C5 - GA

Subspații vectoriale
Morfisme de spații vectoriale

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$(V,+,\cdot)_{/\mathbb{K}}$ sp. vect. finit generat, $V_1, V_2 \subset V$ subspații vect.

$\Rightarrow \dim_{\mathbb{K}}(V_1+V_2) = \dim_{\mathbb{K}} V_1 + \dim_{\mathbb{K}} V_2 - \dim_{\mathbb{K}}(V_1 \cap V_2)$

Dem Not $\dim_{\mathbb{K}} V = n$, $\dim_{\mathbb{K}} V_i = n_i$, $i=\overline{1,2}$, $\dim_{\mathbb{K}}(V_1\cap V_2)=p$

Fie $\mathcal{R}_0=\{e_1,\dots,e_p\}$ reper în $V_1\cap V_2$.

Extindem la $\{e_1,\dots,e_p, f_{p+1},\dots,f_{n_1}\}$ reper în V_1

—"— $\{e_1,\dots,e_p, g_{p+1},\dots,g_{n_2}\}$ —"— V_2.

Dem că $\mathcal{R}=\{e_1,\dots,e_p, f_{p+1},\dots,f_{n_1}, g_{p+1},\dots,g_{n_2}\}$ reper în $V_1+V_2 = \langle V_1\cup V_2\rangle$.

- Dem că $\mathcal{R}$ este SLI.

$\forall a_1,\dots,a_p, b_{p+1},\dots,b_{n_1}, c_{p+1},\dots,c_{n_2}\in\mathbb{K}$ a.î.

$$\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j + \sum_{k=p+1}^{n_2} c_k g_k = 0_V \Rightarrow$$

$$\underbrace{\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j}_{\in V_1} = \underbrace{-\sum_{k=p+1}^{n_2} c_k g_k}_{\in V_2} \in V_1\cap V_2 = \langle \mathcal{R}_0\rangle$$

$\exists a_1',\dots,a_p'\in\mathbb{K}$ a.î. $\displaystyle\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j = -\sum_{k=p+1}^{n_2} c_k g_k = \sum_{i=1}^{p} a_i' e_i$

$$\begin{cases} \displaystyle\sum_{i=1}^{p}(a_i-a_i')e_i + \sum_{j=p+1}^{n_1} b_j f_j = 0_V \xRightarrow{\text{reper în } V_1} \begin{cases} a_i - a_i' = 0, \forall i=\overline{1,p} \\ b_j = 0, \forall j=\overline{p+1,n_1}\end{cases} \\ \displaystyle\sum_{i=1}^{p} a_i' e_i + \sum_{k=p+1}^{n_2} c_k g_k = 0_V \xRightarrow[\text{reper în } V_2]{} \begin{cases} a_i' = 0, \forall i=\overline{1,p} \\ c_k = 0, \forall k=\overline{p+1,n_2}\end{cases}\end{cases}$$

$\Rightarrow a_i=0, \forall i=\overline{1,p}$, $b_j=0, \forall j=\overline{p+1,n_1}$, $c_k=0, \forall k=\overline{p+1,n_2} \Rightarrow$ SLI

- Dem că $\mathcal{R}$ este SG pt V_1+V_2 i.e $V_1+V_2=\langle\mathcal{R}\rangle$

$\forall x\in V_1+V_2 \Rightarrow \exists\, x_1\in V_1,\ x_2\in V_2$ aî $x=x_1+x_2$

$$\Rightarrow x=\Big(\sum_{i=1}^{p}a_ie_i+\sum_{j=p+1}^{n_1}b_jf_j\Big)+\Big(\sum_{i=1}^{p}a'_ie_i+\sum_{k=p+1}^{n_2}c_kg_k\Big)$$

$$=\sum_{i=1}^{p}\underbrace{(a_i+a'_i)}_{a''_i}e_i+\sum_{j=p+1}^{n_1}b_jf_j+\sum_{k=p+1}^{n_2}c_kg_k$$

$\Rightarrow V_1+V_2\subset\langle\mathcal{R}\rangle$

dar $\langle\mathcal{R}\rangle\subset V_1+V_2$ (din constr.)

$\Big\}\Rightarrow V_1+V_2=\langle\mathcal{R}\rangle$

În concluzie $\mathcal{R}$ reper în V_1+V_2

$$\mathcal{R}=\{e_1,\dots,e_p,f_{p+1},\dots,f_{n_1},g_{p+1},\dots,g_{n_2}\}$$

$$\dim_{\mathbb{K}}(V_1+V_2)=\cancel{p}+(n_1-\cancel{p})+(n_2-p)=n_1+n_2-p$$
$$=\dim_{\mathbb{K}}V_1+\dim_{\mathbb{K}}V_2-\dim_{\mathbb{K}}(V_1\cap V_2).$$

$\boxed{\text{OBS}}$ $\dim_{\mathbb{K}}(V_1\oplus V_2)=\dim_{\mathbb{K}}V_1+\dim_{\mathbb{K}}V_2$.

