Nonlinear System

Now let's see what happens with a nonlinear system.

We will focus on systems that can be written in control affine form:

$$\dot{\vec{x}} = \vec{f}(\vec{x},\vec{u}) = \underbrace{\vec{f}_0(\vec{x}) + u_1\vec{f}_1(\vec{x}) + u_2\vec{f}_2(\vec{x}) + \dots + u_n\vec{f}_n(\vec{x})}^{\text{linear in controls!}}$$

$$\vec{y} = \vec{h}(\vec{x},\vec{u})$$

Similar to the linear case, if the # of unique equations (rows) given by h is equal to the number of states, the system is observable.

Though if h is highly nonlinear, it could be quite challenging to extract $\vec{x}$ from $\vec{y}$ and h.

Things get more interesting if we need to consider the derivatives of y.

$$\dot{\vec{y}} = \frac{\partial \vec{h}}{\partial t} = \frac{\partial \vec{h}}{\partial \vec{x}} \frac{\partial \vec{x}}{\partial t} = \frac{\partial \vec{h}}{\partial \vec{x}} \cdot \left[\vec{f}_0(\vec{x}) + u_1 \vec{f}_1(\vec{x}) + u_2 \vec{f}_2(\vec{x}) + \dots + u_n \vec{f}_n(\vec{x}) \right]$$

The major difference btwn this and the linear case is that now the effect of the control inputs u_{1-n} depends on the state x.

Recall in the linear case:

$$y = Cx + \underset{\substack{\uparrow \\ \text{not a} \\ \text{function of} \\ x}}{D}u$$

The implication of this is that the control input can help inform the state value.

If we choose $u_1 = u_2 = \dots = u_n = 0$, we have:

$$\dot{y} = \frac{\partial \vec{h}}{\partial \vec{x}} \vec{f}_0(\vec{x})$$

you can think of this as the directional derivative of h along the vector field $f_0(x)$.

Since we will need many such directional derivatives, including 2nd & 3rd order direc. derivs. it can be helpful to have some more compact notation:

$$\dot{\vec{y}} = \frac{\partial \vec{h}}{\partial \vec{x}} \vec{f}_0(\vec{x}) = L_{\vec{f}_0} \vec{h}$$

let's relate this to the linear by linearizing about some operating point

x_0, u_0:

Linearize

$$\begin{bmatrix} \vec{y} \\ \dot{\vec{y}} \end{bmatrix} = \underbrace{\begin{bmatrix} \left. \mathrm{Jac}(\vec{h}) \right|_{x_0, u_0} \\ \left. \mathrm{Jac}\left(L_{\vec{f}_0} \vec{h}\right) \right|_{x_0, u_0} \end{bmatrix}} \vec{x} \approx \begin{bmatrix} C \\ CA \end{bmatrix} \vec{x}$$

if is full rank, then we can solve for x.

But in this nonlinear case turning on different controls will actually provide new information, which was not the case for the linear case.

For example:

if $u_1 \neq 0$, but all other u_i's $= 0$:

$$\dot{y} = \frac{\partial h}{\partial x} f_0(x) + \frac{\partial h}{\partial x} f_1(x) u_1$$

$$= L_{f_0} h + u_1 L_{f_1} h$$

if we only care about observability, we can set $u_1 = 1$, wlog:

$$\dot{y} = L_{f_0} h + L_{f_1} h$$

repeating this for all the controls we can collect a bunch of terms together to construct the observability Lie algebra:

$$G' = \{ h, \ L_{f_0} h, \ L_{f_0} h + L_{f_1} h, \ L_{f_0} h + L_{f_2} h, \ \ldots, \ L_{f_0} h + L_{f_n} h \}$$

For this set of control options, we can now count the number of unique terms. If there are as many unique terms as there are states, the sys is observable.

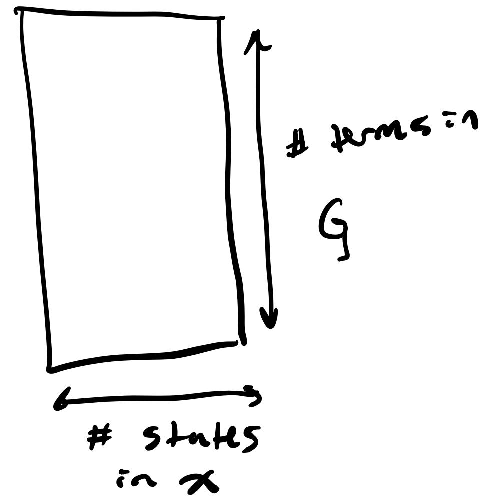
To make this a little easier, for counting purposes we can simplify to:

$$G^1 = \{ h, L_{f_0} h, L_{f_1} h, L_{f_2} h, \dots, L_{f_n} h \}$$

What does unique mean here?
Not talking about linear independence.
for example:
if $h = x_1$ & $L_{f_0} h = x_1^2$ or $\log x_1$,
then $L_{f_0} h$ does not provide new unique information.

A convenient way to count the # of unique terms is to use the Jacobian:

$$\mathcal{O} = \mathrm{Jac}(G)\Big|_{x_0, u_0} = \text{some matrix}$$



if $\mathcal{O}$ is full rank for a given x_0, u_0, then the sys is locally observable at that location.

Just like in the linear case, we may also need to consider 2nd derivatives in order for the sys. to be observable.

For these 2nd derivatives we could choose the same, or a different control:

some examples: $\underbrace{L_{f_0} L_{f_0} h}_{= L_{f_0}^2 h}$, $L_{f_0} L_{f_1} h$, $L_{f_2} L_{f_1} h$

Now you can use this $\mathcal{O}$ as described before.

i.e. analyze eigenvalues, eigenvectors of $\mathcal{O}^T \mathcal{O}$

analyze diagonal of $(\mathcal{O}^T \mathcal{O})^{-1}$