Observability Gramian - linear case

"Gram" matrix is essentially a matrix with the form $A^T A$. Properties:

1. symmetric (and thus square)
2. positive (semi) definite

will be pos-def if full rank.

Let's start by thinking about the output energy of a system:

$$E_0 = \frac{1}{2}\int_0^{\tau} y^T(t)\, y(t)\, dt$$

output energy

$$y(t) = Cx(t) + Du(t)$$

$$x_{k+1} = Ax_k + Bu_k$$

$$x_{k+2} = A\cdot\left[Ax_k + Bu_k\right] + Bu_{k+1}$$

$$= A^2 x_k + ABu_k + Bu_{k+1}$$

$$y(t) = Ce^{At}x_0$$

we ignore controls in the linear case, reconsider them for nonlinear.

$$E_0 = \int_0^{\tau} \left[Ce^{At}x_0\right]^T \left[Ce^{At}x_0\right] dt$$

$$= \int_0^{\tau} x_0^T e^{At^T} C^T C\, e^{At} x_0\, dt$$

$$= x_0^T \underbrace{\left\{\int_0^{\tau} e^{At^T} C^T C\, e^{At}\, dt\right\}}_{\text{Call this } W} x_0$$

Note: it symmetric & positive (semi) definite

if full rank, it is pos-def.

What does this look like?

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & & \vdots \\ \vdots & & \ddots & \\ w_{n1} & \cdots & & w_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} w_{11} x_1 + w_{12} x_2 + \cdots + w_{1n} x_n \\ w_{21} x_1 + w_{22} x_2 + \cdots + w_{2n} x_n \\ \vdots \\ w_{n1} x_1 + w_{n2} x_2 + \cdots + w_{nn} x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

$$= \left[x_1 r_1 + x_2 r_2 + \cdots + x_n r_n \right]$$

Another way to write it:

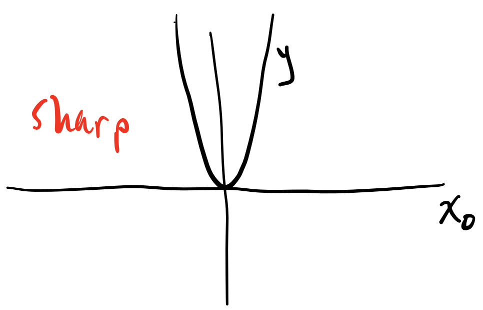
$$\text{sum of all elements} \left\{ \begin{bmatrix} x_1 w_{11} x_1 & x_1 w_{12} x_2 & \cdots & x_1 w_{1n} x_n \\ x_2 w_{21} x_1 & x_2 w_{22} x_2 & \cdots & x_2 w_{2n} x_n \\ \vdots & \vdots & & \vdots \\ x_n w_{n1} x_1 & x_n w_{n2} x_2 & \cdots & x_n w_{nn} x_n \end{bmatrix} \right\}$$

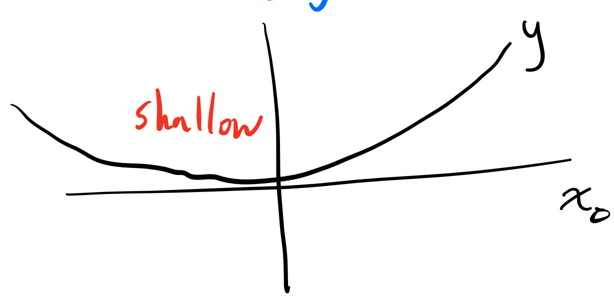
Now let's see how this energy changes with small changes in the initial state.

We are not interested in the direction of the changes in energy, instead, we want to know how quickly those changes will accelerate.

Alternatively, we want to know the curvature of the energy landscape.

How sharp or shallow is the energy?





So we will take the Hessian:

$$\text{Hessian} = \nabla^2 = \frac{\partial^2 f}{\partial x_i \partial x_j} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \frac{\partial^2 f}{\partial x_2\,\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n\,\partial x_1} & \frac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}.$$

$$\frac{\partial^2 E_0}{\partial x_i \partial x_j} = 2 \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{bmatrix}$$

$$\frac{\partial^2}{\partial x_1 \partial x_1} (x_1 w_{11} x_1) = 2 w_{11}$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} (x_1 w_{12} x_2 + x_2 w_{21} x_1) = w_{12} + w_{21}$$

Since W is symmetric, $w_{12} + w_{21} = 2w_{12}$
$= 2w_{21}$

We can get rid of the "2" if we go back and define $E_0 = \frac{1}{2}\int_0^{\tau} y^T y \, dt$

What is W?

Discrete case:

$$E_0 = x_0^T \underbrace{\left[\sum_{k=0}^{\tau} A^{k^T} C^T C A^k \right]}_{W_{0,d}} x_0$$

$$W_{0,d} = C^T C + A^T C^T C A + A^T A^T C^T C A A + \cdots + A^{\tau^T} C^T C A^{\tau}$$

$$= \left[C^T + A^T C^T + A^T A^T C^T + \cdots A^{\tau^T} C^T \right] \cdot \begin{bmatrix} C \\ CA \\ CAA \\ \vdots \\ CA^{\tau} \end{bmatrix}$$

$$= \mathcal{O}^T \mathcal{O}, \quad \text{where} \quad \mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{\tau} \end{bmatrix}$$

Continuous case is similar:

$$W_{o,c} = \mathcal{O}_c^T \mathcal{O}_c, \quad \text{where} \quad \mathcal{O}_c = \begin{bmatrix} C_c \\ C_c A_c \\ C_c A_c^2 \\ \vdots \\ C_c A_c^n \end{bmatrix}$$

So $W_o = \mathcal{O}^T \mathcal{O}$ represents the sensitivity of the output (measurements) to changes in the initial state (assuming controls remain unchanged).