

Endomorfisme. Vectori și valori proprii

Diagonalizare

Ex1. $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ liniară și $A = [f]_{R_0, R_0} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$

a) $f(x) = ?$ b) $A' = [f]_{R_1, R_1}$

unde $R = \{e_1' = 2e_1 - e_2, e_2' = e_1 - 2e_2\}$
 $R_0 = \{e_1, e_2\}$ reperul canonic.

sol

$$\begin{aligned} \text{a) } f(x) = y &\Leftrightarrow Y = AX \Leftrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} 2x_1 - x_2 \\ 3x_1 + 4x_2 \end{pmatrix} \quad f(x_1, x_2) = (2x_1 - x_2, 3x_1 + 4x_2) \end{aligned}$$

OBS

$$f(e_1) = 2e_1 + 3e_2$$

$$f(e_2) = -e_1 + 4e_2$$

$$\begin{aligned} f(x) &= f(x_1 e_1 + x_2 e_2) = x_1 f(e_1) + x_2 f(e_2) \\ &= x_1 (2e_1 + 3e_2) + x_2 (-e_1 + 4e_2) = e_1 (2x_1 - x_2) + e_2 (3x_1 + 4x_2) \\ &= (2x_1 - x_2, 3x_1 + 4x_2) \end{aligned}$$

$$\begin{aligned} \text{b) } f(e_1') &= f(2e_1 - e_2) = f(2, -1) = (5, 2) = a e_1' + b e_2' \\ (5, 2) &= a(2, -1) + b(1, -2) = (2a + b, -a - 2b) \end{aligned}$$

$$\begin{cases} 2a + b = 5 \\ -a - 2b = 2 \end{cases} \quad \text{②}$$

$$\hline -3b = 9$$

$$\begin{aligned} b &= -3 \\ a &= \frac{8}{2} = 4 \end{aligned}$$

$$A' = \begin{pmatrix} 4 & 1 \\ -3 & 2 \end{pmatrix}$$

$[f]_{R_1, R_1}$

$$f(e_2') = f(1, -2) = (4, -5) = c e_1' + d e_2' = (2c + d, -c - 2d)$$

$$\begin{cases} 2c+d=4 \\ -c-2d=-5 \end{cases} \quad \begin{cases} d=2 \\ c=1 \end{cases}$$

$$\hline / -3d=-6$$

(OBS)

$$\mathcal{R}_0 \xrightarrow{A} \mathcal{R}_0$$

$$A' = C^{-1}AC$$

$$\begin{array}{ccc} & & \downarrow C \\ C \downarrow & & \downarrow C \\ \mathcal{R} & \xrightarrow{A'} & \mathcal{R} \end{array}$$

$$C: \begin{cases} e_1' = 2e_1 - e_2 \\ e_2' = e_1 - 2e_2 \end{cases}$$

$$C = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}$$

$$\text{Ex2 } (\mathbb{R}_3[X], +, \cdot) / \mathbb{R}$$

$$V_1 = \langle \{1, 1-x\} \rangle$$

$$V_2 = \langle \{(1-x)^2, (1-x)^3\} \rangle$$

$$a) \mathbb{R}_3[X] = V_1 \oplus V_2$$

$$b) p: V_1 \oplus V_2 \rightarrow V_1 \quad \text{proiectia pe } V_1, \text{ de-a lungul lui } V_2$$

$$p(P) = p(\underbrace{P_1}_{V_1} + \underbrace{P_2}_{V_2}) = P_1$$

$$s: V_1 \oplus V_2 \rightarrow V_1 \oplus V_2 \quad \text{simetria pe } V_1 \quad \dashv \dashv$$

$$s = 2p - \text{id}_{\mathbb{R}_3[X]}$$

$$s(P) = 2p(P) - \text{id}_{\mathbb{R}_3[X]}(P) = 2P_1 - (P_1 + P_2) = P_1 - P_2$$

$$p(1 + 2x + x^2 + x^3) = ?$$

$$s(1 + 2x + x^2 + x^3) = ?$$

sol

$$a) \mathcal{R} = \{1, 1-x, (1-x)^2, (1-x)^3\} \text{ reper in } \mathbb{R}_3[X] \text{ (dem)}$$

$$\mathcal{R}_0 = \{1, x, x^2, x^3\} \text{ reperul canonic}$$

Deoarece $\mathcal{R} \in \text{SLI}$

$$\dim \mathbb{R}_3[X] = 4 = |\mathcal{R}|$$

$$\Rightarrow \mathcal{R} \text{ reper}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

