Sp\{f_1, \dots, f_i\}, \quad \forall i = \overline{1,n}, \quad n = \dim E$$

$$\langle \{e_1, \dots, e_i\} \rangle \qquad \langle \{f_1, \dots, f_i\} \rangle$$

Dem

Met. inductivă.

$e_1 = f_1 \neq 0$

$e_2 = f_2 + \alpha f_1 = f_2 + \alpha e_1$

$\langle e_1, e_2 \rangle = 0 \Rightarrow$

$$\underset{\overset{\|}{0}}{\langle e_1, e_2 \rangle} = \langle f_2 + \alpha e_1, e_1 \rangle = \langle f_2, e_1 \rangle + \alpha \langle e_1, e_1 \rangle$$

$$\Rightarrow \alpha = -\frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle}$$

$$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1$$

$$\begin{cases} f_1 = e_1 \\ f_2 = \dfrac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + e_2 \end{cases} \Rightarrow Sp\{e_1, e_2\} = Sp\{f_1, f_2\}$$

Pp. adev P_k: $\{e_1, \dots, e_k\}$ sist vect mutual ortog și

$$Sp\{e_1, \dots, e_i\} = Sp\{f_1, \dots, f_i\}, \forall i = \overline{1,k}$$

Dem P_{k+1}

Construim

$$e_{k+1} = f_{k+1} + \sum_{i=1}^{k} \alpha_{k+1\, i} e_i \qquad \langle \cdot, e_j \rangle, \ j = \overline{1,k}$$

$$\underset{\overset{\|}{0}}{\langle e_{k+1}, e_j \rangle} = \langle f_{k+1}, e_j \rangle + \underbrace{\sum_{i=1}^{k} \alpha_{k+1\, i} \langle e_i, e_j \rangle}_{\alpha_{k+1\, j} \langle e_j, e_j \rangle}$$

$$\alpha_{k+1\, j} = -\frac{\langle f_{k+1}, e_j \rangle}{\langle e_j, e_j \rangle}$$

$$e_{k+1} = f_{k+1} - \sum_{i=1}^{k} \frac{\langle f_{k+1}, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$

$$\begin{cases} f_1 = e_1 \\ f_2 = \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + e_2 \\ \vdots \\ f_{k+1} = \sum_{i=1}^{k} \frac{\langle f_{k+1}, e_i \rangle}{\langle e_i, e_i \rangle} e_i + e_{k+1} \end{cases} \Rightarrow Sp\{e_1, .., e_i\} = Sp\{f_1, .., f_i\} \quad \forall i = \overline{1, k+1}$$

Se continuă raționamentul

$$\begin{cases} f_1 = e_1 \\ f_2 = \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + e_2 \\ \vdots \\ f_n = \frac{\langle f_n, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \dots + \frac{\langle f_n, e_{n-1} \rangle}{\langle e_{n-1}, e_{n-1} \rangle} e_{n-1} + e_n \end{cases}$$

$Sp\{e_1, \dots e_i\} = Sp\{f_1, .., f_i\}, \forall i = \overline{1, n}$

$\left.\begin{array}{l} \{e_1, .., e_n\} \text{ mutual ortog} \Rightarrow \mathcal{R}' = \{e_1, .., e_n\} \text{ SLI} \\ \text{dar } \dim E = |\mathcal{R}'| = n \end{array}\right\}$

$\Rightarrow \mathcal{R}'$ reper în E

OBS $\mathcal{R} = \{f_1, .., f_n\} \xrightarrow{A} \mathcal{R}' = \{e_1, .., e_n\} \xrightarrow{B} \mathcal{R}'' = \left\{\frac{e_1}{\|e_1\|}, .., \frac{e_n}{\|e_n\|}\right\}$

reper arbitrar — reper ortogonal — reper ortonormat.

$$A^{-1} = \begin{pmatrix} 1 & \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} & \dots & \frac{\langle f_n, e_1 \rangle}{\langle e_1, e_1 \rangle} \\ 0 & 1 & & \vdots \\ \vdots & \vdots & & \frac{\langle f_n, e_{n-1} \rangle}{\langle e_{n-1}, e_{n-1} \rangle} \\ 0 & 0 & & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{\|e_1\|} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\|e_n\|} \end{pmatrix}$$

$\det(A^{-1}) = \frac{1}{\det A} = 1 \Rightarrow \det A = 1 > 0$; $\det B = \frac{1}{\|e_1\|} \cdots \frac{1}{\|e_n\|} > 0$

$\mathcal{R}, \mathcal{R}', \mathcal{R}''$ sunt repere la fel orientate.

