Cursul 9

$\{f_1', f_2'\}$ reper $\perp$ in V $\Rightarrow$ reper ortogonal in V

$g_0(f_1', f_2') = 0$

$R_1^2 = \left\{\frac{1}{\sqrt{6}}(-1,1,2,0), \frac{1}{\sqrt{6}}(1,1,0,2)\right\}$ reper ortonormat in V

$R = R_1 \cup R_2$ reper ortonormat in $\mathbb{R}^4 = V \oplus V^\perp$

Seminarul 9

Spații vectoriale euclidiene. Repere ortonormate

Procedeul Gram-Schmidt

1) $g: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = a x_1 y_1 + b x_1 y_2 + b x_2 y_1 + c x_2 y_2$

a) $g \in L^s(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$

b) g produs scalar $\Leftrightarrow \begin{cases} a > 0 \\ ac - b^2 > 0 \end{cases}$

$$g(x,y) = \underbrace{\sum_{i,j=1}^{2} g_{ij} x_i y_j}_{X^T G Y} \Leftrightarrow g \in L(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$$

$$G = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = G^T \Rightarrow g \text{ simetric}$$

$\Rightarrow g \in L^s(\ldots)$

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Metoda Jacobi

$\Delta_1 = |a| = a \neq 0$

$\Delta_2 = \begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2 \neq 0$

$\not\exists\ \Delta_1, \Delta_2 \neq 0$

$\exists\ \mathcal{R}$ în $\mathbb{R}^2$ ai $g(x,x) = Q(x) = \frac{1}{\Delta_1} x_1' + \frac{\Delta_1}{\Delta_2} x_2'$

Q pozitiv definită $\Leftrightarrow$ signatura este $2, 0 \Leftrightarrow$

$\Leftrightarrow \begin{cases} \Delta_1 > 0 \\ \Delta_2 > 0 \end{cases}$

$$Q(x) = \underline{a x_1^2} + \underline{2b\, x_1 x_2} + c x_2^2$$

$$= \frac{1}{a}\left(a^2 x_1^2 + 2ab\, x_1 x_2 + b^2 x_2^2\right) - \frac{b^2 x_2^2}{a} + c x_2^2$$

$$= \frac{1}{a}(a x_1 + b x_2)^2 - \frac{b^2 x_2^2}{a} + c x_2^2$$

$$= \frac{1}{a}\underbrace{(a x_1 + b x_2)^2}_{x_1'} + \underbrace{\frac{ac - b^2}{a} x_2^2}_{x_2'}$$

$$= \underset{>0}{\frac{1}{a}} x_1'^2 + \underset{>0}{\frac{ac - b^2}{a}} x_2'^2$$

2) $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ formă biliniară

$G = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ este matricea asociată lui g în raport cu reperul canonic

Este $(\mathbb{R}^3, g)$ s.v.e.r.

$\left.\begin{array}{l} G = G^t \Rightarrow g \text{ simetrică} \\ g \text{ formă biliniară} \end{array}\right\} \Rightarrow g \in L^s(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

$Q(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$

$\Delta_1 = |3| = 3 > 0$

$\Delta_2 = \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 6 - 4 = 2 > 0$

$\Delta_3 = \begin{vmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{vmatrix} = 6 - 12 - 4 = -10 < 0$

signatura este 2,1, așadar Q nu este pozitiv definită $\Rightarrow (\mathbb{R}^3, g)$ nu este s.v.e.r.

3). $(\mathbb{R}^3, g_0)$, $g_0: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g_0(x, y) = x_1y_1 + x_2y_2 + x_3y_3$

$V = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\}$

a) $V^\perp$

b) $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ reper ortonormal în $\mathbb{R}^3$,

$\mathcal{R}_1$ = reper ortonormal în V

$\mathcal{R}_2$ = — " — în $V^\perp$

$U = \{x \in \mathbb{R}^3 \mid g_0(x, (1,1,-1)) = 0\}$

$U^\perp = \langle \{(1,1,-1)\} \rangle$

$R_2 = \{ \frac{1}{\sqrt{3}} (1,1,-1) \}$.

$U = \{(x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R}\}$

$= \langle \{ \underbrace{(1,0,1)}_{f_1}, \underbrace{(0,1,1)}_{f_2} \} \rangle$

$\left. \begin{array}{l} \dim U = 2 \\ \{f_1, f_2\} \text{ SG} \end{array} \right\} \Rightarrow \{f_1, f_2\}$ reper.

