C10 - GA.

Transformări ortogonale. Endomorfisme simetrice

Def $(E_i, \langle \cdot, \cdot \rangle_i)$ $i=\overline{1,2}$ s.v.e.r. Aplicația liniară $f: E_1 \to E_2$ s.n. <u>aplicație ortogonală</u> $\Leftrightarrow \langle f(x), f(y)\rangle_2 = \langle x, y\rangle_1$ (*) $\forall x, y \in E_1$

Prop Dacă $f: E_1 \to E_2$ apl. ortogonală $\Rightarrow$

a) $\|f(x)\|_2 = \|x\|_1, \forall x \in E_1$

b) f injectivă

Dem

a) În (*) $y = x$ $\langle f(x), f(x)\rangle_2 = \langle x, x\rangle_1 \Rightarrow \|f(x)\|_2^2 = \|x\|_1^2$

$\Rightarrow \|f(x)\|_2 = \|x\|_1, \forall x \in E_1$

b) f liniară, f inj $\Leftrightarrow \operatorname{Ker} f = \{0_{E_1}\}$

Fie $x \in \operatorname{Ker} f \Rightarrow f(x) = 0_{E_2} \Rightarrow \underbrace{\|f(x)\|_2}_{0} = \|x\|_1 \Rightarrow x = 0_{E_1}$.

Def $(E, \langle \cdot, \cdot \rangle)$ s.v.e.r, $f \in \operatorname{End}(E)$

$f: E \to E$ s.n. <u>transformare ortogonală</u> $\Leftrightarrow \langle f(x), f(y)\rangle = \langle x, y\rangle$ $\forall x, y \in E$.

Not $O(E) = \{f \in \operatorname{End}(E) \mid f$ transf. ortogonale$\}$.

Prop $f \in \operatorname{End}(E)$

$f \in O(E) \Leftrightarrow \|f(x)\| = \|x\|, \forall x \in E$

Dem

"$\Rightarrow$" cf. prop preced

"$\Leftarrow$" $\|f(x+y)\|^2 = \|x+y\|^2 \Leftrightarrow \langle f(x)+f(y), f(x)+f(y)\rangle = \langle x+y, x+y\rangle$

$\underbrace{\langle f(x), f(x)\rangle}_{\|f(x)\|^2 = \|x\|^2} + \underbrace{\langle f(y), f(y)\rangle}_{\|f(y)\|^2 = \|y\|^2} + 2\underline{\langle f(x), f(y)\rangle} = \underbrace{\langle x, x\rangle}_{\|x\|^2} + \underbrace{\langle y, y\rangle}_{\|y\|^2} + 2\underline{\langle x, y\rangle}$

$\Rightarrow \langle f(x), f(y)\rangle = \langle x, y\rangle, \forall x, y \Rightarrow f \in O(E)$.

Prop $f \in O(E)$

$[f]_{R,R} \in O(n)$, $\forall R$ reper ortonormat

Dem $R = \{e_1, .., e_n\}$ reper orton, $[f]_{R,R} = A = (a_{ij})_{i,j=\overline{1,n}}$

$$\langle f(e_i), f(e_j) \rangle = \langle e_i, e_j \rangle \Rightarrow$$

$$\langle \sum_{k=1}^{n} a_{ki} e_k , \sum_{l=1}^{n} a_{lj} e_l \rangle = \langle e_i, e_j \rangle \Rightarrow$$

$$\sum_{k,l=1}^{n} a_{ki} \, a_{lj} \underbrace{\langle e_k, e_l \rangle}_{= \delta_{kl}} = \langle e_i, e_j \rangle = \delta_{ij}$$

$$\sum_{k=1}^{n} a_{ki} a_{kj} = \delta_{ij} \Rightarrow A^T A = I_n \Rightarrow A \in O(n)$$

$R \xrightarrow{C} R' = \{e_1', .., e_n'\}$ repere ortonormate, $C \in O(n)$

$C \cdot C^T = C^T C = I_n$

$C^{-1} = C^T$

$$A' = C^{-1} A C$$

$$A'^T A' = (C^{-1} A C)^T (C^{-1} A C) = C^T A^T \underbrace{C C^T}_{I_n} A C$$

$$= C^T \underbrace{A^T A}_{I_n} C = C^T C = I_n .$$

OBS $f \in O(E) \Leftrightarrow$ o schimbare de repere ortonormate.

