C5

Morfisme de spații vectoriale

Def $(V_i, +, \cdot)_{\mathbb{K}_i}$, $i=\overline{1,2}$ spații vectoriale.

$f: V_1 \to V_2$ s.n. aplicație semi-liniară

$\Leftrightarrow$ 1) $f(x+y) = f(x) + f(y)$

2) $\exists\, \theta : \mathbb{K}_1 \to \mathbb{K}_2$ izomorfism de corpuri

a.î. $f(\alpha x) = \theta(\alpha) f(x)$,

$\forall x, y \in V_1$, $\forall \alpha \in \mathbb{K}$.

- Dacă $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{K}$ și $\theta = id_{\mathbb{K}}$, at f s.n. aplicație liniară sau morfism de spații vectoriale

Exemple

① $(V_i, +, \cdot)/\mathbb{R}$, $i=\overline{1,2}$

Fie $\theta: \mathbb{R} \to \mathbb{R}$ automorfism de corp $\Rightarrow \theta = id$.

$f: V_1 \to V_2$ semiliniară $\Rightarrow$ liniară.

② $(\mathbb{C}^n, +, \cdot)/\mathbb{C}$

$f: \mathbb{C}^n \to \mathbb{C}^n$, $f(z) = \bar{z}$, $\forall z \in \mathbb{C}^n$

$\theta: \mathbb{C} \to \mathbb{C}$, $\theta(z) = \bar{z}$, $\forall z \in \mathbb{C}$

automorfism de corpuri

$f(z+u) = \overline{z+u} = \bar{z} + \bar{u} = f(z) + f(u)$

$f(\alpha z) = \overline{\alpha z} = \bar{\alpha}\, \bar{z} = \theta(\alpha) f(z)$

$\Rightarrow f$ este semi-liniară (și nu e liniară).

Aplicații liniare

- $f: V_1 \to V_2$ aplicație liniară

f s.n. izomorfism $\Leftrightarrow$ f bijecție.

• $(V,+,\cdot)/_{\mathbb{K}}$ sp. vect.

$f: V \to V$ s.n. <u>endomorfism de sp vect</u> $\Leftrightarrow$ f liniară

• $f: V \to V$ s.n. <u>automorfism de sp vect</u> $\Leftrightarrow$ $\begin{cases} 1. & f \text{ liniară} \\ 2. & f \text{ bij.} \end{cases}$

<u>OBS</u>

a) $f: V_1 \to V_2$ apl. liniară $\Rightarrow$

$f: (V_1,+) \to (V_2,+)$ morf. de grupuri $\Rightarrow f(0_{V_1}) = 0_{V_2}$

b) $\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[rr, bend right, "h = g\circ f"'] & V_2 \arrow[r, "g"] & V_3 \end{tikzcd}$

f, g apl. lin $\Rightarrow$ h apl. liniară.

<u>Exemple de apl. lin.</u>

1) $f: V \to V$, $f(x) = 0_V$, $f(x) = x$ apl lin.

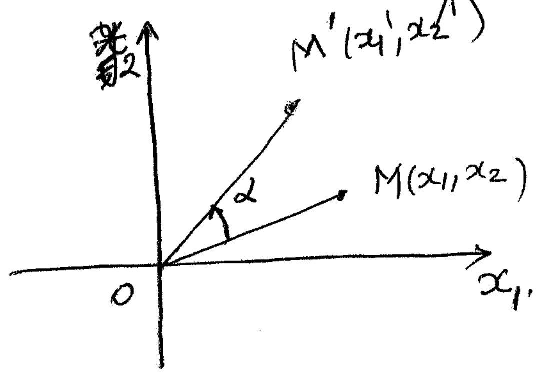
2) $pr_i : \mathbb{R}^n \to \mathbb{R}$, $pr_i(x_1, \dots, x_n) = x_i$ apl. lin.
$\forall i = \overline{1,n}$

3) $f: \mathbb{R}^n \to \mathbb{R}^m$, $f(x) = y$, $\underset{(m,1)}{Y} = \underset{(m,n)}{A}\,\underset{(n,1)}{X}$ apl. lin.

$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$, $A = (a_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$

4) $f: \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$, $f(A) = Tr(A)$ apl. lin.
$f(A) = \det(A)$ nu e apl lin

5) $f: \mathbb{R}^2 \to \mathbb{R}^2$.



