Seminar 6

① $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (2x_1 + 2x_2,$
$x_1 + x_3, x_1 + 3x_2 - 2x_3)$

a) f nu este izomorfism de sp vect

b) $f|_{V'} : V' \to V''$ izomorfism, unde

$V' = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\}$

$V'' = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 3x_1 - 4x_2 - 2x_3 = 0\}$

c) $f(V' \cap V'')$

d) $\mathbb{R}^3 = V' \oplus W$. Dați ex de W

Fie $p: \mathbb{R}^3 \to \mathbb{R}^3$ proiecția pe V'

$s : \mathbb{R}^3 \to \mathbb{R}^3$ simetria față de V'

Să se calculeze $p(1,3,6)$, $s(1,3,6)$

a) $f(x) = y \iff Y = AX$

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix} = [f]_{R_0, R_0}$$

1,5,0)

$$\det A = \begin{vmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{vmatrix} = 0 \Rightarrow A \text{ + inversabilă}, f \text{ + bij}$$

Obs $\operatorname{Ker} f = \{x \in \mathbb{R}^3 \mid f(x) = 0\} = \{x \in \mathbb{R}^3 \mid Ax = O_3, Y = S(A$

$\dim \operatorname{Ker} f = 3 - \operatorname{rang} A = 3 - 2 = 1$

Din teorema dimensiunii:

$$\underbrace{\dim_{\mathbb{R}} \mathbb{R}^3}_{3} = \underbrace{\dim \operatorname{Ker} f}_{1} + \underbrace{\dim \operatorname{Im} f}_{2}$$

b) $x_3 = x_1 + x_2 \Rightarrow V' = \{(x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R}\}$

$= x_1(1,0,1) + x_2(0,1,1)$

$= \langle \underbrace{\{(1,0,1), (0,1,1)\}}_{R'} \rangle$

R' SG $\not\vdash$ V'

$\dim V' = 3 - 1 = 2 = \operatorname{card} R' \Rightarrow R'$ reper în V'

$f(\mathcal{R}') = \{ f(1,0,1), f(0,1,1) \}$

$f(1,0,1) = (2,2,-1) = e_1''$

$f(0,1,1) = (2,1,1) = e_2''$

Verificăm că $e_1'' \in V''$ și $e_2'' \in V''$

$3 \cdot 2 - 4 \cdot 2 - 2 \cdot (-1) = 6 - 8 + 2 = 0$

$3 \cdot 2 - 4 \cdot 1 - 2(1) = 0$

$\dim V'' = 3 - 1 = 2 = \text{card } \mathcal{R}''$

$\text{rang} \begin{pmatrix} 2 & 2 \\ 2 & 1 \\ -1 & 1 \end{pmatrix} = 2 = \max \xrightarrow{\text{CLi}} \mathcal{R}'' \text{ SLi}$ $\Rightarrow \mathcal{R}''$ reper

$f|_{V'} : V' \to V''$ bij $\Rightarrow f|_{V'}$ izom.

c) $V' \cap V'' = \{ x \in \mathbb{R}^3 \mid \begin{cases} x_1 + x_2 - x_3 = 0 \\ 3x_1 - 4x_2 - 2x_3 = 0 \end{cases} \}$

$B = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 3 & -4 & -2 & 0 \end{array}\right)$, $\dim V' \cap V'' = 3 - \text{rang } B = 3 - 2 = 1$.

$\begin{cases} x_1 + x_2 = x_3 \quad | \cdot 4 \\ 3x_1 - 4x_2 = 2x_3 \end{cases} \Rightarrow 7x_1 = 6x_3$

$x_1 = \dfrac{6x_3}{7}$

$x_2 = \dfrac{x_3}{7}$

$V' \cap V'' = \{ (\frac{6x_3}{7}, \frac{x_3}{7}, x_3) \mid x_3 \in \mathbb{R} \}$

$= \{ \frac{x_3}{7} (6, 1, 7) \mid x_3 \in \mathbb{R} \}$

$= \langle \{ (6, 1, 7) \} \rangle$

$f(6, 1, 7) = (14, 13, -5) \neq 0_{\mathbb{R}^3}$

$\dim f(V' \cap V'') = 1$, $f(V' \cap V'') = \langle \{ (14, 13, -15) \} \rangle$

d) $R' = \{ (1,0,1), (0,1,1) \}$ reper in V'

$$rg \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = 3 = \text{maxim} \Rightarrow W = \langle \{ (0,0,1) \} \rangle$$

