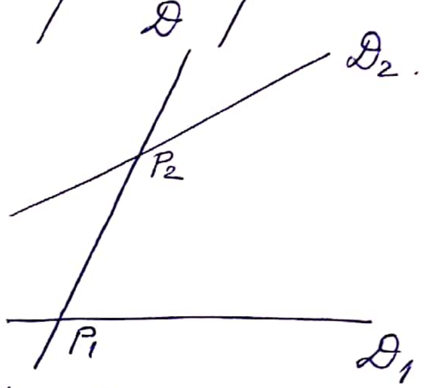
$\boxed{\text{CURSUL 12}}$ 13.05.2020.

Geometrie analitică euclidiană. Izometrii

Perpendiculara comună a 2 drepte necoplanare din $\mathcal{E}_3$



Fie $\mathcal{D}_1 : x_i = a_i + t u_i, \; i=\overline{1,3}$

$\mathcal{D}_2 : x_i = b_i + s v_i, \; i=\overline{1,3}$

$V_{\mathcal{D}_1} = \langle\{u\}\rangle, \; A_1(a_1, a_2, a_3) \in \mathcal{D}_1$

$V_{\mathcal{D}_2} = \langle\{v\}\rangle, \; A_2(b_1, b_2, b_3) \in \mathcal{D}_2$

$\overrightarrow{A_1A_2} = (b_1-a_1, b_2-a_2, b_3-a_3)$.

- $\mathcal{D}_1, \mathcal{D}_2$ necoplanare $\Leftrightarrow \begin{vmatrix} u_1 & v_1 & b_1-a_1 \\ u_2 & v_2 & b_2-a_2 \\ u_3 & v_3 & b_3-a_3 \end{vmatrix} \neq 0$

Fie $\mathcal{D}$ = perpendiculara comună.

(M1) $\mathcal{D} \cap \mathcal{D}_i = \{P_i\}, \; i=\overline{1,2}$.

$P_1(a_1+tu_1, a_2+tu_2, a_3+tu_3)$

$P_2(b_1+sv_1, b_2+sv_2, b_3+sv_3)$.

$\overrightarrow{P_1P_2} = (b_1+sv_1-a_1-tu_1, b_2+sv_2-a_2-tu_2, b_3+sv_3-a_3-tu_3)$

$$\begin{cases} \langle \overrightarrow{P_1P_2}, u \rangle = 0 \\ \langle \overrightarrow{P_1P_2}, v \rangle = 0 \end{cases} \Rightarrow t, s \Rightarrow P_1, P_2.$$

SAU

(M2) Π_1 = planul determinat de $\mathcal{D}, \mathcal{D}_1$

Π_2 = —//— $\mathcal{D}, \mathcal{D}_2$.

$V_{P_1P_2} = \langle\{N\}\rangle, \; N = u \times v$.

$$\Pi_1 : \begin{vmatrix} x_1-a_1 & u_1 & N_1 \\ x_2-a_2 & u_2 & N_2 \\ x_3-a_3 & u_3 & N_3 \end{vmatrix} = 0 \qquad \mathcal{D} = \Pi_1 \cap \Pi_2.$$

$$\Pi_2 : \begin{vmatrix} x_1-b_1 & v_1 & N_1 \\ x_2-b_2 & v_2 & N_2 \\ x_3-b_3 & v_3 & N_3 \end{vmatrix} = 0$$

Aplicație 2 $(\mathcal{E}_3, (E_3, \langle\cdot,\cdot\rangle), \varphi)$. Fie dreptele:

$\mathcal{D}_1 : x_1 = x_3 = 0$; $\mathcal{D}_2 : \begin{cases} x_1 - 1 = 0 \\ x_2 = x_3 . \end{cases}$

a) Să se afle ecuația perpendicularei comune.

b) –//– $dist(\mathcal{D}_1, \mathcal{D}_2)$

SOL.