Gram - Schmidt

$e_1 = f_1 = (1,0,1)$

$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1$

$= (0,1,1) - \frac{1}{2}(1,0,1) = \left(-\frac{1}{2}, 1, \frac{1}{2}\right) =$

$= \frac{1}{2}(-1, 2, 1)$

$\boxed{\begin{array}{l} v = \alpha u, \alpha > 0 \\ \frac{v}{\|v\|} = \frac{\alpha u}{\alpha \|u\|} = \frac{u}{\|u\|} \end{array}}$

$\tilde{e}_1 = \frac{e_1}{\|e_1\|} = \frac{1}{\sqrt{2}}(1,0,1)$

$\tilde{e}_2 = \frac{e_2}{\|e_2\|} = \frac{1}{\sqrt{6}}(-1,2,1)$

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$\{f_1, f_2\} \to \{e_1, e_2\} \to \{\tilde{e}_1, \tilde{e}_2\} = R_1$

reper arbitrar în U — reper ortogonal — reper ortonormat ~~ortonormat~~

$$R = \left\{ \frac{1}{\sqrt{2}}(1,0,1), \frac{1}{\sqrt{6}}(-1,2,1), \frac{1}{\sqrt{3}}(1,1,-1) \right\}.$$

4) $(\mathbb{C} \overset{\mathbb{R}^2}{\|}, +, \cdot)_{\mathbb{R}}$, $g: \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ formă biliniară și

$G = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ matrice asociată lui g în rap cu $R_0 = \{1, i\}$

a) $(\mathbb{C}, g)$ sp. vect. euc. real

b) $u = 2 - i$ este versor în raport cu g

c) $\langle\{u\}\rangle^{\perp}$

d) Să se ortonormeze R_0 în rap cu g

e) Să se afle intersecția dintre cercul unitate în $(\mathbb{C}, g_0)$ și în $(\mathbb{C}, g)$

$$g(x,y) = 1x_1y_1 + 2x_1y_2 + 2x_2y_1 + 5x_2y_2.$$

$G = G^T$

$$Q(x) = g(x,x) = x_1^2 + 4x_1x_2 + 5x_2^2$$

$$= \cancel{x_1^2 + 2x_1x_2 + x_2^2}$$

$$= \underbrace{(x_1 + 2x_2)^2}_{x_1'} + \underbrace{x_2^2}_{x_2'} = x_1'^2 + x_2'^2$$

signatura 2 și 0 $\to$ g produs scalar

a) $\to (\mathbb{C}, g)$ s. v. e. r.

b) $\|u\| = \sqrt{g(u,u)} = \sqrt{Q(u)} = \sqrt{2^2-8+5\cdot(-1)^2} =$

$= \sqrt{4-8+5} = \sqrt{1} = 1$

c) $\langle\{u\}\rangle^{\perp} = \{(y_1,y_2) \mid g(u,y)=0\} =$

$= \{ y \in \mathbb{C} \mid 2y_1+4y_2-2y_1-5y_2 = 0\} = \mathbb{R}$

$-y_2 = 0 \Rightarrow y_2 = 0$

d) $R_0 = \{f_1 = 1, f_2 = i\}$

$x_1 = 0 \quad y_1 = -1$

$x_2 = 1 \quad y_2 = 0$

$e_1 = f_1 = 1$

$e_2 = f_2 - \frac{g(f_2,e_1)}{g(e_1,e_1)} \cdot e_1 = i - \frac{g(i,1)}{g(1,1)} \cdot 1 =$

$= i - \frac{2}{1} = i-2 = -2+i$

$\tilde{e}_1 = \frac{e_1}{\|e_1\|} = \frac{1}{\sqrt{g(1,1)}} = 1$

$\tilde{e}_2 = \frac{e_2}{\|e_2\|} = \cancel{\frac{e_2}{\sqrt{g(-2,+i)}}} = \frac{e_2}{\sqrt{g(-2+i,-2+i)}} = \frac{e_2}{\sqrt{g(u,u)}} = e_2$

