How good are my estimates going to be?
Cramer-Rao inequality provides a lower bound on the expected errors between the estimated quantities & the true values

Cramer Rao inequality for unbiased estimate $\hat{\underline{x}}$:
not doing proof, see C & J 2.3.

$$P \equiv E\{\underbrace{\underbrace{(\hat{\underline{x}} - \underline{x})}_{\text{estimation error}}(\hat{\underline{x}} - \underline{x})^T\}}_{\text{estimation error covariance}} \geq \underbrace{F^{-1}}_{\text{Fisher Information Matrix}}$$

$$F = E\left\{\left[\frac{\partial}{\partial \underline{x}} \ln[\underbrace{p(\tilde{\underline{y}} | \underline{x})}_{\substack{\text{probability of getting}\\ \text{the noisy measurements}\\ \text{given the true parameters}}}]\right]\left[\frac{\partial}{\partial \underline{x}} \ln[p(\tilde{\underline{y}} | \underline{x})]\right]^T\right\}$$

NOT a PDF
This is a likelihood func

For linear system

$$\tilde{\underline{y}} = H\underline{x} + \underline{v}$$

We can show that:

$$F = (H^T R^{-1} H)$$

Then the Cramer-Rao bound is:

$$P \geq F^{-1} \rightarrow P \geq \left(H^T R^{-1} H\right)^{-1}$$

Let us compare to P for LLS, MVE:

Recall our LLS & MVE eqns:

$$\tilde{\underline{y}} = H \underline{x} + \underline{v} \qquad (2.1)$$

$$\hat{\underline{x}} = \left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \tilde{\underline{y}} \qquad (2.29)$$

Plug 2.1 into 2.29:

$$\hat{\underline{x}} = \underbrace{\left(H^T R^{-1} H\right)^{-1} H^T R^{-1} H}_{= I} \underline{x} + \left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \underline{v}$$

$$\Rightarrow \quad \hat{\underline{x}} - \underline{x} = \left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \underline{v}$$

Covariance: $P = E\left\{ (\hat{\underline{x}} - \underline{x})(\hat{\underline{x}} - \underline{x})^T \right\}$

$$P = E\left\{\left[\left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \underline{v}\right]\left[\left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \underline{v}\right]^T\right\}$$

$$= E\left\{\left[\left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \underline{v}\right]\left[\underline{v}^T \left[\left(H^T R^{-1} H\right)^{-1} H^T R^{-1}\right]^T\right]\right\}$$

Since $E\{\underline{v}\,\underline{v}^T\} = R$:

$$= \left(H^T R^{-1} H\right)^{-1} H^T \overbrace{R^{-1} R}^{I} \left[\left(H^T R^{-1} H\right)^{-1} H^T R^{-1}\right]^T$$

$$= \left(H^T R^{-1} H\right)^{-1} \underbrace{H^T R^{-1} H \underbrace{\left(H^T R^{-1} H\right)^{-1^T}}_{\text{symmetric transpose does nothing}}}_{= I}$$

R is symmetric

$$R^{-1} = \left(R^{-1}\right)^T$$

$$P = \left(H^T R^{-1} H\right)^{-1}$$

equality means LLS/MVE are as good

as can be expected. = it is efficient

Practical Application:

If you know R, H, you can determine the lower bound on P.

So you can

(a) determine statistical bounds on estimation errors

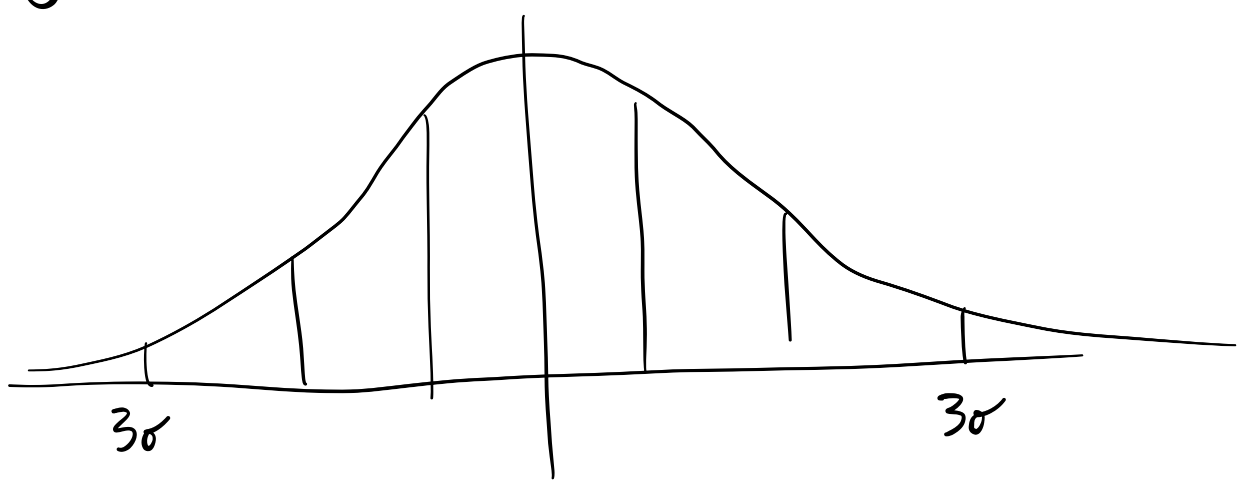
(b) determine if your estimator is "efficient"

A common approach is to look at the "3σ" bounds.

Recall

$$P = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix}$$

For gaussian noise:



99.7% of samples within 3σ bound.

So out of 1000 trials, in ~3 cases should your estimates be off by more than the 3σ bound.

Excercise: verify with example 1