Seminar 9

Spații vectoriale euclidiene.

Repere ortonormate.

Procedeul Gram-Schmidt.

① $g : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad g(x,y) = ax_1y_1 + bx_1y_2 + bx_2y_1 + cx_2y_2$

a) $g \in L^s(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$

b) g produs scalar $\Leftrightarrow \begin{cases} a > 0 \\ ac - b^2 > 0 \end{cases}$

a) $g(x,y) = \underbrace{\sum_{i,j=1}^{2} g_{ij} x_i y_j}_{= X^T G Y} \Leftrightarrow g \in L(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$

$G = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ $\quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \; Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$(g_{ij})_{i,j=\overline{1,2}}$

$G = G^T \Rightarrow g \in L^s(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$

b) $Q : \mathbb{R}^2 \longrightarrow \mathbb{R}$ forma pătratică asociată lui g

$Q(x) = g(x,x) = a x_1^2 + 2b x_1 x_2 + c x_2^2$

g poz. def. $\Leftrightarrow$ Signatura lui Q e $(2,0) \Leftrightarrow$

$\Leftrightarrow \begin{cases} \Delta_1 = |a| = a > 0 \\ \Delta_2 = \begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2 > 0 \end{cases}$

Obs Metoda Jacobi:

$\exists R'$ reper în $\mathbb{R}^2$ a.î. $Q(x) = \frac{1}{\Delta_1} x_1'^2 + \frac{\Delta_1}{\Delta_2} x_2'^2$

(2) $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ formă biliniară

$G = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ matricea asociată în raport cu R_0

Este $(\mathbb{R}^3, g)$ spațiu vectorial euclidian real?

Soluție:

$G = G^T \Rightarrow g \in L^s(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

Este g pozitiv definită?

$\Delta_1 = 3 \qquad \Delta_2 = \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 6 - 4 = 2 > 0$

$\Delta_3 = \begin{vmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{vmatrix} = 6 - 4 - 12 = -10 < 0$

$\exists R'$ a.î. $Q(x) = \frac{1}{3} x_1'^2 + \frac{3}{2} x_2'^2 + \left(-\frac{1}{5}\right) x_3'^2$

signatura este $(2,1) \Rightarrow g$ nu este poz. def $\Rightarrow g$ nu este produs scalar

$Q(x) = 3x_1^2 + 4x_1x_2 + 2x_2^2 + 4x_2x_3 + x_3^2$

(3) $(\mathbb{R}^3, g_0)$, $g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g_0(x,y) = x_1y_1 + x_2y_2 + x_3y_3$

$U = \{ x \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0 \}$

a) $U^\perp$

b) Să se determine $R = R_1 \cup R_2$ reper ortonormat în $\mathbb{R}^3$

R_1 — " — U

R_2 — " — $U^\perp$

a) $U = \{ x \in \mathbb{R}^3 \mid g_0(x, (1,1,-1)) = 0 \} \Rightarrow$

$\Rightarrow U^{\perp} = < \{ \underset{f_1}{\underset{\shortparallel}{(1,1,-1)}} \} >$

b) $R_2 = \{ \frac{1}{\sqrt{3}} (1,1,-1) \}$ reper ortonormat în $U^{\perp}$

$\cancel{\sqrt{g(x,x)} \;\; \langle x,x \rangle}$ $\quad (1, 1, -1)$

$(\boxed{1} \;\; 1 \;\; -1) \mid 0$

$x_3 = x_1 + x_2 \Rightarrow U = \{ (x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R} \} = < \{ \underset{f_1}{\underset{\shortparallel}{(1,0,1)}}, \underset{f_2}{\underset{\shortparallel}{(0,1,1)}} \}$

$\{ f_1, f_2 \}$ reper arbitrar în U (SG + $\dim U = 2$)

Gram - Schmidt $\Rightarrow$ $e_1 = f_1 = (1,0,1)$

$$e_2 = f_2 - \frac{< f_2, e_1 >}{< e_1, e_1 >} \cdot e_1 =$$

$$= (0,1,1) - \frac{1}{2} (1,0,1) = \left(-\frac{1}{2}, 1, \frac{1}{2} \right) = \frac{1}{2} (-1, 2, 1)$$

$e_2 = \alpha (-1, 2, 1) = \alpha v$

$$\frac{e_2}{\| e_2 \|} = \frac{v}{\| v \|} = \frac{\alpha v}{\| \alpha v \|} = \frac{\cancel{\alpha} v}{\cancel{\alpha} \| v \|}$$