"$\Rightarrow$" $f \in O(E) \Rightarrow [f]_{R,R} = A \in O(n)$

" matricea de trecere dintre 2 repere ortonormate.

"$\Leftarrow$" $R \xrightarrow{A} R'$, $A \in O(n)$ $\quad e_i' = \sum_{k=1}^{n} a_{ki} e_k$

$R = \{e_i\}_{i=\overline{1,n}}$ $\quad R' = \{e_i'\}_{i=\overline{1,n}}$

$f : E \to E$, $f(e_i) = e_i'$, $\forall i = \overline{1,n}$

Extindem prin liniaritate

$$f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i f(e_i) = \sum_{i=1}^{n} x_i e_i' = x'$$

Prop $f \in O(E) \Rightarrow$ valorile proprii $\in \{-1, 1\}$

Dem Fie $\lambda \in \mathbb{R}$ valoare proprie $\Rightarrow \exists x \neq 0_E$ aî $f(x) = \lambda x$

$$\|f(x)\|^2 = \|x\|^2 \Rightarrow \underbrace{\|\lambda x\|^2}_{= \langle \lambda x, \lambda x \rangle} = \|x\|^2$$

$$\lambda^2 \|x\|^2 = \|x\|^2 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

Prop $f \in O(E)$, $U \subseteq E$ subsp. invariant (i.e $f(U) \subseteq U$)

a) $f(U) = U$

b) $U^\perp \subseteq E$ subsp invar (i.e $E = U \oplus U^\perp$)

c) $f/_{U^\perp} : U^\perp \longrightarrow U^\perp$ transf. ortogonală

Dem

a) $f : U \longrightarrow f(U)$ bij + liniară $\Rightarrow \dim U = \dim f(U)$ dar $f(U) \subseteq U$

$\Rightarrow f(U) = U$.

b) Dem $f(U^\perp) \subseteq U^\perp$ i.e. $\forall x \in U^\perp \Rightarrow f(x) \in U^\perp$

Fie $z \in U = f(U) \Rightarrow \exists y \in U$ aî $z = f(y)$

$$\langle f(x), \underset{U}{z} \rangle = \langle f(x), f(y) \rangle = \langle \underset{U^\perp}{x}, \underset{U}{y} \rangle = 0 \Rightarrow f(x) \in U^\perp$$

$\overset{a)}{\Rightarrow} f(U^\perp) = U^\perp$

Deci $f/_{U^\perp} : U^\perp \longrightarrow U^\perp$ transf. ortogonală.

Clasificare transformări ortogonale

① $n = 1$, $O(E) = \{id_E, -id_E\}$

② $n = 2$, $f \in O(E)$, $\underset{O(2)}{A} = [f]_{R,R}$

a) $\det A = 1$ (f este de speța 1)

$$A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = A_\varphi, \quad \varphi \in (-\pi, \pi]$$

$f : \mathbb{R}^2 \to \mathbb{R}^2$ $\qquad \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$f(x_1, x_2) = (x_1\cos\varphi - x_2\sin\varphi,\ x_1\sin\varphi + x_2\cos\varphi)$

f este rotație de unghi orientat φ, în plan.

