Seminar 1

16.02.2021

Determinanți. Teorema Laplace. Teorema Hamilton-Cayley

Ex1 $f: \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$, $f(A) = \det A$

Studiați inj/surj funcției f

SOL.

- $\det(I_n) = 1$, $A = \begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & 1 & \\ 0 & & & \ddots 1 \end{pmatrix}$, $\det A = 1$

$\det(A) = \det(I_n) = 1 \Rightarrow f$ nu e inj

- $\forall z \in \mathbb{C}, \exists A \in \mathcal{M}_n(\mathbb{C})$ aî $\det A = z$

$A = \begin{pmatrix} z & & 0 \\ & 1 & \\ 0 & & \ddots 1 \end{pmatrix}$ f e surjectivă $\Leftrightarrow \operatorname{Im} f = \mathbb{C}$

Ex2 $A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$

$a, b, c \in \mathbb{R}$

Dacă $a^2+b^2+c^2 = 1$, atunci $|\det A| \leq 1$

SOL

$$\det A = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ c & a-c & b-c \\ b & c-b & a-b \end{vmatrix} =$$

$$= (a+b+c)(a^2+b^2+c^2-ab-ac-bc)$$

$$a^2+b^2+c^2-ab-ac-bc = \frac{1}{2}\left[(a-b)^2+(a-c)^2+(b-c)^2\right] \geq 0$$

- $x = ab+ac+bc \leq a^2+b^2+c^2 = 1$

$$A\cdot A^T=\begin{pmatrix} a & b & c\\ c & a & b\\ b & c & a\end{pmatrix}\begin{pmatrix} a & c & b\\ b & a & c\\ c & b & a\end{pmatrix}$$

$$=\begin{pmatrix} a^2+b^2+c^2 & ac+ab+bc & ab+bc+ac\\ ac+ab+bc & a^2+b^2+c^2 & bc+ac+ab\\ ab+bc+ac & bc+ac+ab & a^2+b^2+c^2\end{pmatrix}=\begin{pmatrix} 1 & x & x\\ x & 1 & x\\ x & x & 1\end{pmatrix}$$

$$\det(A\cdot A^T)=(\det A)^2=\begin{vmatrix} \boxed{1 \quad x \quad x}\\ x & 1 & x\\ x & x & 1\end{vmatrix}$$

$$=1\begin{vmatrix} 1 & x\\ x & 1\end{vmatrix}-x\begin{vmatrix} x & x\\ x & 1\end{vmatrix}+x\begin{vmatrix} x & 1\\ x & x\end{vmatrix}=$$

$$=(1-x^2)-x(x-x^2)+x(x^2-x)=2x^3-3x^2+1$$

$$(\det A)^2=2x^3-3x^2+1$$

Dem că $|\det A|\le 1 \iff (\det A)^2\le 1$.

Dem $2x^3-3x^2+1\le 1 \iff x^2(2x-3)\le 0$

$$\left.\begin{array}{l} x\le 1\Rightarrow 2x-3<0\\ \text{dar } x^2\geq 0\end{array}\right\}\Rightarrow x^2(2x-3)\le 0$$

OBS
$(\det A)^2=(a+b+c)^2(a^2+b^2+c^2-$
$-ab-bc-ac)^2=$
$=(1+2x)^2(1-x)^2=$
$=2x^3-3x^2+1.$

<u>Ex3</u>. $\forall A,B\in\mathcal{M}_2(\mathbb{R})$ aî $AB=BA$

$\Rightarrow$ a) $\det(A^2+B^2)\geq 0$

b) $\det(A^2+B^2)=0\Rightarrow \det A=\det B$

<u>Sol</u>

a) $\det(A^2+B^2)=\det(\underline{A^2-i^2B^2})=\det((A+iB)(A-iB))$

$$=\det(A+iB)\underbrace{\det(A-iB)}_{\overline{\det(A+iB)}}=|\det(A+iB)|^2\geq 0$$

b) $\det(A^2+B^2)=0\Rightarrow |\det(A+iB)|=0$

$\Rightarrow \det(A+iB)=0$ (1)

$$f(x)=\det(A+xB)=\begin{vmatrix} \overset{1}{a_{11}}+\overset{1'}{xb_{11}} & \overset{2}{a_{12}}+\overset{2'}{xb_{12}} \\ a_{21}+xb_{21} & a_{22}+xb_{22} \end{vmatrix}$$

$$=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}+\begin{vmatrix} a_{11} & xb_{12} \\ a_{21} & xb_{22} \end{vmatrix}+\begin{vmatrix} xb_{11} & a_{12} \\ xb_{22} & a_{22} \end{vmatrix}+\begin{vmatrix} xb_{11} & xb_{12} \\ xb_{21} & xb_{22} \end{vmatrix}=$$

$$=x^2\det B+x\underbrace{\left(\begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix}+\begin{vmatrix} b_{11} & a_{12} \\ b_{22} & a_{22} \end{vmatrix}\right)}_{\alpha\in\mathbb{R}}+\det A.$$

