CURS 11

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Spații vectoriale euclidiene
Endomorfisme simetrice

Teorema Cauchy-Buniakowski-Schwarz

$(E, \langle \cdot, \cdot \rangle)$ s.v.e.r, $x, y \in E$

$\Rightarrow |\langle x, y\rangle| \leq \|x\| \, \|y\|$

Mai mult, „=" $\Leftrightarrow \{x, y\}$ este SLD.

Dem

1) Dacă $x = 0$ sau $y = 0 \Rightarrow 0 \leq 0$ (A)

2) Dacă $x \neq 0$ și $y \neq 0$, fie $\lambda \in \mathbb{R}$ aî

$\langle x + \lambda y, x + \lambda y \rangle \geq 0$

$\|x\|^2 + 2\lambda \langle x, y\rangle + \lambda^2 \|y\|^2 \geq 0, \forall \lambda \in \mathbb{R}$

$\lambda^2 \|y\|^2 + 2\lambda \langle x, y\rangle + \|x\|^2 \geq 0, \forall \lambda \in \mathbb{R} \Rightarrow$

$\Delta_\lambda \leq 0 \Rightarrow 4\langle x, y\rangle^2 - 4\|x\|^2 \cdot \|y\|^2 \leq 0$

$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$

Dem că „=" $\Leftrightarrow \{x, y\}$ e SLD.

„$\Rightarrow$"

„=" $\Leftrightarrow \Delta_\lambda = 0 \Leftrightarrow \exists \lambda_0 \in \mathbb{R}$ aî

$\langle x + \lambda_0 y, x + \lambda_0 y \rangle = 0 \xrightarrow[\text{pozitiv}]{\text{definit}} x + \lambda_0 y = 0$

$\Rightarrow \{x, y\}$ e SLD

„$\Leftarrow$" Ip: $\{x, y\}$ e SLD $\Rightarrow \exists a \in \mathbb{R}^*$ aî $y = ax$

$|\langle x,y\rangle| = |\langle x, ax\rangle| = |a|\cdot\|x\|^2$. $\Rightarrow$ „="

$\|x\|\cdot\|y\| = \|x\|\cdot\|ax\| = |a|\cdot\|x\|^2$

$\|ax\| = \sqrt{\langle ax, ax\rangle} = \sqrt{a^2\|x\|^2} = |a|\cdot\|x\|^2$

<u>Teoremă</u>

$(E, \langle\cdot,\cdot\rangle)$ s.v.e.r

$U \subseteq E$ subsp. vect $\Rightarrow E = U \oplus U^\perp$ (scrierea unică)

$(U^\perp$ = complement ortogonal$)$

<u>Dem</u>

$U, U^\perp = \{x \in E \mid \langle x,y\rangle = 0, \forall y \in U\} \underset{\text{subsp vect}}{\subseteq} E$

$\Rightarrow U + U^\perp \underset{\text{subsp v.}}{\subseteq} E$

Fie $x \in U \cap U^\perp \Rightarrow x \in U$ și $x \in U^\perp$ $\Rightarrow \underset{Q(x)}{\langle x,x\rangle} = 0 \underset{\text{def}}{\overset{\text{poz}}{\Longrightarrow}} x = 0$

$\Rightarrow U \oplus U^\perp \subseteq E$. (1)

Dem că $E \subseteq U \oplus U^\perp$ (2)

Fie $\dim U = k$, $R = \{e_1, .., e_k\}$ reper ortonormat în U.

Fie $v \in E$. Considerăm

$v' = v - \sum_{i=1}^{k} \langle v, e_i\rangle e_i$.