$s, p : V' \oplus W \to V' \oplus W$, $p(v' + w) = v'$

$s(v' + w) = (2p - id_{\mathbb{R}^3})(v' + w)$

$= 2v' - (v' + w) = v' - w$

$(1, 3, 6) = a(1,0,1) + b(0,1,1) + c(0,0,1)$

$= (a, b, a+b+c) \Rightarrow \begin{cases} a = 1 \\ b = 3 \\ c = 2 \end{cases}$

$(1, 3, 6) = \underbrace{(1,0,1) + 3(0,1,1)}_{v' = (1,3,4)} + \underbrace{2(0,0,1)}_{w = (0,0,2)}$

$p(1, 3, 6) = (1, 3, 4)$

$s(1, 3, 6) = v' - w = (1, 3, 4) - (0, 0, 2) = (1, 3, 2)$

③ $f: \mathbb{R}_2[x] \to \mathbb{R}_1[x]$, $f(P) = P'$

a) $[f]_{R,R'} = ?$ $\quad R = \{x^2, 1+x, 2-x\}$ reper in $\mathbb{R}_2[x]$

$R' = \{x, 1+3x\}$ — " — $\mathbb{R}_1[x]$

b) $\mathbb{R}_2[x] = \operatorname{Ker} f \oplus W$

$p_1 : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ proiecția pe $\operatorname{Ker} f$

$p_2 : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ — " — W

$p_1(1 - x + 3x^2)$, $p_2(2x + x^2) = ?$

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a) R reper $\Leftrightarrow rg\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} = 3$

R' reper $\Leftrightarrow rg\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$

$f(x^2) = 2x = 2 \cdot x + 0(1+3x)$

$f(1+x) = 1 = -3x + 1(1+3x) = 1$

$f(2-x) = \cancel{3x} - 1 = 3x - = -1(1+3x) = -1$

$A = \begin{pmatrix} 2 & -3 & 3 \\ 0 & 1 & -1 \end{pmatrix} = [f]_{R,R'}$

a) $\operatorname{Ker} f = \{ g \in \mathbb{R}_2[x] \mid f(g) = 0 \} = \langle \{1\} \rangle$

$W = \langle \{x, x^2\} \rangle$

$$1 - x + 3x^2 = \underbrace{1 \cdot 1}_{\in \operatorname{Ker} g} + \underbrace{(-x + 3x^2)}_{\in W}$$

$$p_1(1 - x + 3x^2) = 1.$$

$$2x + 3x^2 = \underbrace{0 \cdot 1}_{\in \operatorname{Ker} g} + \underbrace{2x + 3x^2}_{\in W}$$

$$p_2(2x + 3x^2) = 2x + 3x^2$$

(4). $f : \mathcal{M}_2(\mathbb{R}) \longrightarrow \mathcal{M}_2^s(\mathbb{R})$, $f(A) = A + A^t$

$R_0 = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ reper în $\mathcal{M}_2(\mathbb{R})$

$R_0' = \{E_{11}, E_{12} + E_{21}, E_{22}\}$ reper canonic în $\mathcal{M}_2^s(\mathbb{R})$

a) $[f]_{R_0, R_0'}$.

$$B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

b) $\ker f$, $\operatorname{Im} f$.

c) $f(V) = ?$, $V = \left\{ \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \middle| \, c, d \in \mathbb{R} \right\}$

a) $f(E_{11}) = f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} =$

$= 2 \cdot E_{11} + 0 \cdot (E_{12} + E_{21}) + 0 \cdot E_{22}$

$f(E_{12}) = f\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =$

$= 0 \cdot E_{11} + 1 \cdot (E_{12} + E_{21}) + 0 \cdot E_{22}$

$$f(E_{21}) = f\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cancel{0}1 \\ 1 & 0 \end{pmatrix} = 0 \cdot E_{11} + 1 \cdot (E_{12} + E_{21}) + 0 \cdot E_{22}$$

$$f(E_{22}) = f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = 0 \cdot E_{11} + 0 \cdot (E_{12} + E_{21}) + 2 \cdot E_{22}$$

<u>Obs</u>: $f\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} + \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 & x_2+x_3 \\ x_2+x_3 & 2x_4 \end{pmatrix}$

$$(x_1, x_2, x_3, x_4) \rightarrow (2x_1, x_2+x_3, x_2+x_3, 2x_4)$$

$$f\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = 2x_1 \cdot E_{11} + (x_2+x_3)(E_{12} + E_{21}) + 2x_4 \cdot E_{22}$$

b) $\ker f = \{A \in \mathcal{M}_2(\mathbb{R}) \mid f(A) = O_2 \Rightarrow A = -A^t\}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix} \Leftrightarrow \begin{cases} a = -a \Rightarrow a = 0 \\ b = -c \\ d = -d \Rightarrow d = 0 \end{cases}$$

$$\ker f = \left\{\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \middle| b \in \mathbb{R}\right\} = \left\langle\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}\right\rangle$$

din Teorema dimensiunii $\Rightarrow \dim \operatorname{Im} f = 3 = \dim \mathcal{M}_2^S(\mathbb{R})$

așadar, f este surjectivă $\Rightarrow \operatorname{Im} f = \mathcal{M}_2^S(\mathbb{R})$

c) $f(V) = \left\{\begin{pmatrix} 0 & c \\ c & 2d \end{pmatrix} \middle| c, d \in \mathbb{R}\right\} = \langle\{E_{12} + E_{21}, E_{22}\}\rangle$

ex : $f : \mathbb{R}^n \to \mathbb{R}^n$ endomorfism a.i. $f^2 = f + id_{\mathbb{R}^n}$

