20.04.2021.

CURS 10

Transformări ortogonale

Def $(E_i, <\cdot,\cdot>_i)_{i=\overline{1,2}}$ sp. vect. euclidiene reale.

Aplicația liniară $f: E_1 \to E_2$ s.n. aplicație ortogonală

$\Leftrightarrow$ $<f(x), f(y)>_2 = <x,y>_1, \forall x,y \in E_1$ (*)

Prop Dc $f: E_1 \to E_2$ este aplicație ortogonală, atunci 1) $\|f(x)\|_2 = \|x\|_1, \forall x \in E_1$

2) f injectivă

Dem

a) Considerăm $x = y$ în (*) $\Rightarrow$

$<f(x), f(x)>_2 = <x,x>_1 \Rightarrow \|f(x)\|_2^2 = \|x\|_1^2 \Rightarrow$

$\|f(x)\|_2 = \|x\|_1, \forall x \in E_1$.

b) f liniară

f inj $\Leftrightarrow$ $\mathrm{Ker} f = \{0_{E_1}\}$.

Fie $x \in \mathrm{Ker} f \Rightarrow f(x) = 0_{E_2}$

$\underbrace{\|f(x)\|_2}_{=0} = \|x\|_1 \Rightarrow x = 0_{E_1} \Rightarrow f$ liniară.

(produsul scalar este pozitiv).

Def $(E, <\cdot,\cdot>)$ sp. vect. euclidian real, $f \in \mathrm{End}(E)$

f s.n. transformare ortogonală $\Leftrightarrow <f(x), f(y)> = <x,y>$ $\forall x,y \in E$.

Prop $f \in O(E) = \{f \in \mathrm{End}(E) \mid f \text{ transf. ortogonală}\}$

$\Leftrightarrow \|f(x)\| = \|x\|, \forall x \in E$.

Dem

"$\Rightarrow$" (cf. prop preced.)

"$\Leftarrow$" $\|f(x+y)\|^2 = \|x+y\|^2 \Rightarrow$

$$\underbrace{\langle f(x+y), f(x+y)\rangle}_{} = \langle x+y, x+y\rangle$$

$$\langle f(x)+f(y), f(x)+f(y)\rangle$$

$$\underline{\|f(x)\|^2} + \underline{\|f(y)\|^2} + 2\langle f(x), f(y)\rangle = \underline{\|x\|^2} + \underline{\|y\|^2} + 2\langle x, y\rangle.$$

$$\Rightarrow \langle f(x), f(y)\rangle = \langle x, y\rangle, \ \forall x, y \in E \Rightarrow f \in O(E)$$

Matricea asociată unei transf. ortogonale

$(E, \langle\cdot,\cdot\rangle)$ s.v.e.r, $R = \{e_1, e_2, \dots, e_n\}$ reper ortonormat

$A = [f]_{R,R}$, $f \in O(E)$

$$\langle f(e_i), f(e_j)\rangle = \langle e_i, e_j\rangle, \ \forall i,j = \overline{1,n}$$

$$\langle \sum_{r=1}^{n} a_{ri} e_r, \sum_{s=1}^{n} a_{sj} e_s\rangle = \langle e_i, e_j\rangle$$

$$\sum_{s,r=1}^{n} a_{ri} a_{sj} \underbrace{\langle e_r, e_s\rangle}_{\delta_{rs}} = \underbrace{\langle e_i, e_j\rangle}_{\delta_{ij}}$$

$$\sum_{r=1}^{n} a_{ri} a_{rj} = \delta_{ij} \Rightarrow A^T A = I_n \Rightarrow A \in O(n)$$

OBS $R = \{e_1, \dots, e_n\} \xrightarrow{C} R' = \{e_1', \dots, e_n'\}$, $C \in O(n)$

repere ortonormate

$A' = [f]_{R',R'}$, $A' = C^{-1}AC = C^TAC$

$$A'^T A' = (C^TAC)^T(C^TAC) = C^TA^T\underbrace{C\,C^T}_{I_n}AC = I_n.$$

Prop $f \in O(E) \Leftrightarrow$ matricea asociată, în raport cu $\forall$ reper ortonormat, este ortogonală

1) Dacă $\det A = 1$, at f s.n. transf. ortog. de speță 1.
$(A \in SO(n))$

2) Dacă $\det A = -1$, at f s.n transf ortog de speță 2.

