SciencesPo
DEPARTMENT OF ECONOMICS

ScPoEconometrics

Simple Linear Regression

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SciencesPo Paris
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Quick "Quiz" on Last Week's Material

1. From your ***computer*** 👉 connect to ***www.wooclap.com/SCPOTIDY2***

   OR

2. From your ***phone*** 👉 flash QR code below

Today - Real 'metrics finally ✌️

- Introduction to the ***Simple Linear Regression Model*** and ***Ordinary Least Squares (OLS)*** *estimation*.
- Empirical application: *class size* and *student performance*
- Keep in mind that we are interested in uncovering **causal** relationships

Class size and student performance

- What policies *lead* to improved student learning?
- Class size reduction has been at the heart of policy debates for *decades*.

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- What policies *lead* to improved student learning?
- Class size reduction has been at the heart of policy debates for *decades*.
- We will be using data from a famous paper by Joshua Angrist and Victor Lavy (1999), obtained from Raj Chetty and Greg Bruich's course.
- Consists of test scores and class/school characteristics for fifth graders (10-11 years old) in Jewish public elementary schools in Israel in 1991.
- National tests measured *mathematics* and (Hebrew) *reading* skills. The raw scores were scaled from 1-100.

Task 1: Getting to know the data

07:00

1. Load the data from here as `grades`. You need to find the function that enables importing *.dta* files. (FYI: *.dta* is the extension for data files used in *Stata*)

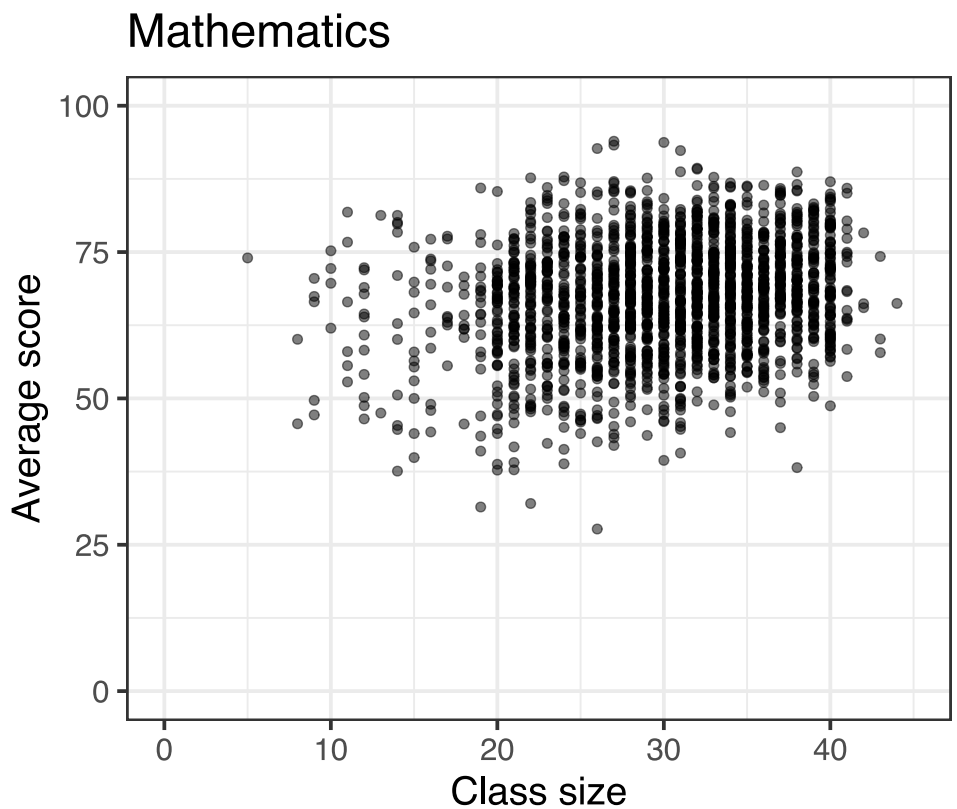
2. Describe the dataset:

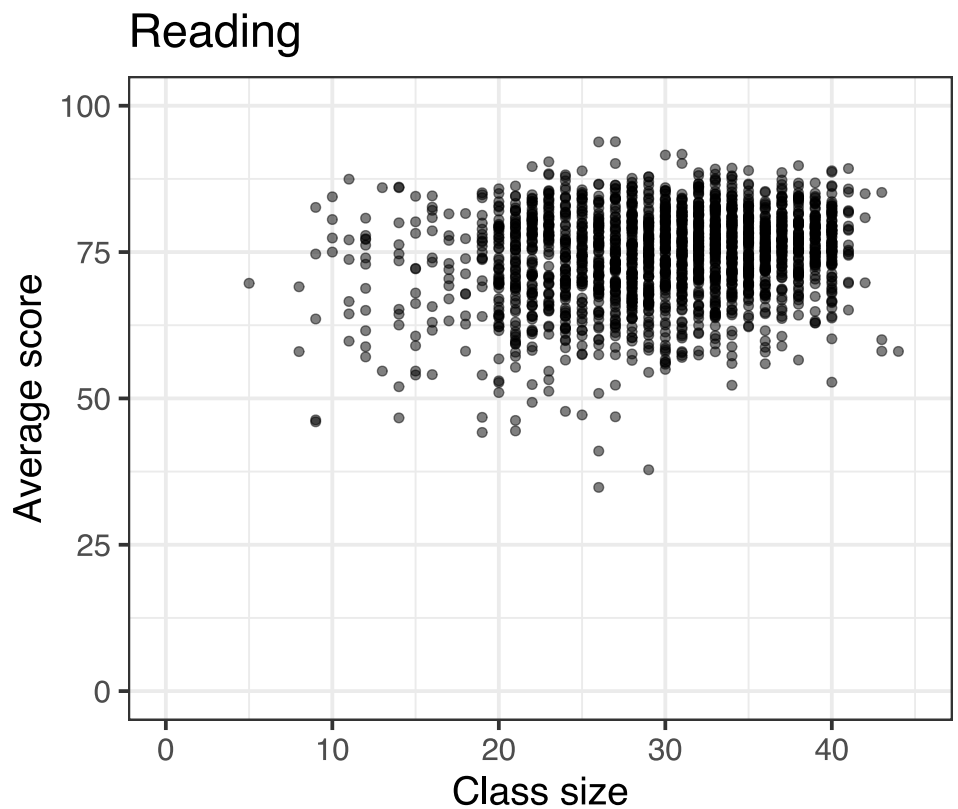
    - What is the unit of observations, i.e. what does each row correspond to?
    - How many observations are there?
    - What variables do we have? View the dataset to see what the variables correspond to.
    - What do the variables `avgmath` and `avgverb` correspond to?
    - Use the `skim` function from the `skimr` package to obtain common summary statistics for the variables `classize`, `avgmath` and `avgverb`. (*Hint: use `dplyr` to `select` the variables and then simply pipe (`%>%`) `skim()`.*)

3. Do you have any priors about the actual (linear) relationship between class size and student achievement? What would you do to get a first insight?

4. Compute the correlation between class size and math and verbal scores. Is the relationship positive/negative, strong/weak?

Class size and student performance: Scatter plot





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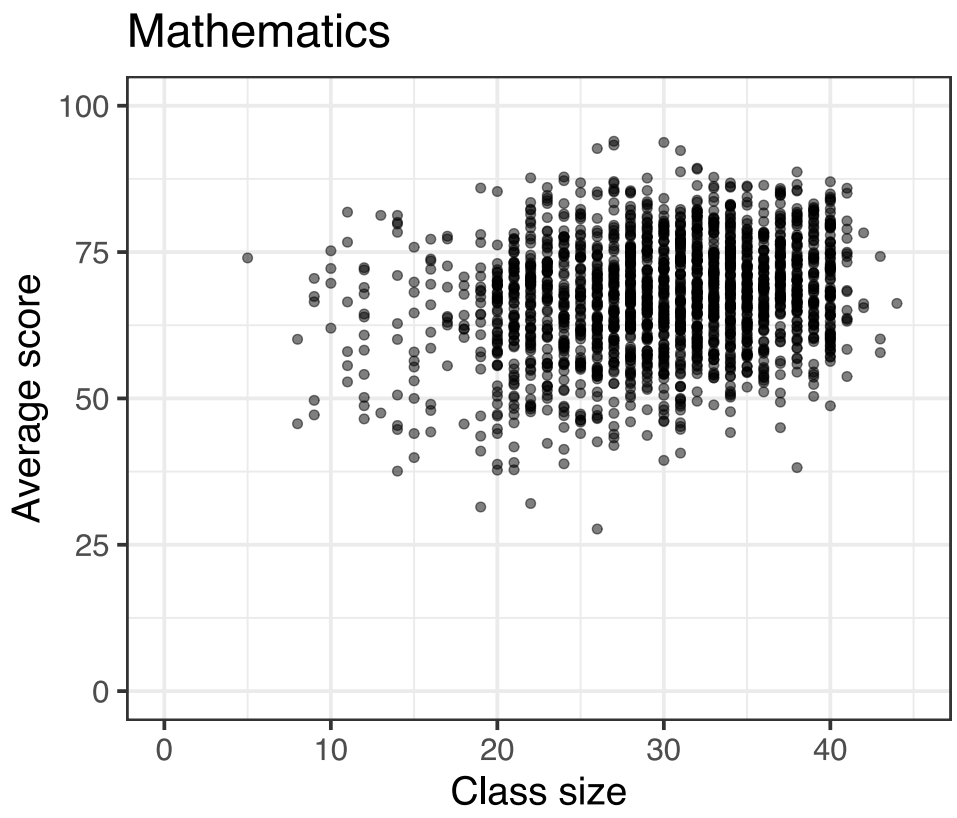
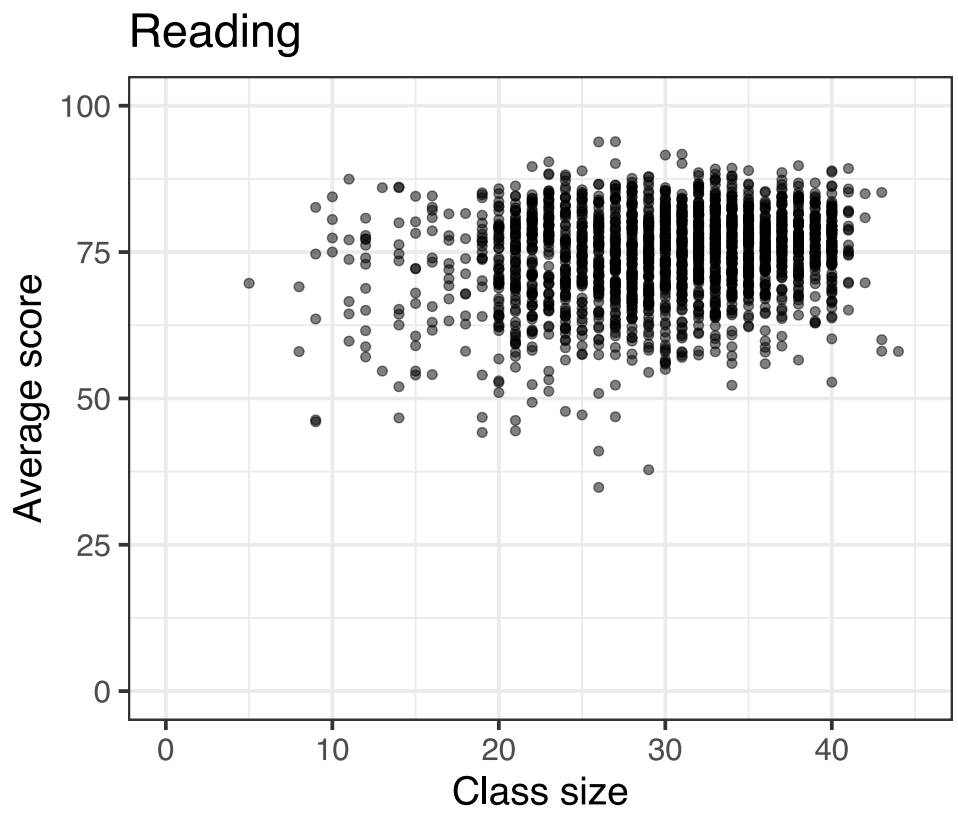
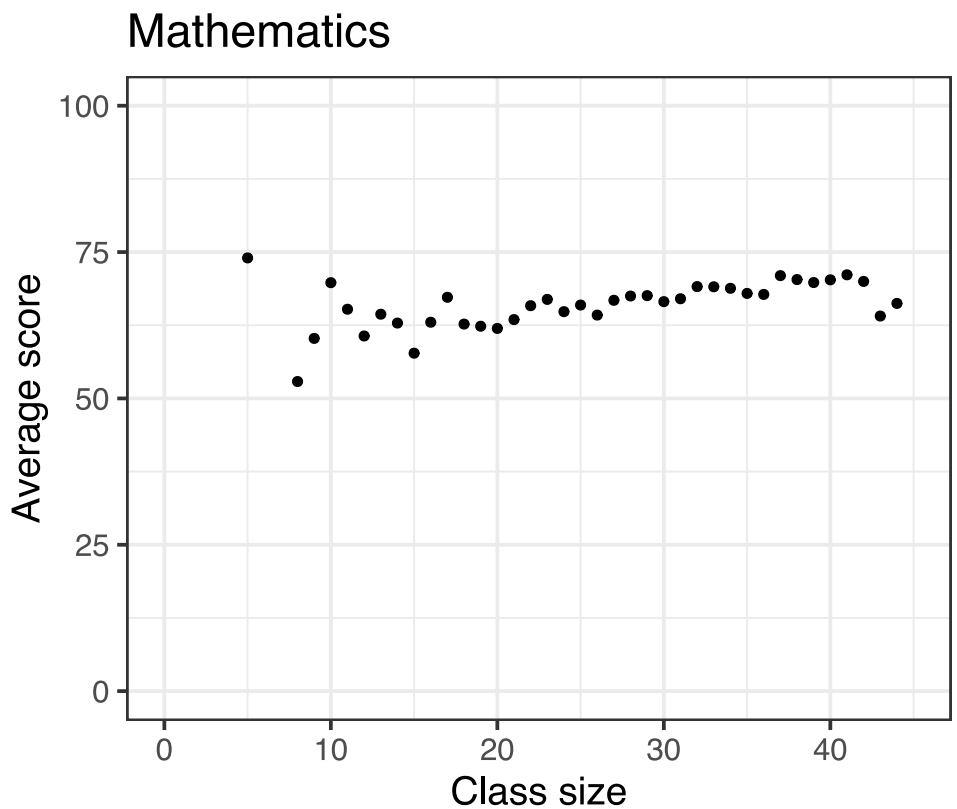
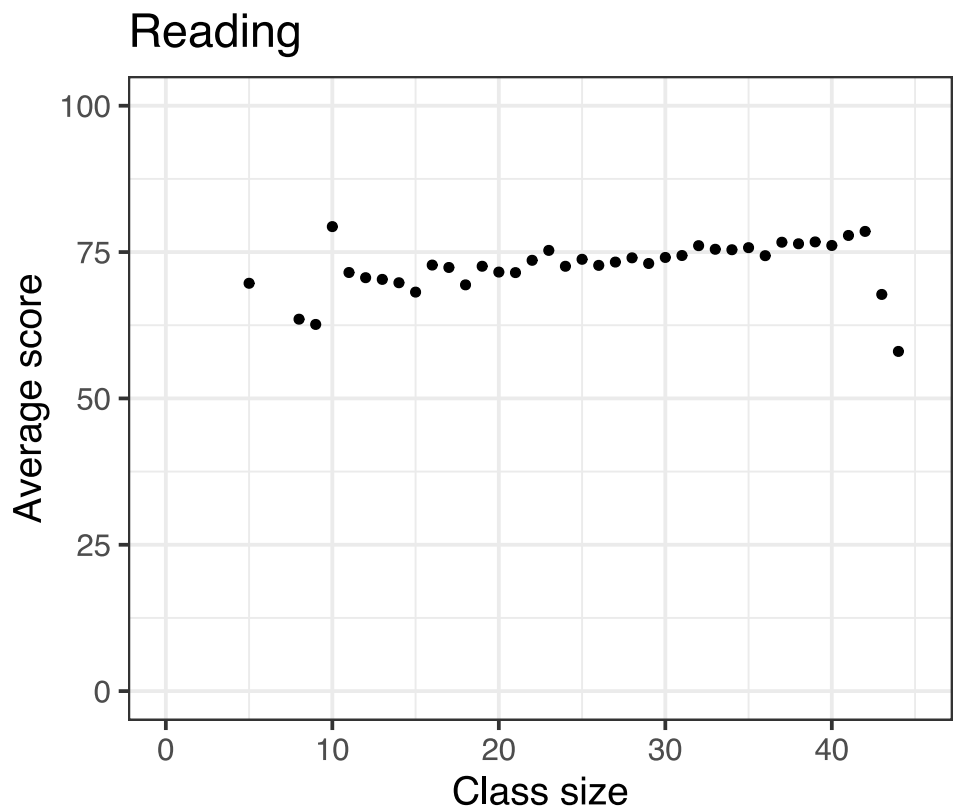