$\Rightarrow f \in Aut(\mathbb{R}^n)$

$f^2 - f = id_{\mathbb{R}^n}$

$A^2 - A = I_n$

$A(A - I_n) = I_n \Rightarrow A$ inversabilă $\Rightarrow f$ bijecție $\Rightarrow$

$\Rightarrow$ automorfism

ex $(\mathbb{R}^3, +, \cdot) / \mathbb{R}$

$R = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\} \xrightarrow{C}$

$R' = \{e_1' = e_1 + e_2 + e_3, e_2' = e_1 + e_2, e_3' = e_1\}$

$((\mathbb{R}^*) = \{f : \mathbb{R}^3 \to \mathbb{R} \mid f$ liniară$\}, +, \cdot) / \mathbb{R}$ sp. vect. dual

$R^* = \{e_1^*, e_2^*, e_3^*\} \xrightarrow{D} (R')^* = \{e_1'^*, e_2'^*, e_3'^*\}$

repere duale în sp. dual

$e_i^*(e_j) = \delta_{ij}$, $e_i'^*(e_j') = \delta_{ij}$, $\forall i,j = \overline{1,3}$

$C, D = ?$

$e_i'^* : \mathbb{R}^3 \to \mathbb{R}$

$e_i^* : \mathbb{R}^3 \to \mathbb{R}$

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$e_1'^*(e_1') = 1$ $\qquad e_1'^* = a e_1^* + b e_2^* + c e_3^*$

$\parallel$

$e_1'^*(e_1 + e_2 + e_3)$

$1 = a(e_1^*(e_1) + e_1^*(e_2) + e_1^*(e_3)) + b(e_2^*(e_1) +$

$+ e_2^*(e_2) + e_2^*(e_3)) + c(e_3^*(e_1) + e_3^*(e_2) + e_3^*(e_3))$

$1 = a(1 + 0 + 0) + b(0 + 1 + 0) + c(0 + 0 + 1)$

$1 = a + b + c$

$0 = e_1'^*(e_2') = a + b$

$e_2' = e_1 + e_2$

$0 = e_1'^*(e_3') = a \Rightarrow \begin{cases} a = 0 \\ b = 0 \\ c = 1 \end{cases}$

$e_3' = e_1$

$$D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$a' + b' + c' = 0$

$a' + b' = 1$

$a' = 0$ $\qquad \begin{cases} a' = 0 \\ b' = 1 \\ c' = -1 \end{cases}$

$a'' + b'' + c'' = 0$
$a'' + b'' = 0$
$a'' = 1$

$\begin{cases} a'' = -1 \\ b'' = -1 \\ c'' = 0 \end{cases}$

e) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1 + x_2 - x_3, -x_1 - x_2 + x_3, x_1 + x_2 + x_3)$

b) $\mathbb{R}^3 = \operatorname{im} f \oplus W$

$s: \mathbb{R}^3 \to \mathbb{R}^3$ simetria față de W

$s(0,1,1) =$

$$\cancel{A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & \boxed{-1} & \boxed{1} \\ 1 & \boxed{1} & \boxed{1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\Delta_c = 0 = \begin{vmatrix} 1 & -1 & y_1 \\ -1 & 1 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & y_1 + y_2 \\ -1 & 1 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = -2(y_1 + y_2)$$

$= 0$

$\operatorname{im} f = \{ y \in \mathbb{R}^3 \mid y_1 + y_2 = 0 \} = \{ (y_1, -y_1, y_3) \mid y_1, y_2, y_3 \in \mathbb{R} \}$

$= \langle \{ (1,-1,0), (0,0,1) \} \rangle$

$\operatorname{rg} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 3 \Rightarrow W = \langle \{ e_1 \} \rangle$

$(0,1,1) = \overbrace{a(1,-1,0) + b(0,0,1)}^{u} + \overbrace{c(1,0,0)}^{w}$
$= (a + c, -a, b)$

$p = 1$

$q = -1$

$r = 1$

$$D(u + w) = -u + w = (1, -1, 0) - (0, 0, 1) + (1, 0, 0)$$
$$= (2, -1, -1).$$