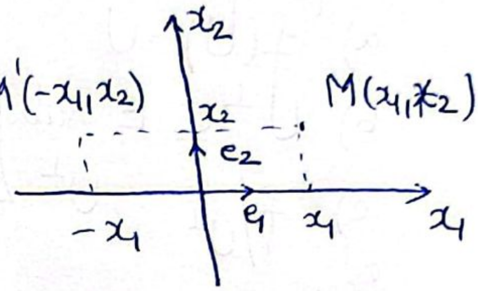
$\operatorname{Tr} A = 2\cos\varphi$ invariant la schimbarea reperelor.

b) $\det A = -1$ (f este de speță 2) $\qquad A = \begin{pmatrix} -\cos\varphi & \sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

$\exists$ un reper $\{e_1, e_2\}$ a.î. $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (-x_1, x_2)$

f = simetrie ortogonală față de $\langle\{e_1\}^{\perp}\rangle$ $\qquad E = \langle e_1\rangle \oplus \langle e_1\rangle^{\perp}$



<u>Teoremă</u> $n = 2$, $\forall f \in O(E)$, $f \neq id_E$ se poate scrie ca o compunere de cel mult 2 simetrii ortogonale (față de drepte)

<u>Dem</u> 1) f de speță 1 $\Rightarrow \det A_f = 1$

Fie s' o simetrie ortogonală $\Rightarrow \det A_{s'} = -1$

$s = s' \circ f$ simetrie ortogonală

$s' \circ s = \underbrace{s' \circ s'}_{id_E} \circ f = f$

2) f este de speța 2 $\Rightarrow f = s$ (sim. ortogonală)

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(3) $n = 3$, $A = [f]_{R,R} \in O(3)$

$P(\lambda) = \det(A - \lambda I_3)$ = polinom de gradul al 3-lea cu coef reali

$\Rightarrow \exists$ cel puțin o răd reală $\lambda \in \{-1, 1\}$

Fie e_1 = versorul propriu coresp. valorii proprii λ.

$f(e_1) = \lambda e_1 \Rightarrow \langle\{e_1\}\rangle$ subsp. invariant al lui f

$\Rightarrow \langle\{e_1\}\rangle^{\perp}$ —//—

$f/_{\langle\{e_1\}\rangle^\perp} : \langle e_1\rangle^\perp \longrightarrow \langle e_1\rangle^\perp$ transf. ortogonală.

(I.) f este de speță 1 ($\det A = 1$)

a) $\lambda = 1$, $f(e_1) = e_1$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

$\tilde{A} = [f_{\langle e_1\rangle^\perp}]_{R'}$; $\det\tilde{A} = 1$

$R' = \{e_2, e_3\}$

b) $\lambda = -1$, $f(e_1) = -e_1$, $\det\tilde{A} = -1$

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \boxed{\begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix}} \\ 0 & \end{pmatrix}$$

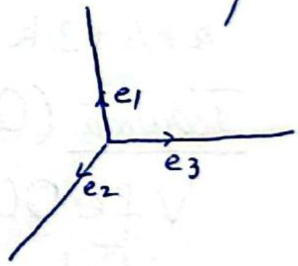
În rap cu $\{e_3, -e_1, -e_2\}$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\pi & -\sin\pi \\ 0 & \sin\pi & \cos\pi \end{pmatrix}$ $\{e_1, e_2, e_3\}$

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$f(x_1, x_2, x_3) = (x_1, x_2\cos\varphi - x_3\sin\varphi, x_2\sin\varphi + x_3\cos\varphi)$

$Tr A = 1 + 2\cos\varphi$ invariant la schimbarea de reper

Axa: $f(x) = x$



f este rotație de unghi orientat φ, în planul $\langle e_1\rangle^\perp$

II f este de speță 2 ($\det A = -1$)

a) $\lambda = 1$, $f(e_1) = e_1$, $\det\tilde{A} = -1$ $\{e_2, e_3\}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

În rap cu $\{\overset{e_1'}{e_2}, \overset{e_2'}{e_1}, \overset{e_3'}{e_3}\}$: $\begin{pmatrix} -1 & 0 & 0 \\ 0 & \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \\ 0 & \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos 0 & -\sin 0 \\ 0 & \sin 0 & \cos 0 \end{pmatrix}$

b) $\lambda = -1$, $f(e_1) = -e_1$, $\det\tilde{A} = 1$

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

$f:\mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1,x_2,x_3) = (-x_1,\ x_2\cos\varphi - x_3\sin\varphi,\ x_2\sin\varphi + x_3\cos\varphi)$

$f = s \circ R_\varphi$ $\quad R_\varphi$ = rotație de unghi orientat φ în planul $\langle e_1\rangle^\perp$ și s = simetrie ortogonală față de $\langle e_1\rangle^\perp$!

