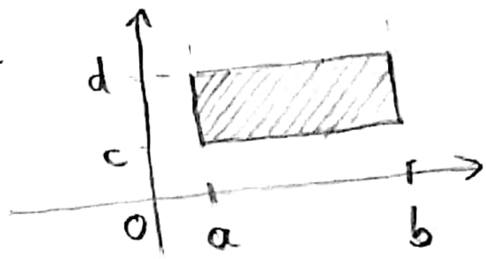
UNIVERSITATEA DIN BUCUREȘTI
FACULTATEA DE MATEMATICĂ ȘI INFORMATICĂ

Seminar 13-14.

1. Calculați $\int_D xy^2\,dxdy$. $D=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1;\ 2\le y\le 3\}$.

Soluție. Aici domeniul D este un dreptunghi

$D=\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b;\ c\le y\le d\}$.



Avem: $\int_D xy^2\,dxdy = \int_0^1\int_2^3 xy^2\,dxdy = \int_0^1\left(\int_2^3 xy^2\,dy\right)dx = \int_0^1 x\cdot\frac{y^3}{3}\Big|_2^3 dx =$

$= \frac{19}{3}\cdot\frac{x^2}{2}\Big|_0^1 = \frac{19}{6}$

Observație: 1. $\int_D xy^2\,dxdy = \int_0^1\left(\int_2^3 xy^2\,dy\right)dx = \int_2^3\left(\int_0^1 xy^2\,dx\right)dy = \frac{19}{6}$.

2. Dacă avem $\int_D f(x,y)\,dxdy$ iar $f(x,y)=g(x)\cdot h(y)$ (i.e. f se scrie ca produsul a două funcții, una care depinde doar de x iar alta doar de y; iar integrala are capete fixe. atunci

$$\int_D f(x,y)\,dxdy = \int_a^b g(x)\,dx\cdot\int_c^d h(y)\,dy.$$

La noi $\int_D xy^2\,dxdy = \int_0^1 x\,dx\cdot\int_2^3 y^2\,dy = \frac{1}{2}\cdot\frac{y^3}{3}\Big|_2^3 = \frac{19}{6}$.

② $\iint_D y^3 e^{xy^2}\,dxdy$ $\quad D=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1,\ 0\le y\le 1\}$.

Soluție: D-este tot un dreptunghi (aici e chiar pătrat)

Avem $\iint_D y^3e^{xy^2}dxdy = \int_0^1\left(\int_0^1 y^3 e^{xy^2}dx\right)dy = \int_0^1 y^3 \cdot \frac{e^{xy^2}}{y^2}\Big|_0^1 dy =$

$$= \int_0^1 y(e^{y^2}-1)\,dy = \int_0^1 ye^{y^2}dy - \int_0^1 dy = \frac{1}{2} e^{y^2}\Big|_0^1 - 1 = \frac{e}{2} - \frac{3}{2} = \frac{e-3}{2}$$

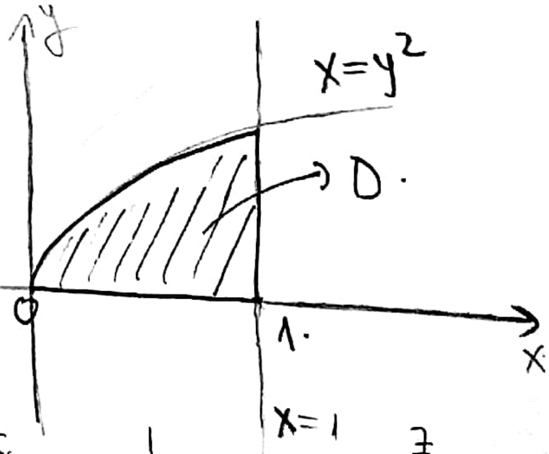
Observație: Încercați $\int_0^1\left(\int_0^1 y^3e^{xy^2}dy\right)dx$. Ce observați?

③ Calculați $\iint_D x^2\,dxdy$. $\quad D=\{(x,y)\in\mathbb{R}^2 \mid x\le 1,\ y\ge 0,\ y^2\le x\}$.

Soluție:

Avem $x\in[0,1]$.

