CURS 10

Transformări ortogonale. Endomorfisme simetrice

Def $(E_i, \langle\cdot,\cdot\rangle_i)$ $i=\overline{1,2}$ s.v.e.r. Aplicația $f: E_1 \to E_2$ s.n. <u>aplicație ortogonală</u> $\iff \langle f(x), f(y)\rangle_2 = \langle x,y\rangle_1$, $\forall x,y \in E_1$.

Prop. f apl. ortogonală $\Rightarrow$ a) $\|f(x)\|_2 = \|x\|_1$, $\forall x \in E_1$

b) f injectivă

Dem

a) Considerăm $x=y$

$\langle f(x), f(x)\rangle_2 = \langle x,x\rangle_1 \Rightarrow (\|f(x)\|_2)^2 = (\|x\|_1)^2 \Rightarrow \|f(x)\|_2 = \|x\|_1$.

b). f lin.

f inj $\iff \operatorname{Ker} f = \{0_{E_1}\}$

Fie $x \in \operatorname{Ker} f \Rightarrow f(x) = 0_{E_2} \Rightarrow \|f(x)\|_2 = \|0_{E_2}\|_2 \Rightarrow$

$\|x\|_1 = 0 \Rightarrow x = 0_{E_1}$ (produsul scalar este pozitiv definit)

Def. $(E, \langle\cdot,\cdot\rangle)$ s.v.e.r, $f \in \operatorname{End}(E)$.

f s.n. <u>transformare ortogonală</u> $\iff \langle f(x), f(y)\rangle = \langle x,y\rangle$, $\forall x,y \in E$.

Prop

$f \in O(E) = \{f \in \operatorname{End}(E) \mid f \text{ transf. ortogonală}\}$

$\iff \|f(x)\| = \|x\|$, $\forall x \in E$.

Dem

"$\Rightarrow$" (cf. prop. precedente)

"$\Leftarrow$" $\|f(x+y)\|^2 = \|x+y\|^2 \Rightarrow$

$\langle f(x)+f(y), f(x)+f(y)\rangle = \langle x+y, x+y\rangle \Rightarrow$

$\|f(x)\|^2 + \|f(y)\|^2 + 2\langle f(x), f(y)\rangle = \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle$

$\Rightarrow \langle f(x), f(y)\rangle = \langle x,y\rangle$, $\forall x,y \in E \Rightarrow f \in O(E)$.

Matricea asociată unei transf. ortogonale

$(E, \langle \cdot, \cdot \rangle)$ s.v.e.r, $\mathcal{R} = \{e_1, .., e_n\}$ reper ortonormat

$A = [f]_{\mathcal{R},\mathcal{R}}$, $f \in O(E)$

$\langle f(e_i), f(e_j) \rangle = \langle e_i, e_j \rangle$, $\forall i,j = \overline{1,n}$

$\langle \sum_{r=1}^{n} a_{ri} e_r, \sum_{s=1}^{n} a_{sj} e_s \rangle = \langle e_i, e_j \rangle$

$\sum_{r,s=1}^{n} a_{ri} a_{sj} \delta_{rs} = \delta_{ij} \Rightarrow \sum_{r=1}^{n} a_{ri} a_{rj} = \delta_{ij}$

$\Rightarrow A^T A = I_n \Rightarrow A \in O(n)$.

Dacă $\mathcal{R} = \{e_1, .., e_n\} \xrightarrow{C} \mathcal{R}' = \{e_1', .., e_n'\}$ repere ortonormate

$C \in O(n)$ (i.e $C^{-1} = C^T$)

$A' = [f]_{\mathcal{R}',\mathcal{R}'}$, $A' = C^{-1} A C = C^T A C$.

$A'^T A' = (C^T A C)^T C^T A C = C^T A^T \underbrace{C \cdot C^T}_{I_n} A C =$

$= C^T \underbrace{A^T A}_{I_n} C = C^T C = I_n$.

