In[289]:=

```
ClearAll["Global`*"]
SetDirectory[NotebookDirectory[]];
SetOptions[$FrontEndSession, NotebookAutoSave → True];
NotebookSave[];
AppendTo[$Path, FileNameJoin[{$HomeDirectory, "Dropbox", "EpidCRNmodels"}]];
Needs["EpidCRN`"];
Get["HopfE`"];
(*Gavish two-strain model*)(*Test cont with real bdAnalEx outputs*)
(*Your setup*)RN = {0 → "S", "S" + "I1" → 2 * "I1", "S" + "Y1" → "Y1" + "I1",
   "I1" → "R1", "S" + "I2" → 2 * "I2", "S" + "Y2" → "Y2" + "I2", "I2" → "R2",
   "R1" + "I2" → "I2" + "Y2", "R1" + "Y2" → 2 * "Y2", "Y2" → "R", "R2" + "I1" → "I1" + "Y1",
   "R2" + "Y1" → 2 * "Y1", "Y1" → "R", "R1" → "S", "R2" → "S", "R" → "S", "S" → 0,
   "I1" → 0, "Y1" → 0, "R1" → 0, "I2" → 0, "Y2" → 0, "R2" → 0, "R" → 0};
rts = {Λ, Subscript[β, 1] * I1 * S, Subscript[β, 1] * Subscript[η, 1] * Y1 * S,
   Subscript[γ, 1] * I1, Subscript[β, 2] * I2 * S, Subscript[β, 2] * Subscript[η, 2] * Y2 * S,
   Subscript[γ, 2] * I2, Subscript[β, 2] * Subscript[σ, 2] * I2 * R1,
   Subscript[β, 2] * Subscript[σ, 2] * Subscript[η, 2] * Y2 * R1,
   Subscript[γ, 2] * Y2, Subscript[β, 1] * Subscript[σ, 1] * I1 * R2,
   Subscript[β, 1] * Subscript[σ, 1] * Subscript[η, 1] * Y1 * R2,
   Subscript[γ, 1] * Y1, Subscript[θ, 1] * R1, Subscript[θ, 2] * R2,
   Subscript[θ, 3] * R, μ * S, μ * I1, μ * Y1, μ * R1, μ * I2, μ * Y2, μ * R2, μ * R};
(*Get bdAnalEx outputs*)


{RHS, var, par, cp, mSi, Jx, Jy, E0, ngm, R0, E1, E2, EA, R0A, R12, R21, coP} =
  bdAnalEx[RN, rts, {(*.0004,0.0004,0.16326,0.43591,1,1,*)0, 0, 0, 1, 2},
   {(*1,2,5,6,7,8,*)9, 10, 11, 12, 13}];
p0Val = par /. coP;
plotInd = {3, 4};
ploR[par_, coP_, R0A_, E0_, E1_, E2_, R12_, R21_, plotInd_ : {1, 2}, sca_ : 5 / 4] :=
  Module[{p0, coPsec, bifP, bifVars, eqs, eqL, p0Values, f1, f2,
     p1, p2, p1Values, p2Values, f0e, p0e, plot, conEx, R1s, R2s, R12s,
     R21s, cp, plotX, plotY, pInt, intResult}, bifP = par〚plotInd〛;
   p0 = coP〚plotInd〛;
   (*Remove β₁ and β₂ rules so ContourPlot can vary them*)
   coPsec = Delete[coP, List /@ plotInd];
   Print["bif param=", bifP, " coPsec = ", coPsec, " p0=", p0];
   (*Define reproduction number expressions*)R1s = (R0A〚1〛 /. E0) // Factor;
   R2s = (R0A〚2〛 /. E0) // Factor;
   R12s = (R12 /. E2) // Factor;
   R21s = (R21 /. E1) // Factor;
   eqL = {R1s - 1, R2s - 1, R12s - 1, R21s - 1};
   bifVars = Variables[eqL /. coPsec];
   cp = Thread[bifVars > 0];
   eqs = Thread[(eqL /. coPsec) == 0];
   (*Extract p0 coordinates*)p0Values = N[bifVars /. p0];
```

```
(*Find intersection of R₀₁=1 and R₀₂=1*)intResult =
 FindInstance[Join[cp, {(R1s /. coPsec) == 1, (R2s /. coPsec) == 1}], bifVars, Reals];
pInt = If[intResult =!= {},
  N[bifVars /. intResult[[1]]], {0.5 * p0Values[[1]], 0.5 * p0Values[[2]]}];
(*Find p1 (strain 1 dominates)*)
f1 = FindInstance[Join[cp, {(R21 /. E1 /. coPsec) < 1, 1 < (R0A[[1]] /. E0 /. coPsec),
    1 < (R0A[[2]] /. E0 /. coPsec)}], bifVars, Reals];
p1Values = If[f1 =!= {}, N[bifVars /. f1[[1]]], {0.3 * p0Values[[1]], 0.7 * p0Values[[2]]}];
(*Find p2 (strain 2 dominates)*)
f2 = FindInstance[Join[cp, {(R12 /. E2 /. coPsec) < 1, 1 < (R0A[[1]] /. E0 /. coPsec),
    1 < (R0A[[2]] /. E0 /. coPsec)}], bifVars, Reals];
p2Values = If[f2 =!= {}, N[bifVars /. f2[[1]]], {0.7 * p0Values[[1]], 0.3 * p0Values[[2]]}];
(*Find point in coexistence zone*)
f0e = FindInstance[Join[cp, {(R12 /. E2 /. coPsec) > 1.01, (R21 /. E1 /. coPsec) > 1.01,
    1 < (R0A[[1]] /. E0 /. coPsec), 1 < (R0A[[2]] /. E0 /. coPsec)}], bifVars, Reals];
p0e = If[f0e =!= {}, N[bifVars /. f0e[[1]]], p0Values + {0.1, 0.1}];
(*Create plot ranges including all points*)
plotX = sca * Max[p0Values[[1]], p1Values[[1]], p2Values[[1]], p0e[[1]], pInt[[1]]];
plotY = sca * Max[p0Values[[2]], p1Values[[2]], p2Values[[2]], p0e[[2]], pInt[[2]]];
Print["All coordinates: p0=", p0Values, " p1=",
 p1Values, " p2=", p2Values, " p0e=", p0e, " pInt=", pInt];
Print["Plot ranges (scale=", sca, "): plotX=", plotX, " plotY=", plotY];
(*Create plot*)plot = ContourPlot[Evaluate[eqs], Evaluate@{bifVars[[1]], 0, plotX},
  Evaluate@{bifVars[[2]], 0, plotY}, ContourStyle → {Blue, Red, Green, Orange},
  PlotLegends → {"R₀₁ = 1", "R₀₂ = 1", "R₁₂ = 1", "R₂₁ = 1"},
  AxesLabel → {ToString[bifP[[1]]], ToString[bifP[[2]]]}, PlotLabel →
   "Analytical Boundaries with Stability Points (scale=" <> ToString[sca] <> ")",
  ImageSize → {500, 400}, Epilog → {PointSize[0.015], Red, Point[p0Values],
    Text[Style["p0", FontSize → 12, FontWeight → Bold, FontColor → Red],
     p0Values + {0.02 * plotX, 0.02 * plotY}], Magenta, Point[p0e],
    Text[Style["p0e", FontSize → 12, FontWeight → Bold, FontColor → Magenta],
     p0e + {0.02 * plotX, -0.02 * plotY}], Blue, Point[p1Values],
    Text[Style["p1", FontSize → 12, FontWeight → Bold, FontColor → Blue],
     p1Values + {-0.02 * plotX, 0.02 * plotY}], Green, Point[p2Values],
    Text[Style["p2", FontSize → 12, FontWeight → Bold, FontColor → Green],
     p2Values + {0.02 * plotX, -0.02 * plotY}], Black, Point[pInt],
    Text[Style["pInt", FontSize → 12, FontWeight → Bold, FontColor → Black],
     pInt + {-0.02 * plotX, -0.02 * plotY}],
    Text[Style["E1 LAS", FontSize → 14, FontWeight → Bold, FontColor → Blue],
     {p1Values[[1]], Max[0.05 * plotY, p1Values[[2]] - 0.1 * plotY]}],
    Text[Style["E2 LAS", FontSize → 14, FontWeight → Bold, FontColor → Green],
     {p2Values[[1]], Max[0.05 * plotY, p2Values[[2]] - 0.1 * plotY]}],
    Text[Style["E1 GAS", FontSize → 14, FontWeight → Bold, FontColor → Blue],
     {0.8 * plotX, 0.05 * plotY}], Text[Style["E2 GAS", FontSize → 12,
      FontWeight → Bold, FontColor → Green], {0.05 * plotX, 0.8 * plotY}, {0, -1}],
    Text[Style["Coexist", FontSize → 14, FontWeight → Bold, FontColor → Red],
     {p0Values[[1]] + 0.05 * plotX, p0Values[[2]] + 0.05 * plotY}]}];
```

