CURS 13 18.05.2021

# Geometrie analitică euclidiană

## Drepte și plane în spațiu

## Conice ca locuri geometrice

### Perpendiculara comună a 2 drepte necoplanare

**Aplicație**

Fie dreptele $\mathcal{D}_1 : x_1 = x_3 = 0$

$$\mathcal{D}_2 : \begin{cases} x_1 - 1 = 0 \\ x_2 = x_3 \end{cases}$$

a) Să se determine ec. perpendicularei comune

b) $dist(\mathcal{D}_1, \mathcal{D}_2)$

SOL

a) $\mathcal{D}_1, \mathcal{D}_2$ necoplanare.

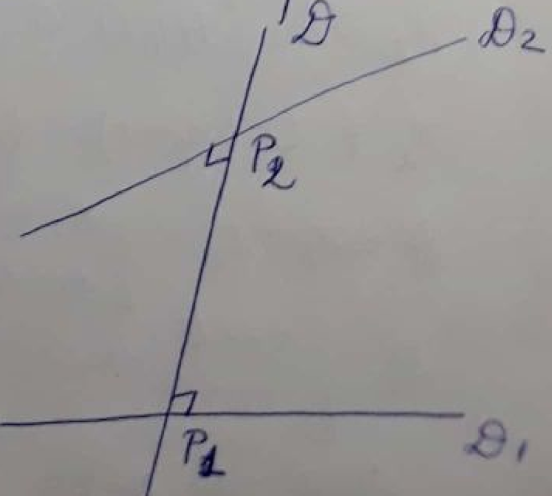
$$\mathcal{D}_1 : \begin{cases} x_1 = 0 \\ x_2 = t \\ x_3 = 0, \ t \in \mathbb{R} \end{cases} \Leftrightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{0} = t \qquad u_1 = (0,1,0), \quad A_1(0,1,0) \in \mathcal{D}_1$$

$$\mathcal{D}_2 : \begin{cases} x_1 = 1 \\ x_2 = s \\ x_3 = s, \ s \in \mathbb{R} \end{cases} \Leftrightarrow \left\{ \frac{x_1 - 1}{0} = \frac{x_2}{1} = \frac{x_3}{1} = s, \right. \qquad u_2 = (0,1,1), \quad A_2(1,1,1) \in \mathcal{D}_2$$

$$\overrightarrow{A_1A_2} = (1,0,1)$$

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \ (\text{necoplanare})$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ u_1 & u_2 & \overrightarrow{A_1A_2} \end{array}$$




$\mathcal{D} \perp \mathcal{D}_k, \ k = \overline{1,2}$

(M1) $P_1(0,t,0) \in \mathcal{D} \cap \mathcal{D}_1$

$P_2(1,s,s) \in \mathcal{D} \cap \mathcal{D}_2$

Scanned with CamScanner

$\overrightarrow{P_1P_2} = (1, s-t, s)$ , $u_1 = (0,1,0)$ , $u_2 = (0,1,1)$

$\begin{cases} \langle \overrightarrow{P_1P_2}, u_1 \rangle = 0 \\ \langle \overrightarrow{P_1P_2}, u_2 \rangle = 0 \end{cases} \Rightarrow \begin{cases} s-t=0 \\ s-t+s=0 \end{cases} \Rightarrow \begin{cases} s-t=0 \\ 2s-t=0 \end{cases} \Rightarrow s=t=0$

$P_1(0,0,0)$ , $\overrightarrow{P_1P_2} = (1,0,0)$ $\quad dist(P_1,P_2) = \sqrt{(1-0)^2+(0-0)^2+(0-0)^2}$

$P_2(1,0,0)$ $\quad \overset{\shortparallel}{\|\overrightarrow{P_1P_2}\|} . = \sqrt{1^2+0^2+0^2}$

$\mathcal{D} = P_1P_2 : \dfrac{x_1}{1} = \dfrac{x_2}{0} = \dfrac{x_3}{0}$

b) $dist(\mathcal{D}_1, \mathcal{D}_2) = dist(P_1,P_2) = 1$.

$M_2$ $\quad N = u_1 \times u_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = (1,0,0)$

$\pi_k$ = plan det. de $A_k$ și vect. directori $u_k, N$.