$Tr A = -1 + 2\cos\varphi$ invariant în rap cu sch. de reper.

Axa: $f(x) = -x$
(pt R_φ)

(4) $n \geq 4$ $\exists$ un reper ortonormat a.î.

$$A = \begin{pmatrix} \underbrace{1 \cdots 1}_{r} & & & & \\ & \underbrace{-1 \cdots -1}_{s} & & & \\ & & A_{\varphi_1} & & \\ & & & \ddots & \\ & & & & A_{\varphi_k} \end{pmatrix} \in O(n)$$

$r + s + 2k = n$.

<u>Teoremă</u> (Cartan) $n \geq 2$

$\forall f \in O(E)$ se poate scrie ca o compunere de
$\ne id_E$
cel mult n simetrii ortogonale față de hiperplane.

<u>Def</u> $(E, \langle\cdot,\cdot\rangle)$ s.v.e.r $\quad f \in End(E)$

f s.n. simetric $\iff \langle x, f(y)\rangle = \langle f(x), y\rangle,\ \forall x,y \in E$.

Not $Sim(E)$ = mulț. endom. simetrice.

<u>Prop</u> $f \in Sim(E) \Rightarrow [f]_{R,R} = A$, $A = A^T$ (simetrică)
$\forall R$ reper ortonormat

<u>Dem</u>

$\langle e_i, f(e_j)\rangle = \langle f(e_i), e_j\rangle \Rightarrow$

$\langle e_i, \sum_{k=1}^{n} a_{kj} e_k\rangle = \langle \sum_{l=1}^{n} a_{li} e_l, e_j\rangle$

$$\sum_{k=1}^{n} a_{kj} \overbrace{\langle e_i, e_j \rangle}^{\delta_{ij}} = \sum_{l=1}^{n} a_{li} \overbrace{\langle e_l, e_j \rangle}^{\delta_{lj}} \Rightarrow a_{ij} = a_{ji} \Rightarrow A = A^T$$

Prop $f \in Sim(E) \Rightarrow$ vectorii proprii coresp. la valori proprii distincte sunt ortogonali

Dem Fie $\lambda \neq \mu$, valori proprii $\Rightarrow \exists\, x, y$ ($x \neq 0_E$, $y \neq 0_E$) aî $f(x) = \lambda x$, $f(y) = \mu y$

$\langle x, f(y) \rangle = \langle f(x), y \rangle \Rightarrow \langle x, \mu y \rangle = \langle \lambda x, y \rangle \Rightarrow$

$\mu \langle x, y \rangle = \lambda \langle x, y \rangle \Rightarrow \underbrace{(\lambda - \mu)}_{\neq 0} \langle x, y \rangle = 0 \Rightarrow \langle x, y \rangle = 0$

Teoremă $f \in Sim(E) \Rightarrow$ toate răd pol. caracteristic sunt reale.

Prop $f \in Sim(E)$, $U \subseteq E$ subsp. invariant al lui f

$\Rightarrow$ a) $U^\perp$ este subspatiu invariant și

b) $f/_{U^\perp} : U^\perp \to U^\perp$ endom. simetric.

Teoremă

$f \in Sim(E)$

$\Rightarrow \exists$ un reper R ortonormat format din versori proprii aî $[f]_{R,R}$ = diagonală

Dem Fie R_0 reper ortonormat arb. în E, $A = [f]_{R_0, R_0}$.

$P(\lambda) = \det(A - \lambda I_n)$ pol caract.

Toate răd. sunt reale.

Fie λ_1 o răd. și e_1 = versor propriu $\Rightarrow \underline{f(e_1) = \lambda_1 e_1}$

$\Rightarrow \langle \{e_1\} \rangle$ subsp. invar $\Rightarrow \langle \{e_1\} \rangle^\perp$ subsp. invariant

$f/_{\langle \{e_1\} \rangle^\perp} : \langle \{e_1\} \rangle^\perp \to \langle \{e_1\} \rangle^\perp$ endom. simetric.