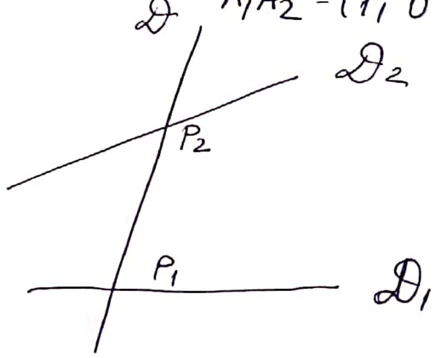
$\mathcal{D}_1 : \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{0} = t : \begin{cases} x_1 = 0 \\ x_2 = t \\ x_3 = 0 \end{cases}$ $A_1(0,0,0)$, $u = (0,1,0)$

$\mathcal{D}_2 : \frac{x_1 - 1}{0} = \frac{x_2}{1} = \frac{x_3}{1} = s : \begin{cases} x_1 = 1 \\ x_2 = s \\ x_3 = s \end{cases}$ $A_2(1,0,0)$, $v = (0,1,1)$

$\overrightarrow{A_1A_2} = (1,0,0)$

- $\mathcal{D}_1, \mathcal{D}_2$ necoplanare

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \neq 0 .$$



(M1) $P_1(0,t,0) \in \mathcal{D}_1 \cap \mathcal{D}$

$P_2(1,s,s) \in \mathcal{D}_2 \cap \mathcal{D}$.

$\overrightarrow{P_1P_2} = (1, s-t, s)$

$\begin{cases} \langle \overrightarrow{P_1P_2}, u \rangle = 0 \\ \langle \overrightarrow{P_1P_2}, v \rangle = 0 \end{cases} \Rightarrow \begin{cases} s - t = 0 \\ s - t + s = 0 \end{cases} \Rightarrow s = t = 0 \Rightarrow P_1(0,0,0)$, $P_2(1,0,0)$

$\mathcal{D} = P_1P_2 : \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$ ec. $\perp$ comune.

$dist(\mathcal{D}_1, \mathcal{D}_2) = dist(P_1, P_2) = \sqrt{1^2 + 0^2 + 0^2} = 1$.

(M2) $N = u \times v = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = (1,0,0)$

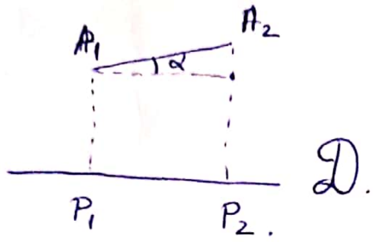
$\pi_1 : \begin{vmatrix} x_1 & 0 & 1 \\ x_2 & 1 & 0 \\ x_3 & 0 & 0 \end{vmatrix} = 0 \Rightarrow -x_3 = 0 .$

$\pi_2 : \begin{vmatrix} x_1 - 1 & 0 & 1 \\ x_2 & 1 & 0 \\ x_3 & 1 & 0 \end{vmatrix} = 0 \Rightarrow x_2 - x_3 = 0$

$\mathcal{D} = \pi_1 \cap \pi_2$

$\begin{cases} x_3 = 0 \\ x_2 - x_3 = 0 \end{cases}$

$\mathcal{D} : \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$



$$pr_{\mathcal{D}} A_1A_2 = \|\overrightarrow{A_1A_2}\| |\cos\alpha|$$

$$= \frac{\|N\|}{\|N\|} \cdot \|\overrightarrow{A_1A_2}\| \cdot |\cos\alpha| =$$

$$= \frac{|\langle \overset{u \times v}{\overset{\parallel}{N}}, \overrightarrow{A_1A_2} \rangle|}{\|N\|} = \frac{|u \wedge v \wedge \overrightarrow{A_1A_2}|}{\|N\|}$$

$$dist(\mathcal{D}_1, \mathcal{D}_2) = \frac{|\langle N, \overrightarrow{A_1A_2} \rangle|}{\|N\|} = \frac{1}{1} = 1.$$

$N = (1,0,0)$

$\overrightarrow{A_1A_2} = (1,0,0)$

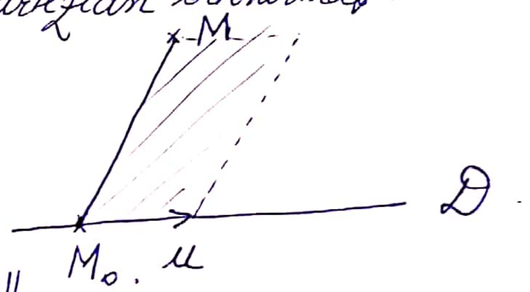
- Distanta de la un punct la o dreaptă în $\mathcal{E}_3$

$\mathcal{R} = \{O; e_1, e_2, e_3\}$ reper cartezian ortonormat.