$N_1 = N \times u_1 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0,0,1)$

$N_2 = N \times u_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = (0,-1,1)$

$\pi_1 \ni A_1(0,1,0)$ și are normala $N_1 = (0,0,1)$

$\pi_1 : 0(x_1-0) + 0(x_2-1) + 1(x_3-0) = 0 \Rightarrow x_3 = 0$

$\pi_2 \ni A_2(1,1,1)$ și are normala $N_2 = (0,-1,1)$

$\pi_2 : 0(x_1-1) - (x_2-1) + (x_3-1) = 0 \Rightarrow -x_2 + x_3 = 0$

$\mathcal{D} = \pi_1 \cap \pi_2 : \begin{cases} x_3 = 0 \\ x_2 - x_3 = 0 \end{cases}$

$\mathcal{D} : \begin{cases} x_1 = t \\ x_2 = 0 \\ x_3 = 0 \end{cases}$ $\qquad \mathcal{D} : \dfrac{x_1}{1} = \dfrac{x_2}{0} = \dfrac{x_3}{0}$

$Dist(\mathcal{D}_1, \mathcal{D}_2) = \dfrac{|\langle N, \overrightarrow{A_1A_2} \rangle|}{\|N\|} = 1$ , $N = (1,0,0)$, $\overrightarrow{A_1A_2} = (1,0,1)$

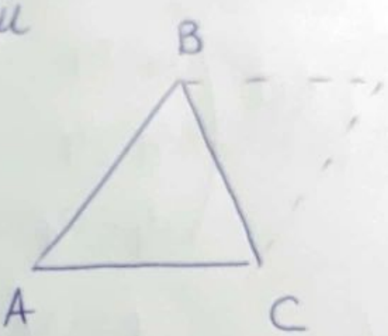
OBS $\{u, v\}$ SLI

$$\|u \times v\|^2 = \begin{vmatrix} \langle u,u\rangle & \langle u,v\rangle \\ \langle v,u\rangle & \langle v,v\rangle \end{vmatrix} = \|u\|^2\|v\|^2 - \|u\|^2\|v\|^2\cos^2\varphi$$

$$= \|u\|^2\|v\|^2\sin^2\varphi \quad , \varphi \in [0,\pi].$$

$$\|u \times v\| = \|u\| \cdot \|v\| \sin\varphi$$

- Aria unui triunghi în spațiu




$$A_{\Delta ABC} = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$$

$$= \frac{1}{2}\left\| \begin{vmatrix} e_1 & e_2 & e_3 \\ b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \end{vmatrix} \right\|$$

$A(a_1, a_2, a_3)$
$B(b_1, b_2, b_3)$
$C(c_1, c_2, c_3)$

$$= \frac{1}{2}\|(\Delta_1, \Delta_2, \Delta_3)\| = \frac{1}{2}\sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}$$

Exemplu

$A(1,2,0)$, $B(0,1,3)$, $C(1,0,1) \Rightarrow A_{\Delta ABC}$

$\overrightarrow{AB} = (-1,-1,3)$

$\overrightarrow{AC} = (0,-2,1)$

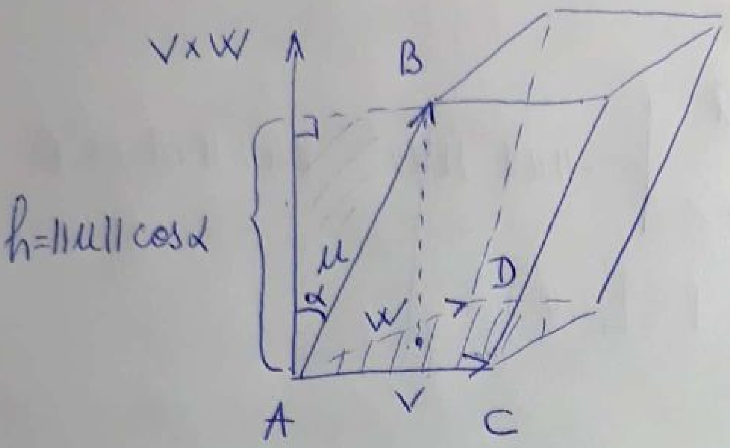
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -1 & -1 & 3 \\ 0 & -2 & 1 \end{vmatrix} = (5, 1, 2)$$

$$A_{\Delta ABC} = \frac{1}{2}\sqrt{25+1+4} = \frac{\sqrt{30}}{2}$$

- Volumul unui tetraedru.