$y\in[0,\sqrt{x}]$



(Domeniul este simplu în raport cu axa Oy)

Avem: $\iint_D x^2dxdy = \int_0^1\left(\int_0^{\sqrt{x}} x^2 dy\right)dx = \int_0^1 x^2 y\Big|_0^{\sqrt{x}} dx = \int_0^1 x^2\sqrt{x}\,dx = \frac{x^{\frac{7}{2}}}{\frac{7}{2}}\Big|_0^1$

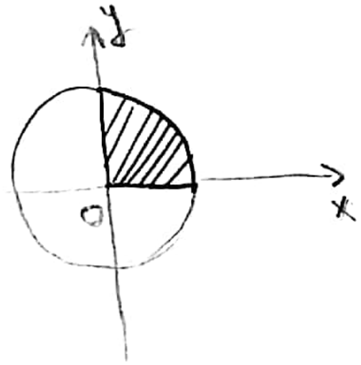
$= \frac{2}{7}$

Obs: Domeniul este simplu și în raport cu axa Ox. Avem $x\in[y^2,1]$

$y\in[0,1]$.

$$\iint_D x^2\,dx\,dy = \int_0^1\left(\int_{y^2}^1 x^2\,dx\right)dy = \int_0^1 \frac{1}{3}x^3\Big|_{y^2}^1 dy = \frac{1}{3}\left(1-\frac{1}{7}\right) = \frac{1}{3}\cdot\frac{6}{7} = \frac{2}{7}$$

4. Calculați $\displaystyle\iint_D \frac{x^2+y^2}{x+\sqrt{x^2+y^2}}\,dx\,dy$. $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1;\ x\ge 0;\ y\ge 0\}$.



Soluție:

Trecem la coordonate polare.

$$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases} \quad r\in[0,\infty);\ t\in[0,2\pi). \qquad (x,y)\xrightarrow{\varphi}(r,t)$$

Jacobianul transformării $J_\varphi(r,t) = \dfrac{D(x,y)}{D(r,t)} = \begin{vmatrix} \frac{\partial x}{\partial r}(r,t) & \frac{\partial x}{\partial t}(r,t) \\ \frac{\partial y}{\partial r}(r,t) & \frac{\partial y}{\partial t}(r,t) \end{vmatrix}$

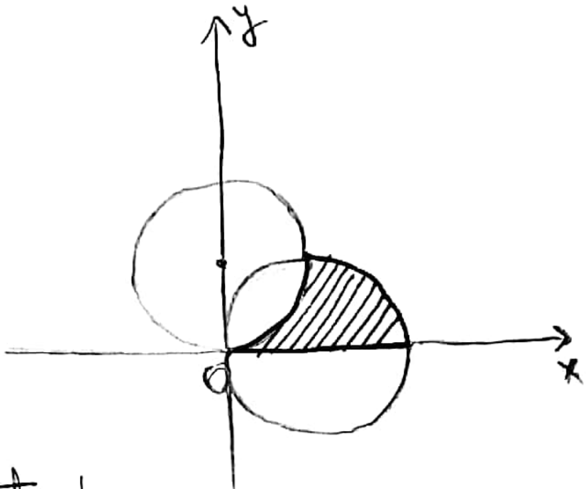
$= \begin{vmatrix} \cos t & -r\sin t \\ \sin t & r\cos t \end{vmatrix} = r$. Prin urmare $|J_\varphi(r,t)| = r$.

La noi $r\in[0,1]$.
$t\in[0;\frac{\pi}{2}]$.

$$\iint_D \frac{x^2+y^2}{x+\sqrt{x^2+y^2}}\,dx\,dy = \int_0^1\int_0^{\frac{\pi}{2}} \frac{r^2}{r\cos t + r}\cdot r\,dr\,dt = \int_0^1 r^2\,dr \int_0^{\frac{\pi}{2}} \frac{dt}{1+\cos t}$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{dt}{2\cos^2\frac{t}{2}} = \frac{1}{6}\cdot 2\cdot \operatorname{tg}\frac{t}{2}\Big|_0^{\frac{\pi}{2}} = \frac{1}{3}\cdot\operatorname{tg}\frac{\pi}{4} = \frac{1}{3}.$$

(5) Calculați $\iint\limits_D (x+y)^2\,dx\,dy$. $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2-x\le 0,\ x^2+y^2-y\ge 0;\ y\ge 0\}$.



Soluție

$$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases} \quad r\in[0,\infty);\ t\in[0,2\pi)$$

$$|J_\varphi(r,t)| = r.$$

Avem: $r^2 - r\cos t \le 0 \Rightarrow r \le \cos t$
$r^2 - r\sin t \ge 0 \Rightarrow r \ge \sin t$ $\Bigg| \Rightarrow r\in[\sin t, \cos t]$.