$f(x_1, x_2) = (x_1', x_2')$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \cos\alpha - x_2 \sin\alpha \\ x_1 \sin\alpha + x_2 \cos\alpha \end{pmatrix}$$

f = rotația în plan de $\measuredangle\alpha$ (apl. lin)

Prop (caracterizare apl. lin).

$f: V_1 \to V_2$ apl.

f liniară $\iff f(ax+by) = af(x)+bf(y)$, $\forall a,b \in \mathbb{K}$, $\forall x,y \in V_1$

Dem

"$\Rightarrow$" Ip: f e liniară

$a \in \mathbb{K}, x \in V_1 \Rightarrow ax \in V_1$
$b \in \mathbb{K}, y \in V_1 \Rightarrow by \in V_1$
$\Rightarrow f(ax+by) = f(ax)+f(by) = af(x)+bf(y)$

"$\Leftarrow$" Ip: $f(ax+by) = af(x)+bf(y)$, $\forall a,b \in \mathbb{K}$, $\forall x,y \in V_1$

- $a = b = 1_{\mathbb{K}}$.
  $f(1_{\mathbb{K}} \cdot x + 1_{\mathbb{K}} y) = f(x+y) = f(x)+f(y)$
- $b = 0_{\mathbb{K}}$ $\quad f(ax) = af(x)$

OBS

a) $f: V_1 \to V_2$ apl. lin.
$V' \subset V_1$ ssp vect $\Rightarrow f(V') \subset V_2$ ssp. vect.

b) $\operatorname{Ker} f = \{x \in V_1 \mid f(x) = 0_{V_2}\} \subset V_1$ (nullspace) (kernel)
subspațiu vect.

$\forall x_1, x_2 \in \operatorname{Ker} f \Rightarrow ax_1 + bx_2 \in \operatorname{Ker} f$

$f(ax_1+bx_2) = a\underbrace{f(x_1)}_{0_{V_2}} + b\underbrace{f(x_2)}_{0_{V_2}} = 0_{V_2}$.

c) $\operatorname{Im} f = \{y \in V_2 \mid \exists x \in V_1 \text{ a.î. } f(x) = y\} \subseteq V_2$
subsp. vect (cf a)
$\operatorname{Im} f = f(V_1) \subseteq V_2$

Prop $f: V_1 \to V_2$ liniară
a) f inj $\iff \operatorname{Ker} f = \{0_{V_1}\}$
b) f surj $\iff \dim \operatorname{Im} f = \dim V_2$

Dem b).

a) "$\Rightarrow$" Ip: f inj

Fie $x \in \operatorname{Ker} f \Rightarrow f(x) = 0_{V_2}$

dar $f(0_{V_1}) = 0_{V_2}$ $\Big\} \xrightarrow{inj} x = 0_{V_1} \Rightarrow$

$\Rightarrow \operatorname{Ker} f = \{0_{V_1}\}$

"$\Leftarrow$" Ip: $\operatorname{Ker} f = \{0_{V_1}\}$.

Fie $x_1, x_2 \in V_1$ aî $f(x_1) = f(x_2) \xrightarrow{f \text{ lin}} f(x_1 - x_2) = 0_{V_2}$

$\Rightarrow x_1 - x_2 \in \operatorname{Ker} f = \{0_{V_1}\} \Rightarrow x_1 = x_2$

b) "$\Rightarrow$" Ip: f surj $\Leftrightarrow \operatorname{Im} f = V_2 \Leftrightarrow \dim_{\mathbb{K}} \operatorname{Im} f = \dim_{\mathbb{K}} V_2$.

"$\Leftarrow$" Ip: $\dim_{\mathbb{K}} \operatorname{Im} f = \dim_{\mathbb{K}} V_2$.

$\operatorname{Im} f \subseteq V_2$. $\Big\} \xrightarrow{OBS} \operatorname{Im} f = V_2 \Rightarrow$

$\Rightarrow f$ surj.

<u>OBS</u> $f : V_1 \to V_2$ liniară

f izomorfism de sp vect $\Leftrightarrow$ ① $\operatorname{Ker} f = \{0_{V_1}\}$ ② $\dim \operatorname{Im} f = \dim V_2$

<u>Teorema dimensiunii pentru aplicații liniare</u>

$f : V_1 \to V_2$ apl. liniară

$\dim V_1 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$.

<u>Dem</u> $\dim_{\mathbb{K}} \operatorname{Ker} f = k$; $\dim_{\mathbb{K}} V_1 = n$.

Fie $\mathcal{R}_0 = \{e_1, \dots, e_k\}$ reper în $\operatorname{Ker} f$.