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$$(1-x)^2 = 1 - 2x + x^2$$

$$(1-x)^3 = 1 - 3x + 3x^2 - x^3$$

$\text{reg } A = 4 \text{ max} \Rightarrow R \text{ e SLI}$

$$R = R_1 \cup R_2, \quad R_1 = \{1, 1-x\}, \quad R_2 = \{(1-x)^2, (1-x)^3\}.$$

$$R_3[X] = V_1 \oplus V_2, \quad V_1 = \langle R_1 \rangle, \quad V_2 = \langle R_2 \rangle.$$

$$b) 1 + 2x + x^2 + x^3 = \underbrace{a \cdot 1 + b(1-x)}_{P_1 \in V_1} + \underbrace{c(1-x)^2 + d(1-x)^3}_{P_2 \in V_2}.$$

$$\bullet x=1 \Rightarrow 5 = a$$

$$2 + 2x + 3x^2 = -b - 2c(1-x) - 3d(1-x)^2$$

$$\bullet x=1 \Rightarrow -7 = b.$$

$$2 + 6x = 2c + 6d(1-x)$$

$$\bullet x=1 \Rightarrow 8 = 2c \Rightarrow c=4$$

$$6 = -6d \Rightarrow d = -1.$$

$$P = 1 + 2x + x^2 + x^3 \text{ are coord } (a, b, c, d) = (5, -7, 4, -1)$$

in raport cu R .

$$P = P_1 + P_2, \quad P_1 = 5 \cdot 1 - 7(1-x) = -2 + 7x \in V_1$$

$$P_2 = 4(1-x)^2 - (1-x)^3 = 3 - 5x + x^2 + x^3 \in V_2$$

$$\rho(P) = P_1 = -2 + 7x$$

$$\Delta(P) = P_1 - P_2 = -2 + 7x - 3 + 5x - x^2 - x^3 = -5 + 12x - x^2 - x^3$$

Ex3 $(\mathbb{R}^2, +, \cdot) / \mathbb{R}$, $\mathcal{R} = \{e_1 = (1, 0), e_2 = (0, 1)\}$ reperul canonic
 $\mathcal{R}' = \{e'_1 = e_1 - e_2, e'_2 = e_1 + 2e_2\}$ reper.

$(\mathbb{R}^2)^* = \{f: \mathbb{R}^2 \rightarrow \mathbb{R} \mid f \text{ liniară}\}, +, \cdot) / \mathbb{R}$ spațiul dual

$\mathcal{R}^* = \{e_1^*, e_2^*\}$, $e_i^*(e_j) = \delta_{ij}$, $\forall i, j = \overline{1, 2}$

$\mathcal{R}'^* = \{e_1'^*, e_2'^*\}$, $e_i'^*(e'_j) = \delta_{ij}$ —

reperele duale în sp. dual.

$\mathcal{R} \xrightarrow{C} \mathcal{R}'$, $\mathcal{R}^* \xrightarrow{D} \mathcal{R}'^*$

Precizați legătura dintre C și D .

SOL $\mathcal{R} \xrightarrow{C} \mathcal{R}'$ $C = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$

$\begin{cases} e'_1 = e_1 - e_2 \\ e'_2 = e_1 + 2e_2 \end{cases}$

$e_1^*: \mathbb{R}^2 \rightarrow \mathbb{R}$ liniară, $e_1^*(e_1) = 1$, $e_1^*(e_2) = 0$

$e_2^*: \mathbb{R}^2 \rightarrow \mathbb{R}$ liniară, $e_2^*(e_1) = 0$, $e_2^*(e_2) = 1$

$e_1^*(x) = e_1^*(x_1 e_1 + x_2 e_2) = x_1 \underbrace{e_1^*(e_1)}_1 + x_2 \underbrace{e_1^*(e_2)}_0 = x_1$