$\mathcal{D}: r = r_0 + tu, \ t \in \mathbb{R}.$

$\overrightarrow{OM_0} = r_0$

$V_{\mathcal{D}} = \langle \{u\} \rangle.$



M_1

$$A_{paralelogram} = \|u \times \overrightarrow{M_0M}\|$$

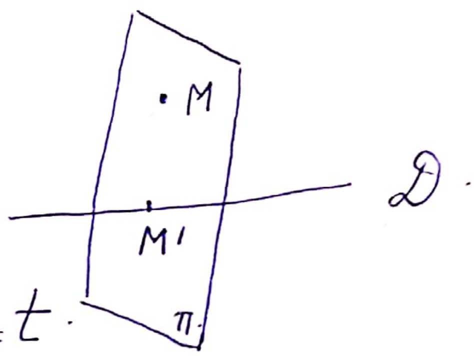
$$= dist(M, \mathcal{D}) \cdot \|u\|$$

$$\Rightarrow dist(M, \mathcal{D}) = \frac{\|u \times \overrightarrow{M_0M}\|}{\|u\|}$$

SAU

M_2 Fie π plan : $\mathcal{D} \perp \pi$

$M \in \pi$.



$N_\pi = u$

$$\mathcal{D}: \frac{x_1 - a_1}{u_1} = \frac{x_2 - a_2}{u_2} = \frac{x_3 - a_3}{u_3} = t.$$

$M(b_1, b_2, b_3)$

$$\pi: u_1(x_1 - b_1) + u_2(x_2 - b_2) + u_3(x_3 - b_3) = 0.$$

$$\mathcal{D} \cap \pi = \{M'\} : u_1(a_1 + tu_1 - b_1) + u_2(a_2 + tu_2 - b_2) +$$

$$+ u_3(a_3 + tu_3 - b_3) = 0 \Rightarrow t \Rightarrow M'$$

$$dist(M, \mathcal{D}) = dist(M, M')$$

Scanned with CamScanner

Aplicatie: $\mathcal{D}: \frac{x_1-1}{2} = \frac{x_2}{1} = \frac{x_3+3}{-1} = t \Rightarrow \begin{cases} x_1 = 1+2t \\ x_2 = t \\ x_3 = -3-t \end{cases}$

$M(1,2,-1)$.

Să se determine $dist(M,\mathcal{D})$ $\qquad u = (2,1,-1)$

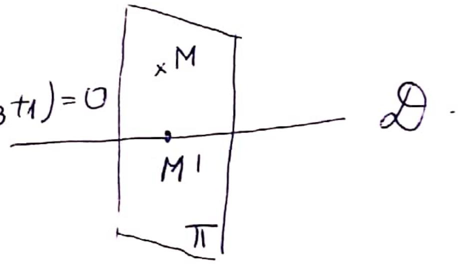
SOL

(M1) $M_0(1,0,-3) \in \mathcal{D}$. ; $\overrightarrow{M_0M} = (0,2,2)$

$$dist(M,\mathcal{D}) = \frac{\| u \times \overrightarrow{M_0M} \|}{\|u\|} = \frac{\sqrt{3\cdot 4^2}}{\sqrt{4+1+1}} = \frac{4\sqrt{3}}{\sqrt{6}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

$$u \times \overrightarrow{M_0M} = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 0 & 2 & 2 \end{vmatrix} = (4,-4,4)$$

(M2)



$\pi: 2(x_1-1) + 1\cdot(x_2-2) - 1\cdot(x_3+1) = 0$

$\pi: 2x_1 + x_2 - x_3 - 2 - 2 - 1 = 0$

$\pi: 2x_1 + x_2 - x_3 - 5 = 0$

$\{M'\} = \pi \cap \mathcal{D}: \quad 2(1+2t) + t - (-3-t) - 5 = 0$

$2 + 4t + t + 3 + t - 5 = 0 \Rightarrow t = 0 \Rightarrow M'(1,0,-3)$

$dist(M,\mathcal{D}) = dist(M,M') = \sqrt{0+4+4} = 2\sqrt{2}.$

$\overrightarrow{MM'} = (0,-2,-2)$

- Unghiul format de 2 drepte orientate de direcțiile vectorilor directori u și v.