$V_1\cap V_2=\{0_V\}$, $\dim_{\mathbb{K}}\{0_V\}=0$

<u>Teoremă</u> $A\in\mathcal{M}_{m,n}(\mathbb{K})$

$S(A)=\{x\in\mathbb{K}^n \mid \underset{(m,n)}{A}\underset{(n,1)}{X}=\underset{(m,1)}{0}\}\subset\mathbb{K}^n$ subsp vect

$\dim_{\mathbb{K}}S(A)=n-rg(A)$

<u>Prop</u> $(V,+,\cdot)/\mathbb{K}$ sp. vect finit generat, $V'\subset V$ subsp vect.
Coordonatele vect din V', în raport cu $\forall$ reper, sunt soluțiile unui SLO
i.e $\exists A$ aî $V'=S(A)$.

Aplicatii

Ex1 $(\mathbb{R}^4,+,\cdot)/\mathbb{R}$, $V' = \langle \{(1,1,0,0), (1,0,1,-1)\} \rangle$

a) Să se descrie V' printr-un sistem de ec. liniare.

b) $\mathbb{R}^4 = V' \oplus V''$, $V'' = ?$ (subsp. vectorial complementar lui V')

SOL

$$rg\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} = 2 = max \xRightarrow{CLI} R' = \{(1,1,0,0), (1,0,1,-1)\} \text{ SLI}$$

R' reper în V'; $V' = \langle R \rangle$

$\forall x = (x_1, x_2, x_3, x_4) \in V'$, $\underline{\exists a, b \in \mathbb{R}}$ aî $x = a(1,1,0,0) + b(1,0,1,-1)$

$x = (a+b, a, b, -b)$

$$\begin{cases} a+b = x_1 \\ a = x_2 \\ b = x_3 \\ -b = x_4. \end{cases} \quad A = \left(\begin{array}{cc|c} \boxed{1 \quad 1} & & x_1 \\ \boxed{1 \quad 0} & & x_2 \\ 0 & 1 & x_3 \\ 0 & -1 & x_4 \end{array}\right) \quad \Delta_p = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0$$

SC $\Rightarrow rg A = rg \bar{A} = 2$

$$\begin{cases} \Delta_{c_1} = \begin{vmatrix} 1 & 1 & x_1 \\ 1 & 0 & x_2 \\ 0 & 1 & x_3 \end{vmatrix} = 0 \\ \Delta_{c_2} = \begin{vmatrix} 1 & 1 & x_1 \\ 1 & 0 & x_2 \\ 0 & -1 & x_4 \end{vmatrix} = 0. \end{cases}$$

$$\Delta_{c_1} = \begin{vmatrix} 1 & 1 & x_1 \\ 0 & -1 & x_2 - x_1 \\ 0 & 1 & x_3 \end{vmatrix} = 0 \Rightarrow -x_2 + x_1 - x_3 = 0$$

$$\Delta_{c_2} = \begin{vmatrix} 1 & 1 & x_1 \\ 0 & -1 & x_2 - x_1 \\ 0 & -1 & x_4 \end{vmatrix} = 0 \Rightarrow -x_1 + x_2 - x_4 = 0$$

$$V' = \left\{ x \in \mathbb{R}^4 \;\middle|\; \begin{cases} x_1 - x_2 - x_3 = 0 \\ -x_1 + x_2 - x_4 = 0 \end{cases} \right\} \quad A' = \begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \end{pmatrix}$$

$V' = S(A')$

b) $$det\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = -\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{vmatrix} = 0 \text{ (nu convine)}$$

$$det\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \neq 0 \quad V'' = \langle \overbrace{\{e_2, e_4\}}^{R''} \rangle$$

R'' e SLI

$R'' \subset R_0$

$\dim(V' \oplus V'') = \underbrace{\dim V'}_{2} + \underbrace{\dim V''}_{2} - \underbrace{\dim(V' \cap V'')}_{0}$

$\mathbb{R}^4 = V' \oplus V''$

$R' \cup R''$ reper în $\mathbb{R}^4 \Rightarrow \mathbb{R}^4 = V' \oplus V''$

$V' \cap V'' = \{0_{\mathbb{R}^4}\}$.