OBS

a) $(O(E), \circ)$ grupul transf. ortogonale.

! **b)** $f \in O(E) \iff$ Schimbare de repere ortonormate.

„$\Rightarrow$" $f \in O(E) \Rightarrow A = [f]_{R,R} \in O(n)$

$R \xrightarrow{A} R'$ repere ortonormate.

„$\Leftarrow$" $\underset{\{e_1, .., e_n\}}{R} \xrightarrow{A} \underset{\{e_1', .., e_n'\}}{R'}$ repere ortonormate $A \in O(n)$

Fie $f \in End(E)$, $f(e_i) = e_i' = \sum_{j=1}^{n} a_{ji} e_j$ $\forall i = \overline{1,n}$

Prelungim prin liniaritate

$f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i f(e_i) = \sum_{i=1}^{n} x_i e_i' = x'$

Aplicație $(\mathbb{R}^3, g_0)$ sp. vect. euclidian canonic

$f : \mathbb{R}^3 \to \mathbb{R}^3$ apl. liniară

$$[f]_{R_0,R_0} = A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

1) $A \cdot A^T = I_3 \Rightarrow A \in O(3) \Rightarrow f \in O(\mathbb{R}^3)$

sau

2) $A \in O(3) \iff R_0 = \{e_1, e_2, e_3\} \xrightarrow{A} R' = \{e_1', e_2', e_3'\}$.
repere ortonormate

$f(e_1) = e_1' = (1, 0, 0)$

$f(e_2) = e_2' = \left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$f(e_3) = e_3' = \left(0, \frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

$\langle e_i', e_j' \rangle = \delta_{ij}$ $\forall i,j = \overline{1,3}$

Scanned with CamScanner

OBS

$A \in O(2)$, $\exists \varphi \in (-\pi, \pi]$ ai

a) $\det A = 1 \Rightarrow A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin & \cos\varphi \end{pmatrix}$

b) $\det A = -1 \Rightarrow A = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$

Prop

$(E, \langle\cdot,\cdot\rangle)$ s.v.e.r, $U \subseteq E$ subspatiu invariant al lui $f \in O(E)$ (i.e. $f(U) \subseteq U$)

$\Rightarrow$ a) $f(U) = U$

b) $U^\perp \subseteq E$ subsp. invariant al lui f

c) $f|_{U^\perp} : U^\perp \to U^\perp$ transf. ortogonala.

Dem

a) $U \subseteq E$ ssp inv $\Rightarrow f(U) \subseteq U$

$f : U \to f(U)$ izom. de sp. vect

$$\left.\begin{array}{l} \Rightarrow \dim U = \dim f(U) \\ \text{dar } f(U) \subseteq U \end{array}\right\} \Rightarrow f(U) = U.$$

b) Dc $U \subseteq E$ subsp. invar, at $U^\perp \subseteq E$ subsp invar.

i.e. $f(U^\perp) \subseteq U^\perp$

Fie $x \in U^\perp$. Dem că $f(x) \in U^\perp$

Fie $y \in \underset{f(U)}{\overset{\|}{U}} \Rightarrow \exists z \in U$ ai $y = f(z)$

$$\langle f(x), \underset{U}{\underset{\cap}{y}} \rangle = \langle f(x), f(z) \rangle \underset{f \in O(E)}{=} \langle \underset{U^\perp}{\underset{\cap}{x}}, \underset{U}{\underset{\cap}{z}} \rangle = 0$$

$\Rightarrow f(x) \in U^\perp$

c) cf a) $f(U^\perp) = U^\perp$ și $f|_{U^\perp} : U^\perp \to U^\perp$ transf. ortog

OBS $(E, \langle\cdot,\cdot\rangle)$ s.v.e.r; $p, s \in End(E)$, $p^2 = p$, $s^2 = id_E$