$(1)\Rightarrow f(i)=0 \Rightarrow -\det B+i\alpha+\det A=0$

$$\det A-\det B+i\alpha=0\Rightarrow\begin{cases}\det A-\det B=0\\ \alpha=0\end{cases}$$

$\Rightarrow \det A=\det B$.

<u>Ex4</u> $A=\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ 1 & -2 & 2 & 4 \end{pmatrix}$

$\det A=?$ (utilizând Th. Laplace $p=2$, ℓ_2,ℓ_3 fixate)

<u>SOL</u>

$$\Delta_A=\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix}(-1)^{2+3+1+2}\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix}+\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}(-1)^{2+3+1+3}\begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix}+$$

$$+\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix}(-1)^{2+3+1+4}\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}+\begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix}(-1)^{2+3+2+3}\begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix}+$$

$$+\begin{vmatrix} 1 & 4 \\ 5 & -1 \end{vmatrix}(-1)^{2+3+2+4}\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix}+\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix}(-1)^{2+3+3+4}\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix}=-5.$$

<u>Ex5</u> $A=\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}\in\mathcal{M}_4(\mathbb{R})$

a) $A\cdot A^T=\alpha I_4$, $\alpha\in\mathbb{R}$

b) Dacă $A\neq O_4$, atunci A = inversabilă.

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SOL a) $A\cdot A^T = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}\begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} =$

$= \begin{pmatrix} a^2+b^2+c^2+d^2 & -ab+ab-dc+dc & 0 & 0 \\ 0 & a^2+b^2+c^2+d^2 & 0 & 0 \\ 0 & 0 & a^2+b^2+c^2+d^2 & 0 \\ 0 & 0 & 0 & a^2+b^2+c^2+d^2 \end{pmatrix}$

$\alpha = a^2+b^2+c^2+d^2$

b) Pp. prin absurd că $\det A = 0$
$\det(A\cdot A^T) = \det(\alpha I_4) \Rightarrow (\det A)^2 = \alpha^4 \cdot 1$ $\Big\} \Rightarrow$

$\alpha = 0 \Rightarrow a^2+b^2+c^2+d^2 = 0 \Rightarrow a=b=c=d=0 \Rightarrow$

$\Rightarrow A = O_4$ Contradicție

Pp este falsă și A este matrice inversabilă

Ex6

Fie $A = \begin{pmatrix} 2 & x & 3 \\ x & -1 & x \\ 3 & x+2 & m+2 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

$m=?$ ai A este matrice inversabilă $\forall x \in \mathbb{R}$

SOL

$\det A \neq 0, \forall x \in \mathbb{R}$.

$\det A = 2(-1)^{1+1}\begin{vmatrix} -1 & x \\ x+2 & m+2 \end{vmatrix} + x(-1)^{1+2}\begin{vmatrix} x & x \\ 3 & m+2 \end{vmatrix} +$

$+ 3(-1)^{1+3}\begin{vmatrix} x & -1 \\ 3 & x+2 \end{vmatrix} =$

$= 2\left[-(m+2) - x^2 - 2x\right] - x^2(m-1) + 3(x^2+2x+3)$

$= x^2(2-m) + 2x - 2m + 5$

$x^2(2-m) + 2x - 2m + 5 \neq 0, \forall x \in \mathbb{R}$

$\Delta = b^2 - 4ac = 4 - 4\cdot(2-m)(5-2m) < 0 \quad |:-4$

$(m-2)(2m-5) - 1 > 0 \Rightarrow$

$2m^2 - 5m - 4m + 10 - 1 > 0$

• $2m^2 - 9m + 9 > 0 \Rightarrow m \in (-\infty, \frac{3}{2}) \cup (3, \infty)$

$\Delta = 9^2 - 4\cdot 2\cdot 9 = 9(9-8) = 9$

$m_{1,2} = \dfrac{9 \pm 3}{4} < \begin{matrix} \frac{12}{4} = 3 \\ \frac{6}{4} = \frac{3}{2} \end{matrix}$