$\sin t \le \cos t \Rightarrow t\in\left[0,\frac{\pi}{4}\right]\cup\left[\frac{5\pi}{4}, 2\pi\right]$
$\sin t \ge 0;\ t\in[0,\pi]$ $\Bigg| \Rightarrow t\in\left[0,\frac{\pi}{4}\right]$

$$\int_D (x+y)^2\,dx\,dy = \int_0^{\frac{\pi}{4}}\int_{\sin t}^{\cos t} (r\cos t + r\sin t)^2 \cdot r\,dr\,dt = \int_0^{\frac{\pi}{4}}\left(\int_{\sin t}^{\cos t} r^3(1+\sin 2t)\,dr\right)dt$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} (\cos^4 t - \sin^4 t)(1+\sin 2t)\,dt = \frac{1}{4}\int_0^{\frac{\pi}{4}} \cos 2t\,(1+\sin 2t)\,dt =$$

$$= \frac{1}{4}\ \frac{\sin 2t}{2}\Bigg|_0^{\frac{\pi}{4}} - \frac{\cos 4t}{8}\Bigg|_0^{\frac{\pi}{4}} = \frac{1}{4}\left(\frac{1}{2}+\frac{1}{8}+\frac{1}{8}\right) = \frac{1}{4}\left(\frac{1}{2}+\frac{1}{4}\right) = \frac{3}{16}.$$

⑥ Calculați $\int_0^{\infty} e^{-x^2}dx$.

Soluție: Notăm $\int_0^{\infty} e^{-x^2}dx$

Considerăm $I \cdot I = \int_0^{\infty} e^{-x^2}dx \int_0^{\infty} e^{-y^2}dy = \int_0^{\infty} e^{-(x^2+y^2)}dxdy$.

Trecând la coordonate polare avem: $\varphi: \begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$ $r \in [0,\infty)$; $t \in [0, 2\pi)$

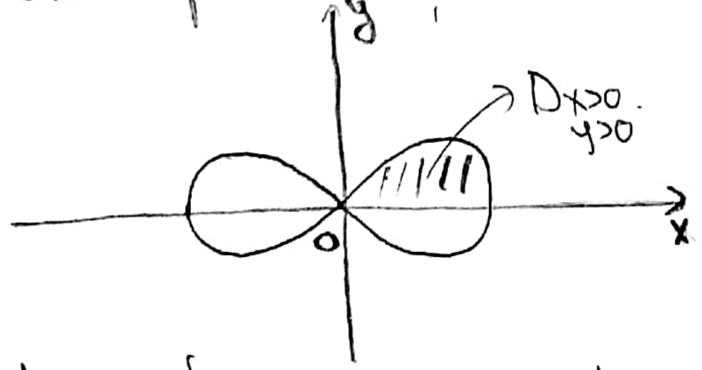
La noi $r \in [0,\infty)$; $t \in [0, \frac{\pi}{2})$ $\quad J\varphi(r,t) = r$

$$I^2 = \int_0^{\infty}\int_0^{\frac{\pi}{2}} re^{-r^2}drdt = \int_0^{\frac{\pi}{2}} dt \int_0^{\infty} re^{-r^2}dr = \frac{\pi}{2}\cdot\left(-\frac{e^{-r^2}}{2}\right)\Big|_0^{\infty} = \frac{\pi}{4}$$

Prin urmare $I^2 = \frac{\pi}{4}$ sau $I = \frac{\sqrt{\pi}}{2}$

⑦ Să se calculeze aria mulțimii plane D - mărginită de curba
$\gamma: (x^2+y^2)^2 = a^2(x^2-y^2)$; $a>0$

Soluție: Curba γ - se numește lemniscata lui Bernoulli



Știm că $\mathcal{A}(D) = \int_D dxdy$. La noi $\mathcal{A}(D) = 4\cdot\mathcal{A}(D_{\substack{x>0\\y>0}})$

Trecem la coordonate polare

$$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$$

$(r^2\cos^2 t + r^2\sin^2 t)^2 = a^2(r^2\cos^2 t - r^2\sin^2 t) \Leftrightarrow r^4 = a^2 r^2 \cos 2t \,\big|: r^2$

$\Leftrightarrow r^2 = a^2\cos 2t \Leftrightarrow r = a\sqrt{\cos 2t}$. Deci $r \in (0, a\sqrt{\cos 2t})$

$\left.\begin{array}{l} x \geq 0 \Rightarrow \cos t \geq 0 \Rightarrow t \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}; 2\pi]. \\ y \geq 0 \Rightarrow \sin t \geq 0 \Rightarrow t \in [0, \pi] \end{array}\right| \Rightarrow t \in [0, \frac{\pi}{2}]. \ (1).$

Cum $\cos 2t = \frac{r^2}{a^2} > 0 \rightarrow \cos 2t \geq 0 \Leftrightarrow 2t \in [0; \frac{\pi}{2}] \cup [\frac{3\pi}{2}; 2\pi]$.

$\Rightarrow t \in [0, \frac{\pi}{4}] \cup [\frac{3\pi}{4}; \pi]. (2)$ Din (1) și (2) $\Rightarrow t \in [0, \frac{\pi}{4}]$.

