Forma biliniară. Forme pătratice

Forma canonică. Metoda Gauss/Jacobi

① $Q:\mathbb{R}^3\to\mathbb{R},\ Q(x)=x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3$

a) $G=$ matr. asociată lui Q în rap cu $R_0=\{e_1,e_2,e_3\}$

b) $g:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ forma polară asociată

c) Să se aducă Q la o formă canonică, utilizând metoda Gauss/Jacobi. Este Q poz. definită?

a) $Q(x)=\sum_{i=1}^{3} g_{ii}x_i^2+2\sum_{i<j} g_{ij}x_ix_j=X^TGX,\ X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$$G=\begin{pmatrix}1&\frac12&\frac12\\ \frac12&1&\frac12\\ \frac12&\frac12&1\end{pmatrix}$$

b) Met. I: $g(x,y)=\frac12(Q(x+y)-Q(x)-Q(y))$

Met. II $g(x,y)=\sum_{i,j=1}^{3} g_{ij}x_iy_j=1\cdot x_1y_1+\frac12x_1y_2+\frac12x_1y_3+$
$+\frac12x_2y_1+1\cdot x_2y_2+\frac12x_2y_3+$
$+\frac12x_3y_1+\frac12x_3y_2+1\cdot x_3y_3=X^TGY,$

$X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\ Y=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$

c) $\det G=\begin{vmatrix}1&\frac12&\frac12\\ \frac12&1&\frac12\\ \frac12&\frac12&1\end{vmatrix}=2\begin{vmatrix}1&1&1\\ \frac12&1&\frac12\\ \frac12&\frac12&1\end{vmatrix}=2\begin{vmatrix}1&0&0\\ \frac12&\frac12&0\\ \frac12&0&\frac12\end{vmatrix}=2\cdot\frac14=\frac12\neq0$

rang $Q=$ rang $G=3$

Met. Gauss:

$$Q(x)=(x_1+\tfrac{1}{2}x_2+\tfrac{1}{2}x_3)^2\underbrace{-\tfrac{1}{4}x_2^2-\tfrac{1}{4}x_3^2-\tfrac{1}{2}x_2x_3+x_2^2+x_3^2+x_2x_3}=$$

$*$

$$\tfrac{3}{4}x_2^2+\tfrac{3}{4}x_3^2+\tfrac{1}{2}x_2x_3$$

$$\tfrac{3}{4}(x_2^2+\tfrac{2}{3}x_2x_3)+\tfrac{3}{4}x_3^2$$

$$\tfrac{3}{4}(x_2+\tfrac{1}{3}x_3)^2-\tfrac{3}{4}\cdot\tfrac{1}{9}x_3^2+\tfrac{3}{4}x_3^2$$

$$Q(x)=(x_1+\tfrac{1}{2}x_2+\tfrac{1}{2}x_3)^2+\tfrac{3}{4}(x_2+\tfrac{1}{3}x_3)^2+\tfrac{2}{3}x_3^2$$

Fie schimbarea de reper $\begin{cases} y_1=x_1+\frac{1}{2}x_2+\frac{1}{2}x_3 \\ y_2=x_2+\frac{1}{3}x_3 \\ y_3=x_3 \end{cases}$

În noul reper $Q(x)=y_1^2+\frac{3}{4}y_2^2+\frac{2}{3}y_3^2$ $\Big| \Rightarrow Q(x)$ pozitiv definit

Signatura: $(3,0)$ → 3: nr de „+", 0: nr de „-"

② $Q:\mathbb{R}^3\to\mathbb{R}$, $Q(x)=2x_1x_2\overbrace{-6x_1x_3-6x_2x_3}^{-6x_3(x_1+x_2)}$