Extindem la $\mathcal{R}_1 = \{e_1, \dots, e_k, \underline{e_{k+1}, \dots, e_n}\}$ reper în V_1.

Fie $\mathcal{R} = \{f(e_{k+1}), \dots, f(e_n)\}$.

Dem că $\mathcal{R}$ este reper în $\operatorname{Im} f$.

- $\mathcal{R}$ este SLI.

Fie $a_{k+1}, \dots, a_n \in \mathbb{K}$ aî $\sum_{j=k+1}^{n} a_j f(e_j) = 0_{V_2} \xrightarrow{f \text{ lin.}}$

$f\Big(\underbrace{\sum_{j=k+1}^{n} a_j e_j}_{\in \operatorname{Ker} f = \langle \mathcal{R}_0 \rangle}\Big) = 0_{V_2} \Rightarrow \exists a_1, \dots, a_k \in \mathbb{K}$ aî $\sum_{j=k+1}^{n} a_j e_j = \sum_{i=1}^{k} a_i e_i$

$$\Rightarrow \sum_{i=1}^{k} a_i e_i - \sum_{j=k+1}^{n} a_j e_j = 0_{V_2} \xRightarrow[SLI]{R_1 \text{ reper}}$$

$$\Rightarrow a_i = 0, \forall i = \overline{1,k}$$
$$a_j = 0, \forall j = \overline{k+1,n} \Rightarrow R \text{ este SLI}$$

• R este SG pentru $\operatorname{Im} f$.

$\forall y \in \operatorname{Im} f$, $\exists a_{k+1}, \dots, a_n \in \mathbb{K}$ aî $y = \sum_{j=k+1}^{n} a_j f(e_j)$ (dem)

$y \in \operatorname{Im} f \Rightarrow \exists x \in \underset{\langle R_1 \rangle}{\overset{\|}{V_1}}$ aî $f(x) = y$

$\exists a_1, \dots, a_k$, $a_{k+1}, \dots, a_n \in \mathbb{K}$ aî $x = \underbrace{\sum_{i=1}^{k} a_i e_i}_{\in \operatorname{Ker} f} + \sum_{j=k+1}^{n} a_j e_j$

$$y = f(x) = f\left(\sum_{i=1}^{k} a_i e_i + \sum_{j=k+1}^{n} a_j e_j\right) \overset{f \text{ lin}}{=}$$
$$= \underbrace{f\left(\sum_{i=1}^{k} a_i e_i\right)}_{= 0_{V_2}} + f\left(\sum_{j=k+1}^{n} a_j e_j\right) \overset{f \text{ lin}}{=} \sum_{j=k+1}^{n} a_j f(e_j)$$

$\Rightarrow R$ este SG pt $\operatorname{Im} f$.

Deci R este reper în $\operatorname{Im} f$.

$$\underset{n}{\overset{\|}{\dim V_1}} = \underset{k}{\overset{\|}{\dim \operatorname{Ker} f}} + \underset{n-k}{\overset{\|}{\dim \operatorname{Im} f}}$$

OBS $f : V_1 \to V_2$ liniară

a) f inj $\Leftrightarrow \operatorname{Ker} f = \{0_{V_1}\} \Leftrightarrow \dim V_1 = \dim \operatorname{Im} f$.

b) f surj $\Leftrightarrow \dim \operatorname{Im} f = \dim V_2 \Leftrightarrow$
$\dim V_1 = \dim \operatorname{Ker} f + \dim V_2$

c) f bij $\Leftrightarrow \dim V_1 = \dim V_2$.
(f izomorfism)

<u>Teorema</u> $V_1 \simeq V_2$ (sp. vect. izomorfe) $\Leftrightarrow \dim V_1 = \dim V_2$

<u>Dem</u>

"$\Rightarrow$" $\exists f : V_1 \to V_2$ izomorfism $\overset{obs}{\Rightarrow} \dim V_1 = \dim V_2$.

"

"$\Leftarrow$" $\dim V_1 = \dim V_2 = n$.

$R_1 = \{e_1, .., e_n\}$ reper in V_1

$R_2 = \{e_1', .., e_n'\}$ —"— V_2.