$\underset{\text{reper arbitrar}}{\{ f_1, f_2 \}} \longrightarrow \underset{\text{reper ortogonal}}{\{ e_1, e_2 \}} \longrightarrow \underset{\text{reper ortonormat în } U}{\{ \frac{1}{\sqrt{2}} (1,0,1), \frac{1}{\sqrt{6}} (-1,2,1) \}} = R_1$

$R = R_1 \cup R_2 = \{ \frac{1}{\sqrt{2}} (1,0,1), \frac{1}{\sqrt{6}} (-1,2,1), \frac{1}{\sqrt{3}} (1,1,-1) \}$

⑥ $(\mathbb{R}^3, g_0)$ $\mathcal{R} = \{f_1 = (1,2,3),\ f_2 = (0,1,1),\ f_3 = (1,2,5)\}$

a) $\mathcal{R}$ reper in $\mathbb{R}^3$. Să se ortonormeze

b) $f_1 \times f_2$

c) $f_1 \wedge f_2 \wedge f_3$

Soluție:

a) $\begin{cases} e_1 = f_1 = (1,2,3) \\ e_2 = f_2 - \dfrac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} \cdot e_1 \\ e_3 = f_3 - \dfrac{\langle f_3, e_1\rangle}{\langle e_1, e_1\rangle} \cdot e_1 - \dfrac{\langle f_3, e_2\rangle}{\langle e_2, e_2\rangle} \cdot e_2 \end{cases}$

$\dim \mathbb{R}^3 = 3 = \dim \langle\{f_1, f_2, f_3\}\rangle$

$\mathcal{R}$, SLi

$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 5 \end{pmatrix}$ $\det A = 5 + 2 - 3 - 2 = 2 > 0$ rang maxim

$e_1 = f_1 = (1,2,3)$

$e_2 = (0,1,1) - \dfrac{5}{14} \cdot (1,2,3) = \dfrac{1}{14}(-5, 4, -1)$

$e_3 = (1,2,5) - \dfrac{20}{14} \cdot (1,2,3) - \dfrac{-\cancel{2}/\cancel{14}}{\left(\cancel{\frac{1}{14}}\right)^2 (25+16+1)} \cdot \cancel{\frac{1}{14}}(-5,4,-1) =$

$= (1,2,5) - \dfrac{10}{7}(1,2,3) - \dfrac{-2}{42_{21}}(-5,4,-1) =$

$= \dfrac{1}{21} \cdot \left[(21, 42, 105) - (30, 60, 90) + (-5, 4, -1)\right] =$

$=$

$$= \frac{1}{21}(-14,-14,14) = \frac{2}{3}(-1,-1,1)$$

$$\tilde{e}_1 = \frac{e_1}{\|e_1\|} = \frac{(1,2,3)}{\sqrt{14}}$$

$$\tilde{e}_2 = \frac{e_2}{\|e_2\|} = \frac{(-5,4,-1)}{\sqrt{42}}$$

$$\tilde{e}_3 = \frac{e_3}{\|e_3\|} = \frac{(-1,-1,1)}{\sqrt{3}}$$

b) $$f_1 \times f_2 = \begin{vmatrix} e_1^0 & e_2^0 & e_3^0 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = e_1^0 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} - e_2^0 \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} + e_3^0 \cdot \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} =$$

$$= -e_1^0 - e_2^0 + e_3^0 = (-1,-1,1)$$

c) $$f_1 \wedge f_2 \wedge f_3 = \langle f_1, f_2 \times f_3 \rangle$$

$$\parallel$$

$$f_3 \wedge f_1 \wedge f_2$$

$$\parallel$$

$$\langle f_3, f_1 \times f_2 \rangle = \langle (1,2,5), (-1,-1,1) \rangle = -1-2+5 = 2$$

11 ~~$C([a,b]) =$~~

⑦ $(\mathbb{R}_2[x], +, \cdot)_{/\mathbb{R}}$, $g : \mathbb{R}_2[x] \times \mathbb{R}_2[x] \to \mathbb{R}$

$g(P, Q) = \sum_{i=0}^{2} a_i b_i$, $P = a_0 + a_1 x + a_2 x^2$

$Q = b_0 + b_1 x + b_2 x^2$

Să se ortonormeze $\{2, 3-2x, 1-2x+x^2\}$

în raport cu produsul scalar g.