- $f \in O(E) \Leftrightarrow$ matricea asociată, în raport cu $\forall$ reper ortonormat, este ortogonală
- Dacă $\det A = 1$ i.e $A \in SO(n)$, atunci f s.n. <u>transf. ortogonală de speța 1.</u>
- Dacă $\det A = -1$, at f s.n. <u>transf. ortogonală de speță 2.</u>

<u>OBS</u> a) $(O(E), \circ)$ grupul transformărilor ortogonale.

b) $f \in O(E) \Leftrightarrow$ Schimbare de repere ortonormate

„$\Rightarrow$" $f \in O(E) \Rightarrow \underset{[f]_{\mathcal{R},\mathcal{R}}}{\overset{\parallel}{A}} \in O(n)$ $\qquad \mathcal{R} \xrightarrow{A} \mathcal{R}'$

$\mathcal{R}, \mathcal{R}'$ repere ortonormate.

„$\Leftarrow$" $\mathcal{R} \xrightarrow{A} \mathcal{R}'$ repere ortonormate, $A \in O(n)$, $f \in End(E)$

$f(e_i) = e_i' = \sum_{j=1}^{n} a_{ji} e_j$

Prelungim f prin liniaritate: $f(x) = \sum_{i,j=1}^{n} x_j a_{ji} e_j$

<u>OBS</u> $A \in O(n)$, $\exists\ \varphi \in (-\pi, \pi]$ a.î.

a) $\det A = 1 \Rightarrow A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

b) $\det A = -1 \Rightarrow A = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$

<u>Prop</u> $(E, \langle\cdot,\cdot\rangle)$ s.v.e.r, $U \subseteq E$ ssp. invariant al lui $f \in O(E) \Rightarrow$

a) $f(U) = U$

b) $U^{\perp} \subseteq U$ subsp. invariant al lui f

c) $f/_{U^{\perp}} : U^{\perp} \longrightarrow U^{\perp}$ transf. ortogonală

<u>Dem</u>

a) $U \subseteq E$ ssp invar $\Rightarrow f(U) \subseteq U$.

$f : U \longrightarrow f(U)$ izomorfism de spații vect.

$\Rightarrow \left.\begin{array}{l} \dim U = \dim f(U) \\ \text{dar } f(U) \subseteq U \end{array}\right\} \Rightarrow f(U) = U$.

b). U invar $\Rightarrow U^{\perp}$ invar. i.e. $f(U^{\perp}) \subseteq U^{\perp}$.

Fie $x \in U^{\perp}$. Dem că $f(x) \in U^{\perp}$.

Fie $y \in \underset{f(U)}{\overset{\shortparallel}{U}}$. $\langle f(x), \underset{U}{\underset{\cap}{y}} \rangle = \langle f(x), f(z) \rangle = \langle \underset{U^{\perp}}{\underset{\cap}{x}}, \underset{U}{\underset{\cap}{z}} \rangle = 0$

$\Rightarrow f(x) \in U^{\perp}$.

c) $U^{\perp}$ invar $\xRightarrow{\text{cf a)}} f(U^{\perp}) = U^{\perp}$

$f/_{U^{\perp}} : U^{\perp} \longrightarrow U^{\perp}$ transf. ortogonală.

<u>Exemplu</u> $(E, \langle\cdot,\cdot\rangle)$ s.v.e.r; $p, s \in End(E)$, $p^2 = p$, $s^2 = id_E$

$s = 2p - id_E$. Not $E' = \ker p$, $E'' = \operatorname{Im} p$, $E = E' \oplus E'' = E' \oplus E'^{\perp} = E''^{\perp} \oplus E''$.

Dacă $E'' = E'^{\perp}$, atunci p s.n. <u>proiecție ortogonală pe E''</u> și s s.n. <u>simetrie ortogonală față de E''</u>. $p(x') = 0$, $p(x'') = x''$, $x = x' + x''$

$s(x') = -x'$, $s(x'') = x''$

$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ reper ortonormat în E, $\mathcal{R}_1$ reper în E', $\mathcal{R}_2$ reper în E''

$A_p = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_{n-k}\end{array}\right)$, $A_s = \left(\begin{array}{c|c} -I_k & 0 \\ \hline 0 & I_{n-k}\end{array}\right)$, $\dim E' = k$, $\dim E'' = n-k$.

