Cuadrice studiate pe ecuații reduse

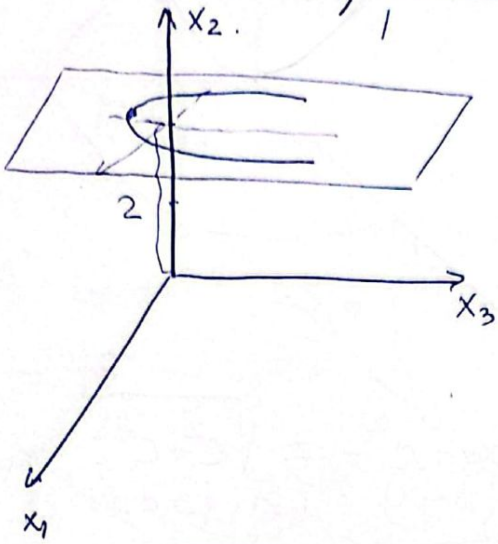
Ex. Să se determine intersecția dintre paraboloidul hiperbolic $\mathcal{P}_h : \frac{x_1^2}{6} - \frac{x_2^2}{4} = 3x_3$

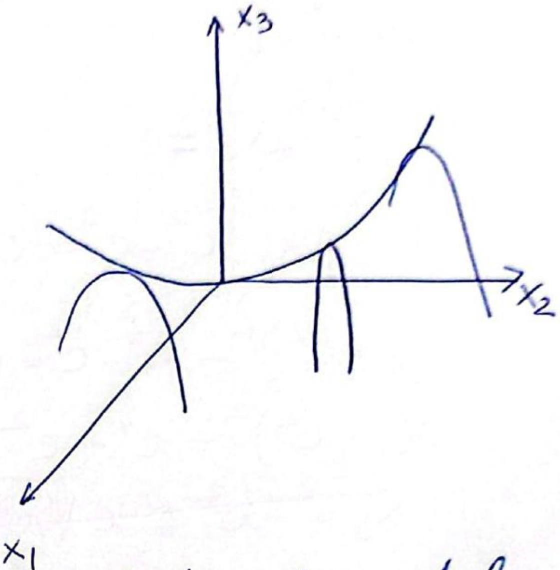
și planul $\pi : x_2 = 2$.

SOL

$$\frac{x_1^2}{6} - \frac{4}{4} = 3x_3 \Rightarrow \frac{x_1^2}{6} = 3x_3 + 1 \Rightarrow$$

$$x_1^2 = 18\left(x_3 + \frac{1}{3}\right) \quad \text{Parabolă.}$$

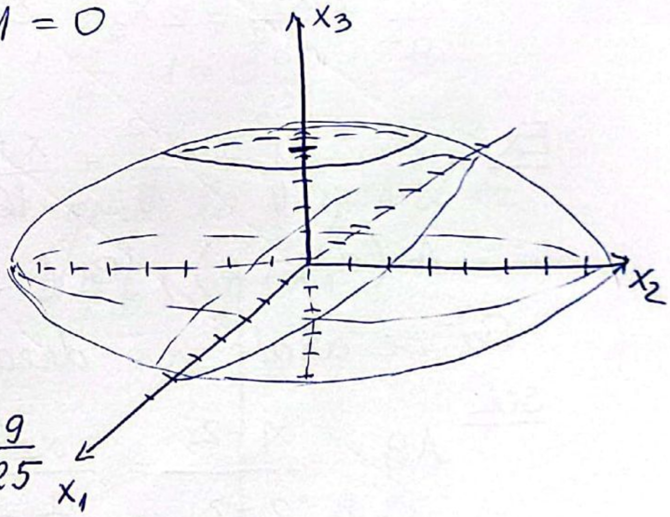




<u>Ex</u>. Să se determine intersecția dintre elipsoidul

$$\mathcal{E}: \frac{x_1^2}{64} + \frac{x_2^2}{49} + \frac{x_3^2}{25} - 1 = 0$$

și planul $\pi: x_3 = 4$.