Soluție:

Observație: Lucrăm cu $(\mathbb{R}^3, g_0)$ și reperul pe care vrem să-l ortonormăm este $\{\underbrace{(2,0,0)}_{f_1}, \underbrace{(3,-2,0)}_{f_2}, \underbrace{(1,-2,1)}_{f_3}\}$

$$rg \begin{pmatrix} 2 & 3 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 1 \end{pmatrix} = 3 \Rightarrow \left.\begin{matrix} R \text{ e SLi} \\ \dim \mathbb{R}^3 = 3 \end{matrix}\right| \Rightarrow R \text{ reper}$$

Gram-Schmidt $\Rightarrow$ $e_1 = f_1 = (2,0,0)$

$$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 = (3,-2,0) - \frac{\cancel{6}\,3}{\cancel{4}\,2}(2,0,0)$$

$$= \frac{1}{2}(6-6, -4-0, 0-0) = \frac{1}{2}(0,-4,0) = (0,-2,0)$$

$$e_3 = f_3 - \frac{\langle f_3, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 - \frac{\langle f_3, e_2 \rangle}{\langle e_2, e_2 \rangle} \cdot e_2 = (1,-2,1) - \underbrace{\frac{2}{4}(2,0,0)}_{(1,0,0)} -$$

$$- \frac{\cancel{4}}{\cancel{4}}(0,-2,0) = (0,0,1)$$

Observație:

$$f_1 \times f_2 = \begin{vmatrix} e_1^0 & e_2^0 & e_3^0 \\ 2 & 0 & 0 \\ 3 & -2 & 0 \end{vmatrix} = e_3^0 \begin{vmatrix} 2 & 0 \\ 3 & -2 \end{vmatrix} = -4 e_3^0$$

$$\tilde{e}_1 = \frac{e_1}{\|e_1\|} = (1,0,0)$$

$$\tilde{e}_2 = \frac{e_2}{\|e_2\|} = (0,1,0)$$

$$\tilde{e}_3 = \frac{e_3}{\|e_3\|} = (0,0,1)$$

11. $C([a,b]) = \{f : [a,b] \longrightarrow \mathbb{R} \mid f \text{ cont}\}$

$g(f,h) = \int_a^b f(t)\, h(t)\, dt \quad (\forall)\ f,h \in C([a,b])$

Este $(C([a,b]), g)$ sp. vect. euclidian?

Soluție:

1) g formă biliniară
2) g simetrică
3) g poz def.

$g(\alpha f + \beta k, h) = \int_a^b (\alpha f(t) + \beta k(t)) \cdot h(t)\, dt =$

$= \alpha \int_a^b f(t) \cdot h(t)\, dt + \beta \int_a^b k(t) \cdot h(t)\, dt =$

$= \alpha g(f,h) + \beta g(k,h)$

$g(f,h) = g(h,f)$

$\Rightarrow g \in L^s([a,b],[a,b];\mathbb{R})$

$g(f,f) = \int_a^b f^2(t)\, dt > 0,\ (\forall)\ f \neq 0$
$g(f,f) = 0 \iff f = 0$ $\Bigg| \Rightarrow g$ poz. definită

Deci g este produs scalar și lucrăm cu un spațiu vectorial euclidian.

(10) $(E, \langle \cdot, \cdot \rangle)$ sp v. e. r

UAE
1) $x \perp y$ $\quad (g(x,y) = 0)$
2) $\|x-y\|^2 = \|x\|^2 + \|y\|^2$
3) $\|x-y\| = \|x+y\|$ $\quad (\forall)\ x, y \in E$

Soluție: $\|x-y\|^2 = g(x-y, x-y) = g(x,x) - 2g(x,y) - g(y,y) =$

$= \|x\|^2 + \|y\|^2 - 2g(x,y)$ $\quad *$

$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2g(x,y)$ $\quad **$

$1 \Rightarrow 2$ $\quad g(x,y) = 0 \Rightarrow \|x-y\|^2 = \|x\|^2 + \|y\|^2$

$2 \Rightarrow 1$ $\quad \|x-y\|^2 = \|x\|^2 + \|y\|^2 \overset{(*)}{\Rightarrow} g(x,y) = 0$

$1 \Rightarrow 3$ $\quad g(x,y) = 0 \overset{(*),(**)}{\Rightarrow} \|x-y\|^2 = \|x+y\|^2 = \|x\|^2 + \|y\|^2 \Rightarrow$

$\Rightarrow \|x-y\| = \|x+y\|$

$3 \Rightarrow 2$ $\quad \|x-y\| = \|x+y\| \Rightarrow \|x-y\|^2 = \|x+y\|^2 \Rightarrow$

$\Rightarrow -g(x,y) = g(x,y) \Rightarrow g(x,y) = 0$