Avem: $\mathcal{A}(D) = 4\mathcal{A}(D_{\substack{x\geq 0 \\ y \geq 0}}) = 4\int_0^{\frac{\pi}{4}}\left(\int_0^{a\sqrt{\cos 2t}} r\,dr\right)dt = 4\int_0^{\frac{\pi}{4}} \frac{r^2}{2}\Big|_0^{a\sqrt{\cos 2t}} dt =$

$= 2\int_0^{\frac{\pi}{4}} a^2\cos 2t\,dt = 2\cdot a^2\,\frac{\sin 2t}{2}\Big|_0^{\frac{\pi}{4}} = a^2(\sin\frac{\pi}{2} - \sin 0) = a^2$

Prin urmare $\mathcal{A}(D) = a^2$

(8) Calculați volumul corpului Ω, $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 2x + 2y - 1\}$.

$\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D;\ 0 \leq z \leq f(x,y)\}$. pentru $f(x,y) = y$.

Soluție: Știu $V(\Omega) = \int_D f(x,y)\,dx\,dy$.

$x^2 + y^2 \leq 2x + 2y - 1 \Leftrightarrow (x-1)^2 + (y-1)^2 \leq 1$ $\quad\begin{cases} x = 1 + r\cos t \\ y = 1 + r\sin t \end{cases}$

avem $r \in [0,1]$; $t \in [0, 2\pi]$ $J_\varphi(r,t) = r$ $\quad (x,y) \xrightarrow{\varphi} (r,t)$

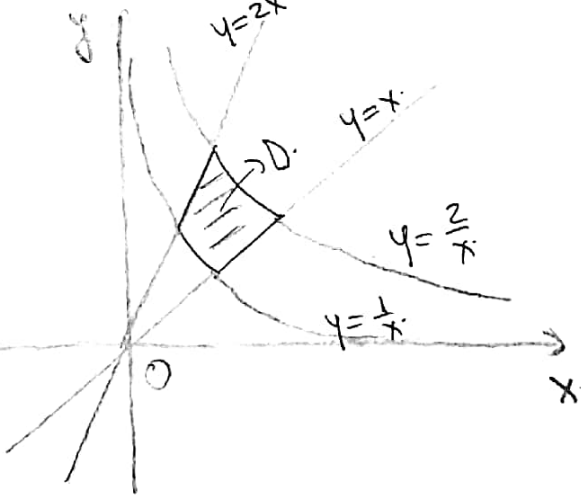
$$V(\Omega) = \iint_D f(x,y)\,dx\,dy = \iint_D y\,dx\,dy = \int_0^1\int_0^{2\pi} r(1+r\sin t)\,dr\,dt =$$

$$= \int_0^1\int_0^{2\pi} r\,dr\,dt + \int_0^1\int_0^{2\pi} r^2\sin t\,dr\,dt = \int_0^1 r\,dr \int_0^{2\pi} dt + \int_0^1 r^2\,dr \int_0^{2\pi} \sin t\,dt =$$

$$= \frac{1}{\cancel{2}}\cdot \cancel{2}\pi + \frac{1}{3}\cdot 0 = \pi$$

Aşadar $V(\Omega) = \pi$

⑨ Calculaţi $I = \iint_D x\,dx\,dy$ unde $D = \{(x,y)\in\mathbb{R}^2 \mid 1 \le xy \le 2,\ 1 \le \frac{y}{x} \le 2;\ x>0\}$.



Soluţie.

Notăm $u = xy$; $v = \frac{y}{x}$; $x>0, y>0$

De aici $y = \sqrt{uv}$; $x = \sqrt{\frac{u}{v}}$

Deci practic facem schimbarea de variabilă

$$\begin{cases} x = \sqrt{\frac{u}{v}} \\ y = \sqrt{uv} \end{cases} \quad u\in[1,2],\ v\in[1,2] \qquad (x,y) \xrightarrow{\varphi} (u,v).$$

$$J_\varphi(u,v) = \frac{D(x,y)}{D(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{1}{2v} \neq 0$$

Avem: $$I = \iint_D x\,dx\,dy = \int_1^2\int_1^2 \sqrt{\frac{u}{v}}\cdot\frac{1}{2v}\,du\,dv = \int_1^2 \sqrt{u}\,du \cdot \int_1^2 \frac{1}{2}\,\frac{1}{v\sqrt{v}}\,dv$$

$$I = \frac{1}{3}(5\sqrt{2}-6).$$

10). Calculați $\int\limits_{A} (x+y)\,dx\,dy$. $A: |x|+|y| \leq 1$.