$e_2^*(x) = e_2^*(x_1 e_1 + x_2 e_2) = x_1 \underbrace{e_2^*(e_1)}_0 + x_2 \underbrace{e_2^*(e_2)}_1 = x_2$

$\mathcal{R}^* \xrightarrow{D} \mathcal{R}'^*$

$D = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$\begin{cases} e_1'^* = a e_1^* + b e_2^* \\ e_2'^* = c e_1^* + d e_2^* \end{cases}$

1) $\underline{e}_1'^*(e'_1) = 1 \Rightarrow (a \underline{e}_1^* + b \underline{e}_2^*)(e_1 - e_2) = 1$

$\underbrace{a e_1^*(e_1)}_1 - \underbrace{a e_1^*(e_2)}_0 + \underbrace{b e_2^*(e_1)}_0 - \underbrace{b e_2^*(e_2)}_1 = 1 \Rightarrow \boxed{a - b = 1}$

$$\underbrace{ae_1^*(e_1)}_1 + \underbrace{2ae_1^*(e_2)}_0 + \underbrace{be_2^*(e_1)}_0 + \underbrace{2be_2^*(e_2)}_1 = 0 \Rightarrow \boxed{a+2b=0}$$

$$b = -\frac{1}{3}$$
$$a = 1 - \frac{1}{3} = \frac{2}{3}$$

$$4) e_2'^* (e_2') = 1 \Rightarrow (ce_1^* + de_2^*)(e_1 + 2e_2) = 1 \Rightarrow \boxed{c + 2d = 1}$$

$$n = d = \frac{1}{3}$$

$$C^T = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}, \quad C^* = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}; \quad \det C = 3$$

$$\mathbb{D} = (C^{-1})^T \quad C_{ij}^* = (-1)^{i+j} \Delta_{ji}$$

\exists un reper R în V ai $[f]_{R,R}$ diagonală

$$2) \dim V_{\lambda_i} = m_i, \quad \forall i = 1, r$$

$$m_1 + \dots + m_k = n = \dim V$$

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x este vector propriu al lui f de. $\exists \lambda \in \mathbb{K}$ (valoare proprie)
 ai $f(x) = \lambda x$
 $V_{\lambda_i} = \{x \in V \mid f(x) = \lambda_i x\}, i = \overline{1, 2}$ subsp. propriu

Ex 4 $f: \mathbb{R}^4 \rightarrow \mathbb{R}^4, f(x) = (x_2 - x_3 + x_4, x_2 - x_3 + x_4, x_4, x_4)$

- a) Sa se afle valorile proprii
 b) ——— subspatiile proprii
 c) \exists un reper R in \mathbb{R}^4 ai $[f]_{R,R}$ diagonală?

SOL

a) $A = [f]_{R_0, R_0} = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$f(x) = y \Leftrightarrow Y = AX$

$P(\lambda) = \det(A - \lambda I_4) = 0$

$$\begin{vmatrix} -\lambda & 1 & -1 & 1 \\ 0 & 1-\lambda & -1 & 1 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^2(1-\lambda)^2 = 0 \Rightarrow \begin{cases} \lambda_1 = 0, m_1 = 2 \\ \lambda_2 = 1, m_2 = 2 \end{cases}$

b) $V_{\lambda_1} = \{x \in \mathbb{R}^4 \mid \underline{f(x) = 0 \cdot x = 0_{\mathbb{R}^4}}\} = \text{Ker } f = S(A)$

$\dim V_{\lambda_1} = 4 - \text{rang } A = 4 - 2 = 2$

$\begin{cases} x_2 - x_3 + x_4 = 0 \\ x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = x_3 \\ x_4 = 0 \end{cases}$

$V_{\lambda_1} = \{ \underset{||}{(x_1, x_2, x_2, 0)}, x_1, x_2 \in \mathbb{R} \} = \underbrace{\langle (1, 0, 0, 0), (0, 1, 1, 0) \rangle}_{R_1}$