Ex7. Fie $A = \begin{pmatrix} m & 1 & 2 \\ 3 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{Z})$

$m = ?$ aî $A^{-1} \in \mathcal{M}_3(\mathbb{Z})$

SOL

$\left.\begin{array}{l} \det(A^{-1}) = \frac{1}{\det A} \\ A, A^{-1} \in \mathcal{M}_3(\mathbb{Z}) \Rightarrow \det A, \det(A^{-1}) \in \mathbb{Z} \end{array}\right\}$

$\Rightarrow \det A = \pm 1.$

$\det A = \begin{vmatrix} m & 1 & 2 \\ 3 & -1 & 2 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} m+2 & 1 & 2 \\ 5 & -1 & 2 \\ 0 & 0 & -1 \end{vmatrix}$

$= (-1)(-1)^{3+3} \begin{vmatrix} m+2 & 1 \\ 5 & -1 \end{vmatrix} = -(-m-2-5) = m+7$

1) $m+7 = 1 \Rightarrow m = -6 \in \mathbb{Z}$

2) $m+7 = -1 \Rightarrow m = -8 \in \mathbb{Z}$

Deci $m \in \{-8, -6\}$.

Ex8 Fie $A \in \mathcal{M}_3(\mathbb{C})$, $A \neq A^T$

Dacă $\exists\, x, y \in \mathbb{C}$ aî $xA + yA^T$ este matrice inversabilă, atunci $x + y \neq 0$.

SOL

Ip. prin absurd că $x+y=0 \Rightarrow y=-x$.

$C = xA + yA^T = xA - xA^T = x(A - A^T)$

$B = A - A^T$ are proprietatea $\det B = 0$

$\det C = \det(x(A-A^T)) = x^3 \det B = 0$ $\Big\}$ $\nexists$

dar C = inversabilă

Ip. este falsă $\Rightarrow x+y \neq 0$.

Ex9 Fie $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Să se calculeze A^{-1}, utilizând th. Hamilton-Cayley

SOL

$P_A(x) = (-1)^3 (X^3 - \sigma_1 X^2 + \sigma_2 X - \sigma_3) = \det(A - xI_3)$

$\sigma_1 = \operatorname{Tr} A = 1$

$\sigma_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = -1$

$\sigma_3 = \det A = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$H\text{-}C \Rightarrow A^3 - A^2 - A + I_3 = O_3$

$-A^3 + A^2 + A = I_3 \Rightarrow A(-A^2 + A + I_3) = I_3$

$\qquad\qquad\qquad\qquad \| $

$\qquad\qquad (-A^2 + A + I_3)A$

$A^{-1} = -A^2 + A + I_3$

Ex10 $A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$

a) $A^{-1} = ?$ (utilizând Th H-C)

b) Dacă $A^4 = aA^2 + bA + cI_3$, at $a, b, c = ?$
$a, b, c \in \mathbb{R}$

SOL

a) $\sigma_1 = Tr\, A = Tr\begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix} = 5$

$\sigma_2 = \begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} =$

$= 2 + 2 + 4 = 8$

$\sigma_3 = \det A = \begin{vmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 1 & -1 \\ -1 & 1 & 0 \\ 2 & 2 & 0 \end{vmatrix}$

$= (-1)(-1)^4 \begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} = -(-2-2) = 4$

TH-C: $A^3 - \sigma_1 A^2 + \sigma_2 A - \sigma_3 J_3 = O_3$

$A^3 - 5A^2 + 8A - 4J_3 = 0$

$A^3 - 5A^2 + 8A = 4J_3 \Rightarrow$

$A \cdot \frac{1}{4}(A^2 - 5A + 8J_3) = J_3 \Rightarrow A^{-1} = \frac{1}{4}(A^2 - 5A + 8J_3)$

$\overset{\shortparallel}{\frac{1}{4}(A^2 - 5A + 8J_3) \cdot A}$

b) $A^4 = aA^2 + bA + cJ_3$

T.H-C: $A^3 = 5A^2 - 8A + 4J_3 \mid \cdot A$

$A^4 = 5(5A^2 - 8A + 4I_3) - 8A^2 + 4A$

$A^4 = 17A^2 - 36A + 20J_3 \Rightarrow a = 17, b = -36, c = 20$

Ex 11 Fie $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$, $B = A^4 - 3A^3 + 3A^2 - 2A + 8J_2$