$A_s^T \cdot A_s = I_n \Rightarrow A_s \in O(n) \Rightarrow s \in O(E)$

$A_p \notin O(n) \Rightarrow p \notin O(E)$.

<u>Prop.</u> $(E, \langle \cdot, \cdot \rangle)$ s.v.e.r, $f \in O(E) \Rightarrow$ valorile proprii sunt ± 1.

<u>Dem</u>

λ = valoare proprie $\Rightarrow \exists x \neq 0_E$ ai $f(x) = \lambda x$

$\|f(x)\| = \|\lambda x\| \Rightarrow \|x\| = |\lambda| \cdot \|x\| \Rightarrow |\lambda| = 1$ dar $\lambda \in \mathbb{R} \Rightarrow \lambda = \pm 1$

<u>OBS</u> $\|\lambda x\|^2 = \langle \lambda x, \lambda x \rangle = \lambda^2 \|x\|^2 \Rightarrow \|\lambda x\| = |\lambda| \cdot \|x\|$.

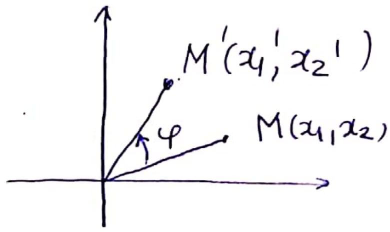
<u>Clasificarea transf. ortogonale</u>

① $\dim E = 1$, $\mathcal{R} = \{e\}$, e versor, $f(e) = \lambda e$

$\lambda = \pm 1. \Rightarrow f(e) = \pm e \Rightarrow f(x) = \pm x$.

$f \in \{id_E, -id_E\}$.

② $\dim E = 2.$, $A \in O(2)$.



a) $\det A = 1$, $A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

$f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (\underbrace{x_1\cos\varphi - x_2 \sin\varphi}_{x_1'}, \underbrace{x_1 \sin\varphi + x_2\cos\varphi}_{x_2'})$

f = rotatie de unghi orientat φ.

<u>OBS.</u> $P(\lambda) = \det(A - \lambda I_2) = 0$, $Tr(A), \det(A)$ invarianți la schimbarea de reper ortonormat.

$Tr(A) = 2\cos\varphi$ $\qquad \lambda^2 - Tr(A)\lambda + \det(A) = 0$

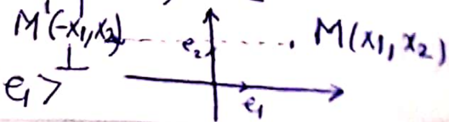
$\lambda^2 - 2\cos\varphi \cdot \lambda + 1 = 0.$, $\Delta = 4\cos^2\varphi - 4 = -4\sin^2\varphi$.

$\lambda_{1,2} = \cos\varphi \pm i\sin\varphi$.

b) $\det A = -1$. $\exists$ o schimbare de reper ortonormat ai $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \mathcal{R} = \{e_1, e_2\}$, $f(e_1) = -e_1$, $f(e_2) = e_2$

$f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (-x_1, x_2)$

$f = s$ = simetrie fata de $\langle e_2 \rangle = \langle e_1 \rangle^{\perp}$



Teoremă $\dim E = 3$. Dacă $f \in SO(E)$, atunci $\exists R = \{e_1, e_2, e_3\}$ reper ortonormat ai $[f]_{R,R} = A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1, \underbrace{x_2\cos\varphi - x_3\sin\varphi}_{x_2'}, \underbrace{x_2\sin\varphi + x_3\cos\varphi}_{x_3'})$

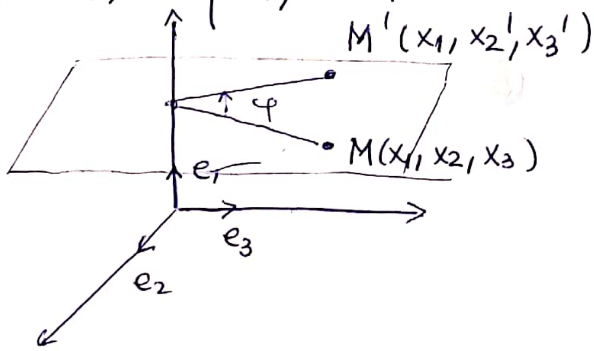
f = este o rotație de $\measuredangle$ orientat φ și axa $\langle\{e_1\}\rangle$.

