$$+(-1)^{1+2+2+3}\begin{vmatrix}1 & 2\\ 1 & 3\end{vmatrix}\begin{vmatrix}2 & 1\\ -1 & 4\end{vmatrix}+(-1)^{1+2+2+4}\begin{vmatrix}1 & 3\\ 1 & 4\end{vmatrix}\cdot$$

$$\cdot\begin{vmatrix}2 & 1\\ -1 & 2\end{vmatrix}+(-1)^{1+2+3+4}\begin{vmatrix}2 & 3\\ 3 & 4\end{vmatrix}\begin{vmatrix}2 & 5\\ -1 & -2\end{vmatrix}=-5$$

## Seminarul 1

1) a) $A=\begin{pmatrix}a & b & c\\ b & c & a\\ c & a & b\end{pmatrix}=$

$$\det A=\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}=\begin{vmatrix}a+b+c & a+b+c & a+b+c\\ b & c & a\\ c & a & b\end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix}1 & 1 & 1\\ b & c & a\\ c & a & b\end{vmatrix}=(a+b+c)\begin{vmatrix}1 & 0 & 0\\ b & c-b & a-b\\ c & a-c & b-c\end{vmatrix}$$

$$=(a+b+c)(-1)^{1+1}\begin{vmatrix}c-b & a-b\\ a-c & b-c\end{vmatrix}=$$

$$=(a+b+c)\left[(c-b)(b-c)-(a-c)(a-b)\right]=$$

$$=(a+b+c)\left[cb-c^2-b^2+bc-(a^2-ab-ca+bc)\right]$$

$$=(a+b+c)(-a^2-b^2-c^2+bc+ab+ac)$$

$$=-(a+b+c)(a^2+b^2+c^2-bc-ab-ac)$$

$$= -\frac{1}{2}(a+b+c)\left[a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ac+a^2\right] = -\frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

b) $A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$

Vandermonde

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} =$$

$$= 1\cdot(-1)^{1+1}\cdot \begin{vmatrix} b-a & c-a \\ (b-a)(b+a) & (c-a)(c+a) \end{vmatrix} =$$

$$= (b-a)(c-a)\cdot \begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} =$$

$$= (b-a)(c-a)(c-b)$$

2) $A = \begin{pmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{Z})$

a) $m = ?$ a.i. $A^{-1} \in \mathcal{M}_3(\mathbb{Z})$

$A, A^{-1} \in \mathcal{M}_3(\mathbb{Z}) \Rightarrow \det(A) \wedge \det(A^{-1}) \in \mathbb{Z}$

dar $\det(A) \cdot \det(A^{-1}) = 1 \Rightarrow \det(A) = \pm 1$

$$\det(A) = \begin{vmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{vmatrix} =$$

$$= \begin{vmatrix} 2 & -3 & 3m+4 \\ 1 & m-1 & 1 \\ -1 & 0 & 0 \end{vmatrix} = -1 \cdot (-1)^4 \begin{vmatrix} -3 & 3m+4 \\ m-1 & 1 \end{vmatrix}$$

$$= -1 \left[ (-3) - (3m+4)(m-1) \right] = 3 + 3m^2 + 4m - 3m - 4$$

$= 3m^2 + m - 1$

$C_1$ : $\det(A) = 1 \Rightarrow 3m^2 + m - 1 = 1 \Rightarrow 3m^2 + m - 2 = 0$

$\Delta = 1 - 4 \cdot (-2) \cdot 3 = 1 + 24 = 25 \Rightarrow \sqrt{\Delta} = 5$

$m_{1,2} = \dfrac{-1 \pm 5}{6} \Rightarrow m = -1 \in \mathbb{Z}$

$C_2$ : $\det(A) = -1 \Rightarrow 3m^2 + m - 1 = 1 \Rightarrow 3m^2 + m = 0 \Rightarrow$

$\Rightarrow m(3m+1) = 0$

$\Rightarrow m = 0 \in \mathbb{Z}$

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$m \in \{0, 1\}$

b) $m = -1 \quad A^{-1} - ?$

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$\det(A) = 3 \cdot (-1)^2 - 1 - 1 = 3 - 2 = 1$$

$$A^T = \begin{pmatrix} 2 & 1 & -1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$A^*_{ij} = (-1)^{i+j} \cdot \Delta_{ij}$$

$$A^*_{11} = (-1)^2 \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} = 1 \cdot 1 = 1$$

$$A^*_{12} = (-1)^3 \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} = 1 (-1) = -1$$

$$A^*_{13} = (-1)^4 \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} = 0$$

$$A^*_{21} = (-1)^3 \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -1$$

$$A^*_{22} = \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

$$A_{23} = - \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

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$$A^*_{31} = \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} = -2$$