Fie λ_2 valoare proprie pt $f/_{\langle \{e_1\} \rangle^\perp}$; și e_2 = versor propriu $\Rightarrow \underline{f(e_2) = \lambda_2 e_2}$ $\quad e_1 \perp e_2$

$\langle \{e_1, e_2\} \rangle$ subsp. invar pt f

$\Rightarrow \langle \{e_1, e_2\} \rangle^\perp$ — // — $\quad f/_{\langle \{e_1, e_2\} \rangle^\perp}$ endom sim.

Repetăm raționamentul și construim versorii $\{e_1, \dots, e_n\}$

mutual ortogonali $\Rightarrow$ $\mathcal{R}$ reper ortonormat

$$[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$\boxed{\text{OBS}}$ a) $f \in \mathrm{Sim}(E) \Rightarrow \dim V_{\lambda_i} = m_i$, $i = \overline{1,k}$

unde $\lambda_1, .., \lambda_k$ răd. dist cu multiplicitatile $m_1, .., m_k$ cu $m_1 + .. + m_k = n$

$E = V_{\lambda_1} \oplus ... \oplus V_{\lambda_k}$

$\mathcal{R} = \mathcal{R}_1 \cup ... \cup \mathcal{R}_k$, $\mathcal{R}_i$ reper ortonormat in V_{λ_i}, $i = \overline{1,k}$

b) $A = A^T$ $\rightarrow$ $f \in \mathrm{Sim}(E)$, $[f]_{R,R} = A$

$\searrow$ $Q : E \to \mathbb{R}$ formă pătratică

$Q(x) = X^T A X$

$\langle x, f(x) \rangle = Q(x)$, $\forall x \in E$.

Aducem Q la formă canonică prin metoda valorilor proprii (efectuând transformări ortogonale)

$$\mathcal{R}_0 \xrightarrow{C} \mathcal{R} \quad \& \quad A = [f]_{R,R}.$$

$\mathcal{R}_0 = \{e_1^0, .., e_n^0\}$, $\mathcal{R} = \{e_1, .., e_n\}$

$h \in O(E)$

$h(e_i^0) = e_i$, $\forall i = \overline{1,n}$

<u>Teoremă</u>

$\forall f \in \mathrm{Aut}(E)$, $\exists h \in \mathrm{Sim}(E)$, $t \in O(E)$ a.î. $f = h \circ t$.

Ex $(\mathbb{R}^3, g_0)$, $f \in End(\mathbb{R}^3)$

$$[f]_{R_0,R_0} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = A$$

a) $f \in Sim(\mathbb{R}^3)$

b) $Q : \mathbb{R}^3 \to \mathbb{R}$ forma pătratică asociată $(Q(x) = \langle f(x), x \rangle)$

Să se aducă Q la o formă canonică printr-o transformare ortogonală.

Sol

a) $A = A^T \Rightarrow f \in Sim(\mathbb{R}^3)$

$f(x) = (x_1 + x_2 - x_3,\ x_1 + x_2 - x_3,\ -x_1 - x_2 + x_3)$

b) $Q : \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 2x_2x_3$

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^3 - \sigma_1 \lambda^2 + \sigma_2 \lambda - \sigma_3 = 0 \Rightarrow \lambda^3 - 3\lambda^2 = 0 \Rightarrow \lambda^2(\lambda - 3) = 0$

$\sigma_1 = Tr A = 3$, $\sigma_2 = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

$\sigma_3 = \det A = 0$

1) $\lambda_1 = 0$, $m_1 = 2$

2) $\lambda_2 = 3$, $m_2 = 1$.

$V_{\lambda_1} = \{X \in \mathbb{R}^3 / AX = 0\} = \{(x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R}\}$.