- Somewhat positive association as suggested by the correlations. Let's compute the average score by class size to see things more clearly!

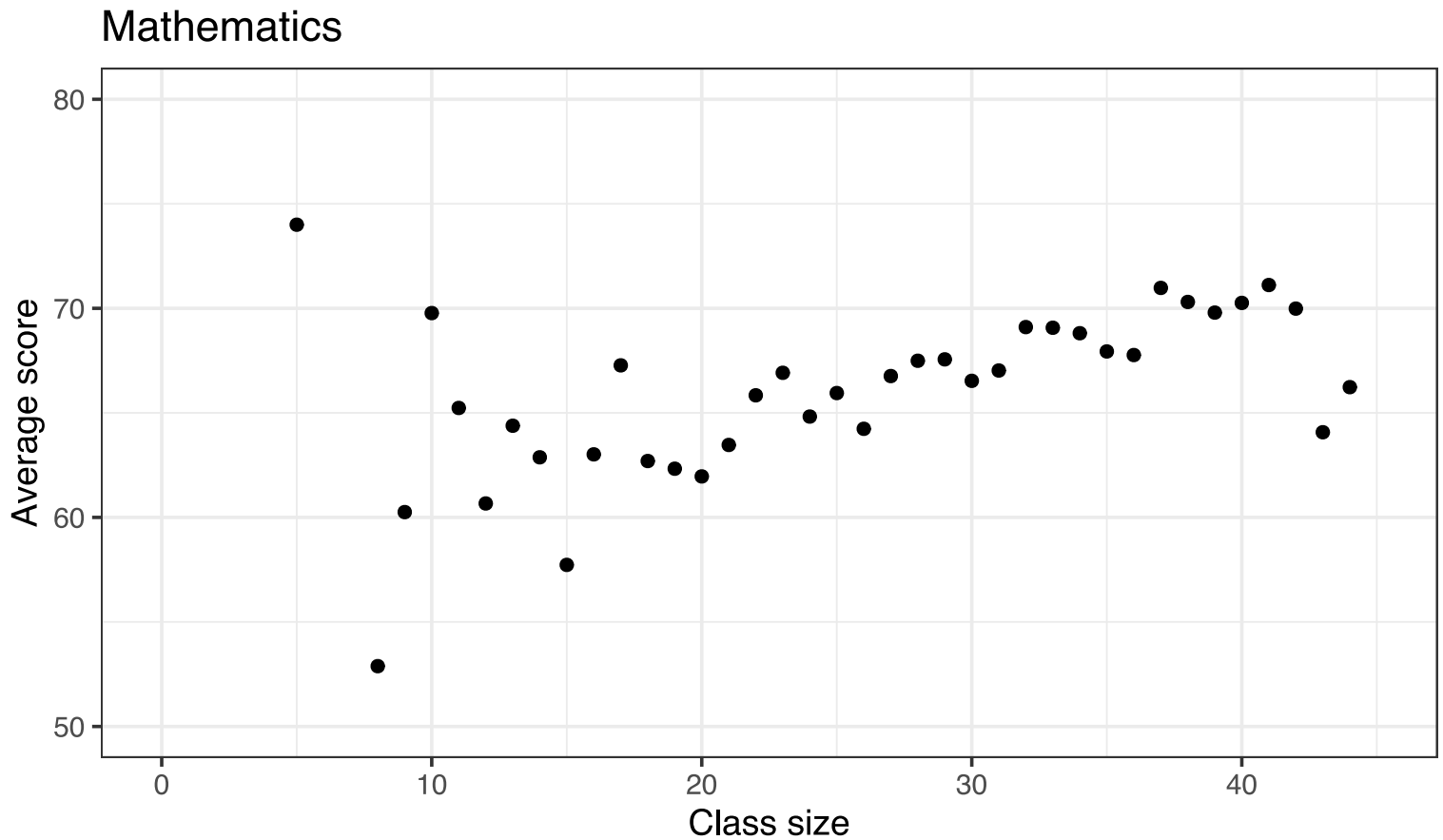
Class size and student performance: Binned scatter plot





Class size and student performance: Binned scatter plot

- We'll first focus on the mathematics scores and for visual simplicity we'll zoom in



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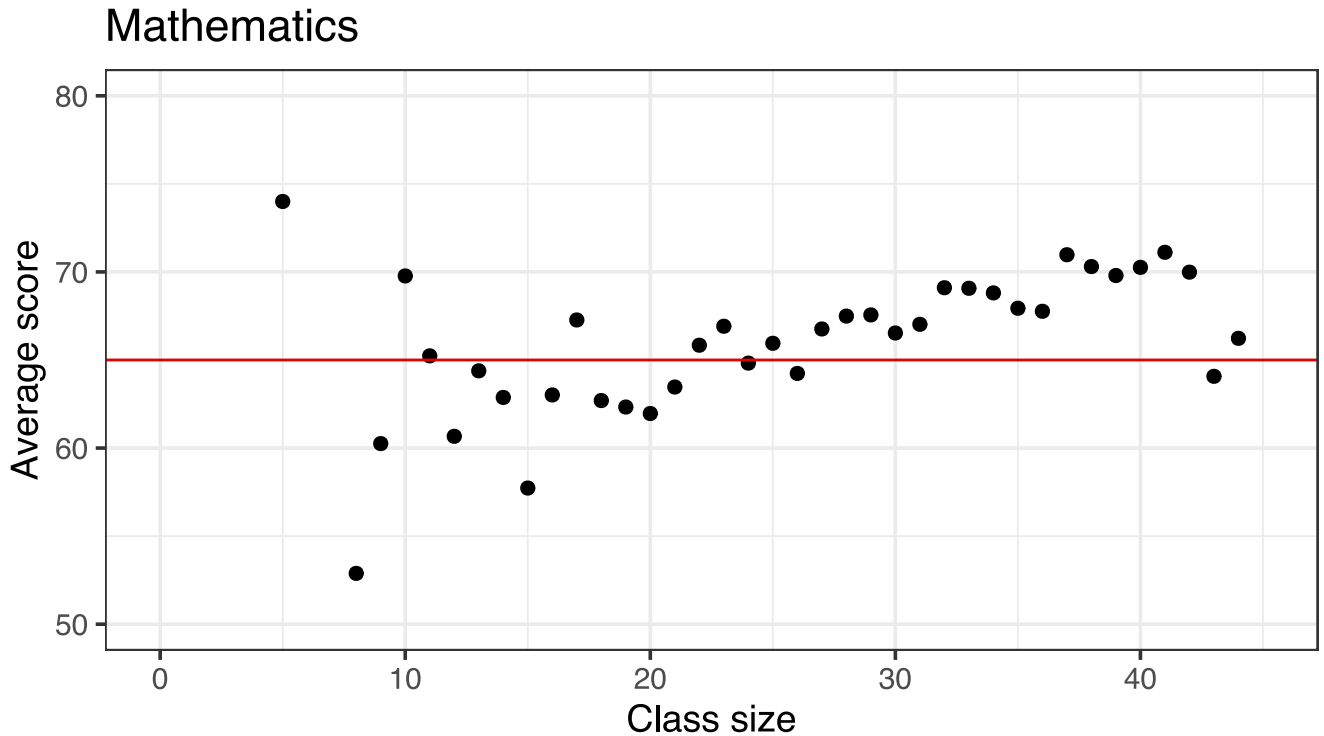