EX2 $(\mathbb{R}^4,+,\cdot)_{/\mathbb{R}}$, $V' = \{(x,y,z,t) \in \mathbb{R}^4 \mid x+y-z-3t=0\}$

$V'' = \{(x,y,z,t) \in \mathbb{R}^4 \mid x+y+z+2t=0\}$

$\mathbb{R}^4 = V'+V''$, dar suma nu e directă.

$\dim V' = 4 - rg(1 \ 1 \ -1 \ -3) = 4-1=3$

$\dim V'' = 4 - rg(1 \ 1 \ 1 \ 2) = 4-1=3.$

$V' \cap V'' = \{(x,y,z,t) \in \mathbb{R}^4 \mid \begin{cases} x+y-z-3t=0 \\ x+y+z+2t=0 \end{cases}\} = S(A)$

$A = \begin{pmatrix} 1 & 1 & -1 & -3 \\ 1 & 1 & 1 & 2 \end{pmatrix}$ $\qquad V' \cap V'' \neq \{0_{\mathbb{R}^4}\}.$

$\dim V' \cap V'' = 4-2=2.$

$\dim(V'+V'') \underset{TG}{=} 3+3-2=4.$

$\left.\begin{array}{l} V'+V'' \subset \mathbb{R}^4 \text{ subsp vect} \\ \dim(V'+V'') = \dim \mathbb{R}^4 = 4 \end{array}\right| \Rightarrow V'+V'' = \mathbb{R}^4.$

Morfisme de spații vectoriale

$(V_i,+,\cdot)_{/\mathbb{K}}$, $i=\overline{1,2}$ spații vectoriale

$f: V_1 \to V_2$ s.n. morfism de sp vect sau aplicație liniară

$\Leftrightarrow$ 1) $f(x+y) = f(x)+f(y)$

2) $f(\alpha x) = \alpha f(x)$, $\forall x,y \in V_1$, $\forall \alpha \in \mathbb{K}$.

OBS a) $V_1 \xrightarrow{g} V_2 \xrightarrow{h} V_3$ $\quad g, h$ aplicații liniare $\Rightarrow$

$\underbrace{\qquad\qquad}_{f}$ $\qquad f = h \circ g$ apl. liniară.

b) $f: V_1 \to V_2$ aplicație liniară $\Rightarrow$

$f: (V_1,+) \to (V_2,+)$ morfism de grupuri $\Rightarrow f(0_{V_1}) = 0_{V_2}$

Def $f: V_1 \to V_2$ s.n. izomorfism de sp vect $\Leftrightarrow$ 1) f liniară
2) f bij.

Not $(V,+,\cdot)_{/\mathbb{K}}$ sp vect, $End(V) = \{f: V \to V \mid f \text{ liniară}\}$

$Aut(V) = \{f \in End(V) \mid f \text{ bij}\}.$

Exemple

1) $f: V \to V$, $f(x) = 0_V$, $f(x) = x$ apl. lin.

2) $f: \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = y$, $Y = AX$ apl. lin.

$$\begin{pmatrix} y_1 \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_n \end{pmatrix}$$

3) $f: \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$, $f(A) = Tr(A)$

$f(A+B) = f(A) + f(B)$, $f(\alpha A) = \alpha f(A)$

OBS $f(A) = \det(A)$ nu este aplicatie liniara.

Prop $f: V_1 \to V_2$ apl. liniara $\Leftrightarrow$ $f(ax+by) = a f(x) + b f(y)$, $\forall x, y \in V_1$, $\forall a, b \in \mathbb{K}$

$\left[\Leftrightarrow f\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i f(x_i)\right.$ $\forall x_1, \dots, x_n \in V_1$, $\forall a_1, \dots, a_n \in \mathbb{K}$ $\left.\right]$

Dem

$\Rightarrow$ Ip: f liniara

$x \in V_1$, $a \in \mathbb{K}$ $\Rightarrow ax \in V_1$

$y \in V_1$, $b \in \mathbb{K}$ $\Rightarrow by \in V_1$.

$f(ax+by) = f(ax) + f(by) = a f(x) + b f(y)$.

"$\Leftarrow$" $f(ax+by) = a f(x) + b f(y)$, $\forall x, y \in V_1$, $\forall a, b \in \mathbb{K}$

Fie $a = b = 1_{\mathbb{K}}$

$f(1_{\mathbb{K}} x + 1_{\mathbb{K}} y) = f(x+y) = 1_{\mathbb{K}} f(x) + 1_{\mathbb{K}} f(y) = f(x) + f(y)$

Fie $b = 0_{\mathbb{K}}$ $\Rightarrow f(ax) = a f(x)$

$\Rightarrow f$ liniara.