Dem că $v' \in U^\perp$

$$\begin{cases} \langle v', e_1\rangle = \langle v, e_1\rangle - \underbrace{\sum_{i=1}^{k} \langle v, e_i\rangle \underbrace{\langle e_i, e_1\rangle}_{\delta_{i1}}}_{\langle v, e_1\rangle} = 0 \\ \vdots \\ \langle v', e_k\rangle = \langle v, e_k\rangle - \langle v, e_k\rangle = 0 \end{cases}$$

Fie $x \in U \Rightarrow x = \sum_{i=1}^{k} x_i e_i$

$\langle v', x \rangle = \sum_{i=1}^{k} x_i \underbrace{\langle v', e_i \rangle}_{= 0} = 0 \Rightarrow v' \in U^{\perp}$

$$\Rightarrow v = \underbrace{\sum_{i=1}^{k} \langle v, e_i \rangle e_i}_{\in U} + \underbrace{v'}_{\in U^{\perp}} \Rightarrow E \subseteq U \oplus U^{\perp} \ (2)$$

Din (1), (2) $\Rightarrow E = U \oplus U^{\perp}$

Aplicatie

$(\mathbb{R}^4, g_0)$

$U = \left\{ x \in \mathbb{R}^4 \mid \begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 + x_2 - x_4 = 0 \end{cases} \right\}$

a) $U^{\perp} = ?$

b) Det un reper ortonormat $R = R_1 \cup R_2$ in $\mathbb{R}^4$, unde R_1, R_2 repere ortonormate in U, resp $U^{\perp}$

SOL

a) $A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}$

$\dim U = 4 - \operatorname{rg} A = 4 - 2 = 2$

$U = \{ \underbrace{(x_1, x_2, -x_1 + x_2, x_1 + x_2)}_{=} \mid x_1, x_2 \in \mathbb{R} \}$

$x_1 \underbrace{(1, 0, -1, 1)}_{f_1} + x_2 \underbrace{(0, 1, 1, 1)}_{f_2}$

$\left.\begin{array}{l} U = \langle \{ f_1, f_2 \} \rangle \\ \dim U = 2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \{ f_1, f_2 \} \text{ reper in } U \\ g_0(f_1, f_2) = 0 \end{array}\right\}$ reper ortogonal.

$U^{\perp} = \left\{ x \in \mathbb{R}^4 \mid \begin{cases} g_0(x, f_1) = 0 \\ g_0(x, f_2) = 0 \end{cases} \right\} = \left\{ x \in \mathbb{R}^4 \mid \begin{cases} x_1 - x_3 + x_4 = 0 \\ x_2 + x_3 + x_4 = 0 \end{cases} \right\}$

$$U^\perp = \{ (x_3 - x_4, -x_3 - x_4, x_3, x_4) \mid x_3, x_4 \in \mathbb{R} \}$$

$$= x_3 \underbrace{(1,-1,1,0)}_{f_3} + x_4 \underbrace{(-1,-1,0,1)}_{f_4}$$

$\{f_3, f_4\}$ reper în $U^\perp$

$g_0(f_3, f_4) = -1+1+0+0 = 0$

$\Rightarrow \{f_3, f_4\}$ reper ortogonal.

$R_1 = \left\{ \frac{1}{\sqrt{3}}(1,0,-1,1), \frac{1}{\sqrt{3}}(0,1,1,1) \right\}$ reper orton. în U.

$R_2 = \left\{ \frac{1}{\sqrt{3}}(1,-1,1,0), \frac{1}{\sqrt{3}}(-1,-1,0,1) \right\}$ — // — $U^\perp$

$R = R_1 \cup R_2$ reper orton în $\mathbb{R}^4$.

Endomorfisme simetrice

Def $(E, \langle\cdot,\cdot\rangle)$ s.v.e.r, $f \in End(E)$

f s.n. endomorfism simetric $\Leftrightarrow \langle x, f(y)\rangle = \langle f(x), y\rangle, \ \forall x, y \in E$

$(f \in Sim(E))$

Prop $f \in Sim(E) \Leftrightarrow$ matricea asoc. în rap cu $\forall$ reper ortonormat este simetrică

Dem

Fie $R = \{e_1, .., e_n\}$ reper ortonormat în E.