$x_1(1, 0, 0, 0) + x_2(0, 1, 1, 0)$

R_1 e SG in V_{λ_1} ; $\dim V_{\lambda_1} = 2 = |R_1| \Rightarrow R_1$ reper in V_{λ_1}

$$\lambda_2 = \{x \in \mathbb{R}^4 \mid f(x) = 1 \cdot x\}$$

$$(x_2 - x_3 + x_4, x_2 - x_3 + x_4, x_4, x_4) = (x_1, x_2, x_3, x_4)$$

$$\begin{cases} x_2 - x_3 + x_4 = x_1 \\ x_2 - x_3 + x_4 = x_2 \\ x_4 = x_3 \end{cases} \Rightarrow \begin{cases} x_1 = x_2 \\ x_4 = x_3 \end{cases}$$

$$V_{\lambda_2} = \{(x_1, x_1, x_3, x_3) \mid x_1, x_3 \in \mathbb{R}\} = \underbrace{\langle (1, 1, 0, 0), (0, 0, 1, 1) \rangle}_{\mathcal{R}_2}$$

\mathcal{R}_2 este SG pt $V_{\lambda_2} \Rightarrow \mathcal{R}_2$ reper in V_{λ_2} .

$$\text{reg} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = 2 \Rightarrow \mathcal{R}_2 \text{ e SLI}$$

$$c) 1) \lambda_1 = 0, \lambda_2 = 1 \in \mathbb{R}$$

$$2) \dim V_{\lambda_1} = 2 = m_1, \dim V_{\lambda_2} = 2 = m_2$$

$$\exists \mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 = \{(1, 0, 0, 0), (0, 1, 1, 0), (1, 1, 0, 0), (0, 0, 1, 1)\}$$

$$\text{reper in } \mathbb{R}^4 \text{ cu } [f]_{\mathcal{R}, \mathcal{R}} = A' = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{R}^4 = V_{\lambda_1} \oplus V_{\lambda_2}$$

Ex 5 $f \in \text{End}(\mathbb{R}^3)$

Fie $\lambda_1 = 3, \lambda_2 = -2, \lambda_3 = 1$ valorile proprii si

$$v_1 = (-3, 2, 1), v_2 = (-2, 1, 0), v_3 = (-6, 3, 1)$$

vectorii proprii corespunzatori

Sa se determine $A = [f]_{\mathcal{R}_0, \mathcal{R}_0}$, \mathcal{R}_0 reperul canonic.

Sol. v_1, v_2, v_3 vectori proprii coresp la valori

propri dist $\xrightarrow{\text{PROP}} \mathcal{R} = \{v_1, v_2, v_3\}$ SLI $\Rightarrow \mathcal{R}$ reper in \mathbb{R}^3
dar $\dim \mathbb{R}^3 = |\mathcal{R}| = 3$

$$\begin{cases} f(v_1) = \lambda_1 v_1 = 3v_1 \\ f(v_2) = \lambda_2 v_2 = -2v_2 \\ f(v_3) = \lambda_3 v_3 = v_3 \end{cases} \quad A' = [f]_{R,R} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_0 = \{e_1, e_2, e_3\} \xrightarrow{C} R = \{v_1, v_2, v_3\}$$

$$\begin{cases} v_1 = (-3, 2, 1) = -3e_1 + 2e_2 + e_3 \\ v_2 = (-2, 1, 0) = -2e_1 + e_2 + 0e_3 \\ v_3 = (-6, 3, 1) = -6e_1 + 3e_2 + e_3 \end{cases}$$

$$C = \begin{pmatrix} -3 & 2 & 1 \\ -2 & 1 & 0 \\ -6 & 3 & 1 \end{pmatrix}$$

$$A' = C^{-1}AC \Rightarrow \boxed{A = CA'C^{-1}}$$

(OBS) $f(v_1) = \lambda_1 v_1 \Rightarrow$
 $f(-3, 2, 1) = f(-3e_1 + 2e_2 + e_3) = -3f(e_1) + 2f(e_2) + f(e_3) = 3(-3, 2, 1)$