"R_φ"

OBS a) $Tr(A) = 1 + 2\cos\varphi$ invariant la schimbarea de reper orientat

b) Axa de rotație: $f(x) = x$.

Dc $x \in \langle\{e_1\}\rangle \Rightarrow f(x) = f(ae_1) = af(e_1) = ae_1 = x$



b) $\det A = -1$

$b_1)$ $\lambda = 1$, $f(e_1) = e_1$, $A = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A} \\ 0 & \end{pmatrix}$

$\Rightarrow \det\tilde{A} = -1 \Rightarrow \exists \{e_2, e_3\}$ reper ortonormat ai $\tilde{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

În raport cu $\{e_2, e_1, e_3\}$ avem $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$

$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos 0 & -\sin 0 \\ 0 & \sin 0 & \cos 0 \end{pmatrix}$

$b_2)$ $\lambda = -1$, $f(e_1) = -e_1$, $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \tilde{A} & \\ 0 & & \end{pmatrix}$

$\det\tilde{A} = 1 \Rightarrow \tilde{A} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

<u>Teoremă</u> $\dim E = 2$

$\forall f \in O(E)\ (f \neq id_E)$ se poate scrie ca o compunere de cel mult 2 simetrii ortogonale (față de drepte)

<u>Dem</u>. ① $f \in SO(E)$ i.e $\det(A_f) = 1$.

Fie s = simetrie ortogonală i.e $\det(A_s) = -1$.

$s \circ f \in O(E)$ de speța 2, $\det(A_{s\circ f}) = \det(A_s \cdot A_f) = (-1)\cdot 1 = -1$.

$s \circ f = s'$ simetrie ortogonală

$\underbrace{s \circ s}_{id_E} \circ f = s \circ s' \Rightarrow f = s \circ s'$

② $f \in O(E)$ de speță $2 \Rightarrow f = s$ simetrie ortog.

B $\dim E = 3$, $f \in O(E)$

$P(\lambda) = \det(A - \lambda I_3) = 0$ (polinom de grad 3 cu coef reali)

$\Rightarrow$ are cel puțin o răd reală $\in \{-1, 1\}$.

Fie e_1 = versor propriu pt $\lambda \in \{-1, 1\}$.

$f(e_1) = \lambda e_1 = \pm e_1 \Rightarrow \langle\{e_1\}\rangle \subset E$ subsp. invar. al lui f

$\Rightarrow \langle\{e_1\}\rangle^\perp \subset E$ subsp. invariant.

$E = \langle\{e_1\}\rangle \oplus \langle\{e_1\}\rangle^\perp$

$f/_{\langle\{e_1\}\rangle^\perp} : \langle\{e_1\}\rangle^\perp \to \langle\{e_1\}\rangle^\perp$ transf. ortogonală

și notăm cu $\tilde{A}$ matricea asociată, $\tilde{A} \in O(2)$

a) $\det A = 1$.

$a_1)$ $\lambda = 1$, $f(e_1) = e_1$ $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\boxed{\tilde{A}}} \\ 0 & & \end{pmatrix}$

$\Rightarrow \det \tilde{A} = 1 \Rightarrow \tilde{A} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

$a_2)$ $\lambda = -1$, $f(e_1) = -e_1$ $\quad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\tilde{A}} \\ 0 & & \end{pmatrix}$

$\Rightarrow \det \tilde{A} = -1 \Rightarrow \tilde{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\exists\, \tilde{\mathcal{R}} = \{e_2, e_3\}$ a.î.

În reperul $\{e_3, e_1, e_2\}$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\pi & -\sin\pi \\ 0 & \sin\pi & \cos\pi \end{pmatrix}$

Teoremă: $\dim E = 3$, $f \in O(E)$ transf. ortog de speța 2.