$s = 2p - id_E$

Not $E' = Ker\, p$ $\quad E = E' \oplus E''$

$E'' = Im\, p$ $\quad \forall x \in E, \exists!\, x' \in E', x'' \in E''$ a.î. $x = x' + x''$

Dacă $E'' = E'^{\perp}$, at p s.n. proiecție ortogonală pe E''

s s.n. simetrie ortogonală față de E''

$p(x') = 0$, $p(x'') = x''$

$s(x') = -x'$; $s(x'') = x''$

$\mathcal{R}_1 = \{e_1, .., e_k\}$, $\mathcal{R}_2 = \{e_{k+1}, .., e_n\}$ repere orton în E', resp E''

$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ reper orton. în E.

$$A_p = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_{n-k} \end{array}\right) \notin O(n), \quad A_s = \left(\begin{array}{c|c} -I_k & 0 \\ \hline 0 & I_{n-k} \end{array}\right) \in O(n)$$

Prop $(E, \langle\cdot,\cdot\rangle)$ s.v.e.r, $f \in O(E)$

$\Rightarrow$ valorile proprii sunt ± 1

SOL λ = valoare proprie a lui $f \Leftrightarrow \exists x \in E$ a.î. $f(x) = \lambda x$, $x \neq 0$ (x vector propriu)

$\|f(x)\| = \|x\| \Rightarrow \|\lambda x\| = \|x\| \Rightarrow |\lambda|\,\|x\| = \|x\|$

$\left.\begin{array}{l} \Rightarrow |\lambda| = 1 \\ \lambda \in \mathbb{R} \end{array}\right\} \Rightarrow \lambda = \pm 1$

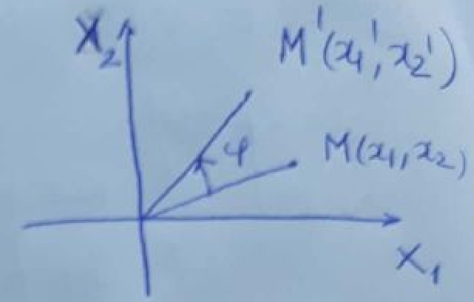
Clasificarea transf. ortogonale

① $\dim E = 1$.

$\mathcal{R} = \{e\}$ reper ortonormat, e = versor $\Rightarrow f(e) = \lambda e$

$\Rightarrow \lambda = \pm 1 \Rightarrow f(e) = \pm e \Rightarrow f(x) = \pm x \Rightarrow f \in \{id_E, -id_E\}$

(2) $\dim E = 2$. $A \in O(2)$



a) $\det A = 1$, $A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

$f: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f \in O(\mathbb{R}^2)$

$f(x_1, x_2) = (\underbrace{x_1\cos\varphi - x_2\sin\varphi}_{x_1'}, \underbrace{x_1\sin\varphi + x_2\cos\varphi}_{x_2'})$

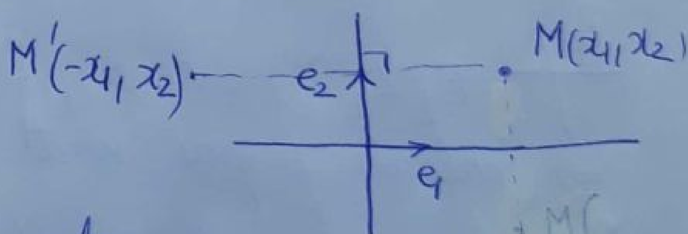
f = rotație de unghi orientat φ.

b) $\det A = -1$. $\exists$ o schimbare de reper ortonormat $\mathcal{R} = \{e_1, e_2\}$

a.î. $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (-x_1, x_2)$

f = simetrie ortogonală față de $\langle\{e_2\}\rangle = \langle\{e_1\}\rangle^{\perp}$



<u>Teoremă</u> $\dim E = 2$

$\forall f \in O(E)$, $f \neq id_E$ se poate scrie ca o compunere de cel mult 2 simetrii ortogonale (față de drepte)

<u>Dem</u>

1) $f \in SO(E)$ i.e $\det(A_f) = 1$.

Fie s o simetrie ortogonală i.e. $\det A_s = -1$.