• $f(-2e_1 + e_2) = -2f(e_1) + f(e_2) = -2(-2, 1, 0)$

• $f(-6e_1 + 3e_2 + e_3) = -6f(e_1) + 3f(e_2) + f(e_3) = (-6, 3, 1)$

ec1 - ec3 : $3f(e_1) - f(e_2) = (-3, 3, 2)$

ec2 : $-2f(e_1) + f(e_2) = (4, -2, 0)$

$f(e_1) = (1, 1, 2)$

$f(e_2) = (4, -2, 0) + (2, 2, 4) = (6, 0, 4)$

$f(e_3) = (-9, 6, 3) + 3(1, 1, 2) - 2(6, 0, 4)$

$= (-9 + 3 - 12, 6 + 3, 3 + 6 - 8) = (-18, 9, 1)$

$$\begin{cases} f(e_1) = e_1 + e_2 + 2e_3 \\ f(e_2) = 6e_1 + 0e_2 + 4e_3 \\ f(e_3) = -18e_1 + 9e_2 + e_3 \end{cases}$$

$$A = [f]_{R_0, R_0} = \begin{pmatrix} 1 & 6 & -18 \\ 1 & 0 & 9 \\ 2 & 4 & 1 \end{pmatrix}$$

$f: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ lin, $A = [f]_{R_0, R_0} = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 2 & -1 & 0 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix}$

a) să se afle valorile proprii și subsp. proprii.

b) $U = \langle \{ e_1 + 2e_2, e_2 + e_3 + 2e_4 \} \rangle$
este un subspațiu invariant al lui f

sol

a) $P(\lambda) = \det(A - \lambda I_4) = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 2 & -1 \\ 0 & 1-\lambda & 4 & -2 \\ 2 & -1 & -\lambda & 1 \\ 2 & -1 & -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow l_3' = l_3 - l_4$$

$$\begin{vmatrix} 1-\lambda & 0 & 2 & -1 \\ 0 & 1-\lambda & 4 & -2 \\ 0 & 0 & 1-\lambda & \lambda-1 \\ 2 & -1 & -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow c_3' = c_3 + c_4$$

$$(\lambda-1) \begin{vmatrix} 1-\lambda & 0 & 1 & -1 \\ 0 & 1-\lambda & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 2 & -1 & 1-\lambda & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(\lambda-1) \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 2 \\ 2 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$-(\lambda-1) \{ (1-\lambda) [(1-\lambda)^2 + 2] - 2(1-\lambda) \} = 0$$

$$(1-\lambda)^2 [(1-\lambda)^2 + 2 - 2] = 0 \Rightarrow (1-\lambda)^4 = 0$$

$$\lambda_1 = 1, m_1 = 4$$

$$V_{\lambda_1} = \{ x \in \mathbb{R}^4 \mid f(x) = x \}$$

$$AX = X \Leftrightarrow (A - I_4)X = 0$$

$$\begin{pmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & 4 & -2 \\ 2 & -1 & -1 & 1 \\ 2 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\dim V_{\lambda_1} = 4 - 2 = 2 \neq m_1$$

Matricea nu se poate diagonaliza.

$$\begin{cases} 4x_3 - 2x_4 = 0 \\ 2x_1 - x_2 - x_3 + x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = \frac{1}{2}x_4 \\ x_2 = 2x_1 - \frac{1}{2}x_4 + x_4 = 2x_1 + \frac{1}{2}x_4 \end{cases}$$

$$V_{\lambda_1} = \left\{ (x_1, 2x_1 + \frac{1}{2}x_4, \frac{1}{2}x_4, x_4) \mid x_1, x_4 \in \mathbb{R} \right\}$$

$$x_1 \underbrace{(1, 2, 0, 0)}_{e_1 + 2e_2} + \frac{x_4}{2} \underbrace{(0, 1, 1, 2)}_{e_2 + e_3 + 2e_4}$$

$$V_{\lambda_1} = \langle \{ e_1 + 2e_2, e_2 + e_3 + 2e_4 \} \rangle = U$$

V_{λ_1} = subspațiu invariant al lui f
 $f(U) \subseteq U$.

$$\bullet V_{\lambda} = \{ x \in V \mid f(x) = \lambda x \}$$

subsp. propriu coresp. valorii proprii λ