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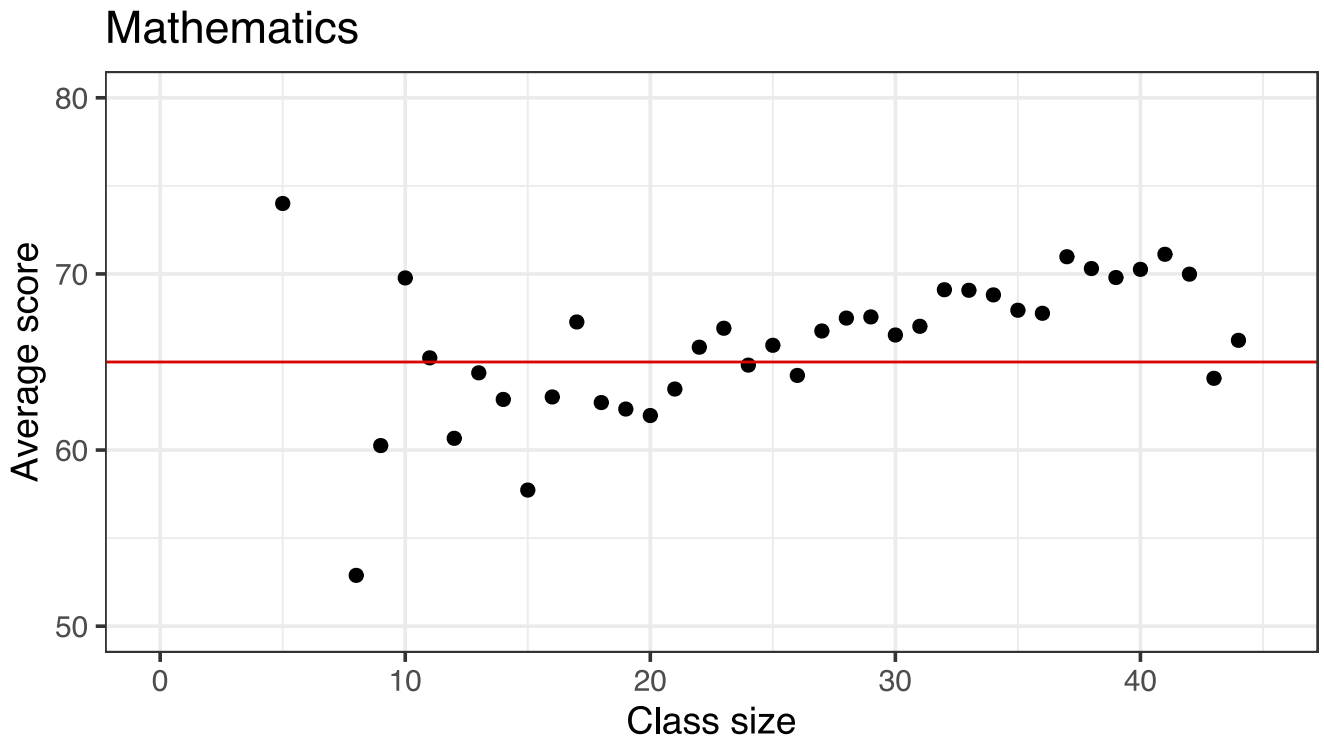

- A *line*! Great. But **which** line? This one?
- That's a *flat* line. But average mathematics score is somewhat *increasing* with class size 😩

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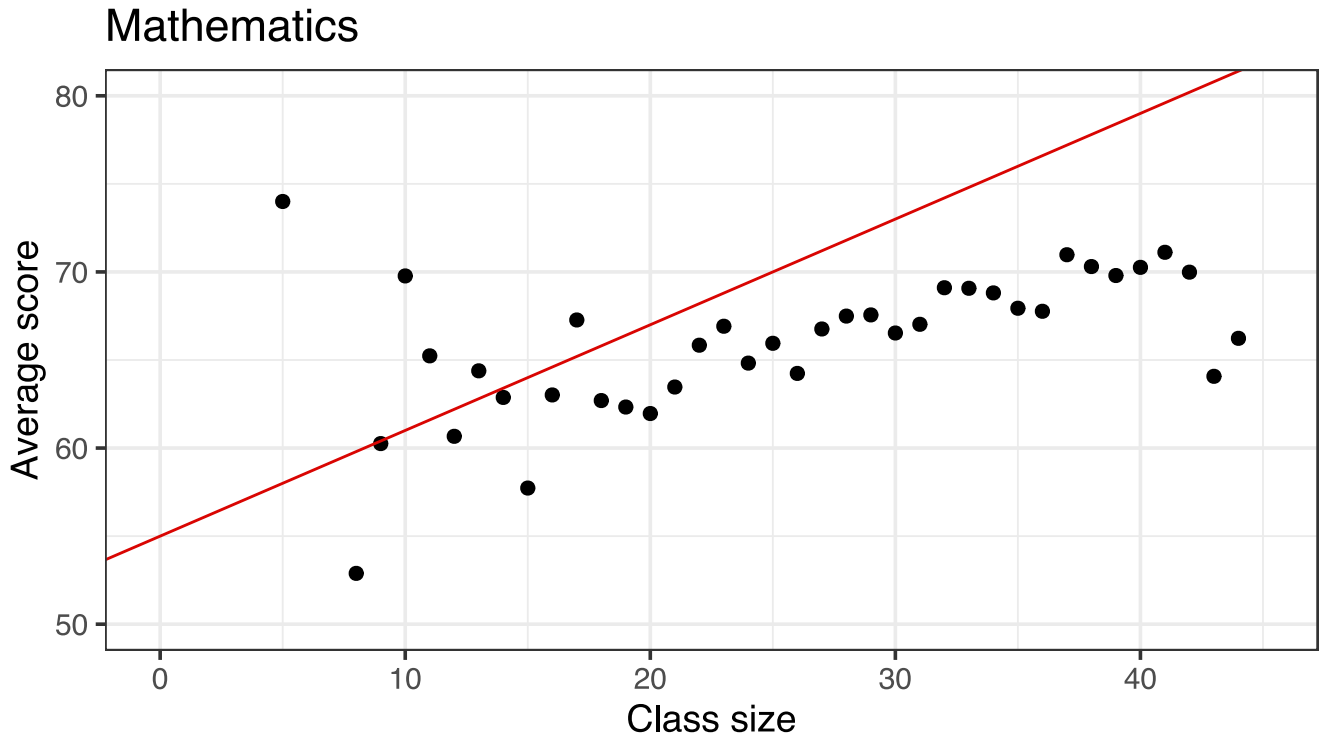

- **That** one?
- Slightly better! Has a **slope** and an **intercept** 😐
- We need a rule to decide!

Simple Linear Regression

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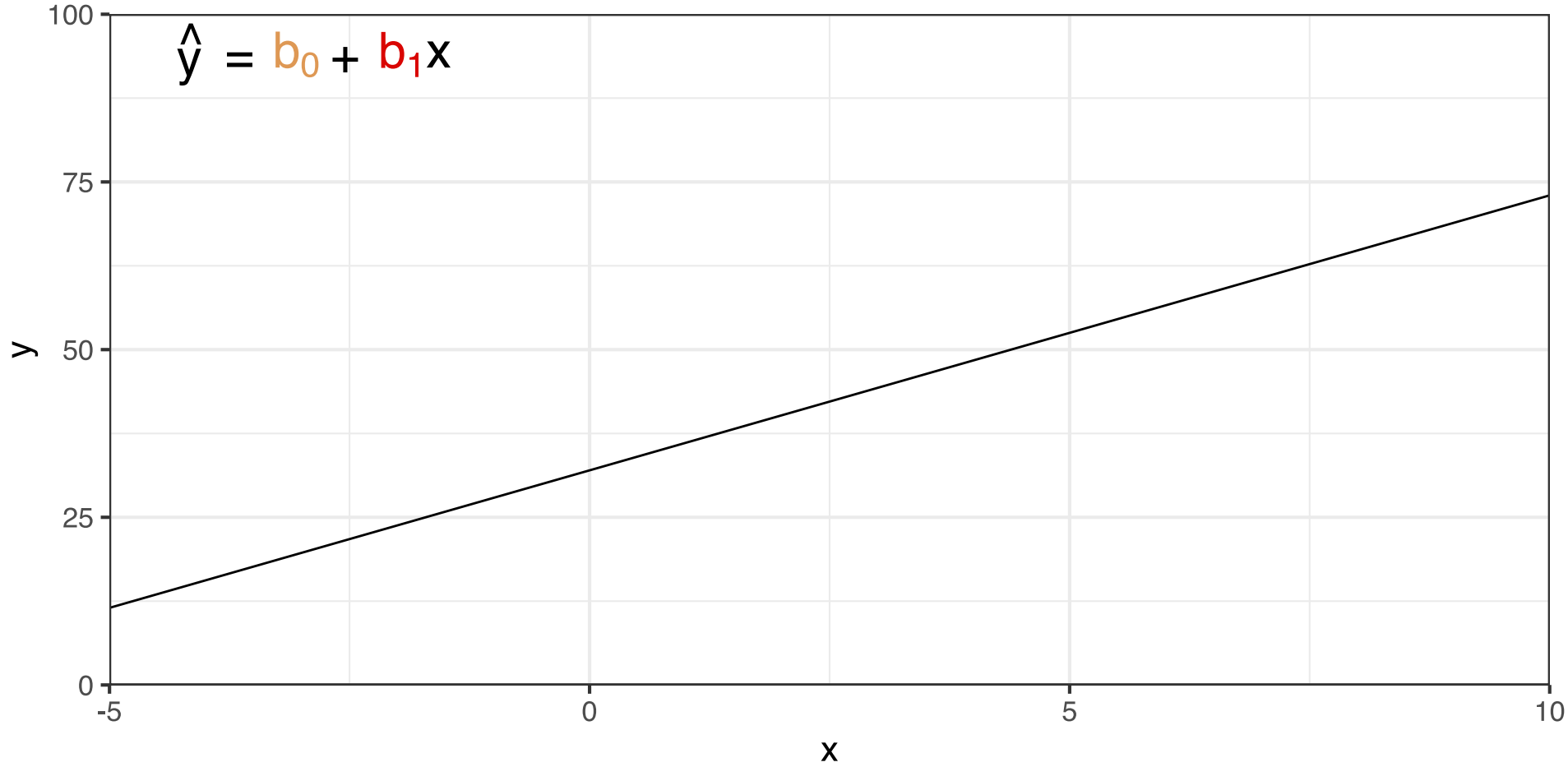
- $\hat{y}_i$ is our *prediction* for y at observation i (y_i) given our model (i.e. the line).