$s \circ f \in O(E)$ de speța 2, $\det(A_s A_f) = -1$

$s \circ f = s'$ simetrie ortogonală.

$\underbrace{s \circ s}_{id_E} \circ f = s \circ s' \Rightarrow f = s \circ s'$

2) $f \in O(E)$ de speța 2 $\Rightarrow f = s$ = simetrie ortog

(3) $\dim E = 3$

$f \in O(E)$, $P(\lambda) = \det(A - \lambda I_3) = 0$ (polinom de gradul al 3-lea cu coef reali) $\Rightarrow$ are cel puțin o răd reală $\lambda \in \{\pm 1\}$.

Fie e_1 = versor propriu pt $\lambda \in \{-1, 1\}$.

$f(e_1) = \lambda e_1 = \pm e_1 \Rightarrow \langle\{e_1\}\rangle \subset E$ subsp. invar al lui f

$\Rightarrow \langle\{e_1\}\rangle^{\perp} \subset E$ subsp. invar. al lui f.

$E = \langle\{e_1\}\rangle \oplus \langle\{e_1\}\rangle^{\perp}$.

$f/_{\langle\{e_1\}\rangle^{\perp}} : \underbrace{\langle\{e_1\}\rangle^{\perp}}_{2-\dim} \longrightarrow \langle\{e_1\}\rangle^{\perp}$ transf. ortog

Not $\tilde{A}$ matricea asociată restricției, $\tilde{A} \in O(2)$.

A) $\det A = 1$ ($f \in O(E)$ de speța 1)

a_1) $\lambda = 1 \Rightarrow f(e_1) = e_1$ $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\tilde{A}} \\ 0 & & \end{pmatrix}$

$\Rightarrow \det \tilde{A} = 1 \Rightarrow \tilde{A} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

a_2) $\lambda = -1 \Rightarrow f(e_1) = -e_1$. $\quad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\tilde{A}} \\ 0 & & \end{pmatrix}$

$\Rightarrow \det \tilde{A} = -1 \Rightarrow \exists\, \tilde{R} = \{e_2, e_3\}$ a.î. $\tilde{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

În reperul $\{e_1, e_2, e_3\}$: $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

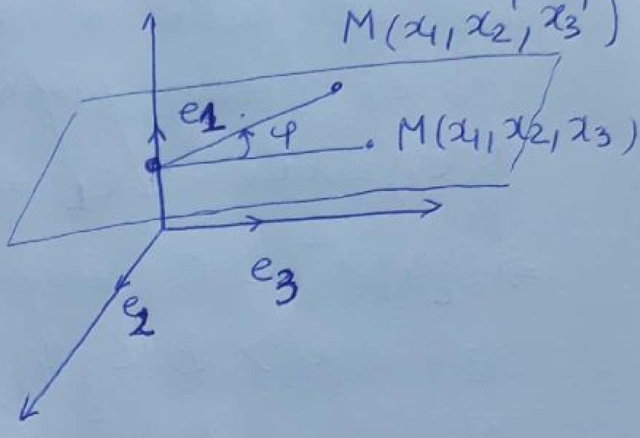
În reperul $\{e_3, e_1, e_2\}$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \boxed{\begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix}} \\ 0 & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\pi & -\sin\pi \\ 0 & \sin\pi & \cos\pi \end{pmatrix}$

<u>Teoremă</u> $\dim E = 3$. Dacă $f \in SO(E)$, at $\exists\, R = \{e_1, e_2, e_3\}$ reper ortonormat a.î. $[f]_{R,R} = A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

$f: E \to E$

$f(x) = (x_1, \underbrace{x_2\cos\varphi - x_3\sin\varphi}_{x_2'}, \underbrace{x_2\sin\varphi + x_3\cos\varphi}_{x_3'})$

f = rotație de ∢ orientat φ și axă $<\{e_1\}>$
$\|$
R_φ



OBS
a) $\mathrm{Tr}\,A = 1 + 2\cos\varphi$
este invariant la schimbarea de repere.

b) Axa de rotație: $f(x) = x$

$x \in <\{e_1\}> \Rightarrow x = \alpha e_1$

$f(x) = \alpha f(e_1) = \alpha e_1 = x$

B) $\det A = -1$.