$\Rightarrow \exists$ un reper ortonormal $\mathcal{R} = \{e_1, e_2, e_3\}$ a.î.

$$[f]_{\mathcal{R},\mathcal{R}} = A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $f(x) = (-x_1, \underbrace{x_2\cos\varphi - x_3\sin\varphi}_{x_2'}, \underbrace{x_2\sin\varphi + x_3\cos\varphi}_{x_3'})$

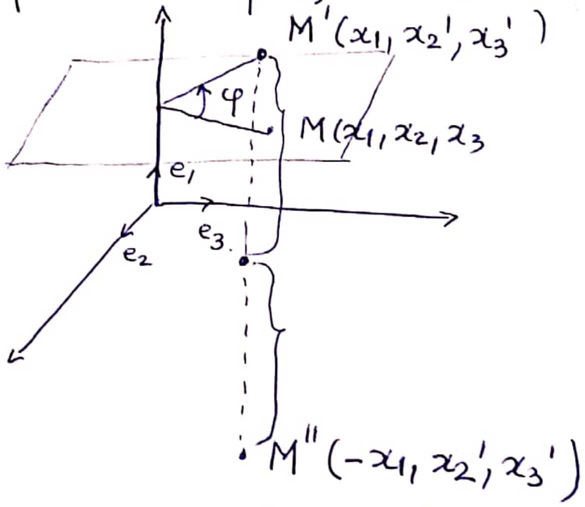
$f = s \circ R_\varphi$, R_φ = rotație de unghi orientat φ și axă $\langle\{e_1\}\rangle$.

s = simetrie ortogonală față de $\langle\{e_1\}\rangle^\perp$

OBS a) $Tr(A) = -1 + 2\cos\varphi$ invariant la schimbarea de reper ortonormat

b) Axa de rotație: $f(x) = -x$.

$x \in \langle\{e_1\}\rangle \Rightarrow f(x) = f(ae_1) = af(e_1) = -ae_1 = -x$.



④ $\dim E \geq 4$. $\exists$ un reper ortonormat a.î.

$$A = \begin{pmatrix} \overbrace{1 \ \cdots \ 1}^{s \text{ ori}} & & & & \\ & \overbrace{-1 \ \cdots \ -1}^{k-s \text{ ori}} & & & \\ & & A_1 & & \\ & & & \ddots & \\ & & & & A_p \end{pmatrix}$$

$k + 2p = n$

$$A_j = \begin{pmatrix} \cos\varphi_j & -\sin\varphi_j \\ \sin\varphi_j & \cos\varphi_j \end{pmatrix}$$

$j = \overline{1,p}$

Prop $f \in O(E) \Rightarrow f$ invariază cel puțin un subspațiu 1-dim sau 2-dim.

<u>Teorema Cartan</u> $\forall f \in O(E)$, $n = \dim E \geq 1$, $f \neq id_E$ se poate scrie ca o compunere de cel mult n simetrii ortogonale față de hiperplane.

<u>Aplicație</u> $f : \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x) = \frac{1}{3}(2x_1 + x_2 - 2x_3, -2x_1 + 2x_2 - x_3; x_1 + 2x_2 + 2x_3)$$

a) $f \in SO(E)$

b) $\exists$ un reper ortonormat $R = \{e_1, e_2, e_3\}$ a.î. $[f]_{R,R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

unde $f = R_\varphi$ rotație de unghi orientat φ și axă $\langle\{e_1\}\rangle$

<u>SOL</u>

a) $A = \frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}$

$$A^T \cdot A = \frac{1}{9}\begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ -2 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix} = I_3$$

$$\det A = \frac{1}{27}(12 + 3 + 12) = \frac{27}{27} = 1 \Rightarrow A \in SO(3)$$

$\Rightarrow f = R_\varphi$.

b) $Tr A = \frac{1}{3} \cdot 6 = 2 = 1 + 2\cos\varphi \Rightarrow 2\cos\varphi = 1 \Rightarrow \cos\varphi = \frac{1}{2}$

$\varphi = \pm\frac{\pi}{3}$.

Axa: $f(x) = x \Rightarrow \begin{cases} 2x_1 + x_2 - 2x_3 = 3x_1 \\ -2x_1 + 2x_2 - x_3 = 3x_2 \\ x_1 + 2x_2 + 2x_3 = 3x_3 \end{cases}$

$\langle\{(-1, 1, 1)\}\rangle$ = axa. $\quad e_1 = \frac{1}{\sqrt{3}}(+1, -1, -1)$.

