Seminar 5 - GA

Subspații vectoriale.

② $(\mathbb{R}^3, +, \cdot)_{/\mathbb{R}}$, $S = \{(1,2,3), (-1,1,5)\}$

$S' = \{(1,5,11), (2,1,-2), \underset{\substack{\shortparallel \\ (1,5,11)+(2,1,-2)}}{(3,6,9)}\}$

a) $\langle S \rangle = \langle S' \rangle = V'$

b) Să se descrie V' printr-un sistem de ec. liniare

c) Să se det. V'' a.î $\mathbb{R}^3 = V' \oplus V''$

Soluție

a) $rg\begin{pmatrix} \boxed{\begin{matrix} 1 & -1 \\ 2 & 1 \end{matrix}} \\ 3 \quad 5 \end{pmatrix} = 2 = \text{maxim} \overset{CLi}{\Rightarrow} S \text{ SLi}$

$rg\begin{pmatrix} 1 & 2 & 3 \\ 5 & 1 & 6 \\ 11 & -2 & 9 \end{pmatrix} = 2 \Rightarrow S'' = \{(1,5,11), (2,1,-2)\}$

SLi maximal în S'

Dem că $\langle S \rangle = \langle S'' \rangle$

$(1,2,3) = a(1,5,11) + b(2,1,-2) =$

$= (a+2b, 5a+b, 11a-2b)$

$\begin{cases} a+2b=1 \\ 5a+b=2 \mid \cdot(-2) \end{cases} \Leftrightarrow \begin{cases} a+2b=1 \\ -10a-2b=-4 \end{cases} \oplus$

$-9a = -3 \Rightarrow a = \frac{1}{3} \Rightarrow b = \frac{1}{3}$

$(-1,1,5) = a(1,5,11) + b(2,1,-2) = (a+2b, 5a+b, 11a-2b)$

$$\begin{cases} a+2b=-1 \\ 5a+b=1 \quad | \cdot (-2) \end{cases} \Leftrightarrow \begin{cases} a+2b=-1 \\ -10a-2b=-2 \quad \oplus \end{cases}$$

$-9a = -3 \Rightarrow a = \frac{1}{3} \Rightarrow b = -\frac{2}{3}$

$$\left.\begin{array}{l} \langle S \rangle \subset \langle S'' \rangle \\ \dim \langle S \rangle = \dim \langle S'' \rangle \end{array}\right| \Rightarrow \langle S \rangle = \langle S'' \rangle = V'$$

c)

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 3 & 5 & 0 \end{pmatrix} \quad rg A = 3$$

$V'' = \langle \{e_1\} \rangle \qquad V' \oplus V'' = \mathbb{R}^3$

b) $V' = \langle \{(1,2,3), (-1,1,5)\} \rangle$

$$A = \begin{pmatrix} \boxed{\begin{matrix} 1 & -1 \\ 2 & 1 \end{matrix}} \\ \begin{matrix} 3 & 5 \end{matrix} \end{pmatrix} \quad rg A = 2$$

$\forall)\ x = (x_1, x_2, x_3)\ \exists$
$a, b \in \mathbb{R}$ a.î.

$x = (a-b, 2a+b, 3a+5b)$

S.C. $\Rightarrow rg A = rg \bar{A} = 2 \Rightarrow \Delta c = 0$

$$\Delta c = \begin{vmatrix} 1 & -1 & x_1 \\ 2 & 1 & x_2 \\ 3 & 5 & x_3 \end{vmatrix} = x_3 + 10x_1 - 3x_2 - 3x_1 + 2x_3 - 5x_2 =$$

$= 7x_1 - 8x_2 + 3x_3 = 0$

$V' = \{x \in \mathbb{R} \mid 7x_1 - 8x_2 + 3x_3 = 0\}$

(3)

$(\mathbb{R}^3, +, \cdot)/\mathbb{R}$ $\quad V' = \left\{ (x,y,z) \in \mathbb{R}^3 \mid \begin{cases} x - y + 2z = 0 \\ 2x + y + z = 0 \end{cases} \right\}$

Să se descompună $x = (-1, 3, 4)$ în raport cu

$\mathbb{R}^3 = V' \oplus V''$

$$\begin{cases} x - y + 2z = 0 \\ 2x + y + z = 0 \end{cases} \qquad A = \left(\boxed{\begin{matrix} 1 & -1 \\ 2 & 1 \end{matrix}} \begin{matrix} 2 \\ 1 \end{matrix} \right) \Bigg| \begin{matrix} 0 \\ 0 \end{matrix}$$

$\mathrm{rg}\, A = 2 \Rightarrow \dim V' = 3 - 2 = 1$

$$\begin{cases} x - y = -2z \\ 2x + y = -z \end{cases} \oplus$$

$3x = -3z \Rightarrow x = -z \Rightarrow y = z$

$$\Rightarrow \{ \underset{\substack{\shortparallel \\ z(-1,1,1)}}{(-z, z, z)} \mid z \in \mathbb{R} \} = < \underbrace{\{(-1,1,1)\}}_{R'} >$$