Să se aducă la o formă canonică

Precizați signatura

$$Q(x)=\underbrace{\sum_{i=1}^{3} g_{ii}x_i^2}_{0}+2\sum_{i<j} g_{ij}x_ix_j$$

$$G=\begin{pmatrix} 0 & 1 & -3 \\ 1 & 0 & -3 \\ -3 & -3 & 0 \end{pmatrix}$$

$g_{12}\neq 0$

Fie schimbarea de reper $\begin{cases} y_1=x_1+x_2 \\ y_2=x_1-x_2 \Rightarrow \\ y_3=x_3 \end{cases}$

$$\Rightarrow \begin{cases} x_1 = \frac{1}{2}(y_1+y_2) \\ x_2 = \frac{1}{2}(y_1-y_2) \\ x_3 = y_3 \end{cases}$$

$$Q(x) = \frac{1}{2}(y_1^2 - y_2^2) - 6y_3y_1 = \frac{1}{2}(y_1^2 - 12y_3y_1) - \frac{1}{2}y_2^2 = \frac{1}{2}(y_1^2 - 12y_3y_1 + 36y_3^2) - \frac{36}{2}y_3^2 - \frac{1}{2}y_2^2 =$$

$$= \frac{1}{2}(y_1 - 6y_3)^2 - \frac{1}{2}y_2^2 - 18y_3^2$$

Fie schimbarea $\begin{cases} z_1 = y_1 - 6y_3 \\ z_2 = y_2 \\ z_3 = y_3 \end{cases}$

În noul reper: $Q(x) = \frac{1}{2}z_1^2 - \frac{1}{2}z_2^2 - 18z_3^2$

<u>Obs</u>: Fie schimbarea de reper: $u_1 = \frac{1}{\sqrt{2}}z_1$

$u_2 = \frac{1}{\sqrt{2}}z_2$

$u_3 = 3\sqrt{2}\,z_3$

$Q(x) = u_1^2 - u_2^2 - u_3^2$ (Forma normală)

Signatura: $(1,2)$

<u>Met. Jacobi</u>

<u>Ex 1</u> $G = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$

Minori diagonali

$\Delta_1 = 1$

$\Delta_2 = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{3}{4}$

$\Delta_3 = \frac{1}{2}$

Dc $\Delta_1, \Delta_2, \Delta_3 \neq 0$, atunci există un reper a.î.:

$$Q(x) = \frac{1}{\Delta_1}(x_1')^2 + \frac{\Delta_1}{\Delta_2}(x_2')^2 + \frac{\Delta_2}{\Delta_3}(x_3')^2$$

Dc $\Delta_k > 0, k = \overline{1,3}$ ~~$\Delta_1, \Delta_2, \Delta_3 > 0$~~, at. Q e poz def

$$Q(x) = (x_1')^2 + \frac{4}{3}(x_2')^2 + \frac{3}{2}(x_3')^2$$

Completare Ex1

Obs: $Q(X) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3$

Generalizare: $Q(X) = \sum_{i=1}^{n} x_i^2 + \sum_{i<j} x_i x_j$ este poz def $\Rightarrow$ Signatura $(n, 0)$

Ex2: $\Delta_1 = 0 \Rightarrow$ nu se poate aplica Jacobi

⑧ $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g(x,y) = x_1y_1 - x_2y_2 - x_1y_3 - x_3y_1 + 2x_2y_3 + 2x_3y_2$

a) $g \in L^s(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

b) G matricea asociată lui g în rap cu $R_0 = \{e_1, e_2, e_3\}$

c) $\operatorname{Ker} g = ?$ Este g nedegenerată

d) G' asociată lui g în rap cu reperul $R' = \{e_1' = (1,1,1), e_2' = (1,2,1), e_3' = (0,0,1)\}$

$$g(x,y) = X^T G Y \Rightarrow g \in L(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$$

$$G = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix}$$

$G = G^T \Rightarrow g$ simetrica

$\Rightarrow g \in L^s(\mathbb{R}_3, \mathbb{R}_3; \mathbb{R})$

c) $\operatorname{Ker} g = \{x \in \mathbb{R}^3 \mid g(x,y) = 0, (\forall) y \in \mathbb{R}^3\}$