What's A Line: A Refresher



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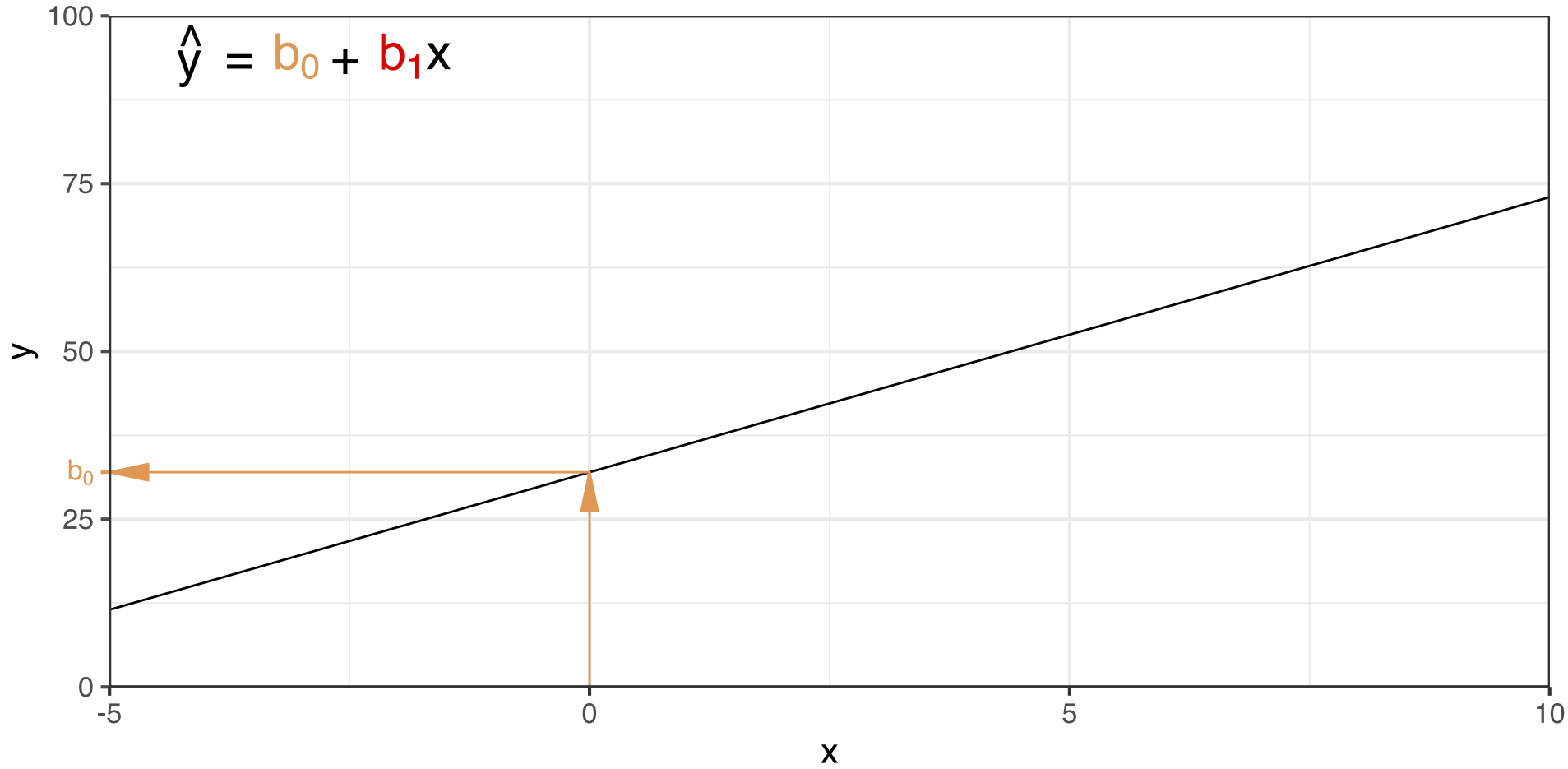

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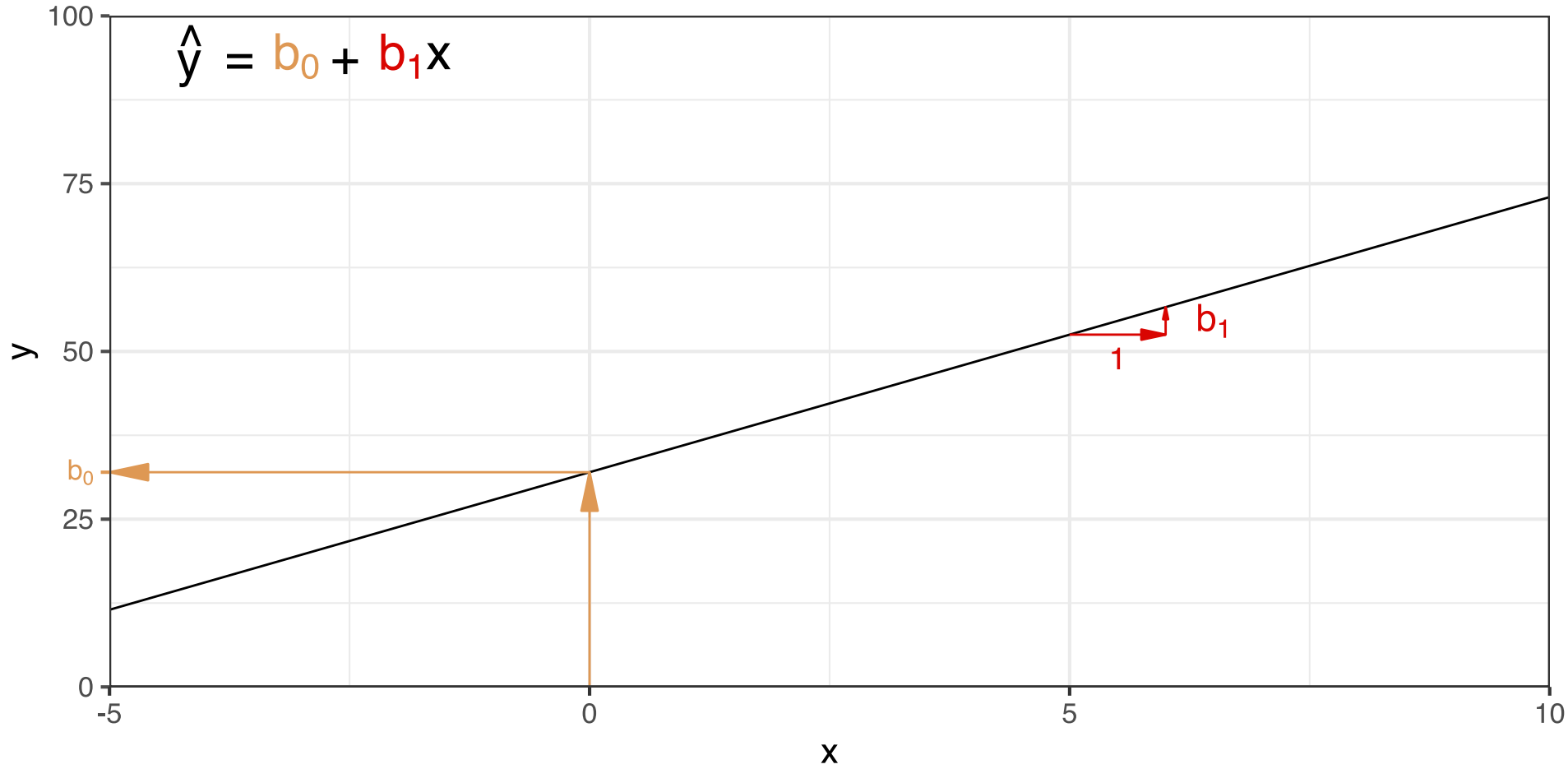

Simple Linear Regression: Residual

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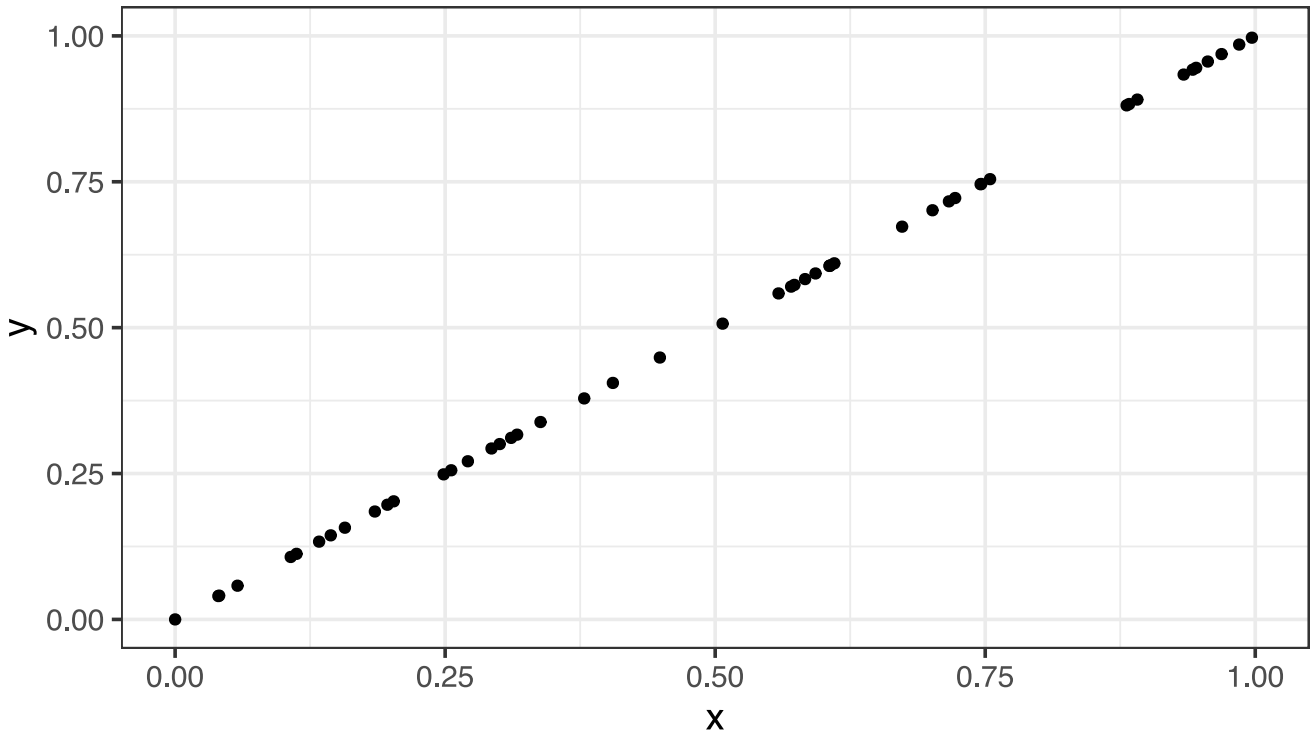

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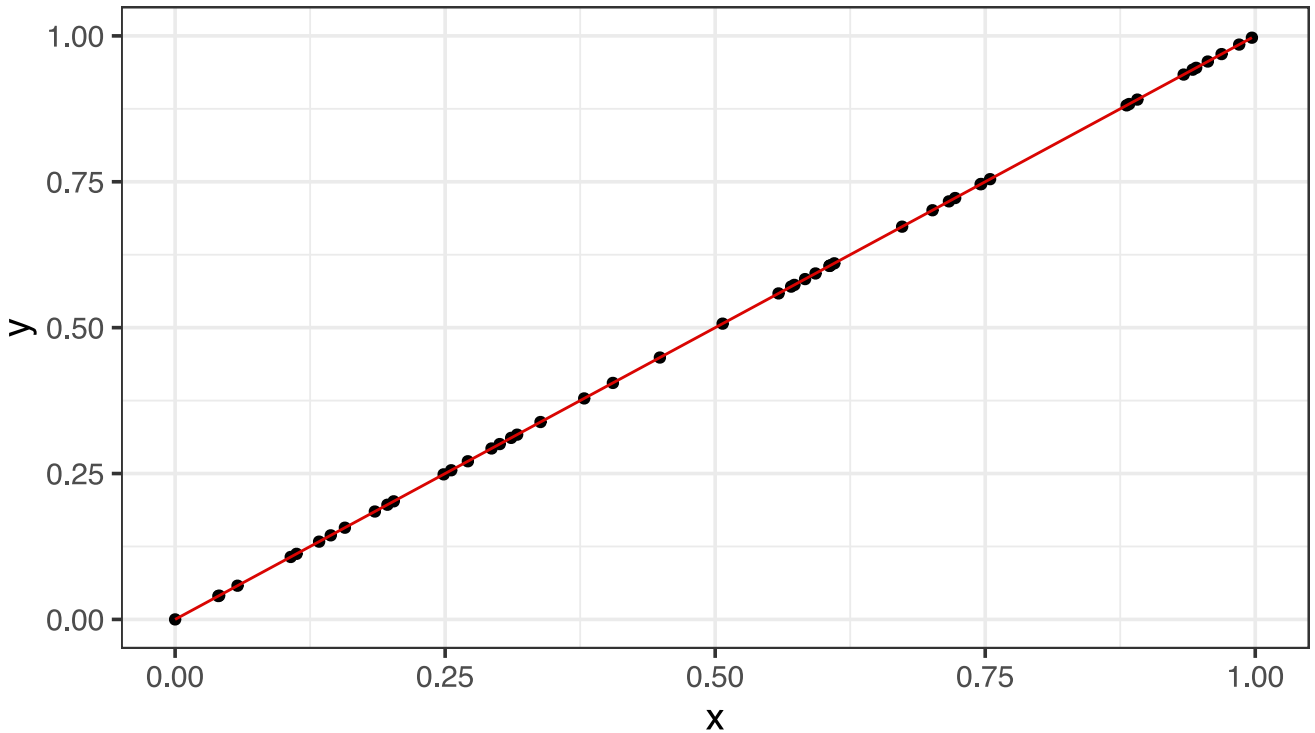

