Seminar 10

Transformări ortogonale. Endomorfisme simetrice

①

$(\mathbb{R}^3, g_0)$ s.v.e.r, cu str. euclidiană canonică

$f \in End(\mathbb{R}^3)$, $A = [f]_{R_0,R_0} = \frac{1}{9}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix}$

R_0 = reperul canonic

a) Să se arate că $f \in O(\mathbb{R}^3)$, de speța 2 i.e. $f = so R_e$ ($f = so R_e$)

b) Să se det ∢ de rotație φ și axa de simetrie

c) Să se det un reper $R = \{e_1, e_2, e_3\}$ ortonormat

a.î. $[f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

$f: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x_1,x_2,x_3) = \frac{1}{9}(\overbrace{8x_1 + x_2 - 4x_3}^{-x_1}, x_1 + 8x_2 + 4x_3, -4x_1 + 4x_2 - 7x_3)$

Soluție:

a) $A \in O(3)$ și $\det A = -1$

$A \cdot A^T = \frac{1}{81}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix} = \frac{1}{81}\begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix} = J_3 \Rightarrow$

$\Rightarrow A \in O(3)$

$\det A = \frac{1}{9^3}\begin{vmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{vmatrix} = -\frac{9^3}{9^3} = -1 \Rightarrow f$ este de speță 2.

b) $Tr A = -1 + 2\cos\varphi = 1 \Rightarrow 2\cos\varphi = 2 \Rightarrow \cos\varphi = 1 \Rightarrow \varphi = 0$

Axa: $f(x) = -x \Leftrightarrow 9f(x) = -9x \Leftrightarrow \begin{cases} 17x_1 + x_2 - 4x_3 = 0 \\ x_1 + 17x_2 + 4x_3 = 0 \\ -4x_1 + 4x_2 + 2x_3 = 0 \end{cases}$

$B = \begin{pmatrix} 17 & 1 & -4 \\ 1 & 17 & 4 \\ -4 & 4 & 2 \end{pmatrix}$

$\det B = 0 \quad rg B = 2$

$\begin{cases} 17x_1 + x_2 = 4x_3 \\ x_1 + 17x_2 = -4x_3 \end{cases} \Big|^{\oplus} \Rightarrow x_2 = -x_1 \Rightarrow$

$\Rightarrow x_1 = \frac{1}{4}x_3$

$\Rightarrow \left\{ \left(\frac{1}{4}x_3, -\frac{1}{4}x_3, x_3 \right) \mid x_3 \in \mathbb{R} \right\}$

$\Rightarrow \frac{x_3}{4}(1,-1,4)$

$e_1 = \frac{1}{3\sqrt{2}}(1,-1,4)$ versorul axei

c)

~~$\langle\{e_1\}\rangle^{\perp} = \{ \mid g_0(x,(1,-1,4)) = $~~

$\langle\{e_1\}\rangle^{\perp} = \{ x \in \mathbb{R}^3 \mid g_0(x,(1,-1,4)) = 0 \} =$

$= \{ x \in \mathbb{R}^3 \mid x_1 - x_2 + 4x_3 = 0 \}$

$\begin{pmatrix} \boxed{1} & -1 & 4 \end{pmatrix}$

$x_1 = x_2 - 4x_3 \Rightarrow \langle\{e_1\}\rangle^{\perp} = \{ (x_2 - 4x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R} \}$

$= \langle \{ \underbrace{(1,1,0)}_{f_2}, \underbrace{(-4,0,1)}_{f_3} \} \rangle$

$\{f_2, f_3\}$ reper în $\langle\{e_1\}\rangle^{\perp}$ (sg + dim)

Aplicăm Gram-Schmidt: $e'_2 = f_2 = (1,1,0)$

$e'_3 = f_3 - \frac{\langle f_3, e'_2 \rangle}{\langle e'_2, e'_2 \rangle} \cdot e'_2 =$

$= (-4,0,1) - \frac{\cancel{-4}^{2}}{\cancel{2}} = (-2,2,1) \Rightarrow e_3 = \frac{1}{3}e'_3$

$R = \left\{ e_1 = \frac{1}{3\sqrt{2}}(1,-1,4),\ e_2 = \frac{1}{\sqrt{2}}(1,1,0),\ e_3 = \frac{1}{3}(-2,2,1) \right\}$

reper ortonormat a.î. matricea asociată $[f]_{R,R}$

$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$u \longrightarrow$ versor $\frac{u}{\|u\|}$ $\quad \|u\| = \sqrt{\langle u,u\rangle} = \sqrt{u_1^2+u_2^2+u_3^2}$

[2] $(\mathbb{R}^3, g_0)$ s.v.e.r , $u=(1,1,0)$

a) $\langle\{u\}\rangle^{\perp} = ?$ Precizați un reper ortonormat.

b) Să se det transf ortogonală de speță 1 care este rotație de $\sphericalangle$ orientat $\rho = \frac{\pi}{2}$ și axă $\langle\{u\}\rangle$

Soluție:

$$\langle\{u\}\rangle^{\perp} = \{x \in \mathbb{R}^3 \mid \underbrace{g_0(x,u)}_{= \langle x,u\rangle} = 0\} = \{x \in \mathbb{R}^3 / x_1 + x_2 = 0\}$$

$$x_2 = -x_1 \quad = \{(x_1, -x_1, x_3) \mid x_1, x_3 \in \mathbb{R}\} = \langle\{\underbrace{(1,-1,0)}_{f_1}, \underbrace{(0,0,1)}_{f_2}\}\rangle$$