$A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, $C(c_1, c_2, c_3)$, $D(d_1, d_2, d_3)$

$\{\overrightarrow{AB} = u, \overrightarrow{AC} = v, \overrightarrow{AD} = w\}$ SLI în $\mathbb{R}^3$




$$V_{paralelipiped} = A_b \cdot h =$$
$$= \|v\times w\| \cdot \|u\| \cdot |\cos\alpha|$$
$$= |\langle u, v\times w\rangle| =$$
$$= |u \wedge v \wedge w| =$$

$$= \left|\det\begin{pmatrix} b_1-a_1 & b_2-a_2 & b_3-a_3 \\ c_1-a_1 & c_2-a_2 & c_3-a_3 \\ d_1-a_1 & d_2-a_2 & d_3-a_3 \end{pmatrix}\right| = \left|\det\begin{pmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{pmatrix}\right| = |\Delta|$$

$$V_{ABC\Delta} = \frac{1}{6} V_{paralelipiped} = \frac{1}{6}|\Delta|$$

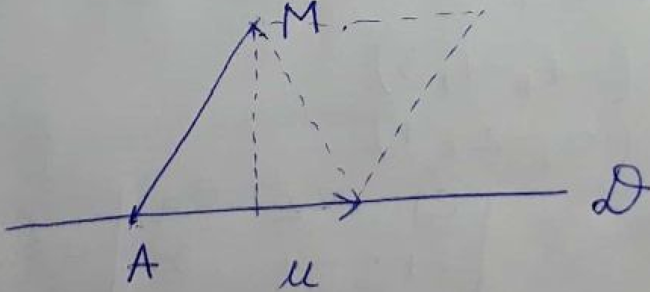
OBS

$A, B, C, \Delta$ puncte coplanare $\Leftrightarrow \Delta = 0$

• $Dist(M, \mathcal{D})$ $\quad M(b_1, b_2, b_3)$

$\mathcal{D}: r = r_0 + tu$, $\overrightarrow{OA} = r_0$

$A(a_1, a_2, a_3)$



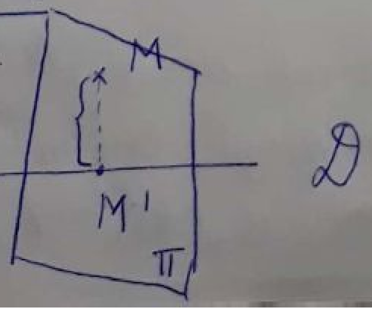

M1

$$A_{paralelogram} = \|u \times \overrightarrow{AM}\| = dist(M, \mathcal{D})\,\|u\|$$

$$dist(M, \mathcal{D}) = \frac{\|u \times \overrightarrow{AM}\|}{\|u\|}$$

M2 $\quad \pi \perp \mathcal{D}$, $M \in \pi$

$N_\pi = u_{\mathcal{D}}$



$\mathcal{D}: \dfrac{x_1-a_1}{u_1} = \dfrac{x_2-a_2}{u_2} = \dfrac{x_3-a_3}{u_3} = t \Rightarrow \begin{cases} x_1 = a_1 + tu_1 \\ x_2 = a_2 + tu_2 \\ x_3 = a_3 + tu_3 \end{cases}$

$\pi: u_1(x_1-b_1) + u_2(x_2-b_2) + u_3(x_3-b_3) = 0.$

$\{M'\} = \mathcal{D} \cap \pi \qquad \Rightarrow t \Rightarrow M' \Longrightarrow dist(M,\mathcal{D}) = dist(M,M')$

Exemplu

$\mathcal{D}: \dfrac{x_1-1}{2} = \dfrac{x_2}{1} = \dfrac{x_3+3}{-1} = t \Rightarrow \begin{cases} x_1 = 1+2t \\ x_2 = t \\ x_3 = -3-t,\ t \in \mathbb{R} \end{cases}$