<u>SOL</u>

$$\mathcal{E}: \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$$

$a = 8$, $b = 7$, $c = 5$.

$$\mathcal{E} \cap \pi: \frac{x_1^2}{64} + \frac{x_2^2}{49} = 1 - \frac{16}{25} = \frac{9}{25}$$

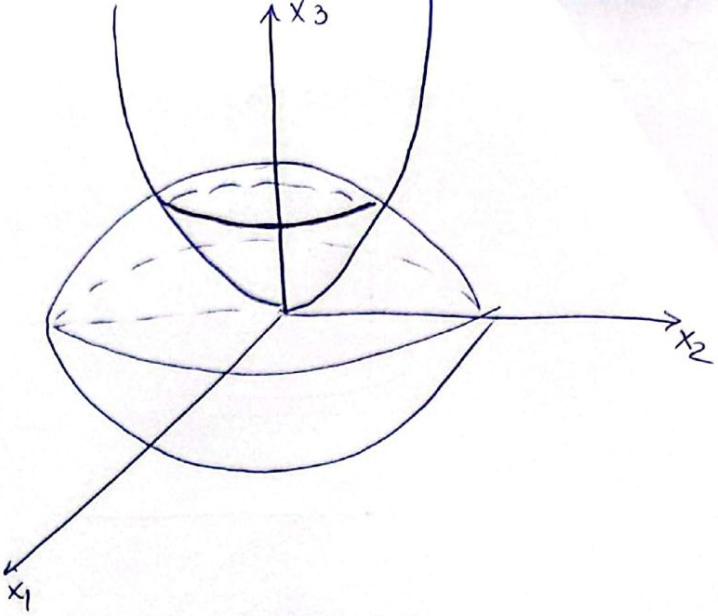
Elipsa: $$\frac{x_1^2}{\frac{64 \cdot 25}{9}} + \frac{x_2^2}{\frac{49 \cdot 25}{9}} = 1 \quad (\text{în planul } x_3 = 4).$$

<u>Ex</u> Să se determine intersecția dintre

Elipsoidul: $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$ și

Paraboloidul eliptic: $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 2x_3$.

$$\begin{cases} \dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} + \dfrac{x_3^2}{c^2} = 1 \\ \dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} = 2x_3 \end{cases}$$



$$\frac{x_3^2}{c^2} + 2x_3 = 1$$

$$x_3^2 + 2 \cdot x_3 \cdot c^2 - c^2 = 0.$$

$$(x_3 + c^2)^2 - c^4 - c^2 = 0$$

$$(x_3 + c^2)^2 = c^4 + c^2 \Rightarrow x_3 + c^2 = \pm \sqrt{c^2 + c^4}.$$

$$x_3 = -c^2 \pm c\sqrt{c^2+1}.$$

$$x_3 \in (-c, c). \; ; \; x_3 = -c^2 + c\sqrt{c^2+1}$$

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 2x_3$$ Elipsă.

<u>Ex</u>. $\mathcal{E} : \dfrac{x_1^2}{4} + \dfrac{x_2^2}{9} + \dfrac{x_3^2}{16} - 1 = 0.$

$A(2, 3, 6)$, $B\left(2, \frac{1}{2}, 1\right)$.

Să se arate că dreapta AB este tg la elipsoid.

<u>Sol</u>

$$AB : \frac{x_1 - 2}{2-2} = \frac{x_2 - 3}{\frac{1}{2} - 3} = \frac{x_3 - 6}{1 - 6} = t$$

$$AB : \begin{cases} x_1 = 2 \\ x_2 = 3 - \frac{5}{2}t \\ x_3 = 6 - 5t. \end{cases}$$ ec. parametrice.

$$\mathcal{E} \cap AB : \cancel{\frac{4}{4}} + \frac{\left(3 - \frac{5}{2}t\right)^2}{9} + \frac{(6-5t)^2}{16} \cancel{-1} = 0$$

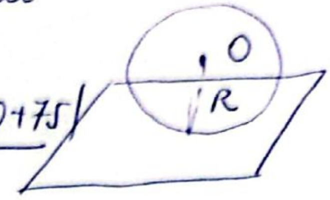
$$\begin{cases} 3 - \frac{5}{2}t = 0 \\ 6 - 5t = 0 \end{cases} \Rightarrow t = \frac{6}{5}$$

$$AB \cap \mathcal{E} = \{M\}, \quad M\left(2, 3 - \frac{5}{2} \cdot \frac{6}{5}, 6 - 5 \cdot \frac{6}{5}\right)$$

AB tg la $\mathcal{E}$. $\quad M(2, 0, 0)$

EX $\mathcal{S}(O(0,0,0), R)$ care este tangentă la planul $\pi : 16x_1 - 15x_2 - 12x_3 + 75 = 0$.
Să se determine ecuația sferei

SOL



$$R = dist(O, \pi) = \frac{|16\cdot 0 - 15\cdot 0 - 12\cdot 0 + 75|}{\sqrt{16^2 + 15^2 + 12^2}}$$