Def $(E, \langle , \rangle)$ s.v.e.r.

a) $x \in E$, $\langle \{x\} \rangle^{\perp} = \{ y \in E \mid \langle x, y \rangle = 0 \}$
subspațiu ortogonal pe x

b) $U \subseteq E$ ssp vect $\Rightarrow U^{\perp} = \{ x \in E \mid \langle x, y \rangle = 0, \forall y \in U \}$
subsp. ortogonal pe U

OBS

$U \subseteq W \subseteq E \Rightarrow W^{\perp} \subseteq U^{\perp} \subseteq E$.

Exercițiu

$(\mathbb{R}^3, g_0)$, $u = (1, 2, -1)$

a) $\langle \{u\} \rangle^{\perp}$; b) Det. un reper ortonormat în $\langle \{u\} \rangle^{\perp}$
(Gram - Schmidt)

SOL

a) $V = \langle \{u\} \rangle^{\perp} = \{ x \in \mathbb{R}^3 \mid g_0(x, u) = 0 \}$
$= \{ x \in \mathbb{R}^3 \mid x_1 + 2x_2 - x_3 = 0 \}$

$\dim V = 3 - 1 = 2$

$V = \{ (x_1, x_2, x_1 + 2x_2) \mid x_1, x_2 \in \mathbb{R} \}$
$\quad = x_1(1,0,1) + x_2(0,1,2)$

$\mathcal{R} = \{ f_1 = (1,0,1), f_2 = (0,1,2) \}$ SG pt V, $\dim V = |\mathcal{R}| = 2$ $\Rightarrow \mathcal{R}$ reper în V

Aplicăm Gram - Schmidt

$e_1 = f_1 = (1,0,1)$

$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 = (0,1,2) - \frac{2}{2}(1,0,1) =$
$= (0,1,2) - (1,0,1) = (-1,1,1)$

$\langle f_2, e_1 \rangle = \langle f_2, f_1 \rangle = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 2 = 2$

$\langle e_1, e_1 \rangle = \langle f_1, f_1 \rangle = 1^2 + 0^2 + 1^2 = 2$

$\mathcal{R}=\{f_1,f_2\} \to \mathcal{R}'=\{e_1,e_2\} \to \mathcal{R}''=\left\{\frac{e_1}{\|e_1\|}, \frac{e_2}{\|e_2\|}\right\}$.

$e_1=(1,0,1) \Rightarrow \|e_1\|=\sqrt{1^2+0^2+1^2}=\sqrt{2}$

$e_2=(-1,1,1) \Rightarrow \|e_2\|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$

$\mathcal{R}''=\left\{\frac{1}{\sqrt{2}}(1,0,1), \frac{1}{\sqrt{3}}(-1,1,1)\right\}$ reper ortonormat în $\langle\{u\}\rangle^{\perp}$

$\boxed{\text{OBS}}$ A da un produs scalar $\Leftrightarrow$ a declara un reper ortonormat

- $g: V\times V \to \mathbb{R}$ produs scalar,
  $\mathcal{R}=\{e_1,..,e_n\}$ reper ortonormat $\Leftrightarrow g(e_i,e_j)=\delta_{ij}, \forall i,j=\overline{1,n}$
- Dacă $\mathcal{R}=\{e_1,..,e_n\}$ reper ortonormat
  Construim $g: V\times V \to \mathbb{R}$ p. scalar a.î. $g(e_i,e_j)=\delta_{ij}$
  Prelungim prin liniaritate

$$g(x,y)=g\Big(\sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{n} y_j e_j\Big)=\sum_{i,j=1}^{n} x_i y_j \underbrace{g(e_i,e_j)}_{\delta_{ij}}$$
$$=\sum_{i=1}^{n} x_i y_i$$