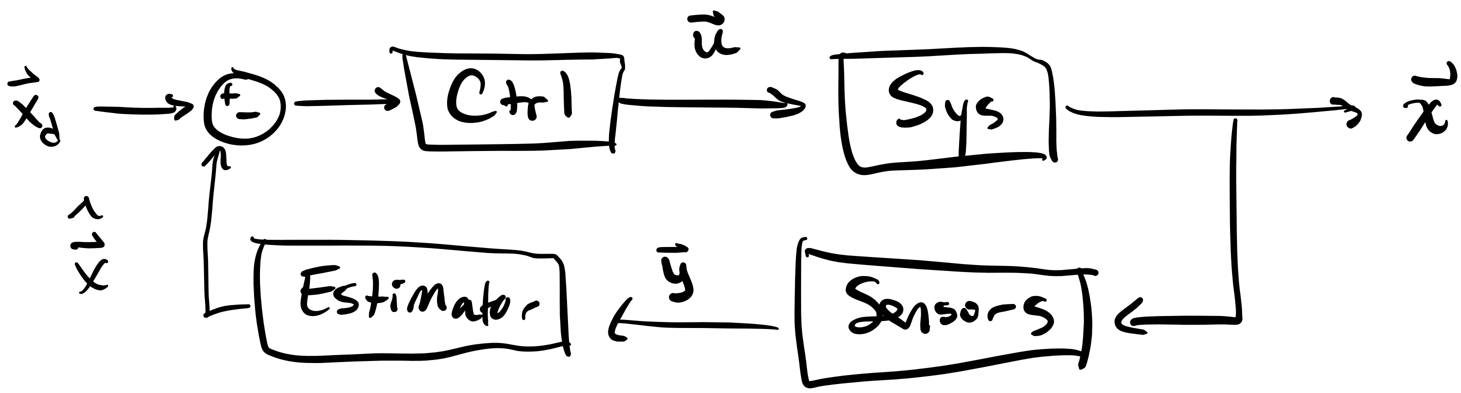
Observability: linear

often times we may be interested in controlling some system using either full or partial state feedback. but when the available sensors don't directly provide measurements of the state, we need to use some kind of estimator to determine the state given the measurements



A common estimator might be a linear or nonlinear kalman filter, for example.

Before designing an estimator, it can be informative to first determine whether or not it is even possible to estimate $\hat{x}$ given y.

System <u>observable</u> if we can uniquely determine $\vec{x}(t_0)$ given $\vec{y}(t_0:t_w)$ and $\vec{u}(t_0:t_w)$ and some information about the system & sensor dynamics.

$$\hat{\vec{x}}(t_0) \longleftrightarrow \vec{y}(t_0:t_w), \vec{u}(t_0:t_w)$$

Before diving into the nonlinear case, let's review linear observability.

Linear System

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

$$\vec{y} = C\vec{x} + D\vec{u}$$

in some cases, we may have enough measurements to determine x quite easily:

$$\hat{\vec{x}} = C^{+}\left[\vec{y} - D\vec{u}\right]$$

psvedo inverse

If C is full rank, then we can solve for $\hat{\vec{x}}$.

If C is not full rank, we need more information.

If we have y(t), then in principle we can determine the derivative of y:

$$\dot{\vec{y}} = \frac{d}{dt}\vec{y} = C\dot{\vec{x}} + D\dot{\vec{u}}, \qquad \dot{\vec{x}} = Ax + Bu$$

assuming of course
that C is
constant, and $\vec{u}$ constant (but

Although we don't know $\dot{\vec{x}}$, we do know:

$$\Rightarrow \quad \dot{\vec{y}} = CAx + CBu + D\dot{\vec{u}}$$

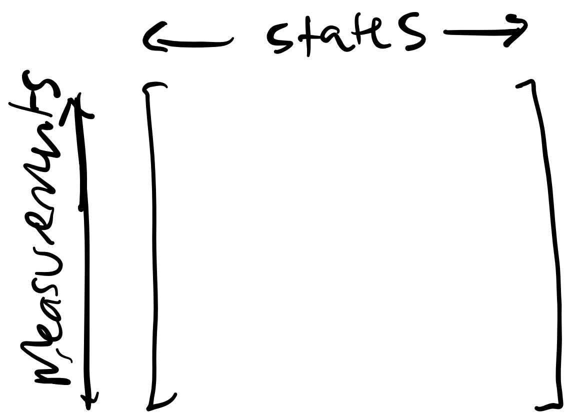
we can now append this to the undifferentiated data to get:

$$\begin{bmatrix} \vec{y} - D\vec{u} \\ \dot{\vec{y}} - CB\vec{u} - D\dot{\vec{u}} \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix} \vec{x} \Rightarrow \hat{\vec{x}} = \begin{bmatrix} C \\ CA \end{bmatrix}^{+} \begin{bmatrix} \vec{y} - D\vec{u} \\ \dot{\vec{y}} - CB\vec{u} - D\dot{\vec{u}} \end{bmatrix}$$

if $\begin{bmatrix} C \\ CA \end{bmatrix}$ is full rank, we can solve for $\vec{x}$.

We can repeat this for higher order derivatives to (potentially) get more information, and therefore more rank. Which gives us the linear observability matrix:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix} \begin{array}{l} \leftarrow 0^{th} \text{ derivative} \\ \leftarrow 1^{st} \text{ derivative} \\ \leftarrow 2^{nd} \text{ derivative} \\ \\ \\ \end{array}$$



$\mathcal{O}$ is the jacobian of the measurements and their time derivatives with respect to the state.

Note that the more higher order derivatives necessary to achieve full rank, the more derivatives of the sensory information will be necessary.

We can repeat this process for a discrete linear system, and get the same result:

$$\mathcal{O}_d = \begin{bmatrix} C_d \\ C_d A_d \\ C_d A_d^2 \\ \vdots \\ C_d A_d^w \end{bmatrix} \begin{matrix} \leftarrow \text{measurement at } t=0 \\ \leftarrow t=1 \\ \leftarrow t=2 \\ \\ t=w \end{matrix}$$

Discrete

$$\vec{x}_{k+1} = A_d \vec{x}_k + B_d \vec{u}_k$$

$$\vec{y}_k = C_d \vec{x}_k + D_d \vec{u}_k$$

drop "d" subscript

$$y_1 = C x_1 + D u_1$$

$$y_2 = C x_2 + D u_2$$

$$x_2 = A x_1 + B u_1$$

$$= C\left[A x_1 + B u_1\right] + D u_2$$

$$y_3 = C x_3 + D u_3$$

$$x_3 = A x_2 + B u_2$$

$$x_3 = A\left[A x_1 + B u_1\right] + B u_2$$

$$= CA^2 x_1 + CAB u_1 + CB u_2 + D u_3$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} x_1 + \begin{bmatrix} Bu_1 \\ CBu_1 + Du_2 \\ CABu_1 + CBu_2 + Du_3 \\ \end{bmatrix}$$

$$Y = \mathcal{O} x_1 + \mathcal{U}$$

$$\mathcal{O} x_1 = Y - \mathcal{U}$$

$$x_1 = \mathcal{O}^+ [Y - \mathcal{U}]$$

$\Rightarrow$ if $\mathcal{O}^+$ is invertable,

Y is known,

$\mathcal{U}$ is known

then the initial state can be determined.