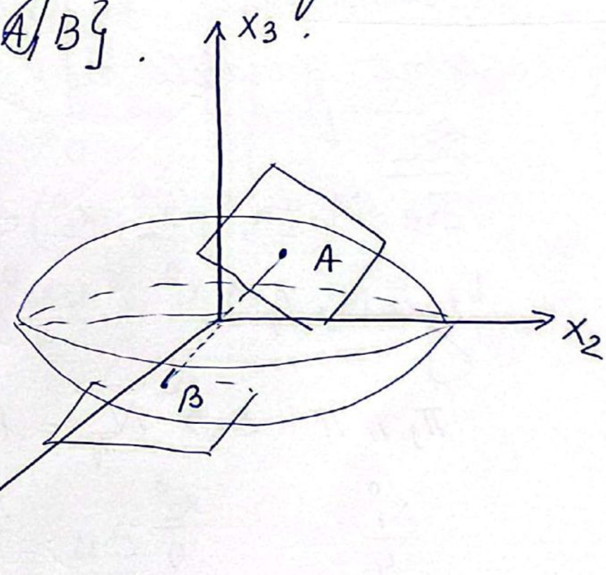
$$16^2 + 15^2 + 12^2 = 256 + 225 + 144 = 625 = 25^2$$

$$R = \frac{75}{25} = 3.$$

$$\mathcal{S}(A(a,b,c), R) : (x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 = R^2$$

$$\mathcal{S}(O(0,0,0), 3) : x_1^2 + x_2^2 + x_3^2 = 9.$$

Ex. Fie elipsoidul : $\frac{x_1^2}{4} + \frac{x_2^2}{3} + \frac{x_3^2}{9} - 1 = 0$ și dreapta $d : x_1 = x_2 = x_3$.
Să se scrie ec. planelor tangente la elipsoid în A și B, unde $\mathcal{E} \cap d = \{A, B\}$.



SOL

$$d : \begin{cases} x_1 = t \\ x_2 = t \\ x_3 = t. \end{cases}$$

$d \cap \mathcal{E}$: 9 12 4

$$t^2\left(\frac{1}{4} + \frac{1}{3} + \frac{1}{9}\right) = 1.$$

$$\frac{t^2 \cdot 25}{36} = 1 \Rightarrow t = \pm\frac{6}{5}.$$

$$A\left(\frac{6}{5}, \frac{6}{5}, \frac{6}{5}\right), \quad B\left(-\frac{6}{5}, -\frac{6}{5}, -\frac{6}{5}\right)$$

$\boxed{\text{OBS}}$ Ec. planului tg. în $M_0(x_1^0, x_2^0, x_3^0)$ la $\mathcal{E}$

$$\mathcal{E} : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 \Rightarrow \pi_{tg} : \frac{x_1 \cdot x_1^0}{a^2} + \frac{x_2 \cdot x_2^0}{b^2} + \frac{x_3 \cdot x_3^0}{c^2} = 1$$

Ec. Π tg în A: $\frac{x_1}{\cancel{4}_2}\cdot\frac{\cancel{6}^{3}}{5} + \frac{x_2}{\cancel{3}}\cdot\frac{\cancel{6}^{2}}{5} + \frac{x_3}{\cancel{9}_3}\cdot\frac{\cancel{6}^{2}}{5} - 1 = 0.$

$$\frac{3x_1}{10} + \frac{2x_2}{5} + \frac{2x_3}{15} - 1 = 0.$$

Ec Π tg în B: $\frac{3x_1}{10} + \frac{2x_2}{5} + \frac{2x_3}{15} + 1 = 0$

Ex Fie paraboloidul eliptic

$$\mathcal{P}_e : \frac{x_1^2}{4} + \frac{x_2^2}{9} = 2x_3$$

Să se scrie ec. planului tangent în $M_0(2,3,1)$

Sol

$M_0 \in \mathcal{P}_e : \frac{4}{4} + \frac{9}{9} = 2\cdot 1$ (A)

Ec planului tg în M_0: $\frac{x_1\cdot\cancel{2}}{\cancel{4}_2} + \frac{x_2\cdot\cancel{3}}{\cancel{9}_3} = x_3 + 1$