$M(1,2,-1)$

$dist(M,\mathcal{D}) = ?$

$u = (2,1,-1)$

SOL

(M1) $A(1,0,-3)$, $\overrightarrow{AM} = (0,2,2)$

$dist(M,\mathcal{D}) = \dfrac{\|u \times \overrightarrow{AM}\|}{\|u\|} = \dfrac{\sqrt{3\cdot 4^2}}{\sqrt{6}} = \dfrac{4\sqrt{3}}{\sqrt{6}} = \dfrac{4}{\sqrt{2}} = 2\sqrt{2}$

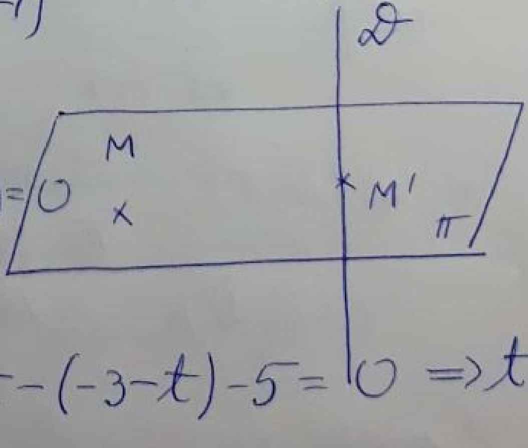
$u \times \overrightarrow{AM} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & 1 & -1 \\ 0 & 2 & 2 \end{vmatrix} = e_1 \underbrace{\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}}_{4} - e_2 \underbrace{\begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix}}_{4} + e_3 \underbrace{\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}}_{4}$

$= (4,-4,4)$

(M2) $\pi \perp \mathcal{D}$, $M \in \pi$, $M(1,2,-1)$

$N_\pi = u = (2,1,-1)$

$\pi: 2(x_1-1) + 1(x_2-2) - 1\cdot(x_3+1) = 0$




$\pi: 2x_1 + x_2 - x_3 - 5 = 0$

$\mathcal{D} \cap \pi = \{M'\}: 2(1+2t) + t - (-3-t) - 5 = 0 \Rightarrow t = 0$

$M'(1,0,-3)$

$dist(M,M') = \sqrt{(1-1)^2 + (2-0)^2 + (-1+3)^2} = \sqrt{0+4+4} = 2\sqrt{2}$

- $\sphericalangle(\mathcal{D}_1, \mathcal{D}_2)$, $\mathcal{D}_k$ = dreaptă orientată de vect $u_k$, $k=\overline{1,2}$

$$\sphericalangle(\mathcal{D}_1, \mathcal{D}_2) = \sphericalangle(u_1, u_2) \overset{not}{=} \hat{\varphi}$$

$$\cos\varphi = \frac{\langle u_1, u_2 \rangle}{\|u_1\| \cdot \|u_2\|}, \quad \varphi \in [0, \pi]$$

Exemplu

$$\mathcal{D}_1 : \frac{x_1 - 1}{1} = \frac{x_2 - 1}{-1} = \frac{x_3}{2} \qquad u_1 = (1, -1, 2)$$

$$\mathcal{D}_2 : \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1} \qquad u_2 = (-1, 1, 1)$$

$$\cos\varphi = \frac{-1-1+2}{\sqrt{6}\cdot\sqrt{3}} = 0 \Rightarrow \varphi = \frac{\pi}{2}.$$

- $\sphericalangle(\pi_1, \pi_2)$, $\pi_k$ = plan orientat de vect. normal $N_k$, $k=\overline{1,2}$

$$\sphericalangle(\pi_1, \pi_2) = \sphericalangle(N_1, N_2) = \hat{\varphi}$$

$$\cos\varphi = \frac{\langle N_1, N_2 \rangle}{\|N_1\| \cdot \|N_2\|}$$

Exemplu

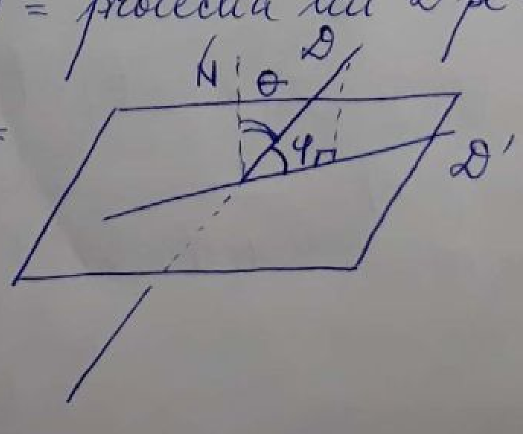
$$\pi_1 : x_1 + x_2 + x_3 - 1 = 0 \qquad N_1 = (1, 1, 1)$$