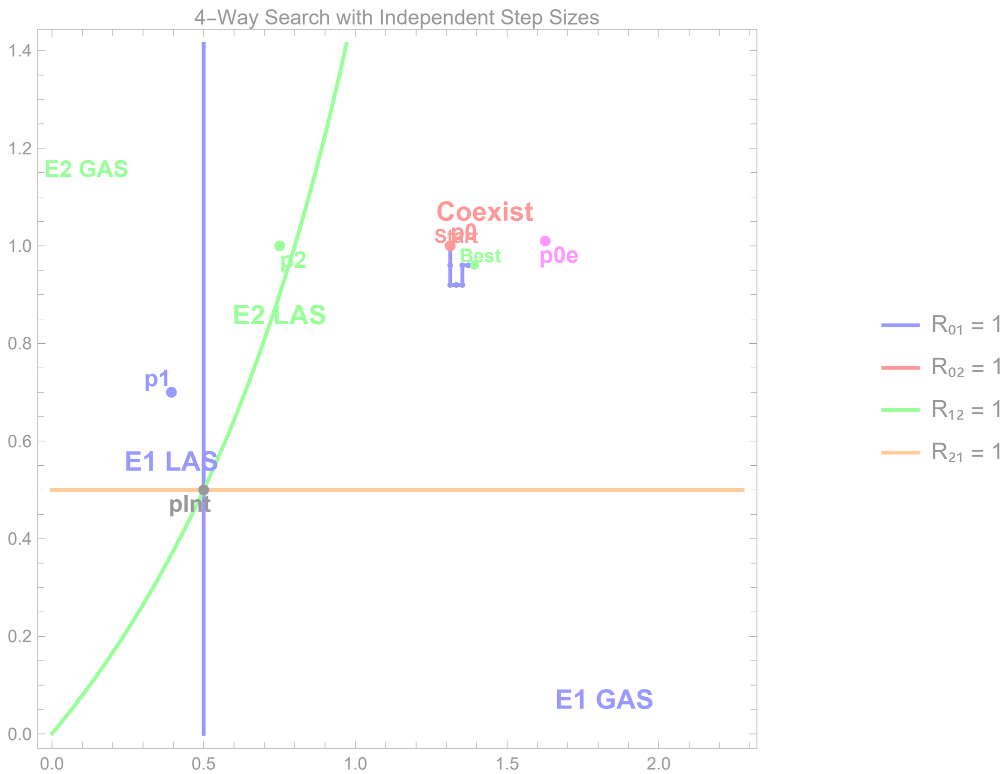
```
   conEx = Join[coPsec, If[f0e =!= {}, f0e[[1]], Thread[bifVars → p0e]]];
   {plot, conEx}];
sca = 1.4;
{plot, conEx} = ploR[par, coP, R0A, E0, E1, E2, R12, R21, plotInd, sca];
fpHopf[RHS, var, par, p0Val]
step = 0.02; step2 = 0.04; tol = 0.001;
{bestAngle, bestP0Val} = cont2[RHS, var, par, p0Val, step, step2, plotInd, tol, plot]
```

bif param={β_1, β_2} coPsec =

$\left\{\Lambda \to 4,\ \mu \to 1,\ \gamma_1 \to 1,\ \gamma_2 \to 1,\ \eta_1 \to \frac{1}{2},\ \eta_2 \to 1,\ \theta_1 \to 0,\ \theta_2 \to 0,\ \theta_3 \to 0,\ \sigma_1 \to 1,\ \sigma_2 \to 2\right\}$ p0=$\left\{\beta_1 \to \frac{21}{16},\ \beta_2 \to 1\right\}$

```
All coordinates: p0={1.3125, 1.} p1={0.39375, 0.7}
  p2={0.75, 1.} p0e={1.62566, 1.01} pInt={0.5, 0.5}

Plot ranges (scale=1.4): plotX=2.27592 plotY=1.414

Out[286]=
{{{1.396, 0.7659, 0.140241, 0.301999, 0.536099, 0.23195, 0.255617, 0.372192}},
 {-2.33468 + 1.01435 i, -2.33468 - 1.01435 i, -1.25916 + 0.803308 i,
  -1.25916 - 0.803308 i, -0.763968 + 0.701959 i, -0.763968 - 0.701959 i}, -66.5165,
 {-2.78308, -2.33468 + 1.01435 i, -2.33468 - 1.01435 i, -1.25916 + 0.803308 i,
  -1.25916 - 0.803308 i, -0.763968 + 0.701959 i, -0.763968 - 0.701959 i, -1.}}

Starting optimization at angle = -66.5165

Optimizing parameters at indices: {3, 4} with steps: {0.02, 0.04}

4-way test: P1+:0.0911268 P1-:-0.111166 P2+:-0.415003 P2-:0.237177

Best direction: Param 4 - step=0.04 improvement=0.237177

Continue: angle = -66.2793

Continue: angle = -66.234

4-way test: P1+:0.0155236 P1-:-0.0301631 P2+:-0.0453464 P2-:-0.170974

Best direction: Param 3 + step=0.02 improvement=0.0155236

Continue: angle = -66.2184

Continue: angle = -66.2163

4-way test: P1+:-0.0101355 P1-:-0.0021202 P2+:0.0237848 P2-:-0.234238

Best direction: Param 4 + step=0.04 improvement=0.0237848

Continue: angle = -66.1925

4-way test: P1+:0.0208106 P1-:-0.0354438 P2+:-0.160193 P2-:-0.0237848

Best direction: Param 3 + step=0.02 improvement=0.0208106

Continue: angle = -66.1717

Continue: angle = -66.1644

4-way test: P1+:-0.00507943 P1-:-0.00733163 P2+:-0.0930675 P2-:-0.0833804

All 4 directions failed (best=0), reducing step sizes to: {0.01, 0.02}

4-way test: P1+:-0.00105204 P1-:-0.00204956 P2+:-0.0256329 P2-:-0.0182827

All 4 directions failed (best=0), reducing step sizes to: {0.005, 0.01}
```

```
4-way test: P1+:-0.000146317 P1-:-0.000629012 P2+:-0.0074745 P2-:-0.00349231

All 4 directions failed (best=0), reducing step sizes to: {0.0025, 0.005}

4-way test: P1+:0.0000227558 P1-:-0.000216583 P2+:-0.00238512 P2-:-0.000355827

All 4 directions failed (best=0.0000227558), reducing step sizes to: {0.00125, 0.0025}

4-way test: P1+:0.000035481 P1-:-0.0000839376 P2+:-0.000852304 P2-:0.000167113

All 4 directions failed (best=0.000167113), reducing step sizes to: {0.000625, 0.00125}

4-way test: P1+:0.0000237819 P1-:-0.000035896 P2+:-0.0003408 P2-:0.000169505

All 4 directions failed (best=0.000169505), reducing step sizes to: {0.0003125, 0.000625}

4-way test: P1+:0.0000134033 P1-:-0.0000164318 P2+:-0.000149025 P2-:0.000106202

All 4 directions failed (best=0.000106202), reducing step sizes to: {0.00015625, 0.0003125}

4-way test: P1+:7.07995×10^-6 P1-:-7.83709×10^-6 P2+:-0.0000691642 P2-:0.0000584584

All 4 directions failed (best=0.0000584584), reducing step sizes to: {0.000078125, 0.00015625}

4-way test: P1+:3.63459×10^-6 P1-:-3.82387×10^-6 P2+:-0.0000332445 P2-:0.000030568

All 4 directions failed (best=0.000030568), reducing step sizes to: {0.0000390625, 0.000078125}

4-way test: P1+:1.84095×10^-6 P1-:-1.88827×10^-6 P2+:-0.0000162878 P2-:0.0000156186

All 4 directions failed (best=0.0000156186
 ), reducing step sizes to: {0.0000195313, 0.0000390625}

4-way test: P1+:9.2639×10^-7 P1-:-9.3822×10^-7 P2+:-8.06024×10^-6 P2-:7.89297×10^-6

All 4 directions failed (best=7.89297×10^-6
 ), reducing step sizes to: {9.76563×10^-6, 0.0000195313}

4-way test: P1+:4.64674×10^-7 P1-:-4.67631×10^-7 P2+:-4.00921×10^-6 P2-:3.96739×10^-6

All 4 directions failed (best=3.96739×10^-6
 ), reducing step sizes to: {4.88281×10^-6, 9.76563×10^-6}

Final result: angle = -66.1644

Best parameters: {1.3925, 0.96}

Final step sizes: {4.88281×10^-6, 9.76563×10^-6}

Iterations: 16

Creating overlay plot with search path...
```



Out[288]=

$$\left\{-66.1644, \left\{4, 1, 1.3925, 0.96, 1, 1, \frac{1}{2}, 1, 0, 0, 0, 1, 2\right\}\right\}$$