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- If all the data points were **on** the line then $\hat{y}_i = y_i$.
- However, since in most cases the *dependent variable* (y) is not *only* explained by the chosen *independent variable* (x), $\hat{y}_i \neq y_i$, i.e. we make an **error**.
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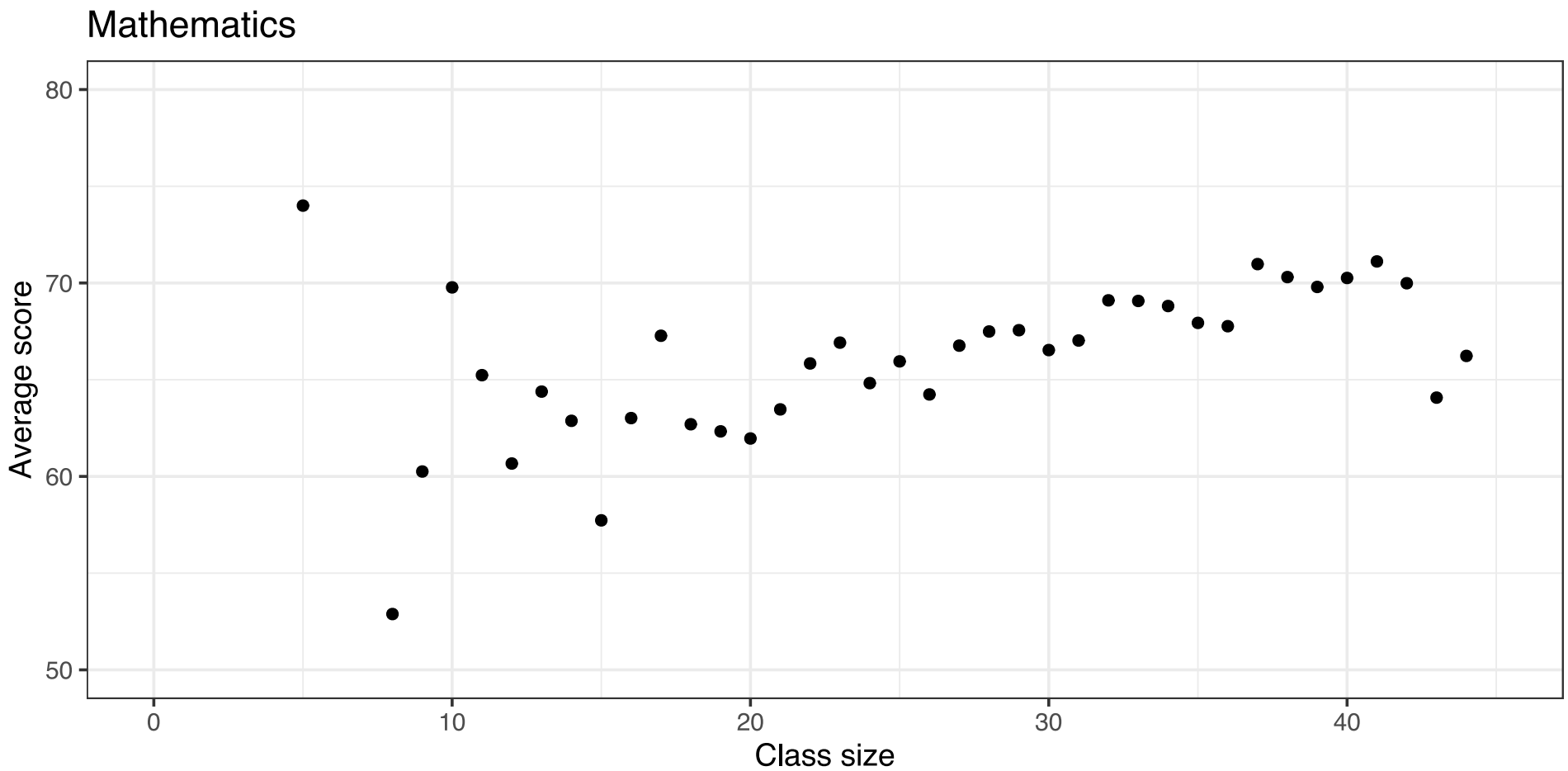
  This **error** is called the **residual**.
- At point (x_i, y_i), we note this residual e_i.
- The *actual data* (x_i, y_i) can thus be written as *prediction* + *residual*:

$$y_i = \hat{y}_i + e_i = b_0 + b_1 x_i + e_i$$

Simple Linear Regression: Graphically



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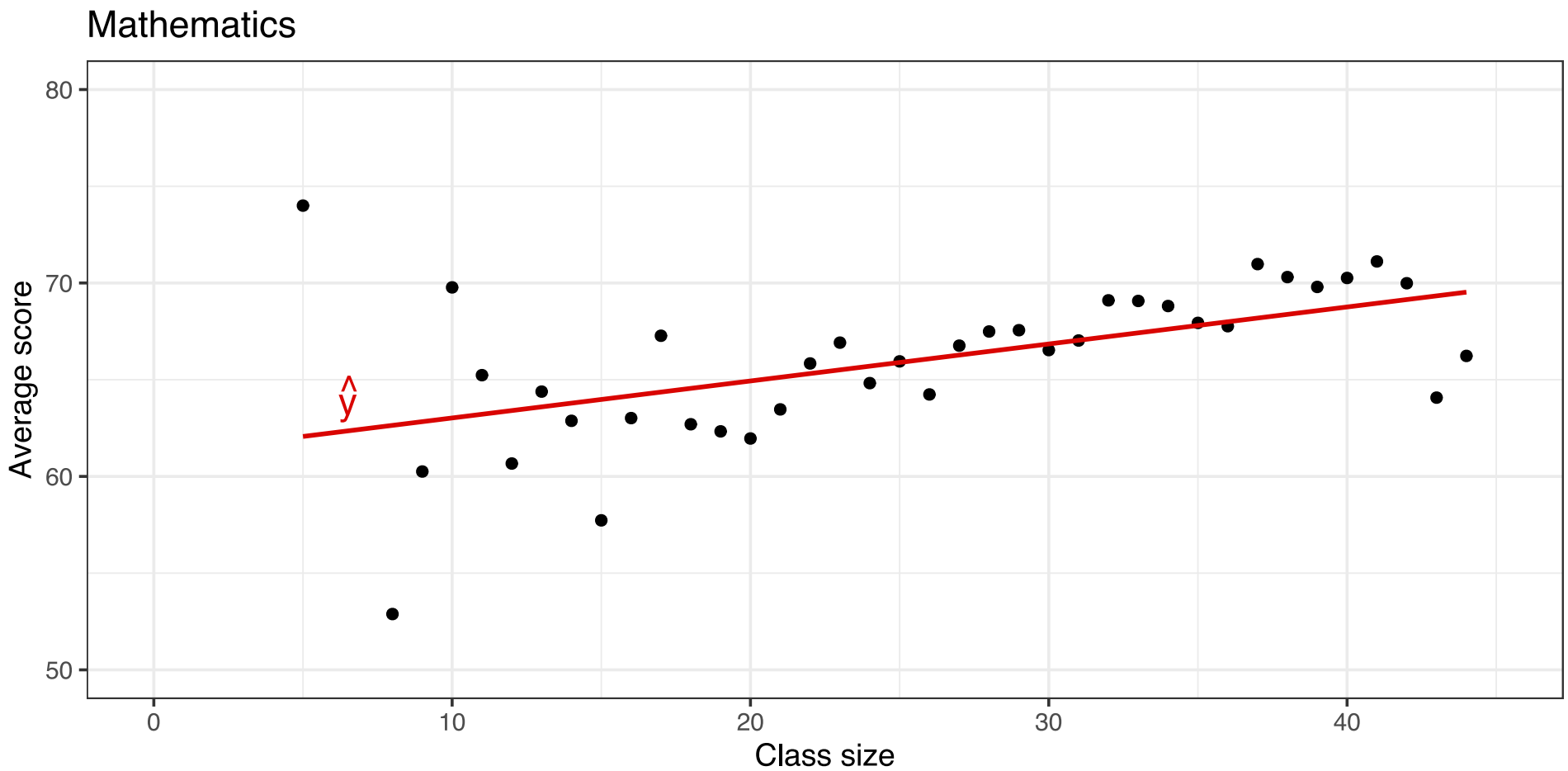

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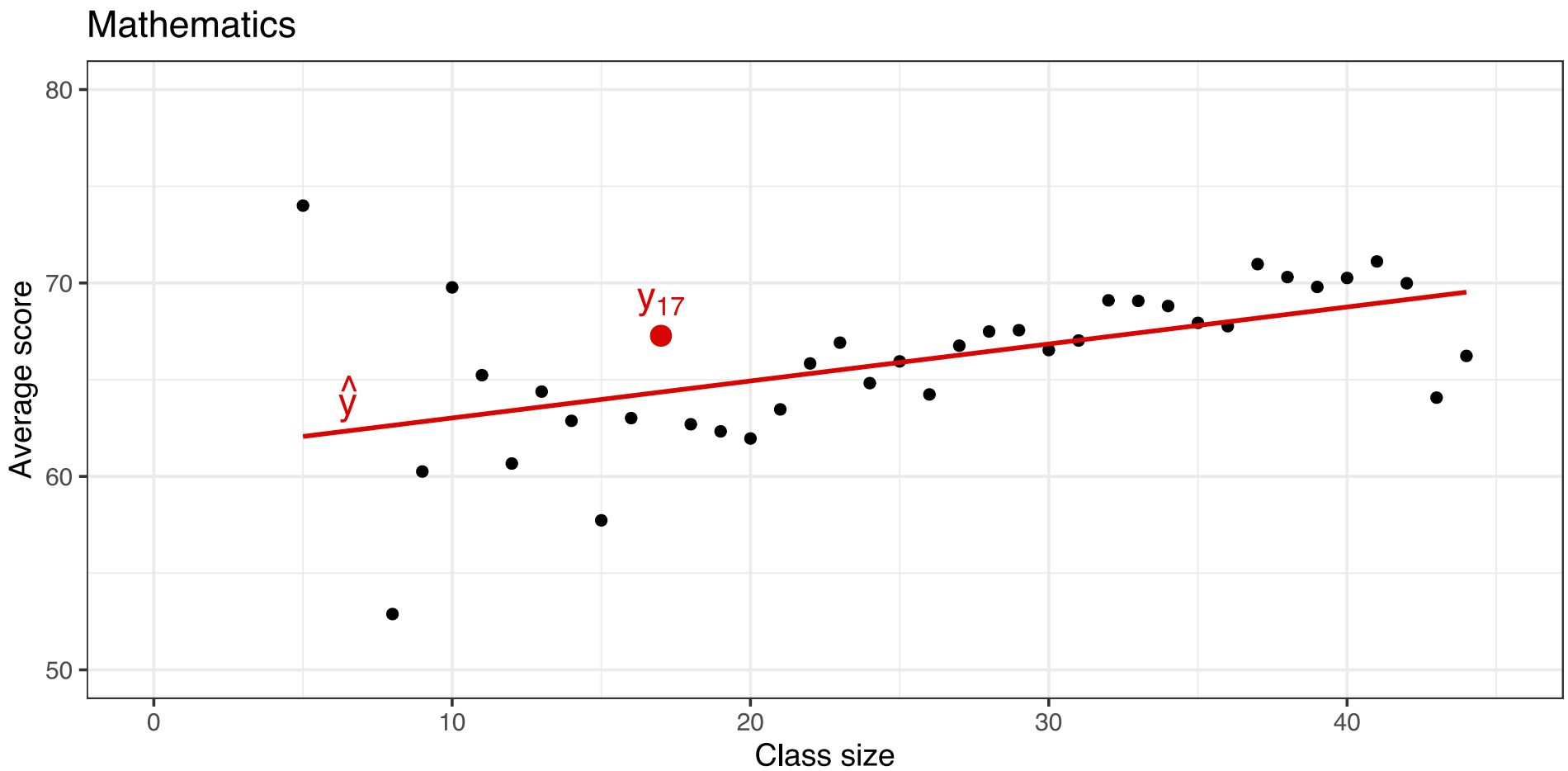