$f : V_1 \longrightarrow V_2$, $f(e_i) = e_i'$, $i = \overline{1,n}$

Prelungim f prin liniaritate

$$f(x) = f\Big(\sum_{i=1}^{n} x_i e_i\Big) = \sum_{i=1}^{n} x_i f(e_i) = \underline{\sum_{i=1}^{n} x_i e_i'}$$

$= x'$

- f lin.
- f bij $\Leftrightarrow$ $\forall\, x' \in V_2$, $\exists!\, x = \sum_{i=1}^{n} x_i e_i$ ai $f(x) = x'$

$\overset{\shortparallel}{\sum_{i=1}^{n} x_i e_i'}$

f izomorfism de sp vect.

<u>OBS</u> $(V, +, \cdot)/_{\mathbb{K}}$ n-dim.

$V \simeq \mathbb{K}^n$

$R = \{e_1, .., e_n\}$ reper in V

$\forall x \in V$, $x = \sum_{i=1}^{n} x_i e_i$

$f(x) = (x_1, .., x_n)$ izom. de sp vect.

$\exists$ o inf. de astfel de izomorfisme ($\exists$ o inf de repere in V)

$f(e_i) = (0, \dots \underset{\underset{i}{\downarrow}}{1} \dots 0) = e_i^0$ $\quad R_0 = \{e_1^0, .., e_n^0\}$

reperul canonic in $\mathbb{K}^n$.

<u>Teorema</u> $f : V_1 \longrightarrow V_2$ liniara

a) f inj $\Leftrightarrow$ f transforma $\forall$ SLI din V_1 intr-un SLI al lui V_2

b) f surj $\Leftrightarrow$ f transf. $\forall$ SG al lui V_1 intr-un SG al lui V_2

c) f bij $\Leftrightarrow$ f transf $\forall$ reper al lui V_1 intr-un reper al lui V_2

<u>Dem</u>

a) "$\Rightarrow$" Ip f inj

"$S = \{v_1, .., v_k\} \subset V_1$ un SLI $\overset{?}{\Longrightarrow}$ $f(S) = \{f(v_1), .., f(v_k)\} \subset V_2$ este SLI

$$\sum_{i=1}^{k} a_i f(v_i) = 0_{V_2} \Rightarrow f\left(\sum_{i=1}^{k} a_i v_i\right) = 0_{V_2} \Rightarrow \sum_{i=1}^{k} a_i v_i = 0_{V_1}$$

$\operatorname{Ker} f = \{0_{V_1}\}$

$\xRightarrow{S \text{ e } SLI} a_i = 0, \forall i = \overline{1,k}$

"$\Leftarrow$" Fie $x \neq 0_{V_1} \Rightarrow S = \{x\}$ e SLI $\xRightarrow{ip} f(S) = \{f(x)\}$ e SLI

Dem că $\operatorname{Ker} f = \{0_{V_1}\}$.

Pp. $\exists\, x \in \operatorname{Ker} f$, $x \neq 0_{V_1} \Rightarrow f(x) \neq 0$ ~~b~~ $\Rightarrow \operatorname{Ker} f = \{0_{V_1}\}$

b) "$\Rightarrow$" $V_1 = \langle S \rangle = \langle \underline{\{v_1, .., v_k\}} \rangle \overset{?}{\Rightarrow}$
$V_2 = \langle f(S) \rangle = \langle \{f(v_1), .., f(v_k)\} \rangle$.

f surj: $\forall y \in V_2, \exists x \in V_1$ aî $f(x) = y$.

$$y = f(x) = f\left(\sum_{i=1}^{k} a_i v_i\right) = \sum_{i=1}^{k} a_i f(v_i) \Rightarrow V_2 = \langle f(S) \rangle$$

"$\Leftarrow$" Ip: $V_1 = \langle S \rangle \Rightarrow V_2 = \langle f(S) \rangle$

Dem că f surj: $\forall y \in V_2, \exists x \in V_1$ aî $f(x) = y$.

$$y \in V_2 \Rightarrow y = \sum_{i=1}^{k} a_i f(v_i) = f\left(\sum_{i=1}^{k} a_i v_i\right)$$

Fie $x = \sum_{i=1}^{k} a_i v_i$

c) f bij $\xRightarrow{a), b)}$ $\mathcal{R}$ reper în $V_1 \Rightarrow f(\mathcal{R})$ reper în V_2.

Matricea asociată unei apl. liniare

$f: V_1 \longrightarrow V_2$ apl liniară, $\dim_{\mathbb{K}} V_1 = n$; $\dim_{\mathbb{K}} V_2 = m$

$\underbrace{\mathcal{R}_1 = \{e_1, .., e_n\}}_{\text{reper în } V_1} \xrightarrow{A} \underbrace{\mathcal{R}_2 = \{\overline{e}_1, .., \overline{e}_m\}}_{\text{reper în } V_2}$.