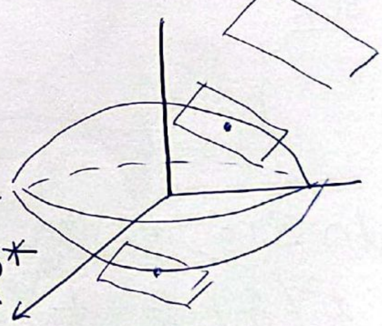
$$\Pi : \frac{x_1}{2} + \frac{x_2}{3} - x_3 - 1 = 0.$$

Ex. Fie elipsoidul $\mathcal{E} : \frac{x_1^2}{4} + \frac{x_2^2}{9} + \frac{x_3^2}{8} - 1 = 0$

Să se scrie ec. planelor tangente la $\mathcal{E}$, care sunt paralele cu $\Pi : 3x_1 - 2x_2 + 5x_3 + 1 = 0$.

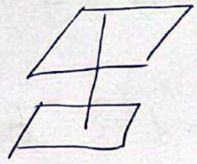
SOL

Fie $M_0(x_1^0, x_2^0, x_3^0) \in \mathcal{E}$



Π tg în M_0 $\Pi_1 : \frac{x_1x_1^0}{4} + \frac{x_2x_2^0}{9} + \frac{x_3x_3^0}{8} = 1.$

$\Pi_1 \parallel \Pi \iff \vec{N}_{\Pi_1} = k\vec{N}_{\Pi}, k\in\mathbb{R}^*$



$$\frac{\frac{x_1^0}{4}}{3} = \frac{\frac{x_2^0}{9}}{-2} = \frac{\frac{x_3^0}{8}}{5} = k$$

$x_1^0 = 12k,\ x_2^0 = -18k,\ x_3^0 = 40k.$

$M_0 \in \mathcal{E} \Rightarrow \frac{12^2k^2}{4} + \frac{18^2k^2}{9} + \frac{40^2k^2}{8} = 1 \Rightarrow$

$k^2(\underbrace{12\cdot 3}_{36} + \underbrace{18\cdot 2}_{36} + \underbrace{40\cdot 5}_{200}) = 1 \Rightarrow k^2 = \frac{1}{272} \Rightarrow k = \pm\frac{1}{4\sqrt{17}}$

$16\cdot 17$

1). $P\left(\frac{8}{3}, \frac{2}{3}, \frac{1}{3}\right)$

$P \in d_\lambda$: $\frac{\cancel{8}^{4}}{3\cdot\cancel{2}\sqrt{2}} - \frac{2}{3\sqrt{2}} = 2\lambda \Rightarrow \frac{2}{3\sqrt{2}} = 2\lambda \Rightarrow \boxed{\lambda = \frac{1}{3\sqrt{2}}}$

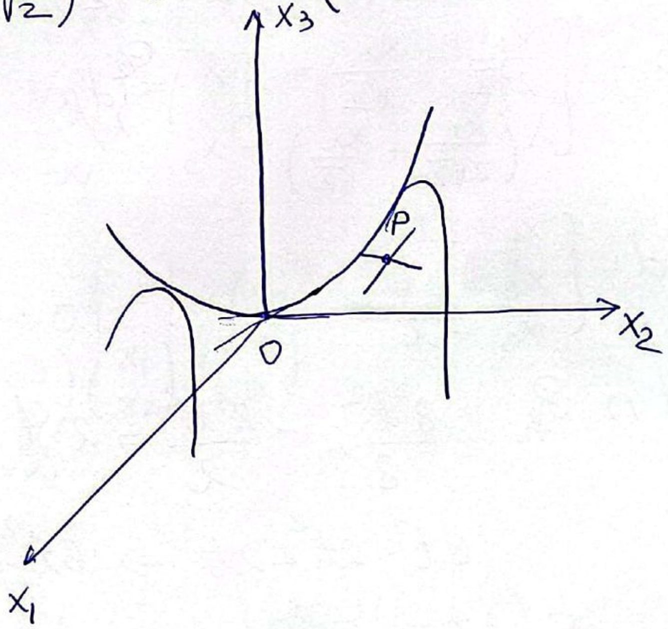
$P \in d_\mu$: $\frac{\cancel{8}^{4}}{3\cdot\cancel{2}\sqrt{2}} - \frac{2}{3\cdot\sqrt{2}} = \frac{1}{3}\mu \Rightarrow \frac{2}{3\sqrt{2}} = \frac{1}{3}\mu \Rightarrow$