$$\pi_2 : 2x_1 - x_2 + 3x_3 + 2 = 0 \qquad N_2 = (2, -1, 3)$$

$$\cos\varphi = \frac{2-1+3}{\sqrt{3}\cdot\sqrt{14}} = \frac{4}{\sqrt{3}\cdot\sqrt{14}}.$$

- $\sphericalangle(\mathcal{D}, \pi)$, $\mathcal{D}$ orientată de $u$, $\pi$ orientat de $N$.

$$\sphericalangle(\mathcal{D}, \pi) = \sphericalangle(\mathcal{D}, \mathcal{D}') \qquad \mathcal{D}' = \text{proiecția lui } \mathcal{D} \text{ pe } \pi$$




$$\sin\varphi = \cos\left(\frac{\pi}{2} - \varphi\right) = \cos\theta = \frac{\langle N, u \rangle}{\|N\| \cdot \|u\|}.$$

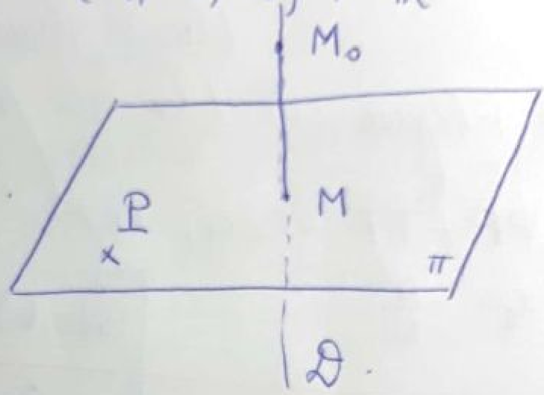
• Dist $(M_0, \pi)$ , $M_0(x_1^0, x_2^0, x_3^0)$

$\pi: A_1x_1 + A_2x_2 + A_3x_3 + A_0 = 0$

$N = (A_1, A_2, A_3) \neq 0_{\mathbb{R}^3}$

$M_0M \perp \pi$

$\overrightarrow{M_0M} = \alpha N$ , $\alpha \in \mathbb{R}$




$|\langle \overrightarrow{M_0M}, N \rangle| = \|\overrightarrow{M_0M}\| \cdot \|N\|$

$$\|\overrightarrow{M_0M}\| = \frac{|\langle \overrightarrow{M_0M}, N\rangle|}{\|N\|} =$$

$$= \frac{|A_1x_1^0 + A_2x_2^0 + A_3x_3^0 + A_0|}{\sqrt{A_1^2 + A_2^2 + A_3^2}}$$

$\pi: \langle \overrightarrow{PM}, N \rangle = 0$

$P(x_1, x_2, x_3) \longrightarrow M_0(x_1^0, x_2^0, x_3^0)$

Exemplu $\pi: 2x_1 - x_2 + x_3 - 1 = 0$, $M_0(1,0,1)$

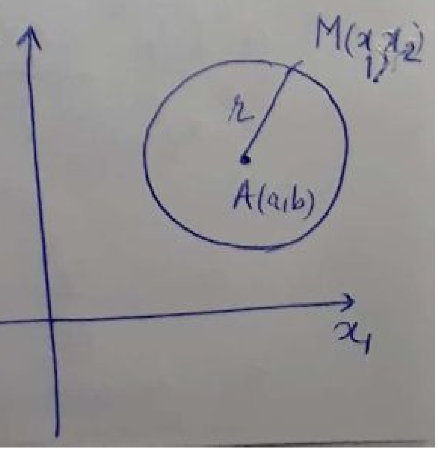
$N = (2, -1, 1)$

$$\text{dist}(M_0, \pi) = \frac{|2\cdot 1 - 0 + 1 - 1|}{\sqrt{4+1+1}} = \frac{2}{\sqrt{6}} = \frac{2\sqrt{6}}{6} = \frac{\sqrt{6}}{3}.$$