$\boxed{\text{OBS}}$ $f: V_1 \to V_2$ liniara

$V' \subseteq V_1$ subsp. vect $\Rightarrow f(V') \subseteq V_2$ subsp. vect

$\forall y_1, y_2 \in f(V')$, $\forall a, b \in \mathbb{K}$ $\overset{?}{\Rightarrow} a y_1 + b y_2 \in f(V')$

$\exists x_1, x_2 \in V'$ ai $y_1 = f(x_1)$, $y_2 = f(x_2)$

$a y_1 + b y_2 = a f(x_1) + b f(x_2) \underset{f\,lin}{=} f(a x_1 + b x_2) = f(\underset{V'}{\underset{\cap}{x}}) \in f(V')$

Def $f: V_1 \to V_2$ aplicație liniară.

$\mathrm{Ker} f = \{x \in V_1 \mid f(x) = 0_{V_2}\} \subseteq V_1$ sup vect (nucleul lui f)

$\mathrm{Im} f = \{y \in V_2 \mid \exists x \in V_1$ aî $f(x) = y\} \subseteq V_2$ supvect (imaginea lui f)

Prop $f: V_1 \to V_2$ apl. liniară

a) f injectivă $\Leftrightarrow$ $\mathrm{Ker} f = \{0_{V_1}\}$

2) f surjectivă $\Leftrightarrow$ $\dim_{\mathbb{K}} \mathrm{Im} f = \dim_{\mathbb{K}} V_2$.

Dem

a) "$\Rightarrow$" f injectivă. Dem că $\mathrm{Ker} f = \{0_{V_1}\}$

Fie $x \in \mathrm{Ker} f \Rightarrow f(x) = 0_{V_2}$, dar $f(0_{V_1}) = 0_{V_2}$ $\Big\} \Rightarrow f(x) = f(0_{V_1})$, dar f inj $\Big\} \Rightarrow x = 0_{V_1}$

$\mathrm{Ker} f = \{0_{V_1}\}$

"$\Leftarrow$" $\mathrm{Ker} f = \{0_{V_1}\}$. Dem că f inj.

Fie $x_1, x_2 \in V_1$ aî $f(x_1) = f(x_2) \Rightarrow \underbrace{f(x_1) - f(x_2)}_{= f(x_1 - x_2)} = 0_{V_2} \Rightarrow$

$x_1 - x_2 \in \mathrm{Ker} f = \{0_{V_1}\} \Rightarrow x_1 = x_2 \Rightarrow f$ inj.

b) "$\Rightarrow$" f surj $\Rightarrow \mathrm{Im} f = V_2 \Rightarrow \dim_{\mathbb{K}} \mathrm{Im} f = \dim_{\mathbb{K}} V_2$

"$\Leftarrow$" $\dim_{\mathbb{K}} \mathrm{Im} f = \dim_{\mathbb{K}} V_2$, dar $\underbrace{\mathrm{Im} f}_{= f(V_1)} \subseteq V_2$ subsp v. $\Big| \Rightarrow \mathrm{Im} f = V_2 \Rightarrow f$ surj.

Consecință

$f: V_1 \to V_2$ lin.

f izomorfism $\Leftrightarrow$ 1) $\mathrm{Ker} f = \{0_{V_1}\}$; 2) $\dim_{\mathbb{K}} \mathrm{Im} f = \dim_{\mathbb{K}} V_2$.

Teorema dimensiunii

$f: V_1 \to V_2$ apl. liniară

$\dim_{\mathbb{K}} V_1 = \dim_{\mathbb{K}} \mathrm{Ker} f + \dim_{\mathbb{K}} \mathrm{Im} f$.

Aplicații

Ex1. $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (x_1 + x_2 - x_3, x_1 + x_2, x_1 + x_2 + x_3)$.

a) f liniară; b) $\mathrm{Ker} f$, $\mathrm{Im} f = ?$ Precizați câte un reper în

fiecare

Scanned with CamScanner

SOL

a) $f(x+y) = (x_1+y_1+x_2+y_2-(x_3+y_3),\ x_1+y_1+x_2+y_2,\ x_1+y_1+x_2+y_2+x_3+y_3)$

$= (x_1+x_2-x_3,\ x_1+x_2,\ x_1+x_2+x_3) + (y_1+y_2-y_3,\ y_1+y_2,\ y_1+y_2+y_3)$