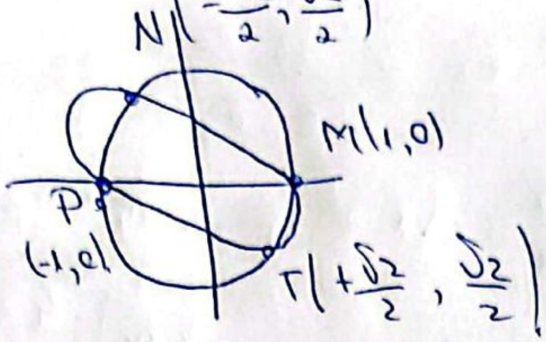
$= -2+i$

e) ~~$\{z \in \mathbb{C}$~~ $A = \{ \underset{x_1+ix_2}{x} \in \mathbb{C} \mid g_0(x,x) = 1 = x_1^2+x_2^2 \}$

$B = \{ x \in \mathbb{C} \mid g(x,x) = 1 = Q(x) = x_1^2+4x_1x_2+5x_2^2 \}$

$4\cos t \sin t + 4\sin^2 t = 0$

$4\sin t(\sin t + \cos t) = 0$



~~5) $(\mathbb{R}^3, g_0)$, $V = <\{(1,0,1), (1,1,2)\}>$~~

~~a) $V^\perp$~~

~~b) Să se det $R = R_1 \cup R_2$ reper ortonormat în $\mathbb{R}^3$~~

~~a.î. R_1 - reper ortonormat în V~~

~~R_2 - " $V^\perp$~~

5) $(\mathbb{R}^3, g_0)$, $R = \{f_1 = (1,2,3), f_2 = (0,1,1), f_3 = (1,2,5)\}$

a) R reper în $\mathbb{R}^3$. Să se ortonormeze

b) $f_1 \times f_2$

c) $f_1 \wedge f_2 \wedge f_3$

$$\det\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 5 \end{pmatrix} = 5 + 2 - 3 - 2 = 2 > 0 \xrightarrow{CLi} R \text{ SLi}, \ \text{card} R = 3 \Rightarrow$$

$\Rightarrow R$ reper

$e_1 = f_1 = (1,2,3)$

$$e_2 = f_2 - \frac{<f_2, e_1>}{<e_1, e_1>} \cdot e_1 = (0,1,1) - \frac{5}{14} \cdot (1,2,3)$$

$$= \frac{1}{14}\Big((0,14,14) - (5,10,15)\Big) = \frac{1}{14}(-5, 4, -1)$$

$$e_3 = f_3 - \frac{\langle f_3, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 - \frac{\langle f_3, e_2 \rangle}{\langle e_2, e_2 \rangle} \cdot e_2$$

$$= (1,2,5) - \frac{20}{14}(1,2,3) - \frac{\frac{1}{14}\cdot(-2)^{(-1)}}{\left(\frac{1}{14}\right)^2 \cdot 42_{21}} \cdot \frac{1}{14} \cdot (-5,4,-1)$$

$$= (1,2,5) - \frac{10}{7}(1,2,3) + \frac{1}{21}(-5,4,-1)$$

$$= \frac{1}{21}\left((21,42,105) - (30,60,90) + (-5,4,-1)\right)$$

$$= \frac{1}{21}\left(\cancel{4.6} -14, -14, 14\right) = \frac{14}{21}(-1,-1,1) = \frac{2}{3}(-1,-1,1)$$

$$\tilde{e}_1 = \frac{e_1}{\|e_1\|} = \frac{\cancel{(1,2,3)}}{\sqrt{14}}(1,2,3)$$

$$\tilde{e}_2 = \frac{1}{\sqrt{42}}(-5,4,-1)$$

$$\tilde{e}_3 = \frac{(-1,-1,1)}{\sqrt{3}}$$

b) $$f_1 \times f_2 = \det \begin{vmatrix} e_1^0 & e_2^0 & e_3^0 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} =$$

$$= e_1^0 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} + (-1) e_2^0 \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} + e_3^0 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} =$$

$$= -e_1^0 - e_2^0 + e_3^0 = (-1,-1,1)$$

c) $$f_1 \wedge f_2 \wedge f_3 = f_3 \wedge f_1 \wedge f_2 = \langle f_3, f_1 \times f_2 \rangle =$$

$$= \cancel{k} -1 -2 + 5 = 2.$$

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