$$A^*_{32} = -\begin{vmatrix} 2 & -1 \\ -1 & -1 \end{vmatrix} = 3$$

$$A^*_{33} = \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} = -1$$

$$A^{-1} = \frac{1}{\det A} \cdot A^* = A^*$$

$$A^* = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ -2 & 3 & -1 \end{pmatrix}$$

$$\cancel{4)\ A = \begin{vmatrix} 1 & 1 & 2 \end{vmatrix}}$$

$$4)\ A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 3 & 0 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 0 & 3 & 4 & 0 \end{pmatrix}$$

$$a)\ \det(A) = (-1)^{1+2+1+2} \cancel{\begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix}} \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} +$$

$$(-1)^{1+2+1+3} \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} \begin{vmatrix} 4 & 1 \\ 3 & 0 \end{vmatrix} + (-1)^{1+2+1+4} \begin{vmatrix} +1 & 1 \\ 3 & 1 \end{vmatrix} \begin{vmatrix} 4 & 1 \\ 3 & 4 \end{vmatrix}$$

$+(-1)^{1+2+2+3}\begin{vmatrix}-1 & 0\\ 0 & 1\end{vmatrix}\begin{vmatrix}2 & 1\\ 0 & 0\end{vmatrix}+(-1)^{1+2+2+4}\begin{vmatrix}-1 & 1\\ 0 & 1\end{vmatrix}\begin{vmatrix}2 & 1\\ 0 & 4\end{vmatrix}$

$+(-1)^{1+2+3+4}\begin{vmatrix}0 & 1\\ 1 & 1\end{vmatrix}\begin{vmatrix}2 & 4\\ 0 & 3\end{vmatrix} = 3\cdot(-4)+3(-2)\cdot 13-0$

$+8-6=-12+3-26+2=-33$

7) Fie $A=\begin{pmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}$

$A^{-1}$ (TH-C)

$A^3-\sigma_1 A^2+\sigma_2\cdot A-\sigma_3 I_3=O_3$

$\sigma_1=Tr(A)=2$

$\sigma_2=\begin{vmatrix}1 & 1\\ 0 & 0\end{vmatrix}+\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}+\cancel{\begin{vmatrix}0 & 0\\ 0 & 1\end{vmatrix}}+\begin{vmatrix}0 & 1\\ 1 & -1\end{vmatrix}=$

$0+1-1=0$

$\sigma_3=\det A=-\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix}=-1$

$A^3-\sigma_1 A^2+\sigma_2 A-\sigma_3 I_3=O_3 \Rightarrow A^3-2A^2+I_3=O_3 \mid \cdot A^{-1}$

$\Rightarrow A^2-2A=-A^{-1} \Rightarrow \cancel{A^2+2A=-A^{-1}}$

$A^2-2A=-A^{-1}$

8) $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$ $\quad B = A^4 - 3A^3 + 3A^2 - 2A + 8I_2$

$a, b \in \mathbb{R}$ ?

$B = aA + bI_2$ (Th H-C)

$A^2 - \sigma_1 A + \sigma_2 I_2 = O_2$

$\sigma_1 = Tr(A) = 1$

$\sigma_2 = \det A = 2$

$A^2 - A + 2I_2 = O_2$

$P_A = X^2 - X + 2$

$P = X^4 - 3X^3 + 3X^2 - 2X + 8$

$$\begin{array}{l|l}
X^4 - 3X^3 + 3X^2 - 2X + 8 & X^2 - X + 2 \\
\hline
-X^4 + X^3 - 2X^2 & X^2 - 2X - 1 = C \\
\hline
-2X^3 + X^2 - 2X + 8 & \\
\cancel{2X^3 + 2X^2} & \\
2X^3 - 2X^2 + 4X & \\
\hline
-X^2 + 2X + 8 & \\
X^2 - X + 2 & \\
\hline
X + 10 = R &
\end{array}$$

$P = P_A \cdot C + R$

$P(A) = P_A(A) \cdot C(A) + R(A) = R(A) = B$

$B = A + 10 i_2 \Rightarrow \begin{cases} a = 1 \\ b = 10 \end{cases}$

<u>Obs</u>: Din H-C: $A^2 = A - 2 i_2$

$A^3 = A^2 - 2A = A - 2 i_2 = -A - 2 i_2$

$A^4 = -A^2 - 2A = -A + 2 i_2 - 2A =$

$= -3A + 2 i_2$.

$B = -3A + 2 i_2 - 3(-A - 2 i_2) + 3(A - 2 i_2) - 2A + 8 i_2 =$

$= A + 10 i_2$.

10) Fie $A \in \mathcal{M}_2(\mathbb{C})$

Dacă $\exists\, k \geq 2$ a.î. $A^k = O_2$, atunci $A^2 = O_2$.