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = (-1)^{1+2} \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0$$

$V'' = < \underbrace{\{e_1, e_2\}}_{R''} > \qquad R = R' \cup R''$ reper în $\mathbb{R}^3 = V' \oplus V''$

$$x = (-1, 3, 4) = \underbrace{a(-1,1,1)}_{\substack{v' \\ \in V'}} + \underbrace{b(1,0,0) + c(0,1,0)}_{\substack{v'' \\ \in V''}} =$$

$= (-a + b, a + c, a)$

$$\begin{cases} -a+b=-1 \\ a+c=3 \\ a=4 \end{cases} \Rightarrow \begin{cases} a=4 \\ b=3 \\ c=-1 \end{cases}$$

Coordonatele lui x în raport cu $\mathcal{R}$ sunt $(4,3,-1)$

$V' = (-4,4,4)$

$V'' = (3,-1,0)$

Matricea asociată unei apl. liniare

(2) $f: \mathbb{R}^3 \to \mathbb{R}^3 \quad f(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, 2x_1 + 5x_2 + 3x_3, -3x_1 - 7x_2 - 4x_3)$.

a) f liniară

b) $\ker f = ?$ Precizați un reper

c) $\operatorname{Im} f = ?$

d) $[f]_{\mathcal{R}_0, \mathcal{R}_0} = A = ?$, $\mathcal{R}_0$ = reperul canonic în $\mathbb{R}^3$

Soluție

a) $f(ax+by) = af(x) + bf(y) \quad (\forall)\ x,y \in \mathbb{R}^3\ \forall a,b \in \mathbb{R}$

$f(ax+by) = (ax_1 + by_1 + 2(ax_2+by_2) + ax_3 + by_3,$

$2(ax_1+by_1) + 5(ax_2+by_2) + 3(ax_3+by_3), -3(ax_1+by_1)$

$-7(ax_1+by_1) - 4(ax_3+by_3)) = f(ax) + f(by) =$

$= af(x) + bf(y)$

b) $\ker f = \{ x \in \mathbb{R}^3 / f(x) = 0_{\mathbb{R}^3} \} \Rightarrow \begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 2x_1 + 5x_2 + 3x_3 = 0 \\ -3x_1 - 7x_2 - 4x_3 = 0 \end{cases}$

$A = \left(\begin{array}{ccc} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{array} \right) \left| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right. \Rightarrow \det(A) = 0 \Rightarrow rg(A) = 2 \Rightarrow$

$\Rightarrow \dim \ker f = 3 - rg A = 3 - 2 = 1$

$\Delta p = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = 1$

$\begin{cases} x_1 + 2x_2 = -x_3 \quad | \cdot (-2) \\ 2x_1 + 5x_2 = -3x_3 \quad \oplus \end{cases}$

$x_2 = -x_3 \Rightarrow x_1 = x_3$

$\ker f = \{ (x_3, -x_3, x_3) / x_3 \in \mathbb{R} \} = < \underbrace{\{(1,-1,1)\}}_{R'} >$

R' reper in Kerf.

c) $\operatorname{Im} f = \{ y \in \mathbb{R}^3 / \exists\, x \in \mathbb{R}^3 \ f(x) = y \} \Rightarrow$

$\Rightarrow \begin{cases} x_1 + 2x_2 + x_3 = y_1 \\ 2x_1 + 5x_2 + 3x_3 = y_2 \\ -3x_1 - 7x_2 - 4x_3 = y_3 \end{cases}$ $A = \left(\begin{array}{ccc} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{array} \right) \left| \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} \right.$

$\Delta c. = \begin{vmatrix} 1 & 2 & y_1 \\ 2 & 5 & y_2 \\ -3 & -7 & y_3 \end{vmatrix} = 0 \overset{L_1 + L_2 + L_3}{\Rightarrow} \begin{vmatrix} 0 & 0 & y_1 + y_2 + y_3 \\ 2 & 5 & y_2 \\ -3 & -7 & y_3 \end{vmatrix}$

$\Rightarrow (y_1 + y_2 + y_3) = 0 \qquad \operatorname{Im} f = \{ y \in \mathbb{R}^3 / y_1 + y_2 + y_3 = 0 \} =$

$= \{ (y_1, y_2, -y_1 - y_2) \mid y_1, y_2 \in \mathbb{R} \} = < \underbrace{\{ (1,0,-1), (0,1,-1) \}}_{R''} >$

$$\left.\begin{array}{l} R'' \text{ SG} \\ \dim \operatorname{Im} f = 2 = |R''| \end{array}\right\} \Rightarrow R'' \text{ reper.}$$

Metoda 2.

$\operatorname{Im} f$ $\quad R' = \{ (1,-1,1) \}$ reper in $\ker f$,

$\ker f \subset \mathbb{R}^3$

Extindem la un reper in $\mathbb{R}^3$

$$rg \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = 3 = \text{maxim}$$

$\{ f(e_1), f(e_2) \}$ reper in $\operatorname{Im} f$.