$= f(x)+f(y)$

$f(\alpha x) = (\alpha x_1+\alpha x_2-\alpha x_3,\ \alpha x_1+\alpha x_2,\ \alpha x_1+\alpha x_2+\alpha x_3) = \alpha f(x).$

b) $\operatorname{Ker} f = \{x\in\mathbb{R}^3 \mid f(x)=0_{\mathbb{R}^3}\} = \left\{x\in\mathbb{R}^3 \,\middle|\, \begin{cases} x_1+x_2-x_3=0 \\ x_1+x_2 = 0 \\ x_1+x_2+x_3=0 \end{cases}\right\} = S(A)$

$A = \begin{pmatrix} 1 & \boxed{1 & -1} \\ 1 & \boxed{1 & 0} \\ 1 & 1 & 1 \end{pmatrix}$ $\quad \Delta_p = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}$

$\det A = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{vmatrix} = 0$; $\dim \operatorname{Ker} f = 3 - rg A = 3-2=1.$

$\begin{cases} x_2 - x_3 = -x_1 \Rightarrow x_3 = 0 \\ x_2 = -x_1 \end{cases}$

$\operatorname{Ker} f = \{(x_1,-x_1,0) \mid x_1\in\mathbb{R}\} = \langle \underbrace{\{(1,-1,0)\}}_{R_1} \rangle$ $\quad R_1$ reper în $\operatorname{Ker} f$.

$\operatorname{Im} f = \{y\in\mathbb{R}^3 \mid \exists \underline{x}\in\mathbb{R}^3$ a.î. $f(x)=y\}$

$\begin{cases} x_1+x_2-x_3 = y_1 \\ x_1+x_2 = y_2 \\ x_1+x_2+x_3 = y_3 \end{cases}$ $\quad A = \left(\begin{array}{ccc|c} 1 & \boxed{1 & -1} & y_1 \\ 1 & \boxed{1 & 0} & y_2 \\ 1 & 1 & 1 & y_3 \end{array}\right)$

$\Delta_c = \begin{vmatrix} 1 & -1 & y_1 \\ 1 & 0 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & -1 & y_1 \\ 1 & 0 & y_2 \\ 2 & 0 & y_1+y_3 \end{vmatrix} = 0$

$\operatorname{Im} f = \{y\in\mathbb{R}^3 \mid y_1 - 2y_2 + y_3 = 0\}$ $\quad \dim \operatorname{Im} f = 3-1 = 2$

$y_1 = 2y_2 - y_3$

$\operatorname{Im} f = \{(2y_2-y_3,\ y_2,\ y_3) \mid y_2,y_3\in\mathbb{R}\} = \langle \overbrace{\{(2,1,0),(-1,0,1)\}}^{R_2} \rangle$

$\left.\begin{array}{l} R_2 \text{ e SG} \\ |R_2| = \dim \operatorname{Im} f = 2 \end{array}\right\} \Rightarrow R_2$ este reper în $\operatorname{Im} f$.

<u>Metoda 2</u> $\quad R_1 = \{(1,-1,0)\}$ reper în $\operatorname{Ker} f$

Extindem R_1 la un reper în $\mathbb{R}^3$, $R_1 \cup \underline{\{e_1, e_3\}}$

$\det\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq 0$ $\quad \{\underset{(1,1,1)}{f(e_1)},\ \underset{(-1,0,1)}{f(e_3)}\}$ reper în $\operatorname{Im} f$

EX2 $f : \mathbb{R}_4[X] \longrightarrow \mathbb{R}_2[X]$, $f(P) = P''$, $\tilde{P} = P$

$P = a_0 + a_1 X + a_2 X^2 + a_3 X^3 + a_4 X^4$. , $P'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$

$f(P) = P''$

$P''(x) = 2a_2 + 6a_3 x + 12a_4 x^2$

$f(aP + bQ) = (aP + bQ)'' = aP'' + bQ''$

$= a f(P) + b f(Q) \Rightarrow f$ lin.

$\operatorname{Ker} f = \{P \in \mathbb{R}_4[X] \mid f(P) = 0\} = \{a_0 + a_1 X ,\ a_0, a_1 \in \mathbb{R}\}$

$= \mathbb{R}_1[X]$

$\underbrace{\dim \mathbb{R}_4[X]}_{5} = \underbrace{\dim \operatorname{Ker} f}_{2} + \dim \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 3$.

$\left.\begin{array}{l} \operatorname{Im} f \subseteq \mathbb{R}_2[X] \\ \dim \operatorname{Im} f = \dim \mathbb{R}_2[X] = 3 \end{array}\right\} \Rightarrow \operatorname{Im} f = \mathbb{R}_2[X] \Rightarrow f$ surj.