$\{f_2, f_3\}$ reper în $u^{\perp}$

aplicăm Gram - Schmidt

$$e_2' = f_2 = (1,-1,0) \Rightarrow e_2 = \frac{1}{\sqrt{2}}(1,-1,0)$$

$$e_3' = f_3 - \frac{\langle f_3, e_2'\rangle}{\langle e_2', e_2'\rangle} \cdot e_2' = (0,0,1) - 0 = (0,0,1)$$

$$e_3 = e_3' = (0,0,1)$$

e_2, e_3 reper ortonormat în $u^{\perp}$

$e_1 = \frac{1}{\sqrt{2}}(1,1,0)$ versorul axei

$$R = \{\{ \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{2}}(1,-1,0), (0,0,1)\}$$

$$A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = [f]_{R,R}$$

$$\begin{array}{ccc} A & & A' \\ R_0 & \xrightarrow{C} & R \\ [f]_{R_0,R_0} & & \end{array}$$

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A' = C^{-1} A C = C^T A C$$

$$A = C A' C^T$$

$\boxed{3}$

$(\mathbb{R}^3, g_0)$, $f \in End(\mathbb{R}^3)$

$$A = [f]_{R_0,R_0} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

a) Dem că $f \in Sim(\mathbb{R}^3)$. Determinați $f(x)$

b) $Q : \mathbb{R}^3 \to \mathbb{R}$ forma pătratică asociată lui f.
($Q(x) = x^T A x$ sau $Q(x) = g_0(x, f(x))$)

c) Să se aducă Q la o formă canonică, efectuând o transformare ortogonală h (i.e. o schimbare de repere ortonormate)

Soluție:

a) $A = A^T \Rightarrow f \in Sim(\mathbb{R}^3)$

$$f(x) = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (x_1 + x_3 , x_2 , x_1 + x_3)$$

b)
$$Q(x) = \sum_{i,j=1}^{3} a_{ij} x_i x_j = (x_1)^2 + x_2^2 + x_3^2 + 2 x_1 x_3$$

c)
$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} =$$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \left((1-\lambda)^2 - 1\right)(1-\lambda) =$$

$$= \lambda(1-\lambda)(\lambda-2)$$

$\lambda_1 = 0 \qquad m\lambda_1 = 1$

$\lambda_2 = 1 \qquad m\lambda_2 = 1$

$\lambda_3 = 2 \qquad m\lambda_3 = 1$

$$V\lambda_1 = \{ x \in \mathbb{R}^3 \mid AX = O_{3,1} \} = \{ (x_1, 0, -x_1) \mid x \in \mathbb{R} \}$$

$$= \langle \{ (1, 0, -1) \} \rangle$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$e_1 = \frac{1}{\sqrt{2}} (1, 0, -1)$$

$$\begin{cases} x_1 + x_3 = 0 \Rightarrow x_3 = -x_1 \\ x_2 = 0 \end{cases}$$

$$V\lambda_2 = \{ x \in \mathbb{R}^3 \mid (A - I_3) X = O_{3,1} \}.$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = 0 \\ x_3 = 0 \end{matrix} \Bigg| \Rightarrow$$

$$\Rightarrow V\lambda_2 = \{ (0, x_2, 0) \mid x_2 \in \mathbb{R} \} = \langle \{ (0,1,0) \} \rangle \Rightarrow$$

$$\Rightarrow e_2 = (0, 1, 0)$$

$$V\lambda_3 = \{ x \in \mathbb{R}^3 \mid AX = 2X$$
$$(A - 2I_3) X = O_{3,1} \}$$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \begin{array}{l} x_1 = x_3 \\ x_2 = 0 \end{array} \Bigg| \Rightarrow V_{\lambda_3} = \{(x_1, 0, x_1) \mid x_1 \in \mathbb{R}\} =$

$= < \{(1,0,1)\} > \Rightarrow e_3 = \frac{1}{\sqrt{2}}(1,0,1)$

$R = \{e_1 = \frac{1}{\sqrt{2}}(1,0-1),\ e_2 = (0,1,0),\ e_3 = \frac{1}{\sqrt{2}}(1,0,1)$

reper ortonormat $\quad [f]_{R_0,R_0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$Q(x) = x_2'^2 + 2x_3'^2$

$(2,0)$ signatura lui Q

$R_0 \xrightarrow{C} R$

$C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix} \qquad C \in O(3)$

$h \in O(\mathbb{R}^3) \quad h(e_i^0) = e_i \quad, i = \overline{1,3}$

$R_0 = \{e_1^0, e_2^0, e_3^0\}$

$\boxed{4}$

$(\mathbb{R}^3, g_0) \quad f : \mathbb{R}^3 \to \mathbb{R}^3 \quad f(x) = g_0(x, u)u,$

unde $u = (1, -1, 2)$

a) Să se arate că $f \in Sim(\mathbb{R}^3)$; $f = ?$

b) Să se afle $Q : \mathbb{R}^3 \to \mathbb{R}$ forma pătratică.

Să se aducă Q la o formă canonică, efectuând o transformare ortogonală h

Soluție:

a) $g_0(x,u) = x_1 - x_2 + 2x_3$

$f(x) = (x_1 - x_2 + 2x_3)\cdot(1,-1,2)$

$= (x_1 - x_2 + 2x_3, -x_1 + x_2 - 2x_3, 2x_1 - 2x_2 + 4x_3)$

$[f]_{R_0,R_0} = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix} = A = A^T \Rightarrow f \in Sim(\mathbb{R}^3)$

b) $Q(x) = x_1^2 + x_2^2 + 4x_3^2 - 2x_1x_2 + 4x_1x_3 - 4x_2x_3$

etapa 1) polinom caracteristic.

2) $V\lambda_1$ $V\lambda_2$ $V\lambda_3$ + repere

3) repere ortonormate

4) $R = R_1 \cup R_2 \cup R_3$ a.î. $A' = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$