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad x_1 + x_2 - x_3 = 0 \qquad x_3 = x_1 + x_2$$

$V_{\lambda_1} = \langle \{ \underset{f_1}{(1,0,1)}, \underset{f_2}{(0,1,1)} \} \rangle$ $\qquad \dim V_{\lambda_1} = 3 - rg A = 3 - 1 = 2$

Aplicăm procedeul Gram-Schmidt

$e_1 = f_1$; $e_2 = f_2 - \dfrac{g_0(f_2, e_1)}{g_0(e_1, e_1)} e_1 =$

$$e_2 = (0,1,1) - \frac{1}{2}(1,0,1) = \left(-\frac{1}{2}, 1, \frac{1}{2}\right) =$$
$$= \frac{1}{2}(-1,2,1)$$
$$e_1' = \frac{1}{\sqrt{2}}(1,0,1), \quad e_2' = \frac{1}{\sqrt{6}}(-1,2,1)$$
$R_1 = \{e_1', e_2'\}$ reper ortonormat în V_{λ_1}

$$V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid AX = 3X\}$$
$$(A - 3I_3)X = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}\boxed{\begin{matrix}-2 & 1\\ 1 & -2\end{matrix}} & \begin{matrix}-1\\-1\end{matrix}\\ \begin{matrix}-1 & -1\end{matrix} & -2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$\dim V_{\lambda_2} = 1$

$$\begin{cases} -2x_1 + x_2 = x_3 & \cdot 2 \\ x_1 - 2x_2 = x_3 & \oplus \end{cases}$$
$$-3x_1 \quad / \quad = 3x_3$$

$x_1 = -x_3$

$x_2 = x_3 - 2x_3 = -x_3$

$$V_{\lambda_2} = \{(-x_3, -x_3, x_3) = x_3(-1,-1,1) \mid x_3 \in \mathbb{R}\}.$$

$R_2 = \{e_3' = \frac{1}{\sqrt{3}}(-1,-1,1)\}$ reper ortonormat în V_{λ_2}.

$$\mathbb{R}^3 = V_{\lambda_1} \oplus V_{\lambda_2}$$

$R = R_1 \cup R_2$ reper ortonormat aî $[f]_{R,R} = \begin{pmatrix}0 & & \\ & 0 & \\ & & 3\end{pmatrix}$

$Q(x) = 3(x_3')^2$ (1,0) signatura

$$R_0 = \{e_1^0, e_2^0, e_3^0\} \xrightarrow{C} R = \Big\{\underbrace{\frac{1}{\sqrt{2}}(1,0,1)}_{e_1'}, \underbrace{\frac{1}{\sqrt{6}}(-1,2,1)}_{e_2'}, \underbrace{\frac{1}{\sqrt{3}}(-1,-1,1)}_{e_3'}$$

$h \in O(\mathbb{R}^3)$

$h(e_i^0) = e_i$, $i = \overline{1,3}$

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$h: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$h(x) = \left(\frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{6}}x_2 - \frac{1}{\sqrt{3}}x_3, \ \frac{2}{\sqrt{6}}x_2 - \frac{1}{\sqrt{3}}x_3, \ \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{6}}x_2 + \frac{1}{\sqrt{3}}x_3\right)$$

$(\mathbb{R}^3, g_0)$ $\quad g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R},\ g_0(x,y) = x_1y_1 + x_2y_2 + x_3y_3.$

Ex Fie $f : \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x_1,x_2,x_3) = \frac{1}{3}\left(2x_1 + x_2 - 2x_3,\ -2x_1 + 2x_2 - x_3,\ x_1 + 2x_2 + 2x_3\right)$$

a) $f \in O(E)$

b) Să se determine $\exists\ R = \{e_1, e_2, e_3\}$ reper ortonormat în $\mathbb{R}^3$ ai $[f]_{R,R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$.

