Week 1

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Alphametic Puzzles

Exercise from http://www.tkcs-collins.com/truman/alphamet/alphamet.shtml:

```
Example: THATS + A + SIDE = EFFECT (2)
```

```
  THATS
      A
   SIDE
 ------
 EFFECT
```

Looking at the alignment we can assume E=1 because it's the beginning letter. We can also assume T=9 because it's not added to anything, so there must be a carry in the previous addition which means F=0;Now we can put these constraints into z3: Input:

```python
S,A,E,T,D,C,H,I,F=Ints("S A E T D C H I F")
x1,x2,x3,x4,x5=Ints("x1 x2 x3 x4 x5")
d = Distinct(S,A,E,T,D,C,H,I,F)
e = (S>=0,S<10,A>=0,A<10,E>=0,E<10,T>=0,T<10,D>=0,D<10,C>=0,C<10,H>=0,H<10,I>=0,I<10,F>=0,F<10)
f = (x1>=0,x1<=1,x2>=0,x2<=1,x3>=0,x3<=1,x4>=0,x4<=1,x5>=0,x5<=1)

s=Solver()
s.add(d)
s.add(e)
s.add(f)

s.add(S+A+E==T)
s.add(Or(T+D+x2==C,T+D+x2==10+C))
s.add(Or(A+I+x3==E,A+I+x3==10+E))
s.add(Or(H+S+x4==F,H+S+x4==10+F))
s.add(Or(T+x5==F,T+x5==F+10))
s.add(E==1)
s.add(F==0)
s.add(T==9)

print(s.check())
print(s.model())
```

Here the x's are carries from previous equations. All letters must be distinct and between 0 and 9. The carry x's must be either 0 or 1.

Once we add all of these constraints we get the output

```
sat
[A = 2,
 D = 5,
 I = 8,
 x3 = 1,
 x2 = 0,
 x1 = 0,
 x4 = 1,
 x5 = 1,
 C = 4,
 H = 3,
 T = 9,
 F = 0,
 E = 1,
 S = 6]
```

So A=2,D=5,I=8,C=4,H=3,T=9,F=0,E=1,S=6

If we input these constraints back in the original equation we will see it is verified:

THAT'S=93296

A=2

SIDE=6851

EFFECT=100149

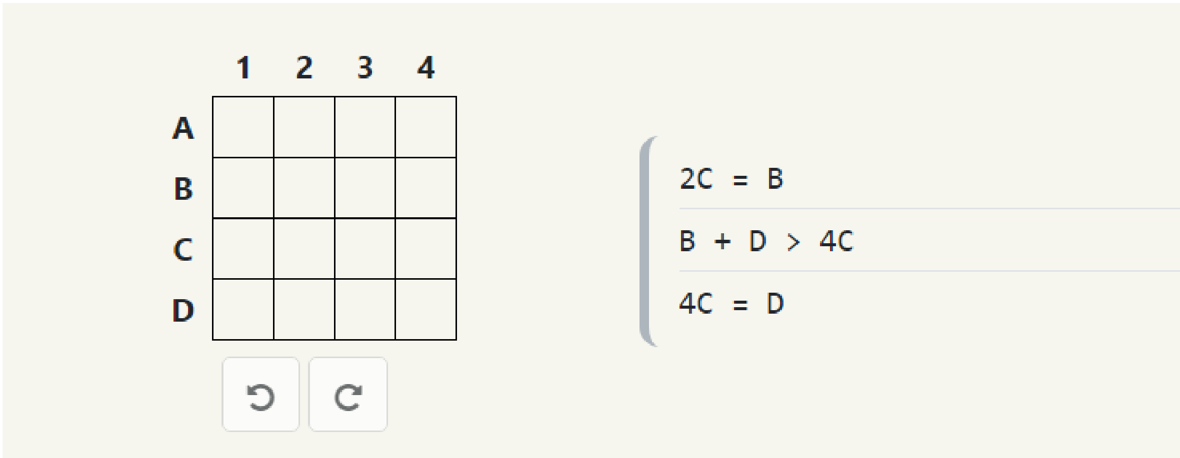
93 296 + 2 + 6851 =

100149

Logic equations

Exercise from : https://www.brainzilla.com/logic/logic-equations/

I chose this example



Input and output for z3:

```
[27]
A,B,C,D = Ints("A B C D")

x=Distinct(A,B,C,D)
y=(A<5,B<5,C<5,D<5, A>0,B>0,C>0,D>0)

s=Solver()
s.add(x)
s.add(y)
s.add(2*C==B)
s.add(B+D>4*C)
s.add(4*C==D)

print(s.check())
print(s.model())
```

```
sat
[A = 3, C = 1, D = 4, B = 2]
```

The solution from above can be visualized in this graph

	1	2	3	4
A	✗	✗	✓	✗
B	✗	✓	✗	✗
C	✓	✗	✗	✗
D	✗	✗	✗	✓

2C = B

B + D > 4C

4C = D

Matrix inverse- direct programming

- calculate the matrix inverse:

$$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$$

We know that for a 2d matrix the inverse has:

- Numbers on main diagonal swapped
- Numbers of second diagonal with – in front

$$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

For example:

If we input this in the code we get

```
# 2x2 matrix of integer variables
# A = [ [ Int("a_%s_%s" % (i, j)) for j in range(2) ]
#       for i in range(2) ]
# print (A)

A=[[3,1],[5,2]]


diag1=A[0][0]*A[1][1]

diag2=A[0][1]*A[1][0]

determinant=diag1-diag2

inverseRow1Col1=A[0][0]
inverseRow0Col0=A[1][1]
inverseRow0Col1=-A[0][1]
inverseRow1Col0=-A[1][0]

InverseA=[ [inverseRow0Col0,inverseRow0Col1],[inverseRow1Col0,inverseRow1Col1] ]


print (1/determinant, " *",InverseA)
```

```
1.0  * [[2, -1], [-5, 3]]
```

The input matrix is A=[[3,1],[5,2]]

The inverse matrix is 1/determinant multiplied by [[2,-1],[-5,3]]

This code would work for any existeing 2d matrix

Matrix inverse- proper z3

In order to solve the first matrix inverse exercise we need to begin by using some math to get the variables in a more convenient form. The math can be seen in the picture below

$$\begin{pmatrix} a11 & a12 \\ a21 & a22 \end{pmatrix} \cdot A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a11 & a12 \\ a21 & a22 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} a11*x+a12*z=1 \\ a11*y+a12*t=0 \\ a21*x+a22*z=0 \\ a21*y+a22*t=1 \end{cases}$$

When we have them in the last form we can input it in z3 to get the result. The input and output can be seen below:

```
a11,a12,a21,a22 = Ints("a11 a12 a21 a22")
a11=3
a12=1
a21=5
a22=2

determinant=a11*a22-a12*a21
x,y,z,t=Ints("x y z t")

s=Solver()
s.add(a11*x+a12*z==1)
s.add(a11*y+a12*t==0)
s.add(a21*x+a22*z==0)
s.add(a21*y+a22*t==1)


print(s.check())
print(s.model())
```

```
sat
[y = -1, x = 2, t = 3, z = -5]
```

As we can see, the result check out and inverse is (2,-1),(-5,3)

Matrix inverse- complicated exercise

This exercise seems similar to the one above, but It provides one problem to the z3 solver. Because the determinant is not 1, the inverse matrix would contain fractions for the variables inside, which z3 can not solve. Thus we will have to use more math to get the variables in a proper form

$$A^{-1} = \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} \frac{x_1}{x_2} & \frac{y_1}{y_2} \\ \frac{z_1}{z_2} & \frac{t_1}{t_2} \end{pmatrix}$$

$$\begin{cases} a_{11} \cdot \frac{x_1}{x_2} + a_{12} \cdot \frac{z_1}{z_2} = 1 \\ a_{11} \cdot \frac{y_1}{y_2} + a_{12} \cdot \frac{t_1}{t_2} = 0 \\ a_{21} \cdot \frac{x_1}{x_2} + a_{22} \cdot \frac{z_1}{z_2} = 0 \\ a_{21} \cdot \frac{y_1}{y_2} + a_{22} \cdot \frac{t_1}{t_2} = 1 \end{cases} \Leftrightarrow \begin{cases} a_{11} x_1 z_2 + a_{12} z_1 x_2 = x_2 z_2 \\ a_{11} y_1 t_2 + a_{12} t_1 y_2 = 0 \\ a_{21} x_1 z_2 + a_{22} z_1 x_2 = 0 \\ a_{21} y_1 t_2 + a_{22} t_1 y_2 = y_2 t_2 \end{cases}$$

But this form is actually more complicated than it should be because instead of writing every number as a fraction we can write it as itself over the determinant(even though it will give fractions that are not simplified

This can be seen in the input for z3

```
a11,a12,a21,a22 = Ints("a11 a12 a21 a22")
a11=4
a12=7
a21=2
a22=6

determinant=a11*a22-a12*a21
x1,y1,z1,t1,x2,y2,z2,t2=Ints("x1 y1 z1 t1 x2 y2 z2 t2")

s=Solver()
s.add(a11*x1*determinant+a12*z1*determinant==determinant*determinant)
s.add(a11*y1*determinant+a12*t1*determinant==0)
s.add(a21*x1*determinant+a22*z1*determinant==0)
s.add(a21*y1*determinant+a22*t1*determinant==determinant*determinant)


print(determinant)
print(s.check())
print(s.model())
```

```
10
sat
[y1 = -7, z1 = -2, x1 = 6, t1 = 4]
```

This is the inverse matrix calculated manually

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 6 & -7 \\ -2 & 4 \end{pmatrix}$$

This are the results from z3 in a more readable form and then simplified

$$A^{-1} = \begin{pmatrix} \frac{-6}{-10} & \frac{7}{-10} \\ \frac{2}{-10} & \frac{-4}{-10} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{-7}{10} \\ \frac{-1}{5} & \frac{2}{5} \end{pmatrix}$$