In[323]:=

```
{RHS, var, par, cp, mSi, Jx, Jy, E0, ngm, R0, E1, E2, EA, R0A, R12, R21, coP} =
  bdAnalEx[RN, rts, {(*.0004,0.0004,0.16326,0.43591,1,1,*)0, 0, 0, .5, 7},
   {(*1,2,5,6,7,8,*)9, 10, 11, 12, 13}];
p0Val = par /. coP;
plotInd = {3, 4};
ploR[par_, coP_, R0A_, E0_, E1_, E2_, R12_, R21_, plotInd_ : {1, 2}, sca_ : 5 / 4] :=
  Module[{p0, coPsec, bifP, bifVars, eqs, eqL, p0Values, f1, f2,
     p1, p2, p1Values, p2Values, f0e, p0e, plot, conEx, R1s, R2s, R12s,
     R21s, cp, plotX, plotY, pInt, intResult}, bifP = par⟦plotInd⟧;
    p0 = coP⟦plotInd⟧;
    (*Remove β₁ and β₂ rules so ContourPlot can vary them*)
    coPsec = Delete[coP, List /@ plotInd];
    Print["bif param=", bifP, " coPsec = ", coPsec, " p0=", p0];
    (*Define reproduction number expressions*)R1s = (R0A⟦1⟧ /. E0) // Factor;
    R2s = (R0A⟦2⟧ /. E0) // Factor;
    R12s = (R12 /. E2) // Factor;
    R21s = (R21 /. E1) // Factor;
    eqL = {R1s - 1, R2s - 1, R12s - 1, R21s - 1};
    bifVars = Variables[eqL /. coPsec];
    cp = Thread[bifVars > 0];
    eqs = Thread[(eqL /. coPsec) == 0];
```

```
(*Extract p0 coordinates*)p0Values = N[bifVars /. p0];
(*Find intersection of R₀₁=1 and R₀₂=1*)intResult =
 FindInstance[Join[cp, {(R1s /. coPsec) == 1, (R2s /. coPsec) == 1}], bifVars, Reals];
pInt = If[intResult =!= {},
  N[bifVars /. intResult[[1]]], {0.5 * p0Values[[1]], 0.5 * p0Values[[2]]}];
(*Find p1 (strain 1 dominates)*)
f1 = FindInstance[Join[cp, {(R21 /. E1 /. coPsec) < 1, 1 < (R0A[[1]] /. E0 /. coPsec),
    1 < (R0A[[2]] /. E0 /. coPsec)}], bifVars, Reals];
p1Values = If[f1 =!= {}, N[bifVars /. f1[[1]]], {0.3 * p0Values[[1]], 0.7 * p0Values[[2]]}];
(*Find p2 (strain 2 dominates)*)
f2 = FindInstance[Join[cp, {(R12 /. E2 /. coPsec) < 1, 1 < (R0A[[1]] /. E0 /. coPsec),
    1 < (R0A[[2]] /. E0 /. coPsec)}], bifVars, Reals];
p2Values = If[f2 =!= {}, N[bifVars /. f2[[1]]], {0.7 * p0Values[[1]], 0.3 * p0Values[[2]]}];
(*Find point in coexistence zone*)
f0e = FindInstance[Join[cp, {(R12 /. E2 /. coPsec) > 1.01, (R21 /. E1 /. coPsec) > 1.01,
    1 < (R0A[[1]] /. E0 /. coPsec), 1 < (R0A[[2]] /. E0 /. coPsec)}], bifVars, Reals];
p0e = If[f0e =!= {}, N[bifVars /. f0e[[1]]], p0Values + {0.1, 0.1}];
(*Create plot ranges including all points*)
plotX = sca * Max[p0Values[[1]], p1Values[[1]], p2Values[[1]], p0e[[1]], pInt[[1]]];
plotY = sca * Max[p0Values[[2]], p1Values[[2]], p2Values[[2]], p0e[[2]], pInt[[2]]];
Print["All coordinates: p0=", p0Values, " p1=",
 p1Values, " p2=", p2Values, " p0e=", p0e, " pInt=", pInt];
Print["Plot ranges (scale=", sca, "): plotX=", plotX, " plotY=", plotY];
(*Create plot*)plot = ContourPlot[Evaluate[eqs], Evaluate@{bifVars[[1]], 0, plotX},
  Evaluate@{bifVars[[2]], 0, plotY}, ContourStyle → {Blue, Red, Green, Orange},
  PlotLegends → {"R₀₁ = 1", "R₀₂ = 1", "R₁₂ = 1", "R₂₁ = 1"},
  AxesLabel → {ToString[bifP[[1]]], ToString[bifP[[2]]]}, PlotLabel →
   "Analytical Boundaries with Stability Points (scale=" <> ToString[sca] <> ")",
  ImageSize → {500, 400}, Epilog → {PointSize[0.015], Red, Point[p0Values],
    Text[Style["p0", FontSize → 12, FontWeight → Bold, FontColor → Red],
     p0Values + {0.02 * plotX, 0.02 * plotY}], Magenta, Point[p0e],
    Text[Style["p0e", FontSize → 12, FontWeight → Bold, FontColor → Magenta],
     p0e + {0.02 * plotX, -0.02 * plotY}], Blue, Point[p1Values],
    Text[Style["p1", FontSize → 12, FontWeight → Bold, FontColor → Blue],
     p1Values + {-0.02 * plotX, 0.02 * plotY}], Green, Point[p2Values],
    Text[Style["p2", FontSize → 12, FontWeight → Bold, FontColor → Green],
     p2Values + {0.02 * plotX, -0.02 * plotY}], Black, Point[pInt],
    Text[Style["pInt", FontSize → 12, FontWeight → Bold, FontColor → Black],
     pInt + {-0.02 * plotX, -0.02 * plotY}],
    Text[Style["E1 LAS", FontSize → 14, FontWeight → Bold, FontColor → Blue],
     {p1Values[[1]], Max[0.05 * plotY, p1Values[[2]] - 0.1 * plotY]}],
    Text[Style["E2 LAS", FontSize → 14, FontWeight → Bold, FontColor → Green],
     {p2Values[[1]], Max[0.05 * plotY, p2Values[[2]] - 0.1 * plotY]}],
    Text[Style["E1 GAS", FontSize → 14, FontWeight → Bold, FontColor → Blue],
     {0.8 * plotX, 0.05 * plotY}], Text[Style["E2 GAS", FontSize → 12,
      FontWeight → Bold, FontColor → Green], {0.05 * plotX, 0.8 * plotY}, {0, -1}],
    Text[Style["Coexist", FontSize → 14, FontWeight → Bold, FontColor → Red],
```

```
        {p0Values⟦1⟧ + 0.05 * plotX, p0Values⟦2⟧ + 0.05 * plotY}}];
    conEx = Join[coPsec, If[f0e =!= {}, f0e⟦1⟧, Thread[bifVars → p0e]]];
    {plot, conEx}];
sca = 1.7;
{plot, conEx} = ploR[par, coP, R0A, E0, E1, E2, R12, R21, plotInd, sca];
fpHopf[RHS, var, par, p0Val]
step = 0.3; step2 = 0.6; tol = 0.001;
{bestAngle, bestP0Val} = cont2[RHS, var, par, p0Val, step, step2, plotInd, tol, plot]
```

RHS has var {S, I1, Y1, R1, I2, Y2, R2, R} par$\{\Lambda, \mu, \beta_1, \beta_2, \gamma_1, \gamma_2, \eta_1, \eta_2, \theta_1, \theta_2, \theta_3, \sigma_1, \sigma_2\}$

minimal siphons {{I1, Y1}, {I2, Y2}} Check siphon={True, True}

Infection species at positions: {2, 3, 5, 6}

DFE solution E0: $\left\{R\to 0, R1\to 0, R2\to 0, S\to \frac{\Lambda}{\mu}, I1\to 0, Y1\to 0, I2\to 0, Y2\to 0\right\}$

NGM K= $\begin{pmatrix} \frac{S\beta_1}{\mu+\gamma_1} & \frac{S\beta_1\eta_1}{\mu+\gamma_1} & 0 & 0 \\ \frac{R2\,\beta_1\sigma_1}{\mu+\gamma_1} & \frac{R2\,\beta_1\eta_1\sigma_1}{\mu+\gamma_1} & 0 & 0 \\ 0 & 0 & \frac{S\beta_2}{\mu+\gamma_2} & \frac{S\beta_2\eta_2}{\mu+\gamma_2} \\ 0 & 0 & \frac{R1\,\beta_2\sigma_2}{\mu+\gamma_2} & \frac{R1\,\beta_2\eta_2\sigma_2}{\mu+\gamma_2} \end{pmatrix} = \begin{pmatrix} \frac{S\beta_1}{\mu+\gamma_1} & \frac{S\beta_1\eta_1}{\mu+\gamma_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{S\beta_2}{\mu+\gamma_2} & \frac{S\beta_2\eta_2}{\mu+\gamma_2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Reproduction functions R0A: $\left\{\frac{\beta_1\,(S + R2\,\eta_1\,\sigma_1)}{\mu + \gamma_1}, \frac{\beta_2\,(S + R1\,\eta_2\,\sigma_2)}{\mu + \gamma_2}\right\}$

R0 at DFE: $\mathrm{Max}\left[\frac{S\beta_1}{\mu + \gamma_1}, \frac{S\beta_2}{\mu + \gamma_2}\right]$

Solve: Equations may not give solutions for all "solve" variables.