## Conice ca locuri geometrice

$(\mathbb{R}^2, (\mathbb{R}^2_{/\mathbb{R}}, g_0), \varphi)$

① Cercul $\mathcal{C}(A(a,b), r)$ = LG al punctelor egal depărtate de punctul fix A




$\mathcal{C}(A(a,b), r): AM = r$

$\sqrt{(x_1-a)^2 + (x_2-b)^2} = r$

$(x_1-a)^2 + (x_2-b)^2 = r^2$

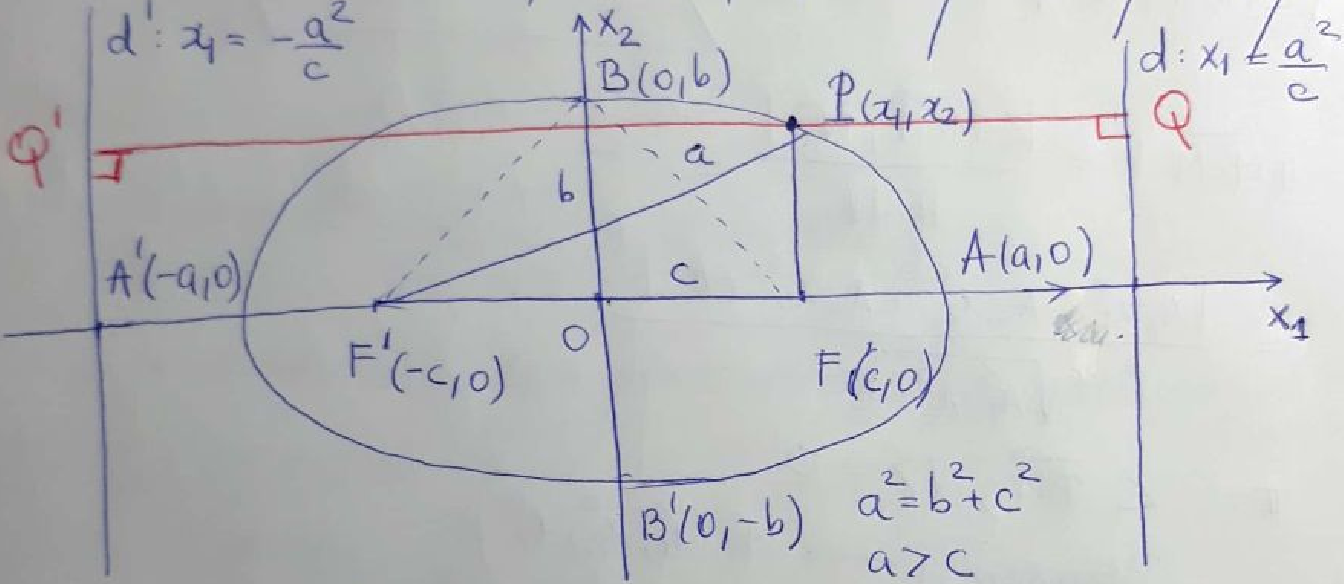
$f(x_1, x_2) = x_1^2 + x_2^2 - 2ax_1 - 2bx_2 + a^2 + b^2 - r^2 = 0$

Ec. param: $\begin{cases} x_1 - a = r\cos t \\ x_2 - b = r\sin t \end{cases}$, $t \in [0, 2\pi)$

(locul geometric)

② Elipsa este LG al punctelor $P \in \mathcal{E}_2$ care verifică

$PF + PF' = 2a$, $a > 0$, $F, F'$ = puncte fixe (focare)




$a^2 = b^2 + c^2$

$a > c$

$PF + PF' = 2a$

$\sqrt{(x_1-c)^2 + x_2^2} + \sqrt{(x_1+c)^2 + x_2^2} = 2a \Rightarrow$

$\sqrt{(x_1+c)^2 + x_2^2} = 2a - \sqrt{(x_1-c)^2 + x_2^2} \geq 0 \quad \uparrow^2$

$x_1^2 + 2x_1c + c^2 + x_2^2 = 4a^2 - 4a\sqrt{(x_1-c)^2 + x_2^2} + x_1^2 - 2x_1c + c^2 + x_2^2$