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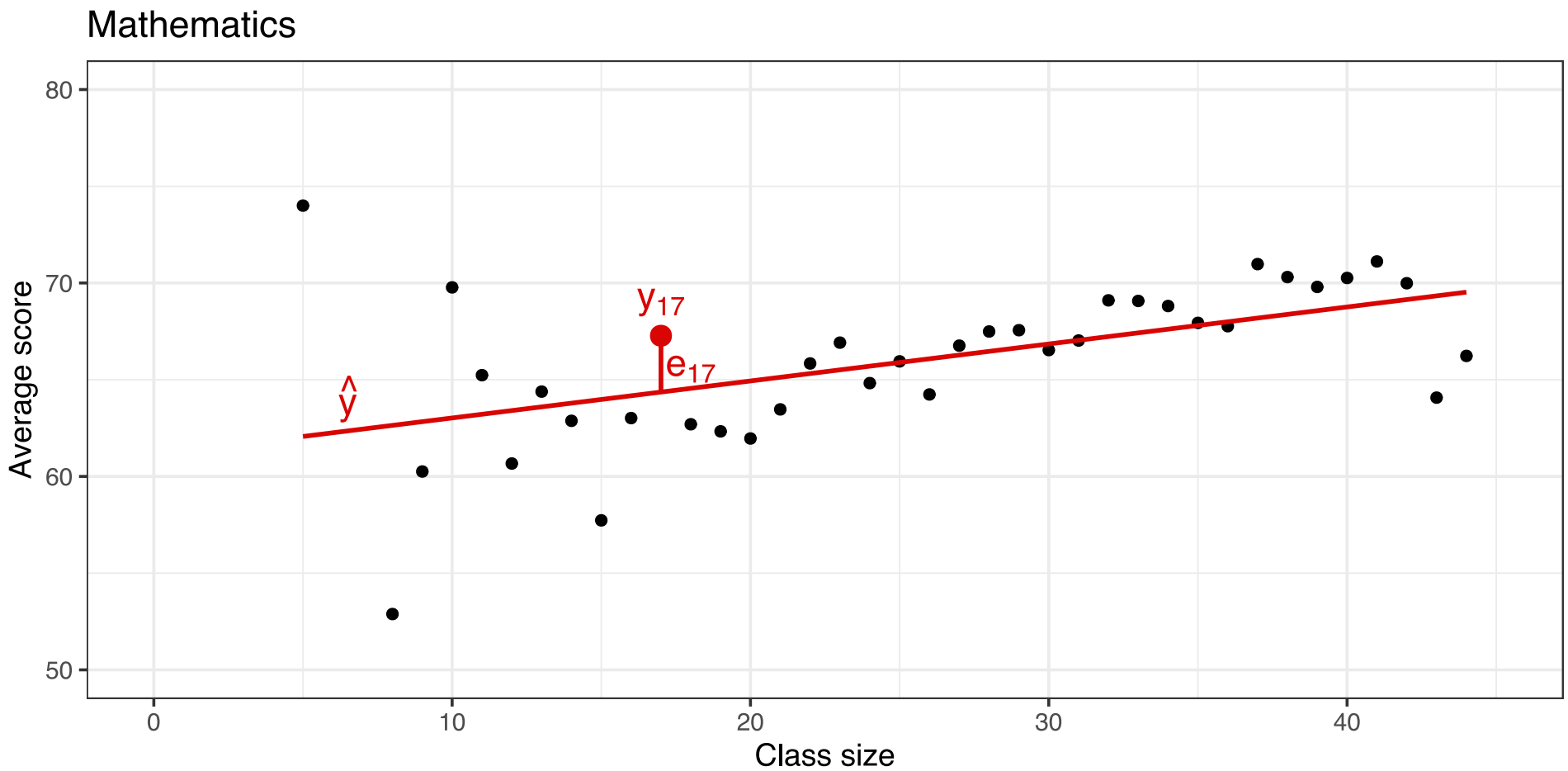

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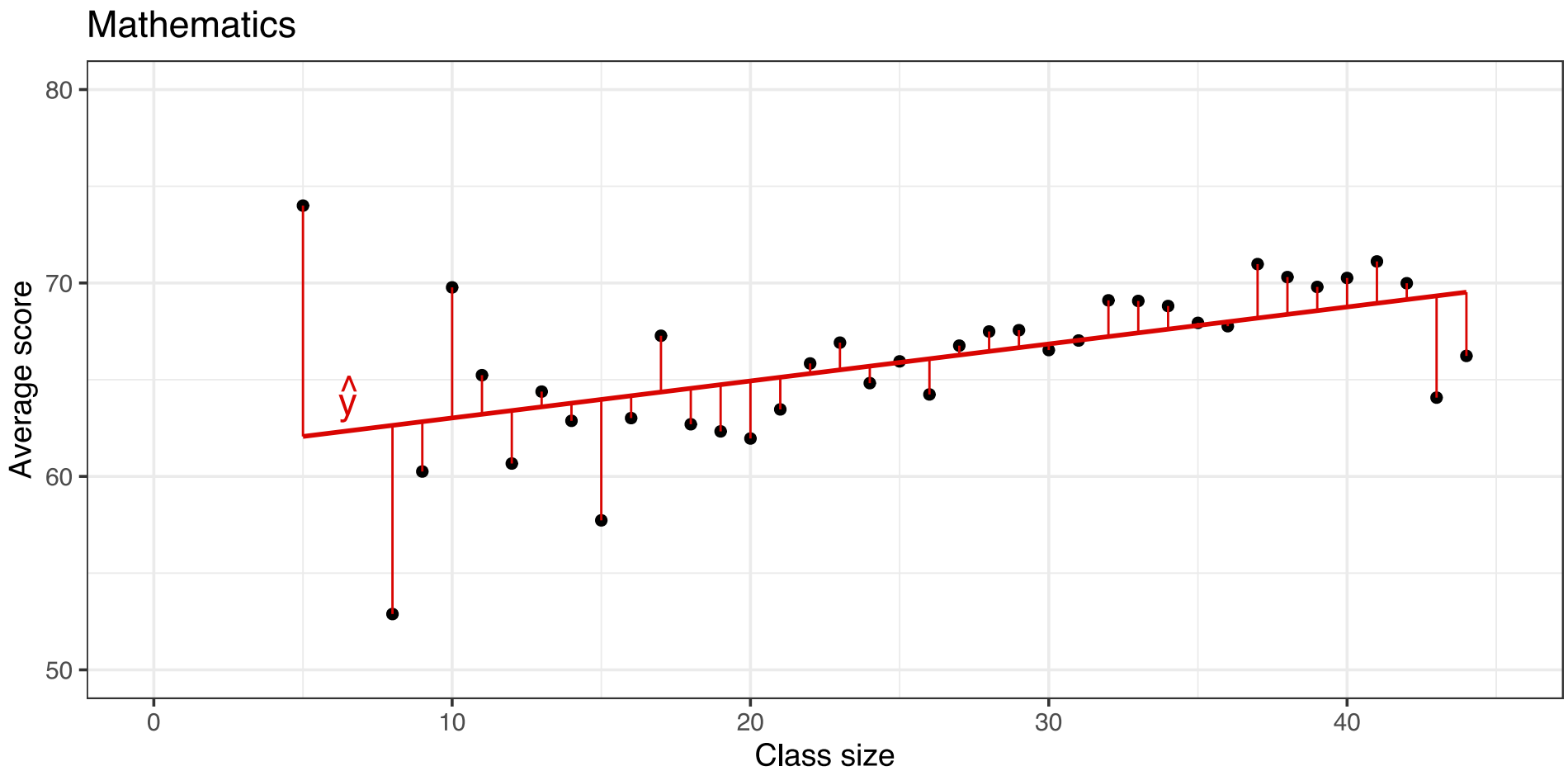

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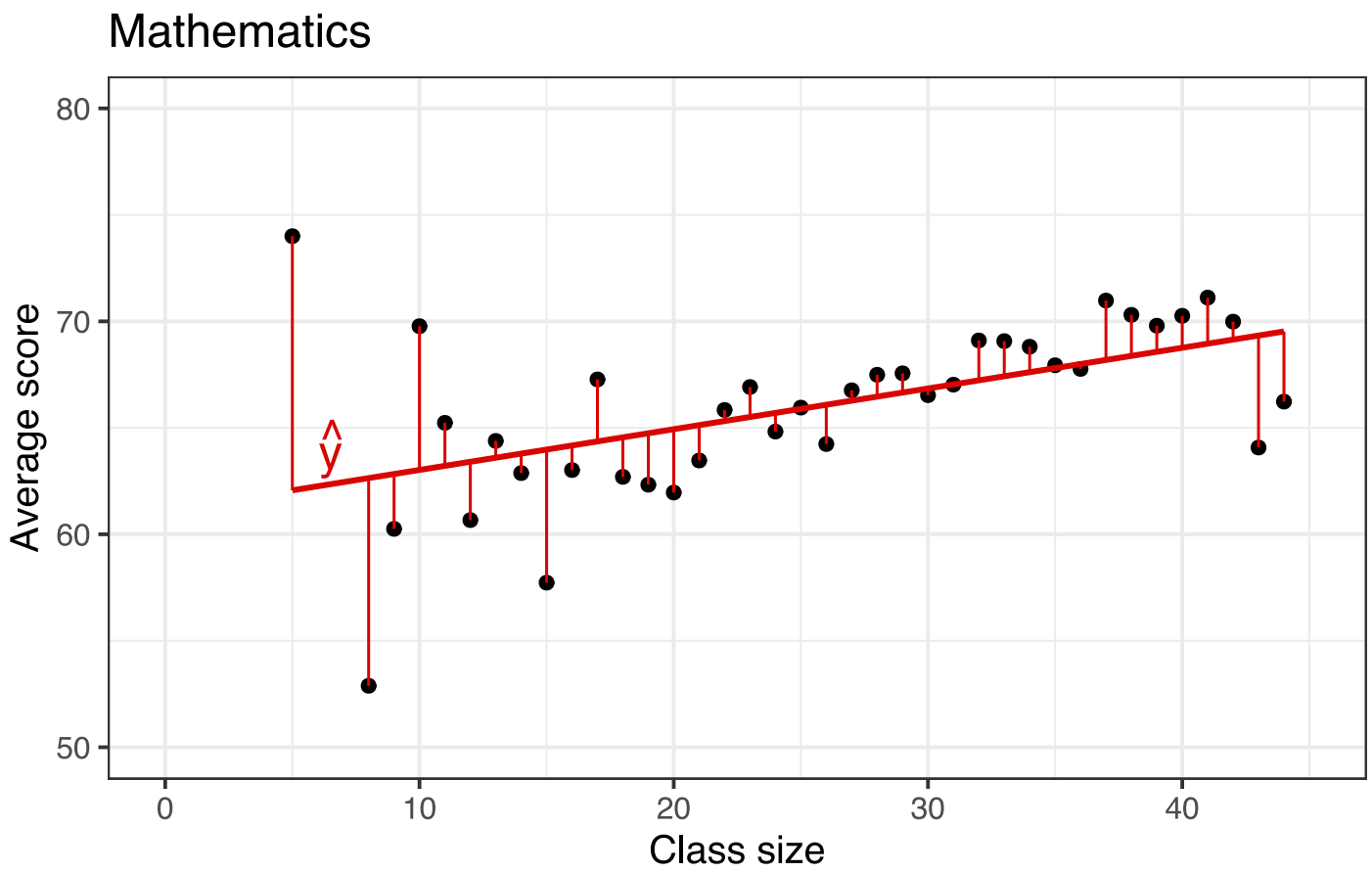

Which "minimisation" criterion should (can) be used?

Ordinary Least Squares (OLS) Estimation

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- Choose (b_0, b_1) such that $\sum_{i=1}^{N} e_1^2 + \cdots + e_N^2$ is **as small as possible**.

