30.03.2021.

CURS 7

Endomorfisme. Vectori proprii. Valori proprii. Diagonalizare

OBS

• $p: V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2$ apl. liniară

Dacă $p(\underset{V_1}{\underset{\cap}{v_1}} + \underset{V_2}{\underset{\cap}{v_2}}) = v_1$, at p s.n. proiecție pe V_1 de-a lungul lui V_2

• $p \in End(V)$

p = proiecție $\Leftrightarrow p \circ p = p$.

Def $s \in End(V)$ s. s.n. simetrie (sau involuție) $\Leftrightarrow s \circ s = id_V$.

Prop $(V,+,\cdot)/\mathbb{K}$ sp. vect, $ch\,\mathbb{K} \neq 2$ (i.e $1+1 \neq 0$)

Fie $p \in End(V)$

p = proiecție $\Leftrightarrow s = 2p - id_V$ este simetrie.

Dem

"$\Rightarrow$" Ip: $p \circ p = p$

$s \circ s = (2p - id_V) \circ (2p - id_V) = 4 p\circ p - 4p + id_V = id_V$

$\Rightarrow s$ simetrie

"$\Leftarrow$" Ip: $s \circ s = id_V$

$s \circ s = 4 p \circ p - 4p + id_V$ $\Bigg\} \Rightarrow p \circ p = p$.

$\boxed{\text{OBS}}$ $V = \operatorname{Im} p \oplus \operatorname{Ker} p = V_1 \oplus V_2$.

$s, p: V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2$

p = proiecția pe V_1 de-a lungul lui V_2

s = simetria față de V_1 —"—

Fie $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ reper în V, $k = \dim \operatorname{Im} p$, $n-k = \dim \operatorname{Ker} p$

$\mathcal{R}_1 = \{e_1, e_k\}$ reper în $\operatorname{Im} p$. $\quad p(e_i) = e_i,\ i = \overline{1,k}$

$\mathcal{R}_2 = \{e_{k+1}, e_n\}$ reper în $\operatorname{Ker} p$ $\quad p(e_j) = 0,\ j = \overline{k+1,n}$

$s = 2p - id_V$; $s(e_i) = 2p(e_i) - e_i = 2e_i - e_i = e_i,\ \forall i = \overline{1,k}$

$s(e_j) = 2p(e_j) - e_j = -e_j,\ j = \overline{k+1,n}$

$$[p]_{\mathcal{R},\mathcal{R}} = A_p = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right) \in \mathcal{M}_n(\mathbb{K})$$

$$[s]_{\mathcal{R},\mathcal{R}} = A_s = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & -I_{n-k} \end{array}\right) \in \mathcal{M}_n(\mathbb{K})$$

$$\Rightarrow A_s = 2A_p - I_n = 2\left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right) - \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & I_{n-k} \end{array}\right) = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & -I_{n-k} \end{array}\right)$$

OBS

a) $A_p \notin O(n)$

b) $A_s \in O(n)$

Aplicatie 2

Fie $V = \langle\{(1,1,0),(1,0,0)\}\rangle$

$\mathbb{R}^3 = V \oplus W$

$p : V \oplus W \to V$ proiectia pe V, de-a lungul lui W

$s : V \oplus W \to V \oplus W$ simetria față de V, — ॥ —

a) $p(1,2,1) = ?$

b) $s(1,2,1) = ?$

SOL

$\mathcal{R}_1 = \{(1,1,0), (1,0,0)\}$ reper în V

$rg\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} = 2 \Rightarrow \mathcal{R}_1$ e SLI

dar $\mathcal{R}_1$ SG pt V

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$$\det\begin{pmatrix} 1 & 1 & \boxed{0} \\ 1 & 0 & \boxed{0} \\ 0 & 0 & \boxed{1} \end{pmatrix} \neq 0 \qquad W = \langle \underbrace{\{(0,0,1)\}}_{R_2 \text{ reper în } W} \rangle$$

$R = R_1 \cup R_2$ reper în $\mathbb{R}^3$.

$$(1,2,1) = \underbrace{a(1,1,0) + b(1,0,0)}_{v \in V} + \underbrace{c(0,0,1)}_{w \in W} = (a+b, a, c)$$

$$\begin{cases} a+b = 1 \Rightarrow b = 1-2 = -1. \\ a = 2 \\ c = 1 \end{cases}$$

coord. lui $(1,2,1)$ în raport cu R: $(a,b,c) = (2,-1,1)$

$$(1,2,1) = \underbrace{2(1,1,0) - (1,0,0)}_{(1,2,0) = v} + \underbrace{(0,0,1)}_{w}.$$