$\measuredangle(\mathcal{D}_1,\mathcal{D}_2) = \measuredangle(u,v) = \measuredangle\varphi$; $\cos\varphi = \frac{\langle u,v\rangle}{\|u\|\cdot\|v\|}$

Aplicatie: $\mathcal{D}_1: \frac{x_1-1}{1} = \frac{x_2-1}{-1} = \frac{x_3}{2}$, $u = (1,-1,2)$

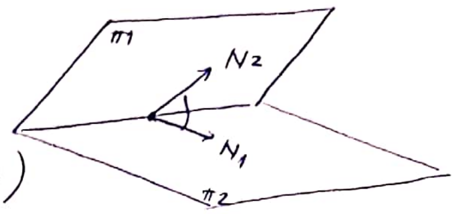
$\mathcal{D}_2: \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$, $v = (-1,1,1)$.

$$\cos\varphi = \frac{-1-1+2}{\sqrt{6}\cdot\sqrt{3}} = 0, \quad \varphi = \frac{\pi}{2}, \quad \varphi \in [0,\pi]$$

• $\measuredangle$ format de planele π_1 și π_2, orientate de normalele N_1, N_2.

$$\measuredangle(\pi_1, \pi_2) = \measuredangle(N_1, N_2) = \measuredangle \varphi \quad , \quad \cos\varphi = \frac{\langle N_1, N_2 \rangle}{\|N_1\| \|N_2\|}$$

<u>Aplicație</u>



$\pi_1 : \; x_1 + x_2 + x_3 - 1 = 0, \; N_1 = (1,1,1)$

$\pi_2 : \; 2x_1 - x_2 + 3x_3 + 2 = 0, \; N_2 = (2,-1,3)$

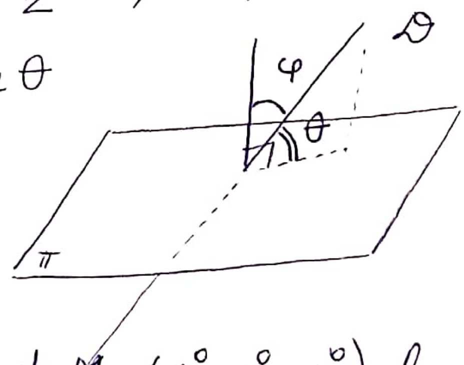
$\measuredangle(\pi_1, \pi_2) = \measuredangle(N_1, N_2)$

$$\cos\varphi = \frac{2-1+3}{\sqrt{3} \cdot \sqrt{4+1+9}} = \frac{4}{\sqrt{3}\cdot\sqrt{14}} = \frac{4}{\sqrt{42}} = \frac{4\sqrt{42}}{42} = \frac{2\sqrt{42}}{21}$$

• $\measuredangle$ format de dreapta $\mathcal{D}$ orientată de vectorul director u și planul π orientat de normala N.

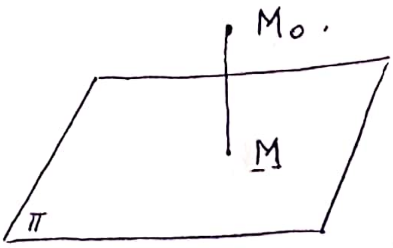
$\measuredangle(\mathcal{D}, \pi) = \measuredangle \theta$ ș. $\theta = \frac{\pi}{2} - \varphi, \quad \hat{\varphi} = \measuredangle(u, N)$.

$$\cos\varphi = \sin\left(\frac{\pi}{2} - \varphi\right) = \sin\theta$$
$$= \frac{\langle u, N \rangle}{\|u\| \cdot \|N\|}.$$



• Distanța de la un pct $M_0(x_1^0, x_2^0, x_3^0)$ la un plan $\pi : \; A_1x_1 + A_2x_2 + A_3x_3 + A_0 = 0 \quad , N = (A_1, A_2, A_3)$.

$M_0M \perp \pi \Rightarrow \overrightarrow{M_0N} = \alpha N$
$\alpha \in \mathbb{R}$.



$$|\langle \overrightarrow{M_0M}, N \rangle| = \|\overrightarrow{M_0M}\| \cdot \|N\|.$$

$$dist(M_0, \pi) = \|\overrightarrow{M_0M}\| = \frac{|\langle \overrightarrow{M_0M}, N \rangle|}{\|N\|} = \frac{|A_1x_1^0 + A_2x_2^0 + A_3x_3^0 + A_0|}{\sqrt{A_1^2 + A_2^2 + A_3^2}}$$

$\pi : \; \langle \overrightarrow{M'M}, N \rangle = 0$

$M' \in \pi$.

Aplicație $\pi: 2x_1 - x_2 + x_3 - 1 = 0$, $M_0(1,0,1)$.

$$\text{dist}(M_0, \pi) = \frac{|2\cdot 1 - 0 + 1 - 1|}{\sqrt{4+1+1}} = \frac{2}{\sqrt{6}} = \frac{2\sqrt{6}}{6} = \frac{\sqrt{6}}{3}$$

$N = (2, -1, 1)$

Izometrii

• $(\mathbb{R}^n, \mathbb{R}^n/_\mathbb{R}, \varphi)$ spațiu afin.