Dacă $B = aA + bJ_2$, at $a, b = ?$

SOL

$x^2 - \sigma_1 x + \sigma_2 = 0$ $\quad \sigma_1 = Tr\, A = 1$, $\sigma_2 = \det A = 2$

$x^2 - x + 2 = 0 \overset{TH-C}{\Rightarrow} A^2 - A + 2J_2 = O_2.$

$$\begin{array}{l|l}
x^4-3x^3+3x^2-2x+8 & x^2-x+2 \\ \cline{2-2}
-x^4+x^3-2x^2 & x^2-2x-1 \\ \cline{1-1}
/ \quad -2x^3+x^2-2x+8 & \\
\quad\;\; 2x^3-2x^2+4x & \\ \cline{1-1}
/ \qquad -x^2+2x+8 & \\
\qquad\quad x^2-x+2 & \\ \cline{1-1}
/ \qquad\quad x+10 &
\end{array}$$

$$x^4-3x^3+3x^2-2x+8=(x^2-x+2)\underbrace{(x^2-2x-1)}_{\text{câtul}}+\underbrace{x+10}_{\text{restul}}$$

$$B=\underbrace{(A^2-A+2J_2)}_{O_2\,(TH-C)}(A^2-2A-J_2)+A+10J_2$$

$$B=A+10I_2=aA+bJ_2 \Rightarrow \begin{cases} a=1 \\ b=10 \end{cases}$$

<u>EX12</u> Fie $A, B \in \mathcal{M}_2(\mathbb{R})$ a.î. $Tr(A)=Tr(B)$ și $Tr(A^2)=Tr(B^2)$

$\Rightarrow \det A = \det B$

<u>SOL</u> Th H-C: $A^2-Tr(A)A+\det(A)J_2=O_2$

$$B^2-Tr(B)B+\det(B)J_2=O_2$$

$$\left.\begin{array}{l} A^2=Tr(A)A-\det(A)J_2 \;|Tr \\ B^2=Tr(B)B-\det(B)J_2 \;|Tr \end{array}\right\} \Rightarrow$$

$$Tr(A^2)=Tr(Tr(A)A)-Tr(\det(A)J_2)=$$

$$\left.\begin{array}{l} Tr(A^2)=Tr^2(A)-2\det A \\ \text{Analog } Tr(B^2)=Tr^2(B)-2\det B \\ \text{dar } \begin{array}{l} Tr(A^2)=Tr(B^2) \\ Tr(A)=Tr B \end{array} \end{array}\right\} \Rightarrow \det A=\det B$$

13 Fie $A \in \mathcal{M}_n(\mathbb{R})$ aî $A^2 = O_n$

$\Rightarrow I_n - A,\ I_n + A$ sunt matrice inversabile

SOL

$$(I_n - A)(I_n + A) = (I_n + A)(I_n - A) = I_n^2 - A^2 = I_n$$

$$(I_n - A)^{-1} = I_n + A, \quad (I_n + A)^{-1} = I_n + A$$

OBS

$n = 2$. Th H-C: $A^2 - Tr(A)A + \det A \cdot I_2 = O_2$

$A^2 = O_2 \mid \det \Rightarrow (\det A)^2 = 0$

$\Rightarrow Tr(A)A = O_2 \quad \mid Tr \Rightarrow Tr^2(A) = 0 \Rightarrow Tr A = 0.$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{aligned} \det A &= ad - bc = 0 \\ Tr A &= a + d = 0 \end{aligned}$

$$I_2 - A = \begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix} \Rightarrow \det(I_2 - A) = (1-a)(1-d) - bc$$
$$= 1 - (a+d) + ad - bc = 1$$

$$I_2 + A = \begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix} \Rightarrow \det(I_2 + A) = 1.$$

EX14 Fie $A \in \mathcal{M}_n(\mathbb{R})$ aî $A^3 = O_n \Rightarrow$

$I_n - A,\ I_n + A$ sunt matrice inversabile

SOL

$$\underset{\displaystyle I_n}{\underset{\parallel}{I_n^3}} - A^3 = (I_n - A)\underbrace{(I_n^2 + I_n A + A^2)}_{I_n + A + A^2} = (I_n + A + A^2)(I_n - A)$$

$$(I_n - A)^{-1} = I_n + A + A^2$$

Analog $(I_n + A)^{-1} = I_n - A + A^2$