Solve: Equations may not give solutions for all "solve" variables.

Number of boundary systems= 2 ;first sys has 3 sols, E1 is

$$\left\{S\to \frac{\mu+\gamma_1}{\beta_1}, I1\to -\frac{(\mu^2 - \Lambda\beta_1 + \mu\gamma_1)\,(\mu+\theta_1)}{\mu\beta_1\,(\mu+\gamma_1+\theta_1)}, Y1\to 0, R1\to -\frac{\gamma_1\,(\mu^2 - \Lambda\beta_1 + \mu\gamma_1)}{\mu\beta_1\,(\mu+\gamma_1+\theta_1)}, R2\to 0, R\to 0\right\}$$

second sys has 3 sols, E2 is

$$\left\{S\to \frac{\mu+\gamma_2}{\beta_2}, R1\to 0, I2\to -\frac{(\mu^2 - \Lambda\beta_2 + \mu\gamma_2)\,(\mu+\theta_2)}{\mu\beta_2\,(\mu+\gamma_2+\theta_2)}, Y2\to 0, R2\to -\frac{\gamma_2\,(\mu^2 - \Lambda\beta_2 + \mu\gamma_2)}{\mu\beta_2\,(\mu+\gamma_2+\theta_2)}, R\to 0\right\}$$

Fixing parameters at positions: {9, 10, 11, 12, 13}
by csi$\{\theta_1\to 0, \theta_2\to 0, \theta_3\to 0, \sigma_1\to 0.5, \sigma_2\to 7\}$ leaves
$\{\Lambda, \mu, \beta_1, \beta_2, \gamma_1, \gamma_2, \eta_1, \eta_2\}$

under coP: $\{\Lambda\to 1., \mu\to 1., \beta_1\to 14., \beta_2\to 6., \gamma_1\to 1., \gamma_2\to 1., \eta_1\to 1., \eta_2\to 0.126984, \theta_1\to 0, \theta_2\to 0, \theta_3\to 0, \sigma_1\to 0.5, \sigma_2\to 7\}$ invasion nrs are{3.5, 1.57143} repr nrs are{7., 3.}

bif param=$\{\beta_1, \beta_2\}$ coPsec = $\{\Lambda\to 1., \mu\to 1., \gamma_1\to 1., \gamma_2\to 1., \eta_1\to 1., \eta_2\to 0.126984, \theta_1\to 0, \theta_2\to 0, \theta_3\to 0, \sigma_1\to 0.5, \sigma_2\to 7\}$ p0=$\{\beta_1\to 14., \beta_2\to 6.\}$

All coordinates: p0={14., 6.} p1={13., 3.25} p2={2.6, 4.} p0e={9.73455, 9.09} pInt={2., 2.}

Plot ranges (scale=1.7): plotX=23.8 plotY=15.453

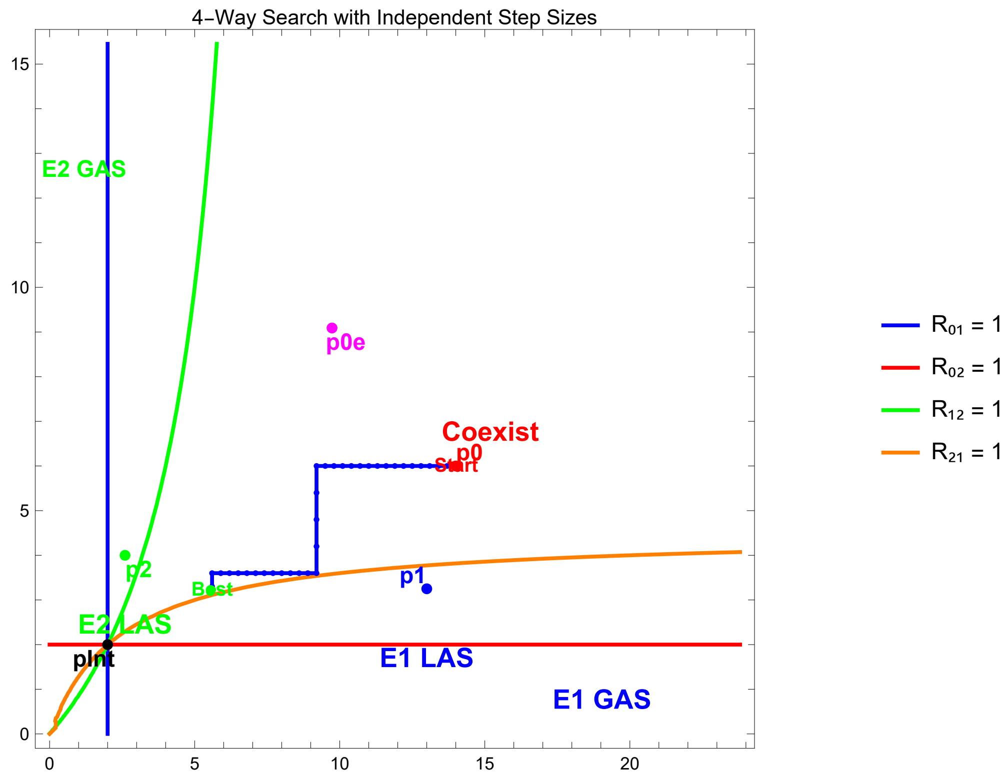
Out[329]=

```
{{{0.14165, 0.419606, 0.00357724, 0.215644, 0.00956967, 0.101981, 0.00241519, 0.105558}},
 {-4.16029 + 0.246507 i, -4.16029 - 0.246507 i, -1.99584 + 0.340388 i,
  -1.99584 - 0.340388 i, -0.929501 + 0.477691 i, -0.929501 - 0.477691 i}, -86.609,
 {-4.16029 + 0.246507 i, -4.16029 - 0.246507 i, -2.79653, -1.99584 + 0.340388 i,
  -1.99584 - 0.340388 i, -0.929501 + 0.477691 i, -0.929501 - 0.477691 i, -1.}}
```

```
Starting optimization at angle = -86.609

Optimizing parameters at indices: {3, 4} with steps: {0.3, 0.6}

4-way test: P1+:-1.492 P1-:1.09819 P2+:-0.0370069 P2-:-0.137848

Best direction: Param 3 - step=0.3 improvement=1.09819

Continue: angle = -85.5109

Continue: angle = -84.5074

Continue: angle = -83.4029

Continue: angle = -81.7427

Continue: angle = -79.3434

Continue: angle = -77.1024

Continue: angle = -75.1894

Continue: angle = -73.5271

Continue: angle = -72.0584

Continue: angle = -70.7507

Continue: angle = -69.5875

Continue: angle = -68.5642

Continue: angle = -67.6864

Continue: angle = -66.9714

Continue: angle = -66.4515

Continue: angle = -66.1818

4-way test: P1+:-0.269663 P1-:-0.0719632 P2+:-3.19854 P2-:2.75508

Best direction: Param 4 - step=0.6 improvement=2.75508

Continue: angle = -63.4268

Continue: angle = -61.4645

Continue: angle = -60.1081

Continue: angle = -59.0904

4-way test: P1+:-1.1029 P1-:1.06854 P2+:-1.01771 P2-:-30.9096

Best direction: Param 3 - step=0.3 improvement=1.06854

Continue: angle = -58.0219

Continue: angle = -56.9862

Continue: angle = -55.9829
```

```
Continue: angle = -55.012
Continue: angle = -54.0748
Continue: angle = -53.1734
Continue: angle = -52.3116
Continue: angle = -51.4947
Continue: angle = -50.7304
Continue: angle = -50.0292
Continue: angle = -49.4053
Continue: angle = -48.8758
4-way test: P1+:-0.52946 P1-:-29.2653 P2+:-23.3783 P2-:-41.1242
All 4 directions failed (best=0), reducing step sizes to: {0.15, 0.3}
4-way test: P1+:-0.25179 P1-:-29.8704 P2+:-26.2809 P2-:0.870868
Best direction: Param 4 - step=0.3 improvement=0.870868
Continue: angle = -48.005
4-way test: P1+:-0.299969 P1-:-35.2113 P2+:-0.870868 P2-:-41.995
All 4 directions failed (best=0), reducing step sizes to: {0.075, 0.15}
4-way test: P1+:-0.146995 P1-:-35.4197 P2+:-0.412035 P2-:-38.8666
All 4 directions failed (best=0), reducing step sizes to: {0.0375, 0.075}
4-way test: P1+:-0.0727249 P1-:0.0711259 P2+:-0.200156 P2-:0.188821
Best direction: Param 4 - step=0.075 improvement=0.188821
Continue: angle = -47.8161
4-way test: P1+:-0.0755742 P1-:-37.0346 P2+:-0.188821 P2-:-39.0555
All 4 directions failed (best=0), reducing step sizes to: {0.01875, 0.0375}
4-way test: P1+:-0.0376014 P1-:-37.091 P2+:-0.0930256 P2-:-37.967
All 4 directions failed (best=0), reducing step sizes to: {0.009375, 0.01875}
4-way test: P1+:-0.0187539 P1-:0.0186594 P2+:-0.0461709 P2-:-37.5371
Best direction: Param 3 - step=0.009375 improvement=0.0186594
Continue: angle = -47.7975
4-way test: P1+:-0.0186594 P1-:-37.1096 P2+:-0.0463479 P2-:-37.5255
All 4 directions failed (best=0), reducing step sizes to: {0.0046875, 0.009375}
4-way test: P1+:-0.00931782 P1-:0.00929398 P2+:-0.0230887 P2-:-37.3277
Best direction: Param 3 - step=0.0046875 improvement=0.00929398
Continue: angle = -47.7882
4-way test: P1+:-0.00929398 P1-:-37.1189 P2+:-0.023133 P2-:-37.3224
All 4 directions failed (best=0), reducing step sizes to: {0.00234375, 0.0046875}
4-way test: P1+:-0.00464401 P1-:-37.126 P2+:-0.0115452 P2-:-37.2268
```

```
All 4 directions failed (best=0), reducing step sizes to: {0.00117188, 0.00234375}

4-way test: P1+:-0.00232126 P1-:-37.1296 P2+:-0.00576728 P2-:-37.1797

All 4 directions failed (best=0), reducing step sizes to: {0.000585938, 0.00117188}

4-way test: P1+:-0.00116044 P1-:-37.1314 P2+:-0.00288231 P2-:-37.1564

All 4 directions failed (best=0), reducing step sizes to: {0.000292969, 0.000585938}

4-way test: P1+:-0.000580174 P1-:-37.1322 P2+:-0.00144082 P2-:-37.1447

All 4 directions failed (best=0), reducing step sizes to: {0.000146484, 0.000292969}

4-way test: P1+:-0.000290075 P1-:-37.1327 P2+:-0.000720329 P2-:-37.1389

All 4 directions failed (best=0), reducing step sizes to: {0.0000732422, 0.000146484}

4-way test: P1+:-0.000145035 P1-:0.000145029 P2+:-0.000360144 P2-:-37.136

All 4 directions failed (best=0.000145029), reducing step sizes to: {0.0000366211, 0.0000732422}

4-way test: P1+:-0.0000725166 P1-:0.0000725151 P2+:-0.000180067 P2-:0.000180056

All 4 directions failed (best=0.000180056), reducing step sizes to: {0.0000183105, 0.0000366211}

4-way test: P1+:-0.0000362581 P1-:0.0000362578 P2+:-0.000090032 P2-:0.0000900294

All 4 directions failed (best=0.0000900294
 ), reducing step sizes to: {9.15527×10^-6, 0.0000183105}

4-way test: P1+:-0.000018129 P1-:0.0000181289 P2+:-0.0000450157 P2-:0.000045015

All 4 directions failed (best=0.000045015), reducing step sizes to: {4.57764×10^-6, 9.15527×10^-6}

Final result: angle = -47.7882

Best parameters: {5.58594, 3.225}

Final step sizes: {4.57764×10^-6, 9.15527×10^-6}

Iterations: 23

Creating overlay plot with search path...
```