Def. $\tau: \mathbb{R}^n \to \mathbb{R}^m$ s.n. aplicație afină $\Leftrightarrow$

$\tau(aP + bQ) = a\tau(P) + b\tau(Q)$, $\forall a, b \in \mathbb{R}$, $a+b=1$
$\forall P, Q \in \mathbb{R}^n$.

Prop. $\tau: \mathbb{R}^n \to \mathbb{R}^m$ aplicație afină $\Leftrightarrow \exists \mathcal{R} = \{O; e_1, .., e_n\}$, $\mathcal{R}' = \{O'; e_1', .., e_m'\}$ repere carteziene a.î. $\tau: X' = AX + B$

$\{e_1, .., e_n\} \xrightarrow{A} \{e_1', .., e_m'\}$, $B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

$\tau(P) = P'$, $\overrightarrow{OP} = \sum_{i=1}^{n} x_i e_i$

$\overrightarrow{O'P'} = \sum_{j=1}^{m} x_j' e_j'$, $\overrightarrow{O'\tau(O)} = \sum_{j=1}^{m} b_j e_j'$

OBS a) Aplicația liniară $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(e_i) = e_i'$, $\forall i = \overline{1,n}$
$T: X' = AX$.

s.n. urma aplicației afine τ

b) τ este unic determinată de $(O, \tau(O))$ și T.

Def. $\tau: \mathbb{R}^n \to \mathbb{R}^n$ s.n. transformare afină $\Leftrightarrow$ τ aplicație afină și bijecție.

Prop. $\tau: \mathbb{R}^n \to \mathbb{R}^n$ transf. afină $\Leftrightarrow \tau: X' = AX + B$
$A \in GL(n, \mathbb{R})$.

OBS $T: \mathbb{R}^n \to \mathbb{R}^n$ urma sa este un izom. de sp vect $T: X' = AX$.

Not $(AGL(\mathbb{R}^n), \circ)$ grupul afin (al transf. afine)

OBS 1) τ $X' = X + B$ translatie

2) $\tau : X' = AX$ centroafinitate de centru O.

Prop. $\forall$ transf. afină este compunerea unei translații cu o centro-afinitate.

$(\mathcal{E}_n, (E_n, \langle \cdot, \cdot \rangle), \varphi)$ sp. afin euclidian.

Def $\tau : \mathcal{E}_n \to \mathcal{E}_n$ s.n. izometrie $\Leftrightarrow$

$d(P,Q) = d(\tau(P), \tau(Q)), \forall P, Q \in \mathcal{E}_n \Leftrightarrow$

$\| \overrightarrow{PQ} \| = \| \overrightarrow{\tau(P)\tau(Q)} \|$. (păstrează distanța)

Prop. $\tau : \mathcal{E}_n \to \mathcal{E}_n$ izometrie $\Leftrightarrow \tau : X' = AX + B$, $A \in O(n)$ i.e. $T : E_n \to E_n$, $T : X' = AX$ este transformare ortogonală.

Not $(Iso(\mathcal{E}_n), \circ)$ grupul izometriilor.

Def. $\tau \in Iso(\mathcal{E}_n)$ s.n. izometrie de speța 1 (resp. speța 2) dacă T transf. ortog. de speța 1 (resp speța 2) i.e. $A \in SO(n)$ (resp $A \in O(n), \det A = -1$)

Clasificarea izometriilor în $\mathcal{E}_2$

(I) $\tau \in Iso(\mathcal{E}_2)$ de speța 1 (sau deplasare).

$\mathcal{R} = \{O; e_1, e_2\}$ reper ortonormat.

(1) $\tau = T_u$ translatie de vector $u = (b_1, b_2)$.

$\tau : X' = X + B$, $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $T : E_2 \to E_2$

$T : X' = X$

$T = id_{E_2}$.

a) τ are pcte fixe $\Leftrightarrow X' = X \Leftrightarrow B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \tau = id_{\mathcal{E}_2}$, $\mathcal{E}_2$ = multime de pcte fixe.

b) τ nu are pcte fixe $\Leftrightarrow u \neq 0_{\mathbb{E}_2}$

② $\tau = \mathcal{R}_{\Omega,\varphi}$ rotație de centru $\Omega(x_1^0, x_2^0)$ și unghi orientat φ.

a) $\tau = \mathcal{R}_{O,\varphi}$: $X' = AX$, $A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

(O = pct fix)

b) $\tau = \mathcal{R}_{\Omega,\varphi}$: $X' - X_0 = A(X - X_0) \Rightarrow X' = A(X - X_0) + X_0$.