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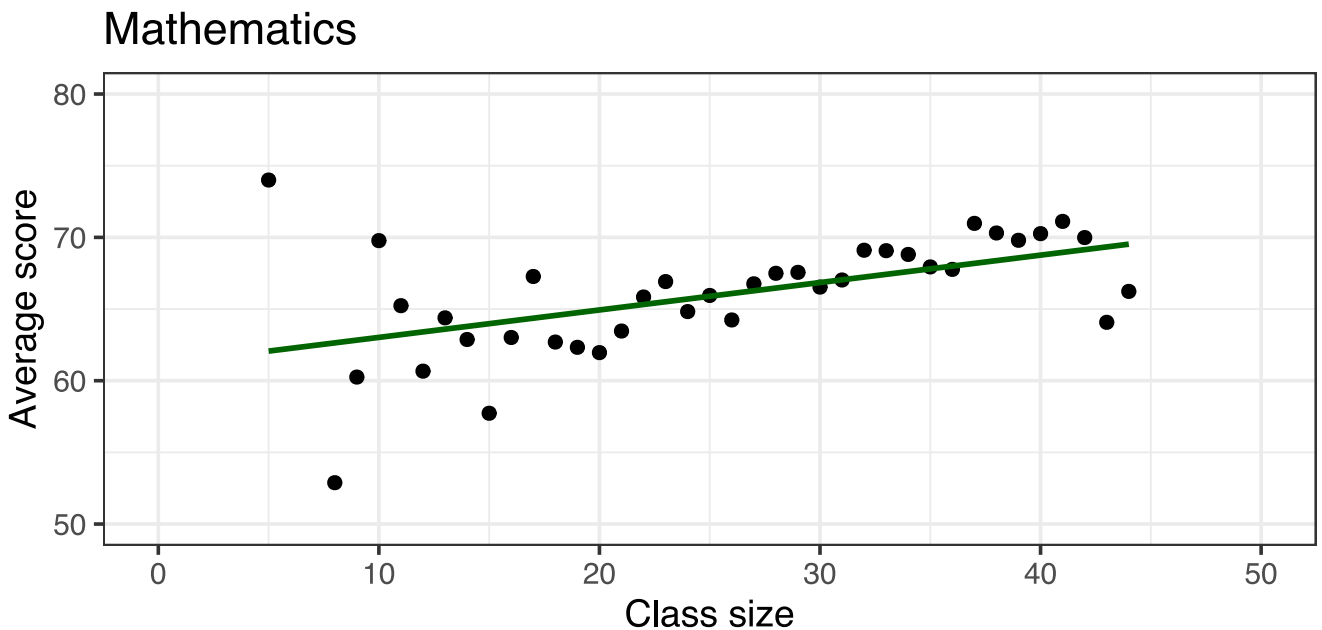

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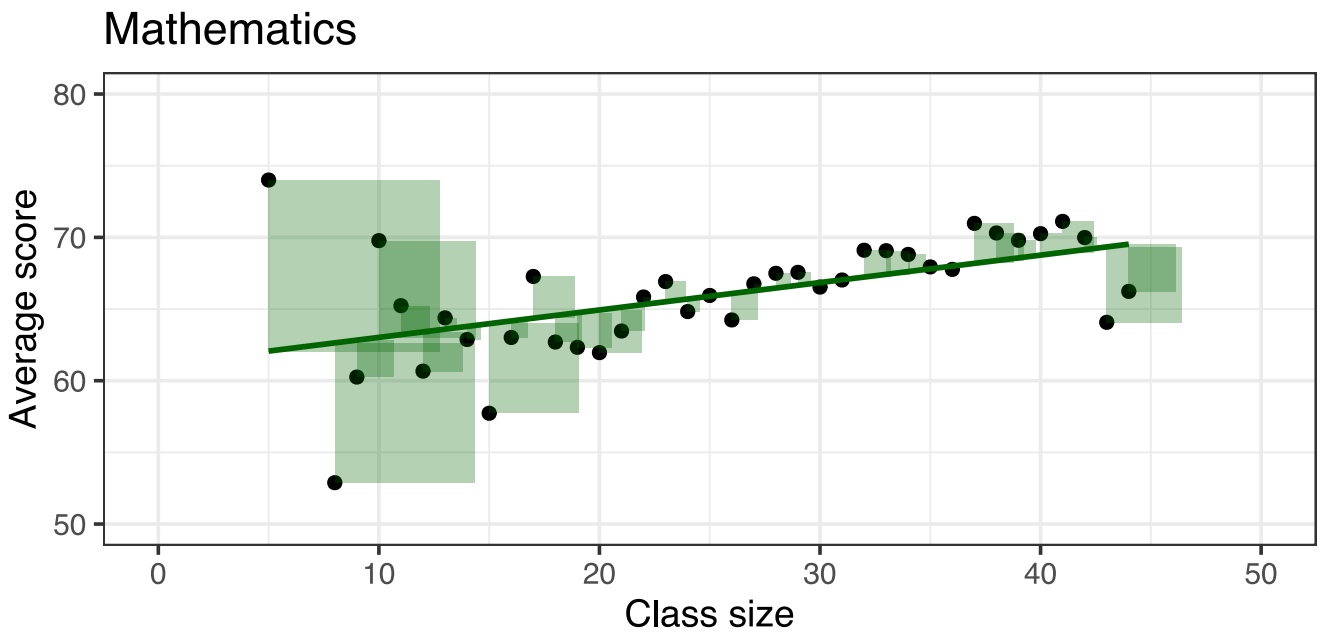

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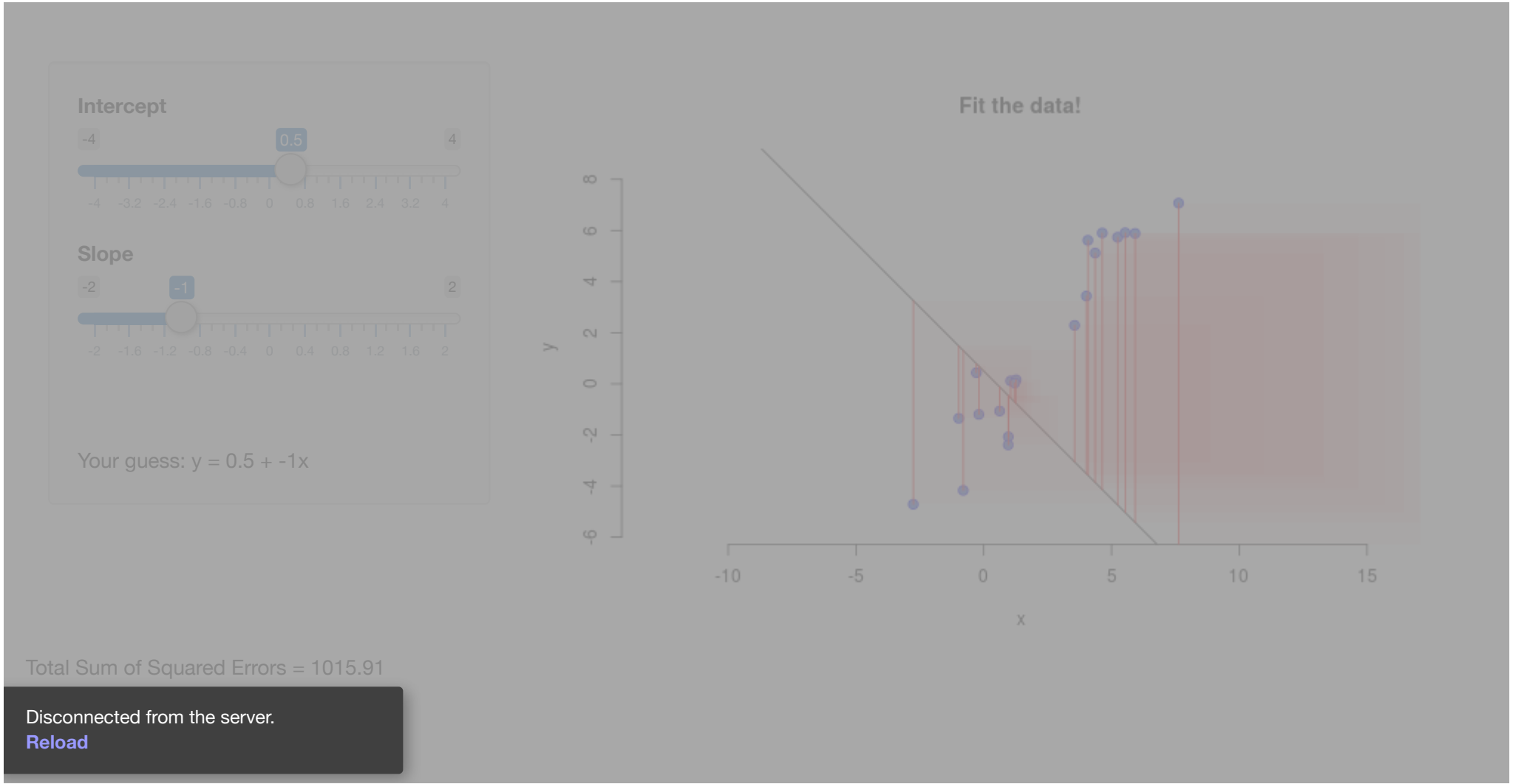

Ordinary Least Squares (OLS) Estimation

Ordinary Least Squares (OLS): Coefficient Formulas

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- ! You should know these formulas, especially the one for b_1^{OLS} !
- These formulas do not appear from magic. They can be found by solving the minimisation of squared errors. The maths can be found here for those who are interested.

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For now assume both the dependent variable (y) and the independent variable (x) are numeric.

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- **You need to be explicit about what the unit of x is!**

OLS with R

- In R, OLS regressions are estimated using the lm function.
- This is how it works:

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lm(formula = dependent variable ~  independent variable, data = data.frame containing the data)
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Class size and student performance

Let's estimate the following model by OLS: $\text{average math score}_i = b_0 + b_1 \text{class size}_i + e_i$

```
# OLS regression of class size on average maths score
lm(avgmath_cs ~ classize, grades_avg_cs)
```

```
##
## Call:
## lm(formula = avgmath_cs ~ classize, data = grades_avg_cs)
##
## Coefficients:
## (Intercept)     classize
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This implies (abstracting the $_i$ subscript for simplicity):

$$\hat{y} = b_0 + b_1 x$$

$$\widehat{\text{average math}}\ \text{score} = b_0 + b_1 \cdot \text{class size}$$

$$\widehat{\text{average math}}\ \text{score} = 61.11 + 0.19 \cdot \text{class size}$$

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##
## Call:
## lm(formula = avgmath_cs ~ classize, data = grades_avg_cs)
##
## Coefficients:
## (Intercept)     classize
##     61.1092       0.1913
```

This implies (abstracting the $_i$ subscript for simplicity):

$$\hat{y} = b_0 + b_1 x$$

$$\widehat{\text{average math score}} = b_0 + b_1 \cdot \text{class size}$$

$$\widehat{\text{average math score}} = 61.11 + 0.19 \cdot \text{class size}$$

What's the predicted average score for a class of 26 students? (Using the *exact* coefficients.)

$$\widehat{\text{average math score}} = 61.11 + 0.19 \cdot 26$$

$$\widehat{\text{average math score}} = 66.08$$

Task 2: OLS Regression

10:00

Run the following code to aggregate the data at the class size level:

```
grades_avg_cs <- grades %>%
  group_by(classize) %>%
  summarise(avgmath_cs = mean(avgmath),
            avgverb_cs = mean(avgverb))
```

1. Compute the OLS coefficients b_0 and b_1 of the previous regression using the formulas on slide 25. (*Hint:* you need to use the `cov`, `var`, and `mean` functions.)

2. Regress average verbal score (dependent variable) on class size (independant variable). Interpret the coefficients.

3. Is the slope coefficient similar to the one found for average math score? Was it expected based on the graphical evidence?