Out[331]=

```
{-47.7882, {1., 1., 5.58594, 3.225, 1., 1., 1., 0.126984, 0, 0, 0, 0.5, 7}}
```

```
maxIt = 500;
accG = 8;
precG = 6;
tiLim = 20;
meth = "NelderMead";
{bestAngle, bestValues, finalP0Val} =
optHopf[RHS, var, par, coP, {3, 4}, tiLim, meth, accG, precG, maxIt];
```

```
Starting Mathematica optimization - angle-based objective
objHopf called - angle-based objective
Initial angle = -75.3357 degrees
```

NMaximize: The function value −\$Failed is not a number at $\{\beta_1, \beta_2\}$ = {0.709311, 1.33176}.

NMaximize: The function value −\$Failed is not a number at $\{\beta_1, \beta_2\}$ = {0.709311, 1.33176}.

NMaximize: The function value −\$Failed is not a number at $\{\beta_1, \beta_2\}$ = {0.709311, 1.33176}.

General: Further output of NMaximize::nnum will be suppressed during this calculation.

```
NMaximize result structure: NMaximize[{$Failed, 0.25 ≤ β1 ≤ 0.75 && 0.5 ≤ β2 ≤ 1.5},
  {β1, β2}, Method → NelderMead, MaxIterations → 500, AccuracyGoal → 8, PrecisionGoal → 6]

Elapsed time: 0. seconds
```

Set: Lists {maxAngle$338111, optParams$338111} and NMaximize[{$Failed, 0.25 ≤ β_1 ≤ 0.75 && 0.5 ≤ β_2 ≤ 1.5}, {β_1, β_2}, Method → NelderMead, MaxIterations → 500, AccuracyGoal → 8, PrecisionGoal → 6] are not the same shape.

Values: The argument optParams$338111 is not a valid Association or a list of rules.

Values: The argument False is not a valid Association or a list of rules.

```
Best values = Values[optParams$338111]
```

In[•]:=
```
simpleOptHopf[RHS, var, par, coP, {3, 4}, 30(*,"DifferentialEvolution"*)]
```

```
Starting simple optimization - angle-based objective

Initial angle = -75.3357 degrees

Improvement: angle = -75.1727 degrees at {0.38801, 1.15989}

Improvement: angle = -64.8654 degrees at {0.246128, 1.6986}

Best angle = -64.8654 degrees

Best values = {0.246128, 1.6986}
```

Out[•]=

$$\left\{-64.8654, \{0.246128, 1.6986\}, \left\{2, \frac{1}{2}, 0.246128, 1.6986, \frac{1}{16}, 1, 1, 1, \frac{1}{2}, 4, 1, \frac{5}{2}, 3\right\}\right\}$$

```
result2 = optHopf[RHS, var, par, coP, {3, 4}, 60, "SimulatedAnnealing"];

result3 = optHopf[RHS, var, par, coP, {3, 4}, 60, "NelderMead"];
```

In[•]:=
```
objFunc = objHopf[RHS, var, par, coP, {3, 4}];
objFunc[{0.5, 1.0}]
```

```
objHopf called - angle-based objective
```

Out[•]=
```
-75.3357
```

```
optHopf[RHS, var, par, coP, {3, 4}, 20]
```

```
Starting Mathematica optimization

Method: NelderMead, Time limit: 20 seconds
```

Initial values: $\left\{\frac{15}{64}, \frac{163}{1024}\right\}$

```
Initial Max[Re] = -0.119138

Smaller search bounds: {{0.117188, 0.46875}, {0.0795898, 0.318359}}

Testing search space with 10 random points...

Random point 1: Max[Re] = -∞ at {0.313496, 0.130374}

Random point 2: Max[Re] = -0.16115 at {0.323052, 0.190254}

Random point 3: Max[Re] = -∞ at {0.183644, 0.109859}

Random point 4: Max[Re] = -0.135016 at {0.238302, 0.173444}

Random point 5: Max[Re] = -0.167588 at {0.337907, 0.193866}

Random point 6: Max[Re] = -∞ at {0.194126, 0.251856}

Random point 7: Max[Re] = -∞ at {0.226782, 0.249007}

Random point 8: Max[Re] = -∞ at {0.161622, 0.243574}

Random point 9: Max[Re] = -∞ at {0.170981, 0.221553}

Random point 10: Max[Re] = -∞ at {0.321969, 0.302673}

Time limit reached!

Elapsed time: 20.1 seconds

Best Max[Re] = -0.119138
```

Best values = $\left\{\frac{15}{64}, \frac{163}{1024}\right\}$

Out[∘]=

$$\left\{-0.119138, \left\{\frac{15}{64}, \frac{163}{1024}\right\}, \left\{\frac{1}{10}, \frac{1}{10}, \frac{15}{64}, \frac{163}{1024}, \frac{1}{64}, \frac{1}{256}, 1, 1, \frac{1}{32}, 1, 1, \frac{5}{2}, 3\right\}\right\}$$

In[∘]:=

In[∘]:= **scan = scanPar[RHS, var, par, coP, {3, 4}, plot, 0.5, 0.8, 5, 0.5]**

```
Scanning 25 points (wRan=0.5, hRan=0.8, hTol=0.5] for Hopf bifurcations...
```

Varying parameters at indices {3, 4} with center values: $\left\{\beta_1 \to \frac{1}{2}, \beta_2 \to 1\right\}$

In[∘]:= **oH = objHopf[RHS, var, par, coP, {3, 4}]; oH[{0.6, 1.2}]**

```
objHopf called
```

Out[∘]=

{{{0.908912, 2.71527, 0.0854035, 0.113696, 0.099498, 0.0373387, 0.0114352, 0.028451}}, -1.32328}