$f(e_1) = (1,2,-3)$, $f(e_2) = (2,5,-7)$

d) Obs $\quad [f]_{R_0, R_0} = A$

$f(x) = y$

$Y = AX$

$$f(e_i) = \sum_{j=1}^{3} a_{ji} e_j, \quad i = \overline{1,3}$$

$f(e_1) = (1,2,-3) = 1 \cdot e_1 + 2 \cdot e_2 + (-3) \cdot e_3$

$f(e_2) = (2,5,-7) = 2e_1 + 5e_2 - 7e_3$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{pmatrix}$$

$$\begin{pmatrix} x_1+2x_2+x_3 \\ 2x_1+5x_2+3x_2 \\ -3x_1-7x_2-4x_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

③ $f: \mathbb{R}^2 \to \mathbb{R}^3 \quad f(x) = (3x_1 - 2x_2,\ 2x_1 - x_2,\ -x_1 + x_2)$

a) f liniară

b) f inj

c) $\operatorname{Im} f = ?$

d) $[f]_{R_0, R_0'} = A = ?$ R_0, R_0' repere canonice în $\mathbb{R}^2$, resp. $\mathbb{R}^3$

Soluție:

a) $f(x) = y \Leftrightarrow Y = AX$

b) f injectivă $\Leftrightarrow \operatorname{Ker} f = \{0_{\mathbb{R}^2}\}$

$f(x) = 0_{\mathbb{R}^3}$

$$\begin{cases} 3x_1 - 2x_2 = 0 \\ 2x_1 - x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases} \qquad A = \left(\begin{array}{cc|c} 3 & -2 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right)$$

$\operatorname{rg} A = 2 = \text{maxim} \Rightarrow$ SCD

$\Rightarrow (x_1, x_2) = (0,0) \Rightarrow f$ injectivă

c) T. Dimensiunii

$$\dim \mathbb{R}^2 = 2 = \underset{\overset{\shortparallel}{0}}{\dim \ker f} + \dim \operatorname{Im} f \Rightarrow$$

$\Rightarrow \dim \operatorname{Im} f = 2$

d)

$$\begin{pmatrix} 3x_1 - 2x_2 \\ 2x_1 - x_2 \\ -x_1 + x_2 \end{pmatrix} = \begin{pmatrix} \boxed{3 \quad -2} \\ 2 \quad -1 \\ -1 \quad 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$f(e_1) = f(1,0) = (3,2,-1) = \boxed{3} e_1' + 2e_2' - e_3'$

$f(e_2) = f(0,1) = (-2,-1,1) = \boxed{-2} e_1' - e_2' + e_3'$

c)

$$\operatorname{Im} f = \left\{ y \in \mathbb{R}^3 \;\middle|\; \begin{cases} 3x_1 - 2x_2 = y_1 \\ 2x_1 - x_2 = y_2 \\ -x_1 + x_2 = y_3 \end{cases} \right\}$$

$$\Rightarrow \begin{vmatrix} 3 & -2 & y_1 \\ 2 & -1 & y_2 \\ -1 & 1 & y_3 \end{vmatrix} = 0 \Rightarrow y_1 - y_2 \cdot 1 + y_3 = 0$$

(11)

$f: \mathbb{R}_2[X] \to \mathbb{R}_1[X]$ liniară

$f(X+2) = X+1$, $f(-X^2+3) = 2X+3$, $f(2X+5) = -X+1$

Determinați f.

Soluție:

$$\begin{cases} 2f(1) + f(x) = x+1 & | \cdot (-2) \\ 3f(1) - f(x^2) = 2x+3 \\ 5f(1) + 2f(x) = -x+1 \end{cases} \Leftrightarrow \begin{cases} 3f(1) - f(x^2) = 2x+3 \\ -4f(1) - 2f(x) = -2x-2 \\ 5f(1) + 2f(x) = -x+1 \end{cases}$$

$\Leftrightarrow$ $\boxed{f(1) = -3x-1}$ $\Rightarrow$ $2(-3x-1) + f(x) = x+1 \Leftrightarrow$

$\Leftrightarrow$ $\boxed{f(x) = 7x+3}$ $\Rightarrow$ $3(-3x-1) - f(x^2) = 2x+3 \Leftrightarrow$

$\Leftrightarrow$ $\boxed{f(x^2) = -11X-6}$

$$[f]_{R_0, R_0'} = A = \begin{pmatrix} -1 & 3 & -6 \\ -3 & 7 & -11 \end{pmatrix}$$

$f(a_0 + a_1X + a_2X^2) = a_0 f(1) + a_1 f(x) + a_2 f(x^2) =$

$= -3a_0X - a_0 + 7a_1X + 3a_1 - 11a_2X - 6a_2 =$

$= -a_0 + 3a_1 - 6a_2 + (-3a_0 + 7a_1 - 11a_2)X$