Given data:

$$\mathbf{X} = \overbrace{\begin{bmatrix} | & | & | & & | \\ x_1 & x_2 & x_3 & \cdots & x_m \\ | & | & | & & | \end{bmatrix}}^{\text{"snapshots in time" dim: } m} \left.\vphantom{\begin{bmatrix} | \\ x \\ | \end{bmatrix}}\right\} \begin{array}{l}\text{spatial dimension} \\ \text{dim: } n\end{array}$$

SVD:

$$\mathbf{X} = U \Sigma V^*$$

complex conjugate transpose for our cases that generallay just means transpose

U: unitary (orthonormal columns)

Σ: diagonal, padded w/ zeros

V: unitary (orthonormal coloms)

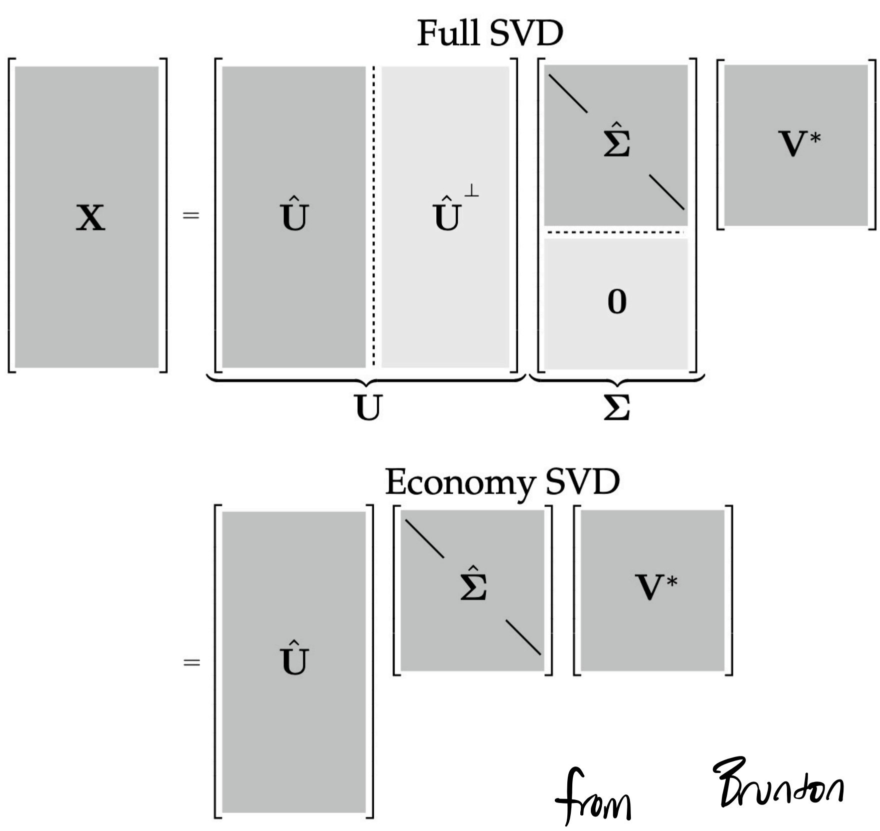
Unitary Matrix: square
rows & columns are orthonormal
inverse = cojugate transpose

When $n \geq m$ (more spatial than temporal dimensions):

$$X = U\Sigma V^* = \underbrace{\begin{bmatrix} \hat{U} & \hat{U}^{\perp} \end{bmatrix} \begin{bmatrix} \hat{\Sigma} \\ 0 \end{bmatrix} V^*}_{\text{'full' SVD}}$$

$\hat{U}^{\perp}$ contains columns perpendicular to $\hat{U}$

Economy SVD: $\hat{U}$, $\hat{\Sigma}$, V^*



from Brunton & Kutz

Columns of U = "left singular vectors" of X

Columns of V = "right singular vectors" of X

diagonal elements of $\hat{\Sigma}$ = singular values, ordered from largest to smallest

SVD is a generalization of Eigenvalue decomp.

Eigenvalue decomp:

$$\mathbf{X}V = V\Lambda \Rightarrow \mathbf{X} = V\Lambda V^{-1}$$

Σ is similar

If σ is a singular value of A,
then σ^2 is an eigenvalue of $\underbrace{A^*A}_{\text{square matrix}}$

Geometric Interpretation

Columns of U provide an orthonormal basis for Columns of X.

Columns of V (= rows of V^*) provide an orthonormal basis for the rows of X.

for $$X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_m \\ | & | & & | \end{bmatrix}$$

Columns of U encode spatial patterns

Rows of V^* encode temporal patterns

Compression

Eckart - Young thm:

A truncated SVD provides the best matrix approximation of a given rank in the Frobenius norm.

Fr norm = sum of squares of all matrix entries.

It is the norm that appears is least squares problem statements.

$$\|A\|_{Fr} = \sqrt{\text{trace}(A^*A)}$$

So the Fr norm also describes the sum of the squares of the singular values of A.