g nedegenerată $\Leftrightarrow \operatorname{Ker} g = \{0_{\mathbb{R}^3}\} \Leftrightarrow \det G \neq 0$

$$x \in \operatorname{Ker} g \Leftrightarrow \begin{cases} g(x, e_1) = 0 \\ g(x, e_2) = 0 \\ g(x, e_3) = 0 \end{cases} \Rightarrow \begin{aligned} x_1 - x_3 &= 0 \\ -x_2 + 2x_3 &= 0 \\ -x_1 + 2x_2 &= 0 \end{aligned}$$

$$\det G = \begin{vmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ -1 & 2 & -1 \end{vmatrix} = -3 \neq 0$$

$\left.\begin{array}{l} \operatorname{Ker} g=\{0_{\mathbb{R}_3}\} \\ \det G\neq 0 \end{array}\right\} \Rightarrow g$ nedeg.

$R_0 \xrightarrow{C} R'$ $\quad C=\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$G \quad\quad G'$

$G'=C^TGC=$

$=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}=$

$=\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 3 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}=\begin{pmatrix} 2 & 3 & 1 \\ 3 & 3 & 3 \\ 1 & 3 & 0 \end{pmatrix}$

În R' $g(x,y)=2x_1'y_1'+3x_1'y_2'+1\cdot x_1'y_3'+$

$+3x_2'y_1'+3x_2'y_2'+3x_2'y_3'+$

$+1\cdot x_3'y_1'+3\cdot x_3'y_2'$

⑧ $g: M_2(\mathbb{R})\times M_2(\mathbb{R})\to\mathbb{R}$

$g(X,Y)=2Tr(X\cdot Y)-Tr(X)Tr(Y), \forall X,Y\in M_2(\mathbb{R})$ $=2(x_1y_1+x_2y_3+x_3y_2+x_4y_4)-\underbrace{(x_1+x_4)(y_1+y_4)}_{x_1y_1+x_1y_4+x_4y_1+x_4y_4}$

a) $g\in L^s(M_2(\mathbb{R}), M_2(\mathbb{R}); \mathbb{R})$

b) $G=?$ matricea în rap cu $R_0=\{E_{ij}\}_{i,j=\overline{1,2}}$

c) $Q: M_2(\mathbb{R})\to\mathbb{R}$ forma pătratică asociată

d) Să se aducă Q la o formă canonică

$X=\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ $\quad Y=\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$

$$X \cdot Y = \begin{pmatrix} x_1y_1 + x_2y_3 & x_1y_2 + x_2y_4 \\ x_3y_1 + x_4y_3 & x_3y_2 + x_4y_4 \end{pmatrix}$$ -o-

$g(X,Y) = x_1y_1 + 2x_2y_3 + 2x_3y_2 + x_4y_4 - x_1y_4 - x_4y_1 \Rightarrow g$ biliniară

b) $G = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} = G^T \Rightarrow g$ simetrică $\Rightarrow$

$\Rightarrow g \in L^s(M_2(\mathbb{R}), M_2(\mathbb{R}); \mathbb{R})$

c) $Q(X) = g(X,X) = x_1^2 + x_4^2 - 2x_1x_4 + 4x_2x_3 = (x_1 - x_4)^2 + 4x_2x_3$

Fie schimbare de reper: $\begin{cases} y_1 = x_1 - x_4 \\ y_2 = x_2 + x_3 \\ y_3 = x_2 - x_3 \\ y_4 = x_4 \end{cases}$ $\quad x_2 = \frac{1}{2}(y_2 + y_3)$, $\quad x_3 = \frac{1}{2}(y_2 - y_3)$

$Q(X) = y_1^2 + y_2^2 - y_3^2$

Signatura: $(2,1)$

<u>Obs</u>: $Q: \mathbb{R}^3 \to \mathbb{R}$

$Q(x) = x_1^2 - x_2^2 + x_3^2$

$R = \{e_1, e_2, e_3\}$

$R' = \{e_1, e_3, e_2\}$

$Q(x) = (x_1')^2 + (x_2')^2 - (x_3')^2$