Sol

$f \in \mathrm{End}(\mathbb{R}^3)$

$f \in O(E) \iff A = [f]_{R_0, R_0} \in O(3)$

1) $A \cdot A^T = I_3$; 2) $\det A = \pm 1$

$$A = \frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}$$

$$A \cdot A^T = \frac{1}{9}\begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ -2 & -1 & 2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = I_3$$

$\Rightarrow A \in O(3)$

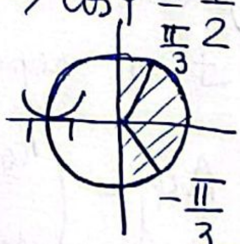
$\det(\alpha A) = \alpha^n \det A,\ A \in \mathcal{M}_n(\mathbb{R})$

$$\det A = \frac{1}{27}\begin{vmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{vmatrix} = 1 \Rightarrow f \in O(E)$$

de speță 1.

b) $\mathrm{Tr}\,A = \frac{6}{3} = 2 = 1 + 2\cos\varphi \Rightarrow 2\cos\varphi = 1 \Rightarrow \cos\varphi = \frac{1}{2}$

$\varphi \in \left\{\frac{\pi}{3}, -\frac{\pi}{3}\right\}$.



Det. axa: $f(x) = x$

$$\begin{cases} \frac{1}{3}(2x_1 + x_2 - 2x_3) = x_1 \\ \frac{1}{3}(-2x_1 + 2x_2 - x_3) = x_2 \\ \frac{1}{3}(x_1 + 2x_2 + 2x_3) = x_3 \end{cases} \Rightarrow \begin{cases} -x_1 + x_2 - 2x_3 = 0 \\ -2x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 - x_3 = 0 \end{cases} \quad \det\begin{pmatrix} -1 & 1 & -2 \\ -2 & -1 & -1 \\ 1 & 2 & -1 \end{pmatrix} = 0$$

$$\begin{cases} -x_1 + x_2 = 2x_3 \\ -2x_1 - x_2 = x_3 \end{cases} \oplus$$

$$-3x_1 \quad / \quad = 3x_3$$

$x_1 = -x_3$

$x_2 = 2x_3 - x_3 = x_3$

$$\{ (x_1, x_2, x_3) = \underset{x_3(-1,1,1)}{\underset{\parallel}{(-x_3, x_3, x_3)}}, x_3 \in \mathbb{R} \}$$

$$\|x\| = \sqrt{g(x,x)}$$

$$g = g_0 \ , \ \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$e_1 = \frac{1}{\sqrt{3}}(-1,1,1)$ versorul axei

$$\langle\{e_1\}\rangle^{\perp} = \{ x \in \mathbb{R}^3 \mid \underset{-x_1+x_2+x_3}{\underset{\parallel}{g_0((x_1,x_2,x_3),(-1,1,1))}} = 0 \}$$

$$= \{ (x_2+x_3, x_2, x_3) = x_2 \underbrace{(1,1,0)}_{f_2} + x_3 \underbrace{(1,0,1)}_{f_3}, \ x_2, x_3 \in \mathbb{R} \}$$

$$\mathbb{R}^3 = \langle\{e_1\}\rangle \oplus \langle\{e_1\}\rangle^{\perp}$$

$\{f_2, f_3\}$ reper arbitrar în $\langle\{e_1\}\rangle^{\perp}$

Aplicăm procedeul Gram - Schmidt

$\{f_2, f_3\} \longrightarrow \{e_2', e_3'\} \longrightarrow \{e_2, e_3\}$

reper $\forall$ — reper ortogonal — reper ortonormat

$$e_2' = f_2 = (1,1,0)$$

$$e_3' = f_3 - \frac{g_0(f_3, e_2')}{g_0(e_2', e_2')} e_2' = (1,0,1) - \frac{1}{2}(1,1,0)$$

$$= \left(\frac{1}{2}, -\frac{1}{2}, 1\right) = \frac{1}{2}(1,-1,2)$$

OBS $u = \alpha u', \ \alpha > 0$

$$\frac{u}{\|u\|} = \frac{\alpha u'}{\|\alpha u'\|} = \frac{\cancel{\alpha} u'}{\cancel{\alpha}\|u'\|}$$

$$\begin{cases} e_2 = \frac{e_2'}{\|e_2'\|} = \frac{1}{\sqrt{2}}(1,1,0) \\ e_3 = \frac{1}{\sqrt{6}}(1,-1,2) \end{cases}$$