$4a\sqrt{(x_1-c)^2 + x_2^2} = 4a^2 - 4x_1c \quad \uparrow^2$

$a^2(x_1^2 - 2x_1c + c^2 + x_2^2) = a^4 + x_1^2c^2 - 2a^2cx_1$

$x_1^2\underbrace{(a^2-c^2)}_{b^2} + x_2^2a^2 = a^4 - a^2c^2 = a^2\underbrace{(a^2-c^2)}_{b^2} \quad | : a^2b^2$

$\mathcal{E}: \dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} = 1$, $a > 0, b > 0$

Ec parametrice: $\begin{cases} x_1 = a\cos t \\ x_2 = b\sin t \end{cases}$, $t \in [0, 2\pi)$

$e = \frac{c}{a} \in (0,1)$ excentricitatea

$a > c$

$d \cup d' : x_1 = \pm \frac{a^2}{c}$ (directoare)

Def. echivalentă

LG al punctelor P care verifică $\frac{dist(P,F)}{dist(P,d)} = e$,

resp $\frac{dist(P,F')}{dist(P,d')} = e$

$\frac{PF}{PQ} = \frac{c}{a} \Rightarrow PF = \frac{c}{a} PQ$

$\frac{PF'}{PQ'} = \frac{c}{a} \Rightarrow PF' = \frac{c}{a} PQ'$ $\oplus$

$PF + PF' = \frac{c}{a}(PQ + PQ') = \frac{c}{a} \cdot 2 \cdot \frac{a^2}{c} = 2a$

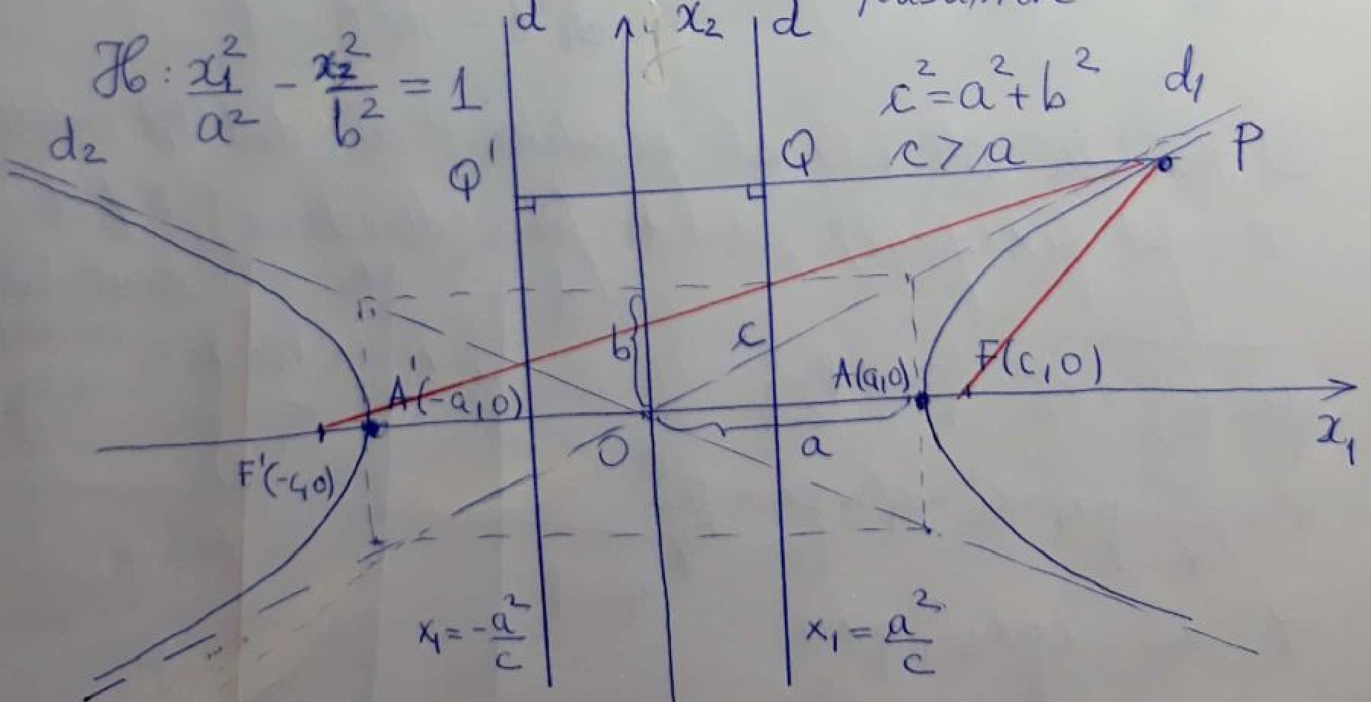
③ Hiperbola = LG al punctelor P care verifică