$p(1,2,1) = p(v+w) = v = (1,2,0)$

$s(1,2,1) = 2p(1,2,1) - (1,2,1) = 2(1,2,0) - (1,2,1) = (1,2,-1)$

Problemă

$f \in End(V)$

Determinăm un reper $R = \{e_1, \dots, e_n\}$ în V a.î.

$$[f]_{R,R} = A_f = \text{diagonală} = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$$

$$\begin{cases} f(e_1) = \alpha_1 e_1 \\ \vdots \\ f(e_n) = \alpha_n e_n \end{cases}$$

Def Fie $f \in End(V)$

$x \neq 0_V$ s.n. vector propriu al lui $f \iff \exists \lambda \in \mathbb{K}$ a.î. $f(x) = \lambda x$.

λ = valoare proprie asociată vect. propriu x

Not $V_\lambda = \{0_V\} \cup \{$vect. proprii asoc. valorii proprii $\lambda\}$

subspațiul propriu coresp. valorii proprii λ.

OBS $f(0_V) = 0_V = \lambda \cdot 0_V$

Prop $f \in End(V)$, $f(x) = \lambda x$, λ = val. proprie

a) $V_\lambda \subseteq V$ subsp. vect

b) V_λ = subspațiu invariant lui f i.e $f(V_\lambda) \subseteq V_\lambda$.

Dem

a) $\forall x, y \in V_\lambda$, $\forall a, b \in \mathbb{K} \Rightarrow ax + by \in V_\lambda$

$$f(ax+by) = a\underbrace{f(x)}_{\lambda x} + b\underbrace{f(y)}_{\lambda y} = \lambda(ax+by) \Rightarrow ax+by \in V_\lambda$$

b) $x \in V_\lambda \Rightarrow f(x) = \lambda x \in V_\lambda$ (subsp. vect)

Polinomul caracteristic $R = \{e_1, .., e_n\}$ reper în V, $[f]_{R,R} = A$.

$f \in End(V)$, $x \neq 0_V$ a.î. $f(x) = \lambda x$, $\lambda \in \mathbb{K}$

$$f(x) = f\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^n x_i f(e_i) = \sum_{i=1}^n x_i \sum_{j=1}^n a_{ji} e_j =$$

$$\| \quad = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ji} x_i\right) e_j$$

$$\lambda x = \lambda \sum_{j=1}^n x_j e_j$$

$\Rightarrow$ R e SLI

$$\sum_{i=1}^n a_{ji} x_i = \lambda x_j \Rightarrow$$

$$(*) \quad \sum_{i=1}^n (a_{ji} - \lambda \delta_{ji}) x_i = 0$$

$(*)$ este un SLO care are și sol. nenule $\Rightarrow$

$$P(\lambda) = \det(A - \lambda I_n) = 0 \quad \text{(polinomul caracteristic)}$$

Prop Polinomul caracteristic este invariant la schimbarea reperului $\quad A' = C^{-1}AC$

Dem

$$\begin{array}{ccc} R = \{e_1, .., e_n\} & \xrightarrow{A_f} & R = \{e_1, .., e_n\} \\ C \downarrow & & \downarrow C \\ R' = \{e_1', .., e_n'\} & \xrightarrow{A_f'} & R' = \{e_1', .., e_n'\} \end{array}$$

$$P(\lambda) = \det(A' - \lambda I_n) = \det(\underline{C}^{-1} A \underline{\underline{C}} - \lambda \underline{C}^{-1} I_n \underline{\underline{C}}) =$$

$$= \det\left[C^{-1}(A - \lambda I_n) C\right] = \det(A - \lambda I_n)$$

$C \in GL(n, \mathbb{K})$

OBS a) Valorile proprii ale lui f sunt rădăcinile <u>din $\mathbb{K}$</u> ale polinomului caracteristic.

b) $P(\lambda) = \det(A - \lambda I_n) =$

$$= (-1)^n \left[\lambda^n - \sigma_1 \lambda^{n-1} + \dots + (-1)^n \sigma_n\right] = 0$$

σ_k = suma minorilor diagonali de ordin k.

$\sigma_1 = Tr(A), \dots, \sigma_n = \det(A)$.

<u>Exemplu</u> $(\mathbb{R}^2, +, \cdot)/\mathbb{R}$, $J \in End(\mathbb{R}^2)$

$J(x) = (-x_2, x_1)$.