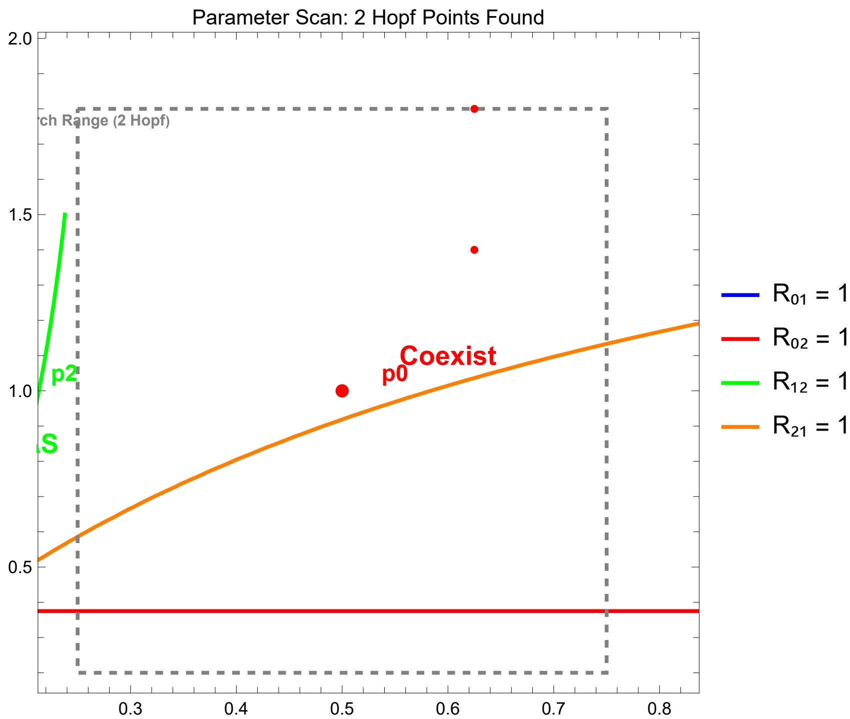
In[∘]:=

In[∘]:=

In[◦]:= `scan = scanPar[RHS, var, par, coP, {3, 4}, plot, 0.5, 0.8, 5, 0.5]`

```
Scanning 25 points (wRan=0.5, hRan=0.8, hTol=0.5) for Hopf bifurcations...
Found 2 Hopf points with real parts: {-0.399038, -0.428811}
```

Out[◦]=



`curve = cont[RHS, var, par, coP, .15, plotInd, 4, plot];`

```
Starting bifParam = β2 (index 4) at 1
Plot parameters: β1 = β1 (index 3), β2 = β2 (index 4)
☑ Will overlay on analytical plot (bifParam ∈ plotInd)
Using equilibrium:
 {1.1059, 2.61286, 0.0451163, 0.131365, 0.0597501, 0.0212923, 0.00763826, 0.0160747}
Result: Complex but not Hopf
Using equilibrium:
 {1.09787, 2.57483, 0.0636321, 0.119481, 0.0846302, 0.0276309, 0.0108527, 0.0210719}
Result: Complex but not Hopf
Using equilibrium:
 {1.08928, 2.53484, 0.0831195, 0.108604, 0.111048, 0.0332154, 0.0142874, 0.0256069}
Result: Complex but not Hopf
Using equilibrium:
 {1.06019, 2.40366, 0.146927, 0.0813845, 0.199296, 0.0458962, 0.0259223, 0.0367194}
Result: Complex but not Hopf
Using equilibrium:
 {1.04198, 2.32455, 0.185205, 0.0693394, 0.253611, 0.0506302, 0.0332073, 0.0414703}
```

Result: Complex but not Hopf

Using equilibrium:
{0.969809, 2.03089, 0.324988, 0.0403669, 0.462151, 0.057709, 0.0620766, 0.0520139}

Result: Complex but not Hopf

Using equilibrium:
{0.936853, 1.90631, 0.382843, 0.0325433, 0.554014, 0.0577341, 0.0752589, 0.0544412}

Result: Complex but not Hopf

Using equilibrium:
{0.705303, 1.17209, 0.697466, 0.00880151, 1.14778, 0.0429696, 0.167879, 0.0577075}

Result: Complex but not Hopf

Using equilibrium: {0.661024, 1.05629, 0.741416, 0.00693139, 1.2522, 0.0393912, 0.18559, 0.0571531}

Result: Complex but not Hopf

Using equilibrium:
{0.621534, 0.958875, 0.776726, 0.00558806, 1.34315, 0.0362278, 0.201387, 0.0565154}

Result: Complex but not Hopf

Using equilibrium:
{1.1134, 2.64891, 0.027588, 0.144329, 0.0363909, 0.0141519, 0.00463836, 0.0105841}

Result: Complex but not Hopf

Using equilibrium:
{1.12039, 2.68303, 0.0110447, 0.158452, 0.0145144, 0.00615816, 0.00184484, 0.00456564}

Result: Complex but not Hopf

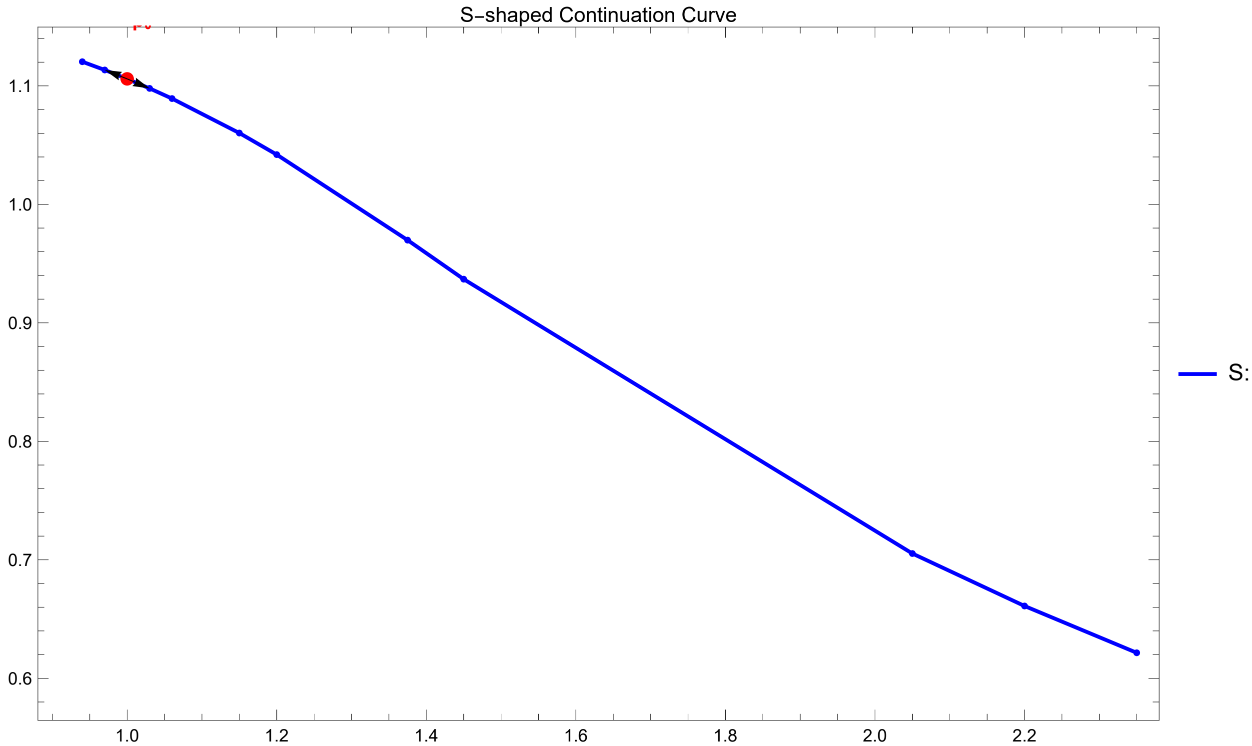
No coexistence equilibrium found

Found 12 points, tested 12 parameters

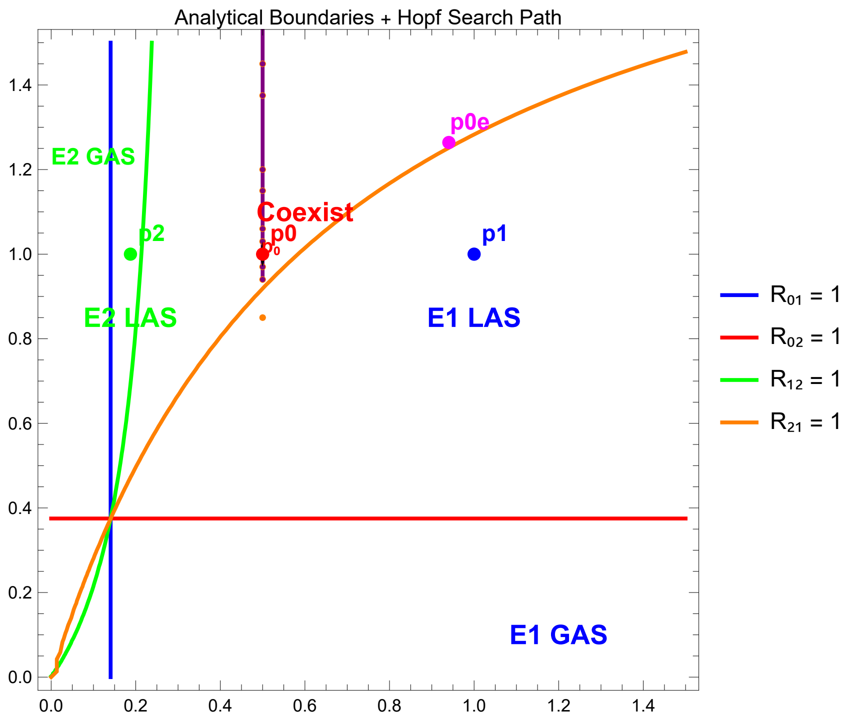
Parameter range: 0.94 to 2.35

=== HOPF BIFURCATION ANALYSIS ===

NO HOPF BIFURCATIONS FOUND in parameter range



Creating overlay on analytical plot...