$\langle e_i, f(e_j)\rangle = \langle f(e_i), e_j\rangle$, $[f]_{R,R} = A = (a_{ij})_{i,j=\overline{1,n}}$

$$\langle e_i, \sum_{r=1}^{n} a_{rj} e_r\rangle = \langle \sum_{k=1}^{n} a_{ki} e_k, e_j\rangle$$

$$\sum_{r=1}^{n} a_{rj} \underbrace{\langle e_i, e_r\rangle}_{\delta_{ir}} = \sum_{k=1}^{n} a_{ki} \underbrace{\langle e_k, e_j\rangle}_{\delta_{kj}} \Rightarrow a_{ij} = a_{ji} \Rightarrow A = A^T$$

Scanned with CamScanner

$\mathcal{R}=\{e_1,\dots,e_n\} \xrightarrow{C} \mathcal{R}'=\{e_1',\dots,e_n'\} \Rightarrow C\in O(n)$

repere ortonormate

$A'=[f]_{\mathcal{R}',\mathcal{R}'}$, $A'=C^{-1}AC=C^TAC$

$A'^T=(C^TAC)^T=C^TA^T(C^T)^T=C^TAC=A'$

$\Rightarrow A'$ simetrică.

<u>OBS</u> $f_i\in Sim(E)$, $i=\overline{1,2}$. În general, $f_1\circ f_2\notin Sim(E)$

$$f_1\circ f_2\in Sim(E) \Leftrightarrow \underbrace{\langle f_1\circ f_2(x), y\rangle}_{\substack{\| \\ \langle f_2(x), f_1(y)\rangle \\ \| \\ \langle x, f_2\circ f_1(y)\rangle}} = \langle x, f_1\circ f_2(y)\rangle$$

$\Leftrightarrow f_1\circ f_2=f_2\circ f_1 \Leftrightarrow A_1A_2=A_2A_1$.

<u>Prop</u>. $f\in Sim(E) \Rightarrow$ vectorii proprii coresp. la valori proprii distincte sunt $\perp$

<u>Dem</u>

Fie $\lambda,\mu\in\mathbb{R}$, $\lambda\neq\mu$ valori proprii

$\Rightarrow \exists\, x,y\in E\setminus\{0_E\}$ a.î. $f(x)=\lambda x$
$f(y)=\mu y$

$\langle x, f(y)\rangle = \langle f(x), y\rangle$

$\langle x, \mu y\rangle = \langle \lambda x, y\rangle \Rightarrow \mu\langle x,y\rangle=\lambda\langle x,y\rangle$

$\langle x,y\rangle \underbrace{(\mu-\lambda)}_{\neq 0}=0 \Rightarrow \langle x,y\rangle=0 \Rightarrow x\perp y$

Teoremă $f \in Sim(E) \Rightarrow$ toate răd. polinomului caracteristic sunt reale.

Prop $f \in End(E)$

Dacă $f \in Sim(E)$ și $U \subseteq E$ subspațiu invariant $\Rightarrow U^{\perp} \subseteq E$ este subspațiu invariant.

Dem

$U \subseteq E$ subsp. invariant al lui $f \Leftrightarrow f(U) \subseteq U$

Dem că $f(U^{\perp}) \subseteq U^{\perp}$

i.e. $\forall x \in U^{\perp} \Rightarrow f(x) \in U^{\perp}$

Fie $y \in U$, $\langle f(x), \underset{U}{\underset{\cap}{y}} \rangle = \langle \underset{U^{\perp}}{\underset{\cap}{x}}, \underset{U}{\underset{\cap}{f(y)}} \rangle = 0$

$\Rightarrow f(x) \in U^{\perp}$

$\boxed{\text{Teoremă}}$ Dacă $f \in Sim(E)$, atunci $\exists R$ un reper ortonormat în E, format din vectori proprii ai f și $[f]_{R,R}$ este diagonală.

Dem Fie R_0 un reper ortonormat arbitrar în E și $A = [f]_{R_0, R_0}$.

$P(\lambda) = \det(A - \lambda I_n) \overset{T}{\Rightarrow}$ are toate răd. reale.

Fie λ_1 valoare proprie și e_1 = <u>versor</u> propriu

$\Rightarrow \|e_1\| = 1$ și $f(e_1) = \lambda_1 e_1 \Rightarrow$

$\langle \{e_1\} \rangle$ subsp. invariant al lui $f \Rightarrow$

$\langle \{e_1\} \rangle^{\perp}$ subsp invariant al lui f.

$\Rightarrow f|_{<\{e_1\}>^\perp}$ e endom simetric.