$R_0 = \{e_1 = (1,0), e_2 = (0,1)\}$

$J(e_1) = J(1,0) = (0,1) = e_2 = 0 \cdot e_1 + 1 \cdot e_2$

$J(e_2) = J(0,1) = (-1,0) = -e_1 + 0 e_2$

$$[J]_{R_0,R_0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

$$\det(A - \lambda I_2) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \notin \mathbb{R}$$

<u>Aplicatie</u>

$f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1 + 2x_2, 2x_1 + x_2)$

a) Să se det. valorile proprii

b) Să se afle subspațiile proprii. Precizați câte un reper în fiecare subspațiu

<u>SOL</u> $R_0 = \{e_1, e_2\}$ reperul canonic $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = [f]_{R_0,R_0}$

$P(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 2^2 =$

$= (1-\lambda-2)(1-\lambda+2) = (-1-\lambda)(3-\lambda) = (\lambda+1)(\lambda-3) = 0$

$\lambda_1 = -1 \in \mathbb{R}$
$\lambda_2 = 3 \in \mathbb{R}$ (valori proprii)

b) $V_{\lambda_1} = \{x \in \mathbb{R}^2 \mid f(x) = -x\}$

$AX = -X \Rightarrow (A + I_2)X = 0$ $\quad \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$2(x_1 + x_2) = 0 \Rightarrow x_2 = -x_1$

$V_{\lambda_1} = \{(x_1, -x_1) = x_1(1,-1),\ x_1 \in \mathbb{R}\} = \langle\{(1,-1)\}\rangle$

$V_{\lambda_2} = \{x \in \mathbb{R}^2 \mid f(x) = 3x\}$

$AX = 3X \Rightarrow (A - 3I_2)X = 0$

$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2(-x_1 + x_2) = 0 \Rightarrow x_1 = x_2$

$V_{\lambda_2} = \{(x_1, x_1) = x_1 \cdot (1,1) \mid x_1 \in \mathbb{R}\} = \langle\{(1,1)\}\rangle$

$\mathcal{R} = \{\underset{e_1'}{(1,-1)}, \underset{e_2'}{(1,1)}\}$ reper in $\mathbb{R}^2$

$\mathbb{R}^2 = V_{\lambda_1} \oplus V_{\lambda_2}$.

$f(1,-1) = (1-2, 2-1) = (-1,1) = -(1,-1) = -e_1' + 0e_2'$

$f(1,1) = (3,3) = 3(1,1) = 3e_2' = 0e_1' + 3e_2'$)

$f(x_1, x_2) = (x_1 + 2x_2, 2x_1 + x_2$

$[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\lambda_1, \lambda_2 \in \mathbb{R}$ $\quad P(\lambda) = (\lambda+1)(\lambda-3)$

$m_1 = 1$
$m_2 = 1$ (multiplicitatile răd)

$\dim V_{\lambda_k} = 1 = m_k,\ k = \overline{1,2}$

Obs $P(\lambda) = 0 \Rightarrow (\lambda-\lambda_1)^{m_1} \cdots (\lambda-\lambda_r)^{m_r} = 0$

$\lambda_1, \dots, \lambda_r$ valorile proprii distincte

$m_1, \dots, m_r$ multiplicitățile lor

$\sigma(f) = \{\lambda_1, \dots, \lambda_r\}$ spectrul

$Spec(f) = \{\underbrace{\lambda_1 = \dots = \lambda_1}_{m_1 \text{ ori}} < \dots < \underbrace{\lambda_r = \dots = \lambda_r}_{m_r \text{ ori}}\}$

Prop. Vectorii proprii coresp. la valori proprii distincte formează un SLI

Dem

Dem. prin ind. după n, nr de vect. proprii

$n = 1$, $\underset{\substack{\neq \\ 0_V}}{x_1}$ = vect. propriu $\Rightarrow \{x_1\}$ e SLI

Pp. prop. adev. pt $n-1$ vect și dem că este adev pt n vect.

Fie $x_1, \dots, x_n$ vect. proprii coresp. la val proprii $\lambda_1, \dots, \lambda_n$ dist.