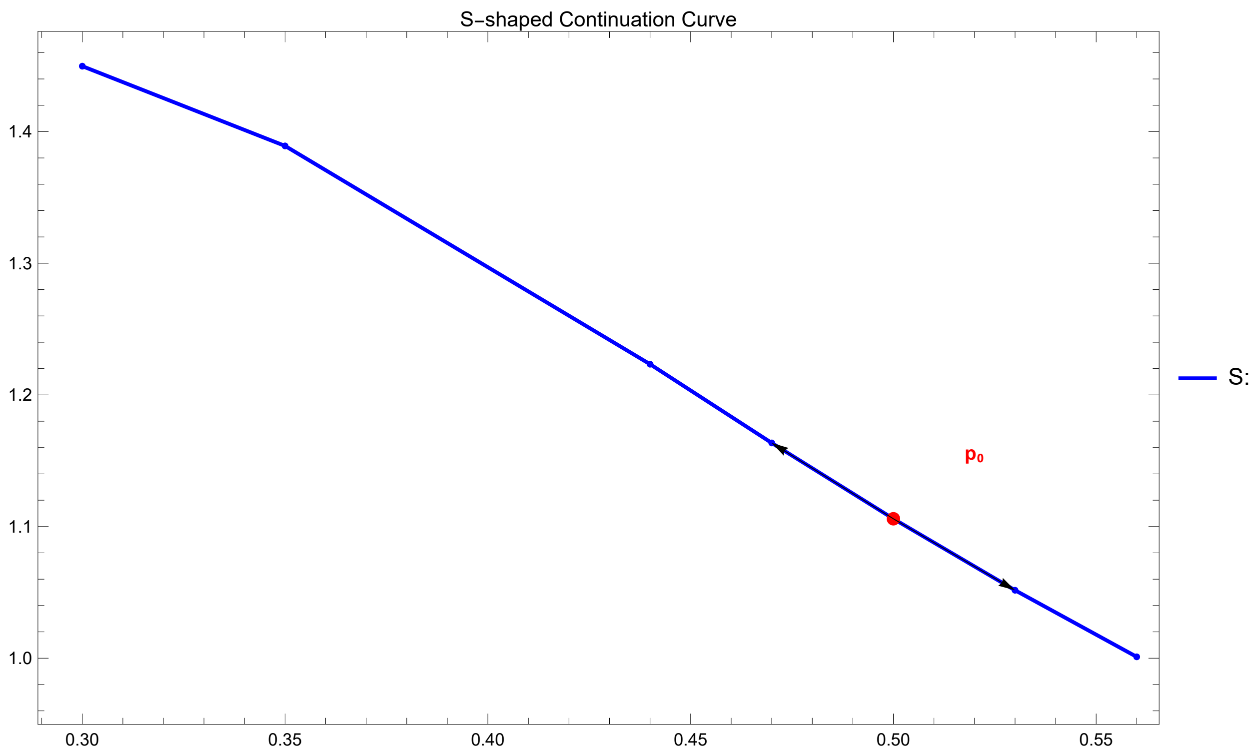



```
In[ ]:= curve = cont2[RHS, var, par, coP, .15, {3, 4}, 3, plot];
```

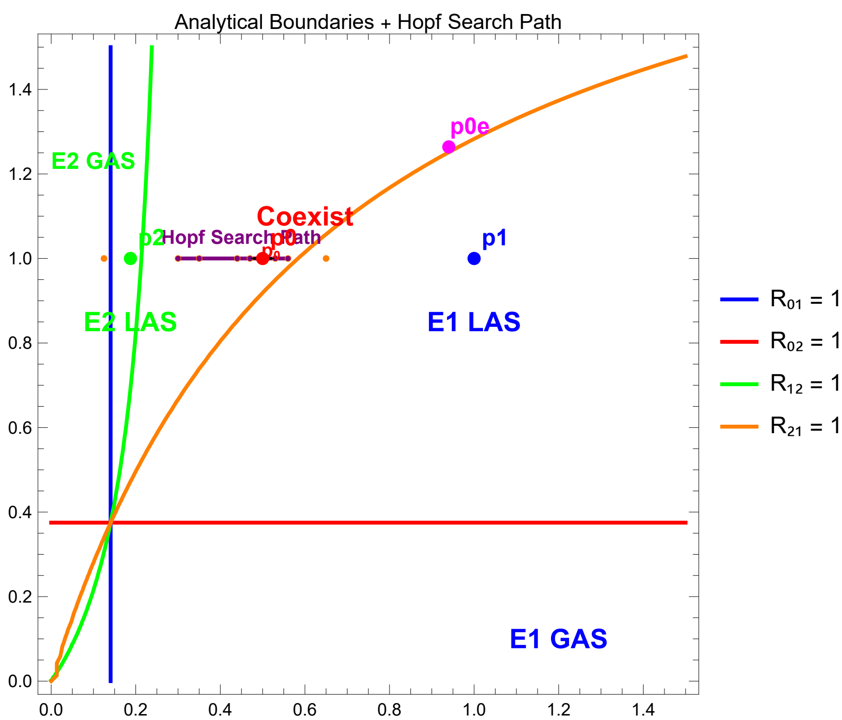
Starting bifParam = β_1 (index 3) at $\frac{1}{2}$

Plot parameters: β_1 = β_1 (index 3), β_2 = β_2 (index 4)

```
☑ Will overlay on analytical plot (bifParam ∈ plotInd)
Using equilibrium:
 {1.1059, 2.61286, 0.0451163, 0.131365, 0.0597501, 0.0212923, 0.00763826, 0.0160747}
Result: Complex but not Hopf
Using equilibrium:
 {1.0516, 2.71485, 0.0250867, 0.149466, 0.0316025, 0.0134751, 0.00388695, 0.0100287}
Result: Complex but not Hopf
Using equilibrium:
 {1.00102, 2.80001, 0.00962096, 0.166325, 0.011603, 0.00578369, 0.00137583, 0.00425667}
Result: Complex but not Hopf
No coexistence equilibrium found
Using equilibrium: {1.16351, 2.48863, 0.0712172, 0.112163, 0.099993, 0.028918, 0.0133185, 0.022246}
Result: Complex but not Hopf
Using equilibrium:
 {1.22332, 2.33491, 0.105146, 0.0922265, 0.158304, 0.0358037, 0.0220354, 0.0282502}
Result: Complex but not Hopf
Using equilibrium:
 {1.38913, 1.61622, 0.253648, 0.0369554, 0.535093, 0.0427056, 0.0872036, 0.0390391}
Result: Complex but not Hopf
Using equilibrium:
 {1.44971, 1.04966, 0.307927, 0.0167619, 0.938723, 0.0325612, 0.170114, 0.0345378}
Result: Complex but not Hopf
No coexistence equilibrium found
Found 7 points, tested 8 parameters
Parameter range: 0.3 to 0.56

=== HOPF BIFURCATION ANALYSIS ===
NO HOPF BIFURCATIONS FOUND in parameter range
```



🎨 Creating overlay on analytical plot...



In[∘]:=
```
var = {x, y};
par = {a, b, c};
RHS = {x (a - x - b y), y (c - b x - y)};

(*Example parameter values-should have coexistence*)
p0Val = {0.771484375, 1.1572265625, 1.080078125};
(*Example parameter values*)
optInd = {2, 3} (*used in search for Hopf*); optVars = par⟦optInd⟧;
{posSols, complexEigs, angle, eigs} = fpHopf[RHS, var, par, p0Val]
```

Out[∘]=
```
{{}, {}, -90, {}}
```

In[∘]:=
```
angleFunc[optInd_] := fpHopf[RHS, var, par, ReplacePart[p0Val, Thread[optInd → #]]]⟦3⟧ &;
angleFunc[optInd][{1 / 3, 1}]
```

Out[∘]=
```
0.
```

```
NMaximize[{angleFunc[optInd][optVars], 0.25 ≤ optVars⟦1⟧ ≤ 0.75 && 0.5 ≤ optVars⟦2⟧ ≤ 1.5},
 optVars, MaxIterations → 2]
```

Out[∘]=
```
0.
```

Union: Application of the SameTest function yielded $\left(\text{Norm}[\#1 - \#2] < \frac{1}{10^{10}} \ \&\right)\left[\left\{\frac{1}{c}, \frac{2.}{b}\right\}, \{0, 0\}\right]$, which evaluates to $\sqrt{\frac{4.}{\text{Abs}[b]^2} + \frac{1}{\text{Abs}[c]^2}} < \frac{1}{10000000000}$. The SameTest function must evaluate to True or False at every pair of elements.