$\langle\{e_1\}\rangle^\perp = \{x \in \mathbb{R}^3 \mid -x_1 + x_2 + x_3 = 0\} = \{(x_2 + x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\}$

$= \langle\{\underbrace{(1, 1, 0)}_{f_2}, \underbrace{(1, 0, 1)}_{f_3}\}\rangle$

$\overline{e_2} = f_2 = (1, 1, 0)$

$$\overline{e_3} = f_3 - \frac{\langle f_3, \overline{e_2}\rangle}{\langle \overline{e_2}, \overline{e_2}\rangle} \cdot \overline{e_2} = (1, 0, 1) - \frac{1}{2}(1, 1, 0) = \left(\frac{1}{2}, -\frac{1}{2}, 1\right)$$

$$= \frac{1}{2}(1, -1, 2)$$

$e_2 = \frac{1}{\sqrt{2}}(1,1,0)$, $e_3 = \frac{1}{\sqrt{6}}(1,-1,2)$; $\begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix}$

$= (2,-2,-2)$

$= 2(1,-1,-1)$.

$\mathcal{R} = \{ e_1 = \frac{1}{\sqrt{3}}(1,-1,-1), e_2 = \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{6}}(1,-1,2) \}$

reperul ortonormat ai $A = [f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

Def. $(E, \langle \cdot, \cdot \rangle)$ sp v.r. , $f \in End(E)$

$f \in Sim(E) \iff \langle x, f(y) \rangle = \langle f(x), y \rangle$
(endomorfism simetric) $\forall x, y \in E$.

OBS

$f \in Sim\, E \iff$ în raport cu $\forall$ reper ortonormat matricea e simetrică (i.e $A = A^T$)

$\mathcal{R} = \{e_1, e_2, .., e_n\}$ reper ortonormat.

$\langle e_i, f(e_j) \rangle = \langle f(e_i), e_j \rangle \Rightarrow \langle e_i, \sum_{r=1}^{n} a_{rj} e_r \rangle = \langle \sum_{s=1}^{n} a_{si} e_s, e_j \rangle$

$\sum_{r=1}^{n} a_{rj}\, \delta_{ir} = \sum_{s=1}^{n} a_{si}\, \delta_{sj} \Rightarrow a_{ij} = a_{ji} \Rightarrow A = A^T$

Dacă $\mathcal{R} \xrightarrow{C} \mathcal{R}'$, $\mathcal{R}, \mathcal{R}'$ repere ortonormate

$A' = C^T A C \Rightarrow A'^T = (C^T A C)^T = C^T A^T C = C^T A C = A'$

OBS $\forall f_1, f_2 \in Sim(E)$. În general, $f_1 \circ f_2 \notin Sim(E)$

Dacă $\langle f_1 \circ f_2(x), y \rangle = \langle x, f_1 \circ f_2(y) \rangle$,

atunci

$\langle f_1 \circ f_2(x), y \rangle = \langle f_2(x), f_1(y) \rangle = \langle x, f_2 \circ f_1(y) \rangle$

$\Rightarrow f_1 \circ f_2 = f_2 \circ f_1$. i.e. $A_1 A_2 = A_2 \circ A_1$.

Tema C_{10} Fie $(\mathbb{R}^3, g_0)$ s.v.e.r, cu str. canonică.

① Fie $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $f(x) = (x_3, x_2, x_1)$

a) Să se arate că f este o transformare ortogonală de speță 2.

b) Să se determine un reper ortonormat (pozitiv orientat) $\mathcal{R} = \{e_1, e_2, e_3\}$ astfel încât

$$[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix},$$

unde $f = s \circ R_\varphi$, R_φ = rotație de unghi orientat φ și s = simetrie ortogonală față de $<\{e_1\}>^\perp$.

② Fie $u = (1,1,-1)$, $v = (0,1,2)$, $w = (0,0,1)$

a) Să se afle volumul paralelipipedului determinat de u, v, w.

b) Să se arate că $\{u, v, w\}$ este un reper în $\mathbb{R}^3$. Să se ortonormeze, utilizând Gram-Schmidt.