Dem $\{x_1, \dots, x_n\}$ este SLI

(*) $a_1 x_1 + \dots + a_n x_n = 0_V \mid f \Rightarrow f(a_1 x_1 + \dots + a_n x_n) = f(0_V)$

(1) $a_1 \underset{\substack{\| \\ \lambda_1 x_1}}{f(x_1)} + \dots + a_n \underset{\substack{\| \\ \lambda_n x_n}}{f(x_n)} = 0_V$

Înm (*) cu λ_n ($\lambda_n \neq 0_{\mathbb{K}}$) $\Rightarrow$

(2) $a_1 \lambda_n x_1 + \dots + a_n \lambda_n x_n = 0_V$

(1) − (2) $\quad a_1 \underbrace{(\lambda_1 - \lambda_n)}_{\neq 0} x_1 + \dots + a_{n-1} \underbrace{(\lambda_{n-1} - \lambda_n)}_{\neq 0} x_{n-1} = 0$

$\{x_1, \dots, x_{n-1}\}$ SLI $\Longrightarrow a_1 = \dots = a_{n-1} = 0 \overset{(*)}{\Longrightarrow} a_n x_n = 0$

$\Rightarrow a_n = 0 \quad \{x_1, \dots, x_n\}$ este un SLI

<u>Prop</u> $f \in End(V)$, λ = valoare proprie cu multiplicitatea
$\Rightarrow \dim V_\lambda \leq m_\lambda$

<u>Dem</u> $V_\lambda = \{x \in V \mid f(x) = \lambda x\} \underset{subsp V}{\subseteq} V$, $\dim V_\lambda = n_\lambda$

$R_\lambda = \{e_1, \dots, e_{n_\lambda}\}$ reper in V_λ. Il extindem
la $R = \{e_1, \dots, e_{n_\lambda}, e_{n_\lambda+1}, \dots, e_n\}$ reper in V, $\dim V = n$

$$A = [f]_{R,R} \quad \begin{cases} f(e_1) = \lambda e_1 \\ \vdots \\ f(e_{n_\lambda}) = \lambda e_{n_\lambda} \\ f(e_j) = \sum_{k=1}^{n} a_{kj} e_k, \ j = \overline{n_\lambda + 1, \dots, n} \end{cases}$$

$$\begin{array}{cc} & \overbrace{}^{n_\lambda \text{ col.}} \\ A = n_\lambda \text{ linii} \Big\{ & \left(\begin{array}{cc|c} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \\ \hline & 0 & \end{array}\right) \end{array} \in \mathcal{M}_n(\mathbb{K})$$

$$P(X) = \det(A - XI_n) = \left|\begin{array}{cc|c} \lambda - X & 0 & \\ 0 & \lambda - X & \\ \hline & 0 & \end{array}\right|$$

$$\left.\begin{array}{l} = (\lambda - X)^{n_\lambda} Q(X) \\ [\lambda \text{ poate fi răd și in } Q] \end{array}\right\} \Rightarrow m_\lambda \geq n_\lambda$$

<u>Teorema de diagonalizare</u>

$(V, +, \cdot)/_{\mathbb{K}}$ sp vect, $f \in End(V)$

$\exists$ un reper $R = \{e_1, \dots, e_n\}$ in V a.i. $A = [f]_{R,R}$ e diagonală

$\Leftrightarrow$ 1) răd. pol. caract. $\in \mathbb{K}$

[i.e. $\lambda_1, \dots, \lambda_r \in \mathbb{K}$, răd. distincte]

2) $\dim V_{\lambda_i} = m_i$, $\forall i = \overline{1, r}$, $m_1 + \dots + m_r = n$

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m_i = multiplicitatea lui λ_i, $\forall i=\overline{1,r}$

Dem

"$\Rightarrow$" Ip: $\exists R=\{e_1,..,e_n\}$ reper a.î.

$A=[f]_{R,R}=\begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix} \in \mathcal{M}_n(\mathbb{K})$

Eventual renumerotăm.

$A=\begin{pmatrix} \overbrace{\lambda_1 \dots \lambda_1}^{m_1 \text{ ori}} & & & 0 \\ & \ddots & & \\ 0 & & \overbrace{\lambda_r \dots \lambda_r}^{m_r \text{ ori}} \end{pmatrix}$

$P(X)=\det(A-XI_n)=\det\begin{pmatrix} \lambda_1-X & & & & 0 \\ & \ddots \lambda_1-X & & & \\ & & \ddots & & \\ & & & \lambda_r-X & \\ 0 & & & & \ddots \lambda_r-X \end{pmatrix}$

$=(\lambda_1-X)^{m_1}\cdot\ldots\cdot(\lambda_r-X)^{m_r}$, $m_1+\ldots+m_r=n$

$\lambda_1,..,\lambda_r$ răd dist (din $\mathbb{K}$) ale pol. caract

$\left.\begin{array}{ll} \begin{cases} f(e_1)=\lambda_1 e_1 \\ \vdots \\ f(e_{m_1})=\lambda_1 e_{m_1} \end{cases} & \begin{array}{l} R_1=\{e_1,..,e_{m_1}\}\subset V_{\lambda_1} \\ R_1 \subset \underset{SLI}{R} \Rightarrow R_1 \text{ SLI} \end{array} \end{array}\right\}$