(*) $f(e_i) = \sum_{j=1}^{m} a_{ji} \overline{e}_j, \forall i = \overline{1,n}$

$A = [f]_{\mathcal{R}_1, \mathcal{R}_2}$ $\quad A \in \mathcal{M}_{m,n}(\mathbb{K})$

Teoremă de caracterizare (apl. lin)

$f : V_1 \to V_2$ apl.

f liniară $\iff \exists\, A \in \mathcal{M}_{m,n}(\mathbb{K})$ a.î. coordonatele lui $x \in V_1$ în rap cu reperul $R_1 = \{e_1, .., e_n\}$ din V_1 și coordonatele lui $f(x) = y \in V_2$ în raport cu $R_2 = \{\bar{e}_1, .., \bar{e}_m\}$ din V_2 verifică relația

$$Y = AX, \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A = (a_{ji})_{\substack{j=\overline{1,m} \\ i=\overline{1,n}}}$$

$$x = \sum_{i=1}^{n} x_i e_i, \quad y = \sum_{j=1}^{m} y_j \bar{e}_j$$

Dem

"$\Rightarrow$" Ip: f liniară

- $f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i f(e_i) \overset{*}{=}$
$\sum_{i=1}^{n} x_i \left(\sum_{j=1}^{m} a_{ji} \bar{e}_j\right) = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ji} x_i\right)\bar{e}_j$

- $f(x) = y = \sum_{j=1}^{m} y_j \bar{e}_j$

$\Rightarrow y_j = \sum_{i=1}^{n} a_{ji} x_i, \ \forall j = \overline{1,m} \Rightarrow Y = AX$

"$\Leftarrow$" $Y = AX$

$f : V_1 \to V_2$. Dem: $f(ax_1 + bx_2) = af(x_1) + bf(x_2)$ (1)

$$\left.\begin{array}{l} Y_1 = AX_1 \\ Y_2 = AX_2 \end{array}\right. \Rightarrow aY_1 + bY_2 = aAX_1 + bAX_2 = A(aX_1 + bX_2)$$

$\Rightarrow (1) \Rightarrow f$ liniară

Modificarea matricei lui f la schimbarea reperelor

$f : V_1 \longrightarrow V_2$ apl. lin.

$$\begin{array}{ccc} \mathcal{R}_1 = \{e_1, \dots, e_n\} & \xrightarrow{A_f} & \mathcal{R}_2 = \{\bar{e}_1, \dots, \bar{e}_m\} \\ C \downarrow & & \downarrow D \\ \mathcal{R}_1' = \{e_1', \dots, e_n'\} & \xrightarrow{A_f'} & \mathcal{R}_2' = \{\bar{e}_1', \dots, \bar{e}_m'\}. \end{array}$$

$$\boxed{A' = D^{-1} A C}$$

$A_f = [f]_{\mathcal{R}_1, \mathcal{R}_2}$

$A_f' = [f]_{\mathcal{R}_1', \mathcal{R}_2'}$

$$f(e_i) = \sum_{j=1}^{m} a_{ji} \bar{e}_j, \quad \forall i = \overline{1,n}$$

$$\boxed{f(e_k')} = \boxed{\sum_{l=1}^{m} a_{lk}' \bar{e}_l'}, \quad \forall k = \overline{1,n}$$

$$f(e_k') = f\left(\sum_{i=1}^{n} c_{ik} e_i\right) = \sum_{i=1}^{n} c_{ik} f(e_i) = \underbrace{\sum_{j=1}^{m} \sum_{i=1}^{n} c_{ik} a_{ji} \bar{e}_j}$$

$$\sum_{l=1}^{m} a_{lk}' \bar{e}_l' = \sum_{l=1}^{m} a_{lk}' \sum_{j=1}^{m} d_{jl} \bar{e}_j = \sum_{j=1}^{m} \underbrace{\left(\sum_{l=1}^{m} d_{jl} a_{lk}'\right)} \bar{e}_j$$

$$\sum_{i=1}^{n} a_{ji} c_{ik} = \sum_{l=1}^{m} d_{jl} a_{lk}', \quad \forall j = \overline{1,m}$$

$$AC = DA' \Longrightarrow \boxed{A' = D^{-1} A C}$$

Prop Rangul matricei asociate lui f este un invariant la schimbarea reperelor

$rg(A') = rg(D^{-1}AC)$, D, C sunt inversabile

OBS. a) $\bullet$ $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$

$$\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[rr, bend right, "h = g\circ f"'] & V_2 \arrow[r, "g"] & V_3 \end{tikzcd}$$