$X_0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$

(Ω = pct fix).

Caz particular $\varphi = \pi$. i.e $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$\tau = \mathcal{R}_{\Omega,\pi} = \mathcal{S}_\Omega$ simetrie centrală (Ω = pct fix)

$X' = A(X - X_0) + X_0 \Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \end{pmatrix} + \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -x_1 + 2x_1^0 \\ -x_2 + 2x_2^0 \end{pmatrix}$

II $\tau \in Iso(\mathcal{E}_2)$ de speță 2 (antideplasări).

$T: E_2 \to E_2$ simetrie ortogonală față de $<\{e_1\}>$,

$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad V_{\mathcal{D}} = <\{e_1\}>$

$\tau : X' = AX + B \Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Rightarrow$

$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1 + b_1 \\ -x_2 + b_2 \end{pmatrix}$

a) Dacă τ are pcte fixe: $X = X' \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + b_1 \\ -x_2 + b_2 \end{pmatrix}$

$\Rightarrow b_1 = 0$, $x_2 = \frac{b_2}{2}$.

$\tau : \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 + b_2 \end{pmatrix}$ $\quad \mathcal{D}: x_2 = \frac{b_2}{2}$ (mulțime de pcte fixe)

$\tau = \mathcal{S}_{\mathcal{D}}$ simetrie axială

b) τ nu are pcte fixe (i.e $b_1 \neq 0$)

$\tau = T_w \circ S_{\mathcal{D}}$, $w \in \langle\{e_1\}\rangle$ $\quad w = (b_1, 0)$

$S_{\mathcal{D}} : \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 + b \end{pmatrix}$, $T_w : \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} x_1' + b_1 \\ x_2' \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \xrightarrow{S_{\mathcal{D}}} \begin{pmatrix} x_1 \\ -x_2 + b_2 \end{pmatrix} \xrightarrow{T_w} \begin{pmatrix} x_1 + b_1 \\ -x_2 + b_2 \end{pmatrix}$$

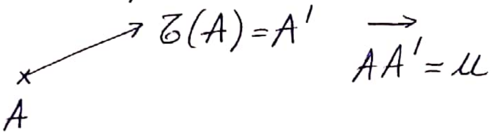
simetrie „alunecată" (glide refection)

<u>Concluzie</u>

(I) $\tau \in Iso(\mathcal{E}_2)$ de speță 1.

a) $\tau = T_u$

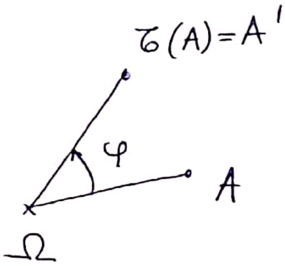
Mulțime de pcte fixe $\begin{cases} u = (0,0) & : \mathcal{E}_2 \\ u \neq (0,0) & : \phi \end{cases}$

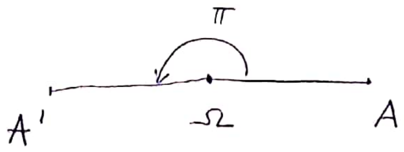


b) $\tau = R_{\Omega, \varphi}$

Mulțime de pcte fixe : Ω.

Dacă $\varphi = \pi \Rightarrow \tau = S_\Omega$.

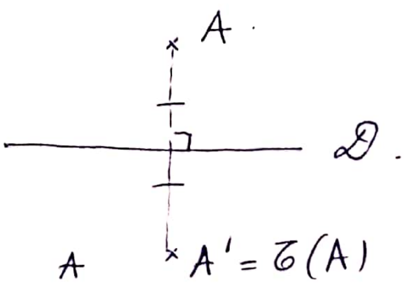




(II) $\tau \in Iso(\mathcal{E}_2)$ de speță 2.

a) $\tau = S_{\mathcal{D}}$

$\mathcal{D}$ = mulțime de pcte fixe.



b) $\tau = T_w \circ S_{\mathcal{D}}$, $w \in V_{\mathcal{D}}$

ϕ = mult de pcte fixe.

$\overrightarrow{A'A''} = w$.