Fie λ_2 valoare proprie a restricției și e_2 versor propriu

$\Rightarrow f(e_2) = \lambda_2 e_2$. $\Rightarrow <\{e_1, e_2\}>$ subsp. invar.

dar $f(e_1) = \lambda_1 e_1$.

$e_1 \perp e_2$

$\Rightarrow <\{e_1, e_2\}>^\perp$ subsp. invar.

Continuăm raționamentul și după n pași construim $R = \{e_1, .., e_n\}$ un sistem de versori, mutual ortogonali $\Rightarrow R$ este SLI $\Big\}\Rightarrow$

$\dim E = n = |R|$

R este reper ortonormat

$$[f]_{R,R} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \lambda_n \end{pmatrix}$$

$\boxed{\text{OBS}}$ a) $f \in Sim(E) \Rightarrow \dim V_{\lambda_i} = m_i$, $i = \overline{1,r}$

unde $\lambda_1, .., \lambda_r$ răd dist. ale pol. caract, (<u>care sunt reale</u>)

$m_1, .., m_r$ = multiplicitățile coresp, $m_1 + ... + m_r = n$

$E = V_{\lambda_1} \oplus ... \oplus V_{\lambda_r}$, $R = R_1 \cup ... \cup R_r$,

R_i reper orton. în V_{λ_i}, $i = \overline{1,r}$

$$A = [f]_{R,R} = \begin{pmatrix} \overbrace{\lambda_1 \ldots \lambda_1}^{m_1 \text{ ori}} & & & O \\ & \overbrace{\lambda_2 \ldots \lambda_2}^{m_2 \text{ ori}} & & \\ & & \ddots & \\ O & & & \overbrace{\lambda_r \ldots \lambda_r}^{m_r \text{ ori}} \end{pmatrix}$$

b) $\mathcal{R}_0=\{e_1^0,\dots,e_n^0\} \xrightarrow{C} \mathcal{R}=\{e_1,\dots,e_n\}$ repere orton

$C\in O(n)$, $h\in O(E)$, $h(e_i^0)=e_i$, $i=\overline{1,n}$

$C=[h]_{\mathcal{R}_0,\mathcal{R}_0}$, $h(e_i^0)=\sum_{k=1}^{n} c_{ki}\, e_k^0$, $\forall i=\overline{1,n}$

c) ! $A=A^T$ → 1) $f\in Sim(E)$

→ 2) $Q: E\to \mathbb{R}$, $Q(x)=X^TAX$

forma pătratică asociată lui f

$\langle x, f(x)\rangle = Q(x)$, $\forall x\in E$.

<u>Def</u> $f\in Sim(E)$ s.n. pozitiv definit dc. Q este poz def.

<u>Prop</u> $f\in Sim(E)$, pozitiv def $\Rightarrow \exists h\in Sim(E)$ pozitiv def a.î. $f=h^2$

<u>Teorema</u> (de descompunere polară)

$\forall f\in Aut(E) \Rightarrow \exists h\in Sim(E)$, $\exists t\in O(E)$ a.î. $f=h\circ t$

<u>Aplicatie</u> $(\mathbb{R}^3, g_0)$

Fie $f\in End(\mathbb{R}^3)$ a.î. $[f]_{\mathcal{R}_0,\mathcal{R}_0}=\begin{pmatrix}1&1&-1\\1&1&-1\\-1&-1&1\end{pmatrix}$,

$\mathcal{R}_0$ = reper canonic

a) $f\in Sim(\mathbb{R}^3)$

b) Să se det $Q:\mathbb{R}^3\to\mathbb{R}$ forma pătratică asociată. Det. un reper $\mathcal{R}$ ortonormat în $\mathbb{R}^3$ în rap cu care Q are formă canonică.

Precizati transf. ortog. ce realizează sch. de repere!