$h: \quad Z = A_h X$

$g: \quad Z = A_g Y$

$f: \quad Y = A_f X$

$Z = A_h X$

$Z = A_g \cdot A_f X$ $\Longrightarrow$ $\boxed{A_{g\circ f} = A_g \cdot A_f}$

$\bullet$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[rr, bend right, "id_V"'] & V \arrow[r, "f^{-1}"] & V \end{tikzcd} \qquad f \in Aut(V)$$

$$\left.\begin{aligned} I_n &= A_{f^{-1}\circ f} = A_{f^{-1}} \cdot A_f \\ I_n &= A_{f\circ f^{-1}} = A_f \cdot A_{f^{-1}} \end{aligned}\right\} \Rightarrow (A_f)^{-1} = A_{f^{-1}}$$

$\bullet$ $\varphi : (GL(V), \circ) \longrightarrow (GL(n, \mathbb{K}), \cdot)$

$f \longrightarrow A$ izom. de grupuri.

$GL(V) = \{ f : V \to V \mid f$ automorfism de sp vect$\}$.

a) $\varphi(f\circ g) = \varphi(f) \cdot \varphi(g)$

b) φ bij :

~~Obs~~ Prop.

$f : V_1 \to V_2$ liniară.

a) f inj $\iff \dim V_1 = rg\, A$

b) f surj $\iff \dim V_2 = rg\, A$

c) f bij $\iff \dim V_1 = \dim V_2 = rg\, A \iff A \in GL(n, \mathbb{K})$

Dem

a) f inj $\iff Ker\, f = \{0_{V_1}\}$

$Ker\, f = \{x \in V_1 \mid AX = 0\}$

$\dim Ker\, f = \dim V_1 - rg\, A \overset{f\ inj}{\iff} \dim V_1 = rg\, A$

b) f surj $\iff \dim Im\, f = \dim V_2$.

$\dim V_1 = \underbrace{\dim Ker\, f}_{\dim V_1 - rg\, A} + \dim Im\, f \iff \dim Im\, f = rg\, A$

$\dim V_2 = rg\, A$.

c) f bij $\iff f$ izom. de sp vect $\overset{a), b)}{\iff} \dim V_1 = \dim V_2 = rg\, A$

$A \in \mathcal{M}_n(\mathbb{K})$

$\Rightarrow A \in GL(n, \mathbb{K})$.

Aplicație

$f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1 + x_2 - x_3, x_1 + x_2, x_1 + x_2 + x_3)$.

a) $A = [f]_{R_0, R_0}$, $R_0 = \{e_1, e_2, e_3\}$ reper canonic în $\mathbb{R}^3$

b) $Ker\, f$, $Im\, f$.

SOL

a) ~~$\forall$~~ $f(x) = y \iff Y = AX \iff f$ liniară

$$\begin{pmatrix} x_1 + x_2 - x_3 \\ x_1 + x_2 \\ x_1 + x_2 + x_3 \end{pmatrix} = \begin{pmatrix} 1 & \boxed{1 \quad -1} \\ 1 & \boxed{1 \quad 0} \\ 1 & 1 \quad 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\det A = 0 \Rightarrow f$ nu e bij.

b) $\operatorname{Ker} f = \{x \in \mathbb{R}^3 \mid AX = \theta\}$.

$\dim \operatorname{Ker} f = 3 - \operatorname{rg} A = 3 - 2 = 1$.

$$\underbrace{\dim \mathbb{R}^3}_{=3} = \underbrace{\dim \operatorname{Ker} f}_{=1} + \dim \operatorname{Im} f \Rightarrow \underline{\dim \operatorname{Im} f = 2}.$$

~~Ker f = {~~

$$\begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 + x_2 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 - x_3 = -x_1 \\ x_2 = -x_1 \end{cases}$$

$x_3 = 0$.

$$\operatorname{Ker} f = \{(\underbrace{x_1, -x_1, 0}_{= x_1(1,-1,0)}) \mid x_1 \in \mathbb{R}\} = \langle\{(+1,-1,0)\}\rangle$$

$\{(1,-1,0)\}$ reper în $\operatorname{Ker} f$.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Reper în $\operatorname{Im} f$: $\{f(0,1,0), f(0,0,1)\}$.