$\mu = \sqrt{2}$

$P \in d_{\frac{1}{3\sqrt{2}}}$, $P \in d_{\sqrt{2}}$

$d_{\frac{1}{3\sqrt{2}}}$: $\begin{cases} \frac{x_1}{2\sqrt{2}} - \frac{x_2}{\sqrt{2}} = 2\cdot\frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}}\left(\frac{x_1}{2\sqrt{2}} + \frac{x_2}{\sqrt{2}}\right) = x_3 \end{cases} \Rightarrow \begin{cases} \frac{x_1}{2} - x_2 = \frac{2}{3} \\ \frac{x_1}{12} + \frac{x_2}{6} = x_3 . \end{cases}$

$d_{\sqrt{2}}$: $\begin{cases} \frac{x_1}{2\sqrt{2}} - \frac{x_2}{\sqrt{2}} = x_3\cdot\sqrt{2} \\ \sqrt{2}\left(\frac{x_1}{2\sqrt{2}} + \frac{x_2}{\sqrt{2}}\right) = 2 \end{cases} \Rightarrow \begin{cases} \frac{x_1}{2} - x_2 = 2x_3 \\ \frac{x_1}{2} + x_2 = 2 \end{cases}$



OBS

$P\cdot Q = RS$

$G_1 : \begin{cases} P = \lambda R \\ \lambda Q = S \end{cases}$ $G_2 : \begin{cases} P = \mu S \\ \mu Q = R \end{cases}$

$M_0\left(\frac{12}{4\sqrt{17}}, \frac{-13}{4\sqrt{17}}, \frac{40}{4\sqrt{17}}\right)$; $M_0'\left(\frac{-3}{\sqrt{17}}, \frac{2}{2\sqrt{17}}, \frac{-10}{\sqrt{17}}\right)$

Planele tg în M_0 și M_0' sunt planele cerute.

Ex În ~~planul~~ spațiul euclidian E_3 se consideră paraboloidul hiperbolic

$$\mathcal{P}_h : \frac{x_1^2}{8} - \frac{x_2^2}{2} = 2x_3.$$

și dreapta $d : \frac{x_1}{8} = \frac{x_2}{2} = \frac{x_3}{1}$

Să se scrie ec. generatoarelor care trec prin punctele de intersecție ale dreptei d cu $\mathcal{P}_h$.

SOL

$$\mathcal{P}_h : \left(\frac{x_1}{2\sqrt{2}} - \frac{x_2}{\sqrt{2}}\right)\left(\frac{x_1}{2\sqrt{2}} + \frac{x_2}{\sqrt{2}}\right) = 2 \cdot x_3$$

$$G_1 : d_\lambda : \begin{cases} \frac{x_1}{2\sqrt{2}} - \frac{x_2}{\sqrt{2}} = 2\cdot\lambda \\ \lambda\left(\frac{x_1}{2\sqrt{2}} + \frac{x_2}{\sqrt{2}}\right) = x_3 \end{cases} \quad G_2 : d_\mu : \begin{cases} \frac{x_1}{2\sqrt{2}} - \frac{x_2}{\sqrt{2}} = x_3\cdot\mu \\ \mu\left(\frac{x_1}{2\sqrt{2}} + \frac{x_2}{\sqrt{2}}\right) = 2 \end{cases}$$

$\mu \neq 0$, $\mu, \lambda \in \mathbb{R}$

$$d : \begin{cases} x_1 = 8t \\ x_2 = 2t \\ x_3 = t \end{cases}$$

$$d \cap \mathcal{P}_h : \frac{8^2t^2}{8} - \frac{2^2t^2}{2} = 2\cdot t \Rightarrow$$

$$8t^2 - 2t^2 = 2t \Rightarrow 6t^2 = 2t \Rightarrow$$

$$t(3t-1) = 0$$

1) $t = 0 \Rightarrow O(0,0,0)$; 2) $t = \frac{1}{3} \Rightarrow P\left(\frac{8}{3}, \frac{2}{3}, \frac{1}{3}\right)$

1 d_λ : $\lambda = 0$ $\quad d_0 : \begin{cases} \frac{x_1}{2\sqrt{2}} - \frac{x_2}{\sqrt{2}} = 0 \\ x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 - 2x_2 = 0 \\ x_3 = 0 \end{cases}$

d_μ : $\mu = \infty$ $\quad d_\infty : \begin{cases} x_3 = 0 \\ \frac{x_1}{2\sqrt{2}} + \frac{x_2}{\sqrt{2}} = 0 \end{cases} \Rightarrow \begin{cases} x_3 = 0 \\ x_1 + 2x_2 = 0 \end{cases}$

2 Analog