4. What is the predicted average verbal score when class size is equal to 0? (Does that even make sense?!)

5. What is the predicted average verbal score when the class size is equal to 30 students?

OLS Variations / Restrictions

- All are described in the book. Optional 🤓
- There is an app for each of them:

type	App
No Intercept, No regressors	`launchApp('reg_constrained')`
Centered Regression	`launchApp('demeaned_reg')`
Standardized Regression	`launchApp('reg_standardized')`

Predictions and Residuals: Properties

- **The average of $\hat{y}_i$ is equal to $\bar{y}$.**

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}\hat{y}_i &= \frac{1}{N}\sum_{i=1}^{N} b_0 + b_1 x_i \\ &= b_0 + b_1\bar{x} = \bar{y}\end{aligned}$$

- **The average (or sum) of residuals is 0.**

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N} e_i &= \frac{1}{N}\sum_{i=1}^{N}(y_i - \hat{y}_i) \\ &= \bar{y} - \frac{1}{N}\sum_{i=1}^{N}\hat{y}_i \\ &= 0\end{aligned}$$

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- **Regressor and residuals are uncorrelated (by definition).**

$$Cov(x_i, e_i) = 0$$

- **Prediction and residuals are uncorrelated.**

$$\begin{aligned}Cov(\hat{y}_i, e_i) &= Cov(b_0 + b_1 x_i, e_i) \\ &= b_1 Cov(x_i, e_i) \\ &= 0\end{aligned}$$

Since $Cov(a + bx, y) = bCov(x, y)$.

Linearity Assumption: Visualize your Data!

- It's important to keep in mind that covariance, correlation and simple OLS regression only measure **linear relationships** between two variables.
- Two datasets with *identical* correlations and regression lines could look *vastly* different.

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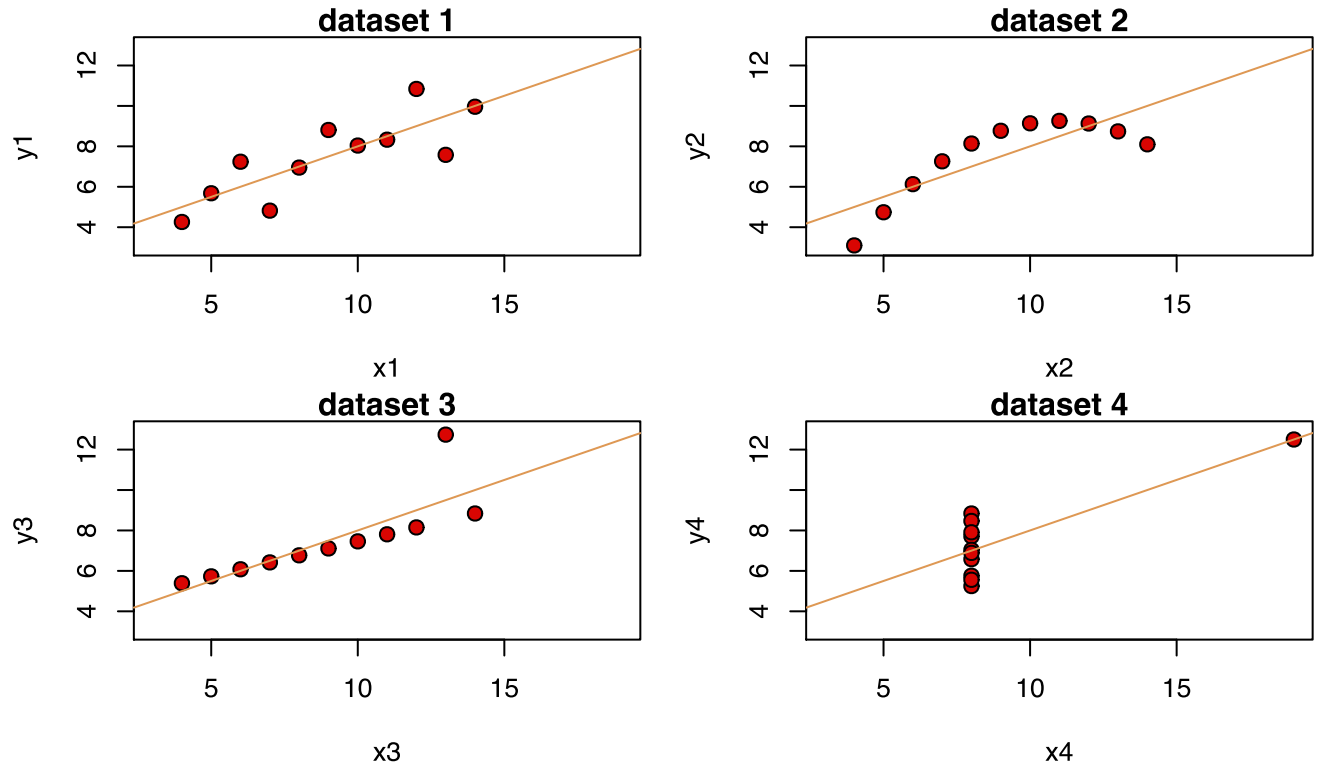
- Two datasets with *identical* correlations and regression lines could look *vastly* different.
- Is that even possible?

Linearity Assumption: Anscombe

- Francis Anscombe (1973) came up with 4 datasets with identical stats. But look!



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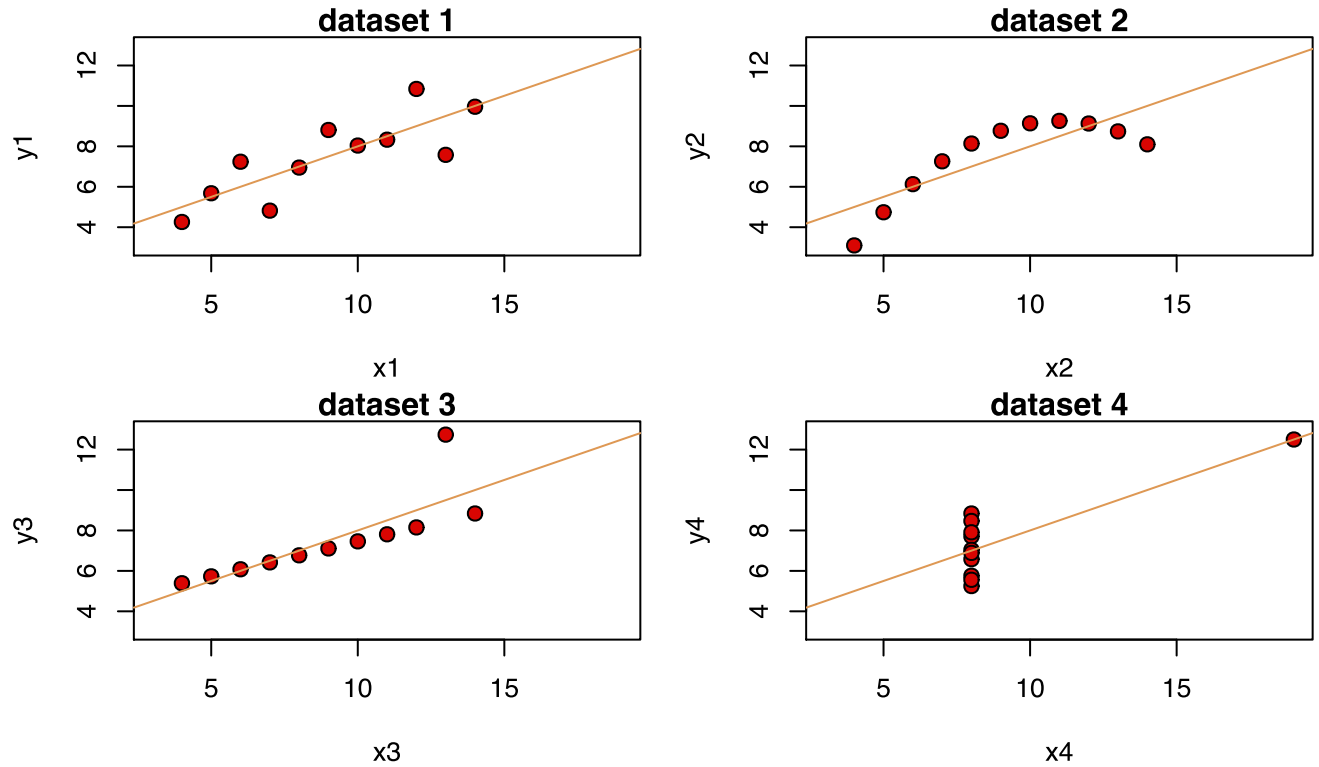

dataset	cov	var(y)	var(x)
1	5.501	4.127	11
2	5.500	4.128	11
3	5.497	4.123	11
4	5.499	4.123	11

Nonlinear Relationships in Data?

- We can accomodate non-linear relationships in regressions.
- Just add a *higher order* term like this:

$$y_i = b_0 + b_1 x_i + b_2 x_i^2 + e_i$$

- This is **multiple regression** (in 2 weeks!)

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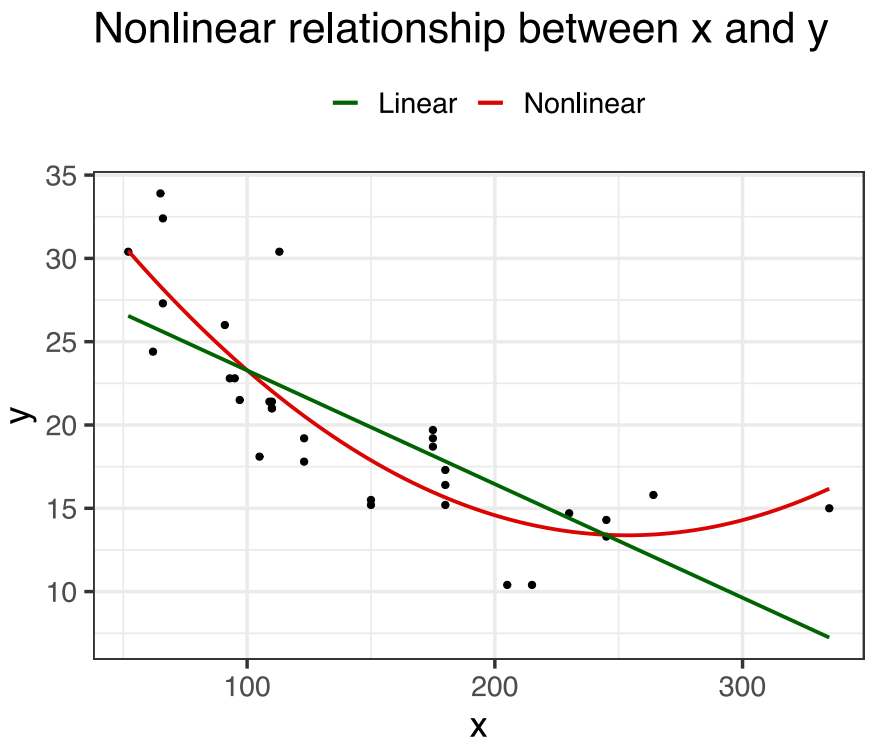
- For example, suppose we had this data and fit the previous regression model:



Analysis of Variance

- Remember that $y_i = \hat{y}_i + e_i$.
- We have the following decomposition:

$$\begin{aligned} Var(y) &= Var(\hat{y} + e) \\ &= Var(\hat{y}) + Var(e) + 2Cov(\hat{y}, e) \\ &= Var(\hat{y}) + Var(e) \end{aligned}$$

- Because:
  - $Var(x + y) = Var(x) + Var(y) + 2Cov(x, y)$
  - $Cov(\hat{y}, e) = 0$
- **Total variation (SST) = Model explained (SSE) + Unexplained (SSR)**

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- *Interpretation:* an R^2 of 0.5, for example, means that the variation in x "explains" 50% of the variation in y.
- ⚠ Low R^2 does **NOT** mean it's a useless model! Remember that econometrics is interested in causal mechanisms, not prediction!

Task 3: R^2 and goodness of fit

10:00

1. Regress `avgmath_cs` on `classize`. Assign to an object `math_reg`.

2. Pass `math_reg` in the `summary()` function. What is the (multiple) R^2 for this regression? How can you interpret it?

3. Compute the squared correlation between `classize` and `avgmath_cs`. What does this tell you about the relationship between R^2 and the correlation in a regression with only one regressor?

4. Install and load the `broom` package. Pass `math_reg` in the `augment()` function and assign it to a new object. Use the variance in `avgmath_cs` (SST) and the variance in `.fitted` (predicted values; SSE) to find the R^2 using the formula on the previous slide.

5. Repeat steps 1 and 2 for `avgverb_cs`. For which exam does the variance in class size explain more of the variance in students' scores?

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