Sol

a) $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = A^T \rightarrow f \in Sim(\mathbb{R}^3)$

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $f(x) = (x_1 + x_2 - x_3,\ x_1 + x_2 - x_3,\ -x_1 - x_2 + x_3)$

b) $Q : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $\langle x, Q(x) \rangle = f(x)$

$Q(x) = X^T A X = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 2x_2x_3$

Met. valorilor proprii

$P(\lambda) = \det(A - \lambda I_3) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{vmatrix} = 0$

$\lambda^3 - \sigma_1 \lambda^2 + \sigma_2 \lambda - \sigma_3 = 0.$

$\sigma_1 = Tr A = 3$

$\sigma_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0$

$\sigma_3 = \det A = 0$

$\lambda^3 - 3\lambda^2 = 0 \Rightarrow \lambda^2(\lambda - 3) = 0.$

$\lambda_1 = 0,\ m_1 = 2$

$\lambda_2 = 3,\ m_1 = 1.$

$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid f(x) = 0 \} = Ker f$

$= \{ x \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0 \} =$

$= \{ (x_1, x_2, x_1 + x_2),\ x_1, x_2 \in \mathbb{R} \} = \langle \{ f_1, f_2 \} \rangle$

$(x_1, x_2, x_1+x_2) = x_1 \underbrace{(1,0,1)}_{f_1} + x_2 \underbrace{(0,1,1)}_{f_2}$

$\dim V_{\lambda_1} = 2 \Rightarrow \{ f_1, f_2 \}$ reper $\forall$ in V_{λ_1}.

Aplicăm Gram - Schmidt

$\begin{cases} e_1 = f_1 \\ e_2 = f_2 - \dfrac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 = (0,1,1) - \dfrac{1}{2}(1,0,1) = \left(-\dfrac{1}{2}, 1, \dfrac{1}{2}\right) = \dfrac{1}{2}(-1,2,1) \end{cases}$

$\left\{ e_1' = \frac{1}{\sqrt{2}}(1,0,1),\ e_2' = \frac{1}{\sqrt{6}}(-1,2,1) \right\}$ reper orton. în V_{λ_1},

$V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid f(x) = 3x \}$

$$\begin{cases} x_1 + x_2 - x_3 = 3x_1 \\ x_1 + x_2 - x_3 = 3x_2 \\ -x_1 - x_2 + x_3 = 3x_3 \end{cases} \Rightarrow \begin{cases} -2x_1 + x_2 - x_3 = 0 \\ x_1 - 2x_2 - x_3 = 0 \\ -x_1 - x_2 - 2x_3 = 0. \end{cases}$$

$$\det \begin{pmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix} = 0$$

$$\begin{cases} -2x_1 + x_2 = x_3 \quad \Big| \cdot 2 \\ x_1 - 2x_2 = x_3 \end{cases}$$

$$-3x_1 \quad / \quad = 3x_3$$

$x_1 = -x_3$

$x_2 = x_3 - 2x_3 = -x_3$

$V_{\lambda_2} = \{ (-x_3, -x_3, x_3) \mid x_3 \in \mathbb{R} \} = \langle \{ (-1,-1,1) \} \rangle$

$\left\{ e_3' = \frac{1}{\sqrt{3}}(-1,-1,1) \right\}$ reper orton în V_{λ_2}.

$R = \left\{ e_1' = \frac{1}{\sqrt{2}}(1,0,1),\ e_2' = \frac{1}{\sqrt{6}}(-1,2,1),\ \frac{1}{\sqrt{3}}(-1,-1,1) \right\}$

reper orton în $\mathbb{R}^3$

$$[f]_{R,R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

$Q(x) = 3x_3'^2$

$(1,0)$ signatura $\Rightarrow Q$ nu e poz. def.

$R_0 \xrightarrow{C} R$

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$h \in O(\mathbb{R}^3)$

$h(e_i^0) = e_i',\ i = \overline{1,3}$

$$h(x) = \left(\frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{6}}x_2 - \frac{1}{\sqrt{3}}x_3,\ \frac{2}{\sqrt{6}}x_2 - \frac{1}{\sqrt{3}}x_3,\ \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{6}}x_2 + \frac{1}{\sqrt{3}}x_3 \right)$$

OBS $Q(x) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 2x_2x_3$

Met Gauss

$$Q(x) = (x_1 + x_2 - x_3)^2$$

$$\begin{cases} x_1' = x_1 + x_2 - x_3 \\ x_2' = x_2 \\ x_3' = x_3 \end{cases} \Rightarrow Q(x) = x_1'^2$$

$(1,0)$ signatura