$b_1)$ $\lambda = 1$, $f(e_1) = e_1$, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\tilde{A}} \\ 0 & & \end{pmatrix}$

$\Rightarrow \det\tilde{A} = -1 \Rightarrow \exists \{e_2, e_3\}$ reper ortonormat a.î. $\tilde{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

În reperul $\{e_1, e_2, e_3\}$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

În reperul $\{e_2, e_1, e_3\}$ $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos 0 & -\sin 0 \\ 0 & \sin 0 & \cos 0 \end{pmatrix}$

$b_2)$ $\lambda = -1$, $f(e_1) = -e_1$ $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\tilde{A}} \\ 0 & & \end{pmatrix}$

$\Rightarrow \det\tilde{A} = 1 \Rightarrow \tilde{A} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

Teoremă $\dim E = 3$, $f \in O(E)$ de speța 2
$\Rightarrow \exists$ un reper ortonormat $R = \{e_1, e_2, e_3\}$ a.î.

$[f]_{R,R} = A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

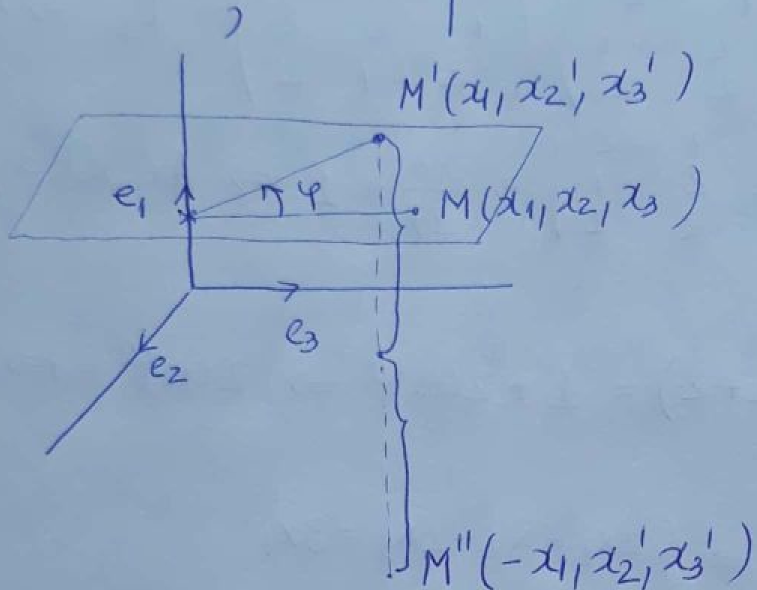
$f: E \to E$, $f(x) = (-x_1, \overbrace{x_2\cos\varphi - x_3\sin\varphi}^{x_2'}, \overbrace{x_2\sin\varphi + x_3\cos\varphi}^{x_3'})$

$f = s \circ R_\varphi$ R_φ = rotație de ∢ orientat φ și axă $<\{e_1\}>$.
s = simetrie ortog. față de $<\{e_1\}^\perp>$

<u>OBS</u>

a) $\operatorname{Tr} A = -1 + 2\cos\varphi$ invar. la sch. de reper ortonormat

b) Axa de rotatie : $f(x) = -x$.



4) $\dim E \geq 4 \Rightarrow \exists$ un reper ortonormat ai

$$A = \begin{pmatrix} \overbrace{1 \ \ldots \ 1}^{s \text{ ori}} & & & & & 0 \\ & \underbrace{-1 \ \ldots \ -1}_{(k-s) \text{ ori}} & & & & \\ & & A_1 & \ldots & & \\ 0 & & & & A_p \end{pmatrix} \qquad k + 2p = n$$

$$A_j = \begin{pmatrix} \cos\varphi_j & -\sin\varphi_j \\ \sin\varphi_j & \cos\varphi_j \end{pmatrix}, \quad j = \overline{1,p}$$

<u>Teorema Cartan</u>

$\forall f \in O(E)$, $f \neq id_E$, $n = \dim E \geq 1$.