$\det(A^k) = 0 \Rightarrow (\det A)^k = 0 \Rightarrow \det A = 0$

$A^2 - Tr(A) \cdot A + \underline{\det(A)} i_2 = O_2 \Rightarrow \boxed{A^2 = Tr(A) \cdot A}$

<u>Obs</u>: Dacă $A^2 = \alpha A \Rightarrow A^n = \alpha^{n-1} \cdot A$, $\forall\, n \geq 2$.

$\Rightarrow (Tr(A))^k = 0 \Rightarrow Tr(A) = 0 \Rightarrow A^2 = O_2$.

12) Rez în $\mathcal{M}_2(\mathbb{C})$ ec:

a) $X^2 = \begin{pmatrix} 1 & 12 \\ -4 & 1 \end{pmatrix} = A$

$|A| = 1 + 48 = 49 \Rightarrow |X|^2 = 49 \Rightarrow |X| = \pm 7$

Caz 1) $|X| = 7$

$$x^2 - Tr(x)\cdot x + |x|\cdot i_2 = O_2 \Rightarrow$$

$$\Rightarrow A - Tr(x)\cdot x + 7\cdot i_2 = O_2$$

$$Tr(x)\cdot x = A + 7 i_2 \quad | \, Tr$$

$$Tr^2(x) = 2 + 14 = 16 \Rightarrow Tr(x) = \pm 4$$

$$x = \frac{1}{\pm 4}(A + 7 i_2)$$

Caz 2) $|x| = -7$

$$x^2 - Tr(x)\cdot x + |x|\cdot i_2 = O_2$$

$$A - Tr(x)\cdot x - 7 i_2 = O_2$$

$$Tr(x)\cdot x = A - 7 i_2 \quad | \, Tr$$

$$Tr^2(x) = 2 - 14 = -12 \Rightarrow Tr(x) = \cancel{8} \pm 2\sqrt{3}\, i$$

$$x = \frac{1}{\pm 2\sqrt{3}\, i}(A - 7 i_2)$$

19) $A = \begin{pmatrix} a & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1-a \end{pmatrix} \in \mathcal{M}_3(\mathbb{R}) \quad rg(A) = ?$

$$\det(A) = \begin{vmatrix} a & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1-a \end{vmatrix} = \begin{vmatrix} a & 1-a & 2-a \\ 1 & 0 & 0 \\ -1 & 2 & 2-a \end{vmatrix} =$$

$$= (-1)^{1+2}\cdot 1 \cdot \begin{vmatrix} 1-a & 2-a \\ 2 & 2-a \end{vmatrix} = (1-a)(2-a) - 2(2-a) =$$

$= -(a-2)(a+1)$

Cazul 1) $\det(A) \neq 0 \Leftrightarrow a \in \mathbb{R} \setminus \{2, -1\}$

$\Rightarrow rg(A) = 3$

Cazul 2) Fie $a = -1$

$$A = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} \quad rg(A) = 2$$

Cazul 3) Fie $a = 2$.

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \quad rg\,A = 2$$

20). $A = \begin{pmatrix} \boxed{1} & 2 & 3 & 1 \\ 2 & 0 & a & 1 \\ 0 & 1 & 3 & b \end{pmatrix} \in \mathcal{M}_{3,4}(\mathbb{R})$

$a, b$? a.i. $rg\,A = 2$.

$|1| \neq 0$

$$\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -4 \neq 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & a \\ 0 & 1 & 3 \end{vmatrix} = 2 \cdot 1 \cdot 3 - a - 2 \cdot 2 \cdot 3$$
$$= 6 - a - 12 = -a - 6 = 0 \Leftrightarrow \boxed{a = -6}$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & b \end{vmatrix} = 2\cdot 1\cdot 1 - 1\cdot 1\cdot 1 - 2\cdot 2\cdot b$$
$$= 2 - 1 - 4b$$
$$= 1 - 4b = 0 \Rightarrow \boxed{b = \frac{1}{4}}$$

15) a) $A \in M_n(\mathbb{R})$, $A^2 = O_n \Rightarrow I_n - A$, $I_n + A$ inversabile

$A^2 = O_2 \mid - I_n \Rightarrow A^2 - I_n = -I_n \Rightarrow$

$\Rightarrow (I_n - A)(I_n + A) = I_n$
$(I_n + A)(I_n - A) = I_n$
$\Rightarrow (I_n + A)^{-1} = I_n - A$
$(I_n - A)^{-1} = I_n + A$

b) $A \in M_2(\mathbb{R})$, $A^3 = O_n \Rightarrow I_n - A$, $I_n + A$ inversabile

$I_n^3 - A^3 = I_n$

$(I_n - A)(I_n^2 + A + A^2) = I_n \Rightarrow \det(I_n - A) \neq 0$

$I_n^3 + A^3 = I_n$

$(I_n + A)(I_n^2 - A + A^2) = I_n \Rightarrow \det(I_n + A) \neq 0$