Out[∘]=
```
{-90., {b → 0.25, c → 0.5}}
```

```
angleFunc[optInd_] := With[{newP0Val = baseP0Val},
   fpHopfAngle[RHS, var, par, ReplacePart[newP0Val, Thread[optInd → #]]]⟦3⟧] &
```

```
angleFunc[p3_, p4_] := fpHopf[RHS, var, par, p0Val][[3]]
angleFunc[{1, 2}]
```

Out[∘]=

$\{\Lambda - S\,\mu - I1\,S\,\beta_1 - I2\,S\,\beta_2 - S\,Y1\,\beta_1\,\eta_1 - S\,Y2\,\beta_2\,\eta_2 + R1\,\Theta_1 + R2\,\Theta_2 + R\,\Theta_3,$
$-I1\,\mu + I1\,S\,\beta_1 - I1\,\gamma_1 + S\,Y1\,\beta_1\,\eta_1,\ -Y1\,\mu - Y1\,\gamma_1 + I1\,R2\,\beta_1\,\sigma_1 + R2\,Y1\,\beta_1\,\eta_1\,\sigma_1,$
$-R1\,\mu + I1\,\gamma_1 - R1\,\Theta_1 - I2\,R1\,\beta_2\,\sigma_2 - R1\,Y2\,\beta_2\,\eta_2\,\sigma_2,$
$-I2\,\mu + I2\,S\,\beta_2 - I2\,\gamma_2 + S\,Y2\,\beta_2\,\eta_2,\ -Y2\,\mu - Y2\,\gamma_2 + I2\,R1\,\beta_2\,\sigma_2 + R1\,Y2\,\beta_2\,\eta_2\,\sigma_2,$
$-R2\,\mu + I2\,\gamma_2 - R2\,\Theta_2 - I1\,R2\,\beta_1\,\sigma_1 - R2\,Y1\,\beta_1\,\eta_1\,\sigma_1,\ -R\,\mu + Y1\,\gamma_1 + Y2\,\gamma_2 - R\,\Theta_3\}$

Out[∘]=

{S, I1, Y1, R1, I2, Y2, R2, R}

Out[∘]=

$\{\Lambda, \mu, \beta_1, \beta_2, \gamma_1, \gamma_2, \eta_1, \eta_2, \Theta_1, \Theta_2, \Theta_3, \sigma_1, \sigma_2\}$

Out[∘]=

$\left\{2, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{16}, 1, 1, 1, \frac{1}{2}, 4, 1, \frac{5}{2}, 3\right\}$

Out[∘]=

angleFunc[{1, 2}]

```
Print["Testing NMaximize with angle function..."];
NMaximize[{angleFunc[{Subscript[β, 1], Subscript[β, 2]}],
   0.25 ≤ Subscript[β, 1] ≤ 0.75 && 0.5 ≤ Subscript[β, 2] ≤ 1.5},
  {Subscript[β, 1], Subscript[β, 2]}]
```

Out[∘]=

$\{\Lambda - S\,\mu - I1\,S\,\beta_1 - I2\,S\,\beta_2 - S\,Y1\,\beta_1\,\eta_1 - S\,Y2\,\beta_2\,\eta_2 + R1\,\Theta_1 + R2\,\Theta_2 + R\,\Theta_3,$
$-I1\,\mu + I1\,S\,\beta_1 - I1\,\gamma_1 + S\,Y1\,\beta_1\,\eta_1,\ -Y1\,\mu - Y1\,\gamma_1 + I1\,R2\,\beta_1\,\sigma_1 + R2\,Y1\,\beta_1\,\eta_1\,\sigma_1,$
$-R1\,\mu + I1\,\gamma_1 - R1\,\Theta_1 - I2\,R1\,\beta_2\,\sigma_2 - R1\,Y2\,\beta_2\,\eta_2\,\sigma_2,$
$-I2\,\mu + I2\,S\,\beta_2 - I2\,\gamma_2 + S\,Y2\,\beta_2\,\eta_2,\ -Y2\,\mu - Y2\,\gamma_2 + I2\,R1\,\beta_2\,\sigma_2 + R1\,Y2\,\beta_2\,\eta_2\,\sigma_2,$
$-R2\,\mu + I2\,\gamma_2 - R2\,\Theta_2 - I1\,R2\,\beta_1\,\sigma_1 - R2\,Y1\,\beta_1\,\eta_1\,\sigma_1,\ -R\,\mu + Y1\,\gamma_1 + Y2\,\gamma_2 - R\,\Theta_3\}$

Out[∘]=

{S, I1, Y1, R1, I2, Y2, R2, R}

Out[∘]=

$\{\Lambda, \mu, \beta_1, \beta_2, \gamma_1, \gamma_2, \eta_1, \eta_2, \Theta_1, \Theta_2, \Theta_3, \sigma_1, \sigma_2\}$

Out[∘]=

$\left\{2, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{16}, 1, 1, 1, \frac{1}{2}, 4, 1, \frac{5}{2}, 3\right\}$

Out[∘]=

```
{{{1.1059, 2.61286, 0.0451163, 0.131365, 0.0597501, 0.0212923, 0.00763826, 0.0160747}},
 {-1.20674 + 0.315779 i, -1.20674 - 0.315779 i}, -75.3357, {-7.81178, -1.68219,
  -1.47706, -1.20674 + 0.315779 i, -1.20674 - 0.315779 i, -0.532873, -0.5, -0.12074}}
```

```
parOpt = par[[{3, 4}]]
```

Out[•]=

$$\{\Lambda - S\,\mu - I1\,S\,\beta_1 - I2\,S\,\beta_2 - S\,Y1\,\beta_1\,\eta_1 - S\,Y2\,\beta_2\,\eta_2 + R1\,\theta_1 + R2\,\theta_2 + R\,\theta_3,\ -I1\,\mu + I1\,S\,\beta_1 - I1\,\gamma_1 + S\,Y1\,\beta_1\,\eta_1,\ -Y1\,\mu - Y1\,\gamma_1 + I1\,R2\,\beta_1\,\sigma_1 + R2\,Y1\,\beta_1\,\eta_1\,\sigma_1,\ -R1\,\mu + I1\,\gamma_1 - R1\,\theta_1 - I2\,R1\,\beta_2\,\sigma_2 - R1\,Y2\,\beta_2\,\eta_2\,\sigma_2,\ -I2\,\mu + I2\,S\,\beta_2 - I2\,\gamma_2 + S\,Y2\,\beta_2\,\eta_2,\ -Y2\,\mu - Y2\,\gamma_2 + I2\,R1\,\beta_2\,\sigma_2 + R1\,Y2\,\beta_2\,\eta_2\,\sigma_2,\ -R2\,\mu + I2\,\gamma_2 - R2\,\theta_2 - I1\,R2\,\beta_1\,\sigma_1 - R2\,Y1\,\beta_1\,\eta_1\,\sigma_1,\ -R\,\mu + Y1\,\gamma_1 + Y2\,\gamma_2 - R\,\theta_3\}$$

Out[•]=

{S, I1, Y1, R1, I2, Y2, R2, R}

Out[•]=

$\{\Lambda, \mu, \beta_1, \beta_2, \gamma_1, \gamma_2, \eta_1, \eta_2, \theta_1, \theta_2, \theta_3, \sigma_1, \sigma_2\}$

Out[•]=

$$\left\{\Lambda \to 2,\ \mu \to \frac{1}{2},\ \beta_1 \to \frac{1}{2},\ \beta_2 \to 1,\ \gamma_1 \to \frac{1}{16},\ \gamma_2 \to 1,\ \theta_1 \to \frac{1}{2},\ \theta_2 \to 4,\ \theta_3 \to 1,\ \eta_1 \to 1,\ \eta_2 \to 1,\ \sigma_1 \to \frac{5}{2},\ \sigma_2 \to 3\right\}$$

Out[•]=

$\{\beta_1, \beta_2\}$

```
angleFunc[{p3_?NumericQ, p4_?NumericQ}] := Module[{newP0Val, result}, newP0Val = p0Val;
   (*Assuming p0Val is defined in your context*)newP0Val[[{3, 4}]] = {p3, p4};
   result =
    result[[3]] (*Return the angle*)];

(*The failing NMaximize call*)
```

Out[•]=

-1.34589

Out[•]=

numericalFunc[{x, y}]

Out[•]=

$\{-1.46464 \times 10^{-17}, \{x \to 0., y \to 3.82706 \times 10^{-9}\}\}$