$|PF - PF'| = 2a$, $a > 0$, $F, F'$ puncte fixe distincte (focare)

$\mathcal{H} : \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1$

$c^2 = a^2 + b^2$

$c > a$




$d_1 \cup d_2 : x_2 = \pm \frac{b}{a} x_1$ (asimptotele)

A, A' = vârfuri

$e = \frac{c}{a} > 1$ (excentricitatea)

$d \cup d'$: $x_1 = \pm\frac{a^2}{c}$ directoarele

Def echivalentă

LG al punctelor P care verif. $\frac{dist(P,F)}{dist(P,d)} = e$

$\frac{dist(P,F')}{dist(P,d')} = e$

$\frac{PF}{PQ} = \frac{c}{a} \Rightarrow PF = \frac{c}{a} PQ$

$\frac{PF'}{PQ'} = \frac{c}{a} \Rightarrow PF' = \frac{c}{a} PQ'$

$|PF - PF'| = \frac{c}{a} \underbrace{|PQ - PQ'|}_{\frac{2a^2}{c}} = 2a$

④ Obs Ec parametrice:

$$\begin{cases} x_1 = a\,ch\,t \\ x_2 = a\,sh\,t, \ t \in \mathbb{R} \end{cases}$$

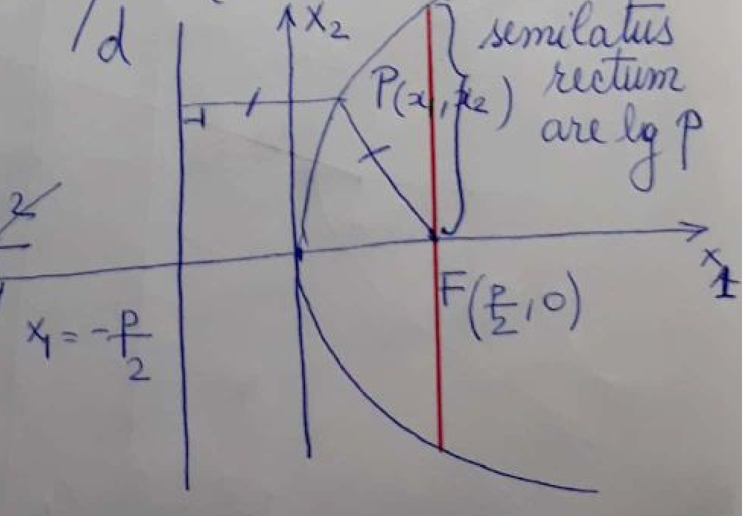
$ch\,t = \frac{e^t + e^{-t}}{2}$

$sh\,t = \frac{e^t - e^{-t}}{2}$

$ch^2 t - sh^2 t = 1$.

④ Parabola este LG al punctelor P a.î. $dist(P,F) = dist(P,d)$,

F = pct fix (focar), d = dr. fixă (directoare), $F \notin d$.




$\sqrt{(x_1 - \frac{p}{2})^2 + x_2^2} = |x_1 + \frac{p}{2}|$

$\cancel{x_1^2} + \cancel{\frac{p^2}{4}} - p x_1 + x_2^2 = \cancel{x_1^2} + p x_1 + \cancel{\frac{p^2}{4}}$

$\mathcal{P}$: $x_2^2 = 2p x_1$, $p > 0$

Ec. param: $\begin{cases} x_1 = \frac{t^2}{2p} \\ x_2 = t, \ t \in \mathbb{R} \end{cases}$

$e = 1$

Def unitară conice (nedegenerate)

LG al punctelor P care verif $\frac{dist(P,F)}{dist(P,d)} = e$

$F$ = pct fix, $d$ = dr fixă, $F \notin d$