$\Rightarrow f$ se poate scrie ca o compunere de cel mult n simetrii ortogonale fata de hiperplane

<u>Aplicatie</u> $f : \mathbb{R}^3 \to \mathbb{R}^3$ $(\mathbb{R}^3, g_0)$

$$f(x) = \frac{1}{3}(2x_1 + x_2 - 2x_3, -2x_1 + 2x_2 - x_3, x_1 + 2x_2 + 2x_3)$$

a) $f \in SO(E)$

b) $\exists$ un reper orton $R = \{e_1, e_2, e_3\}$ ai

$$[f]_{R,R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

SOL

a) $A = \frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}$

$A^T \cdot A = \frac{1}{9}\begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ -2 & -1 & 2 \end{pmatrix} = I_3 \Rightarrow A \in O(3)$

$\det A = \frac{1}{27}\begin{vmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{vmatrix} = \frac{27}{27} = 1 \Rightarrow A \in SO(3)$

$\Rightarrow f = R_\varphi$.

b) $Tr A = \frac{1}{3}(2+2+2) = 1 + 2\cos\varphi \Rightarrow 2\cos\varphi = 1 \Rightarrow \cos\varphi = \frac{1}{2}$

$\varphi = \pm\frac{\pi}{3}$.

Axa: $f(x) = x \Rightarrow \begin{cases} 2x_1 + x_2 - 2x_3 = 3x_1 \\ -2x_1 + 2x_2 - x_3 = 3x_2 \\ x_1 + 2x_2 + 2x_3 = 3x_3. \end{cases}$

$\Rightarrow \langle\{(-1,1,1)\}\rangle$ axa.

$\langle\{e_1\}\rangle^\perp = \{x \in \mathbb{R}^3 \mid g_0(x, \underbrace{(-1,1,1)) = 0}_{-x_1+x_2+x_3}\}$

$= \{(x_2+x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\}$ $\begin{cases} \bar{e}_2 = f_2 \\ \bar{e}_3 = f_3 - \frac{\langle f_3, \bar{e}_2\rangle}{\langle \bar{e}_2, \bar{e}_2\rangle}\bar{e}_2 \end{cases}$

$= x_2\underbrace{(1,1,0)}_{f_2} + x_3\underbrace{(1,0,1)}_{f_3}$

$e_2 = \frac{1}{\|\bar{e}_2\|}\bar{e}_2$; $e_3 = \frac{1}{\|\bar{e}_3\|}\cdot\bar{e}_3$

$\{f_2, f_3\}$ Aplicăm Gram-Schmidt.

$\{e_2 = \frac{1}{\sqrt{2}}(1,1,0),\ e_3 = \frac{1}{\sqrt{6}}(1,-1,2)\}$ reper orton. în $\{e_1\}^\perp$

$\mathcal{R} = \{\frac{1}{\sqrt{3}}(-1,1,1), \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{6}}(1,-1,2)\}$

$[f]_{\mathcal{R},\mathcal{R}} = A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

$\boxed{\text{Tema 5}}$ (curs)

Ex1 Fie $(\mathbb{R}^3, g_0)$ s.v.e.r, cu str. canonică

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_3, x_2, x_1)$

a) Să se arate că f = transf. ortogonală de speța 2

b) Să se determine un reper ortonormat $R = \{e_1, e_2, e_3\}$

a.î. $[f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$.

Ex2 $V = \{A \in \mathcal{M}_2(\mathbb{R}) \mid A = A^T\}$

Fie $g: V \times V \to \mathbb{R}$, $g(A, B) = Tr(AB)$

Este (V, g) spațiu vect. euclidian?

Ex3 Fie $(\mathbb{R}_2[X], +, \cdot)_{/\mathbb{R}}$, $g: \mathbb{R}_2[X] \times \mathbb{R}_2[X] \to \mathbb{R}$

$$g(P, Q) = \int_{-1}^{1} P(x) Q(x) dx.$$

a) g = produs scalar

b) Să se ortonormeze $R_0 = \{1, X, X^2\}$ în raport cu g.