$\left.\begin{array}{l} m_1 \leq \dim V_{\lambda_1} \\ \text{dar } \dim V_{\lambda_1} \leq m_1 \end{array}\right\} \Rightarrow \dim V_{\lambda_1}=m_1$

Analog $\dim V_{\lambda_i}=m_i$, $\forall i=\overline{2,r}$

"$\Leftarrow$" Ip: 1) $\lambda_1,..,\lambda_r$ răd dist ale pol car $\in \mathbb{K}$

2) $\dim V_{\lambda_i}=m_i$, $\forall i=\overline{1,r}$

Fie R_i reper în V_{λ_i}, $i=\overline{1,r}$ $m_1+\ldots+m_r=n$

$R=R_1\cup\ldots\cup R_r$. Dem că R este reper în V și $A=[f]_{R,R}$ diag.

$\dim V=|R|=n$

Dem că R este SLI

$$\underbrace{\sum_{i=1}^{m_1} a_i e_i}_{\substack{\parallel \\ f_1 \in V_{\lambda_1}}} + \dots + \underbrace{\sum_{j=m_1+\dots+m_{r-1}+1}^{n} a_j e_j}_{\substack{\parallel \\ f_r \in V_{\lambda_r}}} = 0$$

Pp abs $\exists\, f_{i_1}, \dots, f_{i_p}$ nenuli dintre $\{f_1, \dots, f_r\}$ vectori proprii coresp. la val pr dist $\lambda_1, \dots, \lambda_r$

$\Rightarrow \{f_{i_1}, \dots, f_{i_p}\}$ e SLI

dar $f_{i_1} + \dots + f_{i_p} = 0_V$

$\Rightarrow f_1 = \dots = f_r = 0 \Rightarrow a_i = 0 \Rightarrow R$ este SLI

$$[f]_{R,R} = \begin{pmatrix} \overbrace{\lambda_1 \dots \lambda_1}^{m_1} & & 0 \\ & \ddots & \\ 0 & & \overbrace{\lambda_r \dots \lambda_r}^{m_r} \end{pmatrix}$$

Aplicatie

$f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1, x_2 + x_3, 2x_3)$

Sa se det un reper $R = \{e_1, e_2, e_3\}$ ai $A' = [f]_{R,R}$ diag.

SOL

$$A = [f]_{R_0,R_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + x_3 \\ 2x_3 \end{pmatrix}$$

$$P(\lambda) = \det(A - \lambda I_3) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$(1-\lambda)^2(2-\lambda) = 0$

- $\boxed{\begin{matrix}\lambda_1 = 1 \\ \lambda_2 = 2\end{matrix}}$, $m_1 = 2$, $m_2 = 1$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = x\}$

$AX = X \Rightarrow (A - I_3)X = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_3 = 0$

$V_{\lambda_1} = \{(x_1, x_2, 0) / x_1, x_2 \in \mathbb{R}\}$

$V_{\lambda_1} = \{x_1(1,0,0) + x_2(0,1,0) \mid x_1, x_2 \in \mathbb{R}\} = \langle \underbrace{\{(1,0,0),(0,1,0)\}}_{R_1} \rangle$

R_1 e SG, SLI $\Rightarrow$ reper în V_{λ_1}

$\boxed{\dim V_{\lambda_1} = 2}$

- $V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid f(x) = 2x\}$ $\begin{cases} x_1 = 2x_1 \\ x_2 + x_3 = 2x_2 \\ 2x_3 = 2x_3 \end{cases} \Rightarrow \begin{cases} -x_1 = 0 \\ -x_2 + x_3 = 0 \end{cases}$

$x_1 = 0$
$x_2 = x_3$ $\quad V_{\lambda_2} = \{(0, x_2, x_2) = x_2(0,1,1) \mid x_2 \in \mathbb{R}\}$

$= \langle \underbrace{\{(0,1,1)\}}_{R_2} \rangle$

R_2 reper în V_{λ_2}

$\boxed{\dim V_{\lambda_2} = 1}$

$\exists R = R_1 \cup R_2 = \{(1,0,0),(0,1,0),(0,1,1)\}$ reper în $\mathbb{R}^3$

$A' = [